Graph Theory

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5. Trees and Spanning Trees

6. Shortest Path Problems

Reference book:

1. Graph Theory with Applications to Engineering and Computer Science by Narsingh Deo, first edition.
2. Graphs and Matrices by Ravindra B. Bapat, second edition.
3. Linear Algebra by Howard Anton and Chris Rorres, applications version, 11th edition.

1.Basic Concept of Graph

1.1 Basic concepts of Graphs
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1.1 Basic concepts of Graphs

DEFINITION:

A graph G consists of a set of objects $V = \{v_1, v_2, v_3, ...\}$ called **vertices** (also called **points** or **nodes**) and other set $E = \{e_1, e_2, e_3, ...\}$ whose elements are called **edges** (also called **lines** or **arcs**).

The set V(G) is called the **vertex set** of G and E(G) is the **edge set.**

Usually the graph is denoted as **G = (V, E)**

Let G be a graph and $\{u, v\}$ an edge of G. Since $\{u, v\}$ is 2-element set, we may write $\{v, u\}$ instead of $\{u, v\}$. It is often more convenient to represent this edge by uv or vu.

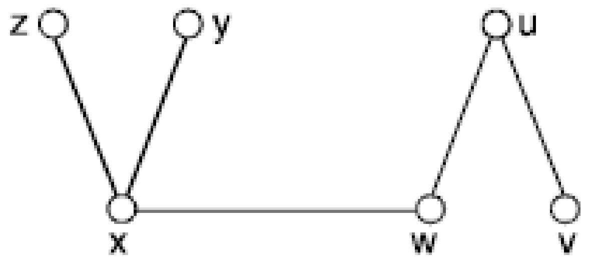
If $e = uv$ is an edge of a graph G, then we say that u and v are **adjacent** in G and that e joins u and v. (We may also say that each that of u and v is adjacent to or with the other).

For example :

A graph G is defined by the sets

$$V(G) = \{u, v, w, x, y, z\} \text{ and } E(G) = \{uv, uw, wx, xy, xz\}.$$

Now we have the following graph by considering these sets.



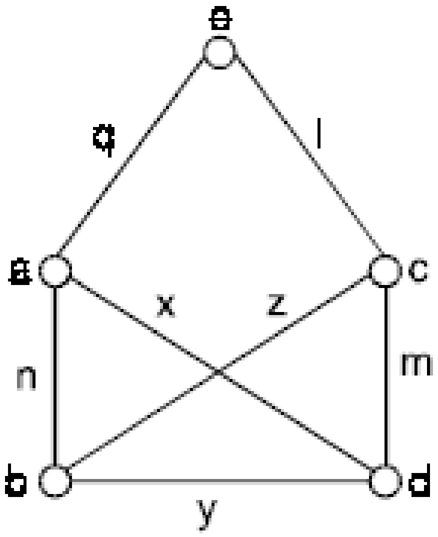
Every graph has a diagram associated with it. The vertex *u* and an edge *e* are **incident** with each other as are *v* and *e*. If two distinct edges say *e* and *f* are **incident** with a common vertex, then they are adjacent edges.

A graph with p-vertices and q-edges is called a **(p, q) graph.**

The (1, 0) graph is called **trivial graph.**

In the following figure the vertices a and b are adjacent but a and c are not. The edges x and y are adjacent but x and z are not.

Although the edges x and z intersect in the diagram, their intersection is not a vertex of the graph.



Examples :

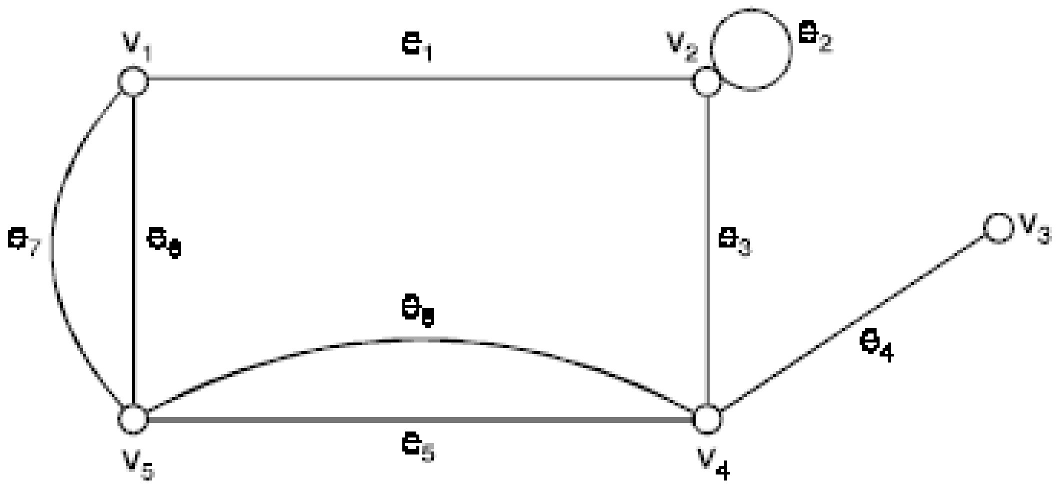
(1) Let $V = \{1, 2, 3, 4\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{3, 2\}. \{4, 4\}\}$.

Then $G(V, E)$ is a graph.

(2) Let $V = \{1, 2, 3, 4\}$ and $E = \{\{1, 5\}, \{2, 3\}\}$.

Then $G(V, E)$ is not a graph, as 5 is not in V.

(3)



A graph with 5-vertices and 8-edges is called a **(5, 8) graph.**

1.2 DIRECTED AND UNDIRECTED GRAPHS

1.2.1. Directed graph

A directed graph or digraph G consists of a set V of vertices and a set E of edges such that $e \in E$ is associated with an ordered pair of vertices.

In other words, if each edge of the graph G has a direction then the graph is called **directed graph.**

In the diagram of directed graph, each edge $e = (u, v)$ is represented by an arrow or directed curve from initial point u of e to the terminal point v.

Figure 1(a) is an example of a directed graph.

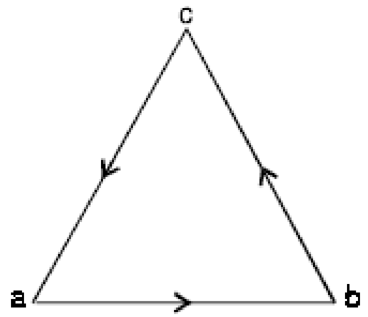


Fig. 1(a). Directed graph.

Suppose $e = (u, v)$ is a directed edge in a digraph, then (i) u is called the **initial vertex** of e and v is the terminal vertex of e

(ii) e is said to be **incident** from u and to be incident to v.

(iii) u is adjacent to v, and v is adjacent from u.

1.2.2. Un-directed graph

An un-directed graph G consists of set V of vertices and a set E of edges such that each edge $e \in E$ is associated with an unordered pair of vertices.

In other words, if each edge of the graph G has no direction then the graph is called **un-directed graph.**

Figure 1(b) is an example of an undirected graph.

We can refer to an edge joining the vertex pair i and j as either (i, j) or (j, i).

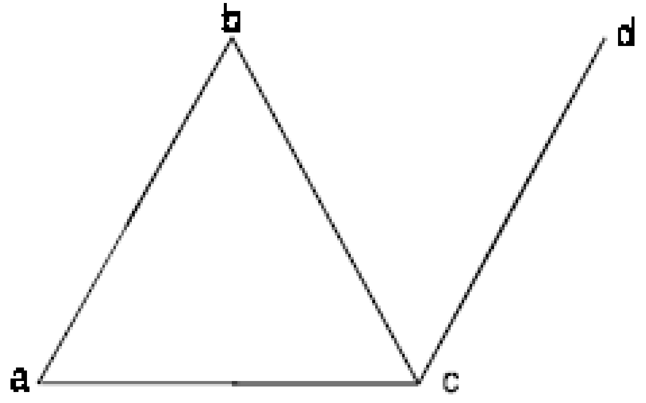


Figure 1(b). Un-directed graph.

