Digital Logic Design

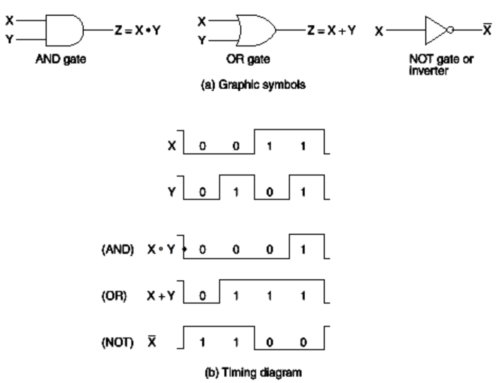
Chapter 2

Boolean Algebra and Logic Gates

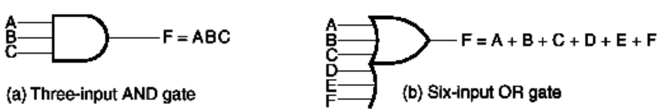
Combinational Logic Circuits

- Logic Gates: Control the flow of information
- Represent Logical Operations (Functions)
  - Inputs are like arguments to a function
  - Outputs are like result of the function
  - Fundamental Set
    - AND
    - OR
    - NOT
    - Transmission Gate
- Truth Tables…

Digital Logic Gates

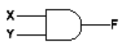
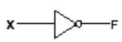
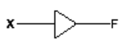


Gates with more than Two Inputs

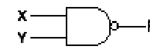
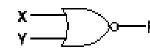
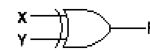
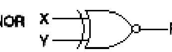


Digital Logic Gates

Graphics Symbols

Name	Distinctive shape	Algebraic equation	Truth table
AND		$F = XY$	X Y F 0 0 0 0 1 0 1 0 0 1 1 1
OR		$F = X + Y$	X Y F 0 0 0 0 1 1 1 0 1 1 1 1
NOT (inverter)		$F = \bar{X}$	X F 0 1 1 0
Buffer		$F = X$	X F 0 0 1 1

Graphics Symbols

Name	Distinctive shape	Rectangular shape	Algebraic equation	Truth table
NAND			$F = \overline{X \cdot Y}$	0 0 1 0 1 1 1 0 1 1 1 0
NOR			$F = \overline{X + Y}$	X Y F 0 0 1 0 1 0 1 0 0 1 1 0
Exclusive-OR (XOR)			$F = X\bar{Y} + \bar{X}Y$ $= X \oplus Y$	X Y F 0 0 0 0 1 1 1 0 1 1 1 0
Exclusive-NOR (XNOR)			$F = XY + \bar{X}\bar{Y}$ $= \overline{X \oplus Y}$	X Y F 0 0 1 0 1 0 1 0 0 1 1 1

Boolean Algebra

- Algebra is a complete set of rules defined on some variables.
- Variables can be Real or Logical: This subject deals with Logical Variables
- A Logical Variable can take one of two values
- A Logical Function is represented by
  - Truth Tables
  - Boolean Expressions

What is Boolean Algebra

- An algebra dealing with
  - Binary variables by alphabetical letters
  - Logic operations: OR, AND, XOR, etc
- Consider the following Boolean equation

$$F(X,Y,Z) = \overline{X \cdot Y} + \overline{Y \cdot \bar{Z} + Z}$$

- A Boolean function can be represented by a truth table which list all combinations of 1's and 0's for each binary value

Fundamental Operators

- NOT
  - Unary operator
  - Complements a Boolean variable represented as A', ~A, or $\bar{A}$
- OR
  - Binary operator
  - A ORed with B is represented as A + B
- AND
  - Binary operator
  - A ANDed with B is represented as AB or A·B
  - Can perform logical multiplication

Binary Boolean Operations

- All possible outcomes of a 2-input Boolean function

A	B	F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
		NULL	A·B		A		B	A⊕B	A+B	$\overline{A+B}$	$\overline{A \oplus B}$	$\bar{B}$		$\bar{A}$		$\overline{A \cdot B}$	Identity

