.

LECTURE NOTES

Control System Analysis and Design

Part 1: Differential Equations (dynamic systems) – Revisions

Contents/concepts:

- Equations of motion of simple biomechanical systems (linear ODEs)
- Time-domain solution (free and forced responses)
- Transient and steady-state solutions
- Standard 2nd-order equation; natural frequency and damping
- Stability of standing
- Frequency-domain solution (via Laplace)
- The transfer-function
- The frequency response
- Direct links between time- and frequency-domain solutions

Contents

The purpose of these notes is to develop mathematical representations of some simple biomechanical systems. The systems are described as linear differential equations. We develop the time-domain solution to these equations, and then contrast this with the (equivalent) frequency-domain solution. The concept of *frequency response* is reviewed. More general theories and concepts are assumed to be covered elsewhere: [1, 2, 3].

Summary (linear ODEs)

The general solution of an n-th order linear, homogeneous, constant-coefficient differential equation depends on the roots $\lambda_1 \dots \lambda_n$ of the characteristic equation. For a second-order equation we have the following cases:

- **Real and distinct roots** λ_1 and λ_2 give the solution

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

- **Repeated real roots** $\lambda_1 = \lambda_2 = \lambda$ give the solution

$$x(t) = (At + B)e^{\lambda t}$$

- **Complex conjugate roots** $\lambda_{1,2} = a \pm jb$ give the solution

$$x(t) = (A \cos bt + B \sin bt)e^{at}$$

  Alternatively, this solution may be expressed in the equivalent form

$$x(t) = C \sin(bt + \phi)e^{at}$$

  where the unknown coefficients C and ϕ are related to A and B by $C = \sqrt{A^2 + B^2}$ and $\phi = \tan^{-1}(B/A)$ (it is left as an exercise to show that this relation between the two equivalent solutions holds).

1 Example: response of the lower leg

1.1 System description

We will investigate the dynamic response of the lower leg. Consider a person seated on a bench with the shank free to move around the knee joint, as shown in figure 1. We approximate

Figure 1: Seated person with shank rotation about the knee joint.

this system using the uniform single link depicted in figure 2. The link has a length l and is free to rotate about O. The mass is m, and the centre of mass is located at the point G, situated a distance $l/2$ from each end of the link. The moment of inertia of the link around the point O is denoted by I, and this is related to the mass and length through the relation $I = m(l/2)^2 = ml^2/4$ (see [3]).

We first consider a situation in which the knee-extensor muscles (principally the quadriceps group of muscles) contract in such a way that the system is held in equilibrium at an angle θ_e by application of a constant equilibrium torque τ_e acting around O. We denote an arbitrary angular displacement from equilibrium as θ, as shown in figure 3. With the system in static

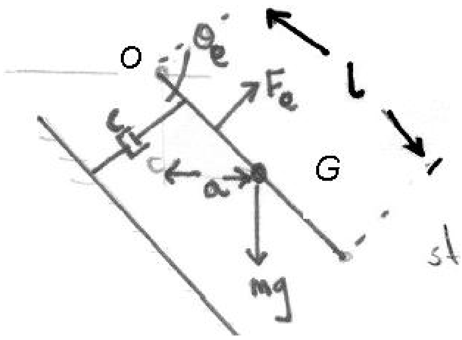

Figure 2: Idealised model of lower leg.

equilibrium the moment arm a for the gravitational force mg is $a = (l/2)\sin\theta_e$. Thus the equilibrium moment is

$$\tau_e = (mgl/2)\sin\theta_e$$

When the system is in motion (i.e. $\dot{\theta} \neq 0$) we consider it to be subject to a viscous damping moment $c\dot{\theta}$, where c is the damping coefficient shown in figure 2. In reality, this damping arises from the physical properties of the contracting muscle and its tendon-bone connection.

1.2 Equations of motion

We now derive the differential equation which describes motion of the lower leg model when it is in motion around the equilibrium.

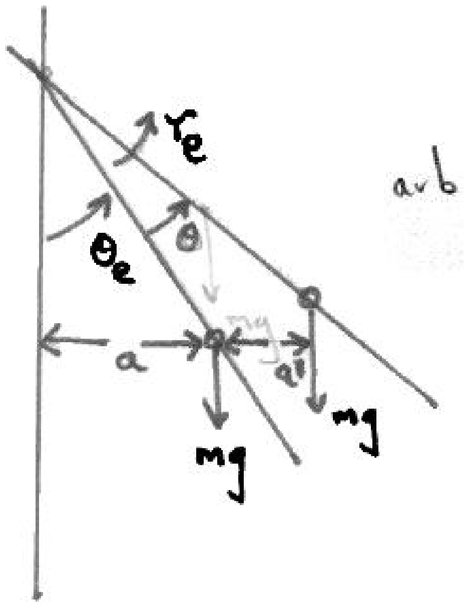

Figure 3: Displacement θ from equilibrium position θ_e.

It is instructive to derive the system equation in two ways: first, by considering all forces acting on the system; second, by ignoring the constant forces associated with equilibrium, and considering only the forces developed in displacement away from equilibrium. We will see that both approaches lead to the same result.

