Classification of Nonlinear Fourier Transform Symbols Based on Symmetry Property

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***Abstract*—In this work, it's revealed that the symmetry property in M-QAM constellation reduces Nonlinear Fourier transform symbol classes, which simplifies the transmitter and receiver design. An algorithm is proposed to find the related symbol classes.**

***Keywords*—M-QAM, Nonlinear Fourier transform, symbol classes**

I. Introduction

Kerr nonlinearity in optical fibers is an inevitable burden to increase the capacity of the high-speed optical communication systems. Nonlinear Fourier transform (NFT) exploits fiber nonlinearity as a merit for signal transmission [1], either through modulation on the continuous spectrum [2] or the discrete spectrum [3]. Many works related to the discrete NFT with higher modulation formats have been conducted. In [4], neural networks (NNs) were utilized to predict the spectrum on 4 eigenvalues modulated with 16-QAM signals at 2 GBaud for 1000 km propagation. Wu et al. optimized the NN structure to demodulate the M-QAM signals modulated on n eigenvalues with two polarizations, which produced M^{2n} symbols by inverse NFT (INFT). In [5], three kinds of NN-based receivers were designed for an on-off keying coded 12-eigenvalue communication system, and an eigenvalue domain (ED)-NN was used to find the eigenvalues of the received symbols, which had 2^{12} symbol types. In [6], geometric shaped 16-APSK signals were modulated on 4 eigenvalues, with the symbol types as 16^4. In [7], S. A. Bogdanov et al. performed an experiment which transmitted 16-PSK signals modulated on 6 eigenvalues with the symbol types as 16^6. The number of symbol types is directly related to the design complexity for the transmitter and receiver.

It has not been revealed that many symbols shared the same amplitude profile and could be classified as one symbol class, and it will be greatly beneficial to obtain the actual number of symbol types after the modulation on the discrete NFT spectrum. Based on two basic properties of NFT, we relate the symbol types to the rotational and mirror symmetry of the M-QAM constellations modulated on the discrete spectrum with polarization multiplexing. An algorithm is proposed to classify these symbols. The agreement between the classification results of the proposed algorithm and those given by the exhaustive search proves validity of our method. This result shows our algorithm can significantly reduce the number of symbol types and will be of great help for the transmitter and receiver design.

II. Principle

A. Nonlinear Fourier transform

The Optical pulse with polarization multiplexing, which propagates in fibers with anomalous dispersion and Kerr nonlinearity, is described by the following normalized Manakov equation:

$$i\mathbf{q}_z = \mathbf{q}_{tt} + 2|\mathbf{q}|^2\mathbf{q} \tag{1}$$

where $\mathbf{q}=(q_1,q_2)^T$ is the Jones vector with two polarization components q_1 and q_2, z and t are the normalized distance and time, which are related to the actual symbol amplitude, propagation distance and time as shown in [1].

One may take INFT by solving the Gelfand–Levitan–Marchenko (GLM) equation, so that the related symbol **q** can be obtained according to the modulation on the discrete spectrum [9]

$$\begin{gathered}\mathbf{K}(x,y)-\mathbf{F}^H(x+y)+\int_x^{+\infty}\int_x^{+\infty}\mathbf{K}(x,s)\mathbf{F}(s+r)\mathbf{F}^H(r+y)dsdr=0\\ \mathbf{F}(x)=\left[-i\sum_{j=1}^{J}Q_1(\lambda_j)e^{i\lambda_j x}\quad -i\sum_{j=1}^{J}Q_2(\lambda_j)e^{i\lambda_j x}\right]^T\end{gathered} \tag{2}$$

where **K** is the vector kernel function to be calculated and $\mathbf{q}=-2\mathbf{K}(t,t)$, **F** is vector Marchenko kernel encoded with the discrete nonlinear spectrum, H stands for conjugate transpose, λ_j is the eigenvalues and $Q_{1/2}(\lambda_j)$ is corresponding discrete nonlinear spectrum of the two polarizations (1 for x and 2 for y). We simplify the representation for NFT and INFT as $NFT(\mathbf{q})=[Q_1(\lambda_j),\ Q_2(\lambda_j)]^T$ and $\mathbf{q}=INFT([Q_1(\lambda_j),\ Q_2(\lambda_j)]^T)$, $j=1,\ldots,N$.

Two important properties exist for the NFT and IFNT with polarizations multiplexing.

