Chapter 1

Universality of multi-particle scattering

Hard, but worthwhile

1.1 Multi-particle quantum walk

Note that this is exactly what I wanted to talk about.

Very difficult in general.

1.1.1 Two-particle scattering on an infinite path

The one thing we can actually compute It might be interesting to talk about what happens with spins.

1.2 Applying an encoded $C\theta$-gate

1.2.1 Finite truncation

Theorem 1. *Let $H^{(2)}$ be a two-particle Hamiltonian of the form (**??**) with interaction range at most C, i.e., $\mathcal{V}(|r|) = 0$ for all $|r| > C$. Let $\theta_{\pm}(p_1, p_2)$ be given by equation (**??**). Define $\theta = \theta_{\pm}(\pi/4, 3\pi/8)$. Let $L \in \natural$, let $M \in \{C+1, C+2, \ldots\}$, and define*

$$|\chi_{z,k}\rangle = \frac{1}{\sqrt{L}} \sum_{x=z-L}^{z-1} e^{ikx} |x\rangle$$

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} \left(|\chi_{-M,-\frac{\pi}{2}}\rangle |\chi_{M+L+1,\frac{\pi}{4}}\rangle \pm |\chi_{M+L+1,\frac{\pi}{4}}\rangle |\chi_{-M,-\frac{\pi}{2}}\rangle \right).$$

Let c_0 be a constant independent of L. Then, for all $0 \le t \le c_0 L$, we have

$$\left\| e^{-iH^{(2)}t} |\psi(0)\rangle - |\alpha(t)\rangle \right\| = \emptyset(L^{-1/8}),$$

where

$$|\alpha(t)\rangle = \sum_{x,y} a_{xy}(t)|x, y\rangle, \tag{1.1}$$

$a_{xy}(t) = \pm a_{yx}(t)$, *and, for* $x \le y$,

$$a_{xy}(t) = \frac{1}{\sqrt{2}L} e^{-\sqrt{2}it} \left[e^{-i\pi x/2} e^{i\pi y/4} F(x, y, t) \pm e^{i\theta} e^{i\pi x/4} e^{-i\pi y/2} F(y, x, t)\right] \tag{1.2}$$

where

$$F(u, v, t) = \begin{cases} 1 & \text{if } u - 2\lfloor t\rfloor \in \{-M - L, \dots, -M - 1\} \text{ and } v + 2\left\lfloor \frac{t}{\sqrt{2}}\right\rfloor \in \{M + 1, \dots, M + L\} \\ 0 & \text{otherwise.} \end{cases}$$

In this section we prove Theorem 1. The main proof appears in Section 1.2.1.1, relying on several technical lemmas proved in Section 1.2.1.2. The proof follows the method used in the single-particle case, which is based on the calculation from the appendix of reference [?].

Recall from (??) that for each $p_1 \in (-\pi, \pi)$ and $p_2 \in (0, \pi)$ there is an eigenstate $|\mathrm{sc}(p_1; p_2)\rangle_\pm$ of $H^{(2)}$ of the form

$$\langle x, y|\mathrm{sc}(p_1; p_2)\rangle_\pm = \frac{e^{-ip_1\left(\frac{x+y}{2}\right)}}{\sqrt{2}} \begin{cases} e^{-ip_2(x-y)} \pm e^{i\theta_\pm(p_1,p_2)} e^{ip_2(x-y)} & \text{if } x - y \le -C \\ e^{-ip_2(x-y)} e^{i\theta_\pm(p_1,p_2)} \pm e^{ip_2(x-y)} & \text{if } x - y \ge C \\ f(p_1, p_2, x - y) \pm f(p_1, p_2, y - x) & \text{if } |x - y| < C \end{cases} \tag{1.3}$$

where

$$e^{i\theta_\pm(p_1,p_2)} = T(p_1, p_2) \pm R(p_1, p_2),$$

C is the range of the interaction, T and R are the transmission and reflection coefficients of the interaction at the chosen momentum, f describes the amplitudes of the scattering state within the interaction range, and the $\pm$ depends on the type of particle ($+$ for bosons, $-$ for fermions). The state $|\mathrm{sc}(p_1; p_2)\rangle_\pm$ satisfies

$$H^{(2)}|\mathrm{sc}(p_1; p_2)\rangle_\pm = 4\cos\frac{p_1}{2}\cos p_2|\mathrm{sc}(p_1; p_2)\rangle_\pm$$

and is delta-function normalized as

$${}_\pm\langle \mathrm{sc}(p_1'; p_2')|\mathrm{sc}(p_1; p_2)\rangle_\pm = 4\pi^2\delta(p_1 - p_1')\delta(p_2 - p_2'). \tag{1.4}$$

Proof. Expand $|\psi(0)\rangle$ in the basis of eigenstates of the Hamiltonian to get

$$|\psi(t)\rangle = e^{-iH^{(2)}t}|\psi(0)\rangle = |\psi_1(t)\rangle + |\psi_2(t)\rangle$$

where

$$|\psi_1(t)\rangle = \iint_{D_\epsilon} \frac{d\phi_1 d\phi_2}{4\pi^2} e^{-it4\cos(\frac{p_1}{2}+\frac{\phi_1}{2})\cos(p_2+\phi_2)}|\mathrm{sc}(p_1+\phi_1; p_2+\phi_2)\rangle_\pm \left({}_\pm\langle \mathrm{sc}(p_1 + \phi_1; p_2 + \phi_2)|\psi(0)\rangle\right)$$

with $D_\epsilon = [-\epsilon,\epsilon]\times[-\epsilon,\epsilon]$, $p_1 = \pi/2 - \pi/4 = \pi/4$, $p_2 = (\pi/2+\pi/4)/2 = 3\pi/8$, and with $|\psi_2(t)\rangle$ orthogonal to $|\psi_1(t)\rangle$. We take $\epsilon = a/\sqrt{L}$ for some constant a. Using equation (1.3) we get

$$|\,\psi_1(t)\,\rangle = |\psi_A(t)\rangle \pm |\psi_B(t)\rangle$$

where

$$|\psi_A(t)\rangle = \iint_{D_\epsilon} \frac{d\phi_1 d\phi_2}{4\pi^2} e^{-it4\cos(\frac{\pi}{8}+\frac{\phi_1}{2})\cos(\frac{3\pi}{8}+\phi_2)} A(\phi_1,\phi_2)|\mathrm{sc}(\tfrac{\pi}{4}+\phi_1;\tfrac{3\pi}{8}+\phi_2)\rangle_\pm \tag{1.5}$$

$$|\,\psi_B(t)\,\rangle = \iint_{D_\epsilon} \frac{d\phi_1 d\phi_2}{4\pi^2} e^{-it4\cos(\frac{\pi}{8}+\frac{\phi_1}{2})\cos(\frac{3\pi}{8}+\phi_2)} e^{-i\theta_\pm(\frac{\pi}{4}+\phi_1,\frac{3\pi}{8}+\phi_2)} B(\phi_1,\phi_2,\tfrac{3\pi}{8})|\mathrm{sc}(\tfrac{\pi}{4}+\phi_1;\tfrac{3\pi}{8}+\phi_2)\rangle_\pm$$

with

$$A(\phi_1,\phi_2) = \frac{1}{L}\sum_{x=-(M+L)}^{-(M+1)}\sum_{y=M+1}^{M+L} e^{i\phi_1\frac{x+y}{2}} e^{i\phi_2(x-y)} \tag{1.6}$$

$$B(\phi_1,\phi_2,k) = \frac{1}{L}\sum_{x=-(M+L)}^{-(M+1)}\sum_{y=M+1}^{M+L} e^{i\phi_1\frac{x+y}{2}} e^{i(\phi_2+2k)(y-x)}.$$