1. **All forces**: The inertial moment of the limb is given by $I\ddot{\theta}$, where I is the moment of inertia. Balancing all moments we obtain

$$I\ddot{\theta} = \tau_e - c\dot{\theta} - mg\frac{l}{2}\sin(\theta_e + \theta)$$

We use the trigonometric identity $\sin(\theta_e + \theta) = \sin\theta_e \cos\theta + \cos\theta_e \sin\theta$. Assuming further that the displacement angle θ is small we have $\cos\theta \approx 1$ and $\sin\theta \approx \theta$. The equation of motion then becomes

$$I\ddot{\theta} = \tau_e - c\dot{\theta} - mg\frac{l}{2}(\sin\theta_e + (\cos\theta_e)\theta)$$

Using the expression derived above for the equilibrium moment, $\tau_e = (mgl/2)\sin\theta_e$, this simplifies to

$$I\ddot{\theta} = -c\dot{\theta} - mg\frac{l}{2}(\cos\theta_e)\theta \quad (1)$$

This expression is finally rearranged as

$$\ddot{\theta} + \frac{c}{I}\dot{\theta} + \frac{mgl\cos\theta_e}{2I}\theta = 0 \tag{2}$$

which we recognise as a 2nd-order linear constant-coefficient differential equation.

2. **Displacement forces**: We now re-derive the equation of motion by neglecting the equilibrium moments. A positive displacement of θ causes the gravitational moment arm to increase by a', as shown in figure 4. We see from the geometric arrangement

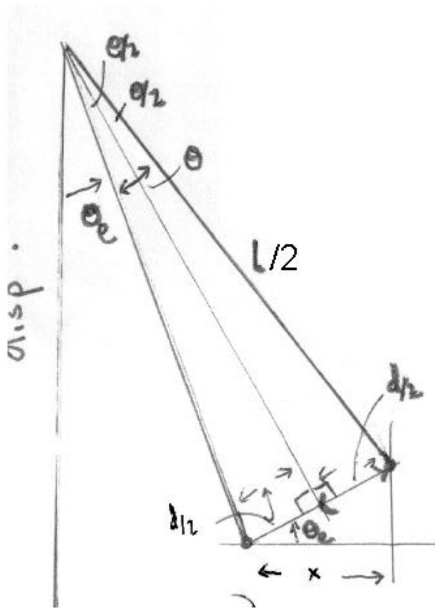

Figure 4: Displacement θ from equilibrium position θ_e.

that $(d/2)/(l/2) = \sin(\theta/2)$, or $d = l\sin(\theta/2)$. Assuming again that θ is small we have $d \approx \frac{l}{2}\theta$. We also see from the diagram that $a'/d = \cos\theta_e$. Substituting for d we obtain $a' = \frac{l\cos\theta_e}{2}\theta$. Thus, the additional gravitational moment τ^d generated by a displacement θ is

$$\tau^d = mga' = mg\frac{l\cos\theta_e}{2}\theta$$

Now, considering this moment within a moment balance for the system leads to the equation of motion

$$I\ddot{\theta} = -c\dot{\theta} - \frac{mgl\cos\theta_e}{2}\theta$$

which on rearrangement gives

$$\ddot{\theta} + \frac{c}{I}\dot{\theta} + \frac{mgl\cos\theta_e}{2I}\theta = 0 \tag{3}$$

We see that this is identical to expression (2). Thus we see that by defining the displacement variable θ to be zero at equilibrium we may ignore the equal and opposite forces associated with equilibrium.

We noted above that the system can be described as a 2nd order linear constant-coefficient differential equation. However, it is important to note that the coefficients of the equation of motion depend not only on the physical properties of the system through the parameters c, I, m and l, but also on the chosen equilibrium point. This is because the equilibrium parameter θ_e appears within one of the coefficients. This means that solutions of the equation of motion obtained at a given equilibrium point θ_e are valid only for "small" angular deviations around that point.

1.3 Solution of equation of motion

It is common practice to rewrite linear 2nd order differential equations of the form (3) in a standard format. To do this we define the variables ω_n and ζ (zeta) as

$$\omega_n = \sqrt{\frac{mgl\cos\theta_e}{2I}} \qquad \text{and} \qquad \zeta = \frac{c}{2I\omega_n} \tag{4}$$

These variables have an important physical significance, as will soon become clear. ω_n is known as the *natural angular frequency of oscillation* (or simply "natural frequency") and ζ is a dimensionless constant known as the *damping factor* (or *damping ratio*), and is seen to be directly proportional to the system's viscous damping c. Substituting for ω_n and ζ in (3), the equation of motion can be written in the standard form

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0 \tag{5}$$

The characteristic equation associated with (5) is

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

The roots of this equation are

$$\lambda_{1,2} = \omega_n(-\zeta \pm \sqrt{\zeta^2 - 1}) \tag{6}$$

Clearly the value of ζ determines the character of the solution: when $0 \le \zeta < 1$ there is a pair of complex conjugate roots; when $\zeta = 1$ there is a pair of equal real roots; and when $\zeta > 1$ the roots are real and distinct. We now consider these cases individually, beginning by considering the special case of $\zeta = 0$.

1. **Undamped system,** $\zeta = 0$. If no viscous damping is present then $c = 0$ and $\zeta = 0$ (equation (4)). In this case the roots of the characteristic equation are $\lambda_{1,2} = \pm j\omega_n$. This leads to the general solution

$$\theta(t) = A\cos\omega_n t + B\sin\omega_n t \tag{7}$$

or, equivalently,

$$\theta(t) = C\sin(\omega_n t + \phi) \tag{8}$$

Particular solutions are obtained by using the initial conditions $\theta_0 = \theta(0)$ and $\dot{\theta}_0 = \dot{\theta}(0)$ to evaluate the constants A and B (or C and ϕ). Evaluating (7) at $t = 0$ we obtain $A = \theta_0$. Differentiating (7) and evaluating at $t = 0$ results in $B = \dot{\theta}_0/\omega_n$. The particular solution is therefore

$$\theta(t) = \theta_0\cos\omega_n t + \frac{\dot{\theta}_0}{\omega_n}\sin\omega_n t$$