1.3 BASIC TERMINOLOGIES

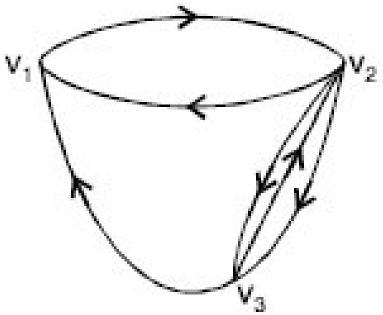
1.3.1 Loop : An edge of a graph that joins a node to itself is called **loop** or **self loop.**

i.e., a loop is an edge (v_i, v_j) where $v_i = v_f$.

1.3.2. Multigraph

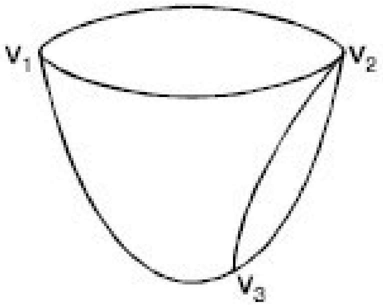
In a multigraph no loops are allowed but more than one edge can join two vertices, these edges are called **multiple edges** or parallel edges and a graph is called **multigraph.**

Two edges (v_i, v_j) and (v_f, v_r) are parallel edges if $v_i = v_r$ and v_j, v_f.



Directed multigraph

Fig. 2(*a*)



Un-directed multigraph

Fig. 2(*b*)

In Figure 1.2(*a*), there are two parallel edges associated with v_2 and v_3.

In Figure 1.2(*b*), there are two parallel edges joining nodes v_1 and v_2 and v_2 and v_3.

1.3.3. Pseudo graph

A graph in which loops and multiple edges are allowed, is called a pseudo graph.

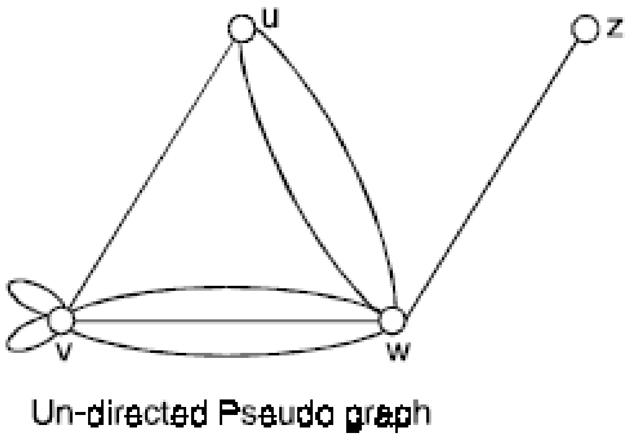


Fig. 3(*a*)

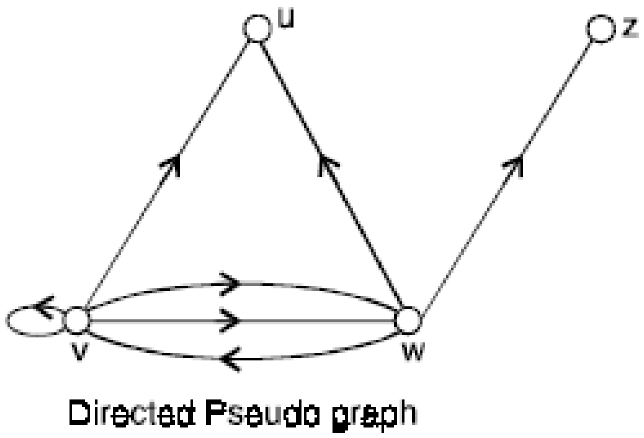


Fig. 3(*b*)

1.3.4. Simple graph

A graph which has neither loops nor multiple edges. *i.e.,* where each edge connects two distinct vertices and no two edges connect the same pair of vertices is called a **simple graph.**

Figure 1.1(*a*) and (*b*) represents simple undirected and directed graph because the graphs do not contain loops and the edges are all distinct.

1.3.5. Finite and Infinite graphs

A graph with finite number of vertices as well as a finite number of edges is called a **finite graph.** Otherwise, it is an **infinite graph.**

1.4 DEGREE OF A VERTEX

The number of edges incident on a vertex v_i with **self-loops counted twice** (is called the **degree of a vertex** v_i and is denoted by $\mathbf{deg_G}(v_i)$ or **deg** v_i or $d(v_i)$.

The degrees of vertices in the graph G and H are shown in Figure 4(a) and 4(b).

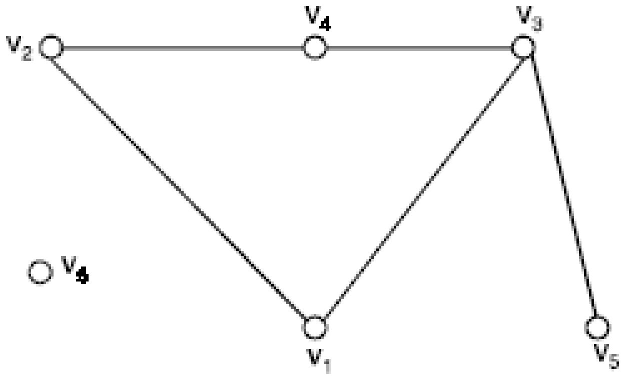


Fig. 4(a)

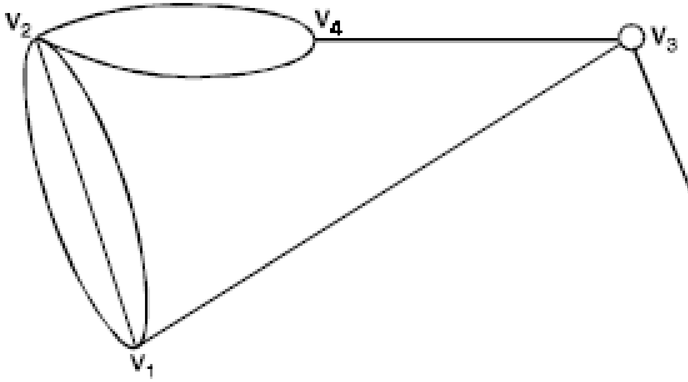


Fig. 4(b)

In G as shown in Figure 4(a),

$\deg_G(v_2) = 2 = \deg_G(v_4) = \deg_G(v_1)$, $\deg_G(v_3) = 3$ and $\deg_G(v_5) = 1$ and

In H as shown in Figure 4(b),

$\deg_H(v_2) = 5$, $\deg_H(v_4) = 3$, $\deg_H(v_3) = 5$, $\deg_H(v_1) = 4$ and $\deg_H(v_5) = 1$.

The degree of a vertex is some times also referred to as its **valency.**

1.5 ISOLATED AND PENDENT VERTICES

1.5.1. Isolated vertex

A vertex having **no incident edge** is called an **isolated vertex.**

In other words, isolated vertices are those with zero degree.

