Karnaugh Maps (K Maps)

Introduction

- Simplification of Boolean functions leads to simpler (and usually faster) digital circuits.
- Simplifying Boolean functions using identities is time-consuming and error-prone.
- K Maps are an easy and systematic method for reducing Boolean expressions.

Introduction (cont.)

- In 1953, Maurice Karnaugh was a Telecommunications Engineer at Bell Labs.
- While exploring the new field of digital logic and its application to the design of telephone circuits, he invented a graphical way of visualizing and then simplifying Boolean expressions.
- This graphical representation, now known as a Karnaugh map, or K Map, is named in his honor.

Description of K Maps and Terminology

- A K Map is a matrix consisting of rows and columns that represent the output values of a Boolean function.
- The output values placed in each cell are derived from the minterms of a Boolean function.
- A *minterm* is a product term that contains all of the function’s variables exactly once, either complemented or uncomplemented.

Description of K Maps and Terminology (cont.)

- For example, the minterms for a function having the inputs *x* and *y* are: $\overline{x}\overline{y}, \overline{x}y, x\overline{y},$ and xy

Minterm	X	Y
$\overline{X}\overline{Y}$	0	0
$\overline{X}Y$	0	1
$X\overline{Y}$	1	0
XY	1	1

Description of K Maps and Terminology (cont.)

- Similarly, a function having three inputs, has the minterms that are shown in this diagram.

Minterm	X	Y	Z
$\bar{X}\bar{Y}\bar{Z}$	0	0	0
$\bar{X}\bar{Y}Z$	0	0	1
$\bar{X}Y\bar{Z}$	0	1	0
$\bar{X}YZ$	0	1	1
$X\bar{Y}\bar{Z}$	1	0	0
$X\bar{Y}Z$	1	0	1
$XY\bar{Z}$	1	1	0
XYZ	1	1	1

Description of K Maps and Terminology (cont.)

- A K Map has a cell for each minterm.
- This means that it has a cell for each line for the truth table of a function.
- The truth table for the function *F(x,y) = xy* is shown at the right along with its corresponding K Map.

F(X,Y) = XY

X	Y	XY
0	0	0
0	1	0
1	0	0
1	1	1

X \ Y	0	1
0	0	0
1	0	1

Description of K Maps and Terminology (cont.)

- As another example, we give the truth table and K Map for the function, $F(x,y) = x + y$ at the right.
- This function is equivalent to the OR of all of the minterms that have a value of 1. Thus:

$$F(x,y) = x + y = \bar{x}y + x\bar{y} + xy$$

F(X,Y) = X+Y

X	Y	X+Y
0	0	0
0	1	1
1	0	1
1	1	1

X \ Y	0	1
0	0	1
1	1	1

K Map Simplification for Two Variables

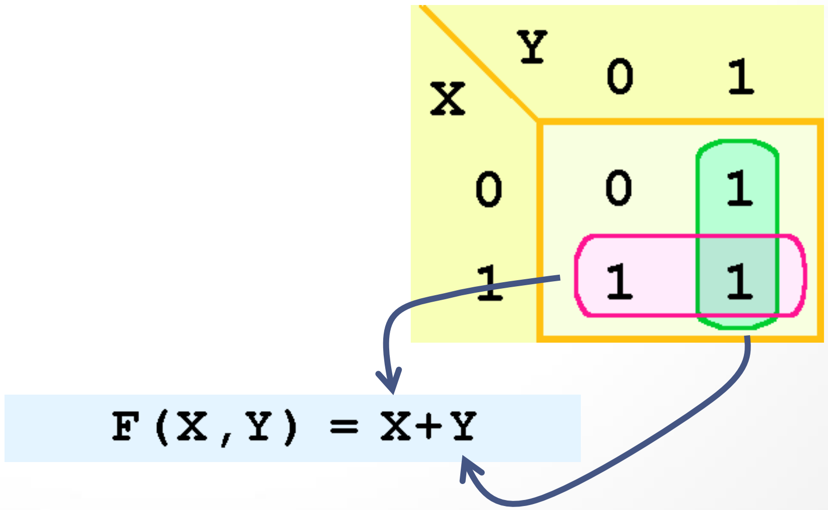
- Of course, the minterm function that we derived from our K Map was not in simplest terms.
- We can, however, reduce our complicated expression to its simplest terms by finding adjacent 1s in the K Map that can be collected into groups that are powers of two.

- In our example, we have two such groups.
  - Can you find them?

X \ Y	0	1
0	0	1
1	1	1

K Map Simplification for Two Variables (cont.)

- The best way of selecting two groups of 1s form our simple K Map is shown below.
- We see that both groups are powers of two and that the groups overlap.
- The next slide gives guidance for selecting K Map groups.



K Map Simplification for Two Variables (cont.)

The rules of K Map simplification are:

- Groupings can contain only 1s (no 0s) or only 0s (no 1s) .
- Groups can be formed only at right angles; diagonal groups are not allowed.
- The number of 1s in a group must be a power of 2 – even if it contains a single 1.
- The groups must be made as large as possible.
- Groups can overlap and wrap around the sides of the K Map.

K Map Simplification for Three Variables

- A K Map for three variables is constructed as shown in the diagram below.
- We have placed each minterm in the cell that will hold its value.
  - o Notice that the values for the *yz* combination at the top of the matrix form a pattern that is not a normal binary sequence.

x \ yz	00	01	11	10
0	$\overline{X}\overline{Y}\overline{Z}$	$\overline{X}\overline{Y}Z$	$\overline{X}YZ$	$\overline{X}Y\overline{Z}$
1	$X\overline{Y}\overline{Z}$	$X\overline{Y}Z$	XYZ	$XY\overline{Z}$

K Map Simplification for Three Variables (cont.)

- Thus, the first row of the K Map contains all minterms where *x* has a value of zero.
- The first column contains all minterms where *y* and *z* both have a value of zero.

x \ yz	00	01	11	10
0	$\overline{x}\,\overline{y}\,\overline{z}$	$\overline{x}\,\overline{y}\,z$	$\overline{x}\,y\,z$	$\overline{x}\,y\,\overline{z}$
1	$x\,\overline{y}\,\overline{z}$	$x\,\overline{y}\,z$	$x\,y\,z$	$x\,y\,\overline{z}$

K Map Simplification for Three Variables (cont.)

