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# Singular Integrals
# and Differentiability Properties
# of Functions

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To Elly

# Preface

This book is an outgrowth of a course which I gave at Orsay during the academic year 1966-67.* My purpose in those lectures was to present some of the required background and at the same time clarify the essential unity that exists between several related areas of analysis. These areas are: the existence and boundedness of singular integral operators; the boundary behavior of harmonic functions; and differentiability properties of functions of several variables. As such the common core of these topics may be said to represent one of the central developments in n-dimensional Fourier analysis during the last twenty years, and it can be expected to have equal influence in the future. These possibilities are further highlighted by listing some of the fields (which are not treated here) where either the particular results of this book, or ideas closely related to the techniques presented here are continuing to find significant application. These include partial differential equations, holomorphic functions of several complex variables, and analysis on other groups, either non-commutative or commutative.

In this connection we should point out that the book *Introduction to Fourier Analysis on Euclidean Spaces*** details some of these applications, as well as much background and related material. It may therefore not be inappropriate to view the present volume as a companion to *Fourier Analysis*. Both books, however, may be read independently. In fact an effort has been made to make the present volume essentially self-contained, requiring only elementary facts from integration theory and Fourier transforms as prerequisites.

A brief description of the organization of the book is as follows. The first three chapters deal primarily with material which, for the most part, is beginning to find its way into several advanced texts and monographs, namely covering lemmas and maximal functions, the Marcinkiewicz interpolation theorem, singular integrals generalizing the Hilbert transform, and harmonic functions represented as Poisson integrals. In the last five chapters the topics are of a more advanced nature, including the Littlewood-Paley theory, multipliers, Sobolov spaces and their variants, extension theorems, further results about harmonic functions, and almost-everywhere differentiability theorems. Here part of the material is systematically organized for the first time, and, for example, the last two chapters contain several results whose details were hitherto unpublished.

* For the published lecture notes of this course see Stein [10].

** This work of G. Weiss and the author is referred to as *Fourier Analysis* in the rest of the text.

In any enterprise of this kind the author is faced with the task of balancing two aims which unfortunately are not always compatible. On the one hand there is the desire to facilitate the task of the serious student by providing all the background material and by presenting proofs in a way so that all details, no matter how unenlightening, are fully given. On the other hand there is the need to get on with the essential job of developing the basic ideas of the subject. In doing the latter it is sometimes best to be brief about certain technical details, and also at times to forego the urge to pursue various possible generalizations which could be formulated. Others may surely find fault with how I have weighed these alternatives. My justification would be based either on the ground of personal predilection (which allows no argument) or, in a more serious vein, in terms of my view of the present subject: that it has advanced to a high degree of sophistication and is still rapidly developing, but has not yet reached the level of maturity that would require it to be enshrined in a edifice of great perfection.

It is my pleasant task to acknowledge with gratitude those who have helped me in writing this book: Norman Weiss, who prepared the lecture notes (unpublished) of a course given at Princeton University in 1964-65, where an earlier version of some of this material was presented; Messrs. Bachvan and A. Somen who wrote the published lecture notes already alluded to; Misses Elizabeth Epstein and Florence Armstrong who typed the bulk of the manuscript; and Messrs. W. Beckner, C. Fefferman, and S. Gelbart who helped both mathematically and in proofreading. To all those and others unnamed, I express my thanks.

*September 1970* E. M. STEIN

# Notation

## Principal Symbols

$dx$—Lebesgue measure on $\mathbf{R}^n$; also $m(E)$—measure of the set $E$
$L^p(\mathbf{R}^n)$—the $L^p$ space with respect to the measure $dx$
$C^k$—the class of functions which have continuous derivatives up to and including total order $k$
$\mathscr{D}$—the space of indefinitely differentiable functions with compact support
$\mathscr{S}$—the space of indefinitely differential functions all of whose derivatives remain bounded when multiplied by any polynomial
${}^cE$—complement of the set $E$

*The symbols that follow are listed according to their first and other principal occurrence.*

*Chapter I, §1*
$B(x, r)$—ball of radius $r$ centered at $x$
$M(f)$—maximal function

§2
$\delta(x) = \delta(x, F)$—distance of $x$ from the set $F$
$I(x)$, $I_*(x)$—integrals of Marcinkiewicz involving the distance function (See also Chapter VIII, §3.)

§3
$Q_1, \ldots, Q_k, \ldots$—cubes; also $\Omega = \bigcup_k Q_k$

*Chapter II, §1*
$C_0(\mathbf{R}^n)$, $\mathscr{B}(\mathbf{R}^n)$—continuous functions on $\mathbf{R}^n$ vanishing at infinity and the dual space of finite Borel measures
$\mu = \mu_1 * \mu_2$—the convolution of measures $\mu_1$ and $\mu_2$
$\hat{f}(y) = \mathscr{F}(f)(y)$—Fourier transform of $f$

§3

$H(f)$—Hilbert transform. (See also Chapter III, §1.)

§4

$S^{n-1}$, $d\sigma$—the unit sphere in $\mathbf{R}^n$ and its induced element of volume

§5

$L^p(\mathbf{R}^n, \mathscr{H})$—$L^p$ space of functions which take their values in $\mathscr{H}$

*Chapter III, §1*

$R_j$—Riesz transforms

$c_n = \Gamma\left(\frac{n+1}{2}\right) \Big/ \pi^{(n+1)/2}$

§2

$\mathbf{R}^{n+1}_+$—the upper half-space $\{(x, y) \in \mathbf{R}^{n+1} : x \in \mathbf{R}^n, y > 0\}$

$P_y(x) = \dfrac{c_n y}{(|x|^2 + y^2)^{(n+1)/2}}$—Poisson kernel

$\Delta$—Laplacean (see also Chapter VII), where it is used in the context of $\mathbf{R}^{n+1}_+$, or in Chapter III, §3, where it is used in the context of $\mathbf{R}^n$

§3

$\mathscr{H}_k$—solid spherical harmonics of degree $k$

$H_k$—surface spherical harmonics of degree $k$

*Chapter IV, §1*

$g$, $g_1$, $g_x$, $g_k$—variants of the Littlewood-Paley $g$ functions

§2

$g^*_\lambda$—another variant. (See also Chapter VII, §3.)

$\Gamma$—cone $\{(x, y) : x \in \mathbf{R}^n, |x| < y\}$ (See also Chapter VII, §1.)

$S$—area integral of Lusin. (See also Chapter VII, §§2 and 3.)

$M_\mu(f) = (M(f^\mu))^{1/\mu}$, $\mu \geq 1$

§3

$T_m$—multiplier transformation with multiplier $m$

$\mathscr{M}_p$—algebra of $L^p$ multipliers

§4

$S_\rho(f)$—"partial sum" operator

$S_{\mathfrak{R}}(f)$—analogue of above corresponding to a family of rectangles

§5

$r_m$—Rademacher function. (See also Appendix D.)

*Chapter V, §1*
$I_\alpha$—Riesz potential

§2
$L_k^p(\mathbf{R}^n)$—Sobolov space

§3
$\mathcal{T}_\alpha(f) = G_\alpha(f)$—Bessel potential
$\mathcal{L}_\alpha^p(\mathbf{R}^n)$—space of Bessel potentials
$\omega_p(t)$—$L^p$ modulus of continuity
$\tilde{\omega}_p(t)$—second-order $L^p$ modulus of continuity

§4
$\Lambda_\alpha(\mathbf{R}^n)$—Lipschitz space

§5
$\Lambda_\alpha^{p,q}(\mathbf{R}^n)$—Besov space

*Chapter VI, §1*
$\Delta(x)$—regularized distance

§2
$\mathscr{E}_k$—Whitney extension operators

§3
$L_k^p(D)$—Sobolov space for the domain $D$
$\mathfrak{E}$—extension operator for the domain $D$

*Chapter VII, §1*
$\Gamma_\alpha(x^0)$—cone $\{(x, y) \in \mathbf{R}_+^{n+1}, |x - x^0| < \alpha y\}$
$\Gamma_\alpha^h$—truncated cone, $\Gamma_\alpha^h(x^0) = \Gamma_\alpha(x^0) \cap \{0 < y < h\}$
$\mathscr{R} = \bigcup_{x^0 \in E} \Gamma_\alpha^h(x^0)$. (See also Chapter VIII, §2.)

§3
$H^p$ space—space of conjugate harmonic functions satisfying an appropriate $L^p$ inequality
$\mathfrak{S}_q$—variant of the area integral

# Contents

# Singular Integrals
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# CHAPTER I

# Some Fundamental Notions of Real-Variable Theory

The basic ideas of the theory of real variables are connected with the concepts of sets and functions, together with the processes of integration and differentiation applied to them. While the essential aspects of these ideas were brought to light in the early part of our century, some of their further applications were developed only more recently. It is from this latter perspective that we shall approach that part of the theory that interests us. In doing so, we distinguish several main features:

(1) The theorem of Lebesgue about the *differentiation of the integral.* The study of properties related to this process is best done in terms of a "maximal function" to which it gives rise; the basic features of the latter are expressed in terms of a "weak-type" inequality which is characteristic of this situation.

(2) Certain *covering* lemmas. In general the idea is to cover an arbitrary (open) set in terms of a disjoint union of cubes or balls, chosen in a manner depending on the problem at hand. One such example is a lemma of Whitney, (Theorem 3). Sometimes, however, it suffices to cover only a portion of the set, as in the simple covering lemma, which is used to prove the weak-type inequality mentioned above.

(3) *Behavior near a "general" point* of an arbitrary set. The simplest notion here is that of point of density. More refined properties are best expressed in terms of certain integrals first studied systematically by Marcinkiewicz.

(4) The *splitting of functions* into their large and small parts. This feature which is more of a technique than an end in itself, recurs often. It is especially useful in proving $L^p$ inequalities, as in the first theorem of this chapter. That part of the proof of the first theorem is systematized in the Marcinkiewicz interpolation theorem discussed in §4 of this chapter and also in Appendix B.

## 1. *The maximal function*

**1.1** According to the fundamental theorem of Lebesgue, the relation

(1) $$\lim_{r\to 0} \frac{1}{m(B(x, r))} \int_{B(x,r)} f(y)\, dy = f(x)$$

holds for almost every $x$, whenever $f$ is locally integrable function defined on $\mathbf{R}^n$. The notation here used is that $B(x, r)$ is the ball of radius $r$, centered at $x$, and $m(B(x, r))$ denotes its measure. In order to study the limit (1) we consider its quantitative analogue, where "$\lim_{r\to 0}$" is replaced by "$\sup_{r>0}$"; this is the *maximal function*, $Mf$. Since the properties of this function are expressed in terms of relative size and do not involve any cancellation of positive and negative values, we replace $f$ by $|f|$. Thus we define

(2) $$M(f)(x) = \sup_{r>0} \frac{1}{m(B(x, r))} \int_{B(x,r)} |f(y)|\, dy$$

It is to be noticed that nothing excludes the possibility that $(Mf)(x)$ is infinite for any given $x$.

The passage from a limiting expression to a corresponding maximal function is a situation that recurs often. Our first example here, (2), will turn out to be the most fundamental one.

**1.2** We shall now be interested in giving a concise expression for the relative size of a function. Thus let $g(x)$ be defined on $\mathbf{R}^n$ and for each $\alpha$ consider the set where $|g|$ is greater than $\alpha$,

$$\{x : |g(x)| > \alpha\}.$$

The function $\lambda(\alpha)$, defined to be the measure of this set, is the sought-for expression. It is the *distribution function* of $|g|$.

In particular, the decrease of $\lambda(\alpha)$ as $\alpha$ grows describes the relative largeness of the function; this is the main concern locally. The increase of $\lambda(\alpha)$ as $\alpha$ tends to zero describes the relative smallness of the function "at infinity"; this is its importance globally, and is of no interest if, for example, the function is supported on a bounded set.

Any quantity dealing solely with the size of $g$ can be expressed in terms of the distribution function $\lambda(\alpha)$. For example, if $g \in L^p$, then

$$\int_{\mathbf{R}^n} |g(y)|^p\, dy = -\int_0^\infty \alpha^p\, d\lambda(\alpha)$$

and if $g \in L^\infty$, then

$$\|g\|_\infty = \inf\{\alpha, \lambda(\alpha) = 0\}.$$

A related fact concerning the distribution function will have immediate application. It is this: If $g$ is integrable, then

$$\lambda(\alpha) \leq A/\alpha \quad \text{where} \quad A = \int_{\mathbf{R}^n} |g(y)|\, dy.$$

In fact

$$\int_{\mathbf{R}^n} |g(y)|\, dy \geq \int_{|g|>\alpha} |g(y)|\, dy \geq \alpha\lambda(\alpha),$$

which proves the assertion.

**1.3** With these definitions we can state our first theorem. It gives the main results for the maximal function, and has as a corollary the differentiability almost everywhere of the integral, expressed in (1).

THEOREM 1. *Let $f$ be a given function defined on $\mathbf{R}^n$*

(a) *If $f \in L^p(\mathbf{R}^n)$, $1 \leq p \leq \infty$, then the function $Mf$ is finite almost everywhere.*

(b) *If $f \in L^1(\mathbf{R}^n)$, then for every $\alpha > 0$*

$$m\{x : (Mf)(x) > \alpha\} \leq \frac{A}{\alpha} \int_{\mathbf{R}^n} |f|\, dx,$$

*where $A$ is a constant which depends only on the dimension $n$ ($A = 5^n$ will do)*

(c) *If $f \in L^p(\mathbf{R}^n)$, with $1 < p \leq \infty$, then $Mf \in L^p(\mathbf{R}^n)$ and*

$$\|Mf\|_p \leq A_p \|f\|_p,$$

*where $A_p$ depends only $p$ and the dimension $n$.*

COROLLARY 1. *If $f \in L^p(\mathbf{R}^n)$, $1 \leq p \leq \infty$, or more generally if $f$ is locally integrable, then*

$$\lim_{r \to 0} \frac{1}{B(x, r)} \int_{B(x,r)} f(y)\, dy = f(x),$$

*for almost every $x$.*

**1.4** Before we come to the proof of the theorem we make some clarifying comments.

(i) In contrast with the case $p > 1$, when $p = 1$ the mapping $f \to M(f)$ is not bounded on $L^1(\mathbf{R}^n)$. Thus if $f$ is not identically zero $Mf$ is *never* integrable on all of $\mathbf{R}^n$. This can be seen by making the simple observation that $Mf(x) \geq c|x|^{-n}$, for $|x| \geq 1$. Moreover even if we limit our considerations to any bounded subset of $\mathbf{R}^n$, then the integrability of $Mf$ holds only if stronger conditions than the integrability of $f$ are required. (See §5.2 below.)

(ii) The result that is obtained, namely estimate (b), is weaker than the statement that $f \to M(f)$ is bounded on $L^1(\mathbf{R}^n)$, as the remarks in §1.2 show; for this reason (b) is referred to as a *weak-type* estimate. This estimate is the best possible (as far as order of magnitude) for the distribution function of $M(f)$, where $f$ is an arbitrary function in $L^1(\mathbf{R}^n)$. That this is so can be seen by replacing the measure $|f(y)|\,dy$ in definition (2) by the measure $d\mu$, whose total measure of one is concentrated at the origin: ($d\mu$ is the "Dirac measure"). Then $M(d\mu)(x) = c|x|^{-n}$, where $c^{-1}$ = volume of the unit ball. In this case the distribution function $\lambda(\alpha)$ is exactly $1/\alpha$. But we can always find a sequence $\{f_m(x)\}$ of positive integrable functions, whose $L^1$ norm is each one, and which converge weakly to the measure $d\mu$. So we cannot expect an estimate essentially stronger than (b), since in the limit a similar stronger version would have to hold for $M(d\mu)(x)$.

**1.5 Proof of Theorem 1 and its corollary.** Here we shall prove the theorem and its corollary, taking for granted the covering lemma of "Vitali-type" stated in §1.6 and proved in §1.7 below. With the definition of $Mf$, and with

$$E_\alpha = \{x : Mf(x) > \alpha\}$$

then for each $x \in E_\alpha$ there exists a ball of center $x$, which we call $B_x$, so that

$$\int_{B_x} |f(y)|\,dy > \alpha m(B_x). \tag{3}$$

But on the one hand (3) gives $m(B_x) < (1/\alpha)\,\|f\|_1$, for all such $x$; on the other hand when $x$ runs through the set $E_\alpha$ the union of the corresponding $B_x$ covers $E_\alpha$. Thus using the covering lemma (1.6) below from this family of balls we can extract a sequence of balls, which we designate by $\{B_k\}$; these balls are mutually disjoint and have the property that

$$\sum_{k=0}^{\infty} m(B_k) \geq Cm(E_\alpha), \tag{4}$$

(e.g. the bound $C = 5^{-n}$ will work). Applying (3) and then (4) to each of the mutually disjoint balls we get

$$\int_{\cup B_k} |f(y)|\,dy > \alpha \sum_k m(B_k) \geq \alpha C m(E_\alpha).$$

But since the first member of this inequality is majorized by $\|f\|_1$, on taking $A = 1/C$ we obtain the assertion (b) of the theorem; (and thus also part (a), when $p = 1$). We shall now prove simultaneously assertion (a) (the finiteness almost everywhere of $M(f)(x)$), and assertion (c) (the $L^p$ inequality), for $1 < p \leq \infty$. The case $p = \infty$ is of course trivial, and here the bound is $A_\infty = 1$. Let us therefore suppose that $1 < p < \infty$. We shall use a simple example of the technique of splitting a function into its large and small parts, alluded to at the beginning of this chapter. Let us define $f_1(x)$ by $f_1(x) = f(x)$, if $|f(x)| \geq \alpha/2$, and $f_1(x) = 0$ otherwise. Then we have successively $|f(x)| \leq |f_1(x)| + \alpha/2$; $M(f)(x) \leq M(f_1)(x) + \alpha/2$, therefore

$$\{x : M(f)(x) > \alpha\} \subset \{x : M(f_1)(x) > \alpha/2\},$$

and finally

$$m(E_\alpha) = m\{x : Mf(x) > \alpha\} \leq \frac{2A}{\alpha} \|f_1\|_1,$$

which is

$$m(E_\alpha) = m\{x : Mf(x) > \alpha\} \leq \frac{2A}{\alpha} \int_{|f| > \alpha/2} |f| \, dx. \tag{5}$$

The last inequality is obtained by applying conclusion (b) of the theorem which we may since $f_1 \in L^1$ whenever $f \in L^p$. We now set $g = M(f)$, and $\lambda$ the distribution function of $g$. Then using the observations in (1.2) together with an integration by parts we have

$$\int_{\mathbf{R}^n} (Mf)^p \, dx = -\int_0^\infty \alpha^p \, d\lambda(\alpha) = p \int_0^\infty \alpha^{p-1} \lambda(\alpha) \, d\alpha$$

In particular, because of (5),

$$\|Mf\|_p^p = p \int_0^\infty \alpha^{p-1} m(E_\alpha) \, d\alpha \leq p \int_0^\infty \alpha^{p-1} \left( \frac{2A}{\alpha} \int_{|f| > \alpha/2} |f(x)| \, dx \right) d\alpha.$$

The double integral is evaluated by interchanging the orders of integration and integrating first with respect to $\alpha$. The inner integral is then

$$\int_0^{2|f(x)|} \alpha^{p-2} \, d\alpha = \left( \frac{1}{p-1} \right) |2f(x)|^{p-1},$$

since $p > 1$. So the double integral has the value

$$\frac{2Ap}{p-1} \int_{\mathbf{R}^n} |f| \, |2f|^{p-1} \, dx = (A_p)^p \int_{\mathbf{R}^n} |f|^p \, dx,$$

which proves conclusion (c). Calculating the constants we get

$$A_p = 2\left( \frac{5^n p}{p-1} \right)^{1/p}, \qquad 1 < p < \infty.$$

It is useful, for certain applications, to observe that

$$A_p = O\left(\frac{1}{p-1}\right), \quad \text{as} \quad p \to 1.$$

We now come to the proof of the corollary. We easily reduce the consideration to the case $p = 1$, by multiplying our original function by the characteristic function of a ball, and then exhausting $\mathbf{R}^n$ by a denumerable union of such balls. Let us denote by $f_r$ the function

$$(6) \qquad f_r(x) = \frac{1}{m(B(x, r))} \int_{B(x,r)} f(y)\, dy, \qquad r > 0.$$

We know* that if $r \to 0$, $\|f_r - f\|_1 \to 0$, whenever $f \in L^1(\mathbf{R}^n)$.

