Observability and Detectability

Consider again the state space model

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t); \quad x(0) = x_0$$
$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

In general, the dimension of the observed output, y, can be less than the dimension of the state, x. However, one might conjecture that, if one observed the output over some nonvanishing time interval, then this might tell us something about the state. The associated properties are called **observability** (*or* ***reconstructability***). A related issue is that of **detectability**.

Observability

Observability is concerned with the issue of what can be said about the state when one is given measurements of the plant output.

A formal definition is as follows:

Definition: The state $x_0 \neq 0$ is said to be unobservable if, given $x(0) = x_0$, and $u(T) = 0$ for $T \geq 0$, then $y(T) = 0$ for $T \geq 0$. The system is said to be completely observable if there exists no nonzero initial state that it is unobservable.

Reconstructability

A concept related to observability is that of reconstructability. This concept is sometimes used in discrete-time systems. Reconstructability is concerned with what can be said about $x(T)$, on the basis of the past values of the output, i.e., $y[k]$ for $0 \leq k \leq T$. For linear time-invariant continuous-time systems, the distinction between observability and reconstructability is unnecessary. However, the following example illustrates that, in discrete time, the two concepts are different.

Consider

$$x[k+1] = 0 \qquad\qquad x[0] = x_o$$
$$y[k] = 0$$

this system is clearly reconstructable for all $T \geq 1$, because we know for certain that $x[T] = 0$ for $T \geq 1$. However, it is completely unobservable, because $y[k] = 0 \; \forall k$, irrespective of the value of x_0.

In view of the subtle difference between observability and reconstructability, we will use the term observability in the sequel to cover the stronger of the two concepts.

Test for Observability

A test for observability of a system is established in the following theorem.

Theorem: Consider the state model

$$\frac{dx(t)}{dt} = \mathbf{A}x(t) + \mathbf{B}u(t); \quad x(0) = x_0$$

$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

Test for Observability (cont.)

The set of all unobservable states is equal to the null space of the observability matrix $\Gamma_0[\mathbf{A}, \mathbf{C}]$, where

$$\Gamma_o[\mathbf{A}, \mathbf{C}] = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA^{n-1}} \end{bmatrix}$$

The system is completely observable if and only if $\Gamma_0[\mathbf{A}, \mathbf{C}]$, has full column rank n (ie. it is non-singular).

Example

Consider the following state space model:

$$\mathbf{A} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Then

$$\mathbf{\Gamma}_o[\mathbf{A}, \mathbf{C}] = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -4 & -2 \end{bmatrix}$$

Hence, rank $\Gamma_0[\mathbf{A}, \mathbf{C}] = 2$, and the system is completely observable.

Example

Consider

$$\mathbf{A} = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Here

$$\boldsymbol{\Gamma}_o[\mathbf{A},\mathbf{C}] = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$$

Hence, rank $\Gamma_0[\mathbf{A}, \mathbf{C}] = 1 < 2$, and the system is not completely observable.

Duality

We see a remarkable similarity between the Theorems for controllability and for observability. We can formalize this as follows:

Theorem (*Duality*). Consider a state space model described by the 4-tuple ($\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$). Then the system is completely controllable if and only if the dual system ($\mathbf{A}^T$, $\mathbf{C}^T$, $\mathbf{B}^T$, $\mathbf{D}^T$) is completely observable.

Note: $\mathbf{A}^T$ denotes the transpose of $\mathbf{A}$.

Observable Decomposition

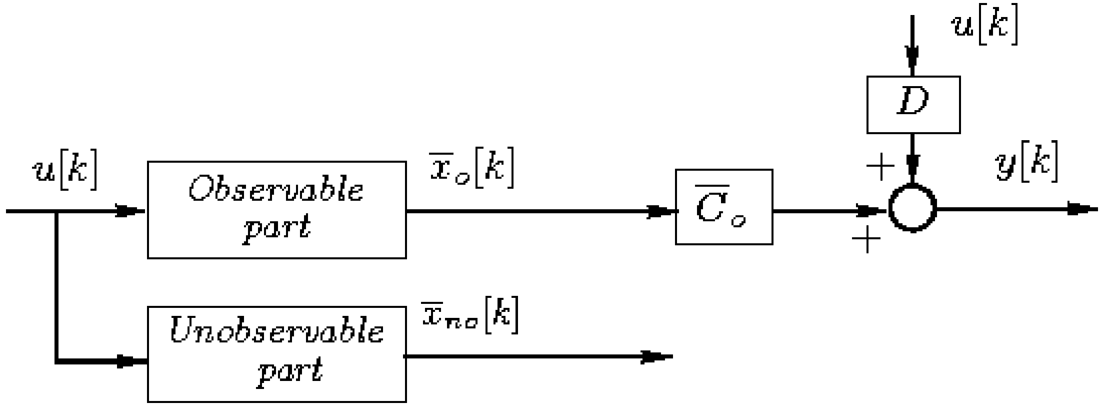
The above theorem can often be used to go from a result on controllability to one on observability, and vice versa. For example, the dual of Lemma 17.1 is the following:

