Past Paper Revision

Paper	Question
DEC 2017	5
DEC 2017	6
DEC 2015	3
DEC 2014	3
DEC 2014	4

Q5 DEC 2017

a) Find discrete equivalent $G(z)$ of plant $G(s)$:

+ Plant has transfer function

$$G(s) = \frac{s+1}{20s+1}$$

+ It is also preceded by a ZOH.

+ 1st we want to find $\frac{G(s)}{s}$

$$\frac{G(s)}{s} = \frac{s+1}{s(20s+1)}$$

+ We use a partial fraction expansion to simplify this!

$$\frac{s+1}{s(20s+1)} = \underbrace{\frac{A}{s} + \frac{B}{20s+1}}_{\underline{A}}$$

Cross multiplying expression $\underline{A}$ and equating numerator to that of $G(s)$:

$$20As + A + Bs = s + 1$$

Equating similar terms on RHS and LHS:

s^0 : $A = 1$

s^1 : $20A + B = 1 \Rightarrow B = -19$

Then $\frac{G(s)}{s}$ becomes:

$$\frac{G(s)}{s} = \frac{1}{s} - \frac{19}{20s+1} = \frac{1}{s} - \frac{19}{20} \cdot \frac{1}{s + 1/20}$$

$$= \frac{1}{s} - 0.95 \cdot \underbrace{\frac{1}{s+0.05}}_{\text{form} \equiv \frac{1}{s+a}}$$

We perform Z-transform on each term separately:

$$Z\{L^{-1}\{\tfrac{1}{s}\}\} = \frac{z}{z-1}$$ (from table of z transforms)

$$\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{1}{s+0.05}\right\}\right\} = \frac{z}{z-e^{-0.05T}}$$

Since $T = 0.1s$,

$$e^{-0.05T} = 0.995$$

So,

$$\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{1}{s+0.05}\right\}\right\} = \frac{z}{z-0.995}$$

Complete discrete TF is then, preceded by ZOH:

$$G(z) = \underbrace{(1-z^{-1})}_{\text{ZOH}}\left[\underbrace{\frac{z}{z-1}}_{\text{from } 1/s} - \underbrace{\frac{0.95z}{z-0.995}}_{\text{from } \frac{1}{s+a}}\right]$$

$$= \underbrace{\left(\frac{z-1}{z}\right)}_{\text{ZOH}}\left[\frac{z}{z-1} - \frac{0.95z}{z-0.995}\right]$$

$$= 1 - \left(\frac{z-1}{\cancel{z}}\right) \cdot \frac{0.95\cancel{z}}{z-0.995}$$

$$= 1 - \frac{(z-1)0.95}{z-0.995}$$

We cross multiply to obtain:

$$G(z) = \frac{z - 0.995 - 0.95z + 0.95}{z - 0.995}$$

$$= \frac{0.05z - 0.045}{z - 0.995}$$

Taking out the term multiplying z in the numerator:

$$G(z) = 0.05 \cdot \frac{(z - 0.9)}{(z - 0.995)}$$

b) Closed Loop TF has the form:

$$C_L(z) = \frac{R(z)G(z)}{1 + R(z)G(z)}$$

{G(z) calculated in part (a)}

Let $F = RG$ then:

$$C_L = \frac{F}{1+F}$$

We obtain F as:

$$F = RG = \frac{\cancel{(z-0.995)}}{z-1} \cdot \frac{(0.05z-0.045)}{\cancel{(z-0.995)}}$$

$$= \frac{0.05z - 0.045}{z-1}$$

So the closed loop TF is:

$$C_L = \frac{\left(\frac{0.05z - 0.045}{z-1}\right)}{1 + \left(\frac{0.05z-0.045}{z-1}\right)}$$

Multiply top and bottom by $(z-1)$:

$$C_L = \frac{0.05z - 0.045}{z - 1 + 0.05z - 0.045}$$

So,
$$C_L = \frac{0.05z - 0.045}{1.05z - 1.045}$$

Take out values multiplying z terms:

$$C_L = \frac{0.05}{1.05} \frac{\left(z - \left(\frac{0.045}{0.05}\right)\right)}{\left(z - \left(\frac{1.045}{1.05}\right)\right)}$$

$$= 0.0476 \frac{(z - 0.9)}{(z - 0.995)}$$

c) Closed loop TF has the form:

$$C_L(z) = \frac{U(z)}{E(z)} = K\frac{z-a}{z-b}$$

+ Cross multiplying:

$$U(z-b) = K(z-a)E$$

$$zU - bU = KzE - KaE$$

+ multiply through by z^{-1}:

$$U - z^{-1}bU = KE - z^{-1}KaE$$

+ Apply inverse L.T.:

$$u_k - bu_{k-1} = Ke_k - Kae_{k-1}$$

$$\Rightarrow u_k = bu_{k-1} + Ke_k - Kae_{k-1}$$

Taking values of a, b, K from part (b):

$$u_k = 0.995u_{k-1} + 0.0476e_k - 0.04286e_{k-1}$$

d) Condition for BIBO stability:

$$\sum_{k=-\infty}^{\infty} |h_k| < \infty$$

where h_k is the inverse L.T. of $H(z)$ (discrete TF).