- Consider the function:

$$F(X,Y) = \overline{X}\,\overline{Y}Z + \overline{X}YZ + X\overline{Y}Z + XYZ$$

- Its K Map is given below.
  - o What is the largest group of 1s that is a power of 2?

X \ YZ	00	01	11	10
0	0	1	1	0
1	0	1	1	0

K Map Simplification for Three Variables (cont.)

- This means that the function,

$$F(X,Y) = \overline{X}\,\overline{Y}Z + \overline{X}YZ + X\overline{Y}Z + XYZ$$

reduces to *F(x) = z.*

You could verify this reduction with identities or a truth table.

X \ YZ	00	01	11	10
0	0	1	1	0
1	0	1	1	0

K Map Simplification for Three Variables (cont.)

- Now for a more complicated K Map. Consider the function:

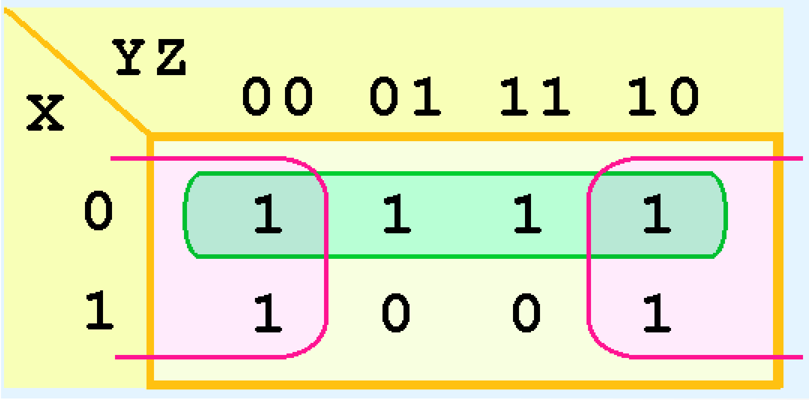
$$F(X,Y,Z)=\bar{X}\bar{Y}\bar{Z}+\bar{X}\bar{Y}Z+\bar{X}YZ+\bar{X}Y\bar{Z}+X\bar{Y}\bar{Z}+XY\bar{Z}$$

- Its K Map is shown below. There are (only) two groupings of 1s.
  - o Can you find them?

X \ YZ	00	01	11	10
0	1	1	1	1
1	1	0	0	1

K Map Simplification for Three Variables (cont.)

- In this K Map, we see an example of a group that wraps around the sides of a K Map.

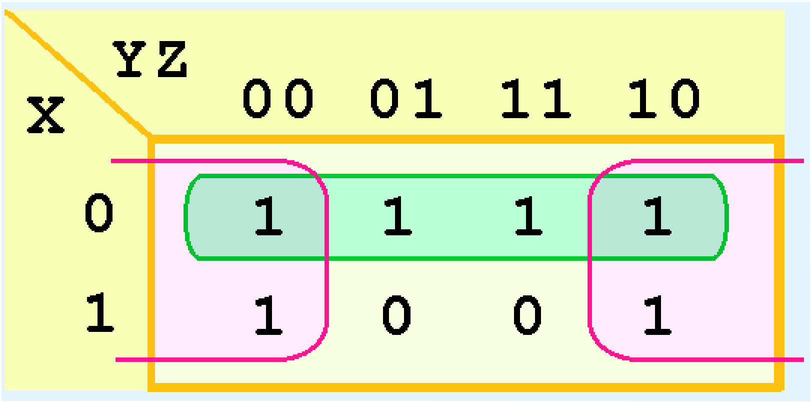


K Map Simplification for Three Variables (cont.)

- Our reduced function is:

$$F(X,Y,Z) = \overline{X} + \overline{Z}$$

Recall that we had six minterms in our original function!



K Map Simplification for Four Variables

- Our model can be extended to accommodate the 16 minterms that are produced by a four-input function.
- This is the format for a 16-minterm K Map.

WX \ YZ	00	01	11	10
00	$\overline{W}\,\overline{X}\,\overline{Y}\,\overline{Z}$	$\overline{W}\,\overline{X}\,\overline{Y}Z$	$\overline{W}\,\overline{X}YZ$	$\overline{W}\,\overline{X}Y\overline{Z}$
01	$\overline{W}X\overline{Y}\,\overline{Z}$	$\overline{W}X\overline{Y}Z$	$\overline{W}XYZ$	$\overline{W}XY\overline{Z}$
11	$WX\overline{Y}\,\overline{Z}$	$WX\overline{Y}Z$	$WXYZ$	$WXY\overline{Z}$
10	$W\overline{X}\,\overline{Y}\,\overline{Z}$	$W\overline{X}\,\overline{Y}Z$	$W\overline{X}YZ$	$W\overline{X}Y\overline{Z}$

K Map Simplification for Four Variables (cont.)

- We have populated the K Map shown below with the nonzero minterms from the function:

$$F(W,X,Y,Z) = \overline{W}\,\overline{X}\,\overline{Y}\,\overline{Z} + \overline{W}\,\overline{X}\,\overline{Y}Z + \overline{W}\,\overline{X}Y\overline{Z} + \overline{W}XY\overline{Z} + W\overline{X}\,\overline{Y}\,\overline{Z} + W\overline{X}\,\overline{Y}Z + W\overline{X}Y\overline{Z}$$

  - Can you identify (only) three groups in this K Map?

