Chapter 1

Ground energy of quantum walk

To get some flavor for QMA-completeness results.

Now that we have shown the universality of these Quantum Walk Hamiltonians via time evolution, we might want to ask related computational questions about these systems. In particular, once the computational universality of a system is shown, people often ask about the related ground energy problem. The reason for this is that many of these systems that allow for universal computation via time evolution also allow for the encoding of a computation in the ground space, which along with some energy penalties, allow one to show that the ground energy problem is QMA-hard.

The point of this chapter is to give a decent introduction to the flavor of QMA-hardness proofs, as well as providing a QMA-complete problem that might be more accessible to classical computer scientists.

1.1 The ground-energy problem

Essentially, we know that the single-particle quantum walk is governed by the adjacency matrix of the underlying graph. In particular, the Hamiltonian is exactly equal to the adjacency matrix, and thus asking questions about the ground energy of a single-particle quantum walk is simply asking a question about the smallest eigenvalue of a particular adjacency matrix.

However, the Hilbert space on which the quantum walk acts is necessarily exponential in size, with efficiently computable matrix entries. As such, this is a question about very specific types of matrices.

Problem 1 (d-sparse graph eigenvalue problem)**.** Given a d-sparse, row-computable graph G, and two constants $a < b$, is the smallest eigenvalue of $A(G)$ below a or above b, with the guarantee that one of these cases occur.

While this problem is definitely inspired from quantum walks, it actually makes on reference to quantum mechanics.

1.1.1 Containment in QMA

The proof that this problem is in QMAfollows many other such Hamiltonian problems. In particular, this proof strategy works for any system in which we can evolve according to a particular Hamiltonian.

The main idea is to be given a particular state, and use phase estimation to determine the energy of the given state, up to some error. In the case that the smallest eigenvalue of the system is below a, the prover can provide the corresponding eigenvector encoded in a quantum state. The phase estimation algorithm will then (with high probability) find this eigenvalue, and the system will accept. If the smallest eigenvalue is above b, then no matter what state the prover provides, the phase estimation algorithm will project onto one of the eigenstates and determine the corresponding eigenvalue, which will necessarily be above b.

More concretely, we have

1.2 QMA-hardness

The main way that this works is that we will use the well known Kitaev Hamiltonian, with some particular changes so that we get taken to a Hamiltonian of a particular form. Once we have that form, we can easily see that the result we want.

1.2.1 Kitaev Hamiltonian

With the definition of the class QMA, the requirement is that for each input there exists some quantum circuit and some particular input state that the circuit either accepts or rejects. When attempting to prove that a particular Hamiltonian has a similar computational power, we need to construct a "circuit-to-Hamiltonian" map. The predominant (and really only) such map is the so-called Kitaev-Hamiltonian.

In this mapping, we attempt to encode the computation into the ground space of the Hamiltonian, in a similar manner to how the proof that 3-SAT is NP-Hard encodes the entire computation of a nondeterministic Turing Machine. [**TO DO:** *NP-hardness of 3-Sat reference*] However, we run into the problem of locality, in that it is hard to enforce that the states corresponding to neighboring time steps are really reachable by a single particular unitary.

Kitaev worked around this problem by having both a clock and a state register, with the computation encoded as an entangled state between these two registers. In this way, by having a projection into those states that evolve correctly for a particular time step, we can have a local check for the correctness of evolution.

In particular, if a given circuit C acts on $\mathbb{C}^{2^m}$ and can be written as $C = U_T U_{T-1} \cdots U_1$, then the Kitaev Hamiltonian H_C acts on the Hilbert space $\mathbb{C}^{2^m} \otimes \mathbb{C}^{T+1}$, and can be written as

$$H_C = \sum_{t=0}^{T-1} \big(\mathbb{I}_{\mathbb{C}^{2^m}} \otimes |t\rangle - U_{t+1} \otimes |t+1\rangle\big)\big(\mathbb{I}_{\mathbb{C}^{2^m}} \otimes \langle t| - U_{t+1}^{\dagger} \otimes \langle t+1|\big) = \sum_{t=0}^{T-1} H_t \tag{1.1}$$

Note that each term H_t is a projector off those states of the form

$$|\psi\rangle \otimes |t\rangle + U_{t+1}|\psi\rangle \otimes |t+1\rangle. \tag{1.2}$$

Hence, we have that the ground state of H_C corresponds to the history states:

$$|\psi_{\text{hist}}\rangle = \sum_{t=0}^{T} U_t U_{t-1} \cdots U_1 |\psi\rangle \otimes |t\rangle. \tag{1.3}$$

These states encode the computation, as for a given initial state $|\psi\rangle$, the projection onto the time register gives the state of the computation at time t. Note that the energy gap for this Hamiltonian is exactly $1 - \cos(\pi/T)$, as the Hamiltonian is unitarily equivalent to a quantum walk on a line of length T.