Using the delta-function normalization of the scattering states (equation (1.4)) we get

$$\begin{aligned}\langle\psi_B(t)|\psi_B(t)\rangle &= \iint_{D_\epsilon} \frac{d\phi_1 d\phi_2}{4\pi^2}\left|B(\phi_1,\phi_2,\tfrac{3\pi}{8})\right|^2\\ &\le \frac{16\pi^2}{L^2\epsilon^2}\end{aligned}$$

by Lemma 3 (as long as $\epsilon < 3\pi/8$, which holds for L sufficiently large). Similarly,

$$\begin{aligned}1 &\ge \langle\psi_A(t)|\psi_A(t)\rangle\\ &= \iint_{D_\epsilon}\frac{d\phi_1 d\phi_2}{4\pi^2}\left|A(\phi_1,\phi_2)\right|^2\\ &\ge 1-\frac{4\pi}{L\epsilon}\end{aligned}$$

(from the first two facts in Lemma 3) and therefore

$$\begin{aligned}\langle\psi_1(t)|\psi_1(t)\rangle &= \langle\psi_A(t)|\psi_A(t)\rangle + \langle\psi_B(t)|\psi_B(t)\rangle + \langle\psi_A(t)|\psi_B(t)\rangle + \langle\psi_B(t)|\psi_A(t)\rangle\\ &\ge 1-\frac{4\pi}{L\epsilon} - 2\left|\langle\psi_A(t)|\psi_B(t)\rangle\right|\\ &\ge 1-\frac{4\pi}{L\epsilon} - 2\left|\langle\psi_A(t)|\psi_A(t)\rangle\right|^{\frac12}\left|\langle\psi_B(t)|\psi_B(t)\rangle\right|^{\frac12}\\ &\ge 1-\frac{12\pi}{L\epsilon}.\end{aligned}$$

Hence

$$\langle\psi_2(t)|\psi_2(t)\rangle \le \frac{12\pi}{L\epsilon}$$

since

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi_1(t)|\psi_1(t)\rangle + \langle\psi_2(t)|\psi_2(t)\rangle = 1.$$

Thus

$$\begin{aligned}\| \, |\psi(t)\rangle - |\psi_A(t)\rangle\| &= \| \, |\psi_B(t)\rangle + |\psi_2(t)\rangle\| \\ &\le \| \, |\psi_B(t)\rangle\| + \| \, |\psi_2(t)\rangle\| \\ &\le \frac{4\pi}{L\epsilon} + \sqrt{\frac{12\pi}{L\epsilon}}.\end{aligned}$$

Now

$$\begin{aligned}\| \, |\psi(t)\rangle - |\alpha(t)\rangle\| &\le \| \, |\psi(t)\rangle - |\psi_A(t)\rangle\| + \| \, |\psi_A(t)\rangle - |\alpha(t)\rangle\| \\ &\le \frac{4\pi}{L\epsilon} + \sqrt{\frac{12\pi}{L\epsilon}} + \| \, |\psi_A(t)\rangle - |\alpha(t)\rangle\| \\ &= \emptyset(L^{-1/4}) + \| \, |\psi_A(t)\rangle - |\alpha(t)\rangle\|\end{aligned}$$

using our choice $\epsilon = a/\sqrt{L}$. To complete the proof, we now show that the second term in this expression is bounded by $\emptyset(L^{-1/8})$.

Lemma 1. *With $|\psi_A(t)\rangle$ and $|\alpha(t)\rangle$ defined through equations (1.5) and (1.1), with $t \le c_0 L$ (for some constant c_0),*

$$\| \, |\psi_A(t)\rangle - |\alpha(t)\rangle\| = \emptyset(L^{-1/8}).$$

Proof. To simplify matters, note that both $|\psi_A(t)\rangle$ and $|\alpha(t)\rangle$ are either symmetric or anti-symmetric (i.e., $\langle x,y|\alpha(t)\rangle = \pm\langle y,x|\alpha(t)\rangle$ and $\langle x,y|\psi_A(t)\rangle = \pm\langle y,x|\psi_A(t)\rangle$). Taking C to be the maximum range of the interaction in our Hamiltonian, we have

$$\| \, |\psi_A(t)\rangle - |\alpha(t)\rangle\| \le 2\, \|P_1|\psi_A(t)\rangle - P_1|\alpha(t)\rangle\| + \|P_2|\,\psi_A(t)\,\rangle\| + \|P_2|\,\alpha(t)\,\rangle\|\,,$$

where

$$P_1 = \sum_{y-x\ge C} |x,y\rangle\langle x,y| \qquad P_2 = \sum_{|x-y|<C} |\,x,y\,\rangle\langle\, x,y\,|.$$

Now, for $y - x \ge C$,

$$\begin{aligned}\langle x,y|\psi_A(t)\rangle &= \iint_{D_\epsilon} \frac{d\phi_1 d\phi_2}{4\pi^2} e^{-it4\cos(\frac{\pi}{8}+\frac{\phi_1}{2})\cos(\frac{3\pi}{8}+\phi_2)} A(\phi_1,\phi_2) \frac{e^{-i\left(\frac{\pi}{4}+\phi_1\right)\left(\frac{x+y}{2}\right)}}{\sqrt{2}} \\ &\quad \left(e^{i\left(\frac{3\pi}{8}+\phi_2\right)(y-x)} \pm e^{-i\left(\frac{3\pi}{8}+\phi_2\right)(y-x)+i\theta_\pm(\frac{\pi}{4}+\phi_1,\frac{3\pi}{8}+\phi_2)}\right) \\ &= \iint_{D_\epsilon} \frac{d\phi_1 d\phi_2}{4\pi^2} \Bigg[\frac{1}{\sqrt{2}} e^{-it4\cos(\frac{\pi}{8}+\frac{\phi_1}{2})\cos(\frac{3\pi}{8}+\phi_2)} A(\phi_1,\phi_2) \\ &\quad \Big(e^{-i\pi x/2} e^{i\pi y/4} e^{-i\phi_1\left(\frac{x+y}{2}\right)} e^{i\phi_2(y-x)} \\ &\quad \pm e^{i\pi x/4} e^{-i\pi y/2} e^{-i\phi_1\left(\frac{x+y}{2}\right)} e^{-i\phi_2(y-x)} e^{i\theta_\pm\left(\frac{\pi}{4}+\phi_1,\frac{3\pi}{8}+\phi_2\right)}\Big)\Bigg].\end{aligned}$$

