Modern Control – State Space Control

Tutorial Solutions

State space and transfer function description

Q1 Taking the Laplace transform yields

$$sX(s) - x(0) = \mathbf{A}X(s) + \mathbf{B}U(s)$$
$$Y(s) = \mathbf{C}X(s) + \mathbf{D}U(s)$$

and hence

$$X(s) = (s\mathbf{I} - \mathbf{A})^{-1}x(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s)$$
$$Y(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]U(s) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}x(0)$$

Thus

$$Y(s) = G(s)U(s)$$
$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Q2

Controller canonical form:

Denominator parameters appear in first row of A-matrix,
numerator parameters appear in C-vector,
B=[1;0]

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & -12 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Observer canonical form:

Denominator parameters appear in first column of A-matrix,
numerator parameters appear in B-vector,
C=[1 0]

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -12 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Q3 The eigenvalues of A must all have negative real parts. Properties of B, C and D are irrelevant. In G(s), all poles (i.e. the roots of the denominator) must have negative real parts.

Controllability and Observability

Q4 Controllability: A state x0 is said to be controllable if there exists a finite interval [0, T] and an input $\{u(t), t \in [0, T]\}$ such that x(T) = 0. If all states are controllable, then the system is said to be completely controllable.

A state space model is said to be stabilisable if its uncontrollable subspace is stable.

Q5 The state $x_0 \neq 0$ is said to be unobservable if, given $x(0) = x_0$, and $u(T) = 0$ for $T \geq 0$, then $y(T) = 0$ for $T \geq 0$. The system is said to be completely observable if there exists no nonzero initial state that it is unobservable.

A plant is said to be detectable if its unobservable subspace is stable.

Q6 The controllability matrix is defined as

$$\boldsymbol{\Gamma}_c[\mathbf{A}, \mathbf{B}] = [\mathbf{B}\ \mathbf{AB}\ \mathbf{A}^2\mathbf{B}\ \ldots \mathbf{A}^{n-1}\mathbf{B}]$$

The set of all controllable states is the range space of the controllability matrix $\boldsymbol{\Gamma}_c[\mathbf{A}, \mathbf{B}]$.

The model is completely controllable if and only if $\boldsymbol{\Gamma}_c[\mathbf{A}, \mathbf{B}]$ has full row rank (ie is non-singular).

The set of all unobservable states is equal to the null space of the observability matrix $\boldsymbol{\Gamma}_o[\mathbf{A}, \mathbf{C}]$, where

$$\boldsymbol{\Gamma}_o[\mathbf{A}, \mathbf{C}] = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$$

The system is completely observable if and only if $\boldsymbol{\Gamma}_o[\mathbf{A}, \mathbf{C}]$ has full column rank n (ie. it is non-singular).

The controllability matrices for the first two systems are

System 1: $\Gamma_c = \begin{bmatrix} 12 & -96 \\ -12 & 96 \end{bmatrix}$ and System 2: $\Gamma_c = \begin{bmatrix} 1 & -11 \\ -1 & -3 \end{bmatrix}$

which have a rank of 1 and of 2. System 2 is therefore controllable, System 1 is not.

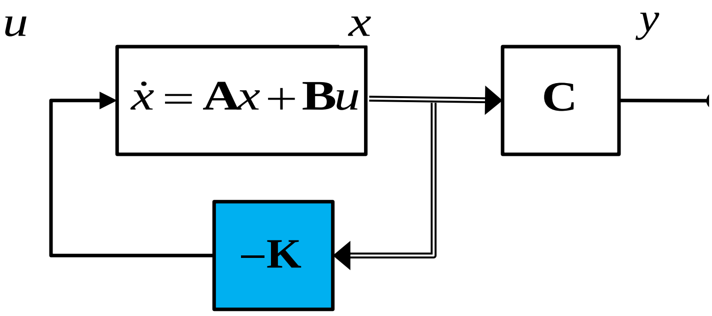
The observability matrix for the two systems are

System 1: $\Gamma_o = \begin{bmatrix} 1 & 0 \\ -4 & 4 \end{bmatrix}$ and System 2: $\Gamma_o = \begin{bmatrix} -1 & 1 \\ 11 & 3 \end{bmatrix}$

which have both a rank of 2. Both systems are therefore fully observable.

