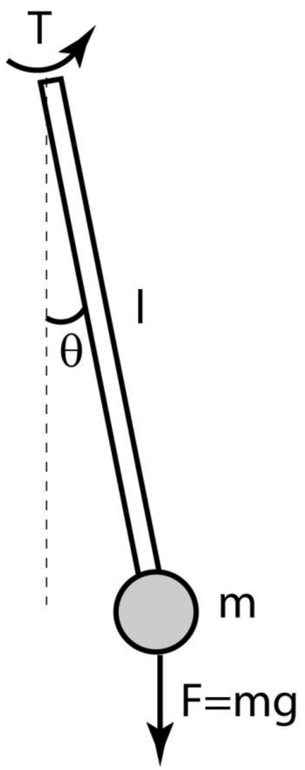
State space model

There are many alternative model formats that can be used for linear dynamic systems. In simple single-input single-output (SISO) problems, any representation is probably as good as any other. However, as we move to more complex problems (*especially multivariable problems*), it is desirable to use special model formats. One of the most flexible and useful structures is the state space model.

We will examine linear state space models in a little more depth for the SISO case. Many of the ideas will carry over to the multi-input multi-output (MIMO) case. In particular we will study

- similarity transformations and equivalent state representations,
- state space model properties:
  - controllability, reachability, and stabilizability,
  - observability, reconstructability, and detectability,
- special (*canonical*) model formats.

Response of a pendulum



$$I = m\left(\frac{l}{2}\right)^2$$

Non-linear differential equation:

$$\ddot{\theta}(t) + \frac{c}{I}\dot{\theta}(t) + \frac{mgl}{2I}\sin\theta(t) = \frac{1}{I}\tau(t)$$

Linearised differential equation:

$$\ddot{\theta}(t) + \frac{c}{I}\dot{\theta}(t) + \frac{mgl}{2I}\theta(t) = \frac{1}{I}\tau(t)$$

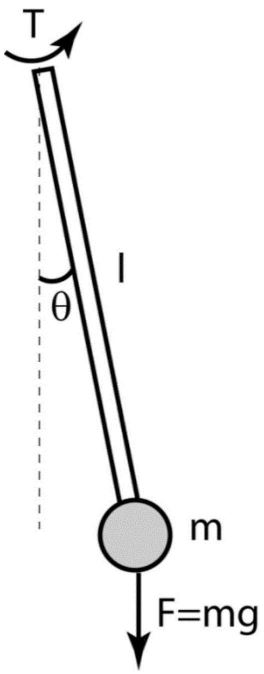
Laplace transform, using

$$\mathcal{L}\left[\frac{dy(t)}{dt}\right] = sY(s) - y(0^-)$$

we obtain

$$s^2\Theta(s) + \frac{c}{I}s\Theta(s) + \frac{mgl}{2I}\Theta(s) = \frac{1}{I}T(s)$$

Response of a pendulum – state space representation



$$\begin{bmatrix} \ddot{\theta}(t) \\ \dot{\theta}(t) \end{bmatrix} = \begin{bmatrix} -\frac{c}{I} & -\frac{mgl}{2I} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}(t) \\ \theta(t) \end{bmatrix} + \begin{bmatrix} 1/I \\ 0 \end{bmatrix} \tau(t)$$

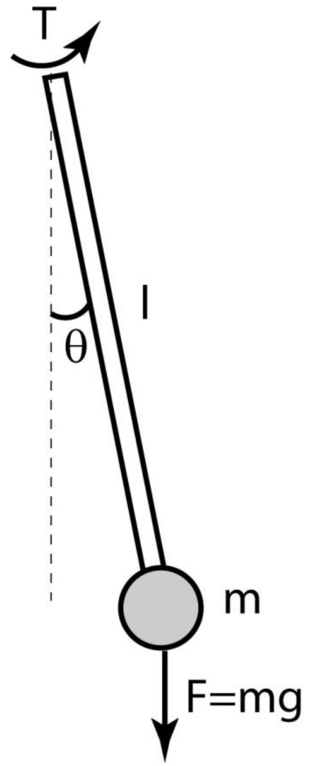
$$\theta(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}(t) \\ \theta(t) \end{bmatrix} + [0]\tau(t)$$

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t) \quad x(0) = x_0$$
$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

$$x(t) = \begin{bmatrix} \dot{\theta}(t) \\ \theta(t) \end{bmatrix} \quad u(t) = \tau(t) \quad y(t) = \theta(t)$$

$$\mathbf{A} = \begin{bmatrix} -\frac{c}{I} & -\frac{mgl}{2I} \\ 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1/I \\ 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad \mathbf{D} = 0$$