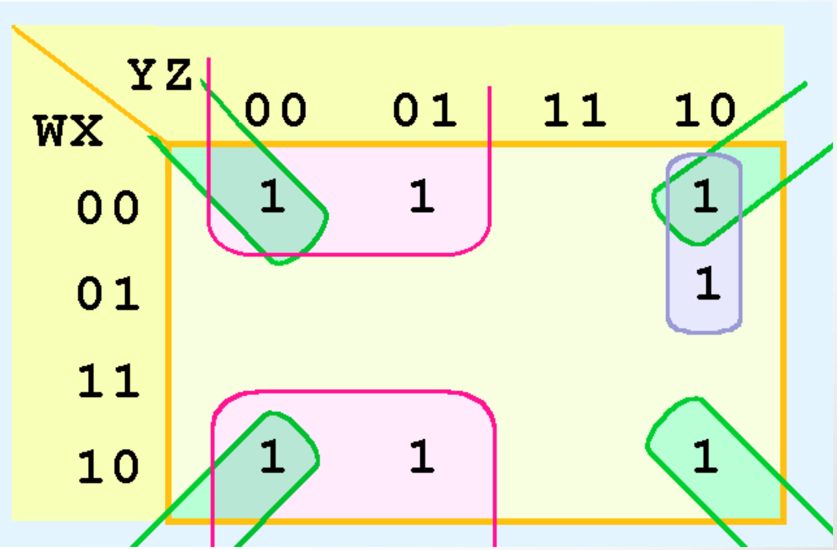
Recall that groups can overlap.

WX \ YZ	00	01	11	10
00	1	1		1
01				1
11				
10	1	1		1

K Map Simplification for Four Variables (cont.)

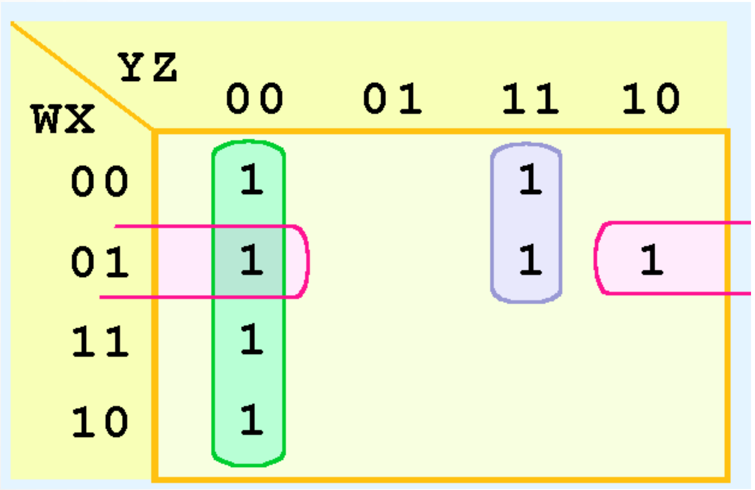
- Our three groups consist of:
  - A purple group entirely within the Kmap at the right.
  - A pink group that wraps the top and bottom.
  - A green group that spans the corners.
- Thus we have three terms in our final function:

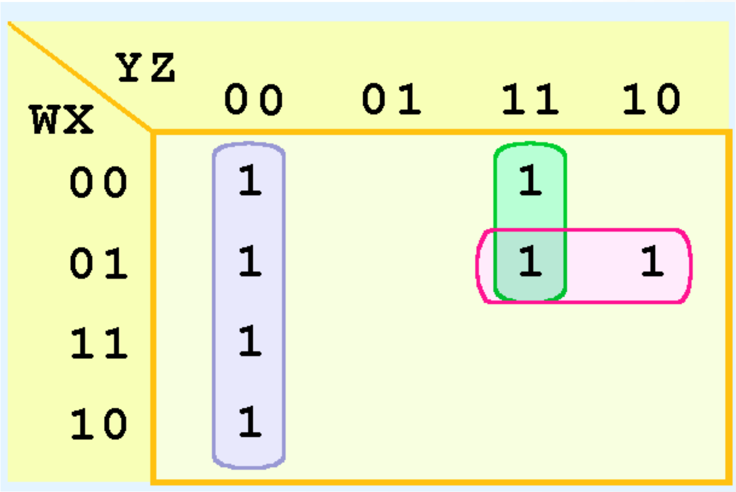
$$F(W,X,Y,Z) = \bar{W}\bar{Y} + \bar{X}\bar{Z} + \bar{W}Y\bar{Z}$$



K Map Simplification for Four Variables (cont.)

- It is possible to have a choice as to how to pick groups within a Kmap, while keeping the groups as large as possible.
- The (different) functions that result from the groupings below are logically equivalent.





Don't Care Conditions

- Real circuits don't always need to have an output defined for every possible input.
  - o For example, some calculator displays consist of 7-segment LEDs. These LEDs can display 2^7 -1 patterns, but only ten of them are useful.
- If a circuit is designed so that a particular set of inputs can never happen, we call this set of inputs a *don't care* condition.
- They are very helpful to us in K Map circuit simplification.

Don’t Care Conditions

- In a K Map, a don’t care condition is identified by an *X or d* in the cell of the minterm(s) for the don’t care inputs, as shown below.
- In performing the simplification, we are free to include or ignore the *X*’s when creating our groups.

WX \ YZ	00	01	11	10
00	X	1	1	X
01		X	1	
11	X		1	
10			1	

Don’t Care Conditions

- In one grouping in the K Map below, we have the function:

$$F(W,X,Y,Z) = \overline{W}\,\overline{Y} + YZ$$

WX \ YZ	00	01	11	10
00	X	1	1	X
01		X	1	
11	X		1	
10			1	

Don’t Care Conditions

- A different grouping gives us the function:

$$F(W,X,Y,Z) = \bar{W}Z + YZ$$

WX \ YZ	00	01	11	10
00	X	1	1	X
01		X	1	
11	X		1	
10			1	

Don't Care Conditions

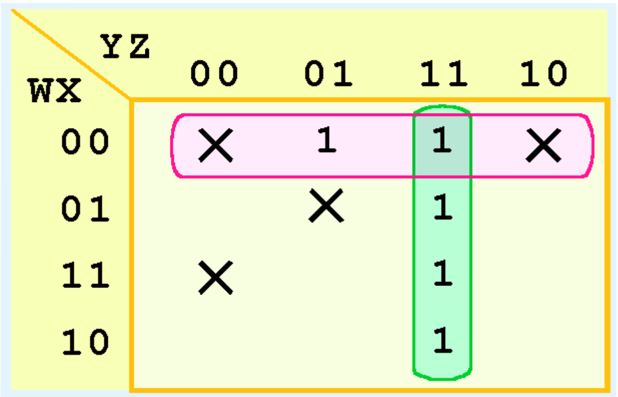
- The truth table of:

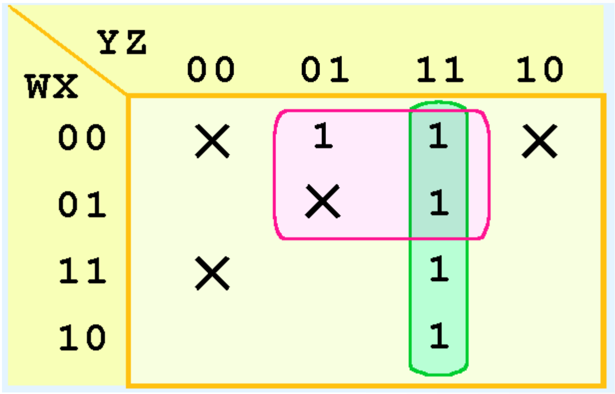
$$F(W,X,Y,Z) = \overline{W}\,\overline{Y} + YZ$$

differs from the truth table of:

$$F(W,X,Y,Z) = \overline{W}Z + YZ$$

- However, the values for which they differ, are the inputs for which we have don't care conditions.





Conclusion

- K Maps provide an easy graphical method of simplifying Boolean expressions.
- A K Map is a matrix consisting of the outputs of the minterms of a Boolean function.
- we have discussed 2- 3- and 4-input K Maps. This method can be extended to any number of inputs through the use of multiple tables.

Conclusion (cont.)

Recapping the rules of Kmap simplification:

- Groupings can contain only 1s (no 0s) or only 0s (no 1s) .
- Groups can be formed only at right angles; diagonal groups are not allowed.
- The number of 1s in a group must be a power of 2 – even if it contains a single 1.
- The groups must be made as large as possible.
- Groups can overlap and wrap around the sides of the Kmap.
- Use don’t care conditions when you can.

More Examples…

Example 1

Derive the minimum-cost product-of-sums expression for the following function

$$f(x_1,x_2,x_3) = \Sigma_m(0,2,4,5,6,7)$$

Minterms and Maxterms (with three variables)

Row number	x_1	x_2	x_3	Minterm	Maxterm
0	0	0	0	$m_0 = \overline{x}_1\overline{x}_2\overline{x}_3$	$M_0 = x_1 + x_2 + x_3$
1	0	0	1	$m_1 = \overline{x}_1\overline{x}_2 x_3$	$M_1 = x_1 + x_2 + \overline{x}_3$
2	0	1	0	$m_2 = \overline{x}_1 x_2\overline{x}_3$	$M_2 = x_1 + \overline{x}_2 + x_3$
3	0	1	1	$m_3 = \overline{x}_1 x_2 x_3$	$M_3 = x_1 + \overline{x}_2 + \overline{x}_3$
4	1	0	0	$m_4 = x_1\overline{x}_2\overline{x}_3$	$M_4 = \overline{x}_1 + x_2 + x_3$
5	1	0	1	$m_5 = x_1\overline{x}_2 x_3$	$M_5 = \overline{x}_1 + x_2 + \overline{x}_3$
6	1	1	0	$m_6 = x_1 x_2\overline{x}_3$	$M_6 = \overline{x}_1 + \overline{x}_2 + x_3$
7	1	1	1	$m_7 = x_1 x_2 x_3$	$M_7 = \overline{x}_1 + \overline{x}_2 + \overline{x}_3$

Minterms and Maxterms (with three variables)

Row number	x_1	x_2	x_3	Minterm	Maxterm
0	0	0	0	$m_0 = \overline{x}_1\overline{x}_2\overline{x}_3$	$M_0 = x_1 + x_2 + x_3$
1	0	0	1	$m_1 = \overline{x}_1\overline{x}_2 x_3$	$M_1 = x_1 + x_2 + \overline{x}_3$
2	0	1	0	$m_2 = \overline{x}_1 x_2\overline{x}_3$	$M_2 = x_1 + \overline{x}_2 + x_3$
3	0	1	1	$m_3 = \overline{x}_1 x_2 x_3$	$M_3 = x_1 + \overline{x}_2 + \overline{x}_3$
4	1	0	0	$m_4 = x_1\overline{x}_2\overline{x}_3$	$M_4 = \overline{x}_1 + x_2 + x_3$
5	1	0	1	$m_5 = x_1\overline{x}_2 x_3$	$M_5 = \overline{x}_1 + x_2 + \overline{x}_3$
6	1	1	0	$m_6 = x_1 x_2\overline{x}_3$	$M_6 = \overline{x}_1 + \overline{x}_2 + x_3$
7	1	1	1	$m_7 = x_1 x_2 x_3$	$M_7 = \overline{x}_1 + \overline{x}_2 + \overline{x}_3$

The function is 1 for these rows

Minterms and Maxterms (with three variables)

Row number	x_1	x_2	x_3	Minterm	Maxterm
0	0	0	0	$m_0 = \overline{x}_1\overline{x}_2\overline{x}_3$	$M_0 = x_1 + x_2 + x_3$
1	0	0	1	$m_1 = \overline{x}_1\overline{x}_2 x_3$	$M_1 = x_1 + x_2 + \overline{x}_3$
2	0	1	0	$m_2 = \overline{x}_1 x_2\overline{x}_3$	$M_2 = x_1 + \overline{x}_2 + x_3$
3	0	1	1	$m_3 = \overline{x}_1 x_2 x_3$	$M_3 = x_1 + \overline{x}_2 + \overline{x}_3$
4	1	0	0	$m_4 = x_1\overline{x}_2\overline{x}_3$	$M_4 = \overline{x}_1 + x_2 + x_3$
5	1	0	1	$m_5 = x_1\overline{x}_2 x_3$	$M_5 = \overline{x}_1 + x_2 + \overline{x}_3$
6	1	1	0	$m_6 = x_1 x_2\overline{x}_3$	$M_6 = \overline{x}_1 + \overline{x}_2 + x_3$
7	1	1	1	$m_7 = x_1 x_2 x_3$	$M_7 = \overline{x}_1 + \overline{x}_2 + \overline{x}_3$