Particular solutions for the equivalent form (8) are similarly obtained by evaluating $\theta(t)$ and $\dot{\theta(t)}$ at $t = 0$. This procedure results in

$$C = \sqrt{\theta_0^2 + \left(\frac{\dot{\theta}_0}{\omega_n}\right)^2} \qquad \text{and} \qquad \phi = \tan^{-1}\left[\frac{\omega_n \theta_0}{\dot{\theta}_0}\right]$$

The particular solution is therefore

$$\theta(t) = \sqrt{\theta_0^2 + \left(\frac{\dot{\theta}_0}{\omega_n}\right)^2} \sin(\omega_n t + \tan^{-1}\left[\frac{\omega_n \theta_0}{\dot{\theta}_0}\right])$$

Thus, in the absence of damping the system will oscillate continuously with the *natural* angular frequency ω_n, which has the units radians per second, and whose value depends on the physical parameters of the system and on the equilibrium position (see equation (4)). The amplitude and phase of the oscillation are dependent upon the initial conditions (and on ω_n). The number of complete cycles per second is the *natural frequency* $f_n = \omega_n/2\pi$ and has the units hertz [Hz]. The time required for one complete cycle of the motion is called the *period* and is given by $T = 1/f_n = 2\pi/\omega_n$. Figure 5 shows a graphical representation of the motion for a system having the parameters $m = 5$ kg, $l = 0.4$ m, $c = 0$ Nm $\cdot$ rad$^{-1} \cdot$ s, thus giving $\omega_n = 7$ rad $\cdot$ s^{-1} and $\zeta = 0$ (see equation (4)). The equilibrium position is $\theta_e = 0$ deg and the initial conditions are $\theta_0 = 10$ deg and $\dot{\theta}_0 = 0$.

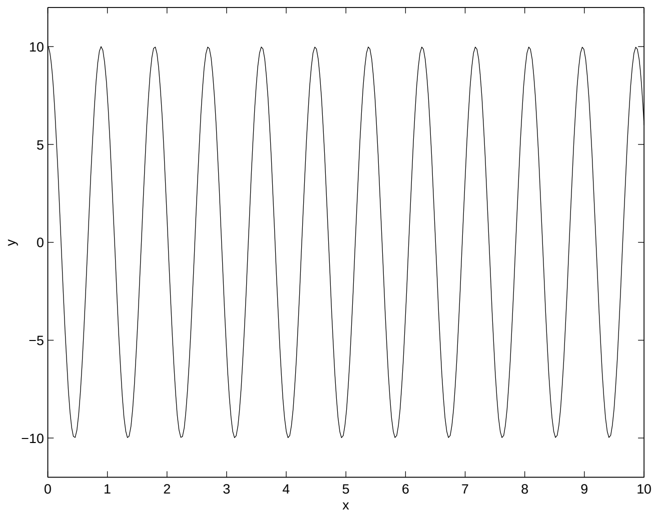

Figure 5: Undamped oscillation ($\zeta = 0$).

2. **Under-damped system,** $0 < \zeta < 1$. In this case the complex-conjugate roots of the characteristic equation are $\lambda_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$. It is convenient to introduce a new variable $\omega_d = \omega_n\sqrt{1-\zeta^2}$, whose physical significance will become clear below. With this definition the characteristic roots are $\lambda_{1,2} = -\zeta\omega_n \pm j\omega_d$. The general solution

can then be expressed as

$$\theta(t) = (A\cos\omega_d t + B\sin\omega_d t)e^{-\zeta\omega_n t} \tag{9}$$

or, equivalently,

$$\theta(t) = C\sin(\omega_d t + \phi)e^{-\zeta\omega_n t} \tag{10}$$

As before, the particular values of the constants A, B, C and ϕ are obtained using the initial conditions. They are found to be

$$A = \theta_0,\ B = \frac{\dot{\theta}_0 + \zeta\omega_n\theta_0}{\omega_d} \quad \text{and} \quad \phi = \tan^{-1}\left[\frac{\theta_0\omega_d}{\dot{\theta}_0 + \zeta\omega_n\theta_0}\right],\ C = \frac{\theta_0}{\sin\phi}$$

The solution (10) shows that the system in this case oscillates with a frequency $\omega_d = \omega_n\sqrt{1-\zeta^2}$, which is known as the *damped natural frequency.* With damping, however, the amplitude of oscillation is seen to decrease exponentially with time due to the factor $e^{-\zeta\omega_n t}$. The *damped period* is $T_d = 2\pi/\omega_d = 2\pi/(\omega_n\sqrt{1-\zeta^2})$. Figure 6 shows a graphical representation of the damped motion for a system having the parameters $m = 5$ kg, $l = 0.4$ m, $c = 0.5$ Nm·rad^{-1}·s, thus giving $\omega_n = 7$ rad·s^{-1} and $\zeta = 0.1785$ (see equation (4)). The equilibrium position is again $\theta_e = 0$ deg and the initial conditions are $\theta_0 = 10$ deg and $\dot{\theta}_0 = 0$.