With this mapping corresponding to a particular circuit, we can then force the initial state to have a particular form by adding in projectors tensored with a projection onto the $|t = 0\rangle$ state, with a similar projection for the requisite form of the final state. Putting everything together we then have a log-local Hamiltonian that will have a polynomial gap depending on whether the initial circuit accepted or rejected.

This is one of the simple ways in which a Hamiltonian is shown to be QMA-hard.

1.2.2 Transformation to Adjacency Matrix

While the above prescription works well for the conversion to local-Hamiltonians in the general case, in the situation we are interested in we want all of the non-zero matrix elements to be the same value. As the matrix elements of H_C are related to the matrix values of the unitaries involved in the circuit C, we thus want to force the matrix values of C to all be of the same form.

To enforce this, we suppose $\mathcal{C}_x$ implements a unitary

$$U_{\mathcal{C}_x} = U_M \dots U_2 U_1 \tag{1.4}$$

where each U_i is from the gate set

$$\mathcal{G} = \{H, HT, (HT)^\dagger, (H \otimes \mathbb{I})\,\text{CNOT}\} \tag{1.5}$$

with

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix} \quad \text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{1.6}$$

Note that this gate set is universal, as we can easily simulate the gate set $\{H, T, \text{CNOT}\}$ with gates from $\mathcal{G}$ since $H^2 = \mathbb{I}$ and we can thus cancel the H terms before the interesting portion of the gates. Further, each non-zero matrix element of these unitaries has norm $2^{-1/2}$, as we wanted.

However, when we look at one of the local terms in the Hamiltonian, we find that not all of the matrix elements have the same norm. In particular, we find that

$$
\begin{aligned}
H_t &= \big(\mathbb{I}_{\mathbb{C}^{2^m}} \otimes |\,t\,\rangle - U_{t+1} \otimes |\,t+1\,\rangle\big)\big(\mathbb{I}_{\mathbb{C}^{2^m}} \otimes \langle\,t\,| - U_{t+1}^\dagger \otimes \langle\,t+1\,|\big) &(1.7)\\
&= \mathbb{I}_{\mathbb{C}^{2^m}} \otimes \big(|\,t\,\rangle\langle\,t\,| + |\,t+1\,\rangle\langle\,t+1\,|\big) - \big(U_{t+1} \otimes |\,t+1\,\rangle\langle\,t\,| + U_{t+1}^\dagger \otimes |\,t\,\rangle\langle\,t+1\,|\big). &(1.8)
\end{aligned}
$$

While each off-diagonal term is either zero or has norm $2^{-1/2}$ in (1.8), the diagonal terms have norm 1. When each term is summed, we almost have that they sum to a term proportional to the identity, which we can deal with, but unfortunately the boundary terms (with $t = 0$ or $t = T$) are only involved in one unitary. However, this problem can be avoided by having circular time, in which we both compute and uncompute the comptutation. With this, each timestep is involved in exactly two local terms, and thus the diagonal term is proportional to the identity.

With this, it will be convenient to consider

$$
U_{\mathcal{C}_x}^\dagger U_{\mathcal{C}_x} = W_{2M} \dots W_2 W_1 \tag{1.9}
$$

where

$$
W_t = \begin{cases} U_t & 1 \le t \le M \\ U_{2M+1-t}^\dagger & M+1 \le t \le 2M. \end{cases} \tag{1.10}
$$

As in Section 1.2.1 we start with a version of the Feynman-Kitaev Hamiltonian [?, ?]