The function is 1 for these rows

The function is 0 for these rows

Two different ways to specify the same function f of three variables

$$f(x_1,x_2,x_3) = \Sigma_m(0,2,4,5,6,7)$$

$$f(x_1,x_2,x_3) = \Pi_M(1,3)$$

The Product of Maxterms Expression

$M_1 = x_1 + x_2 + \overline{x}_3$

$M_3 = x_1 + \overline{x}_2 + \overline{x}_3$

$$f(x_1,x_2,x_3) = \Pi_M(1, 3)$$

$$= M_1 \bullet M_3$$

$$= (x_1 + x_2 + \overline{x_3}) \bullet (x_1 + \overline{x_2} + \overline{x_3})$$

The Minimum POS Expression

$$
\begin{aligned}
f(x_1,x_2,x_3) &= (x_1 + x_2 + \overline{x_3})\bullet(x_1 + \overline{x_2} + \overline{x_3}) \\
&= (x_1 + \overline{x_3} + x_2)\bullet(x_1 + \overline{x_3} + \overline{x_2}) \\
&= (x_1 + \overline{x_3})
\end{aligned}
$$

Hint: Use the following Boolean Algebra theorem

$$(x + y)\bullet(x + \overline{y}) = x$$

Alternative Solution Using K Map

x_1	x_2	x_3	
0	0	0	m_0
0	0	1	m_1
0	1	0	m_2
0	1	1	m_3
1	0	0	m_4
1	0	1	m_5
1	1	0	m_6
1	1	1	m_7

(a) Truth table

x_3 \ x_1x_2	00	01	11	10
0	m_0	m_2	m_6	m_4
1	m_1	m_3	m_7	m_5

(b) Karnaugh map

Alternative Solution Using K Map

x_1	x_2	x_3	
0	0	0	m_0
0	0	1	m_1
0	1	0	m_2
0	1	1	m_3
1	0	0	m_4
1	0	1	m_5
1	1	0	m_6
1	1	1	m_7

(a) Truth table

x_3 \ x_1x_2	00	01	11	10
0	m_0	m_2	m_6	m_4
1	m_1	m_3	m_7	m_5

(b) Karnaugh map

Alternative Solution Using K Map

x_1	x_2	x_3	
0	0	0	m_0
0	0	1	m_1
0	1	0	m_2
0	1	1	m_3
1	0	0	m_4
1	0	1	m_5
1	1	0	m_6
1	1	1	m_7

(a) Truth table

x_3 \ x_1x_2	00	01	11	10
0	m_0	m_2	m_6	m_4
1	m_1	m_3	m_7	m_5

(b) Karnaugh map

Alternative Solution
Using K Map

$x_3 \backslash x_1x_2$	00	01	11	10
0	m_0	m_2	m_6	m_4
1	m_1	m_3	m_7	m_5

Alternative Solution Using K Map

$x_3 \backslash x_1x_2$	00	01	11	10
0	1	1	1	1
1	0	0	1	1

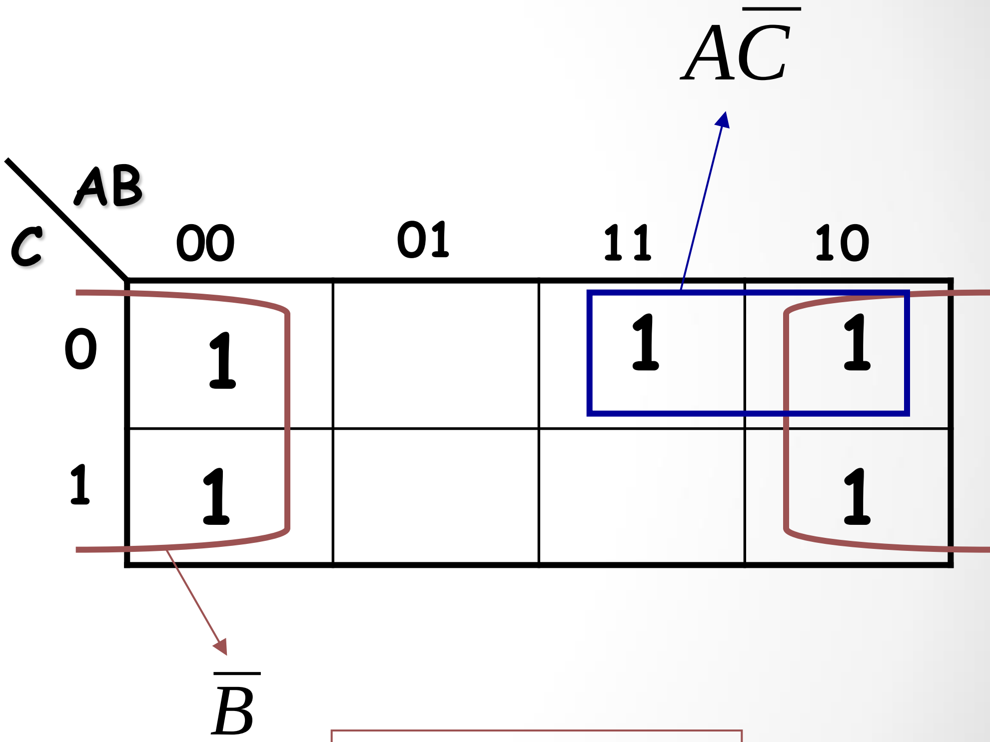
Alternative Solution Using K Map

x_3 \ x_1x_2	00	01	11	10
0	1	1	1	1
1	0	0	1	1

$(x_1 + \overline{x}_3)$

Example 2

A	B	C	Y
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0



$$\overline{B} + A\overline{C}$$

Example 3

Derive the minimum-cost SOP and POS expression for the function

$$f(x_1,x_2,x_3,x_4) = \Sigma_m(4,6,8,10,11,12,15) + d(3,5,7,9)$$

Let's Use a K Map

$$f(x_1,x_2,x_3,x_4) = \Sigma_m(4,6,8,10,11,12,15) + d(3,5,7,9)$$

x_3x_4 \ x_1x_2	00	01	11	10
00	m_0	m_4	m_{12}	m_8
01	m_1	m_5	m_{13}	m_9
11	m_3	m_7	m_{15}	m_{11}
10	m_2	m_6	m_{14}	m_{10}