Sixteen 2-Variable Functions

Boolean Function	Operator Symbol	Name	Comments
$F_0 = 0$		NULL	binary constant 0
$F_1 = xy$	$x \cdot y$	AND	x and y
$F_2 = xy'$	x/y	Inhibition	x, but not y
$F_3 = x$		Transfer	x
$F_4 = x'y$	y/x	Inhibition	y, but not x
$F_5 = y$		Transfer	y
$F_6 = xy' + x'y$	$x \oplus y$	Exclusive-OR	x or y, but not both
$F_7 = x + y$	$x + y$	OR	x or y
$F_8 = (x + y)'$	$x \downarrow y$	NOR	not-OR
$F_9 = xy + x'y'$	$(x \oplus y)'$	Equivalence (XNOR)	x equals y
$F_{10} = y'$	y'	Complement	not y
$F_{11} = x + y'$	$x \subset y$	Implication	if y, then x
$F_{12} = x'$	x'	Complement	not x
$F_{13} = x' + y$	$x \supset y$	Implication	if x, then y
$F_{14} = (xy)'$	$x \uparrow y$	NAND	not-AND
$F_{15} = 1$		Identity	Binary constant 1

Precedence of Operators

- Precedence of Operator Evaluation (Similar to decimal arithmetic)
  - () : Parentheses
  - NOT
  - AND
  - OR

$$F = A \cdot (\overline{B + \overline{C} \cdot D}) + \overline{A} \cdot \overline{\overline{B} + E}$$

❶ ❶ ❷ ❷ ❸ ❸ ❹ ❺

Function Evaluation

$$F = A \cdot \overline{(B + \overline{C} \cdot D)} + \overline{A} \cdot \overline{\overline{B} + E}$$

ABCDE=00000

$$F = 0 \cdot \overline{(0 + \overline{0} \cdot 0)} + \overline{0} \cdot \overline{\overline{0} + 0} = 0 \cdot \overline{(0 + 1 \cdot 0)} + \overline{0} \cdot \overline{1 + 0}$$

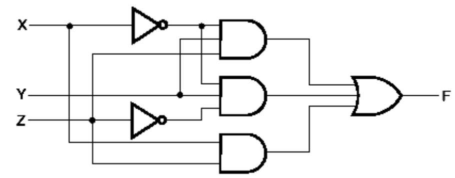
$$= 0 \cdot \overline{(0 + 0)} + 1 \cdot \overline{1} = 0 \cdot 1 + 1 \cdot 0 = 0$$

ABCDE=10000

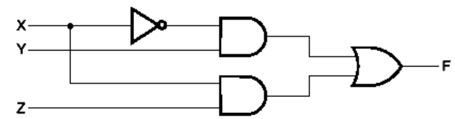
$$F = 1 \cdot \overline{(0 + \overline{0} \cdot 0)} + \overline{1} \cdot \overline{\overline{0} + 0} = 1 \cdot \overline{(0 + 1 \cdot 0)} + 0 \cdot \overline{1 + 0}$$

$$= 1 \cdot \overline{(0 + 0)} + 0 \cdot \overline{1} = 1 \cdot 1 + 0 \cdot 0 = 1$$

Implementation of Boolean Function with Gates



(a) $F = \overline{X}YZ + \overline{X}Y\overline{Z} + XZ$



(b) $F = \overline{X}Y + XZ$

Boolean Variables

- A multi-dimensional space spanned by a set of n Boolean variables is denoted by $\mathcal{B}^n$
- A **literal** is an instance (e.g. A) of a variable or its complement ($\bar{A}$)

Basic Identities of Boolean Algebra

$X + 0 = X$	(Identity)
$X + 1 = 1$	
$X + X = X$	(Idempotent Law)
$X + \overline{X} = 1$	(Complement)
$\overline{\overline{X}} = X$	(Involution Law)
$X + Y = Y + X$	(Commutative)
$X + (Y + Z) = (X + Y) + Z$	(Associative)
$X(Y + Z) = XY + XZ$	(Distributive)
$\overline{X + Y} = \overline{X}\,\overline{Y}$	(DeMorgan's Law)
$X + XY = X$	(Absorption Law)
$X + \overline{X}Y = X + Y$	(Simplification)
$XY + \overline{X}Z + YZ = XY + \overline{X}Z$	(Consensus Theorem)

Derivation of Simplification

$$X+\overline{X}Y$$
$$=X\cdot(1+Y)+\overline{X}Y$$
$$=X+XY+\overline{X}Y$$
$$=X+(X+\overline{X})Y$$
$$=X+Y$$
$$\therefore X+\overline{X}Y=X+Y$$

Derivation of Consensus Theorem

$$XY+\overline{X}Z+YZ$$
$$=XY+\overline{X}Z+YZ\cdot(X+\overline{X})$$
$$=XY+\overline{X}Z+XYZ+\overline{X}YZ$$
$$=XY(1+Z)+\overline{X}Z(1+Y)$$
$$=XY+\overline{X}Z$$
$$\therefore XY+\overline{X}Z+YZ=XY+\overline{X}Z$$

