University of Glasgow

THE JAMES WATT SCHOOL
OF ENGINEERING

CONTROL 4/M

Frequency response design with PID controller

Dr Henrik Gollee
Email: Henrik.Gollee@glasgow.ac.uk

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1 Introduction

Frequency response design is based on finding a controller which results in desired characteristics in the frequency domain. It usually involves **loop-shaping**, i.e. the modification of the shape of the loop gain $L(j\omega)$ to achieve desired properties.

Design targets for feedback control include:

- **Performance** (disturbance attenuation and good command response): This often involves increasing the bandwidth of the closed loop system which is related to increasing the cross-over frequency ω_c of the loop gain, $|L(j\omega_c)|$.
- **Robustness** with respect to **stability** and **performance**.

The loop gain $L(j\omega)$ is related to the closed loop **performance** through the complementary sensitivity function $T(j\omega)$ and the sensitivity function $S(j\omega)$. Robustness and stability can be characterised by the loop gain, using the Nyquist stability criterion which provides stability margins obtained from a plot of $L(j\omega)$, either as a Nyquist plot or as a Bode plot.

Figures 1 and 2 show the frequency response plots as Bode and Nyquist diagrams for the example function

$$L(s) = \frac{3.75}{s(s+36.5)(s+1)} \tag{1}$$

Note that the cross-over frequency ω_c, $|L(j\omega_c)| = 1 = 0_{\text{dB}}$, corresponds to the point at which the phase margin is measured.