From Lemma 4, for $x \le y$, the state $|\,\alpha(t)\,\rangle$ takes the form

$$\langle x,y|\alpha(t)\rangle = \frac{1}{\sqrt{2}}e^{-it\sqrt{2}}\Bigg[e^{-i\pi x/2}e^{i\pi y/4}\Bigg(\iint_{D_\pi}\frac{d\phi_1 d\phi_2}{4\pi^2} A(\phi_1,\phi_2)e^{-i\phi_1\left(-\lfloor t\rfloor+\left\lfloor\frac{t}{\sqrt{2}}\right\rfloor+\frac{x+y}{2}\right)}e^{-2i\phi_2\left(-\lfloor t\rfloor-\left\lfloor\frac{t}{\sqrt{2}}\right\rfloor+\frac{x-y}{2}\right)}\Bigg) \pm e^{i\theta}e^{i\pi x/4}e^{-i\pi y/2}\Bigg(\iint_{D_\pi}\frac{d\phi_1 d\phi_2}{4\pi^2} A(\phi_1,\phi_2)e^{-i\phi_1\left(-\lfloor t\rfloor+\left\lfloor\frac{t}{\sqrt{2}}\right\rfloor+\frac{x+y}{2}\right)}e^{-2i\phi_2\left(-\lfloor t\rfloor-\left\lfloor\frac{t}{\sqrt{2}}\right\rfloor+\frac{y-x}{2}\right)}\Bigg)\Bigg],$$

where $D_\pi = [-\pi,\pi]\times[-\pi,\pi]$. Using these expressions for $|\,\psi_A(t)\,\rangle$ and $|\,\alpha(t)\,\rangle$, we now write

$$P_1|\psi_A(t)\rangle - P_1|\alpha(t)\rangle = \pm|e_1(t)\rangle + |e_2(t)\rangle \pm |f_1(t)\rangle + |f_2(t)\rangle \pm |g_1(t)\rangle + |g_2(t)\rangle \pm |h(t)\rangle$$

where each term in the above equation is supported only on states $|x,y\rangle$ such that $y-x \ge C$, and (for $y-x\ge C$)

$$\langle x,y|e_1(t)\rangle = \frac{e^{i\theta}}{\sqrt{2}}e^{-it\sqrt{2}}e^{i\pi x/4}e^{-i\pi y/2}\iint_{D_\pi}\frac{d\phi_1 d\phi_2}{4\pi^2}A(\phi_1,\phi_2)\Bigg[e^{-i\phi_1\left(-t+\frac{t}{\sqrt{2}}+\frac{x+y}{2}\right)} e^{-2i\phi_2\left(-t-\frac{t}{\sqrt{2}}+\frac{y-x}{2}\right)} - e^{-i\phi_1\left(-\lfloor t\rfloor+\left\lfloor\frac{t}{\sqrt{2}}\right\rfloor+\frac{x+y}{2}\right)}e^{-2i\phi_2\left(-\lfloor t\rfloor-\left\lfloor\frac{t}{\sqrt{2}}\right\rfloor+\frac{y-x}{2}\right)}\Bigg]$$

$$\langle x,y|e_2(t)\rangle = \frac{1}{\sqrt{2}}e^{-it\sqrt{2}}e^{-i\pi x/2}e^{i\pi y/4}\iint_{D_\pi}\frac{d\phi_1 d\phi_2}{4\pi^2}A(\phi_1,\phi_2)\Bigg[e^{-i\phi_1\left(-t+\frac{t}{\sqrt{2}}+\frac{x+y}{2}\right)} e^{-2i\phi_2\left(-t-\frac{t}{\sqrt{2}}+\frac{x-y}{2}\right)} - e^{-i\phi_1\left(-\lfloor t\rfloor+\left\lfloor\frac{t}{\sqrt{2}}\right\rfloor+\frac{x+y}{2}\right)}e^{-2i\phi_2\left(-\lfloor t\rfloor-\left\lfloor\frac{t}{\sqrt{2}}\right\rfloor+\frac{x-y}{2}\right)}\Bigg]$$

$$\langle x,y|f_1(t)\rangle = -\frac{e^{i\theta}}{\sqrt{2}}e^{-it\sqrt{2}}e^{i\pi x/4}e^{-i\pi y/2}\iint_{D_\pi\setminus D_\epsilon}\frac{d\phi_1 d\phi_2}{4\pi^2}A(\phi_1,\phi_2) e^{-i\phi_1\left(-t+\frac{t}{\sqrt{2}}+\frac{x+y}{2}\right)}e^{-2i\phi_2\left(-t-\frac{t}{\sqrt{2}}+\frac{y-x}{2}\right)}$$

$$\langle x,y|f_2(t)\rangle = -\frac{1}{\sqrt{2}}e^{-it\sqrt{2}}e^{-i\pi x/2}e^{i\pi y/4}\iint_{D_\pi\setminus D_\epsilon}\frac{d\phi_1 d\phi_2}{4\pi^2}A(\phi_1,\phi_2) e^{-i\phi_1\left(-t+\frac{t}{\sqrt{2}}+\frac{x+y}{2}\right)}e^{-2i\phi_2\left(-t-\frac{t}{\sqrt{2}}+\frac{x-y}{2}\right)}$$

$$\langle x,y|g_1(t)\rangle = \frac{e^{i\theta}}{\sqrt{2}}e^{i\pi x/4}e^{-i\pi y/2}\iint_{D_\epsilon}\frac{d\phi_1 d\phi_2}{4\pi^2}A(\phi_1,\phi_2)e^{-i\phi_1\left(\frac{x+y}{2}\right)}e^{-2i\phi_2\left(\frac{y-x}{2}\right)} \Big[e^{-it4\cos(\frac{\pi}{8}+\frac{\phi_1}{2})\cos(\frac{3\pi}{8}+\phi_2)} - e^{-it\left(\sqrt{2}+\sqrt{2}\left(\frac{\phi_1}{2}-\phi_2\right)-2\left(\frac{\phi_1}{2}+\phi_2\right)\right)}\Big]$$

$$\langle x,y|g_2(t)\rangle = \frac{1}{\sqrt{2}}e^{-i\pi x/2}e^{i\pi y/4}\iint_{D_\epsilon}\frac{d\phi_1 d\phi_2}{4\pi^2}A(\phi_1,\phi_2)e^{-i\phi_1\left(\frac{x+y}{2}\right)}e^{-2i\phi_2\left(\frac{x-y}{2}\right)} \Big[e^{-it4\cos(\frac{\pi}{8}+\frac{\phi_1}{2})\cos(\frac{3\pi}{8}+\phi_2)} - e^{-it\left(\sqrt{2}+\sqrt{2}\left(\frac{\phi_1}{2}-\phi_2\right)-2\left(\frac{\phi_1}{2}+\phi_2\right)\right)}\Big]$$

$$\langle x,y|h(t)\rangle = \frac{1}{\sqrt{2}}e^{i\pi x/4}e^{-i\pi y/2}\iint_{D_\epsilon}\frac{d\phi_1 d\phi_2}{4\pi^2}A(\phi_1,\phi_2)e^{-i\phi_1\left(\frac{x+y}{2}\right)}e^{-2i\phi_2\left(\frac{y-x}{2}\right)} e^{-it4\cos(\frac{\pi}{8}+\frac{\phi_1}{2})\cos(\frac{3\pi}{8}+\phi_2)}\left(e^{i\theta_\pm(\frac{\pi}{4}+\phi_1,\frac{3\pi}{8}+\phi_2)} - e^{i\theta}\right).$$