Therefore $f_{r_k} \to f$, almost everywhere for a suitable sequence $\{r_k\} \to 0$.

What remains to be seen, therefore, is that $\lim\limits_{r \to 0} f_r(x)$ exists almost everywhere. For this purpose let us denote for each $g \in L^1$, and $x \in \mathbf{R}^n$

$$(7) \qquad \Omega g(x) = \left|\limsup_{r \to 0} g_r(x) - \liminf_{r \to 0} g_r(x)\right|$$

where $g_r$ is defined like $f_r$. $\Omega g$ represents the oscillation of the family $\{g_r\}$, as $r \to 0$.

If $g$ is continuous with compact support, then $g_r \to g$ uniformly, and thus $\Omega g$ is identically zero in this case.

Next if $g$ is in $L^1(\mathbf{R}^n)$, then by conclusion (b) of the theorem

$$m\{x : 2M(g) > \varepsilon\} \le \frac{2A}{\varepsilon} \|g\|_1.$$

However clearly

$$\Omega g(x) \le 2Mg(x),$$

thus

$$(8) \qquad m\{x : \Omega g(x) > \varepsilon\} \le \frac{2A}{\varepsilon} \|g\|_1, \quad g \in L^1(\mathbf{R}^n).$$

Finally any $f \in L^1(\mathbf{R}^n)$ can be written as $f = h + g$, where $h$ is continuous with compact support and where the $L^1$ norm of $g$ is at our disposal. But $\Omega f \le \Omega h + \Omega g$, and $\Omega h \equiv 0$ since $h$ is continuous. Therefore (8) shows that

$$m\{x : \Omega f(x) > \varepsilon\} \le \frac{2A}{\varepsilon} \|g\|_1.$$

* This is a particular property of approximations of the identity. See Chapter III, §2.2 for a detailed discussion; the relevant part of that section can be used without fear of circularity.

Since the norm of $g$ can be chosen to be arbitrarily small we get $\Omega f = 0$ almost everywhere, which means that $\lim_{r \to 0} f_r(x)$ exists almost everywhere.

The following summarizing comment about the proof of the corollary is worth making. The argument used was of a very general nature and occurs often. That is, the almost everywhere convergence is proved as a combination of two parts, one which is deep and already contains the essence of the result; it is expressed in terms of a maximal inequality like part (b) (or (c)) of the theorem. The second fact is usually much simpler to establish but it is just as essential. It is the convergence almost everywhere for a dense subset of the function space, in this case the continuous function on $\mathbf{R}^n$ with compact support.

**1.6 A covering lemma.** We have therefore completed the proof of Theorem 1 and its corollary, save for the crucial step of the covering lemma, which we postponed until now. Not only the simplicity of its statement, or the application we use it for, but also the many variants of it that can be found in the mathematical literature attest to the fundamental character of this lemma. The reader should note that its statement and proof are closely related to a more refined but probably better known lemma of Vitali.*

LEMMA. *Let $E$ be a measurable subset of $\mathbf{R}^n$ which is covered by the union of a family of balls $\{B_j\}$, of bounded diameter. Then from this family we can select a disjoint subsequence, $B_1, B_2, \ldots B_k, \ldots$, (finite or infinite) so that*

$$\sum_k m(B_k) \geq C m(E)$$

*Here $C$ is a positive constant that depends only on the dimension $n$; $C = 5^{-n}$ will do.*

**1.7** We begin the proof of the lemma by describing the choice of $B_1, B_2, \ldots B_k, \ldots$ . We choose $B_1$ so that it is essentially as large as possible; that is so that the diameter of $B_1 \geq \frac{1}{2} \sup \{\text{diameter } B_j\}$. Of course the choice of a $B_1$ satisfying these conditions, as well as the later choices of the other $B_k$, is not unique; but this non-uniqueness is of no consequence to us. Let us now suppose that $B_1, B_2, \ldots B_k$ have already been chosen. We are now forced to select $B_{k+1}$ from those balls in the family $\{B_j\}$ which are disjoint with $B_1, B_2, \ldots B_k$. We choose one that again is essentially as large as possible. That is we take $B_{k+1}$ to be disjoint from $B_1, \ldots B_k$, and the diameter of $B_{k+1} \geq \frac{1}{2}\{\sup_j \text{diameter of } B_j$, with $B_j$ disjoint from $B_1, B_2, \ldots B_k\}$.

* The lemma of Vitali may be found in §5.4 below.

In this way we get the sequence $B_1, B_2, \ldots B_k, \ldots$ of balls. In principle this sequence could be finite, and terminate at $B_k$; this would be the case if there were no balls in $\{B_j\}$ disjoint with $B_1, B_2, \ldots B_k$.

Now two cases present themselves, depending on whether $\sum m(B_k) = \infty$ or $\sum m(B_k) < \infty$. In the first case we have attained our conclusion whether $m(E)$ is infinite or finite. Let us therefore consider the case when $\sum m(B_k) < \infty$.

For this purpose we denote by $B_k^*$ the ball having the same center as $B_k$, but whose diameter is *five* times as large. We claim that

$$\bigcup_k B_k^* \supset E. \tag{9}$$

To prove (9) we have to show that $U_k B_k^* \supset B_j$, for any fixed $B_j$ in our given family which covers $E$. We may certainly assume that our fixed $B_j$ is not one of the sequence $B_1, B_2, \ldots B_k, \ldots$, for otherwise there is nothing to prove. Since $\sum m(B_k) < \infty$, then diam $(B_k) \to 0$, as $k \to \infty$, and so we take the *first* $k$, with the property that diam $(B_{k-1}) < \frac{1}{2}(\text{diam } B_j)$. Now the ball $B_j$ must intersect one of the $k$ previous balls $B_1, B_2, \ldots B_k$, or it should have been picked as the $k + 1^{\text{th}}$ ball instead of $B_{k-1}$, since its diameter is more than twice that of $B_{k-1}$. That is $B_j$ intersects $B_{j_0}$, for some $1 \leq j_0 \leq k$, and $\frac{1}{2}$(diameter of $B_j$) $\leq$ diameter of $B_{j_0}$. From an obvious geometric consideration it is then evident that $B_j$ is contained in the ball that has the same center as $B_{j_0}$, but five times the diameter of $B_{j_0}$; i.e. $B_j \subset B_{j_0}^*$.

Thus we have proved (9), and so

$$m(E) \leq \sum m(B_k^*) = 5^n \sum m(B_k),$$

which proves the lemma.

**1.8 Lebesgue set.** The differentiation theorem just proved refers to the limits of averages taken with respect to balls. But this theorem has, as a rather simple consequence of itself, a generalization where the averages are taken over more general families of sets.

Let $\mathscr{F}$ be a family of measurable subsets of $\mathbf{R}^n$. We shall say that this family is *regular*, if there exists a constant $c > 0$, so that if $S \in \mathscr{F}$, then $S \subset B$, with $m(S) \geq cm(B)$, where $B$ is an appropriate open ball centered at the origin. Examples of such regular families are: (1) the family $\mathscr{F}$ of all sets of the $\delta U$, $0 < \delta < \infty$, (which are the dilations of a fixed set $U$), where $U$ is bounded and $m(U) > 0$. (2) the family of all cubes with the property that their distance from the origin is bounded by a constant multiple of their diameter. (3) any subfamily of such a family $\mathscr{F}$. In analogy with the special case of the family of all balls centered at the origin, we defined the appropriate maximal function

$$M_{\mathscr{F}}(f)(x) = \sup_{S \in \mathscr{F}} \frac{1}{m(S)} \int_S |f(x - y)|\, dy$$

Then clearly $M_{\mathscr{F}}(f)(x) \leq c^{-1} Mf(x)$, and therefore $M_{\mathscr{F}}$ satisfies the same conclusion as those in theorem 1 for $M$. So a repetition of the argument of the corollary leads to the fact that whenever $f$ is locally integrable

$$\lim_{\substack{S \in \mathscr{F} \\ m(S) \to 0}} \frac{1}{m(S)} \int_S f(x - y)\, dy = f(x), \tag{10}$$

for almost every $x$.

All of this is very simple, but is not completely satisfying for the following reason. Given a fixed locally integrable function $f$, we have proved that the relation (10) holds almost everywhere, but the exceptional set (of measure zero) depends on the given family $\mathscr{F}$. It would be better if we could find one exceptional set of measure zero (depending on $f$), so that outside of it the relation (10) would hold for *every* regular family. This is the role of the complement of the *Lebesgue set* of $f$, where the latter set is defined as those $x$ for which

$$\lim_{r \to 0} \frac{1}{m(B(x, r))} \int_{B(x,r)} |f(y) - f(x)|\, dy = 0. \tag{11}$$

(Recall that $B(x, r)$ is the ball of radius $r$ centered at $x$)

To see that the limit (11) is realized almost everywhere, we consider the relation

$$\lim_{r \to 0} \frac{1}{m(B(x, r))} \int_{B(x,r)} |f(y) - c|\, dy = |f(x) - c| \tag{11$'$}$$

which holds for each constant $c$, for almost every $x$. That is, there is an exceptional set $E_c$, with $m(E_c) = 0$, so that (11$'$) is valid whenever $x \notin E_c$. Let $c_1, c_2, \ldots c_n, \ldots$ be an enumeration of the rationals. If $x \notin E = U_n E_{c_n}$, then (11$'$) holds for any rational $c$, and so by continuity for every real $c$. In particular, $x$ in the complement of the set $E$ are in the Lebesgue set of $f$; that is, for those $x$, (11) is valid.

But

$$\begin{aligned}
\left| \frac{1}{m(S)} \int_S f(x - y)\, dy - f(x) \right| &= \left| \frac{1}{m(S)} \int_S [f(x - y) - f(x)]\, dy \right| \\
&\leq \frac{1}{m(S)} \int_S |f(x - y) - f(x)|\, dy \\
&\leq c^{-1} \frac{1}{m(B(x, r))} \int_{B(x,r)} |f(y) - f(x)|\, dy,
\end{aligned}$$

so that differentiability with respect to any regular family is established at every point of the Lebesgue set of $f$.

For a discussion of the case of non-regular families, see §5.3 below.

## 2. *Behavior near general points of measurable sets*

**2.1** In this section we wish to treat various properties of measurable sets of positive measure which confirm the observation that a "general" point of such a set is almost completely surrounded by other points of the set. The simplest concrete example of this heuristic principle is contained in the notion of a *point of density.*

Suppose $E$ is a given measurable set, and $x \in \mathbf{R}^n$. Then we say that $x$ is a point of density of $E$, if

$$\lim_{r \to 0} \frac{m\{E \cap B(x, r)\}}{m\{B(x, r)\}} = 1. \tag{12}$$

Of course for general $x$ the limit need not have value 1, or may not even exist; but if the limit in (12) has value 0, then according to our definition $x$ is a point of density of the complement of $E$. Let us now apply the differentiation theorem (the corollary to theorem 1 stated in §1.3), to the case when $f$ is $\chi_E$, the characteristic function of the set $E$. This gives us immediately the following proposition.

PROPOSITION 1. *For almost every point $x \in E$, the limit* (12) *holds; that is almost every point $x \in E$ is a point of density of $E$, and almost every point of the complement of $E$ is not a point of density of $E$.*

Notice that if we had restricted our attention to the points of the Lebesgue set of $\chi_E$, we should have obtained instead of Proposition 1 a similar but stronger conclusion. The balls in (12) could then have been replaced by regular families, in the sense of §1.8.

**2.2** In order to continue, we shall now limit ourselves to sets $E$ which are closed, but are still otherwise arbitrary. The reason for this limitation is obvious: In what follows the results will be expressed in terms of the distance from $E$; if $E$ is not closed the distance from $E$ is in reality the distance from $\bar{E}$, the closure of $E$, and clearly $E$ and $\bar{E}$ may be quite different measure-theoretically. However, the limitation to closed sets is not a serious obstacle in applications. Closed sets are sufficiently general; in particular, any measurable set may be approximated by the closed sets it contains, so that the respective difference sets have measure as small as we wish.

To reflect our newly imposed restriction we shall denote a general closed subset of $\mathbf{R}^n$ by $F$, and we let $\delta(x) = \delta(x, F)$ represent the distance of the point $x$ from $F$. Of course $\delta(x) = 0$ if and only if $x \in F$. Now it is clear that if $x \in F$, $\delta(x + y) \leq |y|$, since $x$ is a point in $F$ whose distance from $x + y$ is equal to $|y|$. However in general, this estimate of the distance from $F$ can be improved; that is $\delta(x + y) = o(|y|)$, for most $x$ in $F$. The relation of "little $o$" means that given any $\varepsilon > 0$, there exists a $\eta = \eta_\varepsilon$, so that $\delta(x + y) \leq \varepsilon |y|$, if $|y| \leq \eta$.

PROPOSITION 2. *Let $F$ be a closed set. Then for almost every $x \in F$, $\delta(x + y) = o(|y|)$. This holds in particular if $x$ is a point of density of $F$.*

We have formulated this proposition mainly because it is a simple illustration of the notion of point of density. We shall, however, also find an application for this proposition, but this is not until much later.

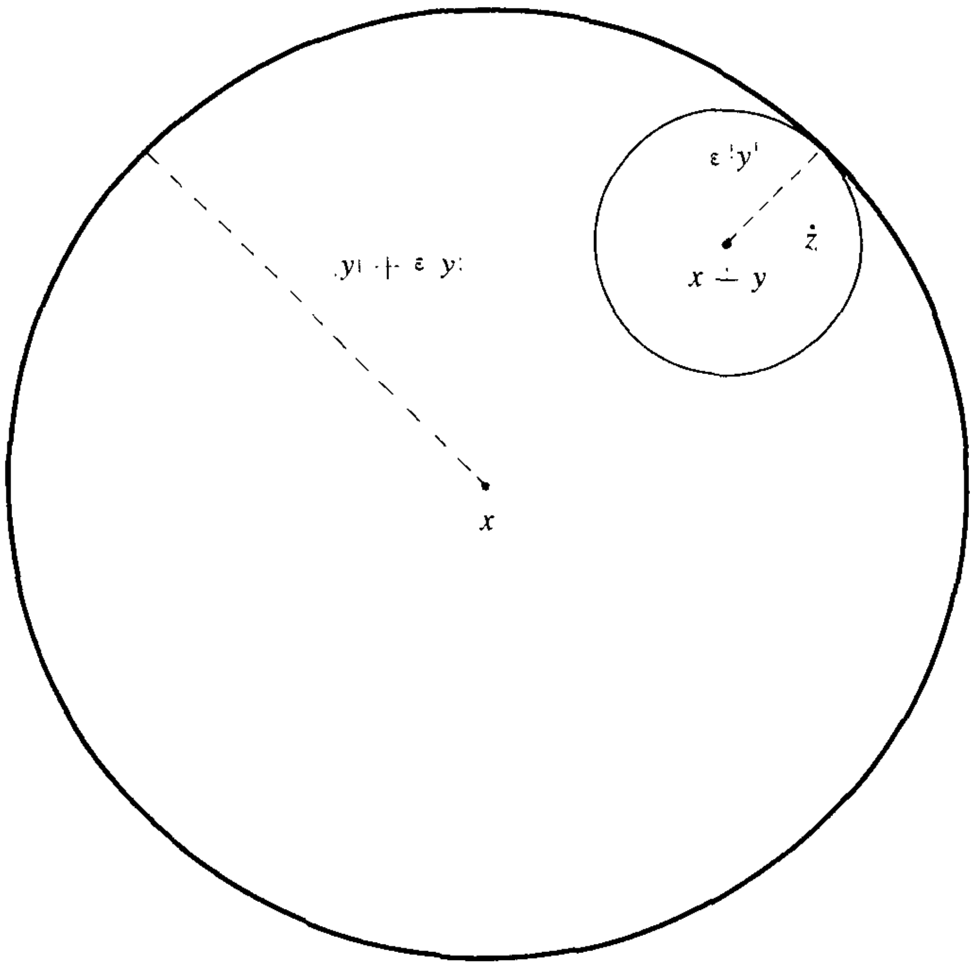



*Figure 1.* The point of density argument. The larger ball is $B(x, |y| + \varepsilon |y|)$, and the smaller ball is $B(x + y, \varepsilon |y|)$.

To prove the proposition, let $x$ be a point of density of $F$ and suppose $\varepsilon$ is given, $\varepsilon > 0$. Consider the "small" ball of center $x + y$, and radius $\varepsilon |y|$; and the "large" ball of center $x$ and radius $|y| + \varepsilon |y|$. Obviously

$B(x+y, \varepsilon|y|) \subset B(x, |y| + \varepsilon|y|)$. We claim that if $|y|$ is sufficiently small then there exists a $z \in F$, so that $z \in B(x+y, \varepsilon|y|)$. For otherwise $F \cap B(x+y, \varepsilon|y|) = \varnothing$, and

$$\frac{m(F \cap B(x, |y| + \varepsilon|y|)}{m(B(x, |y| + \varepsilon|y|)} \leq \frac{m(B(x, |y| + \varepsilon|y|)) - m(B(x+y, \varepsilon|y|))}{m(B(x, |y| + \varepsilon|y|)}$$
$$\leq 1 - \left(\frac{\varepsilon}{1+\varepsilon}\right)^n,$$

which contradicts (12) if $|y|$ is small enough. Thus there exists a $z$ in $F$ which also is in $B(x+y, \varepsilon|y|)$; this means that within a distance of $\varepsilon|y|$ from the point $x+y$ we can find a point of $F$, i.e. $\delta(x+y) \leq \varepsilon|y|$.

**2.3 Integral of Marcinkiewicz.** We shall now present another expression of the principle that a general point of a measurable set is almost completely surrounded by other points of the set. This form will be independent of the theorem of differentiation, but for many problems it will have a significance which is equally important. In fact, the integrals considered below, first treated systematically by Marcinkiewicz, intervene in a decisive way in the theory of singular integrals, as discussed in the following chapter, as well as other problems treated in this book.

We consider as before a closed set, $F$; $\delta(x)$ denotes the distance of $x$ from $F$, and we shall study the integral $I(x)$ given by

$$I(x) = \int_{|y| \leq 1} \frac{\delta(x+y)}{|y|^{n+1}}\, dy. \tag{13}$$

THEOREM 2. (a) *When $x \in$ complement of $F$, then $I(x) = \infty$.*
(b) *For almost every $x \in F$, $I(x) < \infty$.*

The conclusion (a) is evident, since the complement of $F$ is an open set. Then if $x$ belongs to this complement $\delta(x+y) \geq c > 0$, for a neighborhood of the origin in $y$. The conclusion (b) is the interest of this theorem, and it states in effect that the estimate $\delta(x+y) = o(|y|)$ of Proposition 2 can be refined on the average, so as to lead to the convergence of the integral (13). But this is not to say that the convergence of (13) for a given $x$ implies $\delta(x+y) = o(|y|)$.

The theorem will be a simple consequence of the following lemma, which is a more quantitative expression of the same fact.

LEMMA. *Let $F$ be a closed set whose complement has finite measure. With $\delta(x)$ defined as above we let*

$$I_*(x) = \int_{\mathbf{R}^n} \frac{\delta(x+y)}{|y|^{n+1}}\, dy. \tag{14}$$

*Then $I_*(x) < \infty$ for almost every $x \in F$. Moreover*

$$\int_F I_*(x)\,dx \leq c \cdot m({}^cF). \tag{15}$$

**2.4** In proving the lemma, we observe that it suffices to prove (15) since the integrand is positive. Also by the same positivity we can interchange the orders of integration in evaluating the left side of (15). This will accomplish the proof. In detail:

$$\int_F I_*(x)\,dx = \int_F\int_{\mathbf{R}^n} \frac{\delta(x+y)}{|y|^{n+1}}\,dy\,dx = \int_F\int_{\mathbf{R}^n} \frac{\delta(y)}{|x-y|^{n+1}}\,dy\,dx$$

$$= \int_F\int_{{}^cF} \frac{\delta(y)}{|x-y|^{n+1}}\,dy\,dx = \int_{{}^cF}\left(\int_F \frac{dx}{|x-y|^{n+1}}\right)\delta(y)\,dy.$$

Now consider

$$\int_F \frac{dx}{|x-y|^{n+1}} \quad \text{with} \quad y \in {}^cF.$$

The smallest value of $|x-y|$ (as $x$ varies over $F$) is of course $\delta(y)$, which is the distance of $y$ from $F$. Thus

$$\int_F \frac{dx}{|x-y|^{n+1}} \leq \int_{|x| \geq \delta(y)} \frac{dx}{|x|^{n+1}} \leq c(\delta(y))^{-1}.$$

This shows that

$$\int_F I_*(x)\,dx \leq \int_{{}^cF} c(\delta(y))^{-1}\,\delta(y)\,dy = cm({}^cF),$$

and the lemma is proved.

