UNIVERSITY OF GLASGOW

Degrees of MEng, BEng, MSc and BSc in Engineering

CONTROL 4 (ENG4042) [SOLUTIONS]

Wednesday 15 December 2014
09:30 – 11:30

Answer ALL questions in Section A and ONE question from Section B and ONE question from Section C.

The numbers in square brackets in the right-hand margin indicate the marks allotted to the part of the question against which the mark is shown. These marks are for guidance only.

An electronic calculator may be used provided that it does not have a facility for either textual storage or display, or for graphical display.

Data sheet included within paper.

Continued overleaf

SECTION A

Q1 (a) Define the two main differences of a digital signal with respect to an analogue, continuous one. [6]

(b) State the condition under which a difference equation (with no input) is stable. Show it, by means of a simple example of your choice. [7]

(c) Consider a discrete exponential signal $e(k) = r^k$ for $k \geq 0$, $e(k) = 0$ for $k < 0$. Find its z-transform and show the condition for its stability. [7]

Solution

(a)

- Discretisation
  - A digital signal is discretized in time, meaning that the signal only exists at uniformly-spaced instants of time, in terms of samples. The time distance between consecutive samples is the sample time.

[3]

- Quantization
  - A digital signal is quantized, in that it cannot assume any arbitrary value, but instead its value is can only be equal to the value specified by the closest "quantum". In digital computers, quantization is the direct consequence of the binary algebra used by the computer.

[3]

(b)

A difference equation is stable if all the roots of its characteristic equation have magnitude less than one.

[2]

Example [other similar examples are acceptable]:

$$u_k = u_{k-1} + u_{k-2}$$

If we substitute the general solution $u_k = Az^k$, we get the characteristic equation:

$$z^k = z^{k-1} + z^{k-2}$$

[2]

Simplifying:

$$z^2 - z - 1 = 0$$

Which solves to:

$$z_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

Substituting this into the difference equation:

$$u_k = A_1 z_1^k + A_2 z_2^k$$

[2]

This is limited for $k \to \infty$ if both $|z_1| < 1$ and $|z_2| < 1$.

Continued overleaf

[1]

(c)

$$e(k)=\begin{cases} r^k, & k\geq 0 \\ 0, & k<0 \end{cases}$$

[1]

$$E(z)=\sum_{k=-\infty}^{+\infty} e(k)z^{-k}=\sum_{k=-\infty}^{+\infty} e(k)z^{-k}=\sum_{k=-\infty}^{+\infty} r^k z^{-k}=\sum_{k=0}^{+\infty} r^k z^{-k}=$$
$$=\sum_{k=0}^{+\infty}\left(rz^{-1}\right)^k$$

[2]

This is a geometric series, in rz^{-1}, hence:

$$E(z)=\frac{1}{1-rz^{-1}},\qquad \left|rz^{-1}\right|<1$$

Or

$$E(z)=\frac{z}{z-r},\qquad |z|>|r|$$

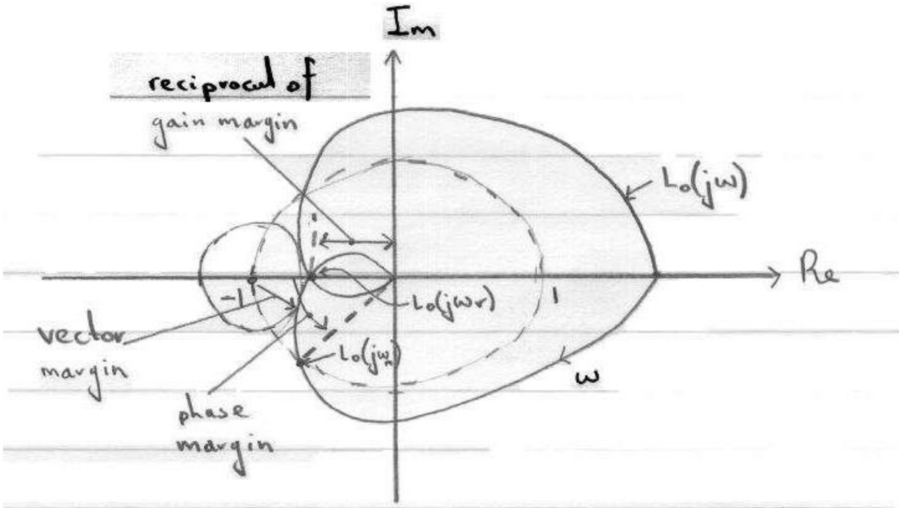
[2]

The signal has one pole at $z=r$, and so it is stable if $|r|<1$.