Duality Principle

- A Boolean equation remains valid if we take the dual of the expressions on both sides of the equals sign
- Dual of expressions
  - Interchange 1's and 0's
  - Interchange AND (•) and OR (+)

Duality Principle

$X+0=X$	$X\cdot 1=X$
$X+1=1$	$X\cdot 0=0$
$X+X=X$	$X\cdot X=X$
$X+\overline{X}=1$	$X\cdot\overline{X}=0$
$X+Y=Y+X$	$X\cdot Y=Y\cdot X$
$X(Y+Z)=XY+XZ$	$X+Y\cdot Z=(X+Y)\cdot(X+Z)$
$\overline{X+Y}=\overline{X}\cdot\overline{Y}$	$\overline{X\cdot Y}=\overline{X}+\overline{Y}$
$X+X\cdot Y=X$	$X\cdot(X+Y)=X$
$X+\overline{X}\cdot Y=X+Y$	$X\cdot(\overline{X}+Y)=X\cdot Y$
$XY+\overline{X}Z+YZ=XY+\overline{X}Z$	$(X+Y)(\overline{X}+Z)(Y+Z)=(X+Y)(\overline{X}+Z)$

DeMorgan's Law

$$\overline{A \cdot B} = \overline{A} + \overline{B}$$

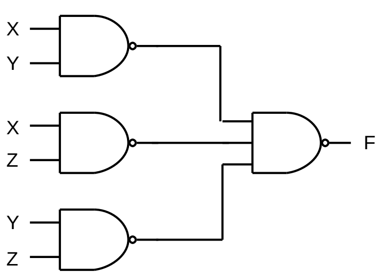
$$\overline{A + B} = \overline{A} \cdot \overline{B}$$

Generalized De Morgan's Theorem

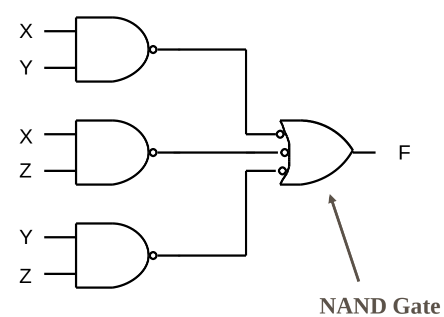
- NOT all variables
- Change . to + and + to .
- NOT the result

- F = X . Y + X . Z + Y . Z
- F = !((!X + !Y) . (!X + !Z) . (!Y + !Z))
- F = !(!(X . Y) . !(X . Z) . !(Y . Z))

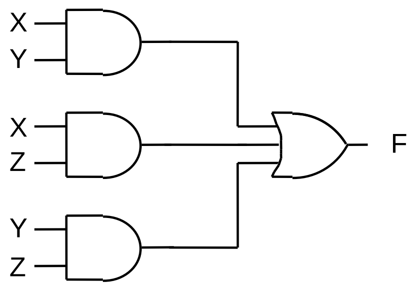
F = !(!(X & Y) & !(X & Z) & !(Y & Z))



F = !(!(X & Y) & !(X & Z) & !(Y & Z))



F = X & Y | X & Z | Y & Z



Sum of Product (SOP) Form

- A product of literals is called a product term (e.g. $\bar{A}{\cdot}B{\cdot}C$ in $\mathcal{B}^3$, or $B{\cdot}C$ in $\mathcal{B}^3$)
- Sum-Of-Product (SOP) Form: OR of product terms is called SOP. e.g. $\bar{A}B+AC$
- A minterm is a product term in which every literal (or variable) appears in $\mathcal{B}^n$
  - $\bar{A}BC$ is a minterm in $\mathcal{B}^3$ but not in $\mathcal{B}^4$. ABCD is a minterm in $\mathcal{B}^4$.
- A canonical (or standard) SOP function:
  - A sum of minterms, corresponding to the input combination of the truth table, for which the function produces a "1" output.

