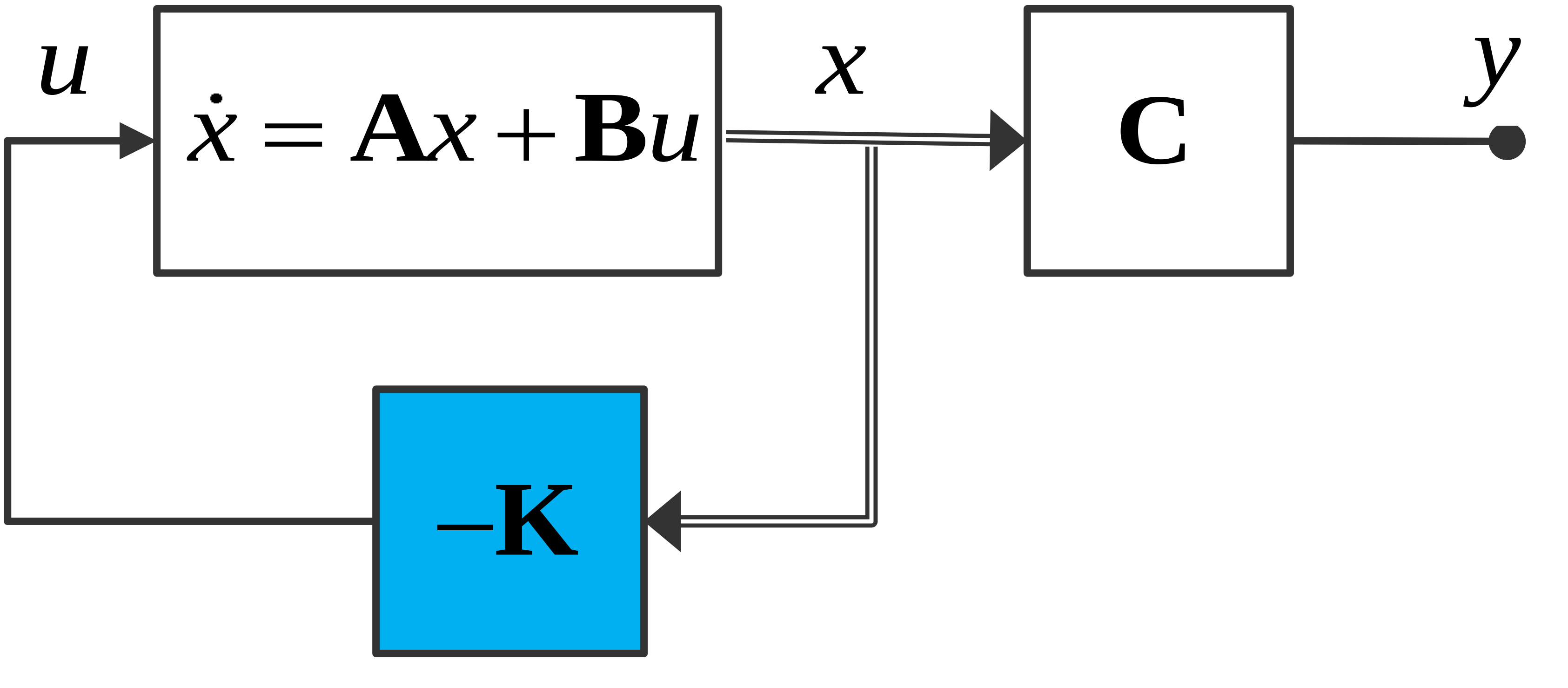
State feedback control



State feedback control

$$\dot{x} = \mathbf{A}x + \mathbf{B}u \quad y = \mathbf{C}x \quad u = -\mathbf{K}x$$

with

$$\mathbf{A} = \begin{bmatrix} -a_1 & -a_2 & \cdots & \cdots & -a_n \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix};$$

$$\mathbf{C} = \begin{bmatrix} b_1 & b_2 & \cdots & \cdots & b_n \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} k_1 & k_2 & \cdots & \cdots & k_n \end{bmatrix}$$

State feedback control

- It is known that a system which is fully controllable (ie. whose controllability matrix $\mathbf{\Gamma}_c[\mathbf{A}, \mathbf{B}]$ has full rank (is non-singular)) can be transformed into the **controller canonical form**.
- It can then be shown that full state feedback using the control law $u=-\mathbf{K}x$ can move the values a_1, $a_2, \ldots, a_n$ of the state matrix $\mathbf{A}$ to arbitrary values $(a_1+k_1), (a_2+k_2), \ldots, (a_n+k_n)$.

