3. Eulerian And Hamaltonian Graphs

3.1 Eulerian Graph
3.2 Hamaltonian Graph

3.1 EULERIAN GRAPHS

3.1.1 Euler path

A path in a graph G is called Euler path if it includes every edges exactly once. Since the path contains every edge exactly once, it is also called Euler trail.

3.1.2 Euler circuit

An Euler path that is circuit is called Euler circuit. A graph which has a Eulerian circuit is called an Eulerian graph.

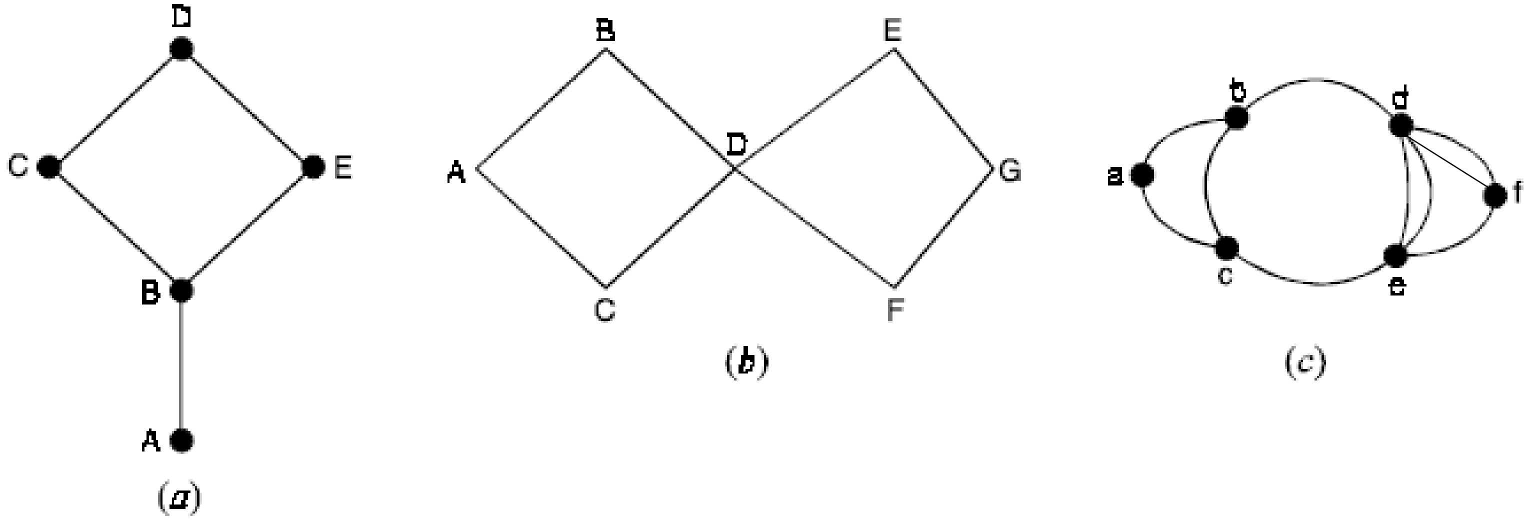


Fig. 36.

The graph of Figure 36(*a*) has an Euler path but no Euler circuit. Note that two vertices A and B are of odd degrees 1 and 3 respectively. That means AB can be used to either arrive at vertex A or leave vertex A but not for both.

Thus an Euler path can be found if we start either from vertex A or from B.

ABCDEB and BCDEBA are two Euler paths. Starting from any vertex no Euler circuit can be found.

The graph of Figure 36(*b*) has both Euler circuit and Euler path. ABDEGFDCA is an Euler path and circuit. Note that all vertices of even degree.

No Euler path and circuit is possible in Figure 36(*c*).

Note that all vertices are not even degree and more than two vertices are of odd degree.

The existence of Euler path and circuit depends on the degree of vertices.

Note : To determine whether a graph G has an Euler circuit, we note the following points :

(*i*) List the degree of all vertices in the graph.

(*ii*) If any value is zero, the graph is not connected and hence it cannot have Euler path or Euler circuit.

(*iii*) If all the degrees are even, then G has both Euler path and Euler circuit.

(*iv*) If exactly two vertices are odd degree, then G has Euler path but no Euler circuit.