We now proceed to bound the norm of each of these states. We repeatedly use the fact that, for $(\phi_1, \phi_2) \in D_\pi$,

$$\sum_{x,y=-\infty}^{\infty} e^{ix\left(\frac{1}{2}(\phi_1-\tilde{\phi}_1)-(\phi_2-\tilde{\phi}_2)\right)} e^{iy\left(\frac{1}{2}(\phi_1-\tilde{\phi}_1)+(\phi_2-\tilde{\phi}_2)\right)} = 4\pi^2\delta(\phi_1-\tilde{\phi}_1)\delta(\phi_2-\tilde{\phi}_2).$$

Using this formula we get

$$\begin{aligned}
\langle e_1(t)|e_1(t)\rangle &= \sum_{y-x\geq C} \langle e_1(t)|x,y\rangle\langle x,y|e_1(t)\rangle \\
&\leq \sum_{x=-\infty}^{\infty}\sum_{y=-\infty}^{\infty} \left| \frac{1}{\sqrt{2}} \iint_{D_\pi} \frac{d\phi_1 d\phi_2}{4\pi^2} A(\phi_1,\phi_2) \Big[e^{-i\phi_1\left(-t+\frac{t}{\sqrt{2}}+\frac{x+y}{2}\right)} \right. \\
&\qquad \left. e^{-2i\phi_2\left(-t-\frac{t}{\sqrt{2}}+\frac{y-x}{2}\right)} - e^{-i\phi_1\left(-\lfloor t\rfloor+\left\lfloor\frac{t}{\sqrt{2}}\right\rfloor+\frac{x+y}{2}\right)} e^{-2i\phi_2\left(-\lfloor t\rfloor-\left\lfloor\frac{t}{\sqrt{2}}\right\rfloor+\frac{y-x}{2}\right)} \Big] \right|^2 \\
&= \frac{1}{2}\iint_{D_\pi} \frac{d\phi_1 d\phi_2}{4\pi^2} |A(\phi_1,\phi_2)|^2 \left| e^{-i\phi_1\left(-t+\frac{t}{\sqrt{2}}\right)} e^{-2i\phi_2\left(-t-\frac{t}{\sqrt{2}}\right)} \right. \\
&\qquad \left. - e^{-i\phi_1\left(-\lfloor t\rfloor+\left\lfloor\frac{t}{\sqrt{2}}\right\rfloor\right)} e^{-2i\phi_2\left(-\lfloor t\rfloor-\left\lfloor\frac{t}{\sqrt{2}}\right\rfloor\right)} \right|^2.
\end{aligned}$$

Now use the fact that $\left|e^{-ic}-1\right|^2 \leq c^2$ for $c \in \mathbb{R}$ to get

$$\begin{aligned}
\langle e_1(t)|e_1(t)\rangle &\leq \frac{1}{2}\iint_{D_\pi} \left(\frac{d\phi_1 d\phi_2}{4\pi^2}\right) |A(\phi_1,\phi_2)|^2 \Bigg(-\phi_1\left(-t+\frac{t}{\sqrt{2}}+\lfloor t\rfloor-\left\lfloor\frac{t}{\sqrt{2}}\right\rfloor\right) \\
&\qquad -2\phi_2\left(-t-\frac{t}{\sqrt{2}}+\lfloor t\rfloor+\left\lfloor\frac{t}{\sqrt{2}}\right\rfloor\right)\Bigg)^2 \\
&\leq 4\iint_{D_\pi} \frac{d\phi_1 d\phi_2}{4\pi^2} |A(\phi_1,\phi_2)|^2 \left(\phi_1^2+4\phi_2^2\right)
\end{aligned}$$

using the Cauchy-Schwarz inequality and the fact that $\left|t-t/\sqrt{2}-\lfloor t\rfloor-\left\lfloor t/\sqrt{2}\right\rfloor\right| \leq 2$. So

$$\begin{aligned}
\langle e_1(t)|e_1(t)\rangle &\leq 4\left(\iint_{D_\pi\setminus D_\epsilon} \frac{d\phi_1 d\phi_2}{4\pi^2} + \iint_{D_\epsilon} \frac{d\phi_1 d\phi_2}{4\pi^2}\right) |A(\phi_1,\phi_2)|^2 \left(\phi_1^2+4\phi_2^2\right) \\
&\leq 4\left(5\pi^2\right)\left(\frac{4\pi}{L\epsilon}\right) + 20\epsilon^2 \\
&= \frac{80\pi^3}{L\epsilon} + 20\epsilon^2
\end{aligned}$$

where we have used Lemma 3 and the fact that $\phi_1^2+4\phi_2^2 \leq 5\epsilon^2$ on D_ϵ. Similarly,

$$\langle e_2(t)|e_2(t)\rangle \leq \frac{80\pi^3}{L\epsilon} + 20\epsilon^2.$$

Now

$$\begin{aligned}\langle f_1(t)|f_1(t)\rangle &\leq \frac{1}{2}\iint_{D_\pi\setminus D_\epsilon}\frac{d\phi_1 d\phi_2}{4\pi^2}\left|A(\phi_1,\phi_2)\right|^2\\&\leq \frac{2\pi}{L\epsilon}\end{aligned}$$

by Lemma 3, and similarly

$$\langle f_2(t)|f_2(t)\rangle \leq \frac{2\pi}{L\epsilon}.$$

Moving on to the next term,

$$\begin{aligned}\langle g_1(t)|g_1(t)\rangle \leq \frac{1}{2}\iint_{D_\epsilon}\frac{d\phi_1 d\phi_2}{4\pi^2}\left|A(\phi_1,\phi_2)\right|^2 &\Big| e^{-it4\cos\left(\frac{\pi}{8}+\frac{\phi_1}{2}\right)\cos\left(\frac{3\pi}{8}+\phi_2\right)}\\&- e^{-it\left(\sqrt{2}+\sqrt{2}\left(\frac{\phi_1}{2}-\phi_2\right)-2\left(\frac{\phi_1}{2}+\phi_2\right)\right)}\Big|^2\\ \leq \frac{1}{2}\iint_{D_\epsilon}\frac{d\phi_1 d\phi_2}{4\pi^2}&\Bigg[\left|A(\phi_1,\phi_2)\right|^2 t^2\left(4\cos\left(\frac{\pi}{8}+\frac{\phi_1}{2}\right)\cos\left(\frac{3\pi}{8}+\phi_2\right)\right.\\&\left.-\sqrt{2}-\sqrt{2}\left(\frac{\phi_1}{2}-\phi_2\right)+2\left(\frac{\phi_1}{2}+\phi_2\right)\right)^2\Bigg] \end{aligned} \tag{1.7}$$

using $\left|e^{-ic}-1\right|^2 \leq c^2$ for $c\in\mathbb{R}$. Now

$$\begin{aligned}4\cos\left(\frac{\pi}{8}+\frac{\phi_1}{2}\right)\cos\left(\frac{3\pi}{8}+\phi_2\right) &= 2\cos\left(\frac{\pi}{2}+\frac{\phi_1}{2}+\phi_2\right)+2\cos\left(-\frac{\pi}{4}+\frac{\phi_1}{2}-\phi_2\right)\\&= -2\sin\left(\frac{\phi_1}{2}+\phi_2\right)+\sqrt{2}\cos\left(\frac{\phi_1}{2}-\phi_2\right)+\sqrt{2}\sin\left(\frac{\phi_1}{2}-\phi_2\right)\end{aligned}$$