Theorem 2 is obtained from this lemma as follows. Let $B_m$ denote the open ball of radius $m$ center at the origin, and let $F_m = F \cup {}^cB_m$. Then $F_m$ is closed but its complement has finite measure (since it is contained in $B_m$). Thus we can apply the lemma to $F_m$. So let $\delta_m$ denote the distance from $F_m$, and $\delta$ the distance from $F$. Observe that $\delta(x+y) = \delta_m(x+y)$, if $|y| \leq 1$ and $x \in B_{m-2}$. Hence the lemma implies that $I(x) < \infty$, for almost every $x \in F \cap B_{m-2}$. Letting $m \to \infty$ we get the desired result.

Among the several variants of the theorem and the lemma we present here one. (Another variant is discussed at the end of this chapter in §5.) We can replace $I(x)$ by

$$I^{(\lambda)}(x) = \int_{|y| \leq 1} \frac{\delta^\lambda(x+y)}{|y|^{n+\lambda}}\,dy,$$

where $\lambda > 0$. Similarly $I_*(x)$ can be replaced by

$$I_*^{(\lambda)}(x) = \int_{\mathbf{R}^n} \frac{\delta^\lambda(x+y)}{|y|^{n+\lambda}}\, dy, \qquad \lambda > 0.$$

In both cases similar conclusions are obtained with the above methods.

### 3. *Decomposition in cubes of open sets in* $\mathbf{R}^n$

**3.1** The decomposition of a given set into a disjoint union of cubes (or balls) is a fundamental tool in the theory described in this chapter. We have already used this type of notion, in very rough form, in the covering lemma, §1.6.

*3.1.1* We now pose ourselves the following related general problem which, however, does not involve measure theory, but deals with the geometric structure of general closed sets $F$ in $\mathbf{R}^n$: Can the complement of $F$ be realized as a disjoint union of cubes in a canonical way? For $n = 1$ the answer is of course *yes*, since every open set is in a unique way the union of disjoint open intervals. For $n \geq 2$, the situation is no longer that simple, since we can realize an arbitrary open set in an infinity of different ways as a disjoint union of cubes (by cubes we now mean *closed* cubes; by disjoint we mean that their *interiors* are disjoint). However there are decompositions, which while not canonical, are very satisfactory and useful substitutes. We have in mind the idea first introduced by Whitney and formulated as follows.

THEOREM 3. *Let $F$ be a non-empty closed set in $\mathbf{R}^n$. Then its complement $\Omega$ is the union of a sequence of cubes $Q_k$, whose sides are parallel to the axes, whose interiors are mutually disjoint, and whose diameters are approximately proportional to their distances from $F$. More explicitly:*

(i) $\Omega = {}^cF = \bigcup_{k=1}^{\infty} Q_k$.

(ii) $Q_j^0 \cap Q_k^0 = \varnothing$ *if* $j \neq k$.

(iii) *There exist two constants* $c_1, c_2 > 0$, (*we can take* $c_1 = 1$, *and* $c_2 = 4$), *so that*

$$c_1\,(\textit{diameter } Q_k) \leq \textit{distance } Q_k \textit{ from } F \leq c_2\,(\textit{diameter } Q_k).$$

*3.1.2* Our intention for stating the theorem at this stage is obviously pedagogical. We shall not, strictly speaking, need to apply it until later (Chapter VI), and since its proof is a little intricate we postpone it until that point.

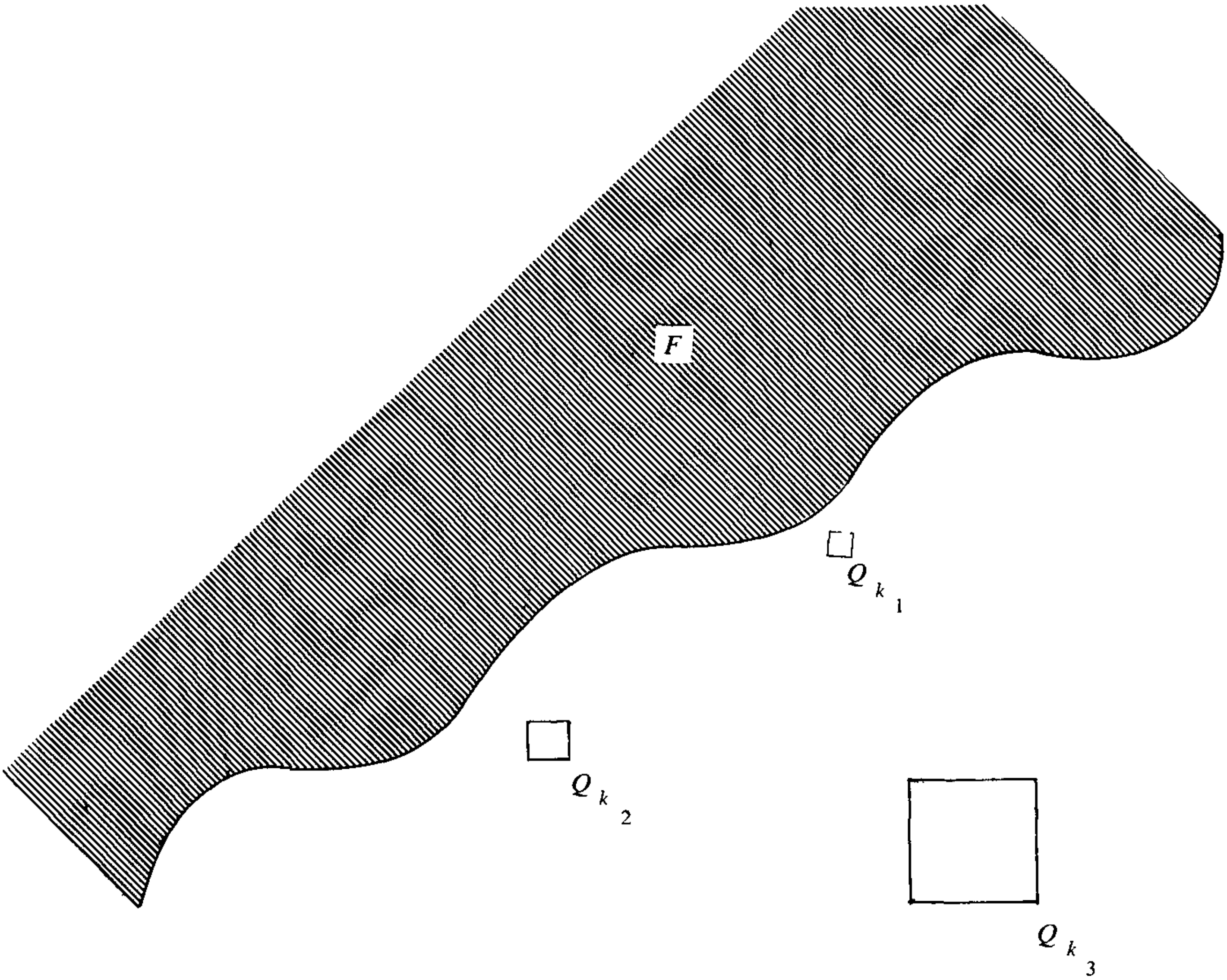



*Figure 2.* The decomposition of the complement of $F$ into cubes, whose diameters equal approximately their respective distances from $F$.

A related reason we have presented the theorem here is because it will help to clarify the meaning of the theorem immediately following. We are referring to a fundamental lemma of Calderón and Zygmund which can be used to give another approach to the theory of the maximal function of §2, but whose main importance for us will be its application to singular integrals in the next chapter.

**3.2** THEOREM 4. *Let $f$ be a non-negative integrable function on $\mathbf{R}^n$, and let $\alpha$ be a positive constant. Then there exists a decomposition of $\mathbf{R}^n$ so that*

(i) $\mathbf{R}^n = F \cup \Omega$, $F \cap \Omega = \varnothing$.

(ii) $f(x) \leq \alpha$ *almost everywhere on* $F$.

(iii) $\Omega$ *is the union of cubes*, $\Omega = \bigcup_k Q_k$, *whose interiors are disjoint, and* *so that for each* $Q_k$

$$\alpha < \frac{1}{m(Q_k)} \int_{Q_k} f(x)\,dx \le 2^n \alpha. \tag{16}$$

**3.3** We decompose $\mathbf{R}^n$ into a mesh of equal cubes, whose interiors are disjoint, and whose common diameter is so large that $\frac{1}{m(Q')} \int_{Q'} f \le \alpha$, for every cube $Q'$ in this mesh.

Let $Q'$ be a fixed cube in this mesh. We divide it into $2^n$ congruent cubes, by bisecting each of the sides of $Q'$. Let $Q''$ be one of these new cubes.

*First case:* $$\frac{1}{m(Q'')} \int_{Q''} f\,dx \le \alpha.$$

*Second case:* $$\frac{1}{m(Q'')} \int_{Q''} f\,dx > \alpha.$$

In the second case one does not sub-divide $Q''$ any further, and $Q''$ is selected as one of the cubes $Q_k$ appearing in the statement of the theorem. One has for it the inequality (16), because

$$\alpha < \frac{1}{m(Q'')} \int_{Q''} f\,dx \le \frac{1}{2^{-n} m(Q')} \int_{Q'} f\,dx \le 2^n \alpha.$$

In the first case we proceed with the sub-division of $Q''$, and repeat this process until (if ever) we are forced into the second case. We denote by $\Omega = \bigcup_k Q_k$, the union of cubes obtained from the second case, where we start the process with all possible cubes $Q'$ of our initial mesh. We claim that $f(x) \le \alpha$ almost everywhere in $F = {}^c\Omega$. In fact for almost every point $x \in F$ we have because of the theorem of differentiation (see the variant in §1.8), that

$$f(x) = \lim_{Q} \frac{1}{m(Q)} \int_Q f(y)\,dy,$$

where the limit is taken over all cubes $Q$ which contain $x$, and the diameter of $Q$ tends to zero. But each of the cubes that enter in our decomposition which contains an $x \in F$, is a cube for which the first alternative holds. This proves the theorem.

**3.4** We now state an immediate corollary, whose interest is that it contains that part of the theorem that we shall apply in the next chapter.

COROLLARY. *Suppose* $f$, $\alpha$, $F$, $\Omega$, *and* $Q_k$ *have the same meaning as in Theorem 4. Then there exists two constants* $A$ *and* $B$ (*depending only on* *the dimension n), so that (i) and (ii) hold and*

(a) $$m(\Omega) \leq \frac{A}{\alpha} \|f\|_1$$

(b) $$\frac{1}{m(Q_k)} \int_{Q_k} f\, dx \leq B\alpha.$$

In fact by (16) we can take $B = 2^n$, and also because of (16)

$$m(\Omega) = \sum m(Q_k) < \frac{1}{\alpha} \int_{\Omega} f(x)\, dx \leq \frac{1}{\alpha} \|f\|_1.$$

This proves the corollary with $A = 1$, and $B = 2^n$.

**3.5** It is possible however to give another proof of the corollary without using Theorem 4 from which it was deduced, but by using Theorem 1, the maximal theorem, and also the theorem about the decomposition of an arbitrary open set as a union of disjoint cubes. This more indirect method of proof has the advantage of clarifying the roles of the sets $F$ and $\Omega$ into which $\mathbf{R}^n$ was divided. We know that in $F$, $f(x) \leq \alpha$, but this fact does not determine $F$. The set $F$ is however determined, in effect, by the fact that the maximal function satisfies $Mf(x) \leq \alpha$ on it. So we choose $F = \{x : Mf(x) \leq \alpha\}$, and $\Omega = E_\alpha = \{x : Mf(x) > \alpha\}$. Then by Theorem 1, part (b) we know that $m(\Omega) \leq \frac{A}{\alpha} \|f\|_1$, with in fact $A = 5^n$. Notice that since by definition $F$ is closed, we can choose cubes $Q_k$ according to Theorem 3, so that $\Omega = \bigcup Q_k$, and whose diameters are approximately proportional to their distances from $F$. Let $Q_k$ then be one of these cubes, and $p_k$ a point of $F$ so that

$$\text{distance } (F, Q_k) = \text{distance } (p_k, Q_k).$$

Let $B_k$ be the smallest ball whose center is $p_k$ and which contains the interior of $Q_k$. Let us set

$$\gamma_k = \frac{m(B_k)}{m(Q_k)}.$$

We have because $p_k \in \{x : Mf(x) \leq \alpha\}$,

$$\alpha \geq (Mf)(p_k) \geq \frac{1}{m(B_k)} \int_{B_k} f\, dx \geq \frac{1}{\gamma_k m(Q_k)} \int_{Q_k} f\, dx.$$

But elementary geometry and the inequality (iii) of Theorem 3 then show that $\gamma_k = \dfrac{m(B_k)}{m(Q_k)} \leq$ constant, for all $k$. Thus we have another proof of the corollary.

Notice that this second proof of the lemma also rewarded us with an unexpected benefit: the cubes $Q_k$ are at a distance from $F$ comparable to their diameters.

**3.6** A final remark about the affinity of the present theorem with Theorem 1. It may be seen that the former also implies the latter, without the use of the covering lemma in §1.6. For this see §5.1 at the end of this chapter.

## 4. *An interpolation theorem for $L^p$*

**4.1** We wish here to formalize a part of the reasoning used in the proof of Theorem 1. What we have in mind is that part of the argument that took us from the inequality (b) to the $L^p$ inequality. This idea will lead us to the Marcinkiewicz interpolation theorem—or more precisely, a basic special case of that theorem. The more extended form of that interpolation theorem will be presented later in Appendix B.

We shall require several definitions. Let $T$ be a mapping from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$. Then $T$ is *of type* $(p, q)$ if

$$\|T(f)\|_q \leq A \, \|f\|_p, \qquad f \in L^p(\mathbf{R}^n) \tag{17}$$

where $A$ does not depend on $f$. Similarly $T$ is of *weak-type* $(p, q)$ if

$$m\{x : |Tf(x)| > \alpha\} \leq \left(\frac{A \, \|f\|_p}{\alpha}\right)^q, \quad \text{when} \quad q < \infty \tag{18}$$

where $A$ does not depend on $f$ or $\alpha$, $\alpha > 0$.

If $q = \infty$ we shall say that $T$ is of weak-type $(p, q)$ if it is of type $(p, q)$. Notice that (17) implies (18) so that the notion of type $(p, q)$ is stronger than the notion of weak-type $(p, q)$. In fact, if $q < \infty$,

$$\alpha^q m\{x : |Tf(x)| > \alpha\} \leq \int_{\mathbf{R}^n} |Tf|^q \, dx = \|Tf\|_q^q \leq (A \, \|f\|_p)^q.$$

It will also be necessary to treat operators $T$ defined on several $L^p$ spaces simultaneously. Thus we define $L^{p_1}(\mathbf{R}^n) + L^{p_2}(\mathbf{R}^n)$ to be the space of all functions $f$, so that $f = f_1 + f_2$, with $f_1 \in L^{p_1}(\mathbf{R}^n)$ and $f_2 \in L^{p_2}(\mathbf{R}^n)$. Suppose now $p_1 < p_2$. Then we observe that $L^p(\mathbf{R}^n) \subset L^{p_1}(\mathbf{R}^n) + L^{p_2}(\mathbf{R}^n)$, for all $p$ such that $p_1 \leq p \leq p_2$. In fact let $f \in L^p(\mathbf{R}^n)$ and let $\gamma$ be a fixed positive constant. Set

$$f_1(x) = \begin{cases} f(x) & \text{if} \quad |f(x)| > \gamma \\ 0 & \text{if} \quad |f(x)| \leq \gamma \end{cases}$$

$$f_2(x) = \begin{cases} f(x) & \text{if} \quad |f(x)| \leq \gamma \\ 0 & \text{if} \quad |f(x)| > \gamma. \end{cases}$$

Then $\int |f_1(x)|^{p_1}\,dx = \int |f_1(x)|^p\,|f_1(x)|^{p_1-p}\,dx \le \gamma^{p_1-p}\int |f(x)|^p\,dx$, since $p_1 - p \le 0$. Similarly

$$\int |f_2(x)|^{p_2}\,dx = \int |f_2(x)|^p\,|f_2(x)|^{p_2-p}\,dx \le \gamma^{p_2-p}\int |f(x)|^p\,dx,$$

so $f_1 \in L^{p_1}$ and $f_2 \in L^{p_2}$, with $f = f_1 + f_2$.

The idea we have just used of splitting $f$ into two parts according to their respective size, is the main idea of the proof of the theorem that follows.

**4.2** THEOREM 5. *Suppose that* $1 < r \le \infty$. *If* $T$ *is a sub-additive mapping from* $L^1(\mathbf{R}^n) + L^r(\mathbf{R}^n)$ *to the space of measurable functions on* $\mathbf{R}^n$ *which is simultaneously of weak type* $(1, 1)$ *and weak-type* $(r, r)$, *then* $T$ *is also of type* $(p, p)$, *for all* $p$ *such that* $1 < p < r$. *More explicitly: Suppose that for all* $f, g \in L^1(\mathbf{R}^n) + L^r(\mathbf{R}^n)$

(i) $|T(f + g)(x)| \le |Tf(x)| + |Tg(x)|$

(ii) $m\{x : |Tf(x)| > \alpha\} \le \dfrac{A_1}{\alpha}\,\|f\|_1, \qquad f \in L^1(\mathbf{R}^n)$

(iii) $m\{x : |Tf(x)| > \alpha\} \le \left(\dfrac{A_r}{\alpha}\,\|f\|_r\right)^r, \qquad f \in L^r(\mathbf{R}^n)$

(*if* $r < \infty$; *when* $r = \infty$ *we assume that the form* (17) *holds*). *Then*

$$\|T(f)\|_p \le A_p\,\|f\|_p, \qquad f \in L^p(\mathbf{R}^n)$$

*for all* $1 < p < r$, *where* $A_p$ *depends only on* $A_1, A_2, p$, *and* $r$.

**4.3** We prove the theorem under the restriction that $r < \infty$. In the case $r = \infty$ the argument presented below needs a slight modification which we leave as an exercise to the interested reader; this case is anyway contained implicitly in the proof given in §1.5.