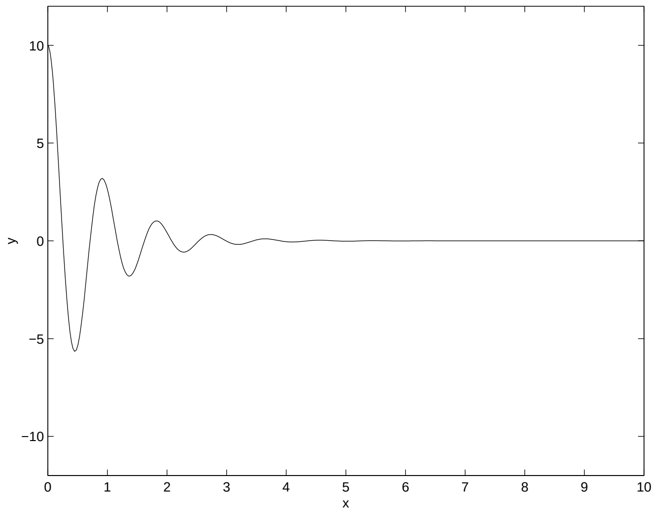

Figure 6: Under-damped oscillation ($\zeta = 0.1785$).

The damping factor ζ can be experimentally determined for under-damped systems. This experimental approach can be useful when the viscous damping coefficient c cannot be easily determined from physical considerations. Assume that the system is set in motion with some initial conditions and two successive amplitudes θ_1 and θ_2 are measured at times t_1 and t_2, respectively. The ratio of the two amplitudes is formed as

$$\frac{\theta_1}{\theta_2} = \frac{Ce^{-\zeta\omega_n t_1}}{Ce^{-\zeta\omega_n(t_1+T_d)}} = e^{\zeta\omega_n T_d}$$

The logarithm of this quantity, known as the *logarithmic decrement*, is then computed as

$$\delta = \ln\left(\frac{\theta_1}{\theta_2}\right) = \zeta\omega_n T_d = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

Solving this equation for ζ we obtain

$$\zeta = \frac{\delta}{\sqrt{4\pi^2+\delta^2}}$$

3. **Critically damped system,** $\zeta = 1$. In this case the roots of the characteristic equation are $\lambda_1 = \lambda_2 = -\omega_n$, which leads to the general solution

$$\theta(t) = (At+B)e^{-\omega_n t} \tag{11}$$

A and B are given in terms of the initial conditions as

$$A = \dot{\theta}_0 + \omega_n\theta_0, \;\; B = \theta_0$$

It is clear from (11) that the motion decays exponentially to zero, i.e. $\theta(t) \to 0$ as $t \to \infty$. Moreover, the motion is non-periodic (non-oscillatory).

4. **Over-damped system,** $\zeta > 1$. For an over-damped system the characteristic roots are real and distinct, i.e. $\lambda_{1,2} = \omega_n(-\zeta \pm \sqrt{\zeta^2-1})$. The general solution is

$$\theta(t) = Ae^{(-\zeta+\sqrt{\zeta^2-1})\omega_n t} + Be^{(-\zeta-\sqrt{\zeta^2-1})\omega_n t} \tag{12}$$

Again, the motion decays in a non-oscillatory fashion towards $\theta(t) = 0$. The rate of decay becomes slower as the damping increases. Figure 7 shows solutions for various values of $\zeta \geq 1$. for a system having the parameters $m = 5$ kg, $l = 0.4$ m, $c = 0.5$ $\text{Nm} \cdot \text{rad}^{-1} \cdot \text{s}$.

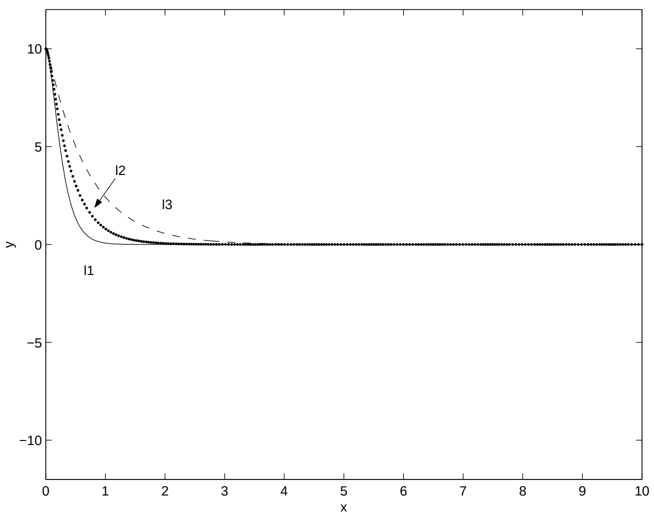

Figure 7: Over-damped response.

1.4 Forced response of the lower leg

We now consider the situation in which motion of the lower leg is continuously excited by a disturbing moment. The moment will typically be generated by contraction of the muscles which extend and flex the knee joint (principally the quadriceps and hamstring muscle groups). Various forms of forcing functions might be applied. However, we consider the special case in which the forcing function is harmonic (sinusoidal). This type of function gives a useful introduction to the analysis of forced motion, but is also very important because, as we shall see later, the response to harmonic excitation defines the *frequency response* of the system.