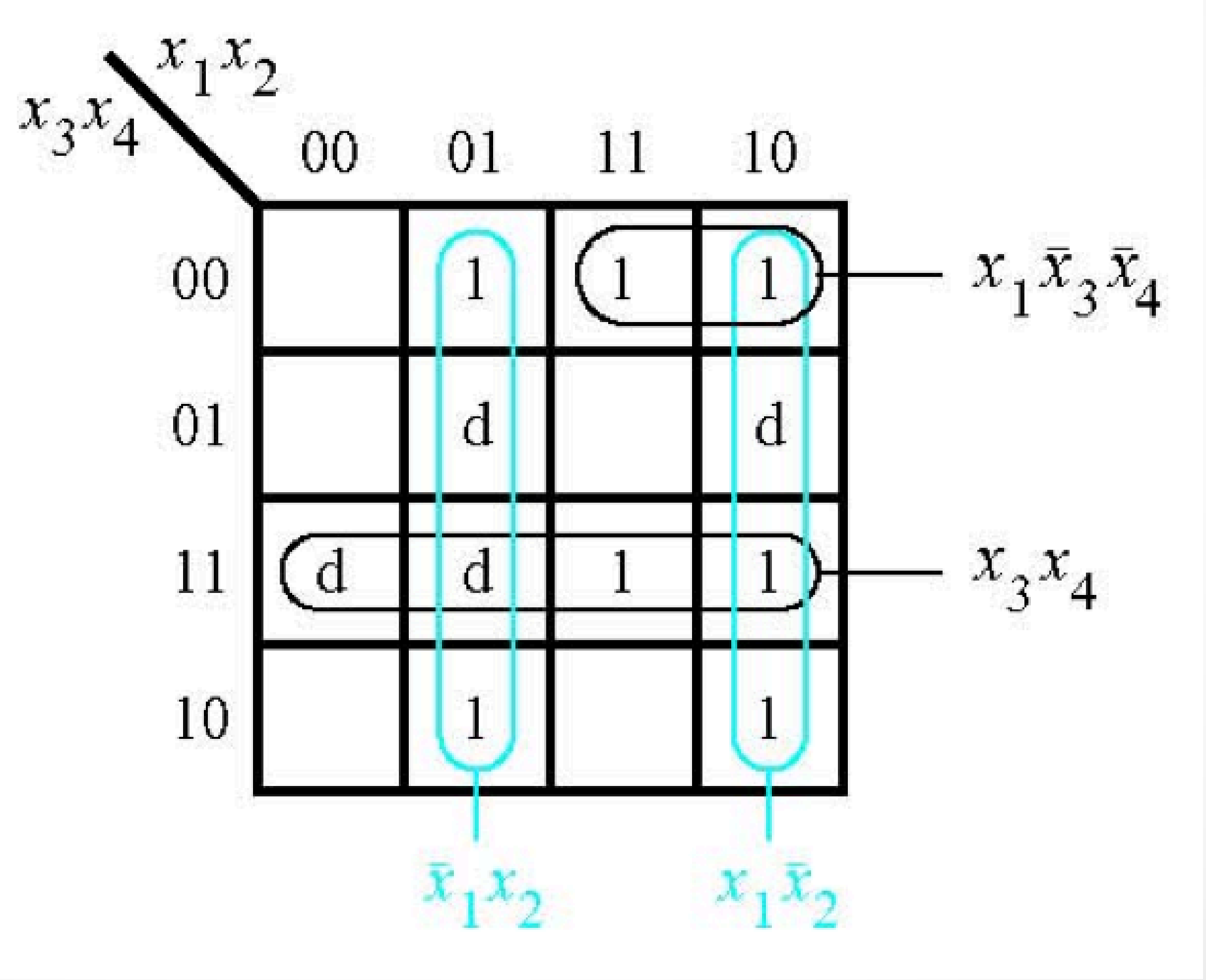
x_1 x_2 x_3 x_4

Let’s Use a K Map

$$f(x_1,x_2,x_3,x_4) = \Sigma_m(4,6,8,10,11,12,15) + d(3,5,7,9)$$

x_3x_4 \ x_1x_2	00	01	11	10
00	0	1	1	1
01	0	d	0	d
11	d	d	1	1
10	0	1	0	1

The SOP Expression

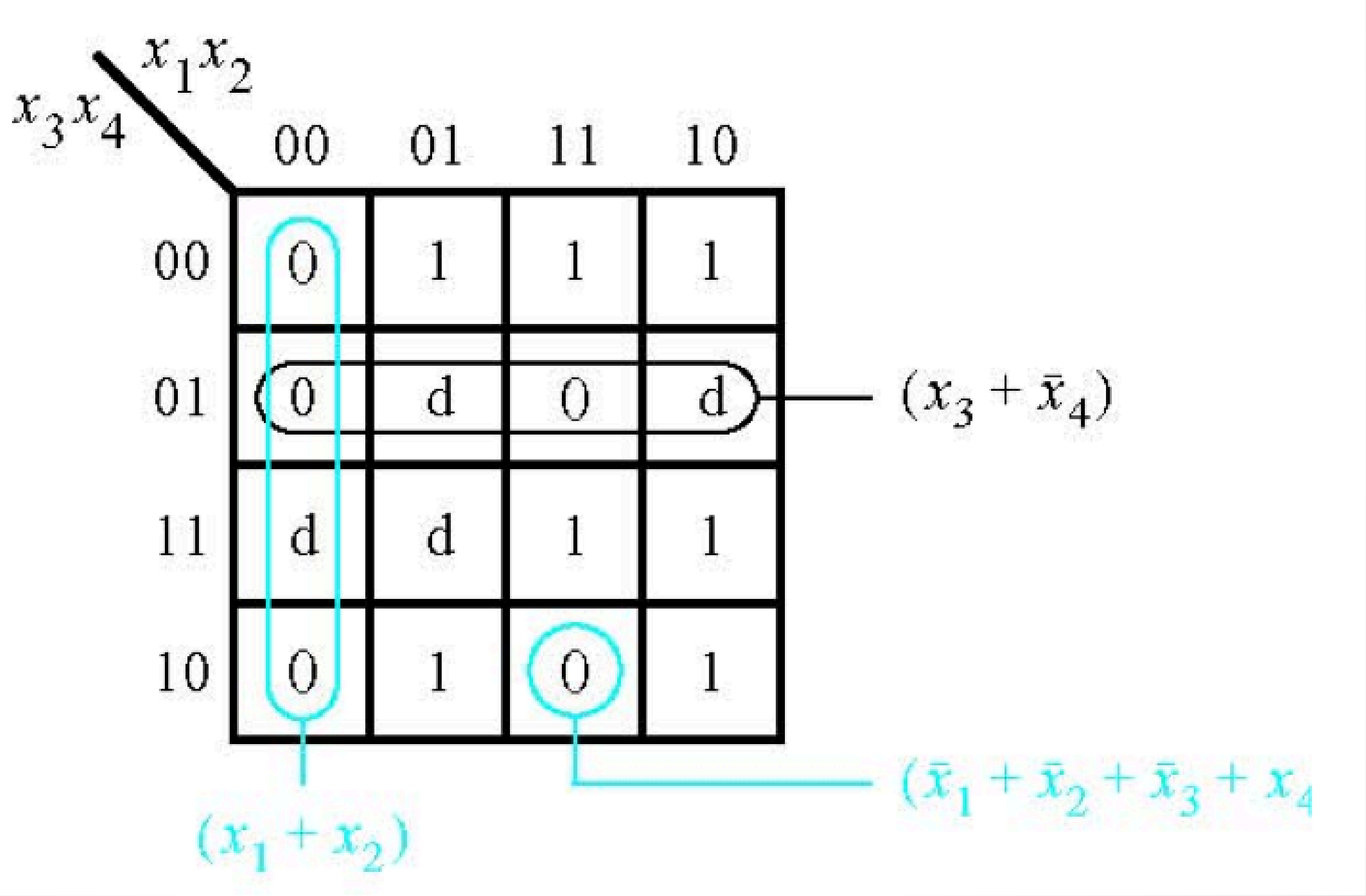


What about the POS Expression?

$$f(x_1,x_2,x_3,x_4) = \Sigma_m(4,6,8,10,11,12,15) + d(3,5,7,9)$$

x_3x_4 \ x_1x_2	00	01	11	10
00	0	1	1	1
01	0	d	0	d
11	d	d	1	1
10	0	1	0	1

The POS Expression



Example 4

Use K Map to find the minimum-cost SOP and POS expression for the function

$$f(x_1, \ldots, x_4) = \bar{x}_1\bar{x}_3\bar{x}_4 + x_3x_4 + \bar{x}_1\bar{x}_2x_4 + x_1x_2\bar{x}_3x_4$$

assuming that there are also don't-cares defined as $D = \sum(9, 12, 14)$.