1.5.2. Pendent or end vertex

A vertex of **degree one,** is called a **pendent vertex** or an **end vertex.**

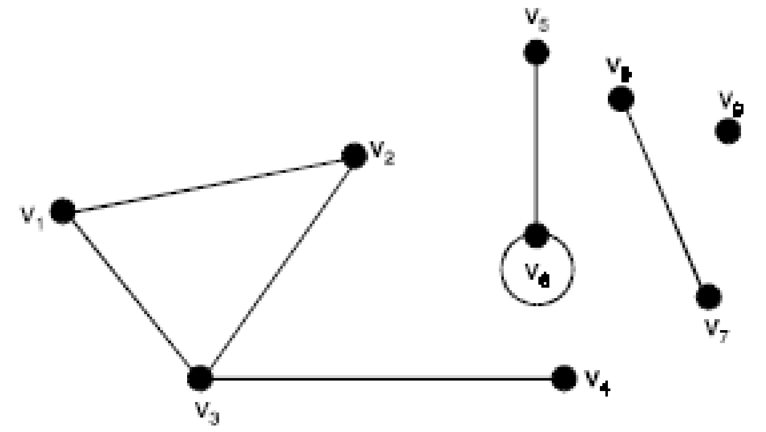
In the above Figure, v_5 is a pendent vertex.

1.5.3. In degree and out degree

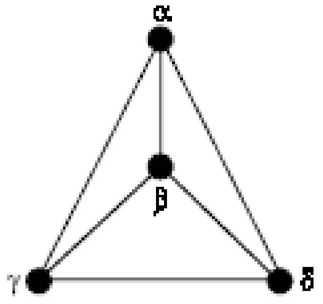
In a graph G, the out degree of a vertex v_i of G, denoted by out $\deg_G(v_i)$ or $\deg_G^+(v_i)$, is the number of edges beginning at v_i and the in degree of v_i, denoted by in $\deg_G(v_i)$ or $\deg_G^{-1}(v_i)$, is the number of edges ending at v_i.

The sum of the in degree and out degree of a vertex is called the **total degree** of the vertex. A vertex with zero in degree is called a **source** and a vertex with zero out degree **is called a sink.** Since each edge has an initial vertex and terminal vertex.

Problem. *Write down the vertex set and edge set of the following graphs shown in Figures below*



(a)



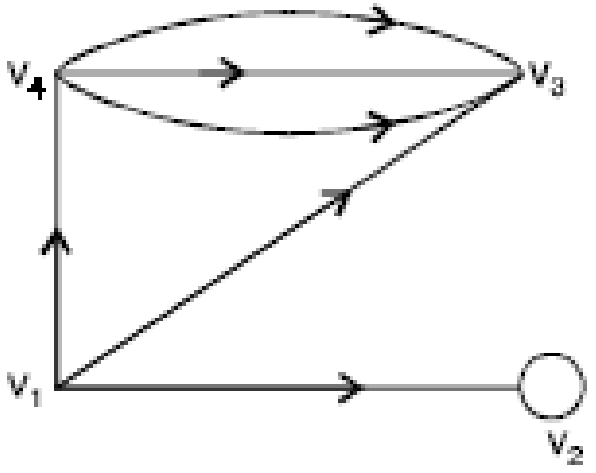
(b)

Problem . *It is possible to draw a simple graph with 4 vertices and 7 edges ? Justify.*

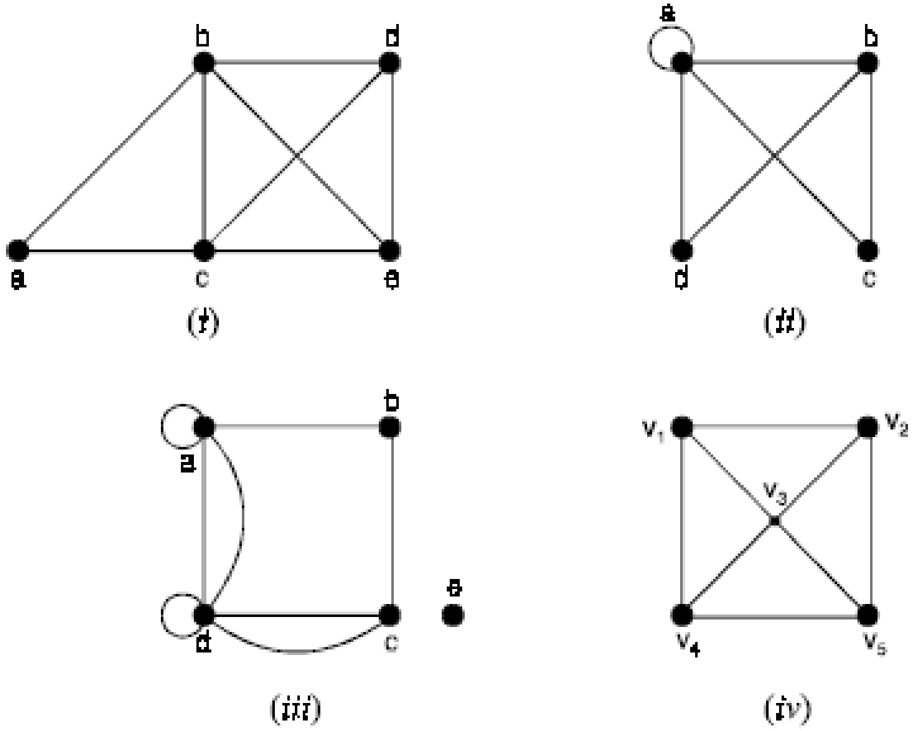
Solution. In a simple graph with P-vertices, the maximum number of edges will be P(P -1)/2.

Hence a simple graph with 4 vertices will have at most
(4 x 3)/2= 6 edges.
Therefore, the simple graph with 4 vertices cannot have 7 edges.
Hence such a graph does not exist.

Problem. *Find the in degree out degree and of total degree of each vertex of the following graph.*



Problem . *Consider the following graphs and determine the degree of each vertex*



1.6 THE HANDSHAKING THEOREM 1.1

If $G = (v, E)$ be an undirected graph with e edges.

Then $\sum_{v \in V} \deg_G(v) = 2e$

i.e., the sum of degrees of the vertices is an undirected graph is even.

Proof : Since the degree of a vertex is the number of edges incident with that vertex, the sum of the degree counts the total number of times an edge is incident with a vertex.

Since every edge is incident with exactly two vertices, each edge gets counted twice, once at each end.

Thus the sum of the degrees equal twice the number of edges.

Note : This theorem applies even if multiple edges and loops are present. The above theorem holds this rule that if several people shake hands, the total number of hands shake must be even that is why the theorem is called handshaking theorem.

Corollary : In a non directed graph, the total number of odd degree vertices is even.

Proof : Let G = (V, E) a non directed graph.

Let U denote the set of even degree vertices in G and W denote the set of odd degree vertices.

Then $\sum_{v_i \in V} \deg_G (v_i) = \sum_{v_i \in U} \deg_G (v_i) + \sum_{v_i \in W} \deg_G (v_i)$

$$\Rightarrow \quad 2e - \sum_{v_i \in U} \deg_G (v_1) = \sum_{v_i \in W} \deg_G (v_1) \qquad ...(1)$$

Now $\sum_{v_i \in W} \deg_G (v_i)$ is also even

Therefore, from (1) $\sum_{v_i \in W} \deg_G (v_i)$ is even

$\therefore$ The no. of odd vertices in G is even.

Problem *: Determine the number of edges in a graph with 6 vertices, 2 of degree 4 and 4 of degree 2. Draw two such graphs.*

Solution. Suppose the graph with 6 vertices has e number of edges. Therefore by Handshaking lemma

$$\sum_{i=1}^{6} \deg(v_i) = 2e$$

$$\Rightarrow \quad d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) + d(v_6) = 2e$$

Now, given 2 vertices are of degree 4 and 4 vertices are of degree 2.

Hence the above equation,

$$(4 + 4) + (2 + 2 + 2 + 2) = 2e$$

$$\Rightarrow \quad 16 = 2e \quad \Rightarrow \quad e = 8.$$

Hence the number of edges in a graph with 6 vertices with given condition is 8.

Two such graphs are shown below in Figure (11).