Consider the system of figure 8 where the lower leg is subject to the applied harmonic moment $\tau = \tau_e + \tau_0 \sin \omega t$. Here, τ_e, as before, is the static moment corresponding to the equilibrium position θ_e. τ_0 is the amplitude of the driving moment and ω is the driving frequency (in radians per second). It is important to distinguish between ω, which is a property of the moment applied to the system, and $\omega_n = \sqrt{mgl \cos \theta_e / 2I}$, which is a physical property of the system. Balancing the moments acting around the knee joint, using the same

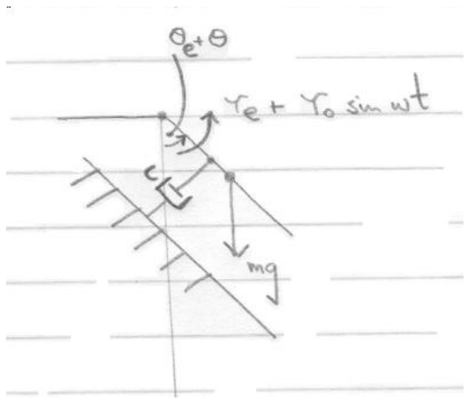

Figure 8: Forced motion of the lower leg.

procedure employed in section 1.2, gives the equation of motion

$$I\ddot{\theta} = \tau_0 \sin \omega t - c\dot{\theta} - \frac{mgl \cos \theta_e}{2} \theta$$

which on rearrangement gives

$$\ddot{\theta} + \frac{c}{I}\dot{\theta} + \frac{mgl \cos \theta_e}{2I} \theta = \frac{\tau_0}{I} \sin \omega t \tag{13}$$

Thus, the equations of motion for the forced system are seen to differ from those for the unforced system (c.f. equations (1)–(2)) through inclusion of the harmonic term $\sin \omega t$.

Making the same variable substitutions as above (see equation (4)) the standard form of the equation of motion is

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = \frac{\tau_0}{I} \sin \omega t \tag{14}$$

We recognise that when the right-hand-side of this equation is set to zero we recover the homogeneous equation (5), which we have studied in detail. The general solution of the non-homogeneous equation (14) consists of the complementary function θ_c (the general solution of

(14) with the right-hand-side set to zero, i.e. the homogeneous equation (5)) and a particular integral θ_p, which is *any* solution to (14), i.e. $\theta(t) = \theta_c(t) + \theta_p(t)$. The solution θ_c for various values of ζ was studied in detail above. We now focus on finding θ_p.

We have previously seen that trial functions for the particular integral have a similar form to the forcing function (the right-hand-side of the equation). Thus, for a harmonic function of the form $\sin \omega t$ we try

$$\theta_p(t) = A \sin \omega t + B \cos \omega t, \qquad \text{or} \qquad \theta_p(t) = D \sin(\omega t - \psi) \tag{15}$$

We proceed by considering the latter form of θ_p. We take the first and second derivatives of this expression, then substitute for θ_p, $\dot{\theta}_p$ and $\ddot{\theta}_p$ in equation (14). The unknowns D and ψ are then obtained by matching coefficients of $\sin \omega t$ and $\cos \omega t$ and solving the resulting expressions. This gives

$$D = \frac{\tau_0/I}{\omega_n^2[(1-(\omega/\omega_n)^2)^2 + (2\zeta\omega/\omega_n)^2]^{1/2}} \tag{16}$$

$$\psi = \tan^{-1}\left[\frac{2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2}\right] \tag{17}$$

The complete solution can now be obtained from $\theta(t) = \theta_c(t) + \theta_p(t)$. We previously considered various forms of θ_c for different values of the damping factor ζ. For example, for an under-damped system the complete solution will be

$$\theta(t) = Ce^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + D\sin(\omega t - \psi) \tag{18}$$

Here, C and ϕ are obtained from $\theta(t)$ and its derivative $\dot{\theta}(t)$, evaluated at the initial conditions $\theta_0 = \theta(0)$ and $\dot{\theta}_0 = \dot{\theta}(0)$. The first term in this expression is seen to decay to zero (for $\zeta > 0$), and for this reason it is known as the *transient solution.* The second term, on the other hand, results from the forcing function and persists as long as the forcing function is applied. For this reason it is know as the *steady-state solution.* We note that all quantities on the right-hand-side of equation (18) are properties of the physical system or of the forcing function, except for the constants C and ϕ in the transient solution, which depend only on the initial conditions. The relationship between the components of the complete solution is summarised as follows:

$$\begin{aligned}\theta(t) = &\theta_c(t) && +\theta_p(t)\\ &= \underbrace{Ce^{-\zeta\omega_n t}\sin(\omega_d t + \phi)}_{\substack{\text{Complementary function, i.e.}\\ \text{general solution of}\\ \text{homogeneous equation.}\\ \rightarrow \textit{Transient Solution}}} && + \underbrace{D\sin(\omega t - \psi)}_{\substack{\text{Particular integral, i.e.}\\ \text{solution of non-}\\ \text{homogeneous equation.}\\ \rightarrow \textit{Steady-state Solution}}}\end{aligned}$$

Since the transient solution diminishes with time (for $\zeta > 0$), the steady-state solution is of particular interest. We see from equation (16) that the magnitude D of the steady-state solution depends strongly on the damping factor ζ and the non-dimensional frequency ratio ω/ω_n. The amplification (or attenuation) of the forcing function is determined by the ratio of the steady-state solution's amplitude and the forcing function's amplitude, i.e. $D/(\tau_0/I)$.

This quantity, denoted by M, is called the *amplitude ratio* or *magnification factor*, and from (16) is given by

$$M = \frac{1}{\omega_n^2[(1-(\omega/\omega_n)^2)^2 + (2\zeta\omega/\omega_n)^2]^{1/2}} \tag{19}$$

We note from above that when the system is undamped ($\zeta = 0$), M approaches infinity as the forcing frequency ω approaches the natural frequency ω_n. The frequency at which M peaks (for undamped systems this is ω_n) is known as the *resonant* or *critical frequency* of the system. The condition of the forcing frequency ω being close to the critical frequency, with associated large values of M, is known as *resonance*.