Minterms in $\mathcal{B}^3$

			m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7
A	B	C	$\bar{A}\bar{B}\bar{C}$	$\bar{A}\bar{B}C$	$\bar{A}B\bar{C}$	$\bar{A}BC$	$A\bar{B}\bar{C}$	$A\bar{B}C$	$AB\bar{C}$	ABC
0	0	0	1	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0	0	0	0
0	1	0	0	0	1	0	0	0	0	0
0	1	1	0	0	0	1	0	0	0	0
1	0	0	0	0	0	0	1	0	0	0
1	0	1	0	0	0	0	0	1	0	0
1	1	0	0	0	0	0	0	0	1	0
1	1	1	0	0	0	0	0	0	0	1

Canonical (Standard) SOP Function

$$F(A,B,C) = \bar{A}\bar{B}\bar{C} + \bar{A}\bar{B}C + A\bar{B}\bar{C} + A\bar{B}C$$
$$= m0 + m1 + m4 + m5$$
$$F(A,B,C) = \sum m(0,1,4,5) = one-set(0,1,4,5)$$
$$F(A,B,C,D) = \bar{A}B\bar{C}\bar{D} + A\bar{B}\bar{C}D + ABC\bar{D}$$
$$= m4 + m9 + m14$$
$$F(A,B,C,D) = \sum m(4,9,14) = one-set(4,9,14)$$

Product of Sums Design

Maxterms:
A maxterm is a NOT minterm
maxterm M0 = NOT minterm m0

```
If m0 = (!X.!Y)
M0 = !m0
   = !(!X . !Y)
   = !!X + !!Y
   = X + Y
```


Product of Sum (POS) form (dual of SOP form)

- A sum of literals is called a sum term (e.g. $\bar{A}+B+C$ in $\mathcal{B}^3$, or $(B+C)$ in $\mathcal{B}^3$)
- Product-Of-Sum (POS) Form: AND of sum terms is called POS. e.g. $(\bar{A}+B)(A+C)$
- A maxterm is a sum term in which every literal (or variable) appears in $\mathcal{B}^n$
  - $(\bar{A}+B+C)$ is a maxterm in $\mathcal{B}^3$ but not in $\mathcal{B}^4$. $A+B+C+D$ is a maxterm in $\mathcal{B}^4$.
- A canonical (or standard) POS function:
  - A product (AND) of maxterms, corresponding to the input combination of the truth table, for which the function produces a "0" output.

Product of Sums Design

X	Y	minterms	maxterms
0	0	m0 = !X . !Y	M0 = !m0 = X + Y
0	1	m1 = !X . Y	M1 = !m1 = X + !Y
1	0	m2 = X . !Y	M2 = !m2 = !X + Y
1	1	m3 = X . Y	M3 = !m3 = !X + !Y

Maxterms in $\mathcal{B}^3$

A	B	C	M_0 $A+B+C$	M_1 $A+B+\bar{C}$	M_2 $A+\bar{B}+C$	M_3 $A+\bar{B}+\bar{C}$	M_4 $\bar{A}+B+C$	M_5 $\bar{A}+B+\bar{C}$	M_6 $\bar{A}+\bar{B}+C$	M_7 $\bar{A}+\bar{B}+\bar{C}$
0	0	0	0	1	1	1	1	1	1	1
0	0	1	1	0	1	1	1	1	1	1
0	1	0	1	1	0	1	1	1	1	1
0	1	1	1	1	1	0	1	1	1	1
1	0	0	1	1	1	1	0	1	1	1
1	0	1	1	1	1	1	1	0	1	1
1	1	0	1	1	1	1	1	1	0	1
1	1	1	1	1	1	1	1	1	1	0

Canonical (Standard) POS Function

$$F(A,B,C) = (\overline{A}+\overline{B}+\overline{C})(\overline{A}+\overline{B}+C)(A+\overline{B}+\overline{C})(A+\overline{B}+C)$$
$$= M7 \cdot M6 \cdot M3 \cdot M2$$

$$F(A,B,C) = \prod M(2,3,6,7) = \text{zero } \mathit{for} \text{ set}(2,3,6,7)$$

$$F(A,B,C,D) = (\overline{A}+B+\overline{C}+\overline{D})(A+\overline{B}+\overline{C}+D)(A+B+C+\overline{D})$$
$$= M11 \cdot M6 \cdot M1$$

$$F(A,B,C,D) = \prod M(1,6,11) = \text{zero } \mathit{for} \text{ set}(1,6,11)$$

Convert a Boolean to Canonical SOP

- Expand the Boolean equation into a SOP
- Take each product term with a missing literal, say A, and "AND" (•) it with (A+Ā)

Convert a Boolean to Canonical SOP

$$F = \overline{A}\,\overline{B} + BC \text{ in } \mathcal{B}^3$$
$$\Rightarrow F(A,B,C) = \overline{A}\,\overline{B}\,\overline{C} + \overline{A}\,\overline{B}C + \overline{A}BC + ABC$$
$$= \sum m(0,1,3,7)$$