so

$$\begin{aligned}&\left|4\cos\left(\frac{\pi}{8}+\frac{\phi_1}{2}\right)\cos\left(\frac{3\pi}{8}+\phi_2\right)-\sqrt{2}-\sqrt{2}\left(\frac{\phi_1}{2}-\phi_2\right)+2\left(\frac{\phi_1}{2}+\phi_2\right)\right|\\&\quad\leq \left|\sqrt{2}\left(\cos\left(\frac{\phi_1}{2}-\phi_2\right)-1\right)\right|+\left|\sqrt{2}\left(\sin\left(\frac{\phi_1}{2}-\phi_2\right)-\left(\frac{\phi_1}{2}-\phi_2\right)\right)\right|\\&\qquad+\left|2\left(\sin\left(\frac{\phi_1}{2}+\phi_2\right)-\left(\frac{\phi_1}{2}+\phi_2\right)\right)\right|\\&\quad\leq \sqrt{2}\left(\frac{\phi_1}{2}-\phi_2\right)^2+\sqrt{2}\left(\frac{\phi_1}{2}-\phi_2\right)^2+2\left(\frac{\phi_1}{2}+\phi_2\right)^2\\&\quad\leq 4\left(\left(\frac{\phi_1}{2}+\phi_2\right)^2+\left(\frac{\phi_1}{2}-\phi_2\right)^2\right),\end{aligned}$$

using $|\cos x - 1| \leq x^2$ and $|\sin x - x| \leq x^2$ for $x \in \mathbb{R}$. Plugging this into equation (1.7) we get

$$\begin{aligned}
\langle g_1(t)|g_1(t)\rangle &\leq \frac{1}{2}\iint_{D_\epsilon} \frac{d\phi_1 d\phi_2}{4\pi^2} 16\,|A(\phi_1,\phi_2)|^2\, t^2 \left(\left(\frac{\phi_1}{2}+\phi_2\right)^2 + \left(\frac{\phi_1}{2}-\phi_2\right)^2\right)^2 \\
&\leq 16t^2 \iint_{D_\epsilon} \frac{d\phi_1 d\phi_2}{4\pi^2} |A(\phi_1,\phi_2)|^2 \left(\left(\frac{\phi_1}{2}+\phi_2\right)^4 + \left(\frac{\phi_1}{2}-\phi_2\right)^4\right) \\
&\leq \frac{16t^2}{L^2} \iint_{D_\epsilon} \frac{d\phi_1 d\phi_2}{4\pi^2} \frac{\sin^2(\frac{L}{2}[\frac{\phi_1}{2}+\phi_2])}{\sin^2(\frac{1}{2}[\frac{\phi_1}{2}+\phi_2])} \frac{\sin^2(\frac{L}{2}[-\frac{\phi_1}{2}+\phi_2])}{\sin^2(\frac{1}{2}[-\frac{\phi_1}{2}+\phi_2])} \\
&\quad \left(\left(\frac{\phi_1}{2}+\phi_2\right)^4 + \left(\frac{\phi_1}{2}-\phi_2\right)^4\right)
\end{aligned}$$

where we used the Cauchy-Schwarz inequality in the second line and equation (1.11) in the last line. Changing coordinates to

$$\alpha_1 = \phi_1 + \frac{\phi_2}{2} \qquad \alpha_2 = \frac{\phi_1}{2} - \phi_2$$

and realizing that $|\alpha_1|, |\alpha_2| < 3\epsilon/2$ for $(\phi_1, \phi_2) \in D_\epsilon$, we see that

$$\begin{aligned}
\langle g_1(t)\,|g_1(t)\rangle &\leq \frac{16t^2}{L^2}\int_{-3\epsilon/2}^{3\epsilon/2} \frac{d\alpha_1}{2\pi} \int_{-3\epsilon/2}^{3\epsilon/2} \frac{d\alpha_1}{2\pi} \frac{\sin^2(\frac{1}{2}L\alpha_1)}{\sin^2(\frac{1}{2}\alpha_1)} \frac{\sin^2(\frac{1}{2}L\alpha_2)}{\sin^2(\frac{1}{2}\alpha_2)} \left(\alpha_1^4 + \alpha_2^4\right) \\
&= \frac{32t^2}{L^2}\int_{-3\epsilon/2}^{3\epsilon/2} \frac{d\alpha_1}{2\pi} \int_{-3\epsilon/2}^{3\epsilon/2} \frac{d\alpha_1}{2\pi} \frac{\sin^2(\frac{1}{2}L\alpha_1)}{\sin^2(\frac{1}{2}\alpha_1)} \frac{\sin^2(\frac{1}{2}L\alpha_2)}{\sin^2(\frac{1}{2}\alpha_2)} \alpha_1^4 \\
&\leq \frac{32t^2}{L}\int_{-3\epsilon/2}^{3\epsilon/2} \frac{d\alpha_1}{2\pi} \frac{\sin^2(\frac{1}{2}L\alpha_1)}{\sin^2(\frac{1}{2}\alpha_1)} \alpha_1^4 \\
&\leq \frac{32t^2}{L}\int_{-3\epsilon/2}^{3\epsilon/2} \frac{d\alpha_1}{2\pi} \frac{\pi^2}{\alpha_1^2} \alpha_1^4 \\
&= \frac{36\pi t^2 \epsilon^3}{L},
\end{aligned}$$

with a similar bound on $\langle g_2(t)|g_2(t)\rangle$.

Finally,

$$\langle h(t)|h(t)\rangle \leq \frac{1}{2}\iint_{D_\epsilon} \frac{d\phi_1 d\phi_2}{4\pi^2} |A(\phi_1,\phi_2)|^2 \left| e^{i\theta_\pm(\frac{\pi}{4}+\phi_1, \frac{3\pi}{8}+\phi_2)} - e^{i\theta}\right|^2.$$

Recall that $e^{i\theta_\pm(p_1,p_2)} = T(p_1,p_2) \pm R(p_1,p_2)$ is obtained by solving for the effective single-particle S-matrix for the Hamiltonian (**??**). For p_1 near $\pi/4$ we divide this Hamiltonian by $2\cos(p_1/2)$ to put it in the form considered in [**?**], where the potential term is now $\mathcal{V}(|r|)/(2\cos(p_1/2))$. The entries $T(p_1,p_2)$ and $R(p_1,p_2)$ of this S-matrix are bounded rational functions of $z = e^{ip_2}$ and $(2\cos(p_1/2))^{-1}$ [**?**], so they are differentiable as a function of p_1 and

p_2 on some neighborhood U of $(\pi/4, 3\pi/8)$ (and have bounded partial derivatives on this neighborhood).