Figure 9 shows a plot of the magnitude of M as a function of the frequency ratio ω/ω_n, for various values of ζ. The figure shows that, even with damping, M (and therefore D) can

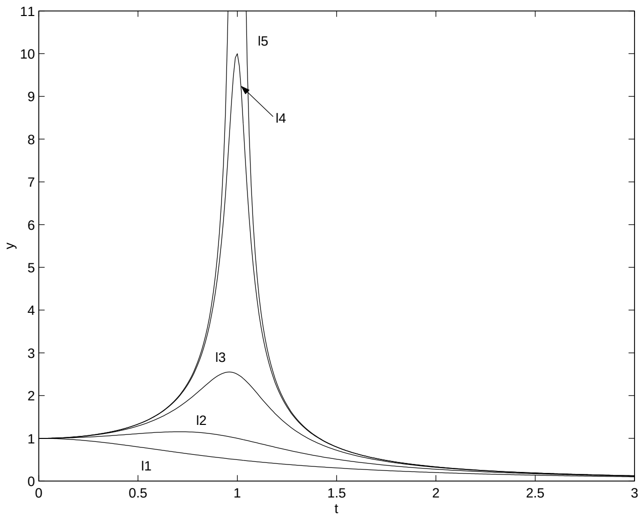

Figure 9: Amplification ratio, various values of damping ζ.

become very large. The figure also illustrates how excessive motion can be avoided: one way is to increase the damping in the system; the other way is to alter the driving frequency ω so that it is not close to ω_n. It is also clear that with damping ($\zeta > 0$) the magnification does not peak at exactly $\omega = \omega_n$. The resonant frequency for any value of ζ can be found from equation (19) by finding the maximum value of M with respect to ω/ω_n.

Finally, the phase angle ψ represents the degree to which the steady-state response θ_p lags behind the forcing function. The lag can vary between 0 and π, depending on the damping ζ. Figure 10 shows how the phase angle varies with ω/ω_n for various values of ζ.

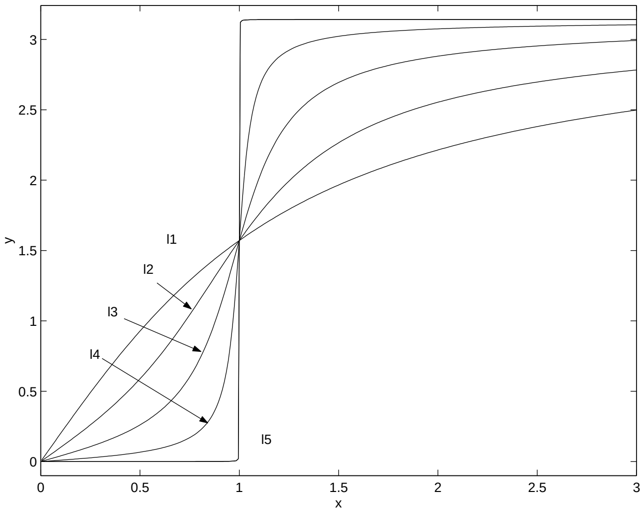

Figure 10: Phase angle, showing lag in forced response.

2 Frequency-domain Solution (Laplace)

The solutions we have thus far obtained to the differential equations describing motion of the lower leg and standing posture have been obtained using time-domain methods. We now consider the solution using the Laplace transform. In the time domain the solution consists of two parts: the complementary function, representing the transient response; and the particular integral, which gives the steady-state response. As we shall see, the Laplace solution immediately gives the complete solution, including initial conditions. However, we shall make clear the links between the equivalent solutions, obtained using the two different methods.

2.1 Solution of the standard equation

The differential equations describing motion of the lower leg and upright standing can both be represented in the standard form (see equations (14))

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = \frac{\tau_0}{I}\sin\omega t \tag{20}$$

where $(\tau_0/I)\sin\omega t$ is the forcing function. The variables ζ and ω_n are given by (4) for the lower leg. Thus the solution derived below is valid for both systems, or for any other system which can be represented in the standard format.

We proceed by taking Laplace transforms throughout equation (20). This gives

$$(s^2\Theta(s) - s\theta_0 - \dot{\theta}_0) + 2\zeta\omega_n(s\Theta(s) - \theta_0) + \omega_n^2\Theta(s) = \frac{\tau_0\omega/I}{s^2+\omega^2}$$

where $\Theta(s) = \mathcal{L}\{\theta(t)\}$, and $\theta_0 = \theta_0(t), \dot{\theta}_0 = \dot{\theta}_0(t)$ are the initial conditions. The above equation can be rearranged in terms of $\Theta(s)$, giving

$$\Theta(s) = \frac{s\theta_0 + 2\zeta\omega_n\theta_0 + \dot{\theta}_0}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{\tau_0\omega/I}{(s^2+\omega^2)(s^2+2\zeta\omega_n s+\omega_n^2)} \tag{21}$$

or,

$$\Theta(s) = \frac{\theta_0 s^3 + (2\zeta\omega_n\theta_0 + \dot{\theta}_0)s^2 + \omega^2\theta_0 s + (\omega^2(2\zeta\omega_n\theta_0 + \dot{\theta}_0) + \tau_0\omega/I)}{(s^2+\omega^2)(s^2+2\zeta\omega_n s+\omega_n^2)} \tag{22}$$

The next step in the procedure is to decompose the above expression for Θ into partial fractions. The form of the partial fractions will depend upon the nature of the roots of the characteristic equation $s^2 + 2\zeta\omega_n s + \omega_n^2$. The roots are given by equation (6), i.e. $s_{1,2} = \omega_n(-\zeta \pm \sqrt{\zeta^2 - 1})$. The nature of the roots depends on the value of ζ, and in section 1.3 we determined the solution (using time-domain methods) for the special cases $\zeta = 0$, $0 < \zeta < 1$, $\zeta = 1$ and $\zeta > 1$. We proceed here by investigating the under-damped response, i.e. $0 < \zeta < 1$.

