Слюсенский Захар 3 курс 2 группа

Лабораторная работа №11

Аппроксимация, устойчивость, сходимость. Спектральный признак устойчивости Неймана

Вариант 14, т.е. 7.

1.

$$\frac{\partial u(x,t)}{\partial t} + a\frac{\partial u(x,t)}{\partial x} = \varphi(x,t)$$

Схема:

$$L_h u^{(h)} = \frac{u_m^n - u_m^{n-1}}{\tau} + a\frac{u_{m+1}^n - u_m^n}{h} = \varphi(x,t);$$

$$\delta f^{(h)} = \frac{u(x_m,t_m) - u(x_m,t_{n-1})}{\tau} + a\frac{u(x_{m+1},t_n) - u(x_m,t_n)}{h} - \varphi(x,t);$$

$$u(x_m,t_{n-1}) = u(x_m,t_n) - \frac{\tau}{1!}\frac{\partial u(x_m,t_n)}{\partial t} + \frac{\tau^2}{2!}\frac{\partial^2 u(x_m,t_n)}{\partial t^2};$$

$$u(x_{m+1},t_n) = u(x_m,t_n) + \frac{h}{1!}\frac{\partial u(x_m,t_n)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u(x_m,t_n)}{\partial x^2};$$

$$\delta f^{(h)} = \frac{u(x_m,t_n) + \frac{\tau}{1!}\frac{\partial u(x_m,t_n)}{\partial t} - \frac{\tau^2}{2!}\frac{\partial^2 u(x_m,t_n)}{\partial t^2} - u(x_m,t_n)}{\tau} - a\frac{u(x_m,t_n) - u(x_m,t_n) - \frac{h}{1!}\frac{\partial u(x_m,t_n)}{\partial x} - \frac{h^2}{2!}\frac{\partial^2 u(x_m,t_n)}{\partial x^2}}{h} - \varphi(x,t);$$

$$\delta f^{(h)} = \frac{\partial u(x_m,t_n)}{\partial t} - \frac{\tau}{2}\frac{\partial^2 u(x_m,t_n)}{\partial t^2} - a\left(\frac{\partial u(x_m,t_n)}{\partial x} + \frac{h}{2}\frac{\partial^2 u(x_m,t_n)}{\partial x^2}\right) - \varphi(x,t);$$

$$\delta f^{(h)} = \frac{\partial u(x_m,t_n)}{\partial t} - \frac{\tau}{2}\frac{\partial^2 u(x_m,t_n)}{\partial t^2} - a\frac{\partial u(x_m,t_n)}{\partial x} - a\frac{h}{2}\frac{\partial^2 u(x_m,t_n)}{\partial x^2} - \varphi(x,t);$$

$$\delta f^{(h)} = -\left(\frac{\tau}{2}\frac{\partial^2 u(x_m,t_n)}{\partial t^2} + a\frac{h}{2}\frac{\partial^2 u(x_m,t_n)}{\partial x^2}\right);$$

$$\|\delta f^{(h)}\|_{F_h} = \max\left|\frac{\tau}{2}\frac{\partial^2 u(x_m,t_n)}{\partial t^2} + a\frac{h}{2}\frac{\partial^2 u(x_m,t_n)}{\partial x^2}\right| \le \frac{\tau}{2}M_\tau + |a|\frac{h}{2}M_x;$$

2.

$$\frac{u_m^{n+1}-u_m^n}{\tau}+a\frac{u_{m+1}^n-u_{m-1}^n}{2h}-\frac{a^2\tau}{2}\frac{u_{m+1}^n-2u_m^n+u_{m-1}^n}{h^2}=0,$$

$$u_m^n=\lambda^n e^{i\alpha m},$$

$$a\frac{u_{m+1}^n-u_{m-1}^n}{2h}+\frac{u_m^{n+1}-u_m^n}{\tau}-\frac{a^2\tau}{2}\frac{u_{m+1}^n-2u_m^n+u_{m-1}^n}{h^2}=0;$$

$$u_{m+1}^n=u_{m-1}^n-\frac{2h}{\tau a}(u_m^{n+1}-u_m^n)+\frac{2a^2h\tau}{2ah^2}(u_{m+1}^n-2u_m^n+u_{m-1}^n);$$