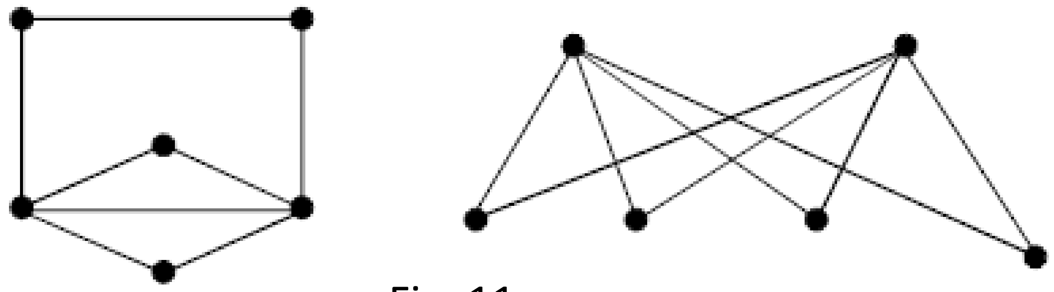


Fig. 11

Theorem 1.2. If V = $\{v_1, v_2, \ldots\ldots v_n\}$ is the vertex set of a non directed graph G,

$$\text{then } \sum_{i=1}^{n} \deg(v_i) = 2\,|\,E\,|$$

If G is a directed graph, then $\sum_{i=1}^{n} \deg^{+}(v_i) = \sum_{i=1}^{n} \deg^{-}(v_i) = |\,E\,|$

Proof : Since when the degrees are summed.

Each edge contributes a count of one to the degree of each of the two vertices on which the edge is incident.

Corollary (1) : In any non directed graph there is an even number of vertices of odd degree.

Proof : Let W be the set of vertices of odd degree and let U be the set of vertices of even degree.

Then $\sum_{v \in V(G)} \deg(v) = \sum_{v \in W} \deg(v) + \sum_{v \in U} \deg(v) = 2\,|\,E\,|$

Certainly, $\sum_{v \in U} \deg(v)$ is even,

Hence $\sum_{v \in W} \deg(v)$ is even,

Implying that $|\,W\,|$ is even.

Corollary (2) : If $k = \delta(G)$ is the minimum degree of all the vertices of a non directed graph G, then

$$k\,|\,V\,| \le \sum_{v \in V(G)} \deg(v) = 2\,|\,E\,|$$

In particular, if G is a k-regular graph, then

$$k\,|\,V\,| = \sum_{v \in V(G)} \deg(v) = 2\,|\,E\,|.$$

1.7 TYPES OF GRAPHS

Some important types of graph are introduced here.

1.7.1. Null graph

A graph which contains only **isolated node,** is called a null graph.

i.e., the set of edges in a null graph is empty.

Null graph is denoted on n vertices by N_n

N_4 is shown in Figure (13), Note that each vertex of a null graph is isolated.

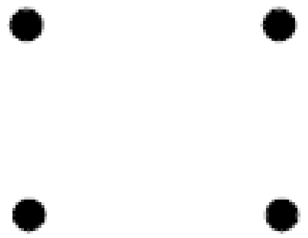


Fig. 13.

1.7.2. Complete graph

A simple graph G is said to be **complete** if every vertex in G is connected with every other vertex.

i.e., if G contains exactly one edge between each pair of distinct vertices.

A comple graph is usually denoted by $\mathbf{K}_n$. It should be noted that K_n has exactly $\dfrac{n(n-1)}{2}$ edges.

The graphs K_n for $n = 1, 2, 3, 4, 5, 6$ are show in Figure 14.

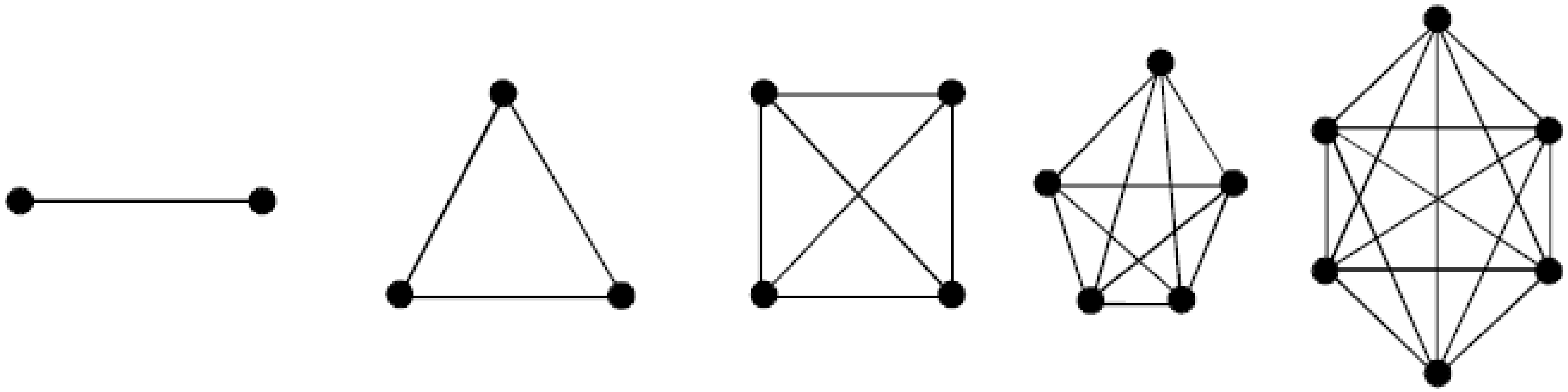


Fig. 14.

1.7.3. Regular graph

A graph in which all vertices are of **equal degree,** is called a **regular graph.**

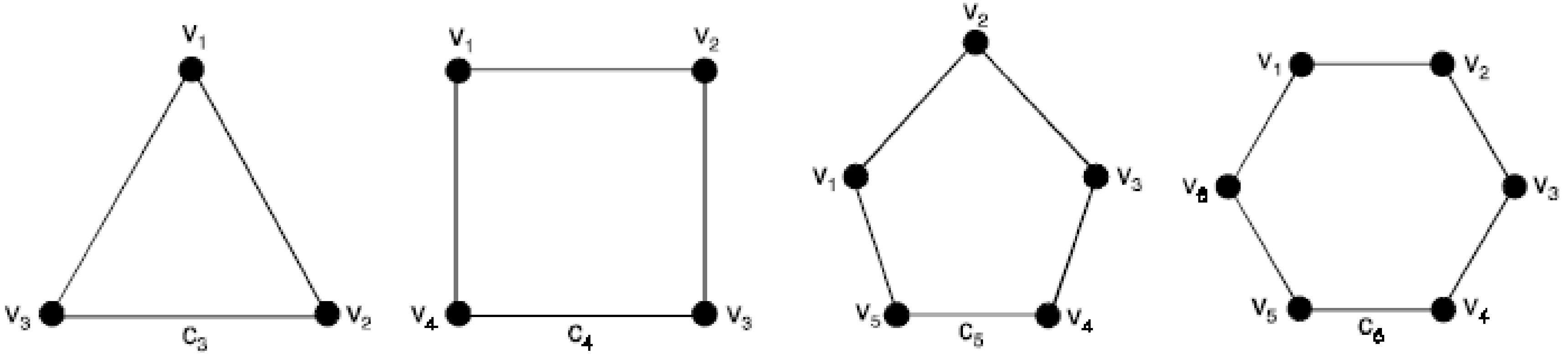
If the degree of each vertex is r, then the graph is called a regular **graph of degree** r**.**

Note that every null graph is regular of degree zero, and that the complete graph K_n is a regular of degree $n - 1$. Also, note that, if G has n vertices and is regular of degree r, then G has $\left(\frac{1}{2}\right) r\, n$ edges.