For this range of ζ the characteristic roots are

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\zeta\omega_n \pm j\omega_d$$

since $\omega_d = \omega_n\sqrt{1-\zeta^2}$. Thus, the characteristic equation factorises as $s^2 + 2\zeta\omega_n s + \omega_n^2 = (s+\zeta\omega_n)^2 + \omega_d^2$, and the appropriate partial fraction expansion is

$$\Theta(s) = \frac{As+B}{(s+\zeta\omega_n)^2+\omega_d^2} + \frac{Cs+D}{s^2+\omega^2} \tag{23}$$

We recognise that the two parts of this expression correspond, respectively, to the transient- and steady-state solutions of the response, and therefore to the complementary function and particular integral of the time-domain solution, i.e.

$$\Theta(s) = \underbrace{\frac{As+B}{(s+\zeta\omega_n)^2+\omega_d^2}}_{\substack{\text{Transform of complementary} \\ \text{function} \\ \rightarrow \textit{Transient Solution}}} + \underbrace{\frac{Cs+D}{s^2+\omega^2}}_{\substack{\text{Transform of particular} \\ \text{integral} \\ \rightarrow \textit{Steady-state Solution}}}$$

The numerical values of the constants A, B, C and D are obtained in the usual way by multiplying equations (22) and (23) by the denominator and then equating the coefficients of corresponding powers of s (this procedure is left as an exercise). The solution for θ is then obtained by inverse transformation. In order to apply standard tables of Laplace transforms it is convenient to rearrange the terms in equation (23) as

$$\Theta(s) = \frac{A(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2+\omega_d^2} + \frac{B'\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2} + \frac{Cs}{s^2+\omega^2} + \frac{D'\omega}{s^2+\omega^2} \tag{24}$$

where $B' = (B - A\zeta\omega_n)/\omega_d$, $D' = D/\omega$. The inverse transform can then be obtained from standard tables as

$$\theta(t) = (A\cos\omega_d t + B'\sin\omega_d t)e^{-\zeta\omega_n t} + C\cos\omega t + D'\sin\omega t$$

or, equivalently,

$$\theta(t) = Ee^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + F\sin(\omega t - \psi) \tag{25}$$

where the constants E, F, ϕ and ψ are related straightforwardly to A, B', C and D':

$$E = \sqrt{A^2 + B'^2}, \quad \phi = \tan^{-1}(B'/A)$$

$$F = \sqrt{C^2 + D'^2}, \quad \psi = \tan^{-1}(-D'/C)$$

Thus, we see that the solution (25) obtained by the Laplace transform procedure is identical to the result obtained previously in the time domain (equation (18)).

2.2 Transfer function and amplitude ratio

From equation (21) we see that the system response can be split into two parts: one containing the initial conditions, and one including the forcing function. This holds for general forcing functions $u(t)$ and not only the sinusoidal functions considered above. Denoting the transform of a general forcing function u as $U(s) = \mathcal{L}\{u(t)\}$ we see from (21) that, in general,

$$\Theta(s) = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}(s\theta_0 + 2\zeta\omega_n\theta_0 + \dot{\theta}_0 + U(s)) \tag{26}$$

The response is strongly dependent upon the function $1/(s^2 + 2\zeta\omega_n s + \omega_n^2)$, which we will denote as $G(s)$, and which is known as the system *transfer function.* With this definition equation (26) becomes

$$\Theta(s) = G(s)(s\theta_0 + 2\zeta\omega_n\theta_0 + \dot{\theta}_0 + U(s)) \tag{27}$$

The transfer function is usually defined as the ratio of the Laplace transform of the system output (here, $\Theta(s)$) and the transform of the system input (or forcing function, $U(s)$) *under the assumption that the initial conditions are zero*, i.e. $G(s) \triangleq U^{-1}(s)\Theta(s)$ when $\theta_0 = \dot{\theta}_0 = 0$.

We shall see below (section 2.3) that the *frequency response* of the system is given by $G(s)$, but with the complex variable s replaced by $j\omega$. It is clear from the relationship $\Theta(s) = G(s)U(s)$, and from the analysis to follow in section 2.3, that the magnitude $|G(j\omega)|$ determines the amount of amplification (or attenuation) of the forcing signal u at the frequency ω. This observation allows us to make an explicit link between $|G(j\omega)|$ and the *amplitude ratio* (or *magnification factor*) M which was derived previously from time-domain considerations (see equation (19)). We will show that these quantities are identical.

Since $G(s) = 1/(s^2 + 2\zeta\omega_n s + \omega_n^2)$ we have

$$G(j\omega) = \frac{1}{(\omega_n^2 - \omega^2) + j2\zeta\omega_n\omega}$$

We multiply the numerator and denominator of this expression by the complex conjugate of the denominator to give

$$G(j\omega) = \frac{(\omega_n^2 - \omega^2) - j2\zeta\omega_n\omega}{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}$$

The magnitude of G is then found to be

$$|G(j\omega)| = \frac{[(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2]^{1/2}}{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2} = \frac{1}{[(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2]^{1/2}}$$

which can be rearranged as

$$|G(j\omega)| = \frac{1}{\omega_n^2[(1 - (\omega/\omega_n)^2)^2 + (2\zeta\omega/\omega_n)^2]^{1/2}} \tag{28}$$

Equations (19) and (28) are seen to be identical, i.e. $|G(j\omega)| = M$.

