Frequency Domain Analysis and Design

Tutorial questions

Differential Equations

Given is a lower leg of mass *m* and length *l*, moving around the knee joint with inertia *I* and damping *c*, while an external torque τ_o is applied at the knee.

Q1 Derive the equation of motion of the lower leg under the assumption that it can be described as a single link pendulum. The equation should be valid for small movements around the vertical position ($\theta = 0$) and should be simplified accordingly to obtain a linear differential equation.

Q2 Derive the transfer function in the s-domain for this model, considering the angle θ to be the output, and the external knee torque τ_o to be the input. Show how the natural frequency ω_n and the damping ξ depend on the physical parameters of the leg.

Q3 Calculate the closed loop transfer function for the following physical values: $m = 5kg$, $I = 0.2kgm^2$, $l = 0.4m$ and $c = 0.5Nms/rad$. What are the values of ω_n and ξ?

Q4 Represent the equations of motion from Q1 in the standard form of a differential equation (with the angle θ as the output, and the external knee torque τ_o as the input), i.e. as

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

Q5 Figure 1 shows the Bode frequency response plot of the plant transfer function for the lower leg response with the physical parameters given in Q3.

(i) Discuss properties of the time domain response (such as steady-state gain, frequency and amplitude of over-shoot) which can be derived from this Bode plot.

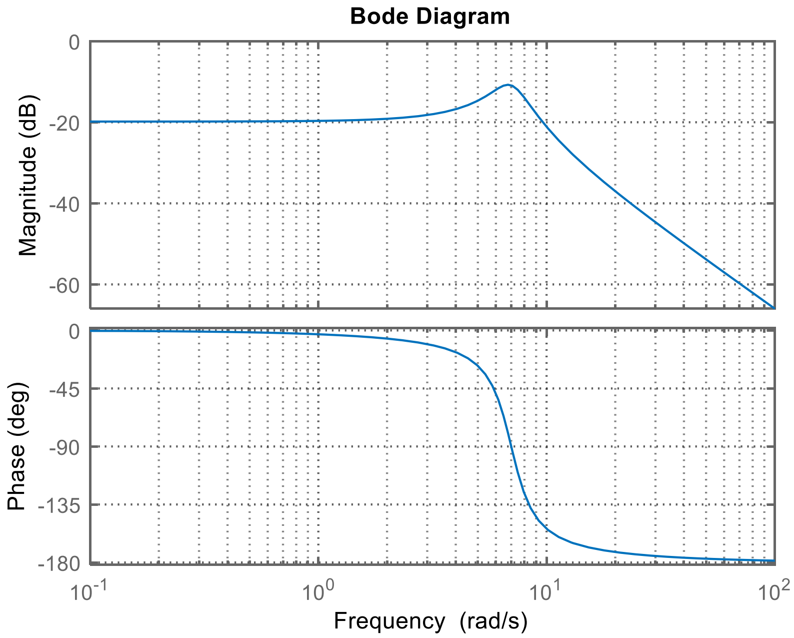

Figure 1

(ii) Figure 2 shows the Bode plot of the plant transfer function with the same parameters, except for a higher damping coefficient, $c = 3\mathrm{Nms/rad}$. Compare the likely time-domain response characteristics with those in (i).