Let $f \in L^p(\mathbf{R}^n)$. We wish to estimate the distribution function $\lambda(\alpha) = m\{x : |Tf(x)| > \alpha\}$. Fix $\alpha$ for the moment. As we saw above we can split $f = f_1 + f_2$, so that $f_1 \in L^1(\mathbf{R}^n)$ and $f_2 \in L^r(\mathbf{R}^n)$ where the splitting was obtained by cutting $|f|$, in effect, at the altitude $\gamma$, $\gamma > 0$. At that stage $\gamma$ was arbitrary but we now fix it to be equal to $\alpha$. Since $|T(f)(x)| \le |Tf_1(x)| + |Tf_2(x)|$, we have

$$\{x : |Tf(x)| > \alpha\} \subset \{x : |Tf_1(x)| > \alpha/2\} \cup \{x : |Tf_2(x)| > \alpha/2\}$$

so

$$\lambda(\alpha) = m\{x : |Tf(x)| > \alpha\} \le m\{x : |Tf_1(x)| > \alpha/2\} + m\{x : |Tf_2(x)| > \alpha/2\},$$

and therefore by the assumptions (ii) and (iii)

$$\lambda(\alpha) \leq \frac{A_1}{\alpha/2} \int |f_1(x)|\,dx + \frac{A_r^r}{(\alpha/2)^r} \int |f_2(x)|^r\,dx.$$

Because of the definitions of $f_1$ and $f_2$ we get

$$\lambda(\alpha) \leq \frac{2A_1}{\alpha} \int_{|f|>\alpha} |f|\,dx + \frac{(2A_r)^r}{\alpha^r} \int_{|f|\leq\alpha} |f(x)|^r\,dx. \tag{19}$$

Now we know that

$$\int_{\mathbf{R}^n} |Tf|^p\,dx = -\int_0^\infty \alpha^p\,d\lambda(\alpha) = p\int_0^\infty \alpha^{p-1}\lambda(\alpha)\,d\alpha,$$

so we need only multiply both sides of (19) by $p\alpha^{p-1}$ and integrate with respect to $\alpha$. To do this observe that

$$\begin{aligned}\int_0^\infty \alpha^{p-1}\alpha^{-1}\left\{\int_{|f|>\alpha} |f|\,dx\right\} d\alpha &= \int_{\mathbf{R}^n} |f| \int_0^{|f|} \alpha^{p-2}\,d\alpha\,dx \\ &= \frac{1}{p-1}\int_{\mathbf{R}^n} |f|\,|f|^{p-1}\,dx\end{aligned}$$

since $p > 1$. Similarly

$$\begin{aligned}\int_0^\infty \alpha^{p-1}\alpha^{-r}\int_{|f|\leq\alpha} |f|^r\,dx\,d\alpha &= \int_{\mathbf{R}^n} |f|^r \int_{|f|}^\infty \alpha^{p-1-r}\,d\alpha\,dx \\ &= \frac{1}{r-p}\int_{\mathbf{R}^n} |f|^r\,|f|^{p-r}\,dx\end{aligned}$$

since $p < r$. Putting the two together we get

$$\|T(f)\|_p \leq A_p\,\|f\|_p, \quad \text{with} \quad (A_p)^p = \frac{2A_1}{p-1} + \frac{(2A_r)^r}{r-p}.$$

One should remark that as in the case of the maximal function, the bound $A_p$ satisfies the inequality $A_p \leq A/(p-1)$, as $p \to 1$.

One example of this theorem is of course Theorem 1, part (c). Part (b) of Theorem 1 tells us that the operator $f \to M(f)$ is of weak-type (1, 1), while the fact that $f \to M(f)$ is of type $(\infty, \infty)$ is obvious. Another important application of the Marcinkiewicz interpolation theorem occurs in the theory of singular integrals, which are the subject of the next chapter.

## 5. *Further results*

**5.1** Theorem 4 (in §3), may be used to give another proof of the fundamental inequality for the maximal function in part (b) of Theorem 1. In fact for $f \geq 0$,

$f \in L^1(\mathbf{R}^n)$ and $\alpha > 0$, let $\Omega = \bigcup_k Q_k$, be as given in Theorem 4. Then

$$m(\Omega) \leq \frac{1}{\alpha} \int f\,dx.$$

Let $Q_k^*$ be the cube with the same center as $Q_k$ but with twice the diameter. Then clearly $m\left(\bigcup_k Q_k^*\right) \leq \frac{2^n}{\alpha} \int f\,dx$, and it can be seen that $Mf(x) \leq c\alpha$, if $x \notin \bigcup Q_k^*$, for an appropriate constant $c$; that is $m\{x : Mf(x) > c\alpha\} \leq \frac{2^n}{\alpha} \int f\,dx$. For details see Calderón and Zygmund [1], pp. 114-115.

**5.2** (a) Suppose that $f$ is supported in a finite ball $B \subset \mathbf{R}^n$. Then $M(f) \in L^1(B)$ if $|f| \log (2 + |f|)$ is integrable over $B$. In fact

$$\int_B Mf\,dx \leq m(B) + \int_{Mf \geq 1} Mf\,dx$$

while

$$\int_{Mf \geq 1} Mf\,dx = \int_1^\infty \lambda(\alpha)\,d\alpha + \lambda(1),$$

where $\lambda(\alpha) = m\{x : Mf(x) > \alpha\} \leq \frac{2A}{\alpha} \int_{|f| > \alpha/2} |f|\,dx$, by (5) in §1. See Wiener [1].

(b) The above inequality for $\lambda(\alpha)$ can be reserved, essentially. In fact for an appropriate positive constant, $c$, $m\{x : Mf(x) > c\alpha\} \geq \frac{2^{-n}}{\alpha} \int_{|f| > \alpha} |f|\,dx$. To prove this apply Theorem 4 to $|f|$ and $\alpha$. This leads to the cubes $Q_k$, where $2^n\alpha \geq \frac{1}{m(Q_k)} \int_{Q_k} |f|\,dx > \alpha$. Thus if $x \in Q_k$, $M(f)(x) > c\alpha$, and so

$$m\{x : M(f)(x) > c\alpha\} \geq \sum m(Q_k) \geq \frac{2^{-n}}{\alpha} \int_{\bigcup_k Q_k} |f|\,dx.$$

But $|f(x)| \leq \alpha$ if $x \notin \bigcup Q_k$; hence $\int_{\bigcup Q_k} |f|\,dx \geq \int_{|f| > \alpha} |f|\,dx$. This establishes the desired inequality.

(c) Part (a) has a converse. If $f$ is supported in a ball $B$, then $M(f) \in L^1(B)$ implies that $|f|$ (log $(2 + |f|)$) is integrable over $B$. To prove this integrate the above inequality for $m(x : Mf(x) > c\alpha\}$ in part (b) as in the direct part of the theorem. For (b) and (c) see Stein [12].

(d) More generally, if $f$ is supported in $B$, then

$$M(f) \log (2 + Mf)^k \in L^1(B) \Leftrightarrow |f| (\log 2 + |f|)^{k+1} \in L^1(B), \qquad k \geq 0.$$

**5.3** We consider the question whether

$$(*) \qquad \lim_{\operatorname{diam}(S) \to 0} \frac{1}{m(S)} \int_S f(x - y)\,dy = f(x), \text{ almost everywhere, with } S \in \mathscr{F},$$

where $\mathscr{F}$ is an appropriate family of *rectangles* containing the origin.

(a) When $\mathscr{F}$ is the family of *all* rectangles, (*) may be false even if $f$ is bounded. See O. Nikodym [1], and Busemann and Feller [1].

(b) When $\mathscr{F}$ is the family of all rectangles with sides parallel to the axes, (*) is false for some integrable $f$. See Saks [1].

(c) However, for $\mathscr{F}$ the family of all rectangles with sides parallel to the axes, (*) holds if $f \in L^p(\mathbf{R}^n)$, $p > 1$. In fact if we define

$$\tilde{M}(f)(x) = \sup_{S \in \mathscr{F}} \frac{1}{m(S)} \int_S |f(x - y)| \, dy,$$

for such a family then $\|\tilde{M}f\|_p \leq A_p \|f\|_p$, $1 < p \leq \infty$. This inequality can be proved by an $n$-fold application of the one-dimensional $L^p$ maximal inequalities of Theorem 1.

(d) Let $\mathscr{F}$ be a one parameter monotonic family of rectangles with sides parallel to the axes, i.e. $\mathscr{F} = \{S_t\}_{0 < t < \infty}$, with $S_{t_1} \subset S_{t_2}$, if $t_1 \leq t_2$.

Then (*) holds for $f \in L^p(\mathbf{R}^n)$, $1 \leq p$. This follows from the fact that for such a monotonic family of rectangles an analogue of the covering lemma (1.6) holds. For (c), (d), and further related results see Zygmund [8], Chapter XVII.

**5.4** *Vitali covering theorem.* Suppose that a measurable set $E$ is covered by a collection of balls $\{B_\alpha\}$, in the sense that for each $x \in E$, and each $\varepsilon > 0$, there exists a $B_{\alpha_0} \in \{B_\alpha\}$, so that $x \in B_{\alpha_0}$, and $m(B_{\alpha_0}) < \varepsilon$. Then there is a disjoint subsequence of these balls $B_1, B_2, \ldots, B_k, \ldots$, so that

$$m(E - \textstyle\bigcup_k B_k) = 0.$$

For this and related generalizations see Saks [2], Chapter 4.

**5.5** (F. Riesz,) Let $F(x)$ be a real-valued continuous and bounded function defined on the line $\mathbf{R}^1$, and suppose $\alpha$ is positive. Let $\Omega$ be the set of those $x$, for which $\sup\limits_{h>0} \dfrac{F(x + h) - F(x)}{h} > \alpha$. Then $\Omega = \bigcup\limits_{k=1}^{\infty} I_k$, with $I_k = (a_k, b_k)$, and $\dfrac{F(b_k) - F(a_k)}{b_k - a_k} = \alpha$. This lemma gives another proof of Theorem 4, in one dimension, if we set $F(x) = \int_0^x f(t)\, dt$. The inequality (16) is then replaced by the identity $\dfrac{1}{m(I_k)} \displaystyle\int_{I_k} f\, dx = \alpha$. See Riesz and Nagy [1], Chapter 1.

**5.6** (a) A strengthened form of the inequality (15) is as follows: Let $\psi \geq 0$, then $\int_F I_*(x)\psi(x)\, dx \leq \int_{cF} (M\psi)(x)\, dx$, where $M\psi$ is the maximal function of $\psi$. This shows that $I_*(x) \in L^p(F)$ for all $1 \leq p < \infty$. If we use (§5.2) for $\psi \log(2 + \psi)$ integrable (but otherwise arbitrary), then it also shows that $\int_F e^{aI_*(x)}\, dx < \infty$, for an appropriate $a > 0$.

(b) A variant of $I_*(x)$ is $\mathscr{I}_*(x) = \displaystyle\int_{\mathbf{R}^n} \frac{\delta(x + y)\, dy}{[|y| + \delta(x+y)]^{n+1}}$. Then (i) $\mathscr{I}_*(x) \geq cI_*(x)$, $x \in F$; (ii) $\mathscr{I}_*(x) < \infty$ for almost every $x \in \mathbf{R}^n$; (iii) Further there exists a positive constant $a$, so that for every finite ball $B$, $\int_B e^{a\mathcal{F}_*(x)}\,dx < \infty$. In this connection see Carleson [3].

**5.7** Suppose that $f \in L^p(\mathbf{R}^n)$, $1 \le p < \infty$. Then a slight modification of the argument of §1.8 shows that

$$\lim_{r\to 0} \frac{1}{m(B(x, r))} \int_{B(x,r)} |f(y) - f(x)|^p \, dy = 0 \quad \text{for almost every } x.$$

**5.8** Let $f_1, f_2, \ldots, f_n, \ldots$ be a sequence of functions in $L^p(\mathbf{R}^n)$ with the property that $\left(\sum_j |f_j(x)|^2\right)^{\frac{1}{2}} \in L^p(\mathbf{R}^n)$. Denote by $Mf_j$ the maximal function of $f_j$. Then $\left\|\left(\sum_j |Mf_j|^2\right)^{\frac{1}{2}}\right\|_p \le A_p \left\|\left(\sum_j |f_j|^2\right)^{\frac{1}{2}}\right\|_p$, $1 < p < \infty$. The estimates for $A_p$ are $A_p = O((p-1)^{-1})$, as $p \to 1$, and $A_p = O(p^{\frac{1}{2}})$ as $p \to \infty$. These are best possible. If the $f_j$ are taken to be the characteristic functions of disjoint cubes, then the above result essentially contains §5.6. See Fefferman and Stein [1].

## *Notes**

*Section 1.* For the basic facts about integration and differentiation, see Saks [2]. The original maximal theorem, for $n = 1$, is due to Hardy-Littlewood [1], and its $n$-dimensional version is in Weiner [1]. For the covering lemma in §1.6 see Weiner [1] and Marcinkiewicz and Zygmund [3]. The reader may find it instructive to compare this lemma with a more refined version found in *Fourier Analysis*, Chapter II, section 3, which is based on further ideas of Besicovitch. See also Edwards and Hewitt [1] and Stein [11] for other generalizations.

*Section 2.* The integral of Marcinkiewicz arose first in Marcinkiewicz [1], [2], and [3]; see also Zygmund [8], Chapter IV. A systematic use for its $n$-dimensional form is found in Calderón and Zygmund [7].

*Section 3.* The decomposition theorem in §3.2 is in Calderón and Zygmund [1]. Its close connection with the Whitney decomposition seems to have been pointed out first in Stein [10].

*Section 5.* The interpolation theorem 5 is due to Marcinkiewicz [5]. The more general version presented in Appendix B is due to Zygmund, but the proof given there is that of R. Hunt [1]. See also the more extended treatment given in *Fourier Analysis*, Chapter V.

* *Fourier Analysis* refers to Stein and Weiss [4].

CHAPTER II

# Singular Integrals

A basic example which lies at the source of the theory of singular integrals is given by the Hilbert transform. This transformation of $f$ is defined by

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x-y)}{y}\,dy, \tag{1}$$

where the non-absolutely convergent integral is interpreted by a suitable limiting process. We shall here single out several features of the theory of the Hilbert transform so that in terms of their aspects we can describe the $n$-dimensional singular integrals treated in this chapter.

(a) *The $L^2$ theory.* We are dealing here, as in the general case, with an operator that commutes with translations. For this reason the tools of convolution, Fourier transforms, and Plancherel theorem—in brief, the basic implements of harmonic analysis in $\mathbf{R}^n$—are unavoidable; it is with a resumé of these that we begin the development in this chapter.

(b) *The $L^p$ theory.* A fundamental property of the operator (1), as well as the generalizations we treat, is that each is a bounded operator on $L^p$, $1 < p < \infty$. In the case of Hilbert transforms this classical theorem was proved by M. Riesz, using complex function theory. This approach is inappropriate in the general context, and there the $L^p$ theory will be obtained as a consequence of the $L^1$ theory.

(c) *The $L^1$ theory.* The Hilbert transform is not a bounded operator on $L^1$. There is for it, however, a substitute result, namely that it is of weak-type (1, 1). There is a similar situation in the general case. The (real-variable) techniques for proving the weak-type result were initiated by Besicovitch and Titchmarsh in the case of the Hilbert transform, and were further developed by Calderón and Zygmund's treatment of the $n$-dimensional theory. It is the presentation of those methods that may be said to represent the core of the present chapter. Needless to say, we shall make decisive use of the general real-variable theory of Chapter **I**.

(d) *Special properties of the Hilbert transform.* Among these are:

(i) The operator (1) commutes not only with translations, but also with dilations $x \to \delta x$, $\delta > 0$. It is therefore not surprising that the theorems describing the $n$-dimensional generalizations are essentially invariant under dilations. Further, those operators which, like (1), are left fixed by dilations represent an important sub-class for which the theory is more explicit and far-reaching. These are the subject matter of §4.

(ii) The connection with analytic functions. There is a special relation between the transform (1) (or certain of its $n$-dimensional variants), and analytic functions (or their generalizations). The meaning of this relation and its concomitant properties of invariance with respect to rotations will be described in the next chapter.

## 1. *Review of certain aspects of harmonic analysis in* $\mathbf{R}^n$

We state here, without proof, certain elementary facts taken from the theory of harmonic analysis in $\mathbf{R}^n$, which incidentally find their natural generality in the setting of locally compact abelian groups.

**1.1** Together with the spaces $L^p(\mathbf{R}^n)$, $1 \leq p \leq \infty$ already used, we consider the space $C_0(\mathbf{R}^n)$ of continuous functions tending to zero at infinity, with the usual supremum norm; also its dual space $\mathscr{B}(\mathbf{R}^n)$, which as is well known can be identified with the Banach space of all finite measures $d\mu$, with norm $\|d\mu\| = \int_{\mathbf{R}^n} |d\mu|$. The space $L^1(\mathbf{R}^n)$ can be identified as a subspace of $\mathscr{B}(\mathbf{R}^n)$ by the isometry $f(x) \to f(x)\, dx$, where $dx$ is Lebesgue measure.

A basic operation is that of that of convolution. Thus if $\mu_1, \mu_2 \in \mathscr{B}$, then $\mu = \mu_1 * \mu_2$ is defined by

$$\mu(f) = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} f(x + y)\, d\mu_1(x)\, d\mu_2(y).$$

We have $\mu_1 * \mu_2 = \mu_2 * \mu_1$ and $\|\mu\| \leq \|\mu_1\| \|\mu_2\|$. The operation of convolution when one of the factors is restricted to $L^1(\mathbf{R}^n)$ has its range also in $L^1(\mathbf{R}^n)$. Hence if $f \in L^1(\mathbf{R}^n)$, $g = f * \mu = \int_{\mathbf{R}^n} f(x - y)\, d\mu(y)$ converges absolutely for almost every $x$, and $g \in L^1(\mathbf{R}^n)$ with

$$\|g\|_1 \leq \|f\|_1 \|\mu\|.$$

Similarly if $f \in L^p(\mathbf{R}^n)$ then $\int_{\mathbf{R}^n} f(x - y)\, d\mu(y)$ is also in $L^p$ and $\|g\|_p \leq \|f\|_p \|\mu\|$. It is to be noted that the transformation

$$f \to \int f(x - y)\, d\mu(y),$$

with $\mu \in \mathscr{B}$, which we have just asserted is bounded in $L^p(\mathbf{R}^n)$, also commutes with translations, $x \to x + h$. This class of transformations is characterized in the following theorem.

**1.2** PROPOSITION 1. *Let $T$ be a bounded linear transformation mapping $L^1(\mathbf{R}^n)$ to itself. Then a necessary and sufficient condition that $T$ commutes with translations is that there exists a measure $\mu$ in $\mathscr{B}(\mathbf{R}^n)$ so that $T(f) = f * \mu$, for all $f \in L^1(\mathbf{R}^n)$. One has then $\|T\| = \|\mu\|$.*

**1.3** For each measure $\mu \in \mathscr{B}(\mathbf{R}^n)$ we can define its Fourier transform $\hat{\mu}(y)$ by

$$\hat{\mu}(y) = \int_{\mathbf{R}^n} e^{2\pi i x\cdot y}\, d\mu(x).*$$

In particular, the Fourier transform is defined for all $f \in L^1(\mathbf{R}^n)$, with $\hat{f}(y) = \int_{\mathbf{R}^n} e^{2\pi i x\cdot y} f(x)\, dx \in C_0(\mathbf{R}^n)$. The Fourier transform has the fundamental property that if $\mu = \mu_1 * \mu_2$, then $\hat{\mu}(y) = \hat{\mu}_1(y)\hat{\mu}_2(y)$.

If $f \in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$, then $\hat{f} \in L^2(\mathbf{R}^n)$ and $\|\hat{f}\|_2 = \|f\|_2$. The Fourier transform can then be extended to all of $L^2(\mathbf{R}^n)$ by continuity, so that it is unitary on $L^2(\mathbf{R}^n)$. By continuity we also have that if $g = f * \mu$, with $f \in L^2(\mathbf{R}^n)$ and $\mu \in \mathscr{B}(\mathbf{R}^n)$, then $\hat{g}(y) = \hat{f}(y)\hat{\mu}(y)$.

**1.4** The $L^2$ analogue of Proposition 1 is the following theorem.

PROPOSITION 2. *Let $T$ be a bounded linear transformation mapping $L^2(\mathbf{R}^n)$ to itself. Then a necessary and sufficient condition that $T$ commutes with translation is that there exists a bounded measurable function $m(y)$* (a "multiplier") *so that $(T\hat{f})(y) = m(y)\hat{f}(y)$, for all $f \in L^2(\mathbf{R}^n)$. One has then $\|T\| = \|m\|_\infty$.*

Notice that in the special case where $T$ is also bounded on $L^1(\mathbf{R}^n)$, then $m(y) = \hat{\mu}(y)$, where $Tf = f * \mu$.

## 2. *Singular integrals: the heart of the matter*

**2.1** The two propositions above show that the structure of translation invariant transformations that are bounded on $L^1$ or $L^2$ is both simple and well understood. The study of translation invariant operators that are bounded on some $L^p$, $p \neq 2$, but not all $p$, is both more arduous and still unfinished. However, for an important class of transformations much has been done. This class consists of convolution operators with a singular kernel, each having its only singularities at a finite point (the origin), and at infinity. The analogous study of kernels whose singularities are situated

* The conventional choice of the factor $-2\pi$ in the exponential should be taken into account when comparing formulae which occur below with those in other texts.

on varieties more general than isolated points is an important problem which it seems must be left to a future theory.

The theorem below represents the essence of the main result. It is stated, however, with somewhat less generality than the fuller development in §3. We have here set ourselves this less exacting task, since it will facilitate the understanding of the main ideas of the theory.

