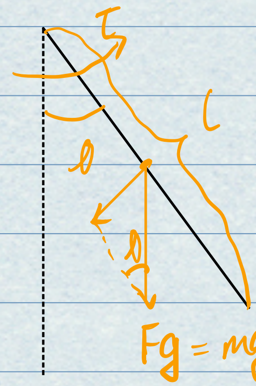

$F_g = mg$, $g = 9.81\,m/s^2$

* : T (Torque) $= F \times L = F \cdot L \cdot \sin\theta$
* $T = I \cdot \ddot{\theta}$.
  $\ddot{\theta}$: angular acceleration
* ODE: ordinary differented equation, 常微分方程
* $\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = K\omega_n^2 x$
  2nd order system
* ω_n: nature frequency

ζ: damping $\begin{cases} \zeta = 0 : \text{undamped} \\ 0<\zeta<1 : \text{underdamped} \\ \zeta \geq 1 : \text{overdamped} \end{cases}$

τ: external torque (外力的力矩)
Com: center of mass
F_g: Force due to gravity
T_g: Torque due to gravity
$$T_g = 0.5\,l \cdot mg\sin\theta$$
T_I: Torque due to inertia (惯性)
I: moment of inertia (惯性矩, 转动惯量)
$$I = \frac{1}{3} ml$$
$$T_I = I \cdot \ddot{\theta} = I \cdot \frac{\partial^2\theta}{\partial t^2}$$
T_D: torque due to damping (阻尼)
$$T_D = c \cdot \dot{\theta} = c \cdot \frac{\partial\theta}{\partial t}$$

$T_g + T_I + T_D = \tau$ (nonlinear)
$$\frac{l}{2} mg\sin\theta(t) + I \cdot \ddot{\theta}(t) + c \cdot \dot{\theta}(t) = \tau(t)$$
assume $\theta \to 0$, $\lim_{\theta \to 0} \sin\theta = \theta$, then
$$\frac{l}{2} \cdot mg\,\theta(t) + I \cdot \ddot{\theta}(t) + c\,\dot{\theta}(t) = \tau(t) \quad \text{(linear)}$$
$\downarrow$
$$\ddot{\theta}(t) + \frac{c}{I}\dot{\theta}(t) + \frac{mgl}{2I}\theta(t) = \frac{1}{I}\tau(t) \quad \text{(2nd Order ODE)}$$
$\downarrow$ $2\zeta\omega_n = \frac{c}{I}$, $\omega_n^2 = \frac{mgl}{2I}$

$$\ddot{\theta}(t) + 2\zeta\omega_n \dot{\theta}(t) + \omega_n^2 \theta(t) = \frac{1}{I} \tau(t)$$

Laplace transform $\downarrow$ $\theta(0) = \dot{\theta}(0) = \ddot{\theta}(0) = 0$

$$s^2\theta(s) + 2\zeta\omega_n \cdot s\,\theta(s) + \omega_n^2 \theta(s) = \frac{1}{I} T(s)$$

$$\underbrace{\theta(s)}_{\text{output}} \left(s^2 + 2\zeta\omega_n s + \omega_n^2\right) = \frac{1}{I} \underbrace{T(s)}_{\text{input}}$$

$$G(s) = \frac{\theta(s)}{T(s)} = \frac{1/I}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

* Laplace transform

$\theta(t) \rightarrow \theta(s)$

$\dot{\theta}(t) \rightarrow s\theta(s) + \theta(0)$

$\ddot{\theta}(t) = s^2\theta(s) + \theta(0) + \dot{\theta}(0)$

* $G(s)$: transfer function

*SSSF: standard state space equation

$$\underbrace{\begin{bmatrix} \ddot{\theta}(t) \\ \dot{\theta}(t) \end{bmatrix}}_{\dot{x}(t)} = \begin{bmatrix} & \\ & \end{bmatrix} \underbrace{\begin{bmatrix} \dot{\theta}(t) \\ \theta(t) \end{bmatrix}}_{x(t)} + \begin{bmatrix} \\ \end{bmatrix} \tau(t)$$

$$\dot{x}(t) = A \cdot x(t) + B \cdot \tau(t)$$

$$\underline{\theta(t)} = \begin{bmatrix} & \end{bmatrix} \begin{bmatrix} \dot{\theta}(t) \\ \theta(t) \end{bmatrix} + (\quad) \tau(t)$$

$$Y(t) = C\,X(t) + D\,u(t)$$

$$\ddot{\theta}(t) = -2\zeta\omega_n\dot{\theta}(t) - \omega_n^2\theta(t) + \frac{1}{I}\tau(t)$$

$$\begin{bmatrix}\ddot{\theta}(t)\\ \dot{\theta}(t)\end{bmatrix} = \begin{bmatrix}-2\zeta\omega_n & -\omega_n^2\\ 1 & 0\end{bmatrix}\begin{bmatrix}\dot{\theta}(t)\\ \theta(t)\end{bmatrix} + \begin{bmatrix}\frac{1}{I}\\ 0\end{bmatrix}\tau(t)$$

$$\dot{\theta}(t) = 1\cdot\dot{\theta}(t) + 0\cdot\theta(t) + 0\cdot\tau(t)$$

$$\theta(t) = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\dot{\theta}(t)\\ \theta(t)\end{bmatrix} + 0\cdot\tau(t)$$

$$\theta(t) = 0\cdot\dot{\theta}(t) + 1\cdot\theta(t) + 0\cdot\tau(t)$$

Let's derive the state-space model for $m\ddot{q} + b\dot{q} + kq = F_c$.

1 Simplify the variables by defining state, input, and output variables

$x_1 = q(t)$
$x_2 = \dot{q}(t)$
$u = F_c$
$y = q(t)$

noting that

$x_2 = \dot{x}_1$

2 Rewrite the equation of motion using the state, input, and output variables from above

$m\dot{x}_2 + bx_2 + kx_1 = u$

3 For each state variable (x_1 and x_2), find its derivative (i.e. $\dot{x}_1$ and $\dot{x}_2$)

$\dot{x}_1 = x_2$
$\dot{x}_2 = \frac{1}{m}(-kx_1 - bx_2) + \frac{1}{m}u$

4 Rewrite the above equations in matrix form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}\begin{bmatrix}u_1\end{bmatrix}$$

noting that

$$\underbrace{\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{bmatrix}}_{\mathrm{A}}\underbrace{\begin{bmatrix}x_1\\ x_2\end{bmatrix}}_{x} + \underbrace{\begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}}_{\mathrm{B}}\underbrace{\begin{bmatrix}u_1\end{bmatrix}}_{u}$$

5 Define the output of the system in terms of the state variables

$y = x_1$

6 Rewrite the output equation in matrix form

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\end{bmatrix}\begin{bmatrix}u_1\end{bmatrix}$$

noting that

$$y = \underbrace{\begin{bmatrix}1 & 0\end{bmatrix}}_{\mathrm{C}}\underbrace{\begin{bmatrix}x_1\\ x_2\end{bmatrix}}_{x} + \underbrace{\begin{bmatrix}0\end{bmatrix}}_{\mathrm{D}}\underbrace{\begin{bmatrix}u_1\end{bmatrix}}_{u}$$