2. Connected and Disconnected Graphs

2.1 connected and disconnected graphs
2.2 Path and cycle graphs
2.3 Rank and Nullity
2.4 Walk, Path and Circuits

2. CONNECTED AND DISCONNECTED GRAPHS

A graph G is said to be a **connected** if every pair of vertices in G are connected. Otherwise, G is called a **disconnected** graph. Two vertices in G are said to be connected if there is at least one path from one vertex to the other.

In other words, a graph G is said to be connected if there is at least one path between every two vertices in G and disconnected if G has at least one pair of vertices between which there is no path.

A graph is **connected** if we can reach any vertex from any other vertex by travelling along the edges and disconnected otherwise.

For example, the graphs in Figure 30(*a, b, c, d, e*) are connected whereas the graphs in Figure 31(*a, b, c*) are disconnected.

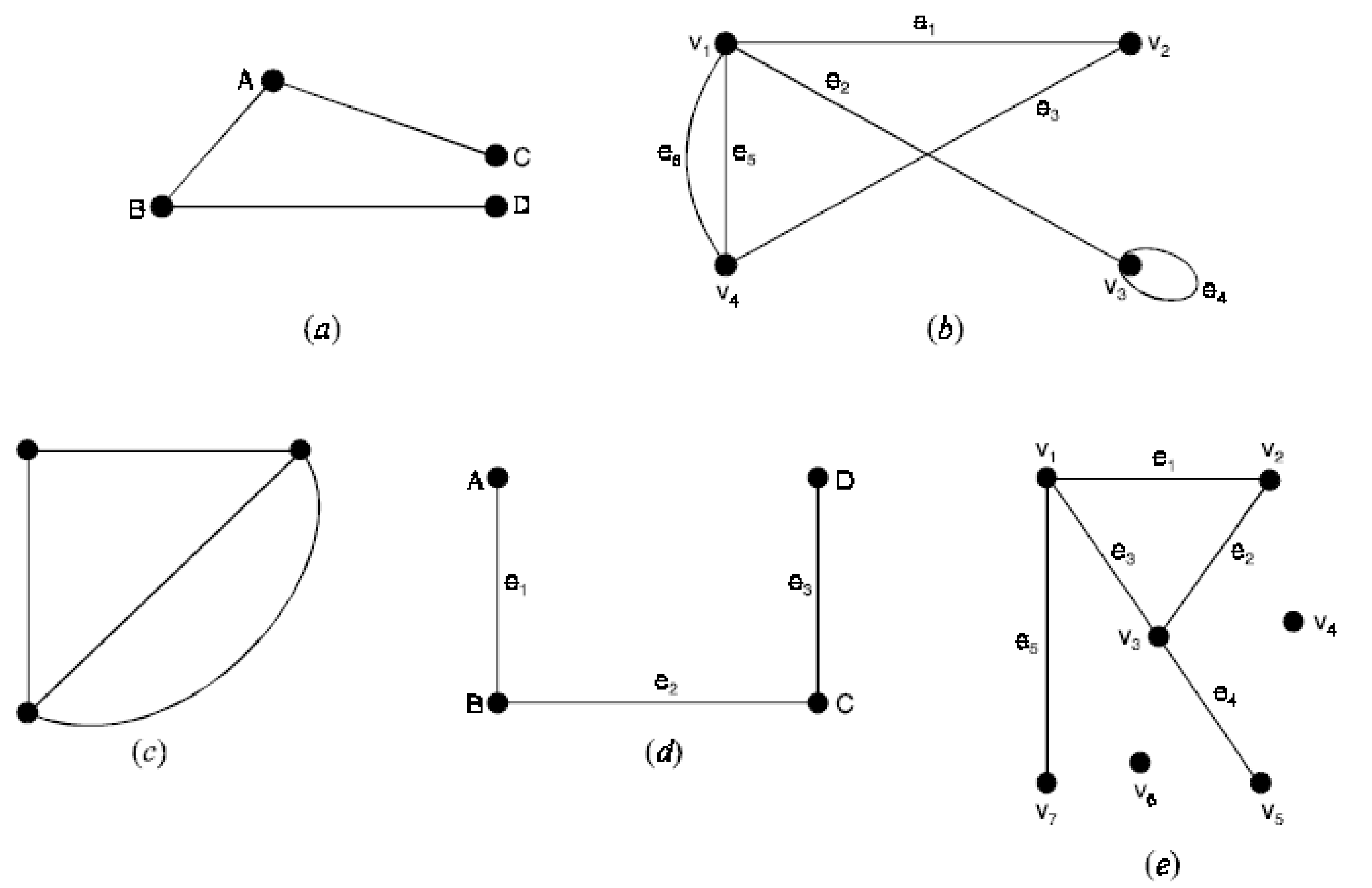