Property 1) $NFT[(e^{i\varphi}q_1,e^{i\psi}q_2)^T]=[e^{-i\varphi}Q_1(\lambda_j),\ e^{-i\psi}Q_2(\lambda_j)]^T$, where φ, ψ are two distinct angles.

lemma 1 (proved in [9]): *If* $\boldsymbol{R}$ *is unitary matrix,* $NFT(\boldsymbol{R}\mathbf{q})=\boldsymbol{R}^*NFT(\mathbf{q})$ while λ_j remains the same.

Proof: Let $\boldsymbol{R}$ to be a 2x2 matrix,

$$\begin{aligned}&NFT\left[(R_{11}q_1+R_{12}q_2,R_{21}q_1+R_{22}q_2)^T\right]\\ &=\left[R_{11}^*Q_1(\lambda_j)+R_{12}^*Q_2(\lambda_j),R_{21}^*Q_1(\lambda_j)+R_{22}^*Q_2(\lambda_j)\right]^T, j=1,\ldots,N.\end{aligned} \tag{3}$$

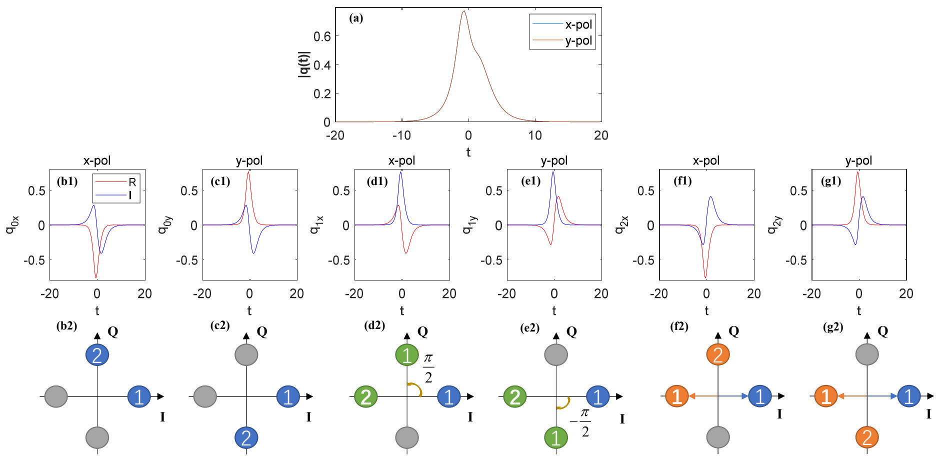

Figure 1. (a) Normalized amplitude of the waveform shared by (b1-g1); (b1-g1) the real parts and imaginary parts of x-pol and y-pol of $\mathbf{q}_0$, $\mathbf{q}_1$ and $\mathbf{q}_2$, which are generated by $\mathbf{Q}_0$, $\mathbf{Q}_1$ and $\mathbf{Q}_2$ respectively; (b2-g2) discrete nonlinear spectrums coded by QPSK constellation, blue for $\mathbf{Q}_0$, green for $\mathbf{Q}_1$, and orange for $\mathbf{Q}_2$. The character '1' stands for spectrums on the 1st eigenvalue and '2' for spectrums on the 2ed eigenvalue.

choose $R_{11}=e^{i\varphi}$, $R_{22}=e^{i\psi}$ while others as zeros, we have property 1.

If the spectrums on x or y polarization differ by a constant phase, the NFT symbols will differ by a constant phase on the corresponding polarization as well. Similar property for the single polarization case has been reported in [10].

Property 2) NFT[$\mathbf{q}^*$]=[$-Q_1^*(-\lambda_j^*)$, $-Q_{2j}^*(-\lambda_j^*)$]^T.

Proof: Taking the complex conjugate of (2), we have $\mathbf{q}^*=-2\mathbf{K}^*(t,t)$. Similar property for the single polarization case has been proved in [10] through a Darboux transform approach.

If the spectrums and the eigenvalues on x pol and y-pol are of mirror symmetry with respect to the imaginary axis, the INFT results will only differ by a conjugated phase profile. Though some works report the eigenvalues on the general complex domain [5,6], pure imaginary eigenvalues are quite commonly used in the discrete NFT communication systems [3-4, 8, 10], which is adopted in following context. In this case, $\lambda_j = -\lambda_j^*$.