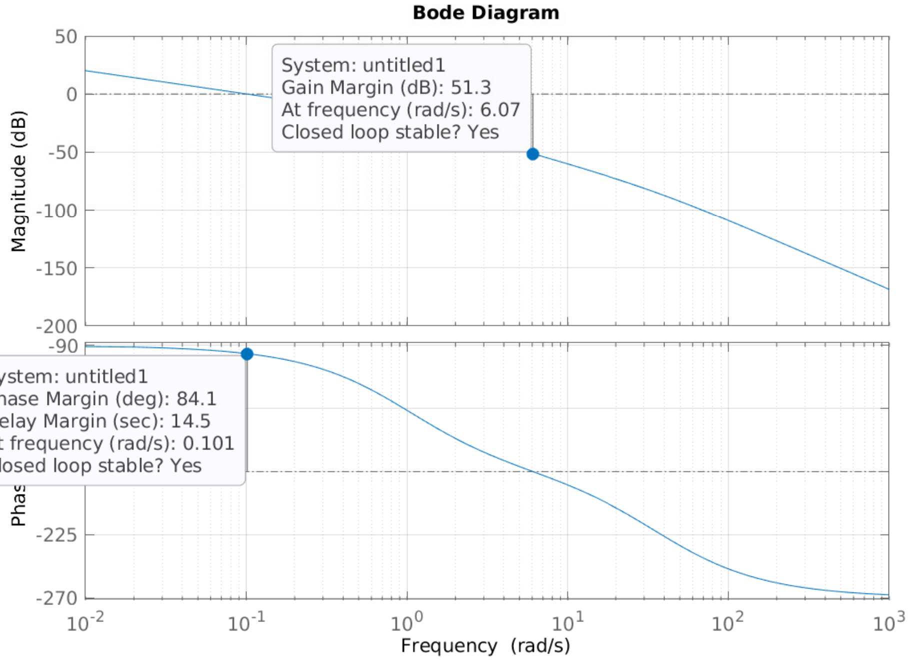

Figure 1: Bode plot

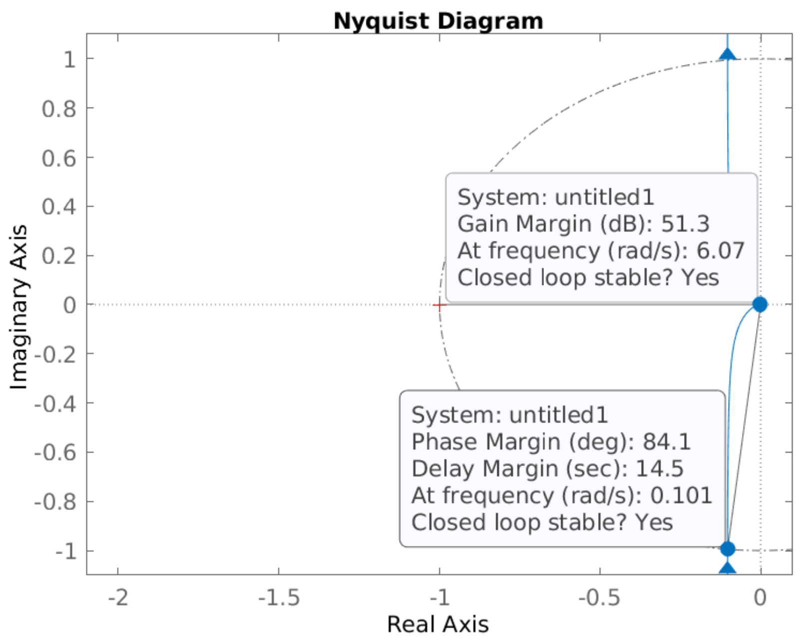

Figure 2: Nyquist plot

We consider the one-degree-of-freedom control structure shown in the figure below, where P is the plant and C denotes the controller, with r the reference input, y the system output, u the control signal, d a disturbance and n measurement noise.

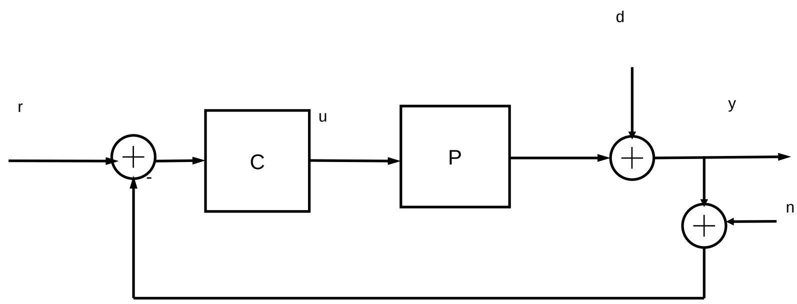

Figure 3: 1-degree-of-freedom control structure.

The loop gain is then given as $L(j\omega) = C(j\omega)P(j\omega)$, and **loop shaping** therefore involves choosing $C(j\omega)$ to achieve a desired shape of $L(j\omega)$, given the plant $P(j\omega)$. This is usually done using the Bode plot of the frequency responses.

2 PID controller

In this section we will use a particular control structure that has become almost universally used in industrial control. It is based on a particular fixed structure controller family, the so-called, PID (proportional-integral-derivative) controller family. They have proven to be robust in the control of many important applications. The simplicity of these controllers is also their weakness, since it limits the range of plants that they can control satisfactorily. Indeed, there exists a set of unstable plants which cannot even be stabilized with any member of the PID family. Nevertheless, the surprising versatility of PID control (really, PID control simply means: control with an up to second-order controller) ensures continued relevance and popularity for this controller.

2.1 Components of the PID controller

The PID controller contains three terms which are expressed in terms of the control error,

$$e(t) = r(t) - y(t) \tag{2}$$

2.1.1 Proportional control

$$u(t) = Ke(t) \quad \Rightarrow \quad C_p(s) = K \tag{3}$$

provides a contribution which depends on the instantaneous value of the control error. A proportional controller can control any stable plant, but it provides limited performance and nonzero steady state errors. This latter limitation is due to the fact that its frequency response is bounded for all frequencies.

2.1.2 Integral control

$$u(t) = \frac{K}{T_i}\int_0^t e(\tau)d\tau \quad \Rightarrow \quad C_I(s) = \frac{K}{T_i s} \tag{4}$$

gives a controller output that is proportional to the accumulated error, which implies that it is a slow reaction control mode. This characteristic is also evident in its low pass frequency response. The integral mode plays a fundamental role in achieving perfect plant inversion at $\omega = 0$. This forces the steady state error to zero in the presence of a step reference and disturbance. The integral mode, viewed in isolation, has a major shortcoming: its pole at the origin is detrimental to loop stability. It also gives rise to the undesirable effect (in the presence of actuator saturation) known as wind-up.