	A	B	C	F	
$\overline{A}\,\overline{B}\,\overline{C}$	0	0	0	1	0
$\overline{A}\,\overline{B}C$	0	0	1	1	1
$\overline{A}B\overline{C}$	0	1	0	0	
$\overline{A}BC$	0	1	1	1	3
$A\overline{B}\,\overline{C}$	1	0	0	0	
$A\overline{B}C$	1	0	1	0	
$AB\overline{C}$	1	1	0	0	
ABC	1	1	1	1	7

Minterms listed as 1's

Convert a Boolean to Canonical SOP

$$F = \overline{A}\,\overline{B} + BC \text{ in } \mathcal{B}^4$$
$$\Rightarrow F(A,B,C,D) = \overline{A}\,\overline{B}\,\overline{C}\,\overline{D} + \overline{A}\,\overline{B}\,\overline{C}D + \overline{A}\,\overline{B}C\overline{D} + \overline{A}\,\overline{B}CD$$
$$+ \overline{A}BC\overline{D} + \overline{A}BCD + ABC\overline{D} + ABCD$$
$$= \sum m(0,1,2,3,6,7,14,15)$$

$$F = AB + \overline{B}(\overline{A} + \overline{C}) \text{ in } \mathcal{B}^3$$
$$\Rightarrow F(A,B,C) = \overline{A}\,\overline{B}\,\overline{C} + \overline{A}\,\overline{B}C + A\overline{B}\,\overline{C} + AB\overline{C} + ABC$$
$$= \sum m(0,1,4,6,7)$$

Convert a Boolean to Canonical POS

- Expand Boolean eqn into a POS
  - Use distributive property
- Take each sum term with a missing literal, say A, and OR it with A·Ā

Convert a Boolean to Canonical POS

$$F = \overline{A}\,\overline{B} + BC \text{ in } B^3$$

$$\text{Use } X + YZ = (X+Y)(X+Z) \quad \text{(Distributive)}$$

$$F = \overline{A}\,\overline{B} + BC$$

$$F = (\overline{A}\,\overline{B} + B)(\overline{A}\,\overline{B} + C)$$

$$F = (\overline{A} + B)(\overline{B} + B)(\overline{A} + C)(\overline{B} + C)$$

$$F = (\overline{A} + B + C\overline{C})(\overline{A} + B\overline{B} + C)(A\overline{A} + \overline{B} + C)$$

$$F = (\overline{A} + B + C)(\overline{A} + B + \overline{C})(\overline{A} + B + C)(\overline{A} + \overline{B} + C)(A + \overline{B} + C)(\overline{A} + \overline{B} + C)$$

$$F = (\overline{A} + B + C)(\overline{A} + B + \overline{C})(\overline{A} + \overline{B} + C)(A + \overline{B} + C)$$

$$= \prod M(2, 4, 5, 6)$$

Convert a Boolean to Canonical POS

$$F = \overline{A}\,\overline{B} + BC \text{ in } \mathcal{B}^3$$

$$F = \overline{A}\,\overline{B} + BC$$

$$F = (A + \overline{B} + C)(\overline{A} + B + C)(\overline{A} + B + \overline{C})(\overline{A} + \overline{B} + C)$$

$$= \prod M(2,4,5,6)$$

	A	B	C	F	
$\overline{A}\,\overline{B}\,\overline{C}$	0	0	0	1	
$\overline{A}\,\overline{B}C$	0	0	1	1	
$\overline{A}B\overline{C}$	0	1	0	0	2
$\overline{A}BC$	0	1	1	1	
$A\overline{B}\,\overline{C}$	1	0	0	0	4
$A\overline{B}C$	1	0	1	0	5
$AB\overline{C}$	1	1	0	0	6
ABC	1	1	1	1	

Maxterms listed as 0's

Convert a Boolean to Canonical SOP

$$F = \overline{A}\,\overline{B} + BC \ \text{ in } \mathcal{B}^3$$

$$\Rightarrow F(A,B,C) = \overline{A}\,\overline{B}\,\overline{C} + \overline{A}\,\overline{B}C + \overline{A}BC + ABC$$

$$= \sum m(0,1,3,7)$$

	A	B	C	F	
$\overline{A}\,\overline{B}\,\overline{C}$	0	0	0	1	0
$\overline{A}\,\overline{B}C$	0	0	1	1	1
$\overline{A}B\overline{C}$	0	1	0	0	
$\overline{A}BC$	0	1	1	1	3
$A\overline{B}\,\overline{C}$	1	0	0	0	
$A\overline{B}C$	1	0	1	0	
$AB\overline{C}$	1	1	0	0	
ABC	1	1	1	1	7