To demonstrate the two properties in Fig.1, INFT is conducted on a discrete spectrum with two pure imaginary eigenvalues [$0.3i$ $0.6i$] on two polarizations. $\mathbf{q}_0$, $\mathbf{q}_1$ and $\mathbf{q}_2$ are the symbols generated with the discrete spectrums $\mathbf{Q}_0$, $\mathbf{Q}_1$ and $\mathbf{Q}_2$ on a standard finite time interval t=[-20,20], with $\mathbf{Q}_0$=[1 i;1 $-i$], $\mathbf{Q}_1$=[i -1; $-i$ -1] , and $\mathbf{Q}_2$=[-1 i;-1 $-i$] respectively. According to property 1 and property 2, $\mathbf{q}_0$ and $\mathbf{q}_1$ share the same amplitude and differ by a constant phase term, while $\mathbf{q}_0$ and $\mathbf{q}_2$ share the same amplitude and the conjugated phase profiles. All these are clearly shown in Fig. 1, where Figs. 1(b1-g1) demonstrate the real and imaginary parts of $\mathbf{q}_0$, $\mathbf{q}_1$ and $\mathbf{q}_2$ and Figs. 1(b2-g2) demonstrate the related discrete spectrum $\mathbf{Q}_0$, $\mathbf{Q}_1$ and $\mathbf{Q}_2$.

B. *Algorithm of symbol classification*

An algorithm is proposed for find the symbol classes by observing the symmetry of M-QAM constellations on the discrete spectrums. The algorithm is given as follows.

Step 1: Consider set S_0 of M-QAM constellations modulated on n eigenvalues, which has M^n symbols.
Step 2: Find an arbitrary symbol $P_i \in S_0$, which is formed by n points on the constellation corresponding to n eigenvalues, and find its rotational symmetry to compose S_i.
Step 3: Remove all the elements in S_i from S_0. if $S_0 \neq \emptyset$, return to 2 to find a new set.
Step 4: Return the number of sets S_i as N.
Step 5: Consider the N classes of Step 4 to find the mirror symmetry with respect to the imaginary axis. If the elements in S_j and S_k are of mirror symmetry with respect to the imaginary axis, a pair is found as (S_i, S_k). If no pair class can be found for S_l, consider (S_l, S_l) as a pair.
Step 6: Return the number of pairs as N'.

Steps 1-4 are based on property 1 to find all the symbol classes, whose elements share the same amplitudes and differ by a constant phase. Steps 5-6 are based on property 2 to find all the symbol classes, whose elements share the same amplitudes and differ by a constant phase or a conjugated phase profile. The above algorithm finds the symbol classes for one polarization.

For the dual polarization case, we have:

$$N_d = N^2, \quad N_d{}' = N_d - \frac{N_d-\left(2N'-N\right)^2}{2} = N^2+2N'^2-2NN' \qquad (4)$$

where N, N' are given by the above algorithm, N_d, $N_d{}'$ are the related symbol class numbers for the dual polarization case.

III. Results and Discuss

With the above algorithm, we find the symbol classes for the signals with M-QAM modulation on n purely imaginary eigenvalues with two polarizations. Given {(M,n)}=[(4, 3),(4,4),(16,2),(16,3),(64,2)], the classification results by the proposed algorithm are listed in Table 1. Exhaustive search is conducted to verify the algorithm and the same results are obtained as Table 1.

TABLE 1. Symbols Types

M	n	N_d'	N_d	M^{2n}	N_d'/M^{2n}	N_d/M^{2n}
4	3	136	256	4^6	3.32%	6.25%
4	4	2080	4096	4^8	3.17%	6.25%
16	2	1850	3600	16^4	2.82%	5.49%
16	3	508680	1016064	16^6	3.03%	6.06%
64	2	473274	944784	64^4	2.82%	5.63%

Compared to the original symbol types M^{2n}, the classified symbol types N_d and N_d' significantly reduce to 6% or 3%. $N_d/N_d' \approx 2$ indicates that in most case, **q** can find its conjugated pair $\mathbf{q}^*$, except that the corresponding spectrums of constellation diagram modulated on the eigenvalues are symmetric with respect to the imaginary axis itself.

It can be expected that the transmitter and receiver design complexity can be significantly reduced. For example, when M=4, n=3, i.e., QPSK signal is modulated on 3 eigen values with two polarizations, the total symbol types should be 4096. The symbol types are only 256 if the rotational symmetry is considered. The symbol types can be further reduced to 136 if the mirror symmetry is considered.

IV. Conclusion

In this paper, NFT symbols classification is performed with an algorithm based on two properties of NFT. Compared with the original symbol types, the classification algorithm reduces to 6% or 3%. Significant reduction of the design complexity for the transmitter and receiver can be expected with many symbols classified as one which share the same amplitude.

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