**2.2** THEOREM 1. *Let $K \in L^2(\mathbf{R}^n)$. We suppose:*
(a) *The Fourier transform of $K$ is essentially bounded*

$$|\hat{K}(x)| \leq B. \tag{1}$$

(b) *$K$ is of class $C^1$ outside the origin and*

$$|\nabla K(x)| \leq B/|x|^{n+1}. \tag{2}$$

*For $f \in L^1 \cap L^p$, let us set*

$$(Tf)(x) = \int_{\mathbf{R}^n} K(x - y) f(y)\, dy. \tag{3}$$

*Then there exists a constant $A_p$, so that*

$$\|T(f)\|_p \leq A_p \|f\|_p, \qquad 1 < p < \infty. \tag{4}$$

*One can thus extend $T$ to all of $L^p$ by continuity. The constant $A_p$ depends only on $p$, $B$, and the dimension $n$. In particular, it does not depend on the $L^2$ norm of $K$.*

It is important to make the following remark. The assumption that $K \in L^2$ is made for the purpose of having a direct definition of $Tf$ on a dense subset of $L^p$ (in this case $L^1 \cap L^p$), and it could be replaced by other assumptions (such as $K \in L^1 + L^2$).

In the applications the hypothesis $K \in L^2$ is of no consequence since it can be dispensed with by appropriate limiting process; and this is because the final bounds in Theorem 1 do not depend on the $L^2$ norm of $K$. See Theorem 2 in §3.2 below.

**2.3** PROOF: FIRST STEP. *$T$ is of weak type* (2, 2).

If we use the Fourier transform we see that $(Tf)\hat{}(y) = \hat{K}(y)\hat{f}(y)$, for $f \in L^1 \cap L^2$, and so by assumption (a) and the Plancherel theorem,

$$\|T(f)\|_2 \leq B\, \|f\|_2. \tag{5}$$

Because of (5) $T$ has unique extension to all of $L^2$, where (5) is still valid. So by the remarks in §4.1 of Chapter I we obtain

$$m\{x : |Tf(x)| > \alpha\} \leq (B^2/\alpha^2) \int_{\mathbf{R}^n} |f|^2\, dx, \qquad f \in L^2(\mathbf{R}^n). \tag{6}$$

**2.4** SECOND STEP: *T is of weak type* (1, 1).

We treat $Tf$, for $f \in L^1(\mathbf{R}^n)$, by decomposing $f$, as $f = g + b$, where $g$ is "good" and is equal to $f$ on the set where $f$ is essentially small; $b$ is the "bad" part, carried on the set where $f$ is essentially large. The good part $g$ turns out to be in $L^2(\mathbf{R}^n)$, and the $L^2$ result above, (6), then gives an appropriate estimate for $T(g)$. One can already perceive the idea that is used for dealing with the large part $b$, when one considers a portion of the Hilbert transform. Thus in the integral

$$\int_{-L}^{L} \frac{b(y)\,dy}{x - y} \tag{7}$$

the principal obstacle to an elementary (but favorable) estimate is the appearance of the logarithm when one integrates $1/x$, that is the fact that $\int_h^L \frac{dx}{x} \sim \log 1/h$, when $h \to 0$. The idea is to replace (7) by

$$\int_{-L}^{L} \left[\frac{1}{x - y} - \frac{1}{x}\right] b(y)\,dy, \tag{8}$$

which we may if $\int_{-L}^{L} b(y)\,dy = 0$. Observe that $\left|\frac{1}{x - y} - \frac{1}{x}\right| \sim \frac{L}{x^2}$, when $x$ is clearly separated from the interval $[-L, L]$, (say, for example, if $|x| \geq 2L$), and on the other hand $L \int_{|x| \geq 2L} \frac{dx}{x^2} \leq 1$.

In this way one can avoid the difficulty of the logarithm if the integrals of $b$ on suitable intervals (cubes in the case of $\mathbf{R}^n$) are zero. This is the property of $b$ expressed in (11) below.

Once we have replaced (7) by (8), we must add the analogues of (8), taken for each cube that occurs. The resulting sum turns out to be majorized by the integral of Marcinkiewicz concerning the distance function ((14) of Chapter I), and an appeal to the lemma of §2.3 in that chapter will then supply the necessary information to complete our estimates.

*2.4.1* We turn to the details. We need to find a constant $C$ so that*

$$m\{x : (Tf(x)) > \alpha\} \leq \frac{C}{\alpha} \int_{\mathbf{R}^n} |f(x)|\,dx; \qquad f \in L(\mathbf{R}^n). \tag{9}$$

* In order to facilitate the writing of inequalities we shall adopt the following convention that until the end of this chapter, $C$ will denote a general constant (not necessarily the same at different occurrences) but which depends only on the constant $B$ of the hypothesis of the theorem and the dimension $n$.

To do this fix $\alpha$, and for this $\alpha$ and $|f(x)|$ apply the corollary of Theorem 4 in Chapter I, §3.4. Then we have $\mathbf{R}^n = F \cup \Omega$, $F \cap \Omega = \varnothing$; $|f(x)| \le \alpha$, $x \in F$; $\Omega = \bigcup_{j=1}^{\infty} Q_j$, with the interiors of the $Q_j$ mutually disjoint; $m(\Omega) \le \frac{C}{\alpha} \int_{\mathbf{R}^n} |f| \, dx$, and $\frac{1}{m(Q_j)} \int_{Q_j} |f(x)| \, dx \le C\alpha$.

We therefore set

$$
(10) \qquad g(x) = \begin{cases} f(x), & \text{for } x \in F \\ \dfrac{1}{m(Q_j)} \displaystyle\int_{Q_j} f(x) \, dx, & \text{for } x \in Q_j^0 \end{cases}
$$

which defines $g(x)$ almost everywhere. This and the fact that $f(x) = g(x) + b(x)$ has as a consequence

$$
(11) \qquad \begin{aligned} & b(x) = 0, && \text{for } x \in F \\ & \int_{Q_j} b(x) \, dx = 0, && \text{for each cube } Q_j. \end{aligned}
$$

Now since $Tf = Tg + Tb$, it follows then that

$$
m\{x : |Tf(x)| > \alpha\} \le m\{x : |Tg(x)| > \alpha/2\} + m\{x : |Tb(x)| > \alpha/2\}
$$

and it suffices to establish separately for both terms of the right-side inequalities analogous to our desired inequality (9).

*2.4.2 Estimate for $Tg$:* One has $g \in L^2(\mathbf{R}^n)$, because by (10)

$$
\begin{aligned}
\|g\|_2^2 = \int_{\mathbf{R}^n} |g(x)|^2 \, dx &= \int_F |g(x)|^2 \, dx + \int_\Omega |g(x)|^2 \, dx \\
&\le \int_F \alpha \, |f(x)| \, dx + C^2\alpha^2 m(Q) \\
&\le (C^2 A + 1)\alpha \, \|f\|_1.
\end{aligned}
$$

If we now apply inequality (6) of the $L^2$ theory to $g$, we obtain

$$
(12) \qquad m\{x : |Tg(x)| > \alpha/2\} \le \frac{C}{\alpha} \|f\|_1.
$$

*2.4.3 Estimate for $Tb$:* Let us write

$$
b_j(x) = \begin{cases} b(x), & x \in Q_j \\ 0, & x \notin Q_j \end{cases}.
$$

Then $b(x) = \sum_j b_j(x)$, and $(Tb)(x) = \sum_j (Tb_j)(x)$, with

(13) $$Tb_j(x) = \int_{Q_j} K(x-y) b_j(y)\, dy.$$

We shall be able to obtain a favorable estimate of (13) when $x \in F$ (= the complement of $\bigcup_j Q_j$). First,

$$Tb_j(x) = \int_{Q_j} [K(x-y) - K(x-y^j)] b_j(y)\, dy,$$

with $y^j$ the center of the cube $Q_j$, since $\int_{Q_j} b_j(y)\, dy = 0$. Because $|\nabla K| \leq B|x|^{-n-1}$, it follows that $|K(x-y) - K(x-y^j)| \leq C \dfrac{\text{diameter } (Q_j)}{|x - \bar{y}^j|^{n+1}}$ where $\bar{y}^j$ is a (variable) point on the straight-line segment connecting $y^j$ with $y$ ($\in Q_j$).

Next we use the remark made in §3.4 of Chapter I that the diameter of $Q_j$ is comparable to its distance from $F$. This means that if $x$ is a fixed point in $F$ the set of distances $\{|x-y|\}$, as $y$ varies over $Q_j$, are all comparable with each other. Hence

$$|Tb_j(x)| \leq C \text{ diameter } (Q_j) \int_{Q_j} \frac{|b(y)|\, dy}{|x-y|^{n+1}}.$$

However $\int_{Q_j} |b(y)|\, dy \leq \int_{Q_j} |f(y)|\, dy + C\alpha \int_{Q_j} dy$, so $\int_{Q_j} |b(y)|\, dy \leq (1 + C)\alpha m(Q_j)$. This has the following consequence. If $\delta(y)$ denotes the distance of $y$ from $F$, because $\text{diameter}(Q_j) m(Q_j) \leq C \int_{Q_j} \delta(y)\, dy$, then

$$|Tb_j(x)| \leq C\alpha \int_{Q_j} \frac{\delta(y)}{|x-y|^{n+1}}\, dy, \qquad x \in F.$$

Finally,

(14) $$|(Tb)(x)| \leq C\alpha \int_{\mathbf{R}^n} \frac{\delta(y)}{|x-y|^{n+1}}\, dy, \qquad x \in F.$$

This majorization of $T$ in terms of the Marcinkiewicz integral is what we promised earlier. The rest is now relatively easy.

Using the lemma in §2.3 we are led to the fact that

(15) $$\int_F |Tb(x)|\, dx \leq C\alpha m(\Omega) \leq C\, \|f\|_1.$$

From the inequality it follows immediately that

(16) $$m\{x \in F : |Tb(x)| > \alpha/2\} \leq \frac{2C}{\alpha}\, \|f\|_1.$$

Since, however, $m({}^cF) = m(\Omega) \leq \frac{C}{\alpha} \|f\|_1$, we have obtained the estimate for $Tb$, that is

$$m\{x : |Tb(x)| > \alpha/2\} \leq \frac{C}{\alpha} \|f\|_1.$$

If we combine it with (12), which is the similar result for $Tg$, we get (9), that is $T$ is of weak-type (1, 1).

**2.5** FINAL STEP: THE $L^p$ INEQUALITIES.

(a) For $p = 2$, see §2.3.

(b) For $1 < p < 2$, it suffices to verify the hypotheses of the interpolation theorem (§4.2 in Chapter I), for the case $r = 2$. $T$ is well defined for $L^1(\mathbf{R}^n) + L^2(\mathbf{R}^n)$ and is also linear. It is of weak-type (1, 1) by §2.4 and of weak-type (2, 2) by §2.3, with bounds that depend only on $B$ and the dimension $n$; ($B$ appears in the hypothesis of the present theorem). Thus the interpolation theorem shows that

$$\|T(f)\|_p \leq A_p \|f\|_p, \qquad 1 < p < 2, \qquad f \in L^p,$$

where $A_p$ depends only on $B$, $p$, and $n$.

(c) For $2 < p < \infty$, we will exploit the duality between $L^p$ and $L^q$, $1/p + 1/q = 1$, and the fact that the theorem is proved for $L^q$. Observe the following: if a function $\psi$ is locally integrable and if $\sup |\int \psi\varphi \, dx| = A < \infty$, where the sup is taken over all continuous $\varphi$ with compact support which verify $\|\varphi\|_q \leq 1$, then $\psi \in L^p$ and $\|\psi\|_p = A$.

This being so, take $f \in L^1 \cap L^p$, $(2 < p < \infty)$, and $\varphi$ of the type described above. Since $K \in L^2$, and because of our choice of $f$ and $\varphi$, the double integral

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} K(x - y) f(y) \varphi(x) \, dx \, dy$$

converges absolutely; its value is therefore

$$I = \int_{\mathbf{R}^n} f(y) \left( \int_{\mathbf{R}^n} K(x - y) \varphi(x) \, dx \right) dy.$$

But the theorem is valid for $1 < q < 2$ (with the kernel $K(-x)$ instead of $K(x)$, but with the same constant $A_q$). Therefore $\int_{\mathbf{R}^n} K(x - y)\varphi(x) \, dx$ belongs to $L^q$, and its $L^q$ norm is majorized by $A_q \|\varphi\|_q = A_q$. Hölder's inequality then shows that $|\int_{\mathbf{R}^n} (Tf)\varphi \, dx| = |I| \leq A_q \|f\|_p$, and taking the supremum of all the $\varphi$'s indicated above gives the result that

$$\|Tf\|_p \leq A_q \|f\|_p, \qquad 2 < p < \infty.$$

We have, therefore, completed the proof of the theorem.

### 3. *Singular integrals: some extensions and variants of the preceding*

**3.1** The hypotheses of Theorem 1 were of two different kinds. One dealt with the $L^2$ theory, that is hypothesis (a); it was already formulated in the most general, but not the most useful, way. The second hypothesis, (b), which is used to deal with the weak-type (1, 1) estimate may be improved somewhat. The interest in formulating this improvement is that it represents what seems to be essentially the weakest condition for which the type of argument used in Theorem 1 holds. The condition is the following

$$(2') \qquad \int_{|x| \geq 2|y|} |K(x-y) - K(x)|\, dx \leq B', \qquad |y| > 0$$

That condition (2) implies (2′) is a simple matter that the reader may verify without difficulty. We have the following corollary of the method of proof of Theorem 1.

COROLLARY. *The results of Theorem 1 hold with condition (b) (equation (2)) replaced by (2′) above, and with the bound $B'$ replacing the bound $B$.*

The argument is the same as that of Theorem 1, except that in the proof of the weak-type (1, 1) inequality certain changes are made which we shall now indicate.

In this variant of the argument we shall not use the observation that diameters of the cubes $Q_j$ are comparable with their distances from $F = {}^c\left(\bigcup_j Q_j\right)$, (if only because we want to show that this fact is not really necessary here!). We get around this point by the simple device of considering for each cube $Q_j$ the cube $Q_j^*$ which has the same center $y^j$, but which is expanded $2n^{\frac{1}{2}}$ times. We have:

(i) $Q_j \subset Q_j^*$; if $\Omega^* = \cup Q_j^*$, $\Omega \subset \Omega^*$, and $m(\Omega^*) \leq (2n^{\frac{1}{2}})^n m(\Omega)$; if $F^* = {}^c\Omega^*$, then $F^* \subset F$.

(ii) If $x \notin Q_j^*$, then $|x - y^j| \geq 2\,|y - y^j|$ for all $y \in Q_j$, as an obvious geometric consideration shows.

The other difference is that we do not majorize $|Tb(x)|$ by the distance integral, but we estimate it directly; as a consequence a favorable estimate is obtained on the set $F^*$, instead of $F$.

As in the theorem,

$$Tb_j(x) = \int_{Q_j} [K(x-y) - K(x-y^j)]b_j(y)\, dy$$

we get

$$\int_{F^*}|Tb(x)|\,dx \le \sum_j \int_{x\notin Q_j^*}\int_{y\in Q_j} |K(x-y)-K(x-y^j)|\,|b(y)|\,dy\,dx.$$

However by (ii) for $y \in Q_j$,

$$\int_{x\in Q_j^*}|K(x-y)-K(x-y^j)|\,dx \le \int_{|x'|\ge 2|y'|} |K(x'-y')-K(x')|\,dx' \le B'$$

if we invoke the hypothesis. So

$$\int_{F^*}|Tb(x)|\,dx \le B'\sum_j \int_{Q_j}|b(y)|\,dy \le C\,\|f\|_1. \tag{17}$$

This brings us back to (15) in the proof of Theorem 1, and the rest is then as before.

**3.2** There is still an element which may be considered unsatisfactory in our formulation, and this is because of the following related points: The $L^2$ boundedness of $T$ has been assumed and not obtained as a consequence of some condition on the kernel $K$; an extraneous condition such as $K \in L^2$ subsists in the hypothesis; and for this reason our results do not directly treat the "principal-value" singular integrals, those which exist because of the cancellation of positive and negative values. However, from what we have done it is now a relatively simple matter to obtain the following theorem which covers the cases of interest.

THEOREM 2. *Suppose the kernel $K(x)$ satisfies the conditions*

$$\begin{aligned} &|K(x)| \le B|x|^{-n}, && 0<|x| \\ &\int_{|x|\ge 2|y|}|K(x-y)-K(x)|\,dx \le B, && 0<|y| \end{aligned} \tag{18}$$

*and*

$$\int_{R_1<|x|<R_2} K(x)\,dx = 0, \qquad 0<R_1<R_2<\infty. \tag{19}$$

*For $f \in L^p(\mathbf{R}^n)$, $1<p<\infty$, let*

$$T_\varepsilon(f)(x) = \int_{|y|\ge\varepsilon} f(x-y)K(y)\,dy, \qquad \varepsilon>0. \tag{20}$$

*Then*

$$\|T_\varepsilon(f)\|_p \le A_p\|f\|_p \tag{21}$$

*with $A_p$ independent of $f$ and $\varepsilon$. Also for each $f \in L^p(\mathbf{R}^n)$, $\lim_{\varepsilon\to0} T_\varepsilon(f) = T(f)$ exists in $L^p$ norm. The operator $T$ so defined also satisfies the inequality (21).*

The cancellation property alluded to is contained in condition (19). This hypothesis, together with (18), allows us to prove the $L^2$ boundedness and from this the $L^p$ convergence of the truncated integrals (21).

**3.3** For the $L^2$ boundedness we have the following lemma.

LEMMA. *Suppose $K$ satisfies the conditions of the above theorem with bound $B$.*

*Let* $K_\varepsilon(x) = \begin{cases} K(x) & \text{if} \quad |x| \geq \varepsilon \\ 0 & \text{if} \quad |x| < \varepsilon \end{cases}$. *Then obviously* $K_\varepsilon \in L^2(\mathbf{R}^n)$; *for the Fourier transforms we have the estimates*

$$\sup_y |\hat{K}_\varepsilon(y)| \leq CB, \qquad \varepsilon > 0 \tag{22}$$

*where $C$ depends only on the dimension $n$.*

We prove the inequality (22) first for the special case $\varepsilon = 1$.

Observe, and this requires only a semi-trivial estimate, that $K_1(x)$ satisfies the same conditions (18) and (19) as $K(x)$, except that the bound $B$ must be replaced by $CB$, where $C$ depends only on the dimension $n$.

Next,

$$\begin{aligned} \hat{K}_1(y) &= \lim_{R\to\infty} \int_{|x|\leq R} e^{2\pi i x\cdot y} K_1(x)\,dx \\ &= \int_{|x|<1/|y|} e^{2i\pi x\cdot y} K_1(x)\,dx + \lim_{R\to\infty} \int_{1/|y|\leq |x|\leq R} e^{2\pi i x\cdot y} K_1(x)\,dx \\ &= I_1 + I_2. \end{aligned}$$

However, $\int_{|x|\leq 1/|y|} e^{2\pi i x\cdot y} K_1(x)\,dx = \int_{|x|<1/|y|} [e^{2\pi i x\cdot y} - 1]\,K_1(x)\,dx$, because $K_1$ satisfies condition (19). Hence $|I_1| \leq C\,|y| \int_{|x|\leq 1/|y|} |x|\,|K_1(x)|\,dx \leq C'B$, in view of (18).