For ϵ small enough that $D_\epsilon \subset U$ we get, using the mean value theorem and the fact that $\theta = \theta_\pm(\pi/4, 3\pi/8)$,

$$\begin{aligned}\left|e^{i\theta_\pm(\frac{\pi}{4}+\phi_1,\frac{3\pi}{8}+\phi_2)} - e^{i\theta}\right| &\leq \sqrt{\phi_1^2+\phi_2^2}\max_U\left|\vec{\nabla}e^{i\theta_\pm}\right| \quad \text{for } (\phi_1,\phi_2)\in D_\epsilon \\ &\leq \epsilon\Gamma\end{aligned}$$

for some constant Γ (independent of L). Therefore

$$\begin{aligned}\langle h(t)|h(t)\rangle &\leq \frac{1}{2}\iint_{D_\epsilon}\frac{d\phi_1 d\phi_2}{4\pi^2}\left|A(\phi_1,\phi_2)\right|^2\epsilon^2\Gamma^2 \\ &\leq \frac{1}{2}\Gamma^2\epsilon^2.\end{aligned}$$

Putting these bounds together, we get

$$\begin{aligned}\|P_1|\psi_A(t)\rangle - P_1|\alpha(t)\rangle\| &\leq \||e_1(t)\rangle\| + \||e_2(t)\rangle\| + \||f_1(t)\rangle\| + \||f_2(t)\rangle\| \\ &\quad + \||g_1(t)\rangle\| + \||g_2(t)\rangle\| + \||h(t)\rangle\| \\ &\leq 2\left(\frac{80\pi^3}{L\epsilon}+20\epsilon^2\right)^{\frac{1}{2}} + 2\left(\frac{2\pi}{L\epsilon}\right)^{\frac{1}{2}} + 2\left(\frac{36\pi t^2\epsilon^3}{L}\right)^{\frac{1}{2}} + \frac{1}{\sqrt{2}}\Gamma\epsilon.\end{aligned}$$

Letting $\epsilon = a/\sqrt{L}$ and $t \leq c_0 L$ we get

$$\|\, P_1|\psi_A(t)\rangle - P_1|\alpha(t)\rangle\| = \emptyset(L^{-1/4}). \tag{1.8}$$

Since $P_2|\,\alpha(t)\,\rangle$ has support on at most $4CL$ basis states $|x,y\rangle$, and since $|\langle x,y|P_2|\alpha(t)\rangle|^2 = \emptyset(L^{-2})$, we get

$$\|P_2|\,\alpha(t)\,\rangle\| = \emptyset(L^{-1/2}). \tag{1.9}$$

We now use the bounds (1.8) and (1.9) and Lemma 2 to show that

$$\||\psi_A(t)\rangle - |\alpha(t)\rangle\| = \emptyset(L^{-1/8}). \tag{1.10}$$

First consider the case where the interaction range is $C = 0$ (as in the Bose-Hubbard model). In this case equation (1.10) follows directly from equation (1.8) and the facts that $\langle x,y|\alpha(t)\rangle = \pm\langle y,x|\alpha(t)\rangle$ and $\langle x,y|\psi_A(t)\rangle = \pm\langle y,x|\psi_A(t)\rangle$.

Now suppose $C \neq 0$. In this case

$$\begin{aligned}\|(1-P_2)\,|\psi_A(t)\rangle\|^2 &= 2\,\|P_1|\psi_A(t)\rangle\|^2 \\ &= 2\left(\|P_1|\alpha(t)\rangle\| + \emptyset(L^{-1/4})\right)^2 \\ &= 2\left(\frac{1}{2}\|(1-P_2)|\alpha(t)\rangle\|^2 + \emptyset(L^{-1/4})\right) \\ &= 1 + \emptyset(L^{-1}) - \langle\alpha(t)|P_2|\alpha(t)\rangle + \emptyset(L^{-1/4}) \\ &= 1 + \emptyset(L^{-1/4})\end{aligned}$$

where in the next-to-last line we have used Lemma 2. So

$$\begin{aligned}
\| |\psi_A(t)\rangle - |\alpha(t)\rangle \| &\leq 2 \left\| P_1 |\psi_A(t)\rangle - P_1 |\alpha(t)\rangle \right\| + \| P_2 |\alpha(t)\rangle \| + \| P_2 |\psi_A(t)\rangle \| \\
&= \emptyset(L^{-1/4}) + \emptyset(L^{-1/2}) + \left(1 - \| (1-P_2) \, |\psi_A(t)\rangle \| \right)^{\frac{1}{2}} \\
&= \emptyset(L^{-1/4}) + \emptyset(L^{-1/2}) + \emptyset(L^{-1/8}) \\
&= \emptyset(L^{-1/8})
\end{aligned}$$

which completes the proof. □

□

1.2.1.1 Technical lemmas

In this section we prove three lemmas that are used in the proof of Theorem 1.

Lemma 2. *Let $|\alpha(t)\rangle$ be defined as in Theorem 1. Then*

$$\langle \alpha(t) | \alpha(t) \rangle = 1 + \emptyset(L^{-1}).$$

Proof. Define

$$\Pi = \sum_{x \leq y} |x, y\rangle \langle x, y|.$$

Note that, since $\langle x, y | \alpha(t) \rangle = \pm \langle y, x | \alpha(t) \rangle$,

$$\begin{aligned}
\langle \alpha(t) | \alpha(t) \rangle &= 2 \langle \alpha(t) | \Pi | \alpha(t) \rangle - \sum_{x=-\infty}^{\infty} \langle \alpha(t) | x, x \rangle \langle x, x | \alpha(t) \rangle \\
&= 2 \langle \alpha(t) | \Pi | \alpha(t) \rangle + \emptyset(L^{-1})
\end{aligned}$$

where the last line follows since $|\langle x, x | \alpha(t) \rangle|^2$ is nonzero for at most L values of x and $|\langle x, x | \alpha(t) \rangle|^2 = \emptyset(L^{-2})$. We now show that

$$\langle \alpha(t) | \Pi | \alpha(t) \rangle = \frac{1}{2} + O(L^{-1}).$$

Note that

$$\begin{aligned}
\langle \alpha(t) | \Pi | \alpha(t) \rangle = \frac{1}{2L^2} \sum_{x \leq y} \Bigg(& F(x, y, t) + F(y, x, t) \\
& \pm e^{i\theta} e^{\frac{3i\pi}{4} x} e^{-\frac{3i\pi}{4} y} F(x, y, t) F(y, x, t) \\
& \pm e^{-i\theta} e^{-\frac{3i\pi}{4} x} e^{\frac{3i\pi}{4} y} F(x, y, t) F(y, x, t) \Bigg).
\end{aligned}$$

Now $F(x, y, t) = 1$ if and only if $x \in \{-M - L + 2\lfloor t \rfloor, \ldots, -M - 1 + 2\lfloor t \rfloor\}$ and $y \in \{M + 1 - 2\lfloor t/\sqrt{2} \rfloor, \ldots, M + L - 2\lfloor t/\sqrt{2} \rfloor\}$. Similarly $F(y, x, t) = 1$ if and only if $x \in \{M + 1 - 2\lfloor t/\sqrt{2} \rfloor, \ldots, M + L - 2\lfloor t/\sqrt{2} \rfloor\}$ and $y \in \{-M - L + 2\lfloor t \rfloor, \ldots, -M - 1 + 2\lfloor t \rfloor\}$. So

$$\sum_{x \leq y} F(y, x, t) = \sum_{y \leq x} F(x, y, t)$$

and

$$\frac{1}{2L^2}\sum_{x\le y}\left[F(x,y,t)+F(y,x,t)\right] = \frac{1}{2L^2}\left(\sum_{x=-\infty}^{\infty}\sum_{y=-\infty}^{\infty}F(x,y,t) - \sum_{x=-\infty}^{\infty}F(x,x,t)\right)$$
$$= \frac{1}{2} + \emptyset(L^{-1}).$$