[2]

Q2 (a) Draw a Nyquist plot for a stable closed loop system and for an unstable system. For the stable system, mark the gain margin, phase margin and the vector margin in the plot. Explain the meaning of gain margin, phase margin and vector margin. [6]

Lecture Notes Part II, page 18, unstable plot similar but encircles -1.



Gain Margin: the gain margin km indicates the additional gain by which Lo may be multiplied so that it reaches the point −1.

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Phase Margin: quantifies the additional phase delay which can be added to Lo to reach the critical condition.
Vector Margin: the gain and phase margins are the traditional indicators of relative stability. However, a much more informative measure is the vector margin, denoted sm. This is the distance from the point −1 to the closest point on the plot of Lo.

(b) Explain disadvantages of low-frequency measurement noise in a feedback system, and explain why high-frequency measurement noise is relatively unimportant. [5]

5.6 Measurement noise

Equation (52) shows that the closed-loop transfer function from measurement noise to output is identical with T_o, the complementary sensitivity function. Above the closed-loop bandwidth the magnitude of T_o will be less than 1, and noise will therefore be attenuated and will have only a small influence on the output. For low frequencies, however, $|T_o|$ is close to 1 and measurement noise fully affects the output. In the crossover region, $|T_o|$ may become greater than 1, in which case measurement noise will be amplified. These facts emphasise the need for low-noise sensors, and the need to avoid peaking of $|T_o|$ in the crossover region.

(c) What are the three terms of a PID controller? Give the control law in the time domain and in the Laplace domain. [4]

Proportional, Integral and Derivative components.

$$u(t) = K\left(1 + \frac{1}{T_I}\int_0^t e(\tau)d\tau + T_D\dot{e}(t)\right)$$

$$C(s) = K\left(1 + \frac{1}{T_I s} + T_D s\right)$$

(d) What is observability in the context of state estimation and what is meant by detectability? [5]

The state $x_0 \neq 0$ is said to be unobservable if, given $x(0) = x_0$, and $u(T) = 0$ for $T \geq 0$, then $y(T) = 0$ for $T \geq 0$. The system is said to be completely observable if there exists no nonzero initial state that it is unobservable.

A plant is said to be detectable if its unobservable subspace is stable.

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SECTION B

Q3 Consider the continuous transfer function of a low pass filter with cutoff frequency (-3 dB) of 10 rad/s.

(a) Design the discrete equivalent of it, using the Tustin rule, considering a sampling time of 0.3 s. Compute the gain (in dB) of the digital filter at the cutoff frequency, and compare it with the analogue version. [10]

(b) Re-design the discrete equivalent of the filter above, but this time apply a pre-warping such that the gain is preserved at the cutoff frequency. Verify it numerically. [10]

Solution

(a)

$$G(s)=\frac{1}{0.1s+1}$$

[3]

$$G(z)=G(s)\Big|_{s=\frac{2}{T}\frac{z-1}{z+1}}=\frac{0.6z+0.6}{z+0.2}$$

[3]

$$G\left(e^{j\omega_0 T}\right)=0.1058$$

$$\left|G\left(e^{j\omega_0 T}\right)\right|_{dB}=20\log_{10}\left|G\left(e^{j\omega_0 T}\right)\right|=-19.5123\,\text{dB}$$

[3]

This gain is much smaller than the gain of the original filter (~ -3dB).

[1]

(b)

A better result can be obtained applying the Tustin's rule with frequency pre-warping. Setting the pre-warping frequency to $\omega_0 = 10 rad/s$:

$$a=\frac{2}{T}\tan\frac{\omega_0 T}{2}=94$$

[3]

We now consider the pre-warped function $G\left(\frac{s}{a}\right)$. Applying the Tustin's rule to this function:

[4]

$$G\left(e^{j\omega_0 T}\right)=0.7071$$

$$\left|G\left(e^{j\omega_0 T}\right)\right|_{dB}=20\log_{10}\left|G\left(e^{j\omega_0 T}\right)\right|=-3.0103\,\text{dB}$$

[3]

Q4 Consider the following continuous-time transfer function:

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$$H(s)=\frac{10(s-1)}{s^2+s-6}$$

(a) Using the zero-pole matching rule, find its discrete equivalent. Select the appropriate parameters such that the discrete transfer function preserves the gain for a constant signal. Consider a sample time $T = 0.5$ seconds. [12]