To estimate $I_2$ choose $z = z(y)$ so that $e^{2\pi i y\cdot z} = -1$. This choice can be realized if $z = \frac{1}{2}\frac{y}{|y|^2}$, with $|z| = \frac{1}{2\,|y|}$. But $\int_{\mathbf{R}^n} K_1(x)e^{2\pi i y\cdot z}dx = \frac{1}{2}\int_{\mathbf{R}^n} [K_1(x) - K_1(x-z)]e^{2\pi i z\cdot y}\,dx$, so

$$\begin{aligned} \lim_{R\to\infty} &\int_{1/|y|<|x|\leq R} K_1(x)e^{2\pi i x\cdot y}\,dx \\ &= \frac{1}{2}\lim_{R\to\infty} \int_{1/|y|\leq|x|\leq R} [K_1(x) - K_1(x-z)]e^{2\pi i x\cdot y}\,dx \\ &\qquad\qquad - \frac{1}{2}\int_{\left\{\substack{1/|y|<|x+z| \\ |x|\leq 1/|y|}\right\}} K_1(x)e^{2\pi i x\cdot y}\,dx. \end{aligned}$$

The last integral is taken over a region contained in the spherical shell, $\frac{1}{2}|y| \leq |x| \leq 1/|y|$, and is bounded since $|K_1(x)| \leq B|x|^{-n}$. The first integral on the right hand side is majorized by $\frac{1}{2}\int_{|x|>1/|y|}|K_1(x - z) - K_1(x)|\,dx$. But since $|z| = (2|y|)^{-1}$, the condition analogous to (18) applied to $K_1$ shows this integral is also bounded by $CB$. If we add the bounds for $I_1$ and $I_2$ we obtain the proof of our lemma for $K_1$. To pass to the case of general $K_\varepsilon$ we use a simple observation whose significance carries over to the whole theory presented in this chapter.

Let $\tau_\varepsilon$ be the dilation by the factor $\varepsilon$, $\varepsilon > 0$, that is $(\tau_\varepsilon f)(x) = f(\varepsilon x)$. Thus if $T$ is a convolution operator $T(f) = \varphi * f = \int_{\mathbf{R}^n}\varphi(x - y)f(y)\,dy$, then $\tau_{\varepsilon^{-1}}T\tau_\varepsilon$ is the convolution operator with the kernel $\varphi_\varepsilon$, where $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\varepsilon^{-1}x)$. In our case if $T$ corresponds to the kernel $K(x)$, $\tau_{\varepsilon^{-1}}T\tau_\varepsilon$ corresponds to the kernel $\varepsilon^{-n}K(\varepsilon^{-1}x)$. Notice that if $K$ satisfies the assumptions of our theorem, then $\varepsilon^{-n}K(\varepsilon^{-1}x)$ also satisfies these assumptions, and with the *same bounds*. (A similar remark holds for the assumptions of all the theorems in this chapter.) Now with our $K$ given, let $K' = \varepsilon^n K(\varepsilon x)$. Then $K'$ satisfies the conditions of our lemma with the same bound $B$, and so if $K_1' = \begin{cases} K_1'(x) & \text{if } |x| > 1 \\ 0 & \text{if } |x| < 1 \end{cases}$, then we know that $|\widehat{K_1'}(y)| \leq CB$. The Fourier transform of $\varepsilon^{-n}K_1'(\varepsilon^{-1}x)$ is $\widehat{K_1'}(\varepsilon y)$ and this is again bounded by $CB$; however $\varepsilon^{-n}K_1'(\varepsilon^{-1}x) = K_\varepsilon(x)$, therefore the lemma is completely proved.

**3.4** We can now prove Theorem 2. Since $K$ satisfies the conditions (18) and (19), then $K_\varepsilon(x)$ satisfies the same conditions with bounds not greater than $CB$. We pointed this out for $K_1$ in the proof of the lemma, and the passage from $K_1$ to $K_\varepsilon$ follows by the dilation argument also given there. Clearly however, each $K_\varepsilon \in L^2(\mathbf{R}^n)$, $\varepsilon > 0$. So an application of the corollary in §3.1 proves (21); that is, the $L^p$ norms of the operators are uniformly bounded. Next suppose $f_1$ is a continuous function with compact support which has one continuous derivative. Then

$$
\begin{aligned}
T_\varepsilon(f_1)(x) &= \int_{|y|>\varepsilon} K(y)f_1(x - y)\,dy \\
&= \int_{|y|>1} K(y)f_1(x - y)\,dy + \int_{1>|y|>\varepsilon} K(y)[f_1(x - y) - f_1(x)]\,dy,
\end{aligned}
$$

because of the cancellation condition (19). The first integral represents an $L^p$ function since it is the convolution of an $L^1$ function, $f_1$, with an $L^p$ function $K(y)$, since $|K(y)| \leq B|y|^{-n}$, if $|y| \geq 1$. The second integral is supported in a fixed compact set of $x$, and converges uniformly in $x$ as $\varepsilon \to 0$ because $|f_1(x - y) - f_1(x)| \leq A|y|$, in view of the differentiability of $f_1$. To summarize, $T_\varepsilon(f_1)$ converges in $L^p$ norm, if $\varepsilon \to 0$.

Finally an arbitrary $f \in L^p$ can be written as $f = f_1 + f_2$ where $f_1$ is of the type described above and $\|f_2\|_p$ is small. If we apply the basic inequality (21) for $f_2$ in place of $f$ we see that $\lim_{\varepsilon \to 0} T_\varepsilon(f)$ exists in $L^p$ norm; that the limiting operator $T$ also satisfies the inequality (21) is then obvious, and this completes the proof of the theorem.

We should point out that the kernel $K(x) = \dfrac{1}{\pi x}$, $x \in \mathbf{R}^1$, clearly satisfies the hypotheses of Theorem 2. We have therefore proved the existence of the Hilbert transform in the sense that if $f \in L^p(\mathbf{R}^1)$, $1 < p < \infty$, then

$$\lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|y| > \varepsilon} \frac{f(x-y)}{y}\, dy$$

exists in the $L^p$ norm and the resulting operator is bounded in $L^p$. A closer study of this example and certain important $n$-dimensional analogues will be taken up in the following section.

## 4. *Singular integral operators which commute with dilations*

**4.1** A basic class of operators in any abelian group is the set of operators that commute with (group) translations. However $\mathbf{R}^n$ is not only an abelian group, and so besides translations possesses certain other distinguished groups of transformations that act on it and which are related to its Euclidean structure. The transformations we have in mind here are the dilations $\tau_\varepsilon : x \to \varepsilon x$, $\varepsilon > 0$ and their corresponding action on functions $\tau_\varepsilon f(x) = f(\varepsilon x)$, discussed before.

We shall be interested in those operators which not only commute with translations but also with dilations. Among these we shall study the class of singular integral operators, falling under the scope of Theorem 2.

If $T$ corresponds to the kernel $K(x)$, then as we have already pointed out, $\tau_{\varepsilon^{-1}} T \tau_\varepsilon$ corresponds to the kernel $\varepsilon^{-n} K(\varepsilon^{-1} x)$. So if $\tau_{\varepsilon^{-1}} T \tau_\varepsilon = T$ we are back to the requirement $K(\varepsilon x) = \varepsilon^{-n} K(x)$, $\varepsilon > 0$; that is, $K$ is homogeneous of degree $-n$. Put another way

$$K(x) = \frac{\Omega(x)}{|x|^n}, \tag{23}$$

with $\Omega$ homogeneous of degree 0, i.e. $\Omega(\varepsilon x) = \Omega(x)$, $\varepsilon > 0$. This condition on $\Omega$ is equivalent with the fact that it is constant on rays emanating from the origin; $\Omega$ is, in particular, completely determined by its restriction to the unit sphere $S^{n-1}$. Let us try to reinterpret the conditions of Theorem 2 in terms of $\Omega$. First, by (18), $\Omega(x)$ must be bounded, and consequently integrable on $S^{n-1}$. The cancellation condition (19) is then the same as the condition

$$\int_{S^{n-1}} \Omega(x)\, d\sigma = 0 \tag{24}$$

where $d\sigma$ is the induced Euclidean measure on $S^{n-1}$. The precise condition (18) is not easily restated in terms of $\Omega$; what is evident, however, is that it requires a certain continuity of $\Omega$. Here we shall content ourselves in treating the case where $\Omega$ satisfies the following "Dini-type" condition suggested by (18):

$$\text{if } \sup_{\substack{|x-x'|\le\delta\\ |x|=|x'|=1}} |\Omega(x) - \Omega(x')| = \omega(\delta), \quad \text{then} \quad \int_0^1 \frac{\omega(\delta)\, d\delta}{\delta} < \infty. \tag{25}$$

Of course any $\Omega$ which is of class $C^1$, or even merely Lipschitz continuous, satisfies the condition (25)*.

**4.2** THEOREM 3. *Let $\Omega$ be homogeneous of degree 0, and suppose that $\Omega$ satisfies the cancellation property (24), and the smoothness property (25) above. For $1 < p < \infty$, and $f \in L^p(\mathbf{R}^n)$ let*

$$T_\varepsilon(f)(x) = \int_{|y|\ge\varepsilon} \frac{\Omega(y)}{|y|^n} f(x-y)\, dy.$$

(a) *Then there exists a bound $A_p$ (independent of $f$ or $\varepsilon$) so that*

$$\|T_\varepsilon(f)\|_p \le A_p \|f\|_p.$$

(b) $\lim_{\varepsilon\to 0} T_\varepsilon(f) = T(f)$ *exists in $L^p$ norm, and*

$$\|T(f)\|_p \le A_p \|f\|_p.$$

(c) *If $f \in L^2(\mathbf{R}^n)$, then the Fourier transforms of $f$ and $T(f)$ are related by $\widehat{(Tf)}(x) = m(x)\hat{f}(x)$, where $m$ is a homogeneous function of degree 0. Explicitly,*

$$m(x) = \int_{S^{n-1}} \left[\frac{\pi i}{2} \operatorname{sign}(x\cdot y) + \log(1/|x\cdot y|)\right] \Omega(y)\, d\sigma(y), \qquad |x| = 1. \tag{26}$$

The conclusions (a) and (b) of the theorem are immediate consequences of Theorem 2, once we have shown that any $K(x)$ of the form $\dfrac{\Omega(x)}{|x|^n}$ satisfies $\int_{|x|\ge 2|y|} |K(x-y) - K(x)|\, dx \le B$, if $\Omega$ is as in condition (25). However

$$K(x-y) - K(x) = \left(\frac{\Omega(x-y) - \Omega(x)}{|x-y|^n}\right) + \Omega(x)\left[\frac{1}{|x-y|^n} - \frac{1}{|x|^n}\right].$$

* For a generalization of this condition see §6.10, at the end of this chapter.

The second group of terms satisfies the correct estimate, since

$$\int_{|x|\geq 2|y|}\left|\frac{1}{|x-y|^n}-\frac{1}{|x|^n}\right|dx \leq C$$

and $\Omega$ is bounded. To estimate the first group of terms, we notice that the distance between the projections of $x-y$ and $x$ on the unit sphere, $\left|\frac{x-y}{|x-y|}-\frac{x}{|x|}\right|$, is bounded by $C\left|\frac{y}{x}\right|$, if $|x|\geq 2\,|y|$. So the integral corresponding to the first group of terms is dominated by

$$C'\int_{|x|\geq 2|y|}\omega\left(C\frac{|y|}{|x|}\right)\frac{dx}{|x|^n} = C''\int_0^{c/2}\frac{\omega(\delta)}{\delta}\,d\delta < \infty.$$

**4.3** Since $T$ is a bounded operator on $L^2$ which commutes with translations, we know by the proposition in §1.4 that $T$ can be realized in terms of a multiplier $m$, so that $(Tf)^\wedge$ is obtained by multiplying $\hat{f}$ by $m$. For such operators, the fact that they commute with dilations is equivalent with the property that the multiplier is homogeneous of degree 0. For our particular operators we have not only the existence of $m$ but an explicit expression of the multiplier in terms of the kernel. This formula is deduced as follows. Let

$$0<\varepsilon<\eta<\infty, \quad \text{and} \quad K_{\varepsilon,\eta}(x)=\begin{cases}\dfrac{\Omega(x)}{|x|^n}, & \text{if } \varepsilon\leq|x|\leq\eta\\ 0 & \text{otherwise}\end{cases}$$

Clearly $K_{\varepsilon,\eta}\in L^1(\mathbf{R}^n)$. If $f\in L^2(\mathbf{R}^n)$ then $(K_{\varepsilon,\eta}*f)^\wedge = K_{\varepsilon,\eta}^\wedge(y)\hat{f}(y)$.

We shall prove two facts about $K_{\varepsilon,\eta}^\wedge(y)$.

(i) $\sup|K_{\varepsilon,\eta}^\wedge(x)|\leq A$, with $A$ independent of $\varepsilon$ and $\eta$,

(ii) if $x\neq 0$, $\lim\limits_{\substack{\varepsilon\to 0\\ \eta\to 0}} K_{\varepsilon,\eta}^\wedge(x)=m(x)$, (see (26)).

For this purpose it is convenient to introduce polar coordinates. Let $x=Rx'$, $R=|x|$, $x'=x/|x|\in S^{n-1}$, and $y=ry'$, $r=|y|$, $y'=y/|y|\in S^{n-1}$. We shall also need the auxiliary integral

$$I_{\varepsilon,\eta}(x,y')=\int_\varepsilon^\eta[\exp(2\pi i\,Rrx'\cdot y')-\cos(2\pi\,Rr)]\,\frac{dr}{r},\quad R>0.$$

Its imaginary part, $\displaystyle\int_\varepsilon^\eta\frac{\sin 2\pi\,Rr(x'\cdot y')}{r}\,dr$, is as an integration by parts shows, uniformly bounded, and converges to

$$\left(\int_0^\infty\frac{\sin t}{t}\,dt\right)\text{sign}\,(x'\cdot y')=\frac{\pi}{2}\cdot\text{sign}\,(x'\cdot y').$$

Its real part is bounded in absolute value by $C \log 1/|x' \cdot y'| + C$, as again an integration by parts shows. Also $\lim_{\substack{\varepsilon\to 0\\ \eta\to\infty}} Re(I_{\varepsilon,\eta}(x\cdot y') = \log 1/|x'\cdot y'|$, since $\lim_{\substack{\varepsilon\to 0\\ \eta\to\infty}} \int_\varepsilon^\eta \frac{h(\lambda r) - h(\mu r)}{r}\, dr = h(0)\log(\mu/\lambda)$, if $\mu, \lambda > 0$, and $h$ is an appropriate function. In this case $h(r) = \cos 2\pi r$, $\lambda = R\,|x'\cdot y'|$, and $\mu = R$.

Now $K_{\varepsilon,\eta}^{\wedge}(x) = \int_{S^{n-1}} \left( \int_\varepsilon^\eta e^{2\pi i R r x'\cdot y'} \Omega(y') \frac{dr}{r} \right) d\sigma(y')$. Since

$$\int_{S^{n-1}} \Omega(y)\, d\sigma(y') = 0$$

we can introduce the factor $\cos 2\pi r R$ (which does not depend on $y'$) in the integral defining $K_{\varepsilon,\eta}^{\wedge}(x)$. This gives

$$K_{\varepsilon,\eta}^{\wedge}(x) = \int_{S^{n-1}} I_{\varepsilon,\eta}(x, y')\Omega(y')\, d\sigma(y').$$

Because of the properties of $I_{\varepsilon,\eta}$ just proved

$$|K_{\varepsilon,\eta}^{\wedge}(x)| \leq A \int_{S^{n-1}} [1 + \log 1/|x', y'|]\, |\Omega(y')|\, d\sigma(y')$$

which proves (i) (the uniform boundedness of the $K_{\varepsilon,\eta}^{\wedge}(x)$), since $\Omega$ is itself bounded. In view of the limit of $I_{\varepsilon,\eta}(x)$, as $\varepsilon \to 0$, $\eta \to \infty$ just ascertained, and the dominated convergence theorem, we get

$$\lim_{\substack{\varepsilon\to 0\\ \eta\to\infty}} K_{\varepsilon,\eta}^{\wedge}(x) = m(x),$$

if $x \neq 0$, that is (ii).

By the Plancherel theorem then, if $f \in L^2(\mathbf{R}^n)$, $K_{\varepsilon,\eta} * f$ converges in $L^2$ norm as $\eta \to \infty$, and $\varepsilon \to 0$, and the Fourier transform of this limit is $m(x)\hat{f}(x)$. However if we keep $\varepsilon$ fixed and let $\eta \to \infty$, then clearly $\int K_{\varepsilon,\eta}(y) f(x-y)\, dy$ converges everywhere to $\int_{|y|\geq\varepsilon} K(y) f(x-y)\, dy$, which is $T_\varepsilon(f)$.

Letting now $\varepsilon \to 0$, we obtain the conclusion (c) and our theorem is completely proved.

**4.4** It is to be noted that the proof of part (c) holds under very general conditions on $\Omega$. Write $\Omega = \Omega_e + \Omega_o$ where $\Omega_e$ is the even part of $\Omega$, $\Omega_e(x) = \Omega_e(-x)$, and $\Omega_o(x)$ is the odd part, $\Omega_o(-x) = -\Omega_o(x)$. Then because of the uniform boundedness of the sine integral we required only $\int_{S^{n-1}} |\Omega_o(y')|\, d\sigma(y') < \infty$, i.e. the integrability of the odd part. For the even part, the proof requires the uniform boundedness of

$$\int_{S^{n-1}} |\Omega_e(y')| \log 1/|x', y'| \, d\sigma(y').$$

This observation is suggestive of certain generalizations of Theorem 2, (see §6.5).

It goes without saying that the transformations described in Theorem 3 are not bounded on either $L^1(\mathbf{R}^n)$ or $L^\infty(\mathbf{R}^n)$. In the case of the Hilbert transform this can be seen immediately by the explicit example of the transform of the characteristic function of the interval $(a, b)$, which has the value $\frac{1}{\pi} \log \left| \frac{x - b}{x - a} \right|$. Other examples are described in §6.1.

**4.5** Theorem 3 guaranteed the existence of the singular integral transformation

$$\lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} \frac{\Omega(y)}{|y|^n} f(x - y) \, dy \tag{27}$$

in the sense of convergence in the $L^p$ norm. The natural counterpart of this result is that of convergence almost everywhere. In the classical case corresponding to the Hilbert transform the results of almost everywhere convergence predated the $L^p$ results, and the former were obtained as a consequence of Fatou's theorem guaranteeing the boundary values almost everywhere of bounded harmonic functions. In our present situation the almost everywhere results will be, in effect, a consequence of the existence of the limit (27) in the $L^p$ norm, already proved. As in other questions involving almost everywhere convergence, it is best to consider also the corresponding maximal function.

THEOREM 4. *Suppose that $\Omega$ satisfies the conditions of the previous theorem. For $f \in L^p(\mathbf{R}^n)$, $1 \leq p < \infty$, consider*

$$T_\varepsilon(f)(x) = \int_{|y| \geq \varepsilon} \frac{\Omega(y)}{|y|^n} f(x - y) \, dy, \qquad \varepsilon > 0.$$

*(The integral converges absolutely for every x.)*

(a) $\lim_{\varepsilon \to 0} T_\varepsilon(f)(x)$ *exists for almost every x.*

(b) *Let* $T^*(f)(x) = \sup_{\varepsilon > 0} |T_\varepsilon(f)(x)|$. *If* $f \in L^1(\mathbf{R}^n)$, *then the mapping* $f \to T^*f$ *is of weak type* $(1, 1)$.

(c) *If* $1 < p < \infty$, *then* $\|T^*(f)\|_p \leq A_p \|f\|_p$.

**4.6** The argument for Theorem 4 presents itself in three stages. At first there is the proof of inequality (c) which can be obtained as a relatively easy consequence of the $L^p$ norm existence of $\lim_{\varepsilon \to 0} T_\varepsilon$, already proved, and certain general properties of "approximations to the identity." We shall therefore postpone the proof of (c) to the next chapter where we deal with these matters more systematically.

*4.6.1* This brings us to the second, and most difficult stage of the proof, leading to conclusion (b). Here the argument proceeds in the main as in the proof of the weak-type (1, 1) result for singular integrals, in particular the variant given in §3.1. We review it with deliberate brevity so as to avoid a repetition of details already examined. For a given $\alpha > 0$, we split $f = g + b$ as in §2.4. We also consider for each cube $Q_j$ its mate $Q_j^*$, which has the same center $y^j$ but which is expanded $2n^{\frac{1}{2}}$ times. The following additional geometric remarks concerning these cubes are nearly obvious.

(iii) Suppose $x \in {}^cQ_j^*$ (in particular this is so if $x \in F^*$) and assume that for some $y \in Q_j$, $|x - y| = \varepsilon$. Then the closed ball centered at $x$, of radius $\gamma_n\varepsilon$, contains $Q_j$; i.e. $\overline{B(x, r)} \supset Q_j$, if $r = \gamma_n\varepsilon$.

(iv) Under the same hypotheses as (iii), we have that $|x - y| \geq \gamma_n'\varepsilon$, for every $y \in Q_j$.