Theorem 3.17 *The following statements are equivalent for a connected graph G :*

(*i*) *G is Eulerian*

(*ii*) *Every point of G has even degree*

(*iii*) *The set of lines of G be partitioned into cycles.*

Proof. (*i*) implies (*ii*)

Let T be an Eulerian trail in G.

Each occurrence of a given point in T contributes 2 to the degree of that point, and since each line of G appears exactly once in T, every point must have even degree.

(*ii*) implies (*iii*)

Since G is connected and non trivial, every point has degree at least 2, so G contains a cycle Z.

The removal of the lines of Z results in a spanning subgraph G_1 in which every point still has even degree.

If G_1 has no lines, then (*iii*) already holds ; otherwise, repetition of the argument applied to G_1 results in a graph G_2 in which again all points are even, etc.

When a totally disconnected graph G_n is obtained, we have a partition of the lines of G into *n* cycles.

(*iii*) implies (*i*)

Let Z_1 be one of the cycles of this partition.

If G consists only of this cycle, then G is obviously Eulerian.

Otherwise, there is another cycle Z_2 with a point v in common with Z_1.

The walk beginning at v and consisting of the cycles Z_1 and Z_2 in succession is a closed trail containing the lines of these two cycles.

By continuing this process, we can construct a closed trail containing all lines of G.

Hence G is Eulerian.

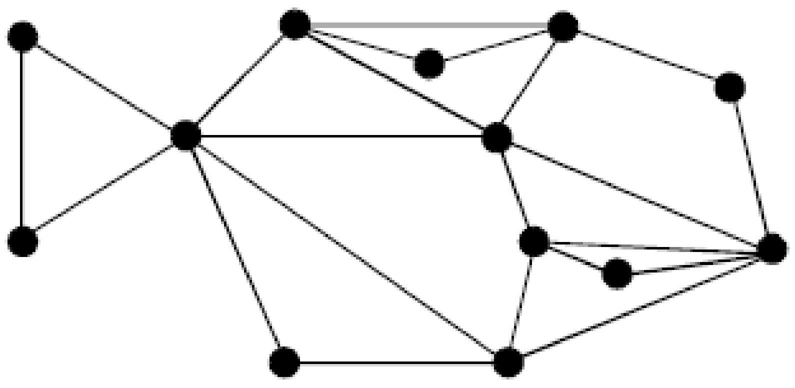


Fig. 37. An Eulerian graph.

For example, the connected graph of Figure 37 in which every point has even degree has an Eulerian trail, and the set of lines can be partitioned into cycles.

Corollary (1) :

Let G be a connected graph with exactly $2n$ odd points, $n \geq 1$, then the set of lines of G can be partitioned into n open trails.

Corollary (2) :

Let G be a connected graph with exactly two odd points. Then G has an open trail containing all the points and lines of G (which begins at one of the odd points and ends at the other).

Problem *A non empty connected graph G is Eulerian if and only if its vertices are all of even degree.*

Proof. Let G be Eulerian.

Then G has an Eulerian trail which begins and ends at u, say.

If we travel along the trail then each time we visit a vertex we use two edges, one in and one out.

This is also true for the start vertex because we also ends there.

Since an Eulerian trial uses every edge once, each occurrence of v represents a contribution of 2 to its degree.

Thus deg(v) is even.

Conversely, suppose that G is connected and every vertex is even.

We construct an Eulerian trail. We begin a trail T_1 at any edge e. We extend T_1 by adding an edge after the other.

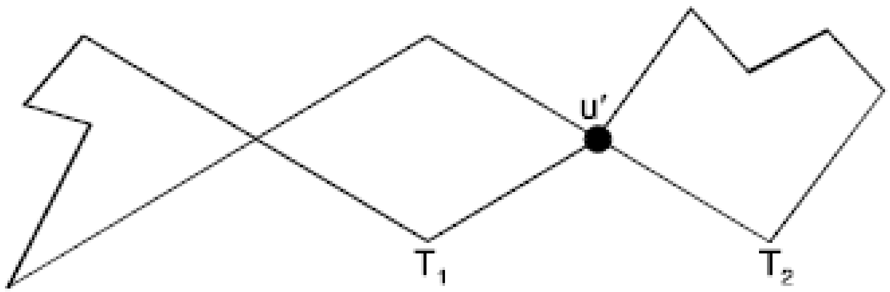
If T_1 is not closed at any step, say T_1 begins at u but ends at $v \neq u$, then only an odd number of the edges incident on v appear in T_1.

