Chapter 1

Mathematical Preliminaries

1.1 Complexity Theory

Definition of QMA, languages, etcetera.

1.2 Various Mathematical Lemmas

1.2.1 Truncation Lemma

Lemma 1 (Truncation Lemma)**.** *Let H be a Hamiltonian acting on a Hilbertspace $\mathcal{H}$ and let $|\Phi\rangle \in \mathcal{H}$ be a normalized state. Let $\mathcal{K}$ be a subspace of $\mathcal{H}$, let P be the projector onto $\mathcal{K}$, and let $\tilde{H} = PHP$ be the Hamiltonian within this subspace. Suppose that, for some $T > 0$, $W \in \{H, \tilde{H}\}$, $N_0 \in \natural$, and $\delta > 0$, we have, for all $0 \leq t \leq T$,*

$$e^{-iWt}|\Phi\rangle = |\gamma(t)\rangle + |\epsilon(t)\rangle \text{ with } \||\epsilon(t)\rangle\| \leq \delta$$

and

$$(1-P)H^r|\gamma(t)\rangle = 0 \text{ for all } r \in \{0, 1, \ldots, N_0 - 1\}.$$

Then, for all $0 \leq t \leq T$,

$$\left\| \left(e^{-iHt} - e^{-i\tilde{H}t} \right) |\Phi\rangle \right\| \leq \left(\frac{4e\|H\|t}{N_0} + 2 \right) \left(\delta + 2^{-N_0}(1+\delta) \right).$$

Proposition 1. *Let H be a Hamiltonian acting on a Hilbert space $\mathcal{H}$, and let $|\Phi\rangle \in \mathcal{H}$ be a normalized state. Let $\mathcal{K}$ be a subspace of $\mathcal{H}$ such that there exists an $N_0 \in \natural$ so that for all $|\alpha\rangle \in \mathcal{K}^\perp$ and for all $n \in \{0, 1, 2, \ldots, N_0 - 1\}$, $\langle \alpha | H^n | \Phi \rangle = 0$. Let P be the projector onto $\mathcal{K}$ and let $\tilde{H} = PHP$ be the Hamiltonian within this subspace. Then*

$$\|e^{-it\tilde{H}}|\Phi\rangle - e^{-itH}|\Phi\rangle\| \leq 2 \left(\frac{e\|H\|t}{N_0} \right)^{N_0}.$$

Proof. Define $|\,\Phi(t)\,\rangle$ and $|\,\tilde{\Phi}(t)\,\rangle$ as

$$|\,\Phi(t)\,\rangle = e^{-itH}|\,\Phi\,\rangle = \sum_{k=0}^{\infty}\frac{(-it)^k}{k!}H^k|\,\Phi\,\rangle \qquad |\,\tilde{\Phi}(t)\,\rangle = e^{-it\tilde{H}}|\,\Phi\,\rangle = \sum_{k=0}^{\infty}\frac{(-it)^k}{k!}\tilde{H}^k|\,\Phi\,\rangle.$$

Note that by assumption, $\tilde{H}^k|\,\Phi\,\rangle = H^k|\,\Phi\,\rangle$ for all $k < N_0$, and thus the first N_0 terms in the two above sums are equal. Looking at the difference between these two states, we have

$$\begin{aligned}
\||\,\Phi(t)\,\rangle - |\,\tilde{\Phi}(t)\,\rangle\| &= \left\|\sum_{k=0}^{\infty}\frac{(-it)^k}{k!}\left(H^k - \tilde{H}^k\right)|\,\Phi\,\rangle\right\| \\
&= \left\|\sum_{k=0}^{N_0-1}\frac{(-it)^k}{k!}\left(H^k - \tilde{H}^k\right)|\,\Phi\,\rangle - \sum_{k=N_0}^{\infty}\frac{(-it)^k}{k!}\left(H^k - \tilde{H}^k\right)|\,\Phi\,\rangle\right\| \\
&\leq \sum_{k=N_0}^{\infty}\frac{t^k}{k!}\left(\|H\|^k + \|\tilde{H}\|^k\right) \\
&\leq 2\sum_{k=N_0}^{\infty}\frac{t^k}{k!}\|H\|^k
\end{aligned}$$

where the last step uses the fact that $\|\tilde{H}\| \leq \|P\|\|H\|\|P\| = \|H\|$. Thus for any $c \geq 1$, we have

$$\begin{aligned}
\||\,\Phi(t)\,\rangle - |\,\tilde{\Phi}(t)\,\rangle\| &\leq \frac{2}{c^{N_0}}\sum_{k=N_0}^{\infty}\frac{(ct)^k}{k!}\|H\|^k \\
&\leq \frac{2}{c^{N_0}}\exp(ct\|H\|).
\end{aligned}$$