Response of a pendulum



$$P(s)=\frac{\Theta(s)}{T(s)}=\frac{1/I}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$\begin{bmatrix}\ddot{\theta}(t)\\ \dot{\theta}(t)\end{bmatrix}=\begin{bmatrix}-2\xi\omega_n & -\omega_n^2\\ 1 & 0\end{bmatrix}\begin{bmatrix}\dot{\theta}(t)\\ \theta(t)\end{bmatrix}+\begin{bmatrix}1/I\\ 0\end{bmatrix}\tau(t)$$

$$\theta(t)=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\dot{\theta}(t)\\ \theta(t)\end{bmatrix}+\begin{bmatrix}0\end{bmatrix}\tau(t)$$

with

$$\omega_n=\sqrt{\frac{mgl}{2I}}$$

$$\xi=\frac{c}{2I\omega_n}$$

Linear Continuous-Time State Space Models

A continuous-time linear time-invariant state space model takes the form

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t); \quad x(0) = x_0$$
$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

where x is the state vector, u is the control signal, y is the output, x_0 is the state vector at time $t = 0$, and **A**, **B**, **C**, and **D** are matrices of appropriate dimensions.

Transfer Functions for Continuous Time State Space Models

State space model:

$$\frac{dx(t)}{dt} = \mathbf{A}x(t) + \mathbf{B}u(t); \quad x(0) = x_0$$

$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

Taking Laplace transform in the state space model equations yields

$$sX(s) - x(0) = \mathbf{A}X(s) + \mathbf{B}U(s)$$

$$Y(s) = \mathbf{C}X(s) + \mathbf{D}U(s)$$

Transfer Functions for Continuous Time State Space Models

and hence

$$X(s) = (s\mathbf{I} - \mathbf{A})^{-1} x(0) + (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} U(s)$$

$$Y(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + \mathbf{D}] U(s) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} x(0)$$

$G(s)$ is the system transfer function.

$$Y(s) = G(s) U(s)$$

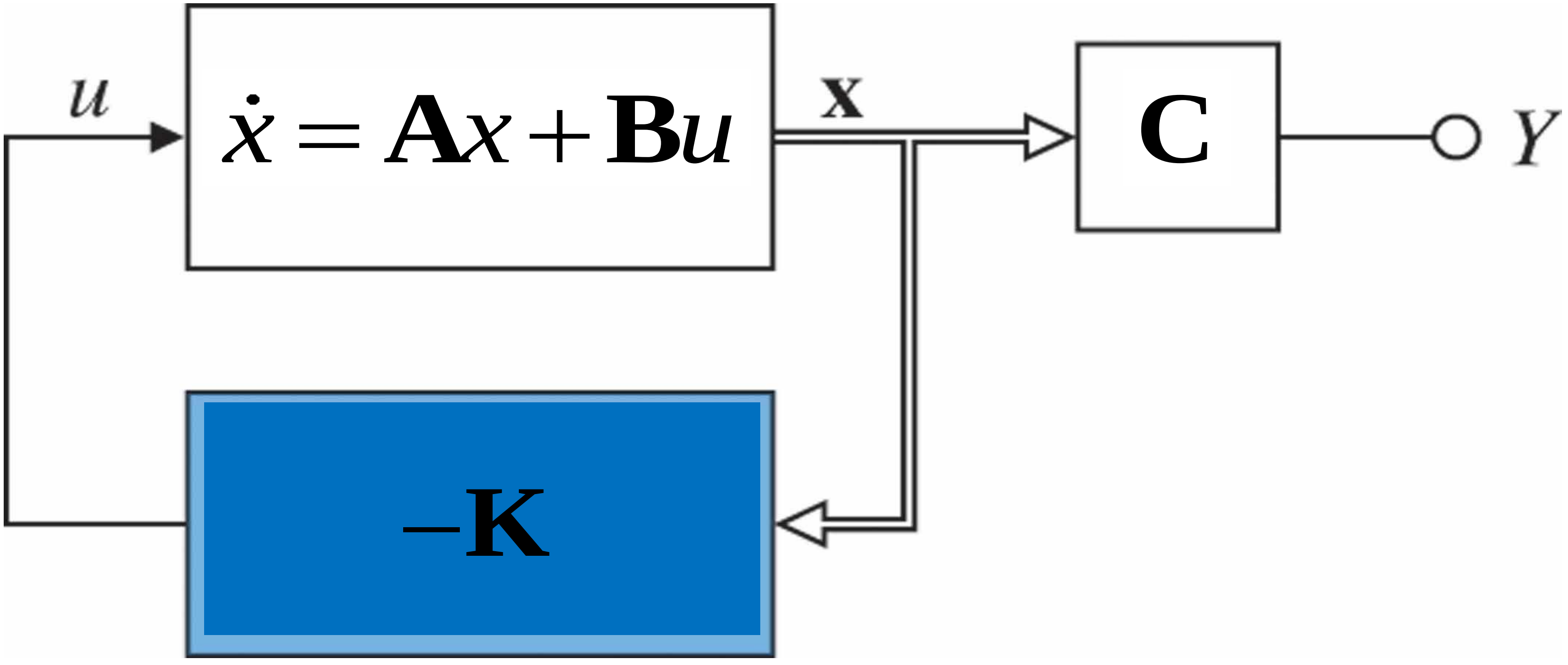
$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + \mathbf{D}$$