We assume an impulse input such that:

$$e_k = \begin{cases} 1 & \text{for } k=0 \\ 0 & \text{elsewhere} \end{cases}$$

and

$$u_k = h_k$$

For $u_0 = K$

{from difference equation obtained in part (c)}

$$u_1 = b u_0 + \cancel{K e_1}^{=0} - Ka\, e_0$$
$$= bK - Ka$$

$$u_2 = b^2 K - bKa$$
$$u_3 = b^3 K - b^2 Ka$$

$$\vdots$$

$$u_k = b^k K - b^{k-1} \cdot Ka$$

For BIBO stability:

$$\sum_{k=1}^{\infty} |u_k| \leq \sum_{k=1}^{\infty} |b^k K - b^{k-1} K a|$$

$$\leq |K - b^{-1} K a| \sum_{k=1}^{\infty} |b^k|$$

$\sum_{k=1}^{\infty} |b^k|$ is a geometric series in b which converges since $b = 0.9952 < 1$

Hence, system is BIBO stable.

Q6 DEC 2017

a) Low pass filter has TF:

$$G(s) = \frac{1}{\tau s + 1}$$

where $\tau = \frac{1}{\omega_f}$ ← cut off frequency

So $\tau = \frac{1}{15} = 0.06667\,s$

+ TF becomes:

$$G(s) = \frac{1}{0.06667 s + 1}$$

+ Tustin's rule: $s \leftarrow \frac{2}{T}\frac{z-1}{z+1}$

$$G(z) = G(s)\Big|_{s=\frac{2}{T}\frac{z-1}{z+1}}$$

$$= \frac{1}{0.06667\left(\frac{2}{T}\frac{z-1}{z+1}\right) + 1}$$

$$= \frac{z+1}{0.06667\left(\frac{2}{T}\right)(z-1) + z + 1}$$

+ For $T = 0.1s$:

$$G(z) = \frac{z+1}{1.3333(z-1) + z + 1}$$

$$= \frac{z+1}{2.3333z - 0.3333}$$

+ Bring out term multiplying z :

$$G(z) = \frac{1}{2.3333} \frac{z+1}{z - 0.1428}$$

$$= \frac{0.4286z + 0.4286}{z - 0.1428}$$

+ To find the gain of the filter at the cut off frequency we use the fact that :

$$z_1 = e^{j\omega_1 T}$$

and find the modulus :

$$|G(z)| = |G(e^{j\omega T})|$$

$$= \left| \frac{0.4286e^{j\omega T} + 0.4286}{e^{j\omega T} - 0.1428} \right|$$

+ using Euler's formula :

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$|G(e^{j\omega_0 T})| = \left| \frac{0.4286\cos\omega_0 T + 0.4286\, j\sin\omega_0 T + 0.4286}{\cos\omega_0 T + j\sin\omega_0 T - 0.1428} \right|$$

$$= \left| \frac{\left((0.45892)^2 + (0.4275)^2\right)}{(-0.0721)^2 + (0.9975)^2} \right|^{1/2}$$

$$= 0.6271 \quad \text{to } 4 \text{ S.F.}$$

+ In dB this is :

$$|G(e^{j\omega_0 T})|_{dB} = 20\log_{10}|G(e^{j\omega_0 T})|$$

$$= -4.0529 \text{ dB}$$

This is less than -3dB of continuous filter.

b) Better results obtained by pre-warping with Tustin's rule.

Setting pre-warping frequency $\omega_0 = 15 \text{ rad s}^{-1}$

$$a = \frac{2}{T}\tan\left(\frac{\omega_0 T}{2}\right) = 18.6319$$

we now look at the pre-warped function $G(s/a)$ and apply the modified Tustin's rule to the function.