(b) Find the corresponding difference equation. [8]

Solution

(a)

$$H(s)=\frac{10(s-1)}{(s-2)(s+3)}$$

[1]

Poles:

$s=2\rightarrow z=e^{sT}=2.7183$

$s=-3\rightarrow z=e^{sT}=0.2231$

[3]

Zeros:

$s=1\rightarrow z=e^{sT}=1.6487$

[1]

$s=\infty\rightarrow z=-1$ (representing the maximum frequency)

[1]

$$H(z)=k\frac{(z-1.649)(z+1)}{(z-2.7183)(z-0.2231)}$$

[1]

For determining the gain k:

$H(s)\big|_{s=0}=\frac{5}{3}$

$H(z)\big|_{z=1}=0.9724k$

[3]

$H(s)\big|_{s=0}=H(z)\big|_{z=1}$, from which:

$k=1.7140$

[1]

Hence the overall TF is:

$$H(z)=1.7140\frac{(z-1.649)(z+1)}{(z-2.7183)(z-0.2231)}$$

[1]

It is also acceptable to have:

$$H(z)=k\frac{(z-1.649)}{(z-2.7183)(z-0.2231)}$$

And hence:

Continued overleaf

$$H(z)\Big|_{z=1} = 0.4862k$$
$$k = 3.4279$$

[1 point is removed for not computing the zero at infinity]

(b)

It is convenient to use symbols for the transfer function:

$$H(z) = k\frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)}$$

The transfer function is defined as:

$$H(z) = \frac{U(z)}{E(z)} = k\frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)}$$

[2]

Hence:

$$U(z)(z-p_1)(z-p_2) = E(z)k(z-z_1)(z-z_2)$$
$$Uz^2 - Up_1z - Up_2z + p_1p_2U = kEz^2 - kEz_1z - kEz_2z + kz_1z_2E$$

[2]

Dividing by z^2:

$$U - Up_1z^{-1} - Up_2z^{-1} + p_1p_2Uz^{-2} = kE - kEz_1z^{-1} - kEz_2z^{-1} + kz_1z_2Ez^{-2}$$

Anti-z-transforming:

$$u_j - p_1u_{j-1} - p_2u_{j-1} + p_1p_2u_{j-2} = ke_j - kz_1e_{j-1} - kz_2e_{j-1} + kz_1z_2e_{j-2}$$

[2]

Solving for u_j:

$$u_j = k\left(e_j - z_1e_{j-1} - z_2e_{j-1} + z_1z_2e_{j-2}\right) + (p_1+p_2)u_{j-1} - p_1p_2u_{j-2}$$

Substituting the poles, zero and gain:

$$u_k = 1.714\left(e_j - 0.649e_{j-1}\text{-}1.6490e_{j-2}\right) + 2.9414u_{j-1} - 0.6065u_{j-2}$$

[2]

A shift in time is also acceptable

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Laplace and Z Transforms for Causal Functions

$f(t)$ $t \geq 0$ *(causal)*	$F(s)$	$F(z)$ *($t=kT$, T = Sample Time, k = Index)*
$\delta\ (t)$	1	$1 = z^{-0}$
$\delta\ (t - kT)$	e^{-kTs}	z^{-k}
$u(t)$	$\dfrac{1}{s}$	$\dfrac{z}{z-1}$
t	$\dfrac{1}{s^2}$	$\dfrac{T \cdot z}{(z-1)^2}$
$\dfrac{t^k}{k!}$	$\left(\dfrac{1}{s}\right)^{k+1}$	$\underset{\alpha \to 0}{Limit} \dfrac{(-1)^k}{k!} \cdot \dfrac{\partial^k}{\partial \alpha^k}\left(\dfrac{z}{z - e^{-\alpha T}}\right)$
e^{-at}	$\dfrac{1}{s+a}$	$\dfrac{z}{z - e^{-aT}}$
$t \cdot e^{-at}$	$\dfrac{1}{(s+a)^2}$	$\dfrac{T \cdot z \cdot e^{-aT}}{(z - e^{-aT})^2}$
$e^{-at} \cdot \cos(bt)$	$\dfrac{(s+a)}{(s+a)^2 + b^2}$	$\dfrac{z \cdot (z - e^{-aT} \cdot \cos(bt))}{z^2 - z \cdot (2 \cdot e^{-aT} \cdot \cos(bt)) + e^{-2aT}}$
$e^{-at} \cdot \sin(bt)$	$\dfrac{b}{(s+a)^2 + b^2}$	$\dfrac{z \cdot e^{-aT} \cdot \sin(bt)}{z^2 - z \cdot (2 \cdot e^{-aT} \cdot \cos(bt)) + e^{-2aT}}$

TABLE 1 (is this needed?)