State feedback control

$$\dot{x} = \left(\mathbf{A} - \mathbf{BK}\right)x \quad y = \mathbf{C}x$$

with

$$\left(\mathbf{A} - \mathbf{BK}\right) = \begin{bmatrix} -\left(a_1 + k_1\right) & -\left(a_2 + k_2\right) & \cdots & \cdots & -\left(a_n + k_n\right) \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix};$$

$$\mathbf{C} = \begin{bmatrix} b_1 & b_2 & \cdots & \cdots & b_n \end{bmatrix}$$

State feedback control

- Remember that $a_1, a_2, \ldots, a_n$ are the elements of the denominator polynomial of the transfer function.

$$G(s) = \frac{b_1 s^{n-1} + \ldots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

- They define the **location of the poles** (or the **eigenvalues** of the state matrix **A**) and therefore the **dynamic behaviour of the system**.
- By modifying these values using state feedback, we change the location of the poles (or the eigenvalues) and hence the dynamic behaviour of the system

Pole Assignment by State Feedback

This method is therefore called closed-loop **pole assignment**. For the moment, we have to make a simplifying assumption that all of the system states are measured. We will remove this assumption later.

We have also assumed that the system is completely controllable. The result then shows that the **closed-loop poles of the system** can be arbitrarily assigned by feeding back the state through a suitably chosen **constant-gain vector**.

Example: Pole assignement

$$G(s)=\frac{B(s)}{A(s)}=\frac{2}{s^2+7s+12}=\frac{2}{(s+4)(s+3)}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\mathbf{A}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\mathbf{B}u=\begin{bmatrix}-7 & -12\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}u$$

$$y=\mathbf{C}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Example: Pole assignment

- Using Matlab, we can verify that both descriptions are equivalent:
- Implement the transfer function:
  - B_tf = [2]; A_tf=[1 7 12];
  - sys1= tf(B_tf, A_tf)
- Implement the state space description
  - A= [-7 -12; 1 0]; B=[1;0]; C=[0 2]; D=0;
  - sys2=ss (A,B,C,D)
- Compare both implementations: ltiview(sys1,sys2)

Example: Pole assignment

- The poles of the original system are at -4 and -3
  - Verify by using roots(A_tf) or eig(A)
- Choose feedback gains such that the closed loop poles are located at -10 and -20
  - $A_{cl}(s)=(s+10)(s+20)=s^2+30s+200$
- Solution: K=[23 188]
- This can be verified by calculating the eigenvalues of the closed loop state matrix:
  - eig(A-BK)

Pole assignment

- Thus, to calculate the state feedback vector **K**, we need to
  1. Calculate the desired closed loop polynomial
  $$A_{cl}(s) = s^n + \alpha_n s^{n-1} + \ldots + \alpha_2 s + \alpha_1$$
  2. transform the system into controller canonical form,
  3. Compare elements of the matrix **A** with the corresponding elements of $A_{cl}(s)$ to calculate the elements of **K**

Ackermann's formula (1972)

- Ackermann's formula combines these steps into a very compact form.
- Given the desired closed loop polynomial $A_{cl}(s)$
- The feedback gains **K** can then be calculated as:

$$\mathbf{K} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \mathbf{\Gamma}_c^{-1} \alpha_c(\mathbf{A})$$

where

$$\mathbf{\Gamma}_c = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} \quad \text{the controllability matrix}$$

$$\alpha_c(\mathbf{A}) = \mathbf{A}^n + \alpha_1 \mathbf{A}^{n-1} + \alpha_2 \mathbf{A}^{n-2} + \ldots + \alpha_n \mathbf{I}$$

Example: Pole assignment (cont.)