Summary

Transfer functions describe the input-output properties of linear systems in algebraic form.

The dynamic behaviour and stability of the system are determined by the location of the **poles of the transfer function**.
For the state space representation, these correspond to the **eigenvalues of the matrix A**

State feedback control



State feedback control

- Consider the standard state space system

$$\frac{dx(t)}{dt} = \mathbf{A}x(t) + \mathbf{B}u(t); \quad x(0) = x_0$$
$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

- Provided the system has certain properties, the eigenvalues (poles) of this system can be moved to arbitrary location using simple proportional feedback of the full state.

State feedback control

- Introducing a feedback law

$$u(t) = -\mathbf{K}x(t);$$

- we can obtain

$$\frac{dx(t)}{dt} = (\mathbf{A} - \mathbf{BK})x(t); \quad x(0) = x_0$$

$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

- Thus, the state matrix of the closed loop is

$$\mathbf{A}_{cl} = \mathbf{A} - \mathbf{BK}$$

State feedback control

- Closed loop system:

$$\frac{dx(t)}{dt} = \mathbf{A}_{cl} x(t); \quad x(0) = x_0$$

$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

with

$$\mathbf{A}_{cl} = \mathbf{A} - \mathbf{BK}$$

State space controller design

- The concepts of **state feedback control** and **state estimation** require some further analysis of the properties of state space models.
- Using similarity transformations, this will lead to the concepts of
  - **Controllability** and
  - **Observability**

Similarity Transformations

It is readily seen that the definition of the state of a system is nonunique. Consider, for example, a linear transformation of $x(t)$ to $\bar{x}(t)$ defined as

$$\bar{x}(t) = \mathbf{T}^{-1}x(t) \qquad\qquad x(t) = \mathbf{T}\bar{x}(t)$$

where $\mathbf{T}$ is any nonsingular matrix, called a **similarity transformation**.

The following alternative state description is obtained

$$\dot{\bar{x}}(t) = \bar{\mathbf{A}}\bar{x}(t) + \bar{\mathbf{B}}u(t); \quad \bar{x}(0) = \mathbf{T}^{-1}x_0$$

$$y(t) = \bar{\mathbf{C}}\bar{x}(t) + \bar{\mathbf{D}}u(t)$$

where

$$\bar{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} \quad \bar{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B} \quad \bar{\mathbf{C}} = \mathbf{C}\mathbf{T} \quad \bar{\mathbf{D}} = \mathbf{D}$$

The above model is an equally valid description of the system. Its characteristics from *u* to *y* are unchanged, only the internal **states** are different.

Transfer Functions Revisited

The solution to the state equation model can be obtained via (assuming that **D**=0):

$$
\begin{aligned}
Y(s) &= \bar{\mathbf{C}}(s\mathbf{I}-\bar{\mathbf{A}})^{-1}\left[\bar{\mathbf{B}}U(s)+\bar{x}(0)\right] \\
&= \mathbf{CT}(s\mathbf{I}-\mathbf{T}^{-1}\mathbf{AT})^{-1}\left[\mathbf{T}^{-1}\mathbf{B}U(s)+\mathbf{T}^{-1}x(0)\right] \\
&= \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\left[\mathbf{B}U(s)+x(0)\right]
\end{aligned}
$$

We thus see that **different choices of state variables** lead to **different internal descriptions** of the model, but to the **same input-output model**, because the system transfer function can be expressed in either of the two equivalent fashions.

$$\bar{\mathbf{C}}(s\mathbf{I}-\bar{\mathbf{A}})^{-1}\bar{\mathbf{B}}+\bar{\mathbf{D}}=\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}$$

for any nonsingular $\mathbf{T}$.