Hence we can extend T_1 by another edge incident on v.

Thus we can continue to extend T_1 until T_1 returns to its initial vertex u.

i.e., until T_1 is closed.

If T_1 includes all the edges of G then T_1 is an Eulerian trail.



Suppose T_1 does not include all edges of G.

Consider the graph H obtained by deleting all edges of T_1 from G.

H may not be connected, but each vertex of H has even degree since T_1 contains an even number of the edges incident on any vertex.

Since G is connected, there is an edge e' of H which has an end point u' in T_1.

We construct a trail T_2 in H beginning at u' and using e'. Since all vertices in H have even degree.

We can continue to extent T_2 until T_2 returns to u' as shown in Figure.

We can clearly put T_1 and T_2 together to form a larger closed trail in G.

We continue this process until all the edges of G are used.

We finally obtain an Eulerian trail, and so G is Eulerian.

Theorem . *A connected graph G has an Eulerian trail if and only if it has at most two odd vertices.*

i.e., it has either no vertices of odd degree or exactly two vertices of odd degree.

Proof. Suppose G has an Eulerian trail which is not closed. Since each vertex in the middle of the trail is associated with two edges and since there is only one edge associated with each end vertex of the trail, these end vertices must be odd and the other vertices must be even.

Conversely, suppose that G is connected with atmost two odd vertices.

If G has no odd vertices then G is Euler and so has Eulerian trail.

The leaves us to treat the case when G has two odd vertices (G cannot have just one odd vertex since in any graph there is an even number of vertices with odd degree).

Corollary (1) :

A directed multigraph G has an Euler path if and only if it is unilaterally connected and the in degree of each vertex is equal to its out degree with the possible exception of two vertices, for which it may be that the in degree of is larger than its out degree and the in degree of the other is oneless than its out degree.

Corollary (2) :

A directed multigraph G has an Euler circuit if and only if G is unilaterally connected and the indegree of every vertex in G is equal to its out degree.

FLEURY'S ALGORITHM

Let G = (V, E) be a connected graph with each vertex of even degree.

Step 1. Select an edge e_1 that is not a bridge in G.

Let its vertices be v_1, v_2.

Let π be specified by $V_\pi : v_1, v_2$ and $E_\pi : e_1$.

Remove e_1 from E and v_1 and v_2 from V to create G_1.

Step 2. Suppose that $V_\pi : v_1, v_2, \ldots\ldots v_k$ and $E_\pi : e_1, e_2, \ldots\ldots e_{k-1}$ have been constructed so far, and that all of these edges and vertices have been removed from v and E to form G_{k-1}.

Since v_k has even degree, and e_{k-1} ends there, there must be an edge e_k in G_{k-1} that also has v_k as a vertex.

If there is more than one such edge, select one that is not a bridge for G_{k-1}.

Denote the vertex of e_k other than v_k by v_{k+1}, and extend V_π and E_π to $V_\pi : v_1, v_2, \ldots\ldots, v_k, v_{k+1}$ and $E_\pi : e_1, e_2, \ldots\ldots, e_{k-1}, e_k$.

Step 3. Repeat step 2 until no edges remain in E.

End of algorithm.

Problem *Use Fleury's algorithm to construct an Euler circuit for the graph in Figure (1).*

