Assignment 3

Problem 1. In a group $(G, *)$, if $(a * b)^2 = a^2 * b^2$ for all $a, b \in G$ then show that G is abelian group.

Problem 2. Show that the set of all positive rational numbers forms an abelian group under the composition * defined by $a * b = (ab)/2$.

Problem 3. Let R be a group of all real numbers under addition and R^+ be a group of all positive real numbers under multiplication. Show that the mapping $f : R^+ \to R$ defined by $f(x) = \log_{10} x$ for all

$x \in R$ is an isomorphism.

Problem 4: Consider the groups $(G_1, *)$ and $(G_2, \oplus)$ with identity elements e_1 and e_2 respectively. If

$f : G_1 \to G_2$ is a group homomorphism, then prove that

(1) If H_1 is a sub group of G_1 and $H_2 = f(H_1)$, then H_2 is a sub group of G_2.

(2) If f is an isomorphism from G_1 onto G_2, then f^{-1} is an isomorphism from G_2 onto G_1.

Problem 5: If G is a group of order p, where p is a prime number. Then what are the number of sub groups of G?

1、 Proof: Given that $(a*b)^2 = a^2*b^2$

$\Rightarrow (a*b)*(a*b) = (a*a)*(b*b)$

$\Rightarrow a*(b*a)*b = a*(a*b)*b$

$\Rightarrow (b*a)*b = (a*b)*b$

$\Rightarrow (b*a) = (a*b)$

Hence, G is abelian group

2、 Let A = set of all postive rational numbers

let a,b,c be any three elemenes of A

$a*b \in A$ for all $a, b \in A$

$(a*b)*c = (ab/2)*c = (abc)/4$

$a*(b*c) = a*(bc/2) = (abc)/4$

Let e be the identity element

We have $a*e = (ae)/2$

$a*e = a$

$(ae)/2 = a \Rightarrow e = 2$ and $2 \in A$

$\therefore$ Identity element exists, and '2' is the identity element in A

3. Let $a, b \in R^+$

Now, $f(a.b) = \log_{10}(a.b)$

$= \log_{10} a + \log_{10} b$

$= f(a) + f(b)$

$\therefore$ f is an homomorphism

For any $a, b \in R$, Let, $f(a) = f(b) \Rightarrow 2^a = 2^b \Rightarrow a = b$

$\therefore$ f is one to one

take any $c \in R^+$

Then $\log_2 c \in R$ and $f(\log_2 c) = 2^{\log_2 c} = c$

$\Rightarrow$ Every element in R^+ has a pre image in R

f is onto

$\therefore$ f is a bijection

Hence, f is an isomorphism

4. (i) $H_2 = f(H_1)$ is the image of H_1 under f; this is a subset of G_2

Let $x, y \in H_2$

Then $x = f(a)$, $y = f(b)$ for some $a, b \in H_1$

Since, H_1 is a subgroup of G_1, we have $a * b^{-1} \in H_1$

Consequently

$$x \oplus y^{-1} = f(a) \oplus [f(b)]^{-1}$$
$$= f(a) \oplus f(b^{-1})$$
$$= f(a * b^{-1}) \in f(H_1) = H_2$$

Hence, H_2 is a subgroup of G_2

(2) Since $f: G_1 \to G_2$ is an isomorphism, f is a bijection

$\therefore f^{-1}: G_2 \to G_1$ exists and is a bijection

Let $x, y \in G_2$ Then $x \oplus y \in G_2$

and there exists $a, b \in G_1$ such that $x = f(a)$ and $y = f(b)$

$$\therefore f^{-1}(x \oplus y) = f^{-1}(f(a) \oplus f(b))$$
$$= f^{-1}(f(a * b))$$
$$= a * b$$
$$= f^{-1}(x) * f^{-1}(y)$$

$\therefore f^{-1}$ is an isomorphism

5. it is 2