$$
H_C = -\sqrt{2} \sum_{t=1}^{2M} \Big(W_t^\dagger \otimes |t\rangle\langle t+1| + W_t \otimes |t+1\rangle\langle t| \Big) \tag{1.11}
$$

acting on the Hilbert space $\mathcal{H}_{\text{comp}} \otimes \mathcal{H}_{\text{clock}}$ where $\mathcal{H}_{\text{comp}} = (\mathbb{C}^2)^{\otimes m}$ is an m-qubit computational register and $\mathcal{H}_{\text{clock}} = \mathbb{C}^{2M}$ is a $2M$-level register with periodic boundary conditions (i.e., we let $|2M+1\rangle = |1\rangle$). Note that

$$
V^\dagger H_x V = -\sqrt{2} \sum_{t=1}^{2M} (\mathbb{I} \otimes |t\rangle\langle t+1| + \mathbb{I} \otimes |t+1\rangle\langle t|) \tag{1.12}
$$

where

$$
V = \sum_{t=1}^{2M} \Bigg(\prod_{j=t-1}^{1} W_j \Bigg) \otimes |t\rangle\langle t|
$$

and $W_0 = 1$. Since V is unitary, the eigenvalues of H_x are the same as the eigenvalues of (1.12), namely

$$
-2\sqrt{2}\cos\left(\frac{\pi \ell}{M}\right)
$$

for $\ell = 0, \dots, 2M-1$. The ground energy of (1.12) is $-2\sqrt{2}$ and its ground space is spanned by

$$
|\phi\rangle \frac{1}{\sqrt{2M}} \sum_{j=1}^{2M} |t\rangle, \quad |\phi\rangle \in \Lambda
$$

where Λ is any orthonormal basis for $\mathcal{H}_{\text{comp}}$. A basis for the ground space of H_x is therefore

$$V\left(|\phi\rangle \frac{1}{\sqrt{2M}} \sum_{j=1}^{2M} |t\rangle\right) = \frac{1}{\sqrt{2M}} \sum_{t=1}^{2M} W_{t-1}W_{t-2}\dots W_1|\phi\rangle|t\rangle$$

where $|\phi\rangle \in \Lambda$. The first excited energy of H_x is

$$\eta = -2\sqrt{2}\cos\left(\frac{\pi}{M}\right)$$

and the gap between ground and first excited energies is lower bounded as

$$\eta + 2\sqrt{2} \geq \sqrt{2}\frac{\pi^2}{M^2} \tag{1.13}$$

(using the fact that $1 - \cos(x) \leq \frac{x^2}{2}$).

The universal set $\mathcal{G}$ is chosen so that each gate has nonzero entries that are integer powers of $\omega = e^{i\frac{\pi}{4}}$. Correspondingly, the nonzero standard basis matrix elements of H_x are also integer powers of $\omega = e^{i\frac{\pi}{4}}$. We consider the 8×8 shift operator

$$S = \sum_{j=0}^{7} |j + 1 \bmod 8\rangle\langle j|$$

and note that ω is an eigenvalue of S with eigenvector

$$|\omega\rangle = \frac{1}{\sqrt{8}} \sum_{j=0}^{7} \omega^{-j}|j\rangle.$$

We modify H_x as follows. For each operator $-\sqrt{2}H$, $-\sqrt{2}HT$, $-\sqrt{2}(HT)^\dagger$, or $-\sqrt{2}\,(H \otimes \mathbb{I})\,\text{CNOT}$ appearing in equation (1.11), define another operator that acts on $\mathbb{C}^2 \otimes \mathbb{C}^8$ or $\mathbb{C}^4 \otimes \mathbb{C}^8$ (as appropriate) by replacing nonzero matrix elements with powers of the operator S:

$$\omega^k \mapsto S^k.$$

Matrix elements that are zero are mapped to the 8×8 all-zeroes matrix. Write $B(W)$ for the operators obtained by making this replacement, e.g.,

$$-\sqrt{2}HT = \begin{pmatrix} \omega^4 & \omega^5 \\ \omega^4 & \omega \end{pmatrix} \mapsto B(HT) = \begin{pmatrix} S^4 & S^5 \\ S^4 & S \end{pmatrix}.$$

Adjoining an 8-level ancilla as a third register and making this replacement in equation (**??**) gives

$$H_{\text{prop}} = \sum_{t=1}^{2M} \left(B(W_t)_{13}^\dagger \otimes |t\rangle\langle t+1|_2 + B(W_t)_{13} \otimes |t+1\rangle\langle t|_2 \right) \tag{1.14}$$

which is a symmetric 0-1 matrix (the subscripts indicate which registers the operators act on). Note that H_{prop} commutes with S (acting on the 8-level ancilla) and therefore is block diagonal with eight sectors. In the sector where S has eigenvalue ω, H_{prop} is identical to the Hamiltonian H_x that we started with (see equation (1.11)). There is also a sector (where S has eigenvalue ω^*) where the Hamiltonian is the complex conjugate of H_x. We will add a term to H_{prop} that introduces an energy penalty for states in any of the other six sectors, ensuring that none of these states lie in the ground space.

