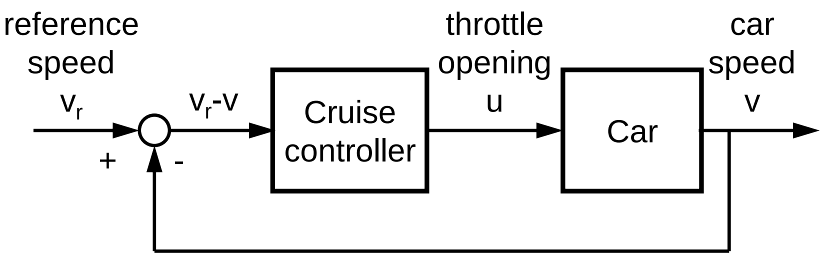
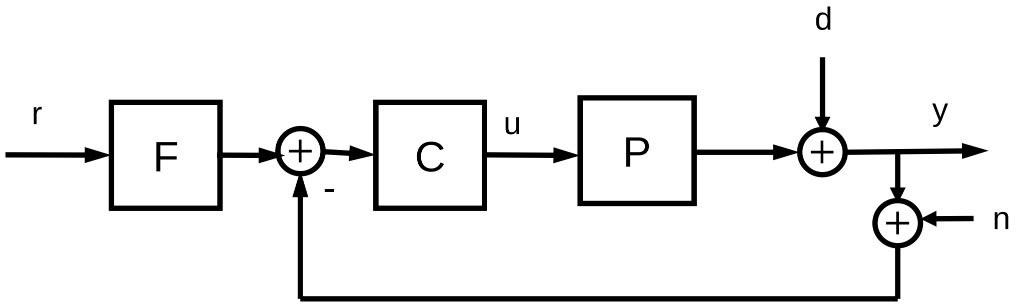
Feedback Control System

- P: the plant
- C: the controller
- F: a prefilter (for command response shaping)

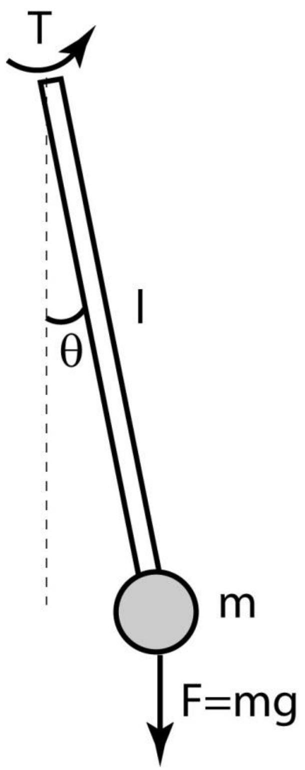


- y: controlled variable
- r: command (or reference) signal
- u: control signal



- d: disturbance signal
- n: measurement noise

Response of a pendulum = stable 2nd order system (e.g. lower leg with muscle stimulation)



$$I = m\left(\frac{l}{2}\right)^2$$

Non-linear differential equation:

$$\ddot{\theta}(t) + \frac{c}{I}\dot{\theta}(t) + \frac{mgl}{2I}\sin\theta(t) = \frac{1}{I}\tau(t)$$

Linear differential equation: valid for small θ such that $\sin\theta \approx \theta$

$$\ddot{\theta}(t) + \frac{c}{I}\dot{\theta}(t) + \frac{mgl}{2I}\theta(t) = \frac{1}{I}\tau(t)$$

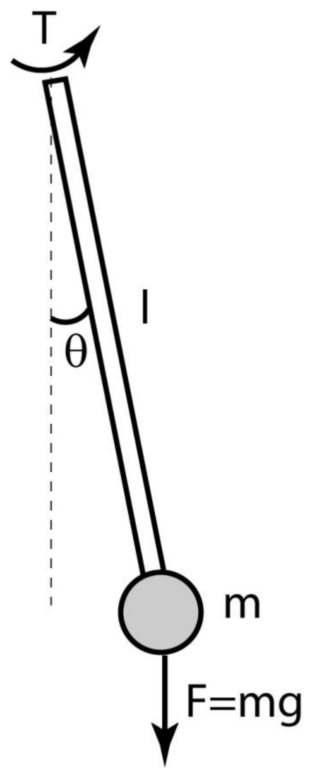
Laplace transform, using

$$\mathcal{L}\left[\frac{dy(t)}{dt}\right] = sY(s) - y(0^-)$$

we obtain

$$s^2\Theta(s) + \frac{c}{I}s\Theta(s) + \frac{mgl}{2I}\Theta(s) = \frac{1}{I}T(s)$$

Response of a pendulum



Transfer function for plant:

$$P(s) = \frac{\Theta(s)}{T(s)} = \frac{1/I}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

with

$$\omega_n = \sqrt{\frac{mgl}{2I}} \qquad \xi = \frac{c}{2I\omega_n}$$

$$g = 9.81 m/s^2$$

$$m = 5kg$$

$$l = 0.4m$$

$$c = 0.5 Nm/rad$$

$$I = \frac{ml^2}{3}$$

Response of the lower leg

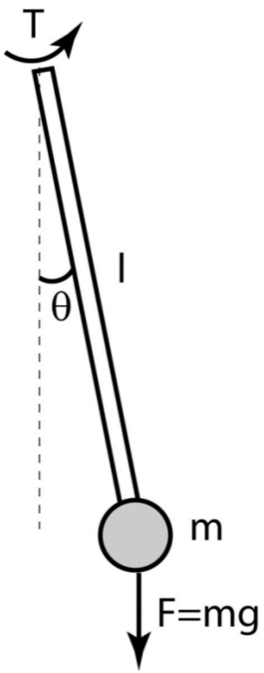
$$P(s) = \frac{1/I}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Steady state step response:

t$\rightarrow\infty$ then s$\rightarrow$0

$$P(s)_{s\rightarrow 0} = \frac{1}{I\omega_n^2} = \frac{2}{mgl}$$ with $$\omega_n = \sqrt{\frac{mgl}{2I}}$$

Response of a pendulum



State space representation:

$$\begin{bmatrix} \ddot{\theta}(t) \\ \dot{\theta}(t) \end{bmatrix} = \begin{bmatrix} -2\xi\omega_n & -\omega_n^2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}(t) \\ \theta(t) \end{bmatrix} + \begin{bmatrix} 1/I \\ 0 \end{bmatrix} \tau(t)$$

$$\theta(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}(t) \\ \theta(t) \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \tau(t)$$

with $\omega_n = \sqrt{\dfrac{mgl}{2I}}$ $\quad \xi = \dfrac{c}{2I\omega_n}$

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t) \qquad x(0) = x_0$$
$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

$$x(t) = \begin{bmatrix} \dot{\theta}(t) \\ \theta(t) \end{bmatrix} \quad u(t) = \tau(t) \quad y(t) = \theta(t)$$

$$\mathbf{A} = \begin{bmatrix} -2\xi\omega_n & -\omega_n^2 \\ 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1/I \\ 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad \mathbf{D} = 0$$