Let's map the expression to the K Map

$$f(x_1, \ldots, x_4) = \bar{x}_1\bar{x}_3\bar{x}_4 + x_3x_4 + \bar{x}_1\bar{x}_2x_4 + x_1x_2\bar{x}_3x_4$$

$$D = \sum(9, 12, 14).$$

x_3x_4 \ x_1x_2	00	01	11	10
00	m_0	m_4	m_{12}	m_8
01	m_1	m_5	m_{13}	m_9
11	m_3	m_7	m_{15}	m_{11}
10	m_2	m_6	m_{14}	m_{10}

Let's map the expression to the K Map

$$f(x_1, \ldots, x_4) = \bar{x}_1\bar{x}_3\bar{x}_4 + x_3x_4 + \bar{x}_1\bar{x}_2x_4 + x_1x_2\bar{x}_3x_4$$

$$D = \sum(9, 12, 14).$$

$x_3x_4 \backslash x_1x_2$	00	01	11	10
00	m_0	m_4	d	m_8
01	m_1	m_5	m_{13}	d
11	m_3	m_7	m_{15}	m_{11}
10	m_2	m_6	d	m_{10}

x_1 (columns 11, 10); x_2 (columns 01, 11); x_3 (rows 11, 10); x_4 (rows 01, 11)