A
B
C
D
(a)
v1
v2
v3
v4
e1
e2
e3
e4
e5
e6
(b)
(c)
A
B
C
D
e1
e2
e3
(d)
v1
v2
v3
v4
v5
v6
v7
e1
e2
e3
e4
e5
(e)

Fig. 30

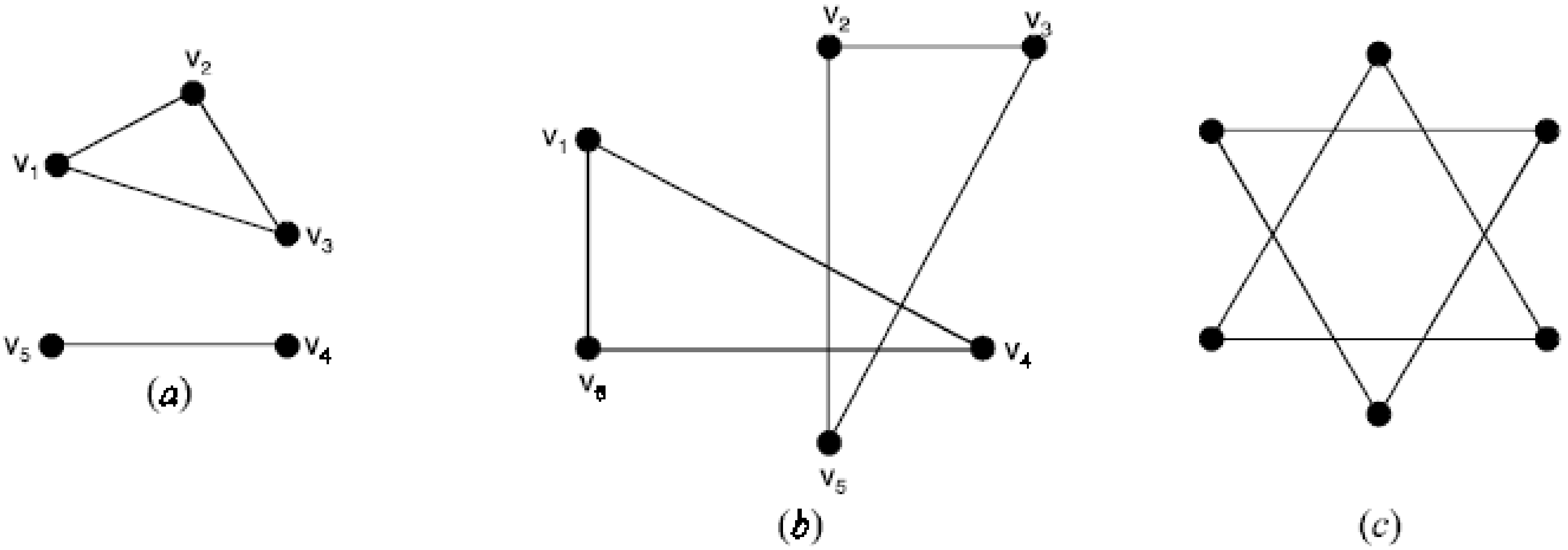


Fig. 31.

A complete graph is always connected, also, a null graph of more than one vertex is disconnected (see Fig. 32). All paths and circuits in a graph G are connected subgraphs of G.

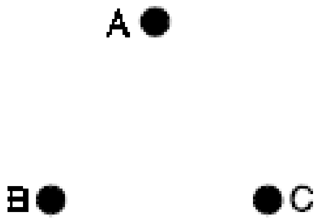


Fig. 32.

Every graph G consists of one or more connected graphs, each such connected graph is a subgraph of G and is called a component of G. A connected graph has only one component and a disconnected graph has two or more components.

For example, the graphs in Figure 31(*a, b*) have two components each.

2.1 Path graphs and cycle graphs

A connected graph that is 2-regular is called a cycle graph. Denote the cycle graph of n vertices by Γ_n. A circuit in a graph, if it exists, is a cycle subgraph of the graph.