We now establish the bound

$$\left|\frac{1}{2L^2}\sum_{x\le y} e^{\frac{3i\pi}{4}x}e^{-\frac{3i\pi}{4}y}F(x,y,t)F(y,x,t)\right| = \emptyset(L^{-1})$$

to complete the proof. To get this bound, note that both $F(x,y,t)=1$ and $F(y,x,t)=1$ if and only if

$$x,y \in \{-M-L+2\lfloor t\rfloor, \ldots, -M-1+2\lfloor t\rfloor\}$$
$$\text{and } x,y \in \left\{M+1-2\left\lfloor\frac{t}{\sqrt{2}}\right\rfloor, \ldots, M+L-2\left\lfloor\frac{t}{\sqrt{2}}\right\rfloor\right\}.$$

Letting

$$B = \{-M-L+2\lfloor t\rfloor, \ldots, -M-1+2\lfloor t\rfloor\} \cap \left\{M+1-2\left\lfloor\frac{t}{\sqrt{2}}\right\rfloor, \ldots, M+L-2\left\lfloor\frac{t}{\sqrt{2}}\right\rfloor\right\},$$

we have

$$B = \{j, j+1, \ldots, j+l\}$$

for some $j,l\in\mathbb{Z}$ with $l<L$. So

$$\begin{aligned}
\frac{1}{2L^2}\left|\sum_{x\le y} e^{\frac{3i\pi}{4}x}e^{-\frac{3i\pi}{4}y}F(x,y,t)F(y,x,t)\right| &= \frac{1}{2L^2}\left|\sum_{x,y\in B,\, x\le y} e^{\frac{3i\pi}{4}x}e^{-\frac{3i\pi}{4}y}\right| \\
&= \frac{1}{2L^2}\left|\sum_{y=j}^{j+l}\sum_{x=j}^{y} e^{\frac{3i\pi}{4}x}e^{-\frac{3i\pi}{4}y}\right| \\
&= \frac{1}{2L^2}\left|\sum_{y=j}^{j+l} e^{-\frac{3i\pi}{4}y}e^{3i\frac{\pi}{4}j}\frac{e^{3i\frac{\pi}{4}(y+1-j)}-1}{e^{3i\frac{\pi}{4}}-1}\right| \\
&\le \frac{(l+1)}{2L^2}\frac{2}{\left|e^{3i\frac{\pi}{4}}-1\right|} \\
&= \emptyset(L^{-1})
\end{aligned}$$

since $l<L$. □

Lemma 3. *Let* $k\in(-\pi,0)\cup(0,\pi)$ *and* $0<\epsilon<\min\{\pi-|k|,|k|\}$. *Let*

$$D_\epsilon = [-\epsilon,\epsilon]\times[-\epsilon,\epsilon]$$
$$D_\pi = [-\pi,\pi]\times[-\pi,\pi].$$

Then

$$\iint_{D_\pi} \frac{d\phi_1 d\phi_2}{4\pi^2} |A(\phi_1,\phi_2)|^2 = 1$$
$$\iint_{D_\pi \setminus D_\epsilon} \frac{d\phi_1 d\phi_2}{4\pi^2} |A(\phi_1,\phi_2)|^2 \leq \frac{4\pi}{L\epsilon}$$
$$\iint_{D_\epsilon} \frac{d\phi_1 d\phi_2}{4\pi^2} |B(\phi_1,\phi_2,k)|^2 \leq \frac{4\pi^2}{L^2\epsilon^2}.$$

where $A(\phi_1,\phi_2)$ and $B(\phi_1,\phi_2,k)$ are given by equation (1.6).

Proof. Using equation (1.6) we get

$$|A(\phi_1,\phi_2)|^2 = \frac{1}{L^2} \sum_{x,\tilde{x}=-(M+L)}^{-(M+1)} \sum_{y,\tilde{y}=M+1}^{M+L} e^{i\frac{\phi_1}{2}(x+y-(\tilde{x}+\tilde{y}))} e^{i\phi_2(x-y-(\tilde{x}-\tilde{y}))}.$$

Now

$$\int_{-\pi}^{\pi} \frac{d\phi_2}{2\pi} e^{i\phi_2(x-y-\tilde{x}+\tilde{y})} = \delta_{x-y,\tilde{x}-\tilde{y}},$$

so (suppressing the limits of summation for readability)

$$\begin{aligned}
\iint_{D_\pi} \frac{d\phi_1 d\phi_2}{4\pi^2} |A(\phi_1,\phi_2)|^2 &= \frac{1}{L^2} \int_{-\pi}^{\pi} \frac{d\phi_1}{2\pi} \sum_{x,\tilde{x}} \sum_{y,\tilde{y}} e^{i\phi_1(y-\tilde{y})} \delta_{x-y,\tilde{x}-\tilde{y}} \\
&= \frac{1}{L^2} \sum_{x,\tilde{x}} \sum_{y,\tilde{y}} \delta_{y,\tilde{y}} \delta_{x-y,\tilde{x}-\tilde{y}} \\
&= 1
\end{aligned}$$

which proves the first part.

By performing the sums in equation (1.6) we get

$$|A(\phi_1,\phi_2)|^2 = \frac{1}{L^2} \frac{\sin^2(\frac{1}{2}L[\frac{\phi_1}{2}+\phi_2])}{\sin^2(\frac{1}{2}[\frac{\phi_1}{2}+\phi_2])} \frac{\sin^2(\frac{1}{2}L[\frac{\phi_1}{2}-\phi_2])}{\sin^2(\frac{1}{2}[\frac{\phi_1}{2}-\phi_2])}. \tag{1.11}$$

Letting $\alpha_1 = \phi_1/2 + \phi_2$ and $\alpha_2 = \phi_1/2 - \phi_2$, we see that $|\alpha_1| \leq 3\pi/2$, $|\alpha_2| \leq 3\pi/2$, and $\alpha_1^2 + \alpha_2^2 \geq 5\epsilon^2/2$ whenever $(\phi_1,\phi_2) \in D_\pi \setminus D_\epsilon$. Defining $D_{3\pi/2} = [-3\pi/2, 3\pi/2]^2$ we get

$(\alpha_1, \alpha_2) \in D_{3\pi/2} \setminus D_\epsilon$ whenever $(\phi_1, \phi_2) \in D_\pi \setminus D_\epsilon$. Hence

$$\begin{aligned}
\iint_{D_\pi \setminus D_\epsilon} \frac{d\phi_1 d\phi_2}{4\pi^2} |A(\phi_1, \phi_2)|^2 &\leq \frac{1}{L^2} \iint_{D_{3\pi/2} \setminus D_\epsilon} \frac{d\alpha_1 d\alpha_2}{4\pi^2} \frac{\sin^2(\frac{1}{2}L\alpha_1)}{\sin^2(\frac{1}{2}\alpha_1)} \frac{\sin^2(\frac{1}{2}L\alpha_2)}{\sin^2(\frac{1}{2}\alpha_2)} \\
&\leq \frac{4}{L} \left(\frac{1}{L} \int_{-\frac{3\pi}{2}}^{\frac{3\pi}{2}} \frac{d\alpha_1}{2\pi} \frac{\sin^2(\frac{1}{2}L\alpha_1)}{\sin^2(\frac{1}{2}\alpha_1)} \right) \left(\int_\epsilon^{3\pi/2} \frac{d\alpha_2}{2\pi} \frac{\sin^2(\frac{1}{2}L\alpha_2)}{\sin^2(\frac{1}{2}\alpha_2)} \right) \\
&\leq \frac{4}{L} \left(\int_{-2\pi}^{2\pi} \frac{d\alpha_1}{2\pi} \frac{1}{L} \frac{\sin^2(\frac{1}{2}L\alpha_1)}{\sin^2(\frac{1}{2}\alpha_1)} \right) \left(\int_\epsilon^{\frac{3\pi}{2}} \frac{d\alpha_2}{2\pi} \frac{1}{\sin^2(\frac{1}{2}\alpha_2)} \right) \\
&= \frac{8}{L} \left(\int_\epsilon^\pi \frac{d\alpha_2}{2\pi} \frac{1}{\sin^2(\frac{1}{2}\alpha_2)} + \int_\pi^{\frac{3\pi}{2}} \frac{d\alpha_2}{2\pi} \frac{1}{\sin^2(\frac{1}{2}\alpha_2)} \right) \\
&\leq \frac{8}{L} \left(\int_\epsilon^\pi \frac{d\alpha_2}{2\pi} \frac{\pi^2}{\alpha_2^2} + 2 \int_\pi^{\frac{3\pi}{2}} \frac{d\alpha_2}{2\pi} \right) \\
&= \frac{4\pi}{L\epsilon}
\end{aligned}$$