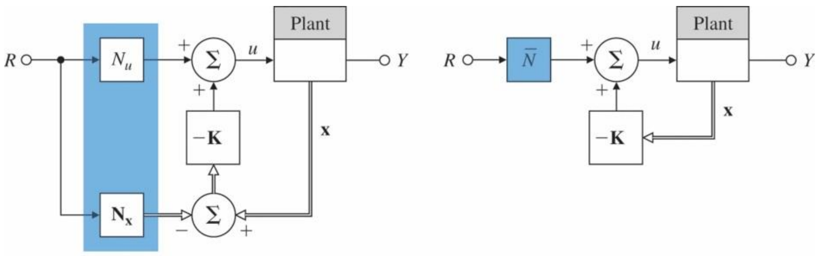
State feedback control

Q7



$$\dot{x}(t) = (A - BK)x(t)$$
$$y(t) = Cx(t)$$

Q8 See handouts “State space design: Pole placement”



(a) (b)

Q9 By modifying the values of the state matrix in controller canonical form using state feedback, we change the location of the poles (or the eigenvalues) and hence the dynamic behaviour of the system. This method is therefore called closed-loop pole assignment. the closed-loop poles of the system can be arbitrarily assigned by feeding back the state through a suitably chosen constant-gain vector.

Q10 Controllability: either calculate controllability matrix and show that rank is 2 **or** note that system is in controller canonical form and therefore controllable.

Q11 Using the equations give, $\omega_n = \frac{9rad}{s}$; $\xi = 0.826$, hence

$$A_{cl} = s^2 + 2\xi\omega_n s + \omega_n^2 = s^2 + 14.87s + 81$$

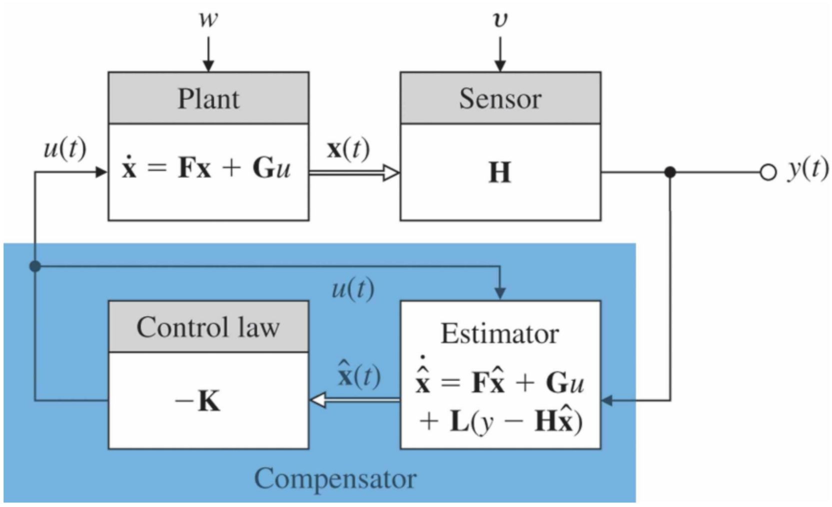
Thus $K = [k_1 \quad k_2]$ with $-14.87 = -3 - k_1$ and $-81 = -12 - k_2$
Hence $k_1 = 14.87 - 3 = 11.87$ and $k_2 = 81 - 12 = 69$ and $K = [11.87 \quad 69]$

For reference input:

$$\begin{bmatrix} \mathbf{N}_x \\ N_u \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \begin{aligned} u &= u_{ss} - \mathbf{K}\left(x - x_{ss}\right) \\ &= N_u r - \mathbf{K}\left(x - \mathbf{N}_x r\right) \\ &= -\mathbf{K}x + \left(N_u + \mathbf{K}\mathbf{N}_x\right) r \\ u &= -\mathbf{K}x + \bar{N} r \end{aligned}$$

State estimator feedback control

Q12



Q13 Observability: either calculate observability matrix and show that rank is 2 or note that system is in observer canonical form and therefore observable.

From location of observer poles it follows that

$A_{obsv} = (s + 10)(s + 20) = s^2 + 30s + 200$

Thus

$$\begin{bmatrix} -30 & 1 \\ -200 & 0 \end{bmatrix} = \mathbf{A} - \mathbf{LC} = \begin{bmatrix} -3 & 1 \\ -12 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -(3 + l_1) & 1 \\ -(12 + l_2) & 0 \end{bmatrix}$$

Hence

$$30 = 3 + l_1 \Rightarrow l_1 = 27$$
$$200 = 12 + l_2 \Rightarrow l_2 = 188$$

Therefore $L = \begin{bmatrix} 27 \\ 188 \end{bmatrix}$.

Q14

Consider the state space model and assume that it is completely controllable and completely observable. Consider also an associated observer and state-variable feedback, where the state estimates are used in lieu of the true states.

Then the closed-loop poles are the combination of the poles from the observer and the poles that would have resulted from using the same feedback on the true states - specifically, the closed-loop polynomial $A_{cl}(s)$ is given by

$$A_{cl}(s) = \det(s\mathbf{I} - (\mathbf{A} - \mathbf{BK}))\det(s\mathbf{I} - (\mathbf{A} - \mathbf{LC}))$$