To see what kind of energy penalty is needed, we lower bound the eigenvalues of H_{prop}. Note that for each $W \in \mathcal{G}$, $B(W)$ contains at most 2 ones in each row or column. Looking at equation (1.14) and using this fact, we see that each row and each column of H_{prop} contains at most four ones (with the remaining entries all zero). Therefore $\|H_{\text{prop}}\| \leq 4$, so every eigenvalue of H_{prop} is at least -4.

The matrix A_x associated with the circuit $\mathcal{C}_x$ acts on the Hilbert space

$$\mathcal{H}_{\text{comp}} \otimes \mathcal{H}_{\text{clock}} \otimes \mathcal{H}_{\text{anc}}$$

where $\mathcal{H}_{\text{anc}} = \mathbb{C}^8$ holds the 8-level ancilla. We define

$$A_x = H_{\text{prop}} + H_{\text{penalty}} + H_{\text{input}} + H_{\text{output}} \tag{1.15}$$

where

$$H_{\text{penalty}} = \mathbb{I} \otimes \mathbb{I} \otimes \left(S^3 + S^4 + S^5\right)$$

is the penalty ensuring that the ancilla register holds either $|\omega\rangle$ or $|\omega^*\rangle$ and the terms

$$H_{\text{input}} = \sum_{j=n_{\text{input}}+1}^{n} |1\rangle\langle 1|_j \otimes |1\rangle\langle 1| \otimes \mathbb{I}$$

$$H_{\text{output}} = |0\rangle\langle 0|_{\text{output}} \otimes |M+1\rangle\langle M+1| \otimes \mathbb{I}$$

ensure that the ancilla qubits are initialized in the state $|0\rangle$ when $t=1$ and that the output qubit is in the state $|1\rangle\langle 1|$ when the circuit $\mathcal{C}_x$ has been applied (i.e., at time $t = M+1$). Observe that A_x is a symmetric 0-1 matrix.

Now consider the ground space of the first two terms $H_{\text{prop}} + H_{\text{penalty}}$ in (1.15). Note that $[H_{\text{prop}}, H_{\text{penalty}}] = 0$, so these operators can be simultaneously diagonalized. Furthermore, H_{penalty} has smallest eigenvalue $-1-\sqrt{2}$, with eigenspace spanned by $|\omega\rangle$ and $|\omega^*\rangle$. One can also easily confirm that the first excited energy of H_{penalty} is -1.

The ground space of $H_{\text{prop}} + H_{\text{penalty}}$ lives in the sector where H_{penalty} has minimal eigenvalue $-1-\sqrt{2}$. To see this, note that within this sector H_{prop} has the same eigenvalues as H_x, and therefore has lowest eigenvalue $-2\sqrt{2}$. The minimum eigenvalue e_1 of $H_{\text{prop}} + H_{\text{penalty}}$ in this sector is

$$e_1 = -2\sqrt{2} + \left(-1-\sqrt{2}\right) = -1 - 3\sqrt{2} = -5.24\ldots, \tag{1.16}$$

whereas in any other sector H_{penalty} has eigenvalue at least -1 and (using the fact that $H_{\text{prop}} \geq -4$) the minimum eigenvalue of $H_{\text{prop}} + H_{\text{penalty}}$ is at least -5. Thus, an orthonormal basis for the ground space of $H_{\text{prop}} + H_{\text{penalty}}$ is furnished by the states

$$\frac{1}{\sqrt{2M}} \sum_{t=1}^{2M} W_{t-1}W_{t-2}\ldots W_1|\phi\rangle|t\rangle|\omega\rangle \tag{1.17}$$

$$\frac{1}{\sqrt{2M}} \sum_{t=1}^{2M} (W_{t-1}W_{t-2}\ldots W_1)^*|\phi^*\rangle|t\rangle|\omega^*\rangle \tag{1.18}$$

where $|\phi\rangle$ ranges over the basis Λ for $\mathcal{H}_{\mathrm{comp}}$ and $*$ denotes (elementwise) complex conjugation.