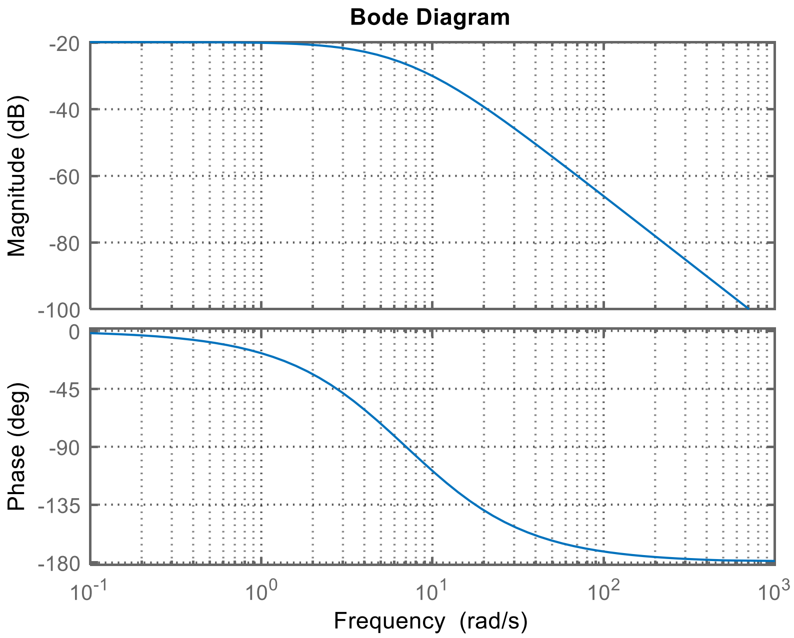

Figure 2

Open-loop and Closed-loop Control

Q1 Draw the structure of a two-degree-of-freedom feedback controller. Show the different signals, including disturbances, and explain their physical meaning.

Q2 Compare and contrast the properties of open- and closed-loop control structures. In your analysis, focus on the characteristics of each structure with respect to plant disturbances, changes in the plant gain, and stabilisation.

Q3 With reference to a compensator having a simple static gain, describe the benefits of a closed-loop control strategy in contrast to an open-loop strategy. In particular, consider the effects of disturbances and the sensitivity of overall open- or closed-loop systems to plant changes.

Q4 Show that for an open-loop structure the ideal compensator should cancel the plant dynamics, and state the drawbacks of this approach. Describe a feedback structure which can provide approximate plant inversion, and show therefore that high-gain closed-loop control implicitly inverts plant dynamics.

Dynamic Closed-loop Control: sensitivity functions

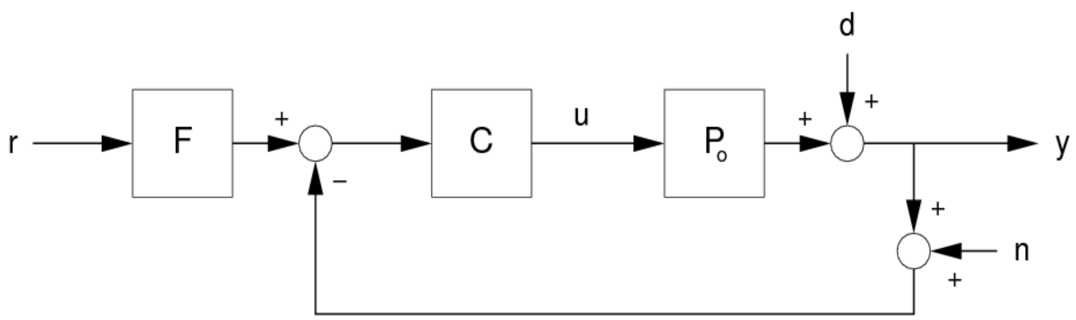

Figure 3

Refer to the closed-loop system shown in Figure 3.

Q1 Derive the closed-loop equations relating (i) the plant output *y* and (ii) the plant input *u* to the signals *r*, *d*, and *n*, respectively.

Q2 Typically, what are the signals *r*, *d* and *n*, and how should the feedback system be designed in order to respond appropriately to each of these signals?

Q3 Define the sensitivity function S_o and the complementary sensitivity function T_o in terms of C and P_o. Sketch the typical plots of $|S_o|$ and $|T_o|$ against frequency, based on the assumption that the plant has a low-pass character. Explain how this results in trade-offs in the achievement of key feedback design goals.

Q4 Show that the following constraint holds: $T_o + S_o = 1$. Discuss the implications of this constraint.

Q5.1 Show that T_o is equivalent to the relative changes in S_o which result from changes in the plant P_o, i.e. show that $T_o = -\frac{\partial S_o / S_o}{\partial P_o / P_o}$. What is the significance of this result?