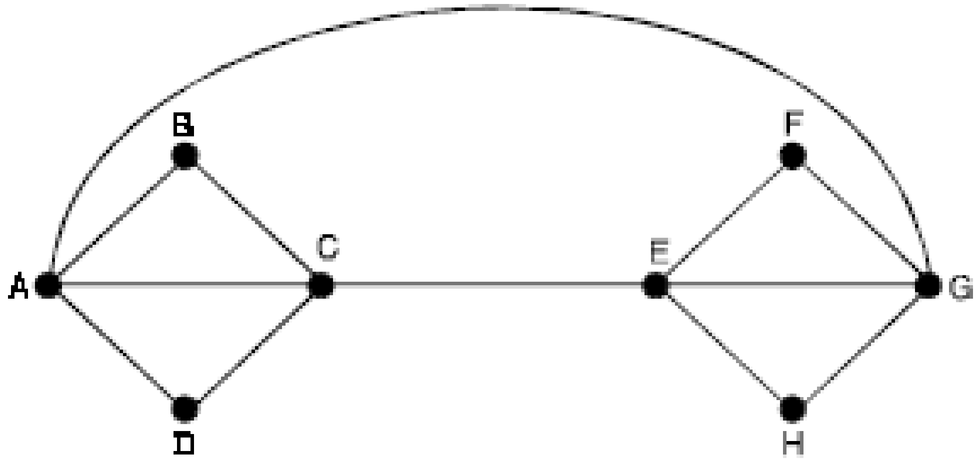


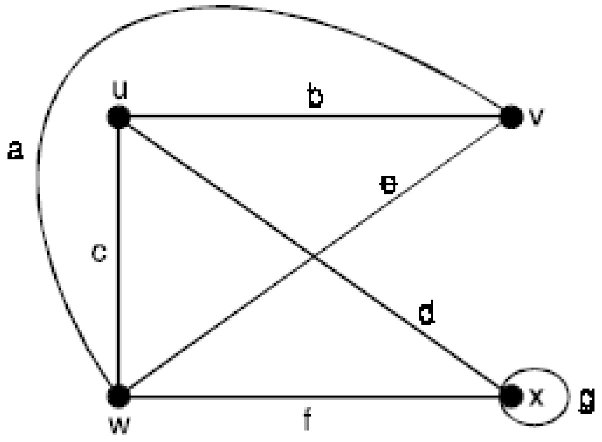
Fig. (1)

Solution. According to step 1, we may begin anywhere.

Arbitrarily choose vertex A. We summarize the results of applying step 2 repeatedly in Table.

Current Path	*Next Edge*	*Reasoning*
π : A	{A, B}	No edge from A is a bridge. Choose any one.
π : A, B	{B, C}	Only one edge from B remains.
π : A, B, C	{C, A}	No edges from C is a bridge. Choose any one.
π : A, B, C, A	{A, D}	No edges from A is a bridge. Choose any one.
π : A, B, C, A, D	{D, C}	Only one edge from D remains.
π : A, B, C, A, D, C	{C, E}	Only one edge from C remains.
π : A, B, C, A, D, C, E	{E, G}	No edge from E is a bridge. Choose any one.
π : A, B, C, A, D, C, E, G	{G, F}	{A, G} is a bridge. Choose {G, F} or {G, H}.
π : A, B, C, A, D, C, E, G, F	{F, E}	Only one edge from F remains.
π : A, B, C, A, D, C, E, G, F, E	{E, H}	Only one edge from E remains.
π : A, B, C, A, D, C, E, G, F, E, H	{H, G}	Only one edge from H remains
π : A, B, C, A, D, C, E, G, F, E, H, G	{G, A}	Only one edge from G remains.
π : A, B, C, A, D, C, E, G, F, E, H, G, A		

Problem *Show that the graph shown in Figure has no Eulerian circuit but has a Eulerian trail.*

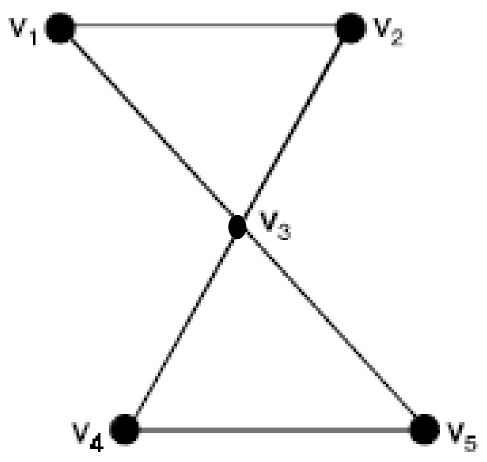


Solution. Here deg (u) = deg (v) = 3 and deg (w) = 4, deg (x) = 4

Since u and v have only two vertices of odd degree, the graph shown in Figure, does not contain Eulerian circuit, but the path.

