Lecture 3

Basic Structures

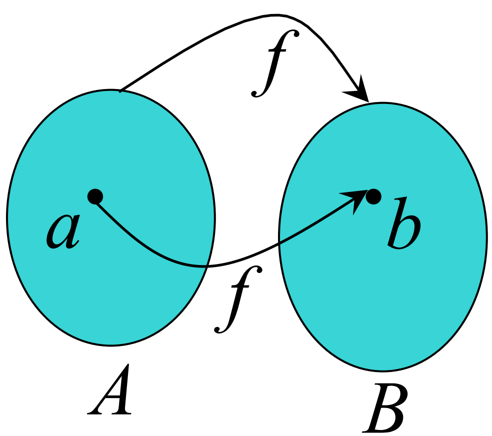
Functions

Functions

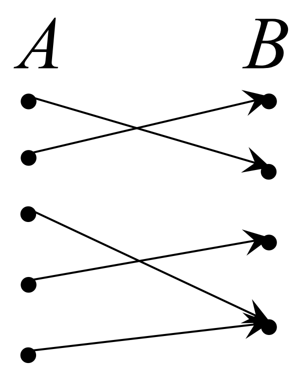
- From calculus, you are familiar with the concept of a real-valued function f, which assigns to each number $x \in \mathbf{R}$ a value $y = f(x)$, where $y \in \mathbf{R}$.
- But, the notion of a function can also be naturally generalized to the concept of assigning elements of *any* set to elements of *any* set. (Also known as a *map*.)

Function: Formal Definition

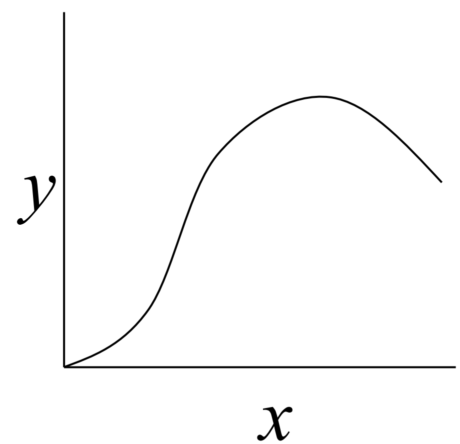
- For any sets A and B, we say that a ***function* (*or "mapping"*) *f from A to B* ($f : A \rightarrow B$)** is a particular assignment of **exactly one element** $f(x) \in B$ to **each element** $x \in A$.
- Functions can be represented graphically in several ways:



Like Venndiagrams



Bipartite Graph



Plot

Some Function Terminology

- If it is written that $f : A \rightarrow B$, and $f(a) = b$ (where $a \in A$ and $b \in B$), then we say:
  - A is the ***domain*** of f
  - B is the ***codomain*** of f
  - b is the ***image*** of a under f
    - a can not have more than 1 image
  - a is a ***pre-image*** of b under f
    - b may have more than 1 pre-image
  - The ***range*** $R \subseteq B$ of f is $R = \{b \mid \exists a\ f(a) = b\}$

Range versus Codomain

- The range of a function might *not* be its whole codomain.
- The codomain is the set that the function is *declared* to map all domain values into.
- The range is the *particular* set of values in the codomain that the function *actually* maps elements of the domain to.

Range vs. Codomain: Example

- Suppose I declare that: "*f* is a function mapping students in this class to the set of grades {A, B, C, D, F}."
- At this point, you know *f* 's codomain is: <u>{A, B, C, D, F}</u>, and its range is <u>unknown!</u>
- Suppose the grades turn out all As and Bs.
- Then the range of *f* is <u>{A, B}</u>, but its codomain is <u>still {A, B, C, D, F}!</u>

Function Operators

- $+$, $\times$ ("plus","times") are binary operators over **R**. (Normal addition & multiplication.)
- Therefore, we can also add and multiply two *real-valued functions* f,g: $\mathbf{R} \to \mathbf{R}$:
  - $(f+g)$: $\mathbf{R} \to \mathbf{R}$, where $(f+g)(x) = f(x) + g(x)$
  - (fg): $\mathbf{R} \to \mathbf{R}$, where $(fg)(x) = f(x)g(x)$
- <u>Example 6</u>:
  Let f and g be functions from **R** to **R** such that $f(x) = x^2$ and $g(x) = x - x^2$.
  What are the functions $f + g$ and fg?