We obtain the best bound by choosing $c = N_0/\|Ht\|$, which gives

$$\||\,\Phi(t)\,\rangle - |\,\tilde{\Phi}(t)\,\rangle\| \leq 2\left(\frac{e\|H\|t}{N_0}\right)^{N_0}$$

as claimed. (If $c < 1$ then the bound is trivial.) $\square$

Proposition 2. *Let $U_1, \ldots, U_n$ and $V_1, \ldots, V_n$ be unitary operators. Then for any $|\,\psi\,\rangle$,*

$$\left\|\left(\prod_{i=n}^{1}U_i - \prod_{i=n}^{1}V_i\right)|\,\psi\,\rangle\right\| \leq \sum_{j=1}^{n}\left\|(U_j - V_j)\prod_{i=j-1}^{1}U_i|\,\psi\,\rangle\right\|. \tag{1.1}$$

Proof. The proof is by induction on n. The case $n = 1$ is obvious. For the induction step, we have

$$\begin{aligned}
\left\|\left(\prod_{i=n}^{1}U_i - \prod_{i=n}^{1}V_i\right)|\,\psi\,\rangle\right\| &= \left\|\left(\prod_{i=n}^{1}U_i - V_n\prod_{i=n-1}^{1}U_i + V_n\prod_{i=n-1}^{1}U_i - \prod_{i=n}^{1}V_i\right)|\,\psi\,\rangle\right\| && (1.2) \\
&\leq \left\|(U_n - V_n)\prod_{i=n-1}^{1}U_i|\,\psi\,\rangle\right\| + \left\|\left(\prod_{i=n-1}^{1}U_i - \prod_{i=n-1}^{1}V_i\right)|\,\psi\,\rangle\right\| && (1.3) \\
&\leq \sum_{j=1}^{n}\left\|(U_j - V_j)\prod_{i=j-1}^{1}U_i|\,\psi\,\rangle\right\| && (1.4)
\end{aligned}$$

where the last step uses the induction hypothesis. □

Proof of Lemma 1. For $M \in \natural$ write

$$\begin{aligned}
\|(e^{-iHt} - e^{-i\tilde{H}t})|\,\Phi\,\rangle\| &= \left\|\left(\left(e^{-iH\frac{t}{M}}\right)^M - \left(e^{-i\tilde{H}\frac{t}{M}}\right)^M\right)|\,\Phi\,\rangle\right\| \\
&\leq \sum_{j=1}^{M}\left\|\left(e^{-iH\frac{t}{M}} - e^{-i\tilde{H}\frac{t}{M}}\right)e^{-iW(j-1)\frac{t}{M}}|\,\Phi\,\rangle\right\| \\
&\leq \sum_{j=1}^{M}\left\|\left(e^{-iH\frac{t}{M}} - e^{-i\tilde{H}\frac{t}{M}}\right)\left(|\,\gamma(\tfrac{(j-1)t}{M})\,\rangle + |\,\epsilon(\tfrac{(j-1)t}{M})\,\rangle\right)\right\| \\
&\leq 2M\delta + \sum_{j=1}^{M}\left\|\left(e^{-iH\frac{t}{M}} - e^{-i\tilde{H}\frac{t}{M}}\right)\frac{|\,\gamma(\frac{(j-1)t}{M})\,\rangle}{\left\||\,\gamma(\frac{(j-1)t}{M})\,\rangle\right\|}\right\|\left\||\,\gamma(\tfrac{(j-1)t}{M})\,\rangle\right\| \\
&\leq 2M\delta + 2M\left(\frac{e\|H\|t}{MN_0}\right)^{N_0}(1+\delta)
\end{aligned}$$

where in the second line we have used Proposition **??** and in the last step we have used Proposition **??** and the fact that $\||\gamma(t)\rangle\| \leq 1+\delta$. Now, for some $\eta > 1$, choose

$$M = \left\lceil \frac{\eta e\|H\|t}{N_0} \right\rceil$$

for $0 < t \leq T$ to get

$$\begin{aligned}
\|(e^{-iHt} - e^{-i\tilde{H}t})|\,\Phi\,\rangle\| &\leq 2M\left(\delta + \eta^{-N_0}(1+\delta)\right) \\
&\leq 2\left(\frac{\eta e\|H\|t}{N_0} + 1\right)\left(\delta + \eta^{-N_0}(1+\delta)\right).
\end{aligned}$$

The choice $\eta = 2$ gives the stated conclusion. □

Note that it would be slightly better to take a smaller value of η. However, this does not significantly improve the final result; the above bound is simpler and sufficient for our purposes.