- For our example, the closed loop polynomial is

$$A_{cl}(s) = s^2 + 30s + 200$$

- It then follows that:

$$\alpha_c(\mathbf{A}) = \mathbf{A}^2 + 30\mathbf{A} + 200\mathbf{I} = \begin{bmatrix} 27 & -276 \\ 23 & 188 \end{bmatrix}$$

$$\mathbf{\Gamma}_c = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ 0 & 1 \end{bmatrix}; \quad \mathbf{\Gamma}_c^{-1} = \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix}$$

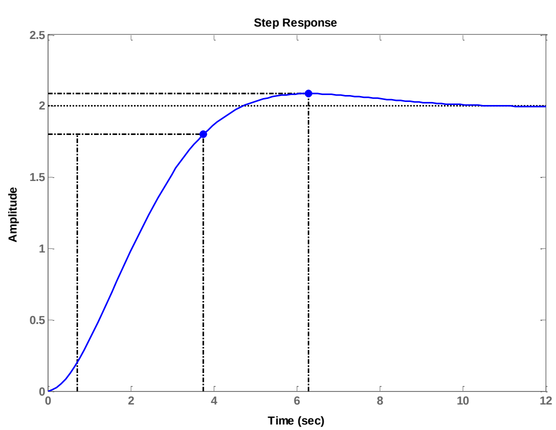
$$\mathbf{K} = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{\Gamma}_c^{-1} \alpha_c(\mathbf{A}) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 27 & -276 \\ 23 & 188 \end{bmatrix} = \begin{bmatrix} 23 & 188 \end{bmatrix}$$

Pole assignment: Matlab commands

- Ackermann's formula is available in Matlab as
  - K = acker(A,B,P);
- *P* is a vector giving the desired locations of the closed loop poles
- An alternative implementation is available as
  - K = place(A,B,P)
  - This implementation is numerically more robust than Ackermann's formula.

Why placing poles?

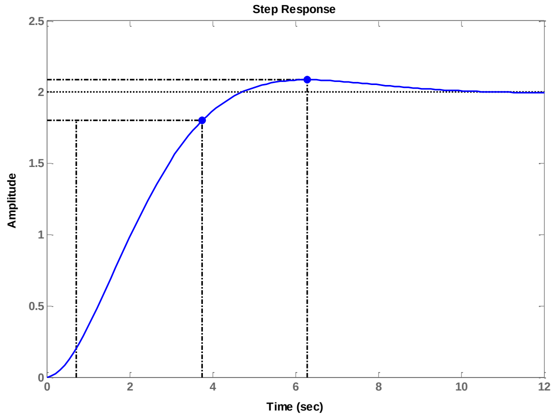
- For a stable 2nd order system, a straightforward relation exist between:
  - time-domain characteristics of its step-respones (rise-time t_r and overshoot M), and
  - parameters of the denominator of its transfer-function (ω_n and ξ), i.e. the location of its poles.



$$G(s) = \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

2^{nd} order response

For system G(s) with **no finite zeros and two complex poles**:



$$G(s) = \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

rise time: $t_r \cong \dfrac{1.8}{\omega_n} \quad \omega_n \cong \dfrac{1.8}{t_r}$

overshoot: $M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \qquad \xi = -\dfrac{\ln M_p}{\sqrt{\pi^2 + \left(\ln M_p\right)^2}};\ 0 \le \xi < 1$

settling time: $t_s \cong \dfrac{4.6}{\xi\omega_n}$

Pole Placement Design

- Given a model of the plant which has the order n
  - Define a **reference performance** specification for rise-time and overshoot, and calculate the corresponding 2nd order polynomial which defines the **two dominant poles** of the characteristic polynomial A_{cl}.
  - If $\deg(A_{cl})>2$, select **additional poles far away from the dominant poles**.
  - Calculate the state feedback vector ***K***
  - Evaluate performance. If necessary, modify specification and re-design controller.

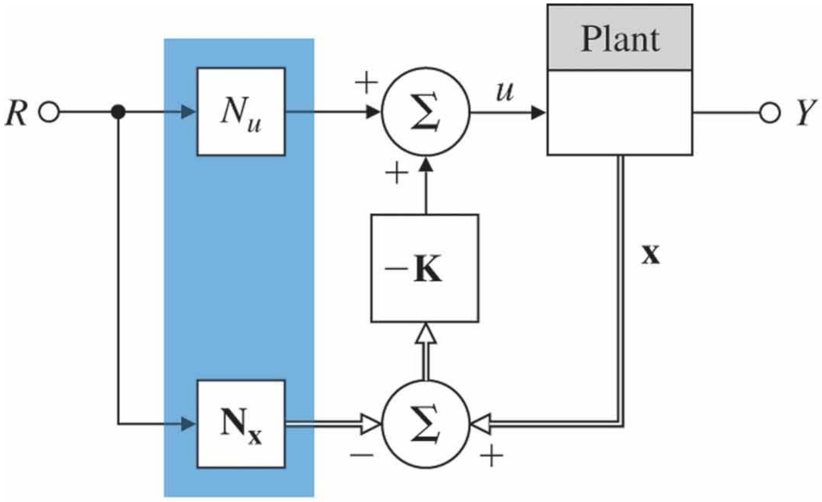
Introducing the reference input

- The feedback loop considered so far drives the state to zero – there is **no reference input**.
- There are several possibilities to introduce a reference input *r*.
- An obvious way would be to amend the control law:

$$u = -\mathbf{K}x + r$$

- This would however result in non-zero steady state error to a step input, so we need to modify this.