1.2.3 Upper bound on the smallest eigenvalue for yes instances

Suppose x is a yes instance; then there exists some n_{input}-qubit state $|\psi_{\mathrm{input}}\rangle$ satisfying $\mathrm{AP}\left(\mathcal{C}_x, |\psi_{\mathrm{input}}\rangle\right) \geq 1 - \frac{1}{2^{|x|}}$. Let

$$|\mathrm{wit}\rangle = \frac{1}{\sqrt{2M}} \sum_{t=1}^{2M} W_{t-1} W_{t-2} \dots W_1 \left(|\psi_{\mathrm{input}}\rangle |0\rangle^{\otimes n - n_{\mathrm{input}}}\right) |t\rangle |\omega\rangle$$

and note that this state is in the e_1-energy ground space of $H_{\mathrm{prop}} + H_{\mathrm{penalty}}$ (since it has the form (1.17)). One can also directly verify that $|\mathrm{wit}\rangle$ has zero energy for H_{input}. Thus

$$\begin{aligned}
\langle \mathrm{wit}|A_x|\mathrm{wit}\rangle &= e_1 + \langle \mathrm{wit}|H_{\mathrm{output}}|\mathrm{wit}\rangle \\
&= e_1 + \frac{1}{2M} \langle \psi_{\mathrm{input}}| \langle 0|^{\otimes n - n_{\mathrm{input}}} U_{\mathcal{C}_x}^{\dagger} |0\rangle\langle 0|_{\mathrm{output}} U_{\mathcal{C}_x} |\psi_{\mathrm{input}}\rangle |0\rangle^{\otimes n - n_{\mathrm{input}}} \\
&= e_1 + \frac{1}{2M} \left(1 - \mathrm{AP}(\mathcal{C}_x, |\psi_{\mathrm{input}}\rangle)\right) \\
&\leq e_1 + \frac{1}{2M} \frac{1}{2^{|x|}}.
\end{aligned}$$

1.2.4 Lower bound on the smallest eigenvalue for no instances

Now suppose x is a no instance. Then the verification circuit $\mathcal{C}_x$ has acceptance probability $\mathrm{AP}\left(\mathcal{C}_x, |\psi\rangle\right) \leq \frac{1}{3}$ for all n_{input}-qubit input states $|\psi\rangle$.

We backtrack slightly to obtain bounds on the eigenvalue gaps of the Hamiltonians $H_{\mathrm{prop}} + H_{\mathrm{penalty}}$ and $H_{\mathrm{prop}} + H_{\mathrm{penalty}} + H_{\mathrm{input}}$. We begin by showing that the energy gap of $H_{\mathrm{prop}} + H_{\mathrm{penalty}}$ is at least an inverse polynomial function of M. Subtracting a constant equal to the ground energy times the identity matrix sets the smallest eigenvalue to zero, and the smallest nonzero eigenvalue satisfies

$$\gamma(H_{\mathrm{prop}} + H_{\mathrm{penalty}} - e_1 \cdot \mathbb{I}) \geq \min \left\{ \sqrt{2} \frac{\pi^2}{M^2}, -5 - e_1 \right\} \geq \frac{1}{5M^2}. \tag{1.19}$$

since $-5 - e_1 \approx 0.24 \ldots > \frac{1}{5}$. The first inequality above follows from the fact that every eigenvalue of H_{prop} in the range $[e_1, -5)$ is also an eigenvalue of H_x (as discussed above) and the bound (1.13) on the energy gap of H_x.

Now use the Nullspace Projection Lemma (Lemma **??**) with

$$H_A = H_{\mathrm{prop}} + H_{\mathrm{penalty}} - e_1 \cdot \mathbb{I} \qquad H_B = H_{\mathrm{input}}.$$

Note that H_A and H_B are positive semidefinite. Let S_A be the ground space of H_A and consider the restriction $H_B|_{S_A}$. Here it is convenient to use the basis for S_A given by (1.17) and (1.18) with $|\phi\rangle$ ranging over the computational basis states of n qubits. In this basis, $H_B|_{S_A}$ is diagonal with all diagonal entries equal to $\frac{1}{2M}$ times an integer, so $\gamma(H_B|_{S_A}) \geq \frac{1}{2M}$.