2.3 Frequency response

The frequency response plays a key role in the design and analysis of systems. The frequency response of the transfer function $G(s)$ is defined as the steady-state response to a sinusoidal input signal of frequency ω. We show here that the frequency response of $G(s)$ is obtained by replacing the complex variable s by its imaginary part $j\omega$.

We consider an input signal $u(t) = (\tau_0/I)\sin\omega t$. From equation (27) the system response, assuming zero initial conditions, is

$$\Theta(s) = G(s)U(s) = G(s)\frac{\tau_0\omega/I}{s^2+\omega^2} \tag{29}$$

The roots of the denominator of $G(s)$ are given by $s_1 = p_1$, $s_2 = p_2$, where p_1, p_2 are in general complex numbers. A partial fraction expansion of $\Theta(s)$ is then

$$\Theta(s) = \frac{a}{s-j\omega} + \frac{b}{s+j\omega} + \frac{c}{s-p_1} + \frac{d}{s-p_2} \tag{30}$$

where a, b, c and d are constants. Note that this expansion is an alternative, but equivalent, expansion to the one previously considered in equation (23). We recognise once more that the first two terms in (30) determine the steady-state response, while the remaining terms are transient. When the real parts of p_1 and p_2 are negative the transient response in the time domain will decay exponentially to zero (i.e. the system is stable). Thus the steady-state response, which we denote as $\theta_{\mathrm{ss}}(t)$, is given by the inverse transform of the first two terms in (30):

$$\theta_{\mathrm{ss}}(t) = ae^{j\omega t} + be^{-j\omega t} \tag{31}$$

Now, combining equations (29) and (30) we have

$$G(s)\frac{\tau_0\omega/I}{s^2+\omega^2} = \frac{a}{s-j\omega} + \frac{b}{s+j\omega} + \frac{c}{s-p_1} + \frac{d}{s-p_2}$$

which we multiply throughout by $s^2+\omega^2 = (s+j\omega)(s-j\omega)$ to give

$$\frac{\tau_0\omega}{I}G(s) = a(s+j\omega) + b(s-j\omega) + \frac{c(s+j\omega)(s-j\omega)}{s-p_1} + \frac{d(s+j\omega)(s-j\omega)}{s-p_2} \tag{32}$$

The constant a can be obtained by evaluating this expression at $s = j\omega$ to give

$$a = \left.\frac{(\tau_0\omega/I)G(s)}{s+j\omega}\right|_{s=j\omega} = \frac{\tau_0}{2Ij}G(j\omega) \tag{33}$$

Similarly we obtain b by evaluating the expression (32) at $s = -j\omega$:

$$b = \left.\frac{(\tau_0\omega/I)G(s)}{s-j\omega}\right|_{s=-j\omega} = \frac{-\tau_0}{2Ij}G(-j\omega) \tag{34}$$

The steady-state response (31) becomes

$$\theta_{\mathrm{ss}}(t) = \frac{\tau_0}{2Ij}G(j\omega)e^{j\omega t} - \frac{\tau_0}{2Ij}G(-j\omega)e^{-j\omega t} \tag{35}$$

The complex number $G(j\omega)$ can be expressed in polar form as $G(j\omega) = |G(j\omega)|e^{j\arg G(j\omega)}$, where $|G(j\omega)|$ denotes the magnitude of $G(j\omega)$ and the term $\arg G(j\omega)$ is its phase. Since $G(-j\omega) = |G(j\omega)|e^{-j\arg G(j\omega)}$ the steady-state response in (35) becomes

$$\theta_{\mathrm{ss}}(t) = \frac{\tau_0}{2Ij}|G(j\omega)|(e^{j(\omega t+\arg G(j\omega))} - e^{-j(\omega t+\arg G(j\omega))}) \tag{36}$$

Using the Euler identity $(e^{jx} - e^{-jx})/2j = \sin x$ we finally obtain

$$\boxed{\theta_{\mathrm{ss}}(t) = \frac{\tau_0}{I}|G(j\omega)|\sin(\omega t + \arg G(j\omega))} \tag{37}$$

This expression defines the system's frequency response, i.e. the response to a sinusoidal input of frequency ω. The frequency response is seen to be characterised by the magnitude, $|G(j\omega)|$, and phase, $\arg G(j\omega)$, of $G(s)$ when evaluated at $s = j\omega$. We have seen in section 2.2 that $|G(j\omega)|$ is equivalent to the *amplitude ratio* (or *magnification factor*) M. It is often referred to simply as the *gain* of G. The quantity $\arg G(j\omega)$ is known as the *phase shift*.

The response to a sinusoidal input signal $u(t) = (\tau_0/I)\sin\omega t$ of frequency ω is also a sinusoid of the same frequency, but its amplitude is scaled by $|G(j\omega)|$ and its phase is shifted by the angle $\arg G(j\omega)$.

It may at first sight appear restrictive to consider only sinusoidal signals. However, it should be recalled that all periodic signals can be exactly represented as the sum of sinusoids of different frequencies (using Fourier's theorem). Moreover, other signals will have a frequency spectrum which allows them to be represented arbitrarily accurately by a sum of sinusoids over a range of frequencies. Superposition then allows us to represent the response of a system as the sum of the responses to the individual sinusoidal inputs. The frequency response therefore provides a very powerful description of the system.

The information characterising the frequency response can be conveniently displayed in a variety of graphical formats. One common format is the *Bode diagram* which consists of two graphs: a plot of $|G(j\omega)|$ against frequency ω; and a plot of $\arg G(j\omega)$ against frequency.

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