$b - a - c - d - g - f - e$ is an Eulerian path.

Problem . *Let G be a graph of Figure. Verify that G has an Eulerian circuit.*



Solution. We observe that G is connected and all the vertices are having even degree

$$\deg(v_1) = \deg(v_2) = \deg(v_4) = \deg(v_5) = 2.$$

Thus G has a Eulerian circuit.

By inspection, we find the Eulerian circuit

$$v_1 - v_3 - v_5 - v_4 - v_3 - v_2 - v_1.$$

3.2 HAMILTONIAN GRAPHS

Hamiltonian graphs are named after Sir William Hamilton, an Irish mathematician who introduced the problems of finding a circuit in which all vertices of a graph appear exactly once.

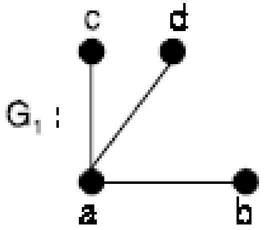
A circuit in a graph G that contains each vertex in G exactly once, except for the starting and ending vertex that **appears twice** is known as **Hamiltonian circuit.**

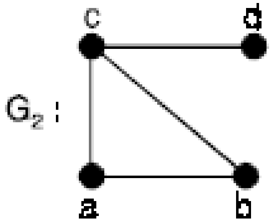
A graph G is called a **Hamiltonian graph,** if it contains a Hamiltonian circuit.

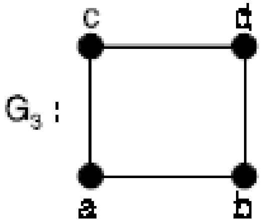
A Hamiltonian path is a simple path that contains all vertices of G where the end points may be distinct.

Note that an Eulerian circuit traverses every edge exactly once, but may repeat vertices, while a Hamiltonian circuit visists each vertex exactly once but may repeat edges. While there is a criterion for determining whether or not a graph contains an Eulerian circuit, a similar criterion does not exist for Hamiltonian ciruits.

In the following figures, hamiltonian path and cycles are shown :





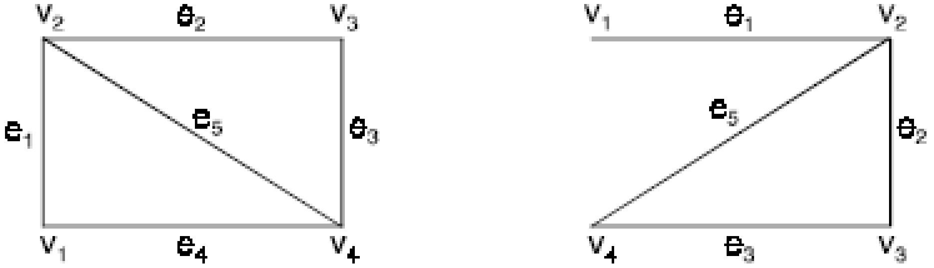


The graph G_1 has no hamiltonian path (and so hamiltonian cycle), G_2 has hamiltonian path *abcd* but no hamiltonian cycle, while G_3 has hamiltonian cycle *abdca*.

The cycle C_n with n distinct (and n edges) is hamiltonian. Moreover given hamiltonian graph G then if G′ is a subgraph obtained by adding in new edges between vertices of G, G′ will also be hamiltonian. Since any hamiltonian cycle in G will also be hamiltonian cycle in G′. In particular k_n, the complete graph on n vertices, in such a supergraph of a cycle, k_n is hamiltonian.

A simple graph G is called maximal non-hamiltonian if it is not hamiltonian but the addition to it any edge connecting two non-adjacent vertices forms a hamiltonian graph. The graph G_2 is a maximal non-hamiltonian since the addition of an edge *bd* gives hamiltonian graph G_3.

Problem *Which of the graphs given in Figure below is Hamiltonian circuit. Give the circuits on the graphs that contain them.*



(*a*) (*b*)

Solution. The graph shown in Figure (*a*) has Hamiltonian circuit given by $v_1e_1v_2e_2v_3e_3v_4e_4v_1$.

Note that all vertices appear in this a circuit but not all edges.