2.1.3 Derivative control

$$u(t) = KT_D\dot{e}(t) \quad \Rightarrow \quad C_D(s) = KT_D s \tag{5}$$

acts on the rate of change of the control error. Consequently, it is a fast mode which ultimately disappears in the presence of constant errors. It is sometimes referred to as a predictive mode because of its dependence on the error trend. The main limitation of the derivative mode, viewed in isolation, is its tendency to yield large control signals in response to high frequency control errors, such as errors induced by setpoint changes or measurement noise. Since implementation requires that the transfer functions are proper, a pole is typically added to the derivative,

$$C_D(s) = \frac{T_D s}{\tau_D s + 1} \tag{6}$$

In the absence of other constraints, the additional time constant is normally chosen such that $0.1T_D \leq \tau_D \leq 0.2T_D$. This constant is called the derivative time constant; the smaller it is, the larger the frequency range over which the filtered derivative approximates the exact derivative.

2.1.4 Summary

In the above equations, T_I is the integral or reset time, T_D is the derivative time, and K is the feedback gain. The combined control law is

$$u(t) = K\left(e(t) + \frac{1}{T_I}\int_0^t e(\tau)d\tau + T_D\dot{e}(t)\right) \tag{7}$$

$$C(s) = K\left(1 + \frac{1}{T_I s} + T_D s\right) \tag{8}$$

2.2 Frequency response

We will now consider the influence of the terms of the PID controller on the shape of $L(j\omega)$.

2.2.1 Proportional control

Proportional control simply changes the magnitude of $L(j\omega)$ equally at all frequency. It does not influence the phase. Thus, proportional control can increase the crossover frequency ω_c, but this will lead to a reduced phase margin.

2.2.2 Integral control

As mentioned above, the main benefit of integral control is that it leads to infinite loop gain at zero frequency, $L(0) = \infty$, which implies

- zero steady state error with respect to a constant reference signal, and
- rejection of constant disturbances.

Integral control adds a pole at the origin, meaning a decrease in loop gain magnitude at -20dB/decade and an additional -90^o phase. The former leads to reduced bandwidth ("slowing down the feedback system"), the latter to reduced phase margin (affecting robustness).

2.2.3 Derivative control

Derivative control leads to a fast response, with the disadvantage of potentially large control signals. It results in an increase in loop gain magnitude by +20dB/decade and a phase advance of $+90^o$. This can increase the crossover frequency and consequently the bandwidth of the closed loop system.

2.2.4 Summary

Combining proportional, integral and derivative terms leads to the following control law

$$C(s) = K\left(1 + \frac{1}{T_I s} + T_D s\right) = \frac{K}{s}\left(T_D s^2 + s + \frac{1}{T_I}\right) \tag{9}$$

with $T_D \ll T_I$, i.e. $\frac{T_D}{T_I} \ll 1$

$$C(s) \approx \frac{K}{s}\left(T_D s^2 + \left(1 + \frac{T_D}{T_I}\right)s + \frac{1}{T_I}\right) = \frac{K}{s}\left(T_D s + 1\right)\left(s + \frac{1}{T_I}\right) \tag{10}$$

$C(s)$ in equation (10) has two zeros (at $\frac{1}{T_D}$ and $\frac{1}{T_I}$) and one pole (at zero). Since the number of zeros is larger than the number of poles, this transfer function is not realisable[1]. To obtain a realisable version $C_r(s)$ of this controller, an additional pole needs to be implemented, as shown in equation (6), leading to

$$C_r(s) \approx K\frac{\left(T_D s + 1\right)\left(s + \frac{1}{T_I}\right)}{s(\tau_D s + 1)} \tag{11}$$

Figure 4 shows example Bode plots of $C(s)$ and $C_r(s)$.