Modified Tustin's rule:

$$s \leftarrow \frac{\omega_0}{\tan\left(\frac{\omega_0 T}{2}\right)} \cdot \frac{z-1}{z+1}$$

$$H(z) = \frac{1}{\frac{0.06667\,\omega_0}{\tan\frac{\omega_0 T}{2}} \cdot \frac{z-1}{z+1} + 1}$$

$$= \frac{z+1}{1.07343(z-1) + (z+1)}$$

$$= \frac{z+1}{2.07343z - 0.07343} = \frac{0.4823(z+1)}{z - 0.035415}$$

we find the gain of the TF by substituting in the following relationship: $\quad z = e^{j\omega T}$

$$|H(e^{j\omega T})| = \left| \frac{0.4823(e^{j\omega T} + 1)}{e^{j\omega T} - 0.035415} \right|$$

$$= \left| \frac{0.4823((\cos\omega T + 1) + j\sin\omega T)}{(\cos\omega T - 0.035415) + j\sin\omega T} \right|$$

$$= \left| \frac{0.51642 + j0.4811}{0.035322 + j0.997} \right|$$

$$= 0.7075$$

$$|H(e^{j\omega T})|_{dB} = 20\log_{10}|H(e^{j\omega T})|$$

$$= -3.0058\,dB$$

Gain at pre-warping frequency is same as continuous cut-off frequency.

c) Backward rule: $s \leftarrow \dfrac{z-1}{Tz}$

$$G(z) = G(s)\Big|_{s=\frac{z-1}{Tz}} = \frac{1}{0.06667 \cdot \frac{z-1}{Tz} + 1}$$

$$= \frac{Tz}{0.06667(z-1) + Tz}$$

$$= \frac{0.1z}{0.06667z - 0.06667 + 0.1z}$$

$$= \frac{z}{0.6667z + z - 0.6667}$$

$$= \frac{z}{1.6667z - 0.6667}$$

$$|G(e^{j\omega T})| = \left| \frac{e^{j\omega T}}{1.6667e^{j\omega T} - 0.6667} \right|$$

$$= \left| \frac{\cos\omega T + j\sin\omega T}{1.6667(\cos\omega T + j\sin\omega T) - 0.6667} \right|$$

$$= \left| \frac{0.07074 + j\,0.9975}{-0.55 + j\,1.663} \right|$$

$$|G(e^{j\omega T})| = 0.571$$

$$|G(e^{j\omega T})|_{dB} = 20 \log_{10} |G(e^{j\omega T})|$$

$$= -4.869\,dB$$

Much lower than continuous cut off frequency due to distortion introduced.

CP3 DEC 2015

$$G(s) = \frac{s+1}{s^2+s+1}$$

a) sample time $T = 0.1s$

To find poles we solve characteristic equation:

$$s^2 + s + 1 = 0$$

$$s = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

Poles at:

$$s_1 = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$$

$$s_2 = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$$

Mapping into the z-plane:

$$z_1 = e^{(-1/2 + j\sqrt{3}/2)T}$$

$$z_2 = e^{(-1/2 - j\sqrt{3}/2)T}$$

For z_1:

$$z_1 = e^{-T/2} \cdot e^{j\frac{\sqrt{3}}{2}T}$$

use Euler's formula for complex exponential:

$$z_1 = \underbrace{e^{-T/2} \cdot \cos\left(\frac{\sqrt{3}}{2}T\right)}_{Re} + \underbrace{j e^{-T/2} \sin\left(\frac{\sqrt{3}}{2}T\right)}_{Im}$$

$$= 0.94766 + j\,0.0823$$

Similary for z_2:

$$z_2 = e^{-T/2} \cdot e^{-j\frac{\sqrt{3}}{2}T}$$

$$= e^{-T/2} \cos\left(-\frac{\sqrt{3}}{2}T\right) + j e^{-T/2} \sin\left(-\frac{\sqrt{3}}{2}T\right)$$

$$= 0.94766 - j\,0.0823$$

<u>For zeros</u>:

$$S_1 = -1 \Rightarrow z = e^{-T} = 0.9048$$

Since order of numerator is less than order of denominator must be the pole at $s = \infty$ added:

So $s = \infty \Rightarrow z = -1$

Transfer function is then:

$$H(z) = K \frac{(z+1)(z-0.9048)}{(z-(0.94766+j0.0823))\cdot(z-(0.94766-j0.0823))}$$

$$= \frac{K(z^2+z-0.9048z-0.9048)}{z^2-(0.94766+j0.0823)z-(0.94766-j0.0823)z+(0.94766+j0.0823)(0.94766-j0.0823)}$$

$$= K\frac{[z^2+0.0952z-0.9048]}{z^2-1.895z+0.89806+6.7733\times10^{-3}}$$

$$= K \cdot \frac{z^2+0.0952z-0.9048}{z^2-1.895z+0.9048}$$

To find the gain:

$$H(s)\big|_{s=0} = 1$$

$$H(z)\big|_{z=1} = K \cdot \frac{1+0.0952-0.9048}{1-1.895+0.9048}$$

$$= K \cdot 19.4286$$

$$\Rightarrow \quad 1 = K \cdot 19.4286$$

$$\Rightarrow \quad K = 0.0515$$

Overall TF is then:

$$H(z) = 0.0515 \cdot \frac{z^2 + 0.0952z - 0.9048}{z^2 - 1.895z + 0.9048}$$

b) Forward rule: $s \Leftarrow \dfrac{z-1}{T}$

$$H(z) = H(s)\Big|_{s=\frac{z-1}{T}}$$

$$= \frac{\frac{z-1}{T} + 1}{\left(\frac{z-1}{T}\right)^2 + \frac{(z-1)}{T} + 1}$$

multiply by T^2:

$$H(z) = \frac{(z-1)T + T^2}{(z-1)^2 + (z-1)T + T^2}$$

$$= \frac{zT + (T^2 - T)}{z^2 - 2z + 1 + zT - T + T^2}$$

for $T = 0.1$

$$H(z) = \frac{0.1z - 0.09}{z^2 - 2z + 1 + 0.1z - 0.1 + 0.01}$$

$\cancel{\text{th}}$

5

$$H(z) = \frac{0.1z - 0.09}{z^2 - 1.9z + 0.91}$$

$$= \frac{0.1(z - 0.9)}{(z^2 - 1.9z + 0.91)}$$

Zero found directly as:

$$z = 0.9$$

Poles found by solving characteristic equation:

$$z^2 - 1.9z + 0.91 = 0$$

$$z = \frac{1.9 \pm \sqrt{1.9^2 - 4.0.91}}{2} = 0.95 \pm j\frac{0.173}{2}$$

$$= 0.95 \pm j0.0866$$

Since magnitude of all poles less than 1 then TF is stable.

Q3 DEC 2014

+ From worked examples, Low pass filter has TF:

$$LPF = \frac{1}{\tau s + 1}$$

where $\omega_f = \frac{1}{\tau}$ is the cut-off frequency.

So $\tau = \frac{1}{10} = 0.1$

+ TF becomes:

$$G(s) = \frac{1}{0.1s + 1}$$

+ Sampling time $T = 0.3\,s$

+ Use Tustin's rule: $s \leftarrow \frac{2}{T}\frac{z-1}{z+1}$

$$G(z) = G(s)\Big|_{s=\frac{2}{T}\frac{z-1}{z+1}} = \frac{1}{0.1.\frac{2}{T}\frac{z-1}{z+1} + 1}$$

$$= \frac{z+1}{\frac{0.2}{0.3}\,z-1 + z+1}$$

$$G(z) = \frac{z+1}{0.6667z - 0.6667 + z + 1}$$

$$= \frac{z+1}{1.6667z + 0.3333}$$

$$= 0.6 \frac{(z+1)}{z+0.2}$$

+ To calculate gain at the cut-off frequency we use the substitution:

$$z = e^{j\omega T}$$

So,

$$G(e^{j\omega T}) = G(z) = 0.6 \cdot \frac{e^{j\omega T} + 1}{e^{j\omega T} + 0.2}$$

We then apply Euler's formula:

$$G(e^{j\omega T}) = \frac{0.6\cos\omega T + j0.6\sin\omega T + 0.6}{\cos\omega T + j\sin\omega T + 0.2}$$

$$= \frac{6.0045\times10^{-3} + 0.085j}{-0.78999 + j0.14}$$

Taking the modulus to obtain gain

$$|G(e^{j\omega T})| = \sqrt{\frac{(6.0045\times10^{-3})^2 + (0.085)^2}{(-0.78999)^2 + (0.14)^2}}$$

$$= 0.1062$$

Convert the gain to dB:

$$|G(e^{j\omega T})|_{dB} = 20\log_{10}|G(e^{j\omega T})|$$
$$= -19.47767 \text{ dB}$$

This is significantly smaller than the gain of the original filter (~ -3dB)

b) Much better result can be obtained by pre-warping with Tustin's rule.

Setting pre-warping frequency $\omega_0 = 10 \text{ rad s}^{-1}$

$$a = \frac{2}{T}\tan\left(\frac{\omega_0 T}{2}\right) = 94.01$$

We now look at the pre-warped function $G\left(\frac{s}{a}\right)$ and apply modified Tustin's rule to the function.