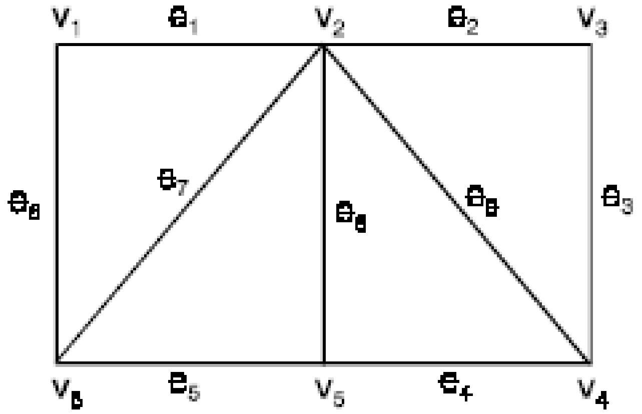
The edge e_5 is not used in the circuit.

The graph shown in Figure (*b*) does not contain circuit since every circuit containing every vertex must contain the e_1 twice.

But the graph does have a Hamiltonian path $v_1 - v_2 - v_3 - v_4$.

Problem *Give an example of a graph which is Hamiltonian but not Eulerian and vice-versa.*

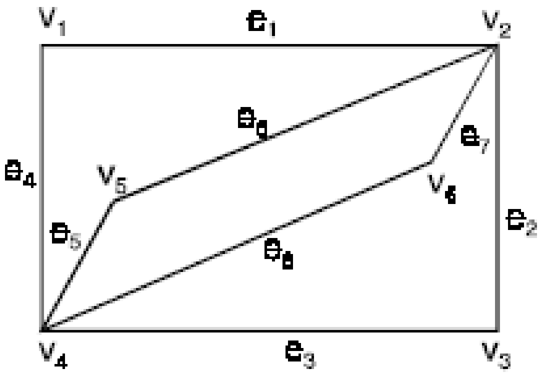
Solution. The following graph shown in Figure below is Hamiltonian but non-Eulerian.



The graph contains a Hamiltonian circuit $v_1e_1v_2e_2v_3e_3v_4e_4v_5e_5v_6e_6v_1$.

Since the degree of each vertex is not *n* even the graph is non-Eulerian.

The graph shown in Figure below is Eulerian but not Hamiltonian.



The graph is Eulerian since the degree of each vertex is even.

It does not contain Hamiltonian circuit.

This can be seen by noting any circuit containing every vertex must contain a vertex twice except starting vertex and ending vertex.

4. Representation of Graphs

4.1 Matrix representation
4.2 Adjacency matrix
4.3 Incidence matrix
4.4 Linked representation

4 REPRESENTATION OF GRAPHS

Although a diagrammatic representation of a graph is very convenient for a visual study but this is only possible when the number of nodes and edges is reasonably small.

Two types of representation are given below :

4.1 Matrix representation

The matrix are commonly used to represent graphs for computer processing. The advantages of representing the graph in matrix form lies on the fact that many results of matrix algebra can be readily applied to study the structural properties of graphs from an algebraic point of view. There are number of matrices which one can associate witch any graph. We shall discuss adjacency matrix and the incidence matrix.

4.2 Adjacency matrix

(*a*) Representation of undirected graph

The adjacency matrix of a graph G with n vertices and no parallel edges is an n by n matrix $A = \{a_{ij}\}$ whose elements are given by $a_{ij} = 1$, if there is an edge between ith and jth vertices, and

$= 0$, if there is no edge between them.

Note that for a given graph, the adjacency matrix is based on ordering chosen for the vertices.

Hence, there are as many as $n\,!$ different adjacency matrices for a graph with n vertices, since there are $n\,!$ different ordering of n vertices.

However, any two such adjacency matrices are closely related in that one can be obtained from the other by simply interchanging rows and columns.

There are a number of observations that one can make about the adjacency matrix A of a graph G are :

Observations :

(*i*) A is symmetric *i.e.* $a_{ij} = a_{ji}$ for all i and j

(*ii*) The entries along the principal diagonal of A all zeros if and only if the graph has no self loops. A self loop at the vertex corresponding to $a_{ij} = 1$.

(*iii*) If the graph is simple (no self loop, no parallel edges), the degree of vertex equals the number of 1's in the corresponding row or column of A.