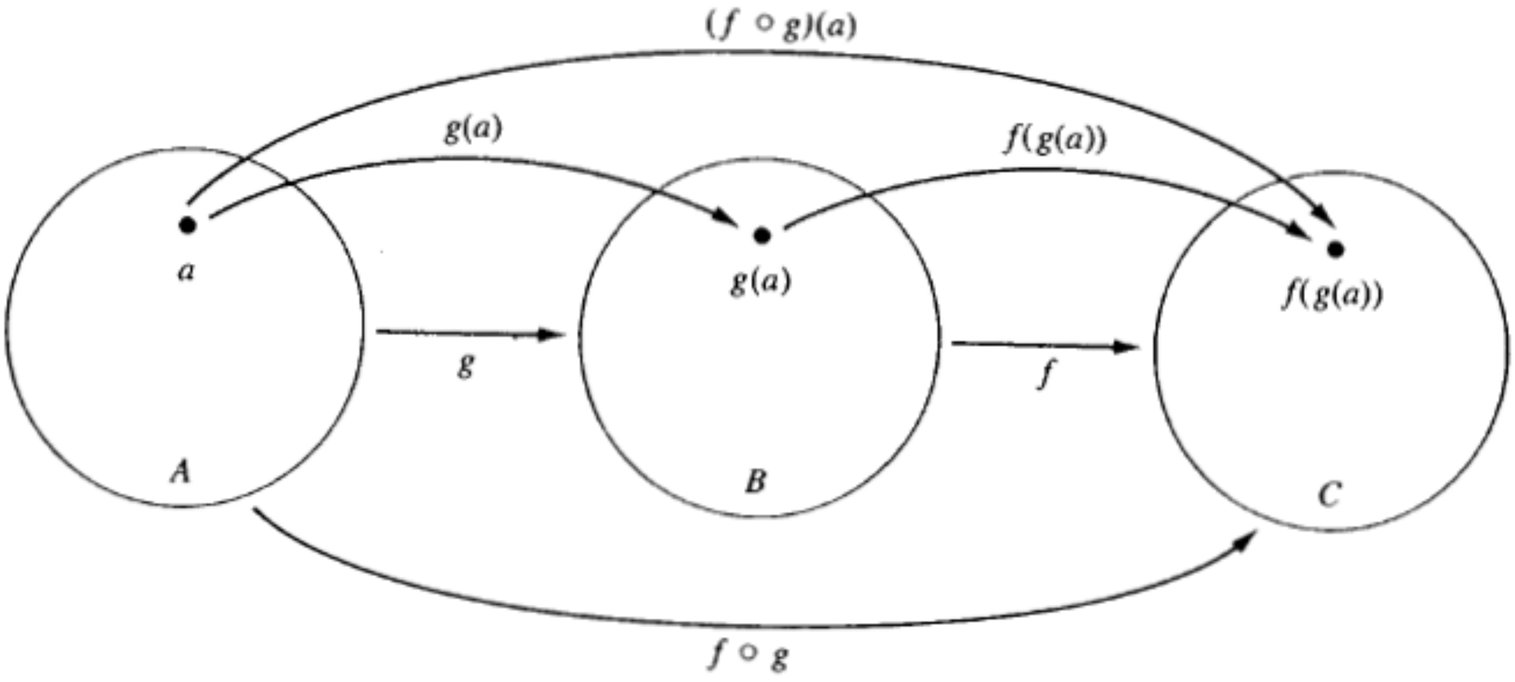
Function Composition Operator

Note the match here. It's necessary!

- For functions g: $A \rightarrow B$ and $f : B \rightarrow C$, there is a special operator called ***compose*** ("$\circ$").
  - It <u>composes</u> (creates) a new function from f and g by applying f to the result of applying g.
  - We say $(f \circ g)$: $A \rightarrow C$, where $(f \circ g)(a) = f(g(a))$.
  - Note: $f \circ g$ cannot be defined unless range of g is a subset of the domain of f.
  - Note $g(a) \in B$, so $f(g(a))$ is defined and $\in C$.
  - Note that $\circ$ is non-commuting. (Like Cartesian $\times$, but unlike $+$, $\wedge$, $\cup$) (Generally, $f \circ g \neq g \circ f$.)