The graph obtained from Γ_n by removing an edge is called the path graph of n vertices, it is denoted by P_n.

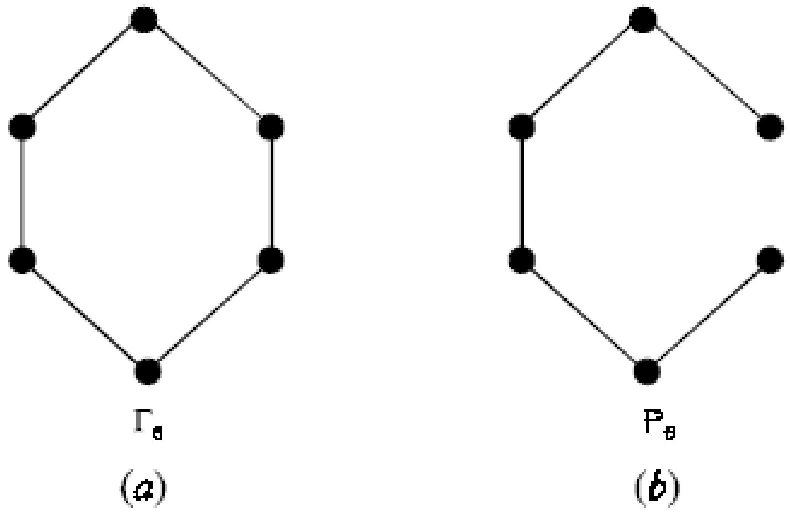


Fig. 33.

The graphs Γ_6 and P_6 are shown in Figure 33(a) and 33(b) respectively.

2.2 Rank and nullity

For a graph G with n vertices, m edges and k components we define the rank of G and is denoted by $\rho(G)$ and the nullity of G is denoted by $\mu(G)$ as follows.

$$\rho(G) = \text{Rank of } G = n - k$$

$$\mu(G) = \text{Nullity of } G = m - \rho(G) = m - n + k$$

If G is connected, then we have

$$\rho(G) = n - 1 \text{ and } \mu(G) = m - n + 1.$$

2.3 WALKS, PATHS AND CIRCUITS

2.3.1 Walk

A **walk** is defined as a finite **alternative sequence** of vertices and edges, of the form

$$v_i e_j, v_{i+1}\, e_{j+1}, v_{i+2}, \ldots\ldots, e_k v_m$$

which **begins** and **ends** with **vertices,** such that

(*i*) **each edge in the sequence is incident on the vertices preceding and following it in the sequence.**

(*ii*) **no edge appears more than once in the sequence, such a sequence is called a walk or a trial in G.**

For example, in the graph shown in Figure 34, the sequences

$v_2e_4v_6e_5v_4e_3v_3$ and $v_7e_8v_2e_4v_6e_6v_5e_7v_5$ are walks.

Note that in the first of these, each vertex and each edge appears only once whereas in the second each edge appears only once but the vertex v_5 appears twice.

These walks may be denoted simply as $v_2v_6v_4v_3$ and $v_7v_2v_6v_5v_5$ respectively.

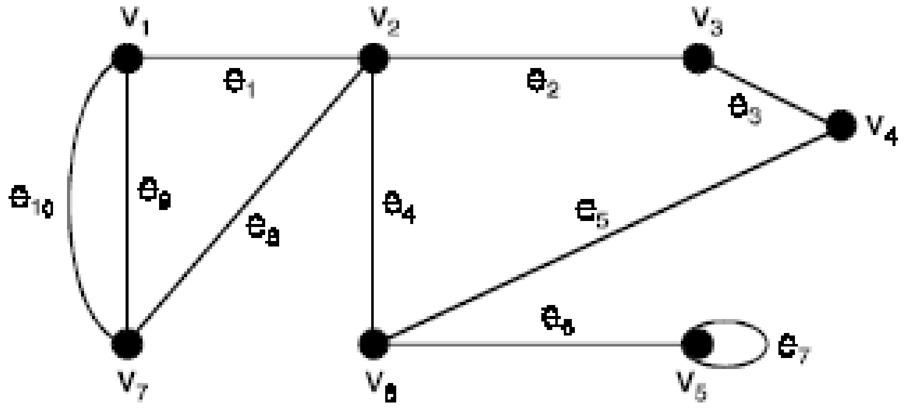


Fig. 34.