For frequency domain design, the parameters K, T_I and T_D are to be chosen in such a way that the frequency response of the controller combined with the plant results in a satisfactory shape of the loop gain.

[1]Converting it to the time domain would lead to a system where the current output depends on future input values, i.e. the system would be non-causal.

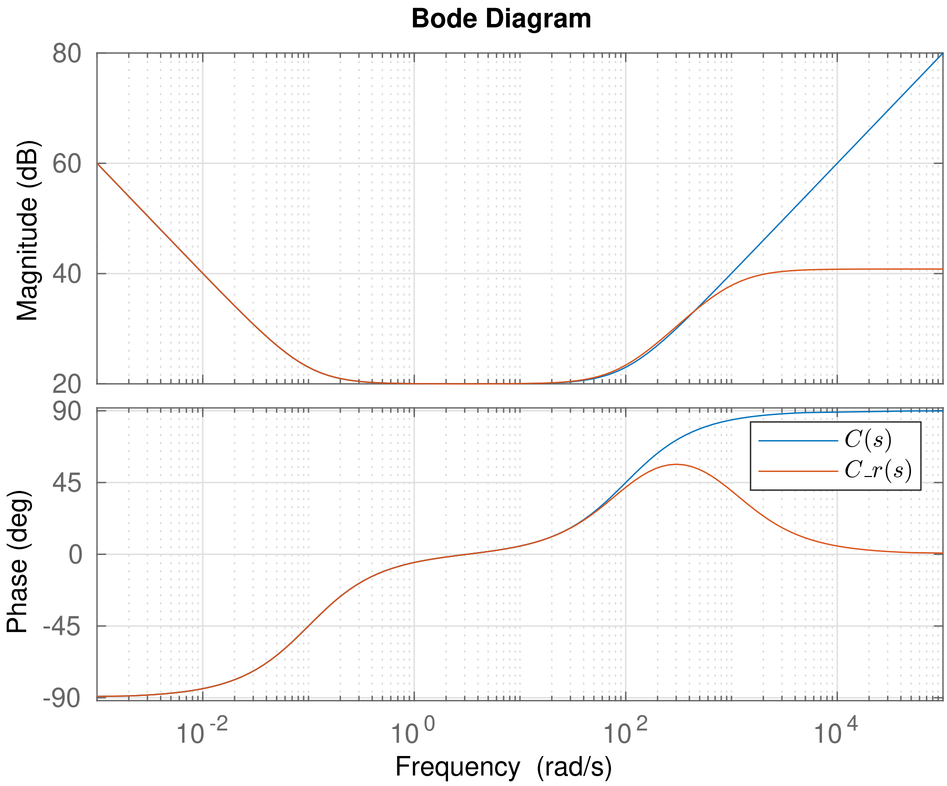

Figure 4: Bode frequency response of PID controller $C(s)$ with $K = 10$, $T_I = 10\,\mathrm{sec}$ (corresponding to a frequency $\omega_I = 0.1\,\frac{\mathrm{rad}}{\mathrm{sec}}$) and $T_D = 0.01\,\mathrm{sec}$ (corresponding to $\omega_D = 100\,\frac{\mathrm{rad}}{\mathrm{sec}}$), together with its realisable version $C_r(s)$ (with $\tau_D = 0.001$).

2.3 Example

Given a pendulum with mass $5\,\mathrm{kg}$, length $0.4\,\mathrm{m}$ and damping of $10\,\frac{\mathrm{Nm}}{\mathrm{rad}}$. The resulting transfer function for the plant is

$$P_o(s) = \frac{3.75}{(s+36.5)(s+1)} \tag{12}$$

The controller parameters are chosen as $K = 100$, $T_I = 1\,\mathrm{sec}$ and $T_D = 0.01\,\mathrm{sec}$. Using equation (10), this results in

$$C(s) = \frac{100}{s}\,(0.01s+1)\,(s+1) \tag{13}$$

This controller is not realisable, since the order of its numerator is larger than that of the denominator. To obtain a realisable version $C_r(s)$ of this controller, we use equation (11) with an additional pole with a time constant of $\tau_D = 0.001\,\mathrm{sec}$,