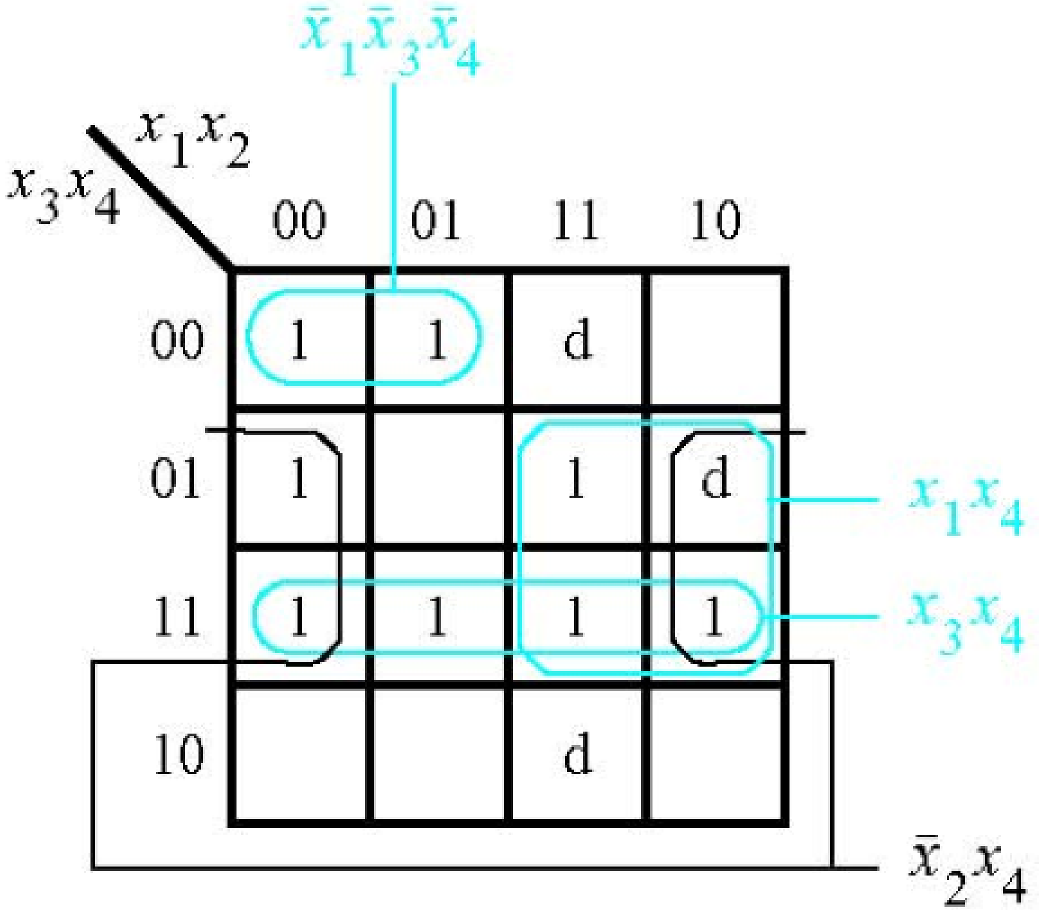
Let’s map the expression to the K Map

$$f(x_1, \ldots, x_4) = \bar{x}_1\bar{x}_3\bar{x}_4 + x_3x_4 + \bar{x}_1\bar{x}_2x_4 + x_1x_2\bar{x}_3x_4$$

$x_3x_4 \backslash x_1x_2$	00	01	11	10
00	m_0	m_4	d	m_8
01	m_1	m_5	m_{13}	d
11	m_3	m_7	m_{15}	m_{11}
10	m_2	m_6	d	m_{10}

The SOP Expression

$$f(x_1, \ldots, x_4) = \bar{x}_1\bar{x}_3\bar{x}_4 + x_3x_4 + \bar{x}_1\bar{x}_2x_4 + x_1x_2\bar{x}_3x_4$$



$$f = x_3x_4 + \bar{x}_1\bar{x}_3\bar{x}_4 + \bar{x}_2x_4 + x_1x_4$$

What about the POS Expression?

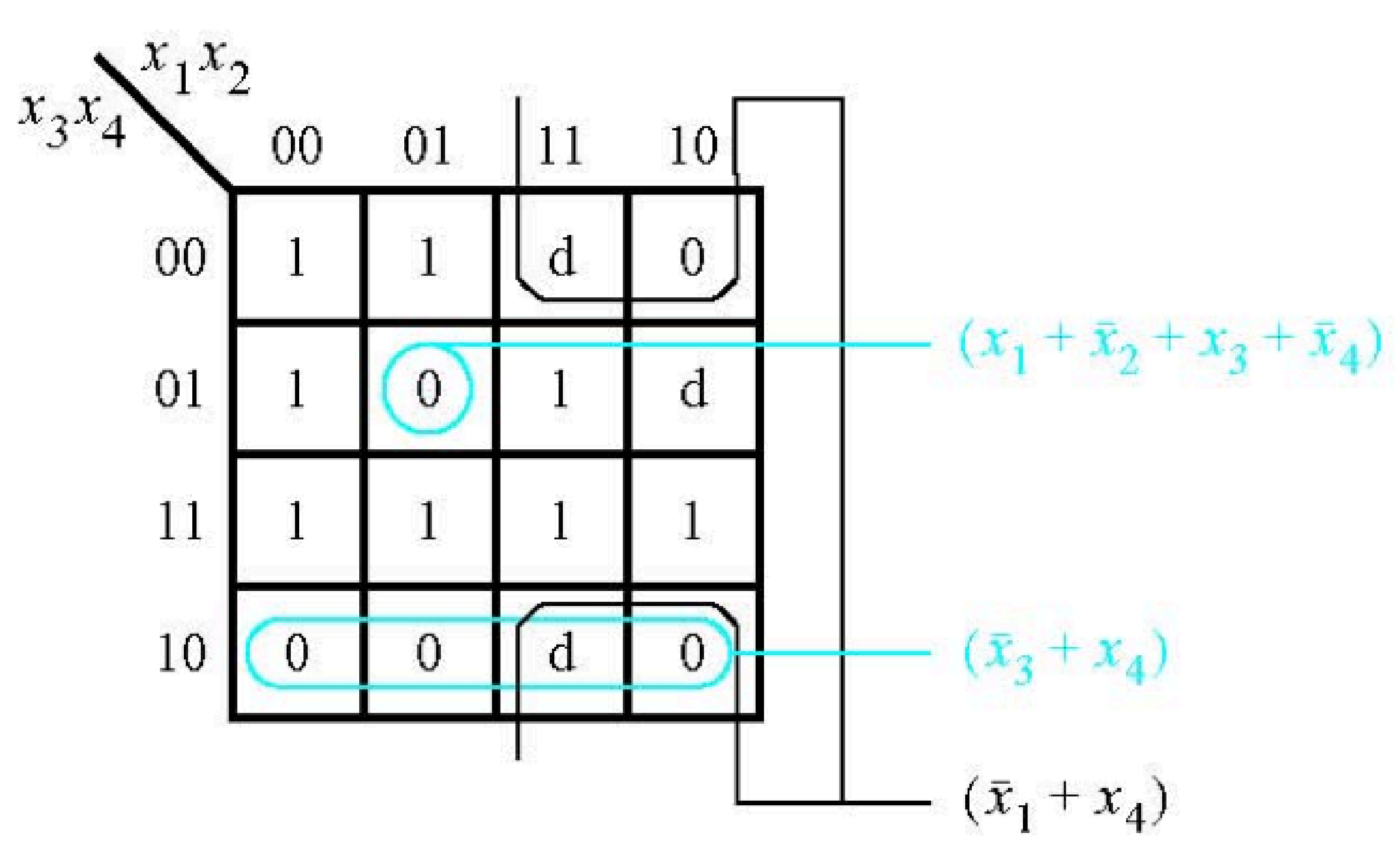
x_1

x_3x_4 \ x_1x_2	00	01	11	10
00	1	1	d	0
01	1	0	1	d
11	1	1	1	1
10	0	0	d	0

x_3 x_4

x_2

The POS Expression



$$f = (\bar{x}_3 + x_4)(\bar{x}_1 + x_4)(x_1 + \bar{x}_2 + x_3 + \bar{x}_4)$$