1.7.4. Cycles

The cycle C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots\ldots, v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots\ldots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$.

The cyles c_3, c_4, c_5 and c_6 are shown in Figure 15.



1.7.5. Wheels

The wheel W_n is obtained when an additional vertex to the cycle c_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in c_n, by new edges. The wheels W_3, W_4, W_5 and W_6 are displayed in Figure 16.

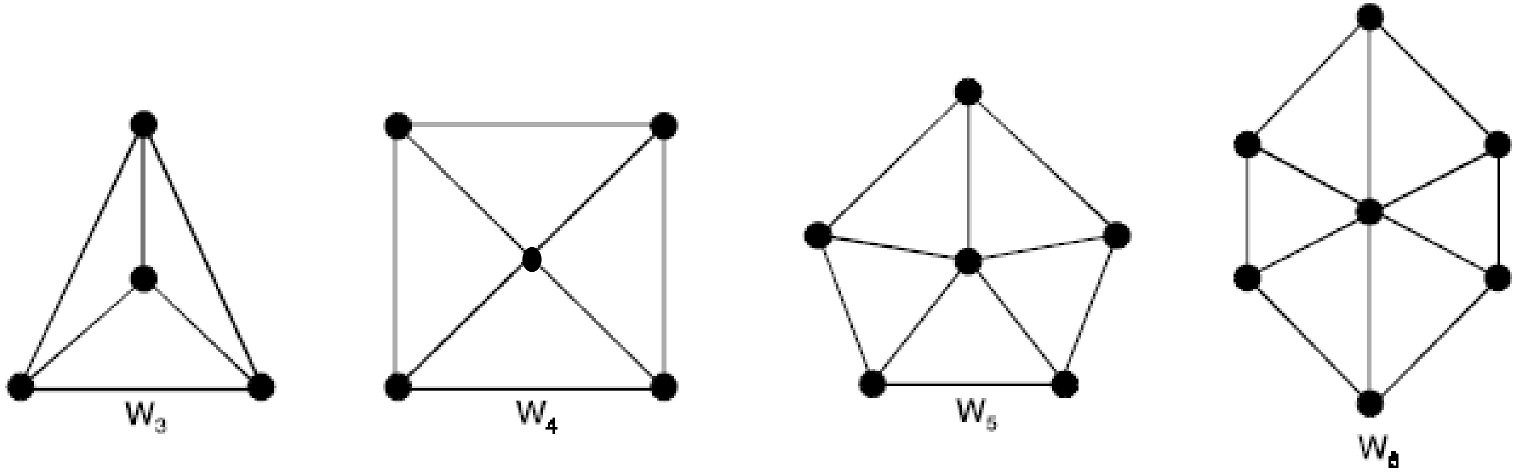


Figure 16. The wheels W_3, W_4, W_5 and W_6

1.7.6. Platonic graph

The graph formed by the vertices and edges of the five regular (platonic) solids—The tetrahedron, octahedron, cube, dodecahedron and icosahedron.

The graphs are shown in Figure 17.

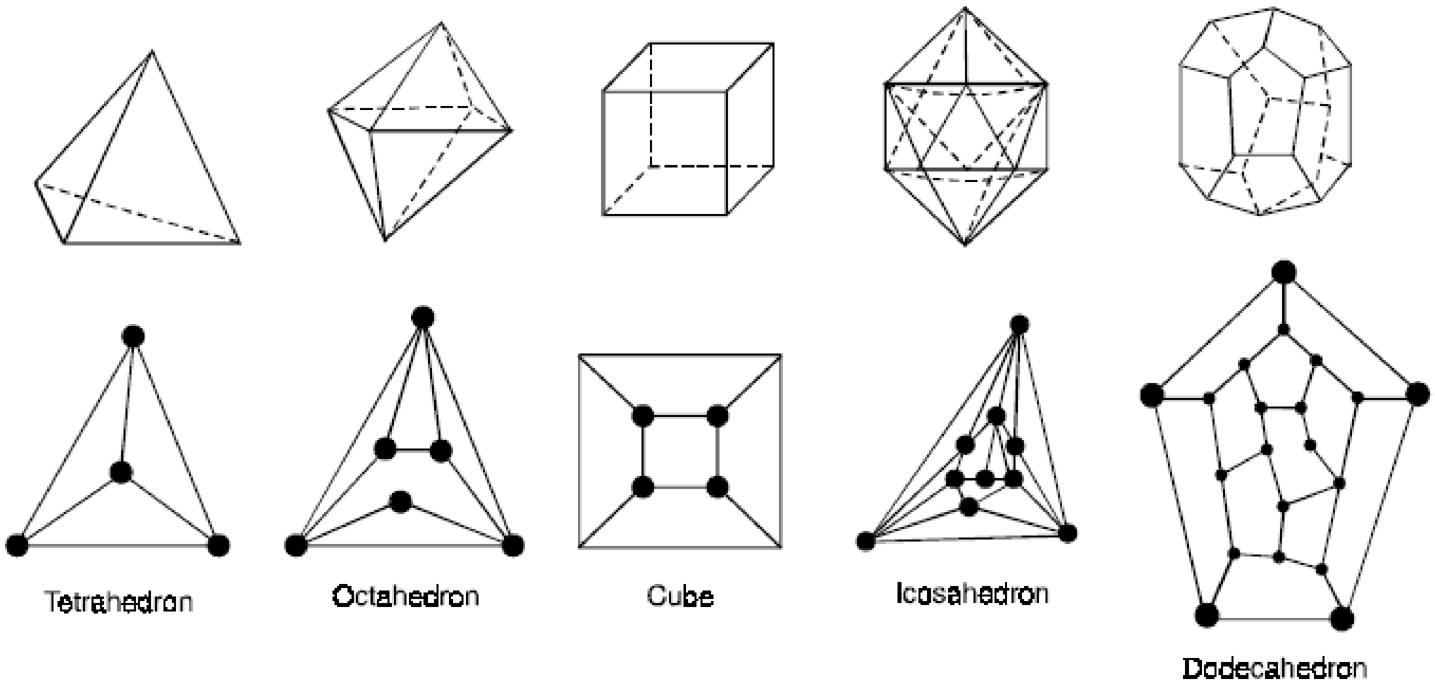


Fig. 17.

1.7.7. N-cube

The N-cube denoted by Q_n, is the graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position. The graphs Q_1, Q_2, Q_3 are displayed in Figure 18. Thus Q_n has 2^n vertices and $n \cdot 2^{n-1}$ edges, and is regular of degree n.