Introducing the reference input



(a)

Introducing the reference input

- The desired steady state output to the reference input is given as y_{ss}=r. The corresponding steady state can be derived as follows

$$\dot{x}_{ss} = 0 = \mathbf{A}x_{ss} + \mathbf{B}u_{ss}$$

$$y_{ss} = r = \mathbf{C}x_{ss} + \mathbf{D}u_{ss}$$

- Substituting x_{ss}=$N_x r$ and u_{ss}=$N_u r$, results in

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{N}_x \\ N_u \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$$

Introducing the reference input

- This system has the solution

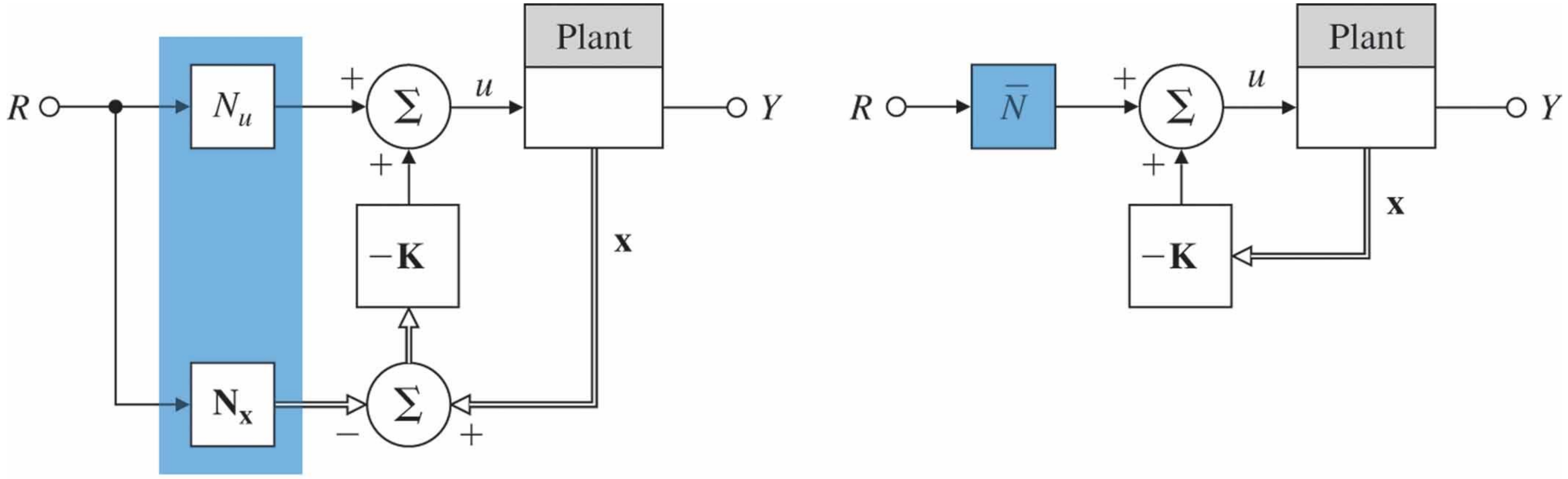
$$\begin{bmatrix} \mathbf{N}_x \\ N_u \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$$

- Thus, the control signal should be

$$\begin{aligned} u &= u_{ss} - \mathbf{K}\left(x - x_{ss}\right) \\ &= N_u r - \mathbf{K}\left(x - \mathbf{N}_x r\right) \\ &= -\mathbf{K}x + \left(N_u + \mathbf{K}\mathbf{N}_x\right) r \end{aligned}$$

$$u = -\mathbf{K}x + \bar{N} r$$

Introducing the reference input



(a) (b)

Introducing the reference input

- The overall closed loop description is (assuming **D=0**)

$$\dot{x} = \left(\mathbf{A} - \mathbf{BK}\right) x + \mathbf{B}\bar{N}r$$

$$y = \mathbf{C}x$$