1.2.2 Nullspace Projection Lemma

Lemma 2 (Nullspace Projection Lemma)**.** *Let H_A and H_B be positive semi-definite matrices. Suppose that the nullspace, S, of H_A is nonempty, and that*

$$\gamma\big(H_B|_S\big) \geq c > 0 \qquad \textit{and} \qquad \gamma(H_A) \geq d > 0. \tag{1.5}$$

Then,

$$\gamma(H_A + H_B) \geq \frac{cd}{d + \|H_B\|}. \tag{1.6}$$

Proof. Let $|\psi\rangle$ be a normalized state satisfying

$$\langle\psi|H_A + H_B|\psi\rangle = \gamma(H_A + H_B). \tag{1.7}$$

Let Π_S be the projector onto the nullspace of H_A. First suppose that $\Pi_S|\psi\rangle = 0$, in which case

$$\langle\psi|H_A + H_B|\psi\rangle \geq \langle\psi|H_A|\psi\rangle \geq \gamma(H_A) \tag{1.8}$$

and the result follows. On the other hand, if $\Pi_S|\psi\rangle \neq 0$ then we can write

$$|\psi\rangle = \alpha|a\rangle + \beta|a^\perp\rangle \tag{1.9}$$

with $|\alpha|^2 + |\beta|^2 = 1$, $\alpha \neq 0$, and two normalized states $|a\rangle$ and $|a^\perp\rangle$ such that $|a\rangle \in S$ and $|a^\perp\rangle \in S^\perp$. (If $\beta = 0$ then we may choose $|a^\perp\rangle$ to be an arbitrary state in $S^\perp$ but in the following we fix one specific choice for concreteness.) Note that any state $|\phi\rangle$ in the nullspace of $H_A + H_B$ satisfies $H_A|\phi\rangle = 0$ and hence $\langle\phi|a^\perp\rangle = 0$. Since $\langle\phi|\psi\rangle = 0$ and $\alpha \neq 0$ we also see that $\langle\phi|a\rangle = 0$. Hence any state

$$|f(q,r)\rangle = q|a\rangle + r|a^\perp\rangle \tag{1.10}$$

is orthogonal to the nullspace of $H_A + H_B$, and

$$\gamma(H_A + H_B) = \min_{|q|^2+|r|^2=1} \langle f(q,r)|H_A + H_B|f(q,r)\rangle. \tag{1.11}$$

Within the subspace Q spanned by $|\,a\,\rangle$ and $|\,a^\perp\,\rangle$, note that

$$H_A|_Q = \begin{pmatrix} w & v^* \\ v & z \end{pmatrix} \qquad H_B|_Q = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} \tag{1.12}$$

where $w = \langle a|H_B|a\rangle$, $v = \langle a^\perp|H_B|a\rangle$, $y = \langle a^\perp|H_A|a^\perp\rangle$, and $z = \langle a^\perp|H_B|a^\perp\rangle$, and that we are interested in the smaller eigenvalue of

$$M = H_A|_Q + H_B|_Q = \begin{pmatrix} w & v^* \\ v & y+z \end{pmatrix}. \tag{1.13}$$

Letting ϵ_+ and ϵ_- be the two eigenvalues of M with $\epsilon_+ \geq \epsilon_-$, note that

$$\epsilon_+ = \|M\| \leq \|H_A|_Q\| + \|H_B|_Q\| \leq y + \|H_B|_Q\| \leq y + \|H_B\|, \tag{1.14}$$

where we have used the Cauchy interlacing theorem to note that $\|H_B|_Q\| \leq \|H_B\|$. Additionally, we have that

$$\epsilon_+\epsilon_- = \det(M) = w(y+z) - |v|^2 \geq wy \tag{1.15}$$

where we used the fact that $H_B|_Q$ is positive-semidefinite. Putting this together, we have that

$$\gamma(H_A + H_B) = \min_{|q|^2+|r|^2=1} \langle f(q,r)|H_A + H_B|f(q,r)\rangle = \epsilon_- \geq \frac{wy}{y + \|H_B\|}. \tag{1.16}$$

As the right hand side increased monotonically with both w and y, and as $w \geq \gamma(H_B|_S) \geq c$ and $y \geq \gamma(H_A) \geq d$, we have

$$\gamma(H_A + H_B) \geq \frac{cd}{d + \|H_B\|} \tag{1.17}$$

as required. $\square$