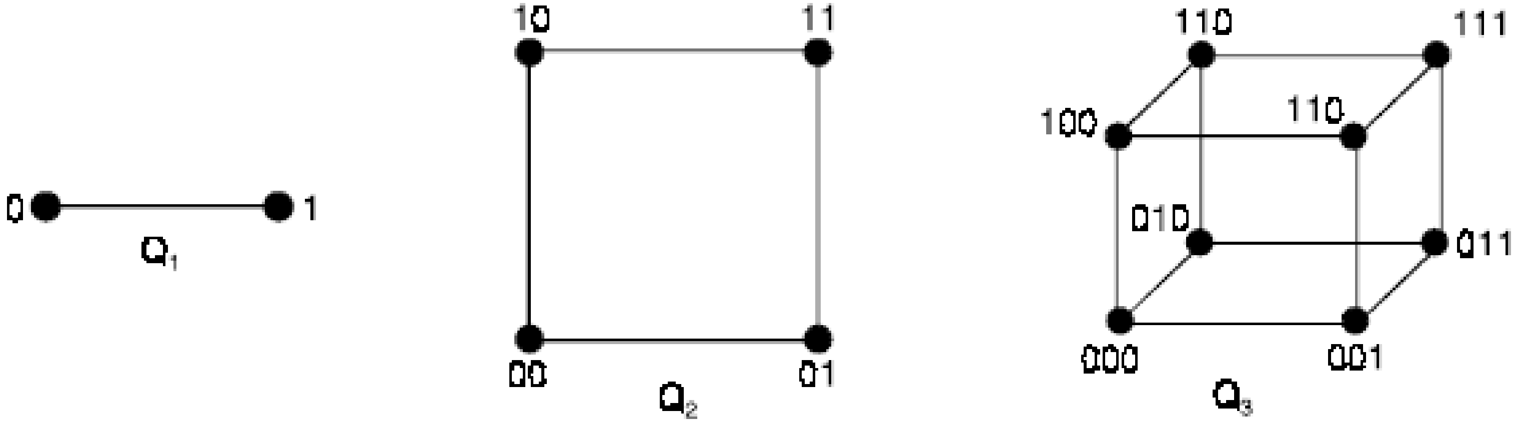
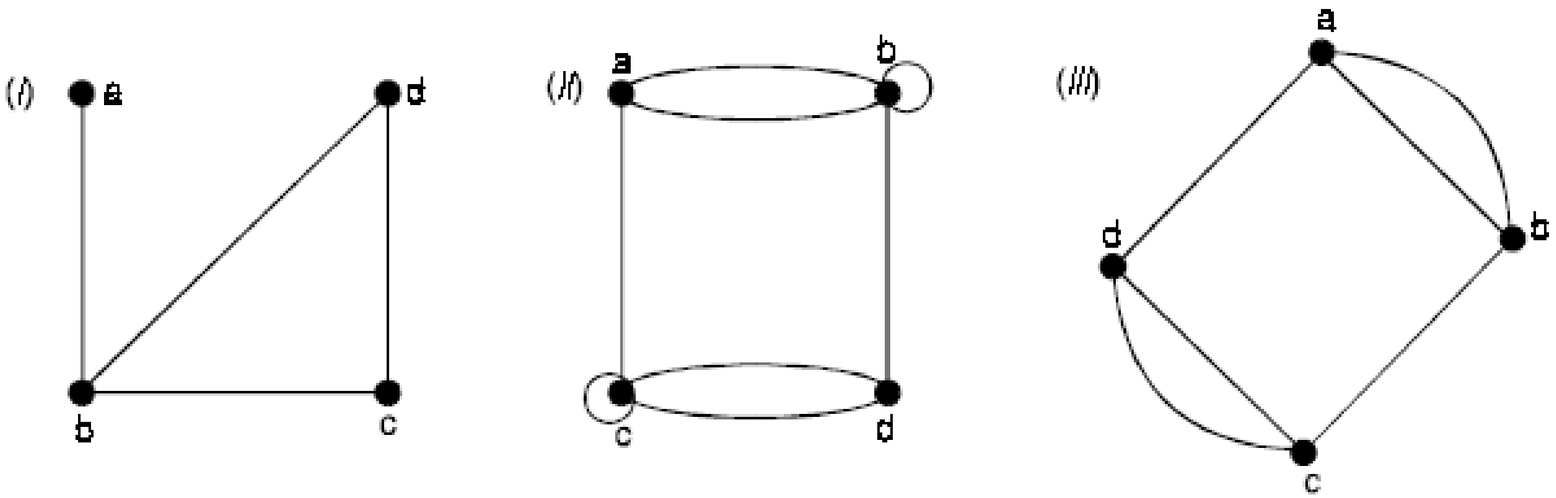


Fig. 18. The n-cube Q_n for $n = 1, 2, 3$.

Problem 1.34. *Determine whether the graphs shown is a simple graph, a multigraph, a pseudograph.*



Solution. (*i*) Simple graph

(*ii*) Pseudograph

(*iii*) Multigraph.

1.8 SUBGRAPH

A subgraph of G is a graph having all of its vertices and edges in G. If G_1 is a subgraph of G, then G is a super graph of G_1.

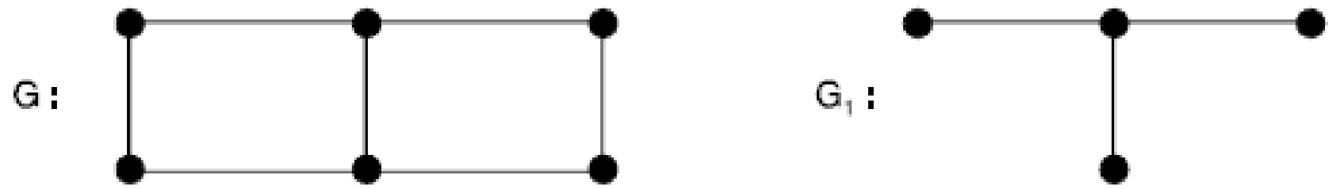


Fig. 19. G_1 is a subgraph of G.

In other words. If G and H are two graphs with vertex sets V(H), V(G) and edge sets E(H) and E(G) respectively such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ then we call H as a subgraph of G or G as a supergraph of H.

1.8.1. Spanning subgraph

A spanning subgraph is a subgraph containing all the vertices of G.

In other words, if $V(H) \subset V(G)$ and $E(H) \subseteq E(G)$ then H is a proper subgraph of G and if $V(H) = V(G)$ then we say that H is a spanning subgraph of G.

A spanning subgraph need not contain all the edges in G.

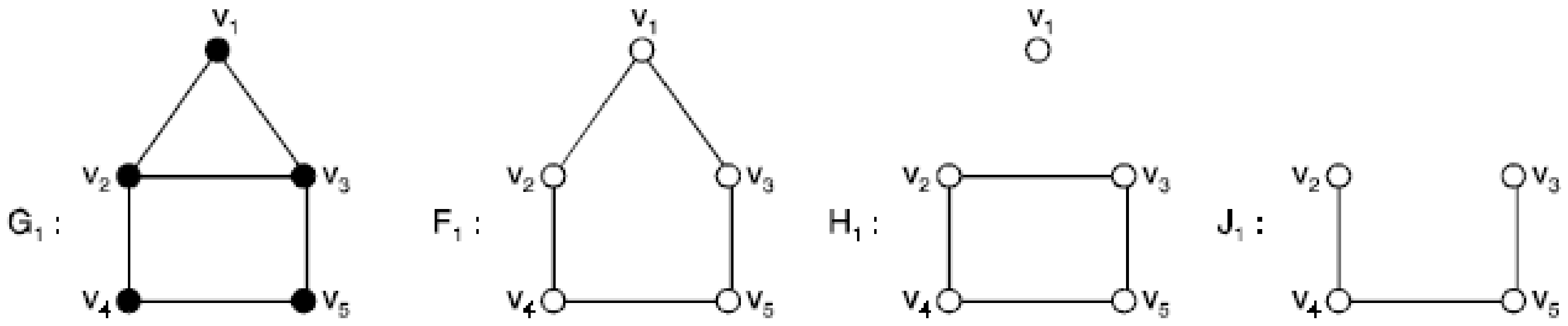


Fig. 20.

The graphs F_1 and H_1 of the above Fig. 20 are spanning subgraphs of G_1, but J_1 is not a spanning subgraph of G_1.

Since $V_1 \in V(G_1) - V(J_1)$. If E is a set of edges of a graph G, then G – E is a spanning subgraph of G obtained by deleting the edges in E from E(G).

In fact, H is a spanning subgraph of G if and only if $H = G - E$, where $E = E(G) - E(H)$. If e is an edge of a graph G, then we write $G - e$ instead of $G - \{e\}$. For the graphs G_1, F_1 and H_1 of the Fig. 20, we have $F_1 = G_1 - v_2v_3$ and $H_1 = G_1 - \{v_1v_2, v_2v_3\}$.