The vertex with which a walk begins is called the initial vertex and the vertex with which a walk ends is called the final vertex of the walk. The initial vertex and the final vertex are together called terminal vertices. Non-terminal vertices of a walk are called its internal vertices.

A walk having u as the initial vertex and v as the final vertex is called a walk from u to v or briefly a $u-v$ **walk.** A walk that begins and ends at the same vertex is called a **closed walk. In other words,** a closed walk is a walk in which the terminal vertices are coincident.

A walk that is not closed is called an **open walk.**

In other words, an open walk is a walk that begins and ends at two different vertices.

For example, in the graph shown in Figure 34.

$v_1e_9v_7e_8v_2e_1v_1$ is a closed walk and $v_5e_7v_5e_6v_6e_5v_4$ is an open walk.

2.3.2 Path

In a walk, a vertex can appear more than once. An open walk in which no vertex appears more than once is called a **simple path** or a **path.**

For example, in the graph shown in Figure 34.

$v_6e_5v_4e_3v_3e_2v_2$ is a path whereas $v_5e_7v_5e_6v_6$ is an open walk but not a path.

2.3.3 Circuit

A closed walk with atleast one edge in which no vertex except the terminal vertices appears more than once is called **a circuit** or a cycle.

For example, in the graph shown in Figure 34,

$v_1e_1v_2e_8v_7e_9v_1$ and $v_2e_4v_6e_5v_4e_3v_3e_2v_2$ are circuits.

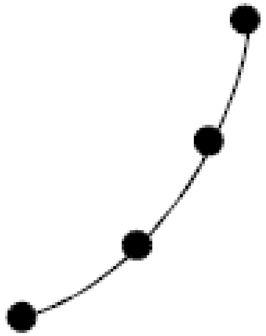
But $v_1e_9v_7e_8v_2e_4v_6e_5v_4e_3v_3e_2v_2e_1v_1$ is a closed walk but not a circuit.

Note : (*i*) In walks, path and circuit, no edge can appears more than once.

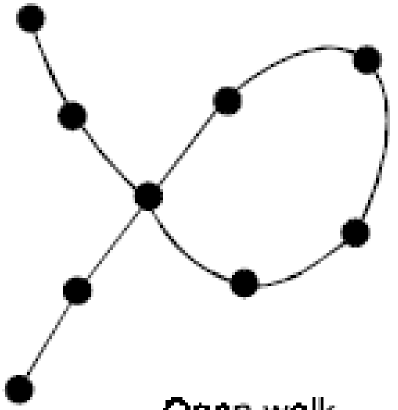
(*ii*) A vertex can appear more than once in a walk but not in a path.

(*iii*) A path is an open walk, but an open walk need not be a path.

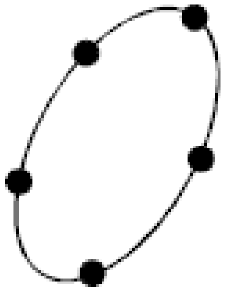
(*iv*) A circuit is a closed walk, but a closed walk need not be a circuit.



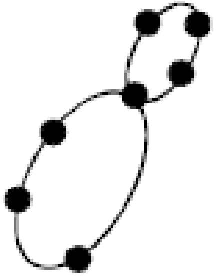
Open walk
which is a path



Open walk
which is not a path



Closed walk
which is a circuit



Closed walk
which is not a circuit.

2.3.4 **Length**

The number of edges in a walk is called its length. Since paths and circuits are walks, it follows that the length of a path is the number of edges in the path and the length of a circuit is the number of edges in the circuit.

A circuit or cycle of length k, (with k edges) is called a k-circuit or a k-cycle. A k-circuit is called odd or even according as k is odd or even. A 3-cycle is called a triangle.

For example, in the graph shown in Figure 34,

The length of the open walk $v_6e_6v_5e_7v_5$ is 2

The length of the closed walk $v_1e_9v_7e_8v_2e_1v_1$ is 3

The length of the circuit $v_2e_4v_6e_5v_4e_3v_3e_2v_2$ is 4

The length of the path $v_6e_5v_4e_3v_3e_2v_2e_1v_1$ is 4

The circuit $v_1e_1v_2e_8v_7e_{10}v_1$ is a triangle.

Note : (*i*) A self-loop is a 1-cycle.

(*ii*) A pair of parallel edges form a cycle of length 2.

(*iii*) The edges in a 2-cycle are parallel edges.