We also have $\gamma(H_A) \geq \frac{1}{5M^2}$ from equation (1.19), and clearly $\|H_B\| \leq n$. Thus Lemma **??** gives

$$\gamma(H_{\text{prop}} + H_{\text{penalty}} + H_{\text{input}} - e_1 \cdot \mathbb{I}) \geq \frac{\left(\frac{1}{5M^2}\right)\left(\frac{1}{2M}\right)}{\frac{1}{5M^2} + \frac{1}{2M} + n} \geq \frac{1}{10M^3\,(1+n)} \geq \frac{1}{20M^3 n}. \tag{1.20}$$

Now consider adding the final term H_{output}. We use Lemma **??** again, now setting

$$H_A = H_{\text{prop}} + H_{\text{penalty}} + H_{\text{input}} - e_1 \cdot \mathbb{I} \qquad H_B = H_{\text{output}}.$$

Let S_A be the ground space of H_A. Note that it is spanned by states of the form (1.17) and (1.18) where $|\phi\rangle = |\psi\rangle|0\rangle^{\otimes n-n_{\text{input}}}$ and $|\psi\rangle$ ranges over any orthonormal basis of the n_{input}-qubit input register. The restriction $H_B|_{S_A}$ is block diagonal, with one block for states of the form

$$\frac{1}{\sqrt{2M}}\sum_{t=1}^{2M} W_{t-1}W_{t-2}\ldots W_1\left(|\psi\rangle|0\rangle^{\otimes n-n_{\text{input}}}\right)|t\rangle|\omega\rangle \tag{1.21}$$

and another block for states of the form

$$\frac{1}{\sqrt{2M}}\sum_{t=1}^{2M} (W_{t-1}W_{t-2}\ldots W_1)^*\left(|\psi\rangle^*|0\rangle^{\otimes n-n_{\text{input}}}\right)|t\rangle|\omega^*\rangle. \tag{1.22}$$

We now show that the minimum eigenvalue of $H_B|_{S_A}$ is nonzero, and we lower bound it. We consider the two blocks separately. By linearity, every state in the first block can be written in the form (1.21) for some state $|\psi\rangle$. Thus the minimum eigenvalue within this block is the minimum expectation of H_{output} in a state (1.21), where the minimum is taken over all n_{input}-qubit states $|\psi\rangle$. This is equal to

$$\min_{|\psi\rangle}\frac{1}{2M}\left(1 - \mathrm{AP}(\mathcal{C}_x, |\psi\rangle)\right) \geq \frac{1}{3M}$$

where we used the fact that $\mathrm{AP}\left(\mathcal{C}_x, |\psi\rangle\right) \leq \frac{1}{3}$ for all $|\psi\rangle$. Likewise, every state in the second block can be written as (1.22) for some state $|\psi\rangle$, and the minimum eigenvalue within this block is

$$\min_{|\psi\rangle}\frac{1}{2M}\left(1 - \mathrm{AP}(\mathcal{C}_x, |\psi\rangle)^*\right) \geq \frac{1}{3M}$$

(since $\mathrm{AP}(\mathcal{C}_x, |\psi\rangle)^* = \mathrm{AP}(\mathcal{C}_x, |\psi\rangle) \leq \frac{1}{3}$). Thus we see that $H_B|_{S_A}$ has an empty nullspace, so its smallest eigenvalue is equal to its smallest nonzero eigenvalue, namely

$$\gamma(H_B|_{S_A}) \geq \frac{1}{3M}.$$

Now applying Lemma **??** using this bound, the fact that $\|H_B\| = 1$, and the fact that $\gamma(H_A) \geq \frac{1}{20M^3 n}$ (from equation (1.20)), we get

$$\gamma(A_x - e_1 \cdot \mathbb{I}) \geq \frac{\frac{1}{60M^4 n}}{\frac{1}{20M^3 n} + \frac{1}{3M} + 1} \geq \frac{1}{120M^4 n}.$$

Since $H_B|_{S_A}$ has an empty nullspace, $A_x - e_1 \cdot \mathbb{I}$ has an empty nullspace, and this is a lower bound on its smallest eigenvalue.

1.3 Extensions and Discussion

While this result is interesting in its own right, as it shows that finding the ground energy of a sparse, row-computable matrix is QMA-complete, perhaps the most interesting result is that nothing particularly quantum is involved in the definition of the problem. In particular, the only condition we have on the matrix is that it is sparse, and row-computable. This condition might allow for a more natural understanding for more classically-minded computer scientists, as a QMA-complete problem could be stated without having to delve into any quantum computing.

As an additional problem, since the circuit-to-Hamiltonian map creates a 7-regular, simple graph, one might wonder if the removal of these conditions are necessary when the boundary terms are added. This is obviously going to be necessary, as otherwise we would have that determining the lowest eigenvalue of a Laplacian is QMA-complete, but it is a well known fact that the smallest eigenvalue of a Laplacian is zero.

[**TO DO:** *Write more*]