$$C_r(s) = 100\frac{(0.01s+1)\,(s+1)}{s\,(0.001s+1)} \tag{14}$$

Figure 5 shows the corresponding bode plots of $P_o(j\omega)$, $C(j\omega)$ and $L(j\omega) = C(j\omega)P_o(j\omega)$, together with the realisable versions $C_r(j\omega)$ and $L_r(j\omega) = C_r(j\omega)P_o(j\omega)$.

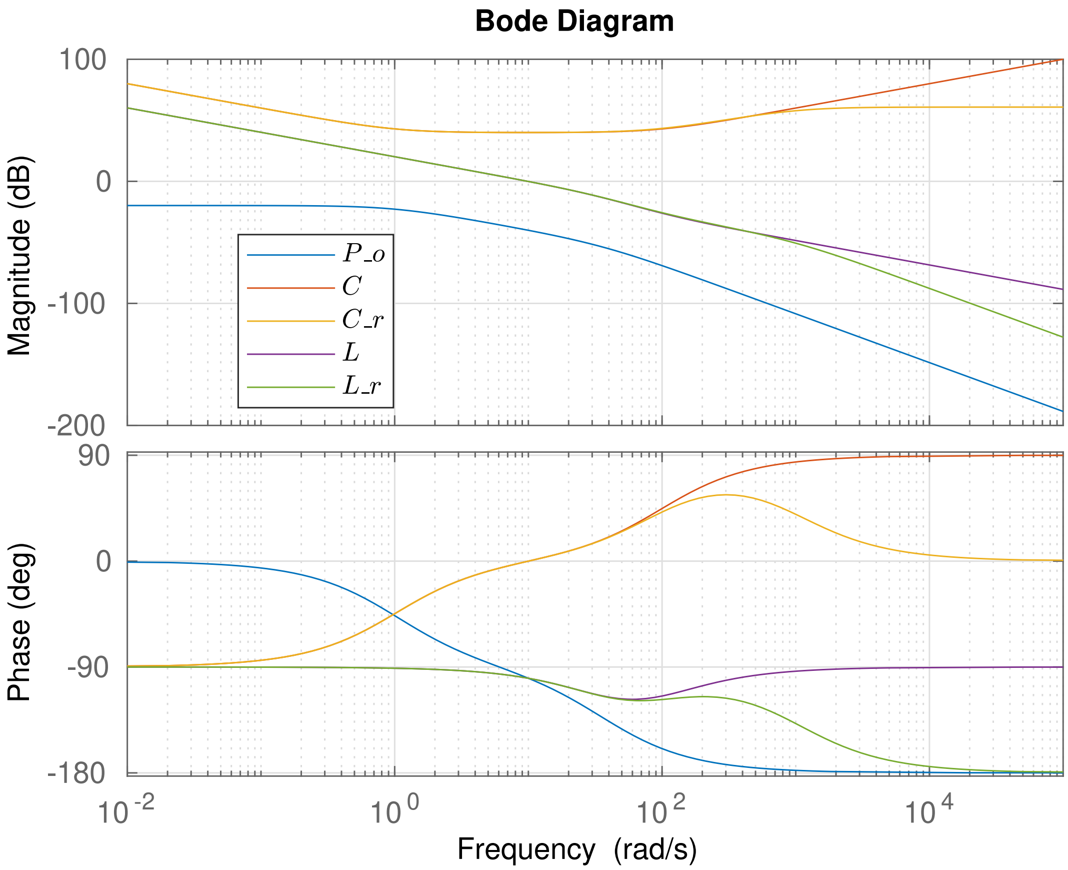

Figure 5: Frequency response of plant, controller and loop gain for PID controller with $K = 100$, $T_I = 1\,\mathrm{sec}$ and $T_D = 0.01\,\mathrm{sec}$, and $\tau_D = 0.001$ for the realisable version.