$\gamma_n$ and $\gamma_n'$ depend only on the dimension $n$, and not the particular cube $Q_j$.

*4.6.2* With these observations, and following the development in §2.4 we shall prove that if $x \in F^*$,

$$(28) \quad \sup_{\varepsilon > 0} |T_\varepsilon(b(x))| \leq \sum_j \int_{Q_j} |K(x - y) - K(x - y^j)|\,|b(y)|\, dy + C \sup_{r > 0} \frac{1}{m(B(x, r))} \int_{B(x,r)} |b(y)|\, dy,$$

with $\quad K(x) = \dfrac{\Omega(x)}{|x|^n}$.

Thus the addition of the maximal function to the right side of (28) is the main new element of the proof. To prove (28), fix $x \in F^*$, and $\varepsilon > 0$. Now the cubes $Q_j$ fall into three classes:

(a) for all $y \in Q_j$, $|x - y| < \varepsilon$

(b) for all $y \in Q_j$, $|x - y| > \varepsilon$

(c) there is a $y \in Q_j$, so that $|x - y| = \varepsilon$.

We now examine $T_\varepsilon b(x)$.

$$(29) \qquad T_\varepsilon b(x) = \sum_j \int_{Q_j} K_\varepsilon(x - y) b(y)\, dy.$$

*Case* (a). $K_\varepsilon(x - y) = 0$ if $|x - y| < \varepsilon$, and so the integral over the cube $Q_j$ in (29) is zero.

*Case* (b). $K_\varepsilon(x - y) = K(x - y)$, if $|x - y| > \varepsilon$, and therefore this integral over $Q_j$ equals

$$\int_{Q_j} K(x - y)b(y)\,dy = \int_{Q_j} [K(x - y) - K(x - y^j)]b(y)\,dy.$$

This term is majorized in absolute value by

$$\int_{Q_j} |K(x - y) - K(x - y^j)|\,|b(y)|\,dy,$$

which expression appears in the right side of (28).

*Case* (c). We write simply

$$\left|\int_{Q_j} K_\varepsilon(x - y)b(y)\,dy\right| \leq \int_{Q_j} |K_\varepsilon(x - y)|\,|b(y)|\,dy$$
$$= \int_{B(x,r)\cap Q_j} |K_\varepsilon(x - y)|\,|b(y)|\,dy,$$

by (iii), with $r = \gamma_n\varepsilon$. However $|K_\varepsilon(x - y)| \leq \left|\dfrac{\Omega(x - y)}{|x - y|^n}\right| \leq \dfrac{B}{(\gamma_n')^n\varepsilon^n}$, by (iv) and the fact that $\Omega$ is bounded. So

$$\left|\int_{Q_j} K_\varepsilon(x - y)b(y)\,dy\right| \leq C\,\frac{1}{m[B(x, r)]} \int_{B(x,r)\cap Q_j} |b(y)|\,dy$$

in this case. If we add over all cubes $Q_j$ we finally obtain

$$|T_\varepsilon b(x)| \leq \sum_j \int_{Q_j} |K(x - y) - K(x - y^j)|\,|b(y)|\,dy$$
$$+ C\,\frac{1}{B(x, r)} \int_{B(x,r)} |b(y)|\,dy, \qquad r = \gamma_n\varepsilon.$$

The taking of the supremum over $\varepsilon$ gives (28).

This inequality can be written in the form

$$|T^*b(x)| \leq \Sigma + CMb(x), \qquad x \in F^*,$$

and so

$$m\{x \in F^* : |T^*b(x)| > \alpha/2\} \leq m\{x \in F^* : \Sigma > \alpha/4\}$$
$$+ m\{x \in F^* : CMb(x) > \alpha/4\}.$$

The measures of both sets appearing in the right-hand side of the just-written inequality are bounded by $\dfrac{c}{\alpha}\,\|b\|_1$. In the first case this is because an inequality similar to (17) of §3.1 holds for $\Sigma$; for the second set it is because of the weak-type estimate for the maximal function $M$ (theorem in §1.3, Chapter I). The weak type (1, 1) property of $T^*$ then follows as in the proof of the same property for $T$, in §3.1 (or in greater detail in §2.4 following equation (15)).

*4.6.3* The final stage of the proof of the theorem, the passage from the inequalities of $T^*$ to the existence of the limits almost everywhere, follows the familiar pattern described in §1.5 of Chapter I. More precisely, for any $f \in L^p(\mathbf{R}^n)$, $1 \leq p < \infty$, let

$$\Lambda(f)(x) = |\limsup_{\varepsilon \to 0} T_\varepsilon(f)(x) - \limsup_{\varepsilon \to 0} T_\varepsilon(f)(x)|.$$

Clearly $\Lambda(f)(x) \leq 2(T^*f)(x)$. Now write $f = f_1 + f_2$ where $f_1$ has compact support and is of class $C^1$, and $\|f_2\|_p \leq \delta$. We have already remarked in §3.4 that $T_\varepsilon f_1$ converges uniformly as $\varepsilon \to 0$, so $\Lambda f_1(x) \equiv 0$. But $\Lambda(f)(x) \leq \Lambda(f_1)(x) + \Lambda(f_2)(x)$, so

$$\|\Lambda(f_2)\|_p \leq 2A_p \|f_2\|_p \leq 2A_p\delta, \quad \text{if} \quad 1 < p < \infty.$$

This shows $\Lambda f_2 = 0$, almost everywhere, thus $\Lambda f = 0$ almost everywhere, and so $\lim_{\varepsilon \to 0} T_\varepsilon f$ exists almost everywhere if $1 < p < \infty$. In the case $p = 1$, we get similarly

$$m\{x : \Lambda f(x) > \alpha\} \leq \frac{A}{\alpha} \|f_2\|_1 \leq \frac{A\delta}{\alpha},$$

and so again $\Lambda f(x) = 0$ almost everywhere, which implies that $\lim_{\varepsilon \to 0} T_\varepsilon(f)(x)$ exists almost everywhere.

## 5. *Vector-valued analogues*

**5.1** It is interesting to point out that the results of this chapter, where our functions were assumed to take real or complex values, can be extended to the case of functions taking their values in a Hilbert space. We present this generalization because it can be put to good use in several problems. An indication of this usefulness is given in the Littlewood-Paley theory in Chapter IV.

We begin by reviewing quickly certain aspects of integration theory in this context.

Let $\mathscr{H}$ be a separable Hilbert space. Then a function $f(x)$, from $\mathbf{R}^n$ to $\mathscr{H}$ is *measurable* if the scalar valued functions $(f(x), \varphi)$ are measurable, where $(\cdot, \cdot)$ denotes the inner product of $\mathscr{H}$, and $\varphi$ denotes an arbitrary vector of $\mathscr{H}$. If $f(x)$ is such a measurable function, then $|f(x)|$ is also measurable (as a function with non-negative values), where $|\cdot|$ denotes the norm of $\mathscr{H}$. Thus $L^p(\mathbf{R}^n, \mathscr{H})$ is defined as the equivalence classes of measurable functions $f(x)$ from $\mathbf{R}^n$ to $\mathscr{H}$, with the property that the norm $\|f\|_p = (\int_{\mathbf{R}^n} |f(x)|^p \, dx)^{1/p}$ is finite, when $p < \infty$; when $p = \infty$ there is a similar definition, except $\|f\|_\infty = \text{ess sup } |f(x)|$.

Next, let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two separable Hilbert spaces, and let $B(\mathscr{H}_1, \mathscr{H}_2)$ denote the Banach space of bounded linear operators from $\mathscr{H}_1$ to $\mathscr{H}_2$, with the usual operator norm. We say that a function $f(x)$, from $\mathbf{R}^n$ to $B(\mathscr{H}_1, \mathscr{H}_2)$ is measurable if $f(x)\varphi$ is an $\mathscr{H}_2$-valued measurable function for every $\varphi \in \mathscr{H}_1$. In this case also $|f(x)|$ is measurable and we can define the space $L^p(\mathbf{R}^n, B(\mathscr{H}_1, \mathscr{H}_2))$, as before; (here again $|\cdot|$ denotes the norm, this time in $B(\mathscr{H}_1, \mathscr{H}_2)$). The usual facts about convolution hold in this setting. For example, suppose $K(x) \in L^q(\mathbf{R}^n, B(\mathscr{H}_1, \mathscr{H}_2)$ and $f(x) \in L^p(\mathbf{R}^n, \mathscr{H}_1)$. Then $g(x) = \int_{\mathbf{R}^n} K(x-y) f(y) \, dy$ converges in the norm of $\mathscr{H}_2$ for almost every $x$, and

$$|g(x)| \leq \int_{\mathbf{R}^n} |K(x-y)f(y)| \, dy \leq \int_{\mathbf{R}^n} |K(x-y)| \, |f(y)| \, dy.$$

Also $\|g\|_r \leq \|K\|_q \|f\|_p$, if $1/r = 1/p + 1/q - 1$, with $1 \leq r \leq \infty$.

**5.2** Suppose that $f(x) \in L^1(\mathbf{R}^n, \mathscr{H})$. Then we can define its Fourier transform $\hat{f}(y) = \int_{\mathbf{R}^n} e^{2\pi i x \cdot y} f(x) \, dx$ which is an element of $L^\infty(\mathbf{R}^n, \mathscr{H})$. If $f \in L^1(\mathbf{R}^n, \mathscr{H}) \cap L^2(\mathbf{R}^n, \mathscr{H})$, then $\hat{f}(y) \in L^2(\mathbf{R}^n, \mathscr{H})$ with $\|\hat{f}\|_2 = \|f\|_2$. The Fourier transform can then be extended by continuity to a unitary mapping of the Hilbert space $L^2(\mathbf{R}^n, \mathscr{H})$ to itself.

These facts can be obtained easily from the scalar-valued case by introducing an arbitrary orthonormal basis in $\mathscr{H}$.

**5.3** Now suppose that $\mathscr{H}_1$ and $\mathscr{H}_2$ are two given Hilbert spaces. Assume $f(x)$ takes values in $\mathscr{H}_1$, and $K(x)$ takes values in $B(\mathscr{H}_1, \mathscr{H}_2)$. Then

$$Tf(x) = \int_{\mathbf{R}^n} K(y) f(x-y) \, dy,$$

whenever defined, takes values in $\mathscr{H}_2$.

THEOREM 5. *The results in this chapter, in particular Theorem 1, its corollary, and Theorems 2 to 4 are valid in the more general context where $f$ takes its value in $\mathscr{H}_1$, $K$ takes its values in $B(\mathscr{H}_1, \mathscr{H}_2)$ and $(Tf)$ and $T_\varepsilon(f)$ take their value in $\mathscr{H}_2$, and where throughout the absolute value $|\cdot|$ is replaced by the appropriate norm in $\mathscr{H}_1$, $B(\mathscr{H}_1, \mathscr{H}_2)$ or $\mathscr{H}_2$ respectively.*

This theorem is not in any *obvious* way a corollary of the scalar-valued case treated. Its proof, however, consists in nothing but an identical repetition of the arguments given for the scalar-valued case, if we take into account the remarks made in the above paragraphs. This seemingly bold assertion may, in fact, be verified without difficulty by a patient review of the proofs; but if the reader is not so disposed, he may find the necessary details in the literature cited at the end of the chapter.

Several clarifying observations emerge from such a verification:

(a) The final bounds obtained do not depend on the Hilbert spaces $\mathcal{H}_1$ or $\mathcal{H}_2$, but only on $B$, $p$, and $n$, as in the scalar-valued case.

(b) Most of the argument goes through in the even greater generality of Banach space-valued functions, appropriately defined. The Hilbert space structure is used only in the $L^2$ theory when applying the variant of Plancherel's formula described in §5.2.

The Hilbert space structure also enters in the following corollary:

COROLLARY. *With the same assumptions as in Theorem 5, if in addition*

$$\|T(f)\|_2 = c\,\|f\|_2, \qquad c > 0, \qquad f \in L^2(\mathbf{R}^n, \mathcal{H}_1)$$

*then* $\|f\|_p \leq A_p' \,\|T(f)\|_p$, *if* $f \in L^p(\mathbf{R}^n, \mathcal{H}_1)$, *if* $1 < p < \infty$.

*Proof.* We remark that the $L^2(\mathbf{R}^n, \mathcal{H}_j)$ are Hilbert spaces. In fact, let $(,)_j$ denote the inner product of $\mathcal{H}_j$, $j = 1, 2$, and let $\langle , \rangle_j$ denote the corresponding inner product in $L^2(\mathbf{R}^n, \mathcal{H}_j)$; that is

$$\langle f, g\rangle_j = \int_{\mathbf{R}^n} (f(x), g(x))_j \, dx.$$

Now $T$ is a bounded linear transformation from the Hilbert space $L^2(\mathbf{R}^n, \mathcal{H}_1)$ to the Hilbert space $L^2(\mathbf{R}^n, \mathcal{H}_2)$, and so by the general theory of inner products there exists a unique adjoint transformation $\tilde{T}$, from $L^2(\mathbf{R}^n, \mathcal{H}_2)$ to $L^2(\mathbf{R}^n, \mathcal{H}_1)$, which satisfies the characterizing property

$$\langle Tf_1, f_2\rangle_2 = \langle f_1, \tilde{T}f_2\rangle_1, \quad \text{with} \quad f_j \in L^2(\mathbf{R}^n, \mathcal{H}_j).$$

But our assumption is equivalent with the identity

$$\langle Tf, Tg\rangle_2 = c^2\langle f, g\rangle_1, \quad \text{for all} \quad f, g \in L^2(\mathbf{R}^n, \mathcal{H}_1).$$

Thus using the definition of the adjoint, $\langle \tilde{T}Tf, g\rangle_1 = c^2\langle f, g\rangle_1$, and so the assumption can be restated as

$$\tilde{T}Tf = c^2 f, \qquad f \in L^2(\mathbf{R}^n, \mathcal{H}_1). \tag{30}$$

$\tilde{T}$ is again an operator of the same kind as $T$ but it takes function with values in $\mathcal{H}_2$ to functions with values in $\mathcal{H}_1$, and its kernel $\tilde{K}(x)$, is

$\tilde{K}(x) = K^*(-x)$, where here * denotes the adjoint of an element in $B(\mathscr{H}_1, \mathscr{H}_2)$.

This is obvious on the formal level since

$$\begin{aligned}\langle Tf_1, f_2\rangle_2 &= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} (K(x-y)f_1(y), f_2(x))_2\, dy\, dx \\ &= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} (f_1(y), K^*(-(y-x))f_2(x))_1\, dx\, dy \\ &= \langle f_1, \tilde{T}f_2\rangle_1.\end{aligned}$$

The rigorous justification of this identity is achieved by a simple limiting argument. We will not tire the reader with the routine details.

This being said we have only to add the remark that $K^*(-x)$ satisfies the same conditions as $K(x)$, and so we have for it similar conclusions as for $K$ (with the same bounds). Thus by (30),

$$c^2 \|f\|_p = \|\tilde{T}Tf\|_p \le A_p \|Tf\|_p.$$

This proves the corollary with $A_p' = A_p/c^2$.

This corollary applies in particular to the singular integrals of §4; then the condition required is that the multiplier $m(x)$ have constant absolute value. This is the case, for example, when $T$ is the Hilbert transform, $K(x) = \dfrac{1}{\pi x}$, and $m(x) = i \operatorname{sign} x$. For a generalization of this remark see §6.6 below.

## 6. *Further results*

**6.1** Let $K(x) = \dfrac{\Omega(x)}{|x|^n}$ be as in Theorem 3, with $\Omega \not\equiv 0$.

(a) If $f \in L^1(\mathbf{R}^n)$, $f \ge 0$, then $Tf \notin L^1(\mathbf{R}^n)$, if $f \not\equiv 0$. *Hint:* $m(x)\hat{f}(x)$ cannot be continuous at 0, since $m(x)$ is homogeneous of degree 0 and non-constant, and $\hat{f}(0) > 0$.

(b) There exists a continuous $f$, which vanishes outside the unit ball $B$, such that $T(f)$ is unbounded near every point of $B$.

**6.2** (a) If $A_p$ is the $L^p$ bound for $T$ in Theorem 1, 2, or 3, then $A_p \le \dfrac{A}{p-1}$ for $1 < p \le 2$ and $A_p \le Ap$, for $2 \le p < \infty$. (See the remark at the end of §4, Chapter I.)

(b) If $f$ is supported in a ball $B$, and $|f| \log (2 + |f|)$ is integrable over $B$, then $Tf$ is integrable over $B$.

(c) If $f$ is bounded and supported on $B$, then $e^{a|Tf|}$ is integrable over $B$, for suitable $a > 0$. *Hint:* Write $\int e^{a|Tf|}\, dx = \sum \dfrac{a^n}{n!} \|Tf\|_n^n$ and use part (a).

(d) The same result holds for the maximal operator $T^*$ of Theorem 4. For these results see Calderón and Zygmund [1], and Zygmund [8], Chapter XII.

**6.3** Let $(Tf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} K(x, y) f(y)\, dy$, with $|K(x,y)| \le \dfrac{A}{|x-y|^n}$. Suppose $T$ is bounded on $L^p(\mathbf{R}^n)$. Then $T$ is bounded on the $L^p$ space taken with respect to the measure $|x|^\alpha\, dx$, (instead of $dx$) where $-n < \alpha < n(p-1)$. See Stein [2]; this easily implies the same result for $(1 + |x|)^\alpha\, dx$.

**6.4** The following approach unifies the maximal function and differentiation theorems of Chapter I, and many of the singular integrals of the present chapter.

Let $L(x)$ be integrable on $\mathbf{R}^n$, and suppose $L(x) = 0$, if $|x| \ge 2$,

$$\int_{\mathbf{R}^n} L(x)\, dx = 0,$$

and

$$\int_{\mathbf{R}^n} |L(x-y) - L(x)|\, dx \le B\,|y|.$$

For any pair of integers $i, j$, define $L_{i,j}$ by $L_{i,j}(x) = \sum_{k=-i}^{k=j} 2^{nk} L(2^k x)$. Write $T_{i,j}f = L_{i,j} * f$. Then if $T_* f = \sup_{i,j} |T_{i,j}f(x)|$ we have

(a) $f \to T_* f$ is of weak-type $(1, 1)$

(b) $f \to T_* f$ is bounded on $L^p$, $1 < p < \infty$

(c) if $f \in L^p$, $1 \le p < \infty$, $\lim_{\substack{i \to \infty \\ j \to \infty}} T_{i,j}f$ exists almost everywhere, and also in $L^p$ norm, if $1 < p$.

Two interesting examples are:

(i) $L(x) = 1 - 2^n$, for $|x| \le 1$ and $L(x) = 1$, for $1 \le |x| \le 2$, 0 otherwise. Then

$$(T_{i,0}f)(x) = 2^{-ni} \int_{|y| \le 2^{i+1}} f(x-y)\, dy - 2^n \int_{|y| \le 1} f(x-y)\, dy$$

(ii) $L(x) = \dfrac{\Omega(x)}{|x|^n}$, for $1 \le |x| \le 2$, 0 otherwise. Then

$$(T_{i,j}f)(x) = \int_{2^{-i} \le |y| \le 2^{j-1}} \frac{\Omega(y)}{|y|^n} f(x-y)\, dy.$$

(For details see Cotlar [2].)

**6.5** (a) Let $y'$ be a unit vector in $\mathbf{R}^n$. The Hilbert transform in the direction $y'$ can be defined as $\lim_{\varepsilon \to 0} H_{y'}^{(\varepsilon)}(f)(x)$, where

$$H_{y'}^{(\varepsilon)}(f)(x) = \int_{|t| > \varepsilon} \frac{f(x - y't)\, dt}{t} = \int_\varepsilon^\infty \frac{[f(x - y't) - f(x + y't)]\, dt}{t}.$$

Then $\|H_{y'}^{(\varepsilon)} f\|_p \le A_p \|f\|_p$ for $f \in L^p(\mathbf{R}^n)$, $1 < p < \infty$, with $A_p$ independent of $y'$ and $\varepsilon$.