Q5.2 Prove that S_o is equal to the sensitivity of the closed-loop transfer function T_o with respect to changes in the plant P_o, i.e. show that $S_o = (\delta T_o / T_o)/(\delta P_o / P_o)$.

Closed-loop Stability and Stability Robustness

Q1 Explain the concept of "internal stability" and relate this to BIBO stability, using a block-diagram similar to that shown in Figure 3. Why is it important to consider internal stability in a feedback control system?

Q2 Given is the feedback control loop shown in Figure 3. Formulate the Nyquist stability criterion. Under which conditions does this criterion apply.

Q3.1 Draw a Nyquist plot for a stable closed loop system and for an unstable system. For the stable system, mark the gain margin, phase margin and the vector margin in the plot. Explain the meaning of gain margin, phase margin and vector margin.

Q3.2 By means of a sketch of a typical Nyquist diagram, indicate the vector margin for the loop gain, and the alternative vector margin for the inverse loop gain, and explain how these can be used to characterise the relative stability of the closed-loop.

Q4 From the Nyquist plot for a closed loop system, a phase margin of 120deg at a frequency of 3Hz, and a gain margin of 11.6dB was derived.

(i) By how much can the plant gain increase before the closed loop becomes unstable?

(ii) What additional delay can be added to the closed loop before stability is lost?

Q5 Derive conditions in terms of loop-gain and inverse-loop-gain perturbations involving $|S_o|$ and $|T_o|$, which characterise the robustness of closed-loop stability when the loop gain changes from L_0 to L.

Q6.1 Explain why 'peaking' in the sensitivity function S_0 and the complementary sensitivity function T_0 should be avoided, by performing the following analysis:

i. Derive the closed-loop equations of the system, and discuss how peaking would affect the system's response to *r*, *d* and *n*, as defined in Figure 3.

ii. Show that the complementary vector margin for the inverse loop gain is equal to the inverse of the peak value of $|T_0|$. By a symmetry argument, briefly describe the effect of $|S_0|$ on the vector margin for the loop gain.

Q6.2 Assume that the nominal plant P_0 changes to some other value P. If this change causes the Nyquist plot of the loop gain *L* to move substantially closer to the -1 point, explain how this may affect the closed-loop sensitivity function *S* and the closed-loop complementary sensitivity function *T*, and how this in turn may adversely affect the various closed-loop performance targets.

Design Goals

Q1 Explain why it is highly undesirable to have low-frequency measurement noise in a feedback system, and explain why high-frequency measurement noise is relatively unimportant.

Q2 Describe the design goals which one attempts to achieve in the design of closed-loop feedback systems, and describe the factors which limit the extent to which these goals can be achieved.

Frequency Domain Controller Design – Loop Shaping

(Revised 2021: removed questions on pole-placement design)

Q1 Briefly analyse the advantages and disadvantages of each term in the standard PID control scheme.

Q2 What are the components of a PID controller? Give the control law in the time domain and in the Laplace domain.

Consider the plant $P_0(s) = \frac{4}{s+3}$

Q3 Sketch the frequency response of this plant in form of a Bode plot. What is the bandwidth of the open loop system?

Q4 Describe how a proportional controller can be used to increase the bandwidth of the closed loop system by considering how it affects the loop gain. Mention two limitations associated with using only a proportional component in the controller and how the other terms of the PID controller could be used to overcome these limitations.

Q5 Consider a standard feedback control structure with a proportional controller $C(s) = 100$. Calculate the loop gain $L(s)$ and sketch the Bode plot of its frequency response. Comment on the characteristics of the closed loop system (such as steady state gain/error, bandwidth, stability margins and damping) which you can derive from the plot of $L(s)$.

Q6 The controller is amended by a derivative term $C_D(s) = 100\frac{0.1s}{0.01s+1}$, so that the new PD controller is $C(s) = 100 + 100\frac{0.1s}{0.01s+1}$. How does the loop gain $L(s)$ change, and what is influence of this term on the characteristics of the close loop