1.8.2. Removal of a vertex and an edge

The removal of a vertex v_i from a graph G result in that subgraph G – v_i of G containing of all vertices in G except v_i and all edges not incident with v_i. Thus G – v_i is the maximal subgraph of G not containing v_i. On the otherhand, the removal of an edge x_j from G yields the spanning subgraph G – x_j containing all edges of G except x_j.

Thus G – x_j is the maximal subgraph of G not containing x_j.

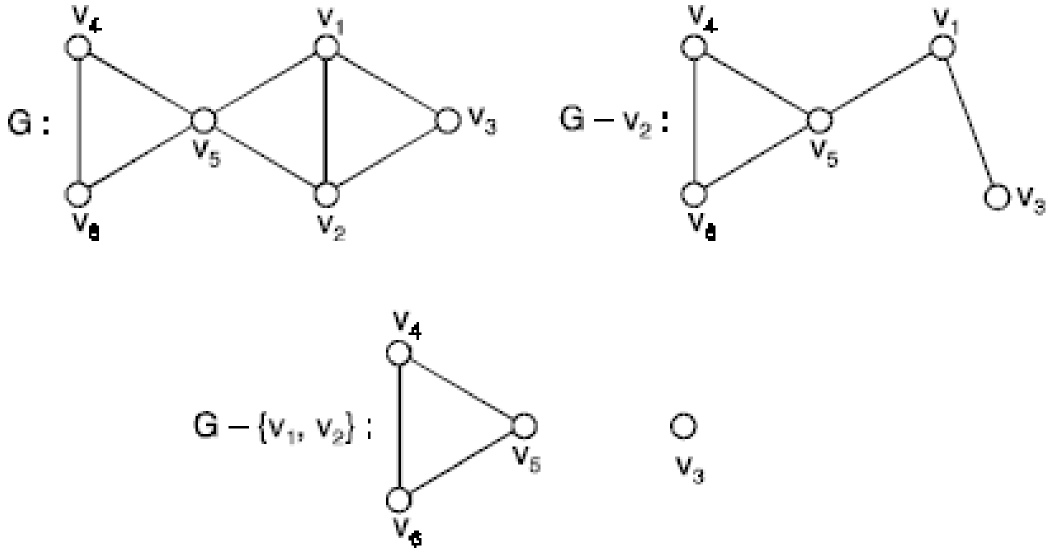


Fig. 21(*a*). Deleting vertices from a graph.

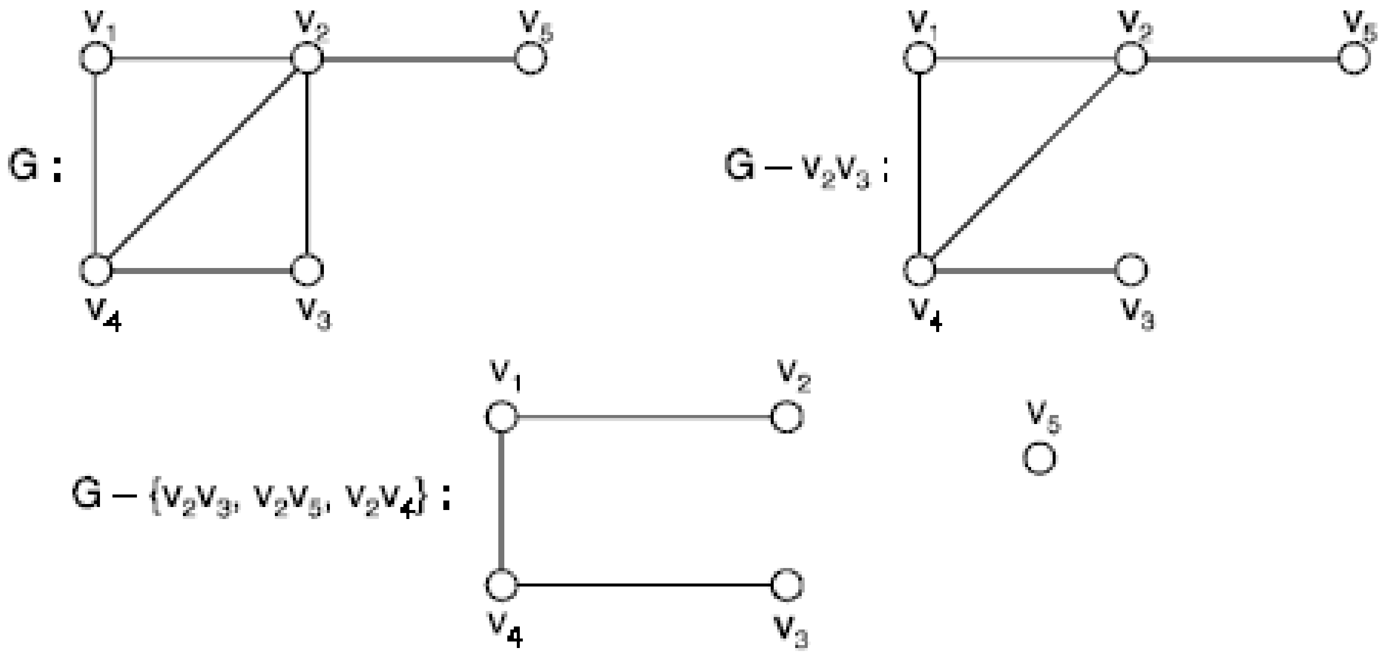
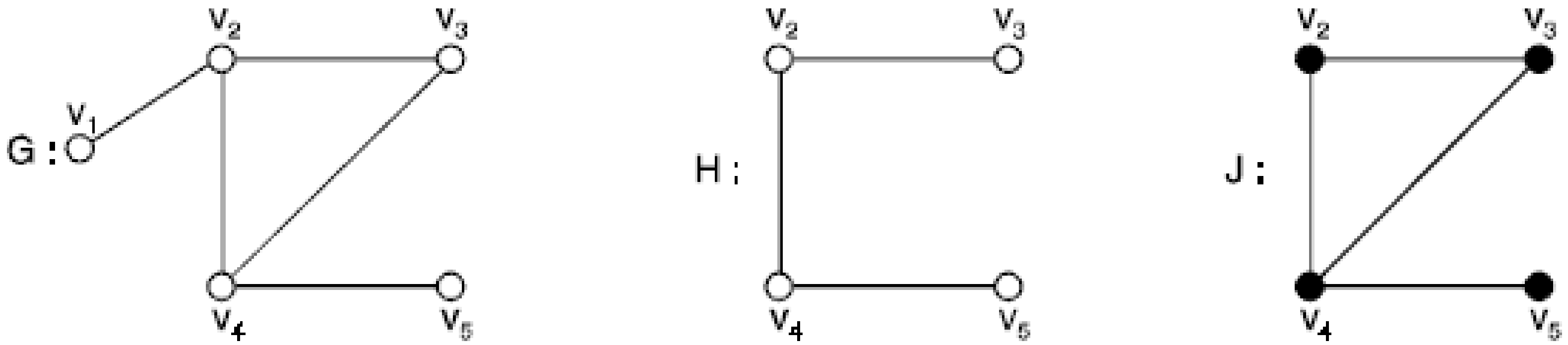


Fig. 21(*b*). Deleting edges from a graph.

1.8.3. Induced subgraph

For any set S of vertices of G, the vertex induced subgraph or simply an induced subgraph <S> is the maximal subgraph of G with vertex set S. Thus two vertices of S are adjacent in <S> if and only if they are adjacent in G.

In other words, if G is a graph with vertex set V and U is a subset of V then the subgraph G(U) of G whose vertex set is U and whose edge set comprises exactly the edges of E which join vertices in U is termed as induced subgraph of G.



Here H is not an induced subgraph since $v_4v_1 \in E(G)$, but $v_4v_3 \notin E(H)$.