Canonical forms

- Similarity transformation can be used to bring a system into a special form which are called *canonical forms.*
- A transformation with a matrix of **eigenvectors** results in a system in **modal canonical form** which has the **eigenvalues** as diagonal values in the state matrix.
- Other canonical forms include the *controller canonical form* and the *observer canonical form*

Controller Canonical Form

We may be able to choose a similarity transformation which leads to the following state space model for the system.

$$\mathbf{A}_c = \begin{bmatrix} -a_1 & -a_2 & \cdots & \cdots & -a_n \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ \vdots & & \ddots & & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \quad \mathbf{B}_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

$$\mathbf{C}_c = \begin{bmatrix} b_1 & b_2 & \cdots & \cdots & b_n \end{bmatrix} \quad \mathbf{D}_c = 0$$

The above model has a special form. We will see later that any ***completely controllable*** system can be expressed in this way. Before we do this, we need to introduce the idea of **controllability**.

Controller Canonical Form

Note that by re-arranging the states, the controller canonical form can also be expressed as:

$$\mathbf{A}_c = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & & \cdots & \cdots & 0 \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ \vdots & & \ddots & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{bmatrix} \quad \mathbf{B}_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}$$

$$\mathbf{C}_c = \begin{bmatrix} b_n & b_{n-1} & \cdots & \cdots & b_1 \end{bmatrix} \quad \mathbf{D}_c = 0$$

The equivalent transfer function form is

$$G(s) = \frac{b_1 s^{n-1} + \ldots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

Controllability and Stabilizability

An important question that lies at the heart of control using state space models is whether we can steer the state via the control input to certain locations in the state space. Technically, this property is called controllability or reachability. A closely related issue is that of stabilizability. We will begin with controllability.

Controllability

The issue of controllability concerns whether a given initial state x_0 can be steered to the origin in finite time using the input $u(t)$.

Formally, we have the following:

Definition: A state x is said to be **controllable** if, given any $x_0 \neq 0$ and any finite time interval $[0, T]$, there exists an input $\{u(t), t \in [0, T]\}$ such that $x(T) = 0$. If all states are controllable, then the system is said to be **completely controllable**.

Reachability

A related concept is that of reachability. This concept is sometimes used in discrete-time systems. It is formally defined as follows:

Definition: A state $\bar{x} \neq 0$ is said to be **reachable** (*from the origin*) if, given $x(0) = 0$, there exist a finite time interval $[0, T]$ and an input $\{u(t), \ t \in [0, T]\}$ such that $x(T) = \bar{x}$. If all states are reachable, the system is said to be **completely reachable**.

Remark

For continuous, time-invariant, linear systems, there is no distinction between complete controllability and complete reachability. However, the following example illustrates that there is a subtle difference in discrete time.

Consider the following shift-operator state space model:

$$x[k+1] = 0$$

This system is obviously completely controllable: the state immediately goes to the origin. However, no nonzero state is reachable.

In view of the subtle distinction between controllability and reachability in discrete time, we will use the term *controllability* in the sequel to cover the stronger of the two concepts.

Test for Controllability

Consider the state space model

$$\frac{dx(t)}{dt} = \mathbf{A}x(t) + \mathbf{B}u(t); \quad x(0) = x_0$$

$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

The **controllability matrix** is defined as

$$\Gamma_c[\mathbf{A}, \mathbf{B}] = \left[\mathbf{B}\ \mathbf{AB}\ \mathbf{A^2B}\ \ldots \mathbf{A^{n-1}B}\right]$$

The model is **completely controllable** if and only if $\boldsymbol{\Gamma}_c[\mathbf{A}, \mathbf{B}]$ has full row rank (ie is non-singular).

Controller - Canonical Form

Consider a **completely controllable** state space model for a SISO system. Then there exists a similarity transformation that converts the state space model into the following controller-canonical form:

$$A_c = \begin{bmatrix} -a_1 & -a_2 & \cdots & \cdots & -a_n \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}; \quad B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix};$$

Example

Consider the state space model

$$\mathbf{A} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The controllability matrix is given by

$$\mathbf{\Gamma}_c[\mathbf{A}, \mathbf{B}] = [\mathbf{B}, \mathbf{AB}] = \begin{bmatrix} 1 & -4 \\ -1 & -2 \end{bmatrix}$$

Clearly, rank $\Gamma_c[\mathbf{A}, \mathbf{B}] = 2$; thus, the system is completely controllable.