which proves the second inequality (in the next-to-last line we have used the fact that $\sin(x/2) > x/\pi$ for $x \in (0, \pi)$ and $\sin^2(x/2) > 1/2$ for $x \in (\pi, 3\pi/2)$).

Now

$$\begin{aligned}
|B(\phi_1, \phi_2, k)|^2 &= |A(\phi_1, -\phi_2 - 2k)|^2 \\
&\leq \frac{1}{L^2} \frac{1}{\sin^2\left(\frac{1}{2}\left[\frac{\phi_1}{2} + \phi_2 + 2k\right]\right)} \frac{1}{\sin^2\left(\frac{1}{2}\left[-\frac{\phi_1}{2} + \phi_2 + 2k\right]\right)}.
\end{aligned}$$

If $(\phi_1, \phi_2) \in D_\epsilon$ then $|k| - 3\epsilon/4 \leq |\pm\phi_1/4 + \phi_2/2 + k| \leq |k| + 3\epsilon/4$. Noting that ϵ is chosen such that $0 < \epsilon < \min\{\pi - |k|, |k|\}$, we get

$$\frac{\epsilon}{4} \leq \left| \pm \frac{\phi_1}{4} + \frac{\phi_2}{2} + k \right| \leq \pi - \frac{\epsilon}{4}$$

so

$$\begin{aligned}
|B(\phi_1, \phi_2, k)|^2 &\leq \frac{1}{L^2} \frac{1}{\sin^4(\frac{\epsilon}{4})} \\
&\leq \frac{16\pi^4}{L^2\epsilon^4}
\end{aligned}$$

and

$$\begin{aligned}
\iint_{D_\epsilon} \frac{d\phi_1 d\phi_2}{4\pi^2} |B(\phi_1, \phi_2, k)|^2 &\leq \frac{1}{4\pi^2} (2\epsilon)^2 \left(\frac{16\pi^4}{L^2\epsilon^4} \right) \\
&= \frac{16\pi^2}{L^2\epsilon^2}.
\end{aligned}$$

□

Lemma 4. *Let $a_{xy}(t)$ be as in Theorem 1. For $x \leq y$,*

$$a_{xy}(t) = \frac{1}{\sqrt{2}} e^{-it\sqrt{2}} \Bigg[e^{-i\pi x/2} e^{i\pi y/4} \Bigg(\iint_{D_\pi} \frac{d\phi_1 d\phi_2}{4\pi^2} A(\phi_1, \phi_2) \\ e^{-i\phi_1\left(-\lfloor t \rfloor + \lfloor \frac{t}{\sqrt{2}} \rfloor + \frac{x+y}{2}\right)} e^{-2i\phi_2\left(-\lfloor t \rfloor - \lfloor \frac{t}{\sqrt{2}} \rfloor + \frac{x-y}{2}\right)} \Bigg) \\ \pm e^{i\theta} e^{i\pi x/4} e^{-i\pi y/2} \Bigg(\iint_{D_\pi} \frac{d\phi_1 d\phi_2}{4\pi^2} A(\phi_1, \phi_2) \\ e^{-i\phi_1\left(-\lfloor t \rfloor + \lfloor \frac{t}{\sqrt{2}} \rfloor + \frac{x+y}{2}\right)} e^{-2i\phi_2\left(-\lfloor t \rfloor - \lfloor \frac{t}{\sqrt{2}} \rfloor + \frac{y-x}{2}\right)} \Bigg) \Bigg].$$

Proof. The lemma follows from (1.2) and the fact that, for any two numbers γ_1, γ_2 such that $\gamma_1 + \gamma_2, \gamma_1 - \gamma_2 \in \mathbb{Z}$,

$$\iint_{D_\pi} \frac{d\phi_1 d\phi_2}{4\pi^2} A(\phi_1, \phi_2) e^{i\gamma_1\phi_1 + 2i\gamma_2\phi_2} = \begin{cases} \frac{1}{L} & \text{if } (-\gamma_1 - \gamma_2, -\gamma_1 + \gamma_2) \in S \\ 0 & \text{otherwise} \end{cases}$$

where $S = \{-M-L, \ldots, -M-1\} \times \{M+1, \ldots, M+L\}$. To establish this formula, observe that

$$\iint_{D_\pi} \frac{d\phi_1 d\phi_2}{4\pi^2} A(\phi_1, \phi_2) e^{i\gamma_1\phi_1 + 2i\gamma_2\phi_2} = \frac{1}{L} \sum_{x=-M-L}^{-M-1} \sum_{y=M+1}^{M+L} \iint_{D_\pi} \frac{d\phi_1 d\phi_2}{4\pi^2} e^{i\phi_1\left(\gamma_1 + \frac{x+y}{2}\right)} e^{i\phi_2(x-y+2\gamma_2)} \\ = \frac{1}{L} \sum_{x=-M-L}^{-M-1} \sum_{y=M+1}^{M+L} \int_{-\pi}^{\pi} \frac{d\phi_1}{2\pi} e^{i\phi_1\left(\gamma_1 + \frac{x-y}{2}\right)} \delta_{y,-x-2\gamma_2}.$$

Here we have performed the integral over ϕ_2 using the fact that $2\gamma_2$ is an integer. We then have

$$\iint_{D_\pi} \frac{d\phi_1 d\phi_2}{4\pi^2} A(\phi_1, \phi_2) e^{i\gamma_1\phi_1 + 2i\gamma_2\phi_2} = \frac{1}{L} \sum_{x=-M-L}^{-M-1} \sum_{y=M+1}^{M+L} \int_{-\pi}^{\pi} \frac{d\phi_1}{2\pi} e^{i\phi_1(\gamma_1 + x + \gamma_2)} \delta_{y,-x-2\gamma_1} \\ = \frac{1}{L} \sum_{x=-M-L}^{-M-1} \sum_{y=M+1}^{M+L} \delta_{x,-\gamma_1-\gamma_2} \delta_{y,\gamma_2-\gamma_1}$$

as claimed. ∎

1.2.2 Construction of $C\theta$-gate

1.3 Impossibility of some momentum switches

1.4 Universal Computation

1.4.1 Two-qubit blocks

1.4.2 Combining blocks

1.5 Improvements and Modifications

What about long-range interactions, but where the interactions die off? Additionally, what about error correction?