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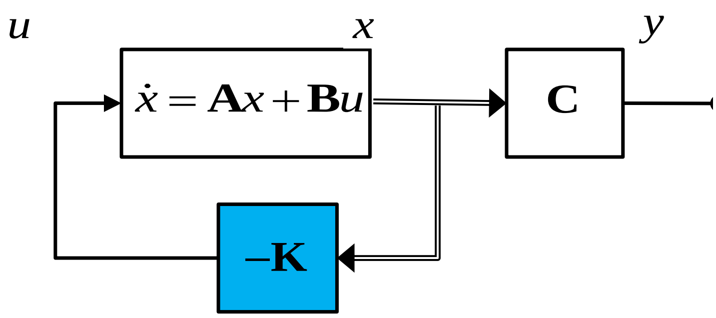
Section C

Q5 (a) Describe the concept of state feedback control using a block diagram and derive the differential equation of the closed loop system. [5]



$$\dot{x}(t) = (A - BK)x(t)$$
$$y(t) = Cx(t)$$

(b) For the state feedback system from (a), what is controller design by pole assignment and how can it be used to design a state feedback controller? [5]

By modifying the values of the state matrix in controller canonical form using state feedback, we change the location of the poles (or the eigenvalues) and hence the dynamic behaviour of the system. This method is therefore called closed-loop pole assignment. The closed-loop poles of the system can be arbitrarily assigned by feeding back the state through a suitably chosen constant-gain vector.

(c) Design a state feedback controller K for the state space system given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & -12 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

using the pole placement method in such a way that the closed system has an overshoot of Mp=1% and a rise time of tr=0.2 seconds. You can use the following equations to derive the natural frequency and the damping of the desired closed loop system:

$$\omega_n \cong \frac{1.8}{t_r} \quad \xi = -\frac{\ln M_p}{\sqrt{\pi^2 + (\ln M_p)^2}}; \; 0 \le \xi < 1 \qquad [5]$$

Continued overleaf

$\omega_n = 9$, $\xi = 0.826$, hence $A_{cl} = s^2 + 2\xi\omega_n + \omega_n^2 = s^2 + 14.87s + 81$

Thus $K = \begin{bmatrix} 11.87 & 69 \end{bmatrix}$

(d) Consider a reference input r and show how the corresponding gain can be derived so that the output reaches the reference in steady state. [5]

For reference input:

$$\begin{bmatrix} \mathbf{N}_x \\ N_u \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$\begin{aligned} u &= u_{ss} - \mathbf{K}(x - x_{ss}) \\ &= N_u r - \mathbf{K}(x - \mathbf{N}_x r) \\ &= -\mathbf{K}x + (N_u + \mathbf{K}\mathbf{N}_x) r \\ u &= -\mathbf{K}x + \bar{N} r \end{aligned}$$

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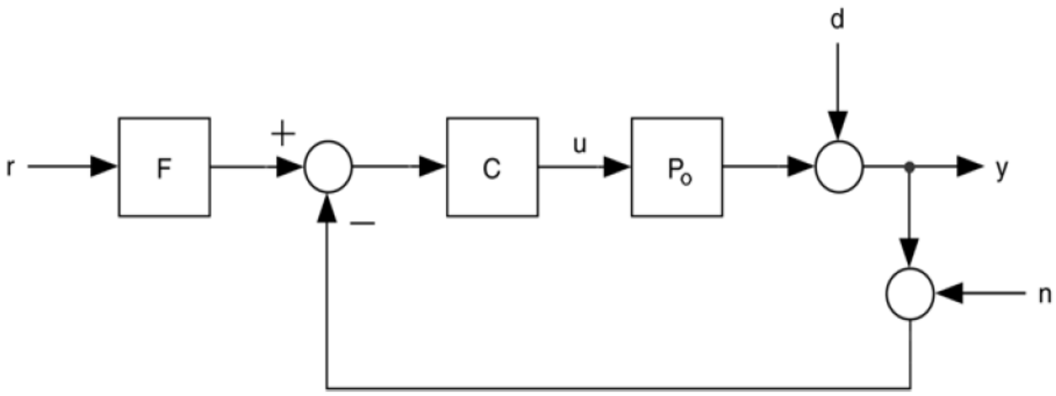