Modified Tustin's rule is then:

$$s \leftarrow \frac{\omega_0}{\tan\left(\frac{\omega_0 T}{2}\right)} \cdot \frac{z-1}{z+1}$$

$$G(z) = \frac{1}{\frac{0.1\,\omega_0}{\tan\frac{\omega_0 T}{2}} \cdot \frac{z-1}{z+1} + 1}$$

$$= \frac{z+1}{0.07091(z-1) + z + 1}$$

$$= \frac{z+1}{1.07091 z + 0.929085}$$

To find gain we again substitute

$$z = e^{j\omega T}$$

$$\left| G(e^{j\omega T}) \right| = \left| \frac{e^{j\omega T} + 1}{1.07091(e^{j\omega T}) + 0.929085} \right|$$

$$= \left| \frac{(\cos\omega T + 1) + j\sin\omega T}{(1.07091\cos\omega T + 0.929085) + (j1.07091\sin\omega T)} \right|$$

$$|G(e^{j\omega T})| = \sqrt{\frac{(0.01)^2 + (0.1411)^2}{(-0.1311)^2 + (0.15113)^2}}$$

$$= 0.70703$$

To find the gain in dB:

$$|G(e^{j\omega T})|_{dB} = 20 \log_{10} |G(e^{j\omega T})|$$

$$= -3.011$$

Same as that for the continuous filter.

Q4 DEC 2014

a) $$H(s) = \frac{10(s-1)}{s^2+s-6} = \frac{10(s-1)}{(s-2)(s+3)}$$

Sample period, $T = 0.5\,s$

+ **Poles:**

$$s = 2 \rightarrow z = e^{sT} = e^{(2 \cdot 0.5)} = 2.7183$$

$$s = -3 \Rightarrow z = e^{-1.5} = 0.2231$$

+ **Zeros:**

$$s = 1 \Rightarrow z = e^{sT} = e^{0.5} = 1.64872$$

Order of numerator is less than order of denominator by s^1 so infinite zero must be added:

$$s \rightarrow \infty \Rightarrow z = -1$$

+ **Discrete TF is then:**

$$H(z) = K\frac{(z-1.64872)(z+1)}{(z-2.7183)(z-0.2231)}$$

+ To determine gain K :

$$H(s)|_{s=0} = \frac{10(-1)}{-6} = \frac{10}{6} = \frac{5}{3}$$

$$H(z)|_{z=1} = K\frac{(-0.64872)(2)}{(-1.7183)(0.7769)} = K.0.9719$$

+ So $H(s)|_{s=0} = H(z)|_{z=1}$

$$\frac{5}{3} = K.0.9719$$

$$\Rightarrow K = 1.71485$$

+ Overall TF is then:

$$H(z) = 1.71485 . \frac{(z-1.64872)(z+1)}{(z-2.7183)(z-0.2231)}$$

b) Expressing the TF as:

$$\frac{U(z)}{E(z)} = H(z) = K \frac{(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)}$$

cross multiplying:

$$U(z)(z-p_1)(z-p_2) = K(z-z_1)(z-z_2)E(z)$$

$$U(z)\left[z^2 - (p_1+p_2)z + p_1 p_2\right] = KE(z)\left[z^2 - (z_1+z_2)z + z_1 z_2\right]$$

multiplying through by z^{-2}:

$$U(z)\left[1 - (p_1+p_2)z^{-1} + p_1 p_2 z^{-2}\right] = KE(z)\left[1 - (z_1+z_2)z^{-1} + z_1 z_2 z^{-2}\right]$$

multiplying out:

$$U - (p_1+p_2)z^{-1}U + p_1 p_2 z^{-2} U = KE - K(z_1+z_2)z^{-1}E + Kz_1 z_2 z^{-2}E$$

Applying inverse transform:

$$u_k - (p_1+p_2)u_{k-1} + p_1 p_2 u_{k-2} = Ke_k - K(z_1+z_2)e_{k-1} + Kz_1 z_2 e_{k-2}$$

$$u_k = (p_1+p_2)u_{k-1} - p_1 p_2 u_{k-2} + K e_k$$
$$- K(z_1+z_2)e_{k-1} + K z_1 z_2 e_{k-2}$$

Substituting in values:

$$u_k = 2.9414\, u_{k-1} - 0.6065\, u_{k-2}$$
$$+ 1.71485(e_k - 0.649\, e_{k-1} - 1.64872\, e_{k-2})$$