(*iv*) The (i, j) entry of A^m is the number of paths of length (no. of occurence of edges) m from vertex v_i vertex v_j.

(*v*) If G be a graph with n vertices $v_1, v_2, \ldots\ldots\ v_n$ and let A denote the adjacency matrix of G with respect to this listing of the vertices. Let B be the matrix.

$$B = A + A^2 + A^3 + \ldots\ldots + A^{n-1}$$

Then G is a connected graph if B has no zero entries of the main diagonal.

This result can be also used to check the connectedness of a graph by using its adjacency matrix.

Adjacency can also be used to represent undirected graphs with loops and multiple edges. A loop at the vertex v_1 is represented by a 1 at the (i, j)th position of the adjacency matrix. When multiple edges are present, the adjacency matrix is no longer a zero-one matrix, since the (i, j)th entry equals the number of edges these are associated to $\{v_i - v_j\}$.

All undirected graphs, including multigraphs and pseudographs, have symmetric adjacency matrices.

(*b*) Representation of directed graph

The adjacency matrix of a diagonal D, with n vertices is the matrix A = $\{a_{ij}\}_{n \times n}$ in which

$$\begin{aligned} a_{ij} &= 1 \text{ if arc } \{v_i - v_j\} \text{ is in D} \\ &= 0 \text{ otherwise.} \end{aligned}$$

One can make a number of observations about the adjacency matrix of a diagonal.

Observations

(*i*) A is not necessary symmetric, since there may not be an edges from v_i to v_j when there is an edge from v_i to v_j.

(*ii*) The sum of any column of j of A is equal to the number of arcs directed towards v_j.

(*iii*) The sum of entries in row i is equal to the number of arcs directed away from vertex v_i (out degree of vertex v_i)

(*iv*) The (i, j) entry of A^m is equal to the number of path of length m from vertex v_i to vertex v_j entries of A^T. A shows the in degree of the vertices.

The adjacency matrices can also be used to represent directed multigraphs. Again such matrices are not zero-one matrices when there are multiple edges in the same direction connecting two vertices.

In the adjacency matrix for a directed multigraph a_{ij} equals the number of edges that are associated to (v_i, v_j).

4.3 . Incidence matrix

. (*a*) Representation of undirected graph

Consider a undirected graph G = (V, E) which has n vertices and m edges all labelled. The incidence matrix B = $\{b_{ij}\}$, is then $n \times m$ matrix,

where $\quad b_{ij} = 1$ when edge e_j is incident with v_i

$\quad\quad = 0$ otherwise

We can make a number of observations about the incidence matrix B of G.

Observations :

(*i*) Each column of B comprises exactly two unit entries.

(*ii*) A row with all O entries corresponds to an isolated vertex.

(*iii*) A row with a single unit entry corresponds to a pendent vertex.

(*iv*) The number of unit entries in row i of B is equal to the degree of the corresponding vertex v_i.

(*v*) The permutation of any two rows (any two columns) of B corresponds to a labelling of the vertices (edges) of G.

(*vi*) Two graphs are isomorphic if and only if their corresponding incidence matrices differ only by a permutation of rows and columns.

(*vii*) If G is connected with n vertices then the rank of B is $n - 1$.

Incidence matrices can also be used to represent multiple edges and loops. Multiple edges are represented in the incidence matrix using columns with identical entries. Since these edges are incident with the same pair of vertices. Loops are represented using a column with exactly one entry equal to 1, corresponding to the vertex that is incident with this loop.