Lemma : If rank$\{\Gamma_0[\mathbf{A}, \mathbf{C}]\} = k < n$, there exists a similarity transformation T such that with $\bar{x} = \mathbf{T}^{-1}x$, then $\overline{\mathbf{C}}$ and $\overline{\mathbf{A}}$ take the form $\overline{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}, \overline{\mathbf{C}} = \mathbf{C}\mathbf{T}$,

$$\overline{\mathbf{A}} = \begin{bmatrix} \overline{\mathbf{A}}_o & 0 \\ \overline{\mathbf{A}}_{21} & \overline{\mathbf{A}}_{no} \end{bmatrix} \qquad \overline{\mathbf{C}} = \begin{bmatrix} \overline{\mathbf{C}}_o & 0 \end{bmatrix}$$

where $\overline{\mathbf{A}}_0$ has dimension k and the pair $(\overline{\mathbf{C}}_0 \overline{\mathbf{A}}_0)$ is completely observable.

Figure: *Observable-unobservable decomposition*



Detectability

A plant is said to be *detectable* if its unobservable subspace is stable.

We remarked earlier that noncontrollable (*indeed nonstabilizable*) models are frequently used in control-system design. This is not true for nondetectable models. Essentially all models used in the sequel can be taken to be detectable, without loss of generality.

Observer Canonical Form

Consider a **completely observable** SISO system given by

$$\frac{dx(t)}{dt} = \mathbf{A}x(t) + \mathbf{B}u(t); \quad x(0) = x_0$$

$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

Then there exists a similarity transformation that converts the model to the **observer-canonical form**

$$\mathbf{A}_o = \begin{bmatrix} -a_1 & 1 & 0 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & 0 & \cdots & \vdots \\ \vdots & & \ddots & & & 1 \\ -a_n & 0 & & 0 & & 0 \end{bmatrix}; \quad \mathbf{B}_o = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix};$$

$$\mathbf{C}_o = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Summary

- State variables are system internal variables, upon which a full model for the system behavior can be built. The state variables can be ordered in a state vector.
- Given a linear system, the choice of state variables is not unique - however,
  - the minimal dimension of the state vector is a system invariant,
  - there exists a nonsingular matrix that defines a similarity transformation between any two state vectors, and
  - any designed system output can be expressed as a linear combination of the state variables and the inputs.

- For linear, time-invariant systems, the state space model is expressed in the following equations:

continuous-time systems

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

- Stability and natural response characteristics of the system can be studied from the eigenvalues of the matrix **A.**
- State space models facilitate the study of certain system properties that are paramount in the solution to the control-design problem. These properties relate to the following questions:
  - By proper choice of the input u, can we steer the system state to a desired state (*point value*)? (***controllability***)
  - If some states are uncontrollable, will these states generate a time-decaying component? (***stabilizability***)
  - If one knows the input, $u(t)$, for $t \geq t_0$, can we infer the state at time $t = t_0$ by measuring the system output, $y(t)$, for $t \geq t_0$? (***observability***)
  - If some of the states are unobservable, do these states generate a time-decaying signal? (***detectability***)

- Controllability tells us about the feasibility of attempting to control a plant.
- Observability tells us about whether it is possible to know what is happening inside a given system by observing its outputs.
- The above system properties are system invariants. However, changes in the number of inputs, in their injection points, in the number of measurements, and in the choice of variables to be measured can yield different properties.

- A transfer function can always be derived from a state space model.

- A state space model can be built from a transfer-function model. However, only the completely controllable and observable part of the system is described in that state space model. Thus ***the transfer-function model might be only a partial description of the system***.