$$r=a\frac{\tau}{h};$$

$$\lambda^n e^{i\alpha(m+1)}=\lambda^n e^{i\alpha(m-1)}-2r^{-1}(\lambda^{n+1}e^{i\alpha m}-\lambda^n e^{i\alpha m})+r(\lambda^n e^{i\alpha(m+1)}-2\lambda^n e^{i\alpha m}+$$
$$+\lambda^n e^{i\alpha(m-1)});$$

$$\lambda^n=\frac{\lambda^n e^{i\alpha(m-1)}}{e^{i\alpha(m+1)}}-\frac{2\lambda^{n+1}e^{i\alpha m}}{re^{i\alpha(m+1)}}+\frac{2\lambda^n e^{i\alpha m}}{re^{i\alpha(m+1)}}+\frac{r\lambda^n e^{i\alpha(m+1)}}{e^{i\alpha(m+1)}}-\frac{2r\lambda^n e^{i\alpha m}}{e^{i\alpha(m+1)}}+\frac{r\lambda^n e^{i\alpha(m-1)}}{e^{i\alpha(m+1)}};$$

$$1=\frac{e^{i\alpha(m-1)}}{e^{i\alpha(m+1)}}-\frac{2\lambda e^{i\alpha m}}{re^{i\alpha(m+1)}}+\frac{2e^{i\alpha m}}{re^{i\alpha(m+1)}}+\frac{re^{i\alpha(m+1)}}{e^{i\alpha(m+1)}}-\frac{2re^{i\alpha m}}{e^{i\alpha(m+1)}}+\frac{re^{i\alpha(m-1)}}{e^{i\alpha(m+1)}};$$

$$\lambda=\frac{re^{i\alpha(m+1)}e^{i\alpha(m-1)}}{2e^{i\alpha(m+1)}e^{i\alpha m}}+\frac{2e^{i\alpha m}re^{i\alpha(m+1)}}{2e^{i\alpha m}re^{i\alpha(m+1)}}+\frac{r^2e^{i\alpha(m+1)}e^{i\alpha(m+1)}}{2e^{i\alpha m}e^{i\alpha(m+1)}}-\frac{2r^2e^{i\alpha m}*}{2e^{i\alpha m}*}$$

$$\frac{*e^{i\alpha(m+1)}}{*e^{i\alpha(m+1)}}+\frac{r^2e^{i\alpha(m-1)}e^{i\alpha(m+1)}}{2e^{i\alpha m}e^{i\alpha(m+1)}}-\frac{re^{i\alpha(m+1)}}{2e^{i\alpha m}};$$

$$\lambda=\frac{re^{i\alpha(m+1)}}{2e^{i\alpha m}}+1+\frac{r^2e^{i\alpha(m+1)}}{2e^{i\alpha m}}-r^2+\frac{r^2e^{i\alpha(m-1)}}{2e^{i\alpha m}}-\frac{re^{i\alpha(m+1)}}{2e^{i\alpha m}};$$

$$\lambda=\frac{re^{i\alpha(m-1)}}{2e^{i\alpha m}}(r+1)+\frac{re^{i\alpha(m+1)}}{2e^{i\alpha m}}(r-1)+1-r^2=\frac{re^{i\alpha m}}{2e^{i\alpha}e^{i\alpha m}}(1+r)+\frac{re^{i\alpha}e^{i\alpha m}}{2e^{i\alpha m}}(r-1)+1-$$
$$-r^2=\frac{r}{2e^{i\alpha}}(1+r)+\frac{re^{i\alpha}}{2}(r-1)+1-r^2;$$

$$\lambda=\frac{r}{2e^{i\alpha}}(1+r)+\frac{re^{i\alpha}}{2}(r-1)+1-r^2=\frac{r^2e^{-i\alpha}}{2}+\frac{re^{-i\alpha}}{2}+\frac{r^2e^{i\alpha}}{2}-\frac{re^{i\alpha}}{2}+1-r^2;$$

$$\lambda=r^2\cos(\alpha)+ri\sin(\alpha)+1-r^2=1-2r^2\sin^2(\tfrac{\alpha}{2})-ri\sin(\alpha);$$

$$|\lambda|^2=(1-2r^2\sin^2(\tfrac{\alpha}{2}))^2+r^2\sin^2(\alpha)=1+4r^2\sin^4(\tfrac{\alpha}{2})(r^2-1);$$

Условие $|\lambda|\le 1 \Leftrightarrow 1-|\lambda|^2\ge 0$ выполнено, если $r\le 1$. Оно не выполнено, т.е. схема не устойчива, если $r>1$.