Minterms listed as 1's

Convert a Boolean to Canonical POS

$F = AB + \overline{B}(\overline{A} + \overline{C})$ in $\mathcal{B}^3$

Use $X + YZ = (X + Y)(X + Z)$ (Distributive)

$$F = AB + \overline{B}(\overline{A} + \overline{C})$$
$$F = (AB + \overline{B})\,(AB + \overline{A} + \overline{C})$$
$$F = (A + \overline{B})(B + \overline{B})(A + \overline{A} + \overline{C})(B + \overline{A} + \overline{C})$$
$$F = (A + \overline{B})(\overline{A} + B + \overline{C})$$
$$F = (A + \overline{B} + C\overline{C})(\overline{A} + B + \overline{C})$$
$$F = (A + \overline{B} + C)(A + \overline{B} + \overline{C})(\overline{A} + B + \overline{C})$$
$$= \prod M(2,3,5)$$

Convert a Boolean to Canonical SOP

$F = AB + \overline{B}(\overline{A} + \overline{C})$ in $\mathcal{B}^3$

$$\Rightarrow F(A,B,C) = \overline{A}\,\overline{B}\,\overline{C} + \overline{A}\,\overline{B}C + A\overline{B}\,\overline{C} + AB\overline{C} + ABC$$
$$= \sum m(0,1,4,6,7)$$

Interchange Canonical SOP and POS

- For the same Boolean equation
  - Canonical SOP form is complementary to its canonical POS form
  - Use missing terms to interchange Σ and Π
- Examples
  - F(A,B,C) = Σ m(0,1,4,6,7)

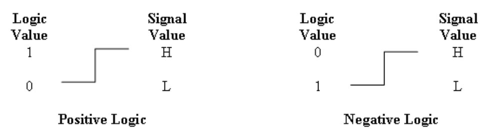
  Can be re-expressed by

  - F(A,B,C) = Π M(2,3,5)

    Where 2, 3, 5 are the missing minterms in the canonical SOP form

Positive and Negative Logic

- Binary signals in a circuit can have one of two values.
  - One signal represents logic-1 and the other logic-0.
- A circuit input or output will hold either a high or low signal.
  - Choosing the high level, H, to represent logic-1 is called a positive logic system.
  - Choosing the low level, L, to represent logic-1 is called a negative logic system



2-8 Integrated Circuits

- An integrated circuit (IC) is a silicon semiconductor crystal, called a chip, containing the electronic components for constructing digital gates.
  - Gates are interconnected within the chip to form the required circuit
  - The IC is housed inside a ceramic or plastic container with connections welded to external pins
  - There can be 14 to several thousand pins on a logic chip
  - Each IC has a numeric designation printed on the surface for identification. The number can be looked up in catalogs (paper and electronic) that contain descriptions and information about the IC

Levels of Integration

- ICs are categorized by the number of gates that they contain in them:
  - Small-scale integration (SSI) devices contain several (usually less than 10) independent gates in a single package.
  - Medium-scale integration (MSI) devices include 10 to 1000 gates in a single package, used to perform elementary digital operations.
  - Large-scale integration (LSI) devices contain thousands of gates in a single package, used in processors, memory chips, and programmable logic devices.
  - Very Large-scale integration (VLSI) devices contain hundreds of thousands of gates in a single package, used in large memory arrays and complex microcomputer chips.

Logic Families

- ICs are also classified by the specific circuit technology (digital logic family) that they belong to:
  - Transistor-transistor logic (TTL) is a standard.
  - Emitter-coupled logic (ECL) is used in high-speed operation.
  - Metal-oxide semiconductor (MOS) is used for high component density.
  - Complementary metal-oxide semiconductor (CMOS) is used in low power consumption.

Logic Family Characteristics

- Digital logic families are usually compared by the following characteristics:
  - Fan-out specifies the amount of current that an output needs to drive many input pins on other gates.
  - Fan-in is the number of inputs available in a gate.
  - Power dissipation is the power consumed by the gate.
  - Propagation delay is the average delay time for the signal to propagate from input to output.
  - Noise margin is the maximum external noise voltage added to an input signal that does not cause an undesirable change in the circuit output.
  - Real estate is the amount of space required to implement the IC.
  - Reliability is the long-term success factor of the IC.