Example

For

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The controllability matrix is given by:

$$\mathbf{\Gamma}_c[\mathbf{A}, \mathbf{B}] = [\mathbf{B}, \mathbf{AB}] = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

Rank $\Gamma_c[\mathbf{A}, \mathbf{B}] = 1 < 2$; thus, the system is not completely controllable.

If a system is not completely controllable, it can be decomposed into a controllable and a completely uncontrollable subsystem, as explained below.

Controllable Decompositon

Lemma 17.1: Consider a system having rank$\{\Gamma_c[\mathbf{A}, \mathbf{B}]\}$ = k < n; then there exists a similarity transformation T such that $\bar{x} = \mathbf{T}^{-1}x$,

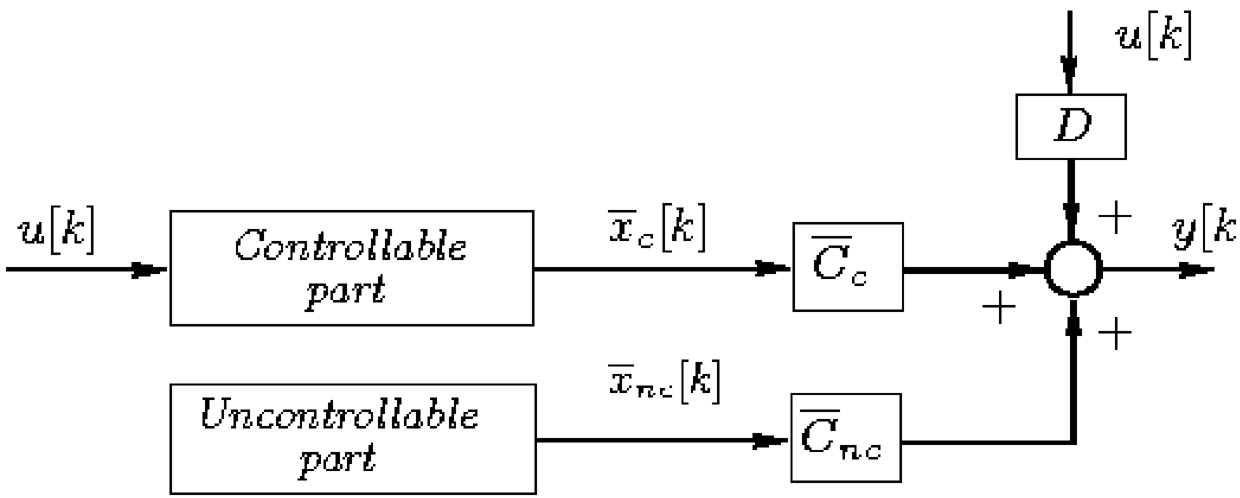
$$\overline{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}; \qquad \overline{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B}$$

and $\overline{\mathbf{A}}, \overline{\mathbf{B}}$ have the form

$$\overline{\mathbf{A}} = \begin{bmatrix} \overline{\mathbf{A}}_c & \overline{\mathbf{A}}_{12} \\ 0 & \overline{\mathbf{A}}_{nc} \end{bmatrix}; \qquad \overline{\mathbf{B}} = \begin{bmatrix} \overline{\mathbf{B}}_c \\ 0 \end{bmatrix}$$

where $\overline{\mathbf{A}}_c$ has dimension k and $(\overline{\mathbf{A}}_c, \overline{\mathbf{B}}_c)$ is completely controllable.

We see that caution must be exercised when controlling a system (*or designing a controller with a model that is not completely controllable*), because the output has a component $\mathbf{C}_{nc}\overline{x}_{nc}[k]$ that does not depend on the manipulated input $u[k]$.



The **controllable subspace** of a state space model is composed of all states generated through every possible linear combination of the states in $\bar{x}_c$. The stability of this subspace is determined by the location of the eigenvalues of $\mathbf{A}_c$.

The **uncontrollable subspace** of a state space model is composed of all states generated through every possible linear combination of the states in $\bar{x}_{nc}$. The stability of this subspace is determined by the location of the eigenvalues of $\mathbf{A}_{nc}$.

Stabilizability

A state space model is said to be *stabilizable* if its uncontrollable subspace is stable.

Finally, we remark that, it is very common indeed to employ uncontrollable models in control-system design. This is because they are a convenient way of describing various commonly occurring disturbances. For example, a constant disturbance can be modeled by the following state space model:

$$\dot{x}_d = 0$$

which is readily seen to be uncontrollable and, indeed, nonstabilizable.