(*b*) Representation of directed graph

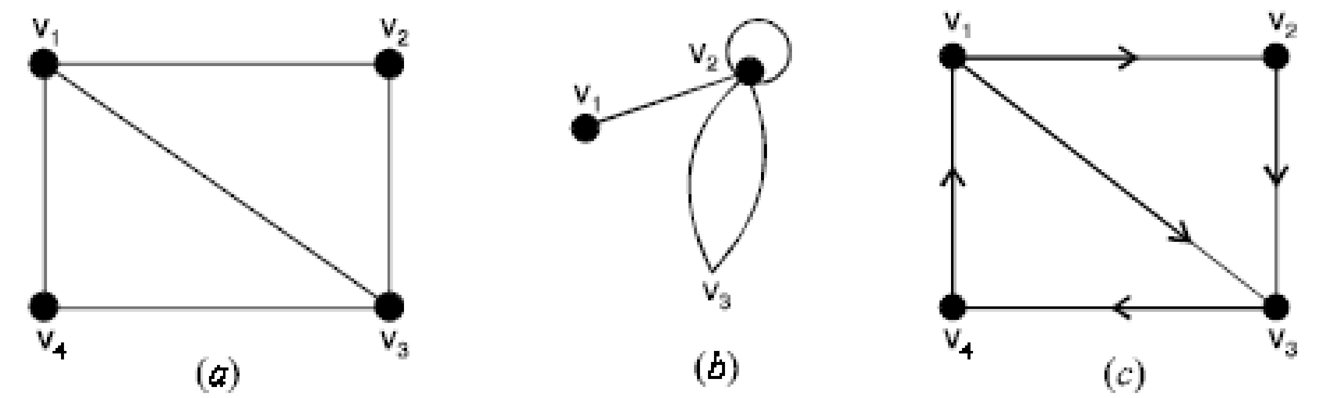
The incidence matrix $B = \{b_{ij}\}$ of digraph D with n vertices and m edges is the $n \times m$ matrix in which

$$
\begin{aligned}
b_{ij} &= 1 \text{ if arc } j \text{ is directed away from a vertex } v_i \\
&= -1 \text{ if arc } j \text{ directed towards vertex } v_i \\
&= 0 \text{ otherwise.}
\end{aligned}
$$

4.4 Linked representation

In this representation, a list of vertices adjacent to each vertex is maintained. This representation is also called adjacency structure representation. In case of a directed graph, a case has to be taken, according to the direction of an edge, while placing a vertex in the adjacent structure representation of another vertex.

Problem *Use adjacency matrix to represent the graphs shown in Figure below*



Solution. We order the vertices in Figure (1)(a) as v_1, v_2, v_3 and v_4.

Since there are four vertices, the adjacency matrix representing the graph will be a square matrix of order four. The required adjacency matrix A is

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

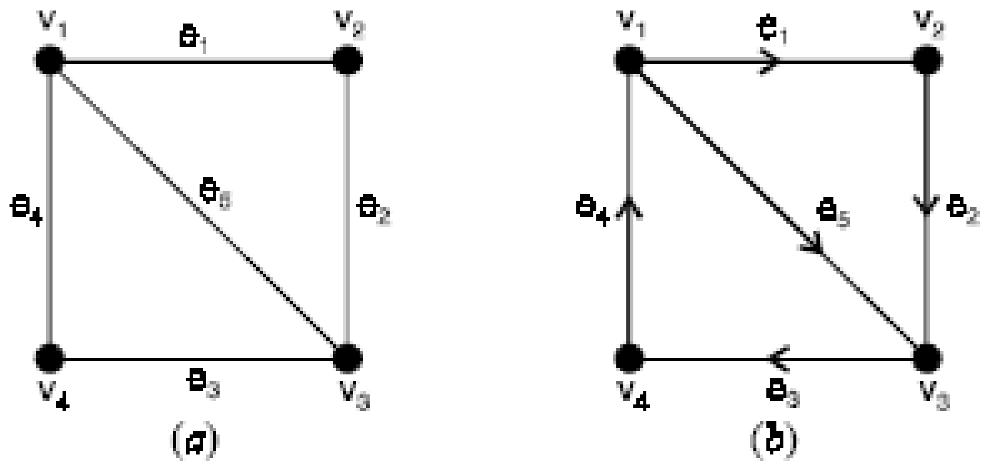
We order the vertices in Figure (1)(b) as v_1, v_2 and v_3. The adjacency matrix representing the graph is given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

Taking the order of the vertices in Figure(1)(c) as v_1, v_2, v_3 and v_4. The adjacency matrix representing the graph is given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Problem . *Find the incidence matrix to represent the graph shown in Figure below :*



Solution. The incidence matrix of Figure (*a*) is obtained by entering for row v and column e is 1 if e is incident on v and 0 otherwise. The incidence matrix is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The incidence matrix of the graph of Figure (*b*) is

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$