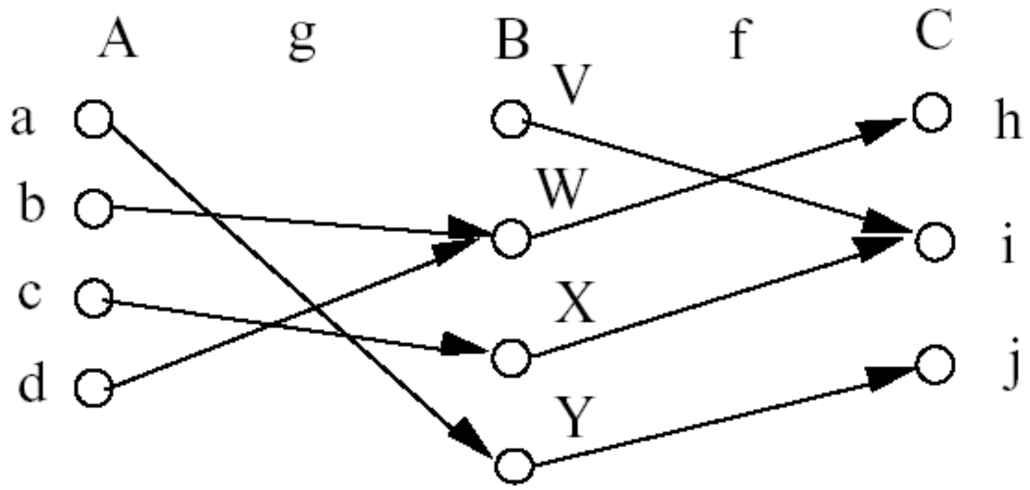
Function Composition Illustration

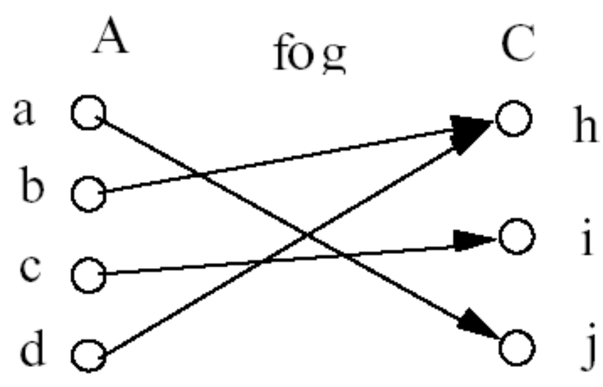
- $g: A \rightarrow B,\ f: B \rightarrow C$



Function Composition: Example

- $g: A \rightarrow B,\ \ f: B \rightarrow C$





Function Composition: Example

- Example 20: Let g: $\{a, b, c\} \rightarrow \{a, b, c\}$ such that

  $$g(a) = b,\ g(b) = c,\ g(c) = a.$$

  Let f : $\{a, b, c\} \rightarrow \{1, 2, 3\}$ such that

  $$f(a) = 3,\ f(b) = 2,\ f(c) = 1.$$

  What is the composition of f and g, and what is the composition of g and f ?

  - $f \circ g$: $\{a, b, c\} \rightarrow \{1, 2, 3\}$ such that

    $(f \circ g)(a) = 2,\ (f \circ g)(b) = 1,\ (f \circ g)(c) = 3.$

  - $g \circ f$ is not defined (why?)

Function Composition: Example

- If $f(x) = x^2$ and $g(x) = 2x + 1$, then what is the composition of f and g, and what is the composition of g and f?
  - $(f \circ g)(x) = f(g(x))$
    $= f(2x+1)$
    $= (2x+1)^2$
  - $(g \circ f)(x) = g(f(x))$
    $= g(x^2)$
    $= 2x^2 + 1$

Note that $f \circ g \neq g \circ f$. $(4x^2+4x+1 \neq 2x^2+1)$

1. For each of the following sets, determine whether 2 is an element of that set.

a) $\{x \in \mathbf{R} \mid x \text{ is an integer greater than } 1\}$
b) $\{x \in \mathbf{R} \mid x \text{ is the square of an integer}\}$
c) $\{2, \{2\}\}$
d) $\{\{2\}, \{\{2\}\}\}$
e) $\{\{2\}, \{2, \{2\}\}\}$
f) $\{\{\{2\}\}\}$

2. Determine whether each of these statements is true or false.

a) $x \in \{x\}$
b) $\{x\} \subseteq \{x\}$
c) $\{x\} \in \{x\}$

Solution 1:

a) Since 2 is an integer greater than 1, 2 is an element of this set.

b) Since 2 is not a perfect square ($1^2 < 2$, but $n^2 > 2$ for $n > 1$), 2 is not an element of this set.

c) This set has two elements, and as we can clearly see, one of those elements is 2.

d) This set has two elements, and as we can clearly see, neither of those elements is 2. Both of the elements of this set are sets; 2 is a number, not a set.

e) This set has two elements, and as we can clearly see, neither of those elements is 2. Both of the elements of this set are sets; 2 is a number, not a set.

f) This set has just one element, namely the set $\{\{2\}\}$. So 2 is not an element of this set. Note that $\{2\}$ is not an element either, since $\{2\} \neq \{\{2\}\}$.

Solution 2:

a) T (in fact x is the only element) **b)** T (every set is a subset of itself)

c) F (the only element of $\{x\}$ is a letter, not a set)