Figure 1

Q6 (a) Describe potential benefits and drawbacks of a closed-loop control strategy and contrast these to an open-loop strategy. In your analysis, focus on the characteristics of each structure with respect to plant disturbances, changes in the plant gain, and stabilisation. [5]

- Open-loop control provides no compensation for disturbances.
- The overall system is highly sensitive to possible changes in the plant and in operating conditions.
- A further key issue, not considered above, is that the plant may be unstable. Clearly, with open-loop control the overall system remains unstable: open-loop control cannot stabilise an unstable plant.

- With feedback control the system errors can be made insensitive to disturbances, i.e. disturbances can be significantly attenuated. The steady-state tracking error caused by disturbances will be less than that for the open-loop case by a factor of $\mathcal{S}^{-1} = 1 + cP_{ss}$.
- System errors are less sensitive to possible changes in the plant and in operating conditions. Compared to open-loop control, the steady-state error is less sensitive by a factor of $\mathcal{S}^{-1} = 1 + cP_{ss}$.
- A further key issue, not considered above, is that unstable systems can be stabilised by feedback. The dynamic closed-loop transfer function is $C(s)P(s)/(1 + C(s)P(s))$. Even if $P(s)$ is unstable, the closed-loop transfer function may be stable. Stability is considered in more detail later.

Continued overleaf

- High gain feedback reduces the effect of disturbances and plant variations. However, high gain may result in large control inputs to the plant which exceed the plant capacity. Thus, the beneficial effects of feedback are limited by available plant capacity.

- Despite the fact that feedback may stabilise an unstable system, making the gain of the system large may in itself result in an unstable feedback system.

- Feedback requires using a sensor to measure the output. All sensors have limited accuracy, resulting in the injection of *measurement noise* into the loop. Measurement noise amplification may result, and this is undesirable.

Refer to the closed-loop system shown in Figure 1

(b) Derive the closed-loop equation relating the plant output *y* and the plant input *u* to the signals *r*, *d*, and *n*. Typically, what are the signals *r*, *d* and *n*, and how should the feedback system be designed in order to respond appropriately to each of these signals? [5]

$$y = \frac{CP_o}{1+CP_o}Fr + \frac{1}{1+CP_o}d - \frac{CP_o}{1+CP_o}n \tag{14}$$

r = reference signal, d=disturbance, n=measurement noise

output should follow the reference, especially in steady state (i.e. at low frequencies).

disturbance should be attenuated, especially at low frequencies (eliminating constant disturbances)

measurement noise should be attenuated, especially at higher frequencies (where this noise is more likely to occur)

(c) Define the sensitivity function S_o and the complementary sensitivity function T_o and show explicitly how these transfer functions determine the properties of the system with respect to reference, disturbance and measurement noise signals. [5]

The *nominal sensitivity function* S_o and the *nominal complementary sensitivity function* T_o are defined as

$$S_o = \frac{1}{1+CP_o}, \qquad T_o = \frac{CP_o}{1+CP_o} \tag{15}$$

We see that S_o determines the effect of disturbances on the output, since it is the transfer function from d to y. The function T_o describes the effect of measurement noise, since it is the transfer function from n to y. T_o additionally determines the response from the filtered reference signal Fr to y. For this reason T_o is commonly referred to as the *closed-loop transfer function*. The pre-filter F allows the overall command response to be shaped largely arbitrarily.

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(d) Show that the complementary sensitivity T_o is equivalent to the relative changes in the sensitivity S_o which result from changes in the plant P_o, i.e. show that $T_o = -\dfrac{\partial S_o / S_o}{\partial P_o / P_o}$. What is the significance of this result? [5]

$$S_{P_o}^{S_o} = \frac{P_o}{S_o}\left(\frac{\partial S_o}{\partial P_o}\right) \tag{25}$$

Using equation (20) and differentiating we obtain

$$S_{P_o}^{S_o} = \frac{-CP_o}{1 + CP_o} \tag{26}$$

which is equivalent to $-T_o$ from equation (21), i.e. $S_{P_o}^{S_o} = -T_o$. Thus, in frequency ranges where $|T_o|$ is small, which means having small loop gain ($|CP_o| \ll 1$), S_o is relatively insensitive to variations in the plant.

End of question paper