On the otherhand the graph J is an induced subgraph of G. Thus every induced subgraph of a graph G is obtained by deleting a subset of vertices from G.

Note : Let $|V| = m$ and $|E| = n$. The total non-empty subsets of V is $2^m - 1$ and total subsets of E is 2^n.

Thus, number of subgraphs is equal to $(2^m - 1) \times 2^n$.

The number of spanning subgraphs is equal to 2^n.

1.9 GRAPHS ISOMORPHISM

Let $G_1 = (v_1, E_1)$ and $G_2 = (v_2, E_2)$ be two graphs. A function $f : v_1 \rightarrow v_1$ is called a graphs isomorphism if

(*i*) f is one-to-one and onto.

(*ii*) for all $a, b \in v_1$, $\{a, b\} \in E_1$ if and only if $\{f(a), f(b)\} \in E_2$ when such a function exists, G_1 and G_2 are called isomorphic graphs and is written as $G_1 \equiv G_2$.

In other words, two graphs G_1 and G_2 are said to be isomorphic to each other if there is a one-to-one correspondence between their vertices and between edges such that incidence relationship is preserve. Written as $G_1 \equiv G_2$ or $G_1 = G_2$.

The necessary conditions for two graphs to be isomorphic are

1. Both must have the **same number of vertices**
2. Both must have the **same number of edges**
3. Both must have **equal number of vertices** with the **same degree.**
4. They must have the same degree sequence and same cycle vector $(c_1, ..., c_n)$, where c_i is the number of cycles of length i.

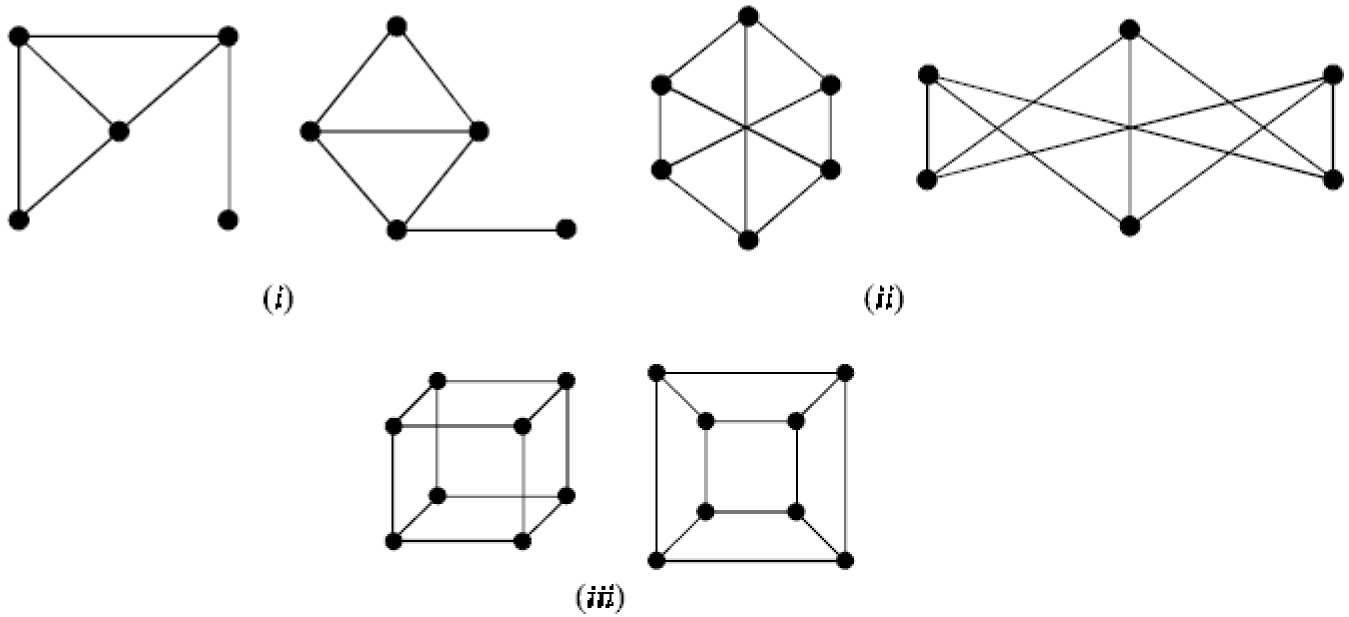


Fig. 22(*i*), (*ii*) (*iii*) Isomorphic pair of graphs

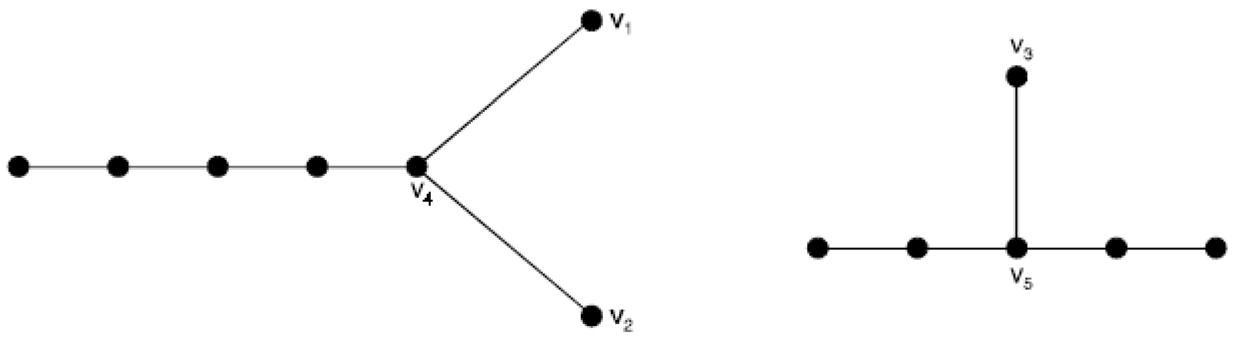
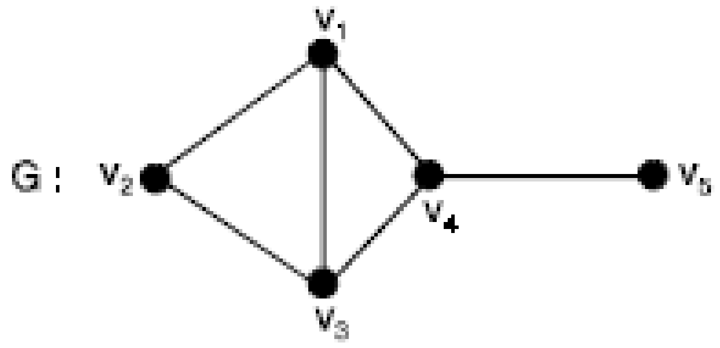
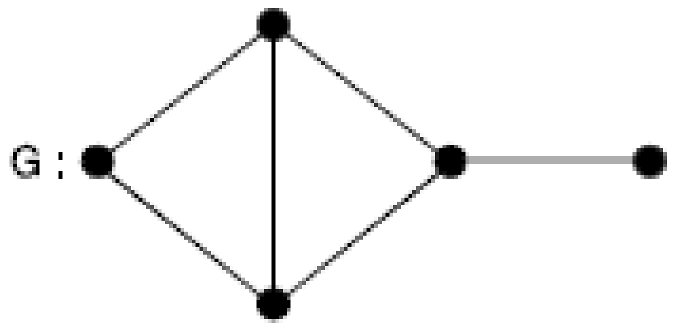


Fig. 23. Two graphs that are not isomorphic.

Problem. *Construct two edge-disjoint subgraphs and two vertex disjoint subgraphs of a graph G shown below*



Problem. *Construct three non-isomorphic spanning subgraphs of the graph G shown below :*



Problem. *Show that the following graphs are isomorphic*