(b) Suppose $\Omega(y')$ is homogeneous of degree 0, is integrable over the unit sphere $S^{n-1}$ and is odd, i.e. $\Omega(y') = -\Omega(-y')$. Let

$$T_\varepsilon(f)(x) = \int_{|y| \geq \varepsilon} \frac{\Omega(y)}{|y|^n} f(x-y)\,dy.$$

Then

$$T_\varepsilon = \tfrac{1}{2} \int_{S^{n-1}} \Omega(y') H_{y'}^{(\varepsilon)}\,d\sigma(y'),$$

and so

$$\|T_\varepsilon(f)\|_p \leq \left(\tfrac{1}{2} A_p \int_{S^{n-1}} |\Omega(y')|\,d\sigma(y')\right) \|f\|_p.$$

(c) A similar but more difficult result holds if $\Omega$ is even. It is then required that $|\Omega(y')| \log (2 + |\Omega(y')|)$ be integrable over $S^{n-1}$.

(d) The behavior of $T_\varepsilon(f)$ for $f \in L^1(\mathbf{R}^n)$, for these general $\Omega$ considered here, remains open.

For details on (a), (b), and (c) above see Calderón and Zygmund [3]. This part of the theory is also presented in *Fourier Analysis*, Chapter VI, in somewhat lesser generality.

**6.6** Suppose $m(x)$ is homogeneous of degree 0, and continuous on $S^{n-1}$. For $f \in L^2(\mathbf{R}^n)$ define $Tf$ by $(Tf)^\wedge(x) = m(x)\hat{f}(x)$. Suppose $\|T(f)\|_p \leq A_p \|f\|_p$, for $f \in L^2 \cap L^p$ for some $p$, $1 < p < \infty$. If $|m(x)| \geq c > 0$, then also $\|f\|_p \leq B_p \|Tf\|_p$. (See Calderón and Zygmund [4], Hörmander [1], Benedek, Calderón and Panzone [1].

**6.7** Let $T$ be the Hilbert transform (1), and $\chi_E$ denote the characteristic function of a subset $E$ of $\mathbf{R}^1$, of finite measure. Then the distribution function of $T\chi_E$ depends only on the measure of $E$; more precisely if $\lambda(\alpha)$ is this distribution function, then $\lambda(\alpha) = \dfrac{2m(E)}{\sin h\ \pi\alpha}$. See Stein and Weiss [1].

**6.8** As already pointed out, the dilations $x \to \varepsilon x = (\varepsilon x_1, \varepsilon x_2, \ldots, \varepsilon x_n)$ play an important role in this chapter. There are variants of many of the results of this chapter where this type of homogeneity is replaced by a non-isotropic one, i.e. $x \to \varepsilon \circ x = (\varepsilon^{a_1} x_1, \varepsilon^{a_2} x_2, \ldots, \varepsilon^{a_n} x_n)$, where $a_1, a_2, \ldots, a_n$ are fixed positive exponents, and $\varepsilon > 0$. Then the action on the kernels $K(x) \to \varepsilon^n K(\varepsilon x)$ is replaced by $K(x) \to \varepsilon^a K(\varepsilon \circ x)$, where $a = a_1 + a_2 \cdots + a_n$. For details see Jones [1], Fabes and Rivière [1], Kree [1], and Besov, Il'in, and Lizorkin [1].

**6.9** Let $T$ and $K(x) = \dfrac{\Omega(x)}{|x|^n}$ be as in Theorem 3. Suppose that $0 < \alpha < 1$, and $f$ is a continuous function of compact support which satisfies

$$|f(x+t) - f(x)| \leq A|t|^\alpha.$$

Then if $g(x) = T(f)$, we also have $|g(x + t) - g(x)| \leq B|t|^\alpha$. (Hint: If $\Omega$ is sufficiently smooth the proof is "elementary"; see Privalov [1], Calderón and Zygmund [2], and in general, Taibleson [1].

**6.10** Let $\Omega$ be homogeneous of degree 0, integrable on the unit sphere and $\int_{S^{n-1}} \Omega \, d\sigma = 0$. Suppose that $\sup_{|r| \leq \delta} \int_{S^{n-1}} |\Omega(r(x')) - \Omega(x')| \, d\sigma \leq \omega(\delta)$, with $\displaystyle\int_0^\varepsilon \frac{\omega(\delta)\, d\delta}{\delta} < \infty$. Here $r$ designates rotations about the origin, and $|r|$ denotes the distance of $r$ from the identity rotation, measured by any smooth Riemannian metric on the group of rotations. Then

(a) $\int_{S^{n-1}} |\Omega| \log^+ |\Omega| \, d\sigma < \infty$. Therefore the $L^p$ theory of §6.5 applies.

(b) If $K(x) = \dfrac{\Omega(x)}{|x|^n}$, then $\int_{|x| \geq 2|y|} |K(x - y) - K(x)| \, dx \leq B$, and so the $L^1$ theory of §3.1 also applies. See Calderón, M. Weiss, and Zygmund [1].

**6.11** A slight modification of the argument of §3.3 proves the following. Suppose $K(x)$ is a given function which satisfies the assumptions

(i) $\int_{|x| \leq R} |x| \, |K(x)| \, dx \leq BR, \qquad 0 < R < \infty$

(ii) $\int_{|x| \geq 2|y|} |K(x - y) - K(x)| \, dx \leq B$

(iii) $\displaystyle \Big| \int_{R_1 < |x| < R_2} K(x) \, dx \Big| \leq B, \qquad 0 < R_1 < R_2 < \infty.$

Let $K_{\varepsilon,\eta}(x) = K(x)$ if $\varepsilon < |x| < \eta$,
$= 0$ otherwise.

Then $|\hat{K}_{\varepsilon,\eta}(x)| \leq CB$, with $C$ independent of $\varepsilon$ and $\eta$. See Benedek, Calderón, and Panzone [1].

**6.12** Suppose $f$ is bounded and has bounded support. Let $I^s(f)(x) = \int_{\mathbf{R}^n} f(x - y) \, |y|^{-n-\sigma+it} \, dy$, $\sigma > 0$, $s = \sigma + it$. Then for each $\varepsilon > 0$

$$I^s(f)(x) = \int_{|y| \leq \varepsilon} \{f(x - y) - f(x)\} \, |y|^{-n+s} \, dy + \int_{|y| > \varepsilon} f(x - y) \, |y|^{-n+s} \, dy + \omega_{n-1}(\varepsilon^s/s) f(x).$$

This shows that $I^{it}f(x) = \lim_{\sigma \to 0} I^{\sigma - it}(f)(x)$, exists if $t \neq 0$ and if in addition $f$ is of class $C^1$. Finally we may apply §6.11 above with $K(x) = |x|^{-n+it}$, and Theorem 1 of the present chapter. This shows that the operator

$$f \to I_\varepsilon^{it}(f) = \int_{|y| > \varepsilon} f(x - y) \, |y|^{-n+it} \, dy,$$

is bounded on $L^p(\mathbf{R}^n)$, $1 < p < \infty$, uniformly in $\varepsilon$. By choosing an appropriate sequence of $\varepsilon$ tending to zero (i.e. such that $\varepsilon^{it} \to 0$, for fixed $t \neq 0$) we see that $I^{it}$ can be extended to be a bounded operator on $L^p(\mathbf{R}^n)$, $1 < p < \infty$. It can also be seen that by the use of §3.3 of Chapter III we have

$$(I^{it}(f))\hat{}\,(x) = \gamma_{0,it} |x|^{-it} \hat{f}(x),$$

with $\gamma_{0,it} = \pi^{n/2-it}\dfrac{\Gamma(it/2)}{\Gamma(n/2-it/2)}$.

For related results see Muckenhoupt [1].

**6.13** Let $K_1(x) = \dfrac{\Omega_1(x)}{|x|^n}$, and $K_2(x) = \dfrac{\Omega_2(x)}{|x|^m}$ be two kernels of the type considered in §4, defined respectively for $\mathbf{R}^n$ and $\mathbf{R}^m$. On $L^p(\mathbf{R}^{n+m})$ define the transformation $f \to T_{\varepsilon,\delta}(f)$, by $T_{\varepsilon,\delta} = T^1_\varepsilon \otimes T^2_\delta$ where

$$T^1_\varepsilon(f)(x^1) = \int_{|y|^1\geq\varepsilon} \frac{\Omega_1(y^1)}{|y^1|^n} f(x^1 - y^1)\,dy^1,$$

and $T^2_\delta(f)(x^2) = \displaystyle\int_{|y^2|\geq\delta} \frac{\Omega_2(y^2)f(x^2-y^2)}{|y^2|^m}\,dy^2$. ($T_{\varepsilon,\delta}$ can be taken to be the composition of $T^1_\varepsilon$ acting on the first $n$ variables, and $T^2_\delta$ acting on the last $m$ variables of functions on $\mathbf{R}^{n+m}$.)

(a) If $f \in L^p(\mathbf{R}^{n+m})$, then $\lim\limits_{\substack{\varepsilon\to 0\\ \delta\to 0}} T_{\varepsilon,\delta}(f)(x) = T(f)(x)$ exists almost everywhere and in $L^p$ norm, when $1 < p < \infty$.

(See Sokol-Sokolowski [1], and Cotlar [1]. The latter paper however contains an inaccurate deduction in the case of $L \log L$).

(b) There is also a similar but more refined result which holds if $f$ is in the class $L \log L$. For the case when $T^1$ and $T^2$ are one-dimensional Hilbert transforms, see Zygmund [3]; his method, however, uses complex function-theory. For the general case see Fefferman [2].

**6.14** Let $K$ be a distribution of compact support, which equals a locally integrable function away from the origin, and suppose that its Fourier transform $\hat{K}$ is a function. Assume that for a fixed $\theta$, $0 \leq \theta < 1$, we have

(i) $$|\hat{K}(x)| \leq A(1 + |x|)^{-n\theta/2}$$

(ii) $$\int_{|x|\geq 2|y|^{1-\theta}} |K(x-y) - K(x)|\,dx \leq A.$$

Then the operator $f \to K * f$, initially defined on $C^\infty$ functions with compact support, extends to a transformation which is of weak-type (1, 1) and is bounded on $L^p$, $1 < p < \infty$. Fefferman [1].

An example arises for the operator $\lim\limits_{\varepsilon\to 0}\displaystyle\int_{\varepsilon\leq|y|\leq 1} K(y)f(x-y)\,dy$, where $K(x) = |x|^{-n}\exp\{i\,|x|^{-\gamma}\}$, with $\gamma > 0$. This is closely related to the multiplier transformations described in §7.4 of Chapter IV.

*Notes*

*Section 1.* A detailed exposition of the material reviewed in this section may be found in *Fourier Analysis*, Chapter I. For a treatment from the point of view of abstract locally compact abelian groups, see Rudin [1], and Hewitt and Ross [1].

*Sections 2, 3, 4, and 5.* The first approach (in one-dimension) was by complex function-theory methods. For details see Zygmund [7, Chapter VII] and [8, Chapter VII], where further historical references may be found. The real-variable theory for the Hilbert transform goes back to Besicovitch [1] and [2], Titchmarsh [1], and Marcinkiewicz [1]. The present $n$-dimensional theory originates in Calderón-Zygmund [1]. It was elaborated by Cotlar [2], Stein [3], Hörmander [1], Schwartz [1], and Benedeck, Calderón, and Panzone [1], among others. The reader is also referred to the survey papers of Zygmund [5] and Calderón [7].

CHAPTER III

# Riesz Transforms, Poisson Integrals, and Spherical Harmonics

The reader who has followed us to this point has already had to deal with some of the more technical aspects of the theory. He has had to climb, step by step, in a direction that might well have seemed dry and unrewarding. It is understandable if at several places he has possibly felt reluctant to continue.

The purpose of the present chapter is in part to reassure the reader by scanning with him the landscape he has already mastered. At the same time we will take the opportunity to introduce him to some of the tools we will need in our further efforts.

Thus our presentation here will naturally differ in manner from that of the first two chapters. In fact, major stress will be laid on the significant formal aspects of the theory, and certain important examples will be studied in detail. In back of these formal aspects and special examples lie two considerations, which we briefly indicate. It is in the nature of things that the group of rotations that acts on $\mathbf{R}^n$ should play a decisive role in its harmonic analysis, as do the groups of translations and dilations. If from this point of view we consider the simplest, non-trivial, "invariant" operators we are lead to the Riesz transforms. Related to this is the intimate connection of classical harmonic analysis (that of $\mathbf{R}^1$) with complex function theory. The attempts to extend this as far as possible to $\mathbf{R}^n$ via the theory of harmonic functions leads us back to the Riesz transforms.

## 1. *The Riesz transforms*

**1.1** We begin by some observations about the Hilbert transform

$$Hf(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|y| > \varepsilon} \frac{f(x-y)}{y} \, dy.$$

Here we are in the case of $\mathbf{R}^1$, with $K(x) = 1/\pi x$, $\Omega(x) = \frac{1}{\pi} \operatorname{sign} x = \frac{1}{\pi} \frac{x}{|x|}$. Then according to formula (26) of §4.2 of the previous chapter, we see immediately that in terms of the Fourier transform, $(Hf)^\wedge(x) = m(x)\hat{f}(x)$, where the multiplier $m(x)$ is given by $m(x) = i \operatorname{sign} x$. From this, it is clear that $H$ is unitary on $L^2(\mathbf{R}^1)$, and $H^2 = -I$.

Recall now the operation of dilation $(\tau_\delta f)(x) = f(\delta x)$, which in the case of one variable we find convenient to define for all non-zero $\delta$, positive and negative. Then as is obvious, if $\delta > 0$,

$$(H\tau_\delta)f(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|y| > \varepsilon} \frac{f(\delta x - \delta y)}{y}\, dy$$
$$= \lim_{\varepsilon' \to 0} \frac{1}{\pi} \int_{|y| > \varepsilon'} \frac{f(\delta x - y)}{y}\, dy = (\tau_\delta H) f(x),$$

so $H\tau_\delta = \tau_\delta H$; and it is equally obvious that $\tau_\delta H = -H\tau_\delta$, if $\delta < 0$.

These simple considerations of dilation "invariance" and the obvious translation invariance in fact characterize the Hilbert transform.

Proposition 1. *Suppose $T$ is a bounded operator on $L^2(\mathbf{R}^1)$ which satisfies the following properties:*

(a) *$T$ commutes with translations*

(b) *$T$ commutes with positive dilations*

(c) *$T$ anticommutes with the reflection $f(x) \to f(-x)$.*

*Then $T$ is a constant multiple of the Hilbert transform.*

The proof involves no difficulties. In fact since $T$ commutes with translations, then according to the proposition in §1.4 of the previous chapter there is a bounded function $m(x)$, so that $(Tf)^\wedge(x) = m(x)\hat{f}(x)$. Let us also denote by $\mathscr{F}$ the Fourier transform, $\mathscr{F}f = \hat{f}$. Then

$$(\mathscr{F}\tau_\delta f)(y) = \int_{-\infty}^{\infty} e^{2\pi i x y} f(\delta x)\, dx$$
$$= |\delta|^{-1} \int_{-\infty}^{\infty} e^{2\pi i x y/\delta} f(x)\, dx = |\delta|^{-1} (\tau_{\delta^{-1}} \mathscr{F} f)(y),$$

so $\mathscr{F}\tau_\delta = |\delta|^{-1}\tau_{\delta^{-1}}\mathscr{F}$.

The definition of the multiplier may be written symbolically as $\mathscr{F}T = m\mathscr{F}$, (where by $m$ we mean the operator of multiplication by $m$!). However the assumptions (b) and (c) may be rewritten as $T\tau_\delta = \operatorname{sign}(\delta)\tau_\delta T$, which when inserted in the above gives

$$\tau_\delta m = \tau_\delta \mathscr{F} T \mathscr{F}^{-1} = |\delta|^{-1} \mathscr{F} \tau_{\delta^{-1}} T \mathscr{F}^{-1} = \delta^{-1} \mathscr{F} T \tau_{\delta^{-1}} \mathscr{F}^{-1}$$
$$= \operatorname{sign}(\delta) \mathscr{F} T \mathscr{F}^{-1} \tau_\delta = \operatorname{sign}(\delta) m \tau_\delta.$$

So $\tau_\delta m = \text{sign}\,(\delta) m \tau_\delta$, which means $m(\delta x) = \text{sign}\,(\delta) m(x)$, if $\delta \neq 0$.

This shows that $m(x) = \text{constant} \times \text{sign}\,(x)$, and the proposition is proved.

The proof of the proposition shows incidentally that the only bounded linear transformations on $L^2(\mathbf{R}^1)$ which *commute* with all the operations described, namely translations and positive and negative dilations, are constant multiples of the identity operator. This remark together with the proposition attest graphically to the special role of the Hilbert transform in harmonic analysis of $\mathbf{R}^1$. We now look for the operators in $\mathbf{R}^n$ which have the analogous structural characterization.

**1.2** We begin by making a few remarks about the interaction of dilations and rotations with the $n$-dimensional Fourier transform. With $\mathscr{F}f = \hat{f}$, and $\delta > 0$

$$(\mathscr{F}\tau_\delta)f(x) = \int_{\mathbf{R}^n} e^{2\pi i x\cdot y} f(\delta y)\, dy$$

$$= \delta^{-n} \int_{\mathbf{R}^n} e^{2\pi i x\cdot y/\delta} f(y)\, dy = \delta^{-n}(\tau_{\delta^{-1}}\mathscr{F})f(x).$$

So, symbolically,

$$\mathscr{F}\tau_\delta = \delta^{-n}\tau_{\delta^{-1}}\mathscr{F}. \tag{1}$$

Next let $\rho$ denote any rotation (proper or improper) about the origin in $\mathbf{R}^n$. Denote also by $\rho$ its induced action on functions, $\rho(f)(x) = f(\rho^{-1}x)$ Then

$$(\mathscr{F}\rho)f(x) = \int_{\mathbf{R}^n} e^{2\pi i x\cdot y} f(\rho^{-1}y)\, dy$$

$$= \int_{\mathbf{R}^n} e^{2\pi i x\cdot \rho y} f(y)\, dy = \int_{\mathbf{R}^n} e^{2\pi i \rho^{-1}x\cdot y} f(y)\, dy = (\rho\mathscr{F})f(x),$$

and

$$\mathscr{F}\rho = \rho\mathscr{F}. \tag{2}$$

We shall also need the following elementary observation. Let $m(x) = (m_1(x), m_2(x), \ldots, m_n(x))$ be an $n$-tuple of functions defined on $\mathbf{R}^n$. For any rotation $\rho$, write $\rho = (\rho_{jk})$ for its matrix realization. Suppose that $m$ transforms like a vector. Symbolically this can be written as

$$m(\rho^{-1}x) = \rho(m(x)),$$

or more explicitly

$$m_j(\rho^{-1}x) = \sum_k \rho_{jk} m_k(x), \qquad \text{for every rotation } \rho. \tag{3}$$

LEMMA. *Suppose $m$ is homogeneous of degree 0, i.e. $m(\delta x) = m(x)$, for $\delta > 0$. If $m$ transforms according to (3) then $m(x) = c\dfrac{x}{|x|}$ for some constant $c$; that is*

$$m_j(x) = c\frac{x_j}{|x|}. \tag{4}$$

To prove the assertion we notice that it suffices to consider $x$ on the unit sphere. Now let $e_1, e_2, \ldots, e_n$ denote the usual unit vectors along the axes. Set $c = m_1(e_1)$. We can see that $m_j(e_1) = 0$, if $j \neq 1$. In fact, for any rotation $\rho$ having $e_1$ fixed (3) gives us that $m_j(e_1) = \sum_{k=2}^{n} \rho_{jk} m_k(e_1)$, $j = 2, \ldots, n$. That is, the $n - 1$ dimensional vector $(m_2(e_1), m_3(e_1), \ldots, m_n(e_1))$ is left fixed by all the rotations on this $n - 1$ dimensional vector space. So $m_2(e_1) = m_3(e_1) \cdots = m_n(e_1) = 0$. Inserting this again in (3) gives $m_j(\rho^{-1}e_1) = \rho_{j1}m_1(e_1) = c\rho_{j1}$. But if $\rho^{-1}e_1 = x$, then $\rho_{j1} = x_j$, so $m_j(x) = cx_j$, $(|x| = 1)$, which proves the lemma.

It is curious to observe that the full group of rotations is needed only in the case $n = 1$, and $n = 2$. In the case $n \geq 3$ the proper rotations would have sufficed, since then the subgroup of proper rotations in one less dimension is still transitive on its unit sphere $(S^{n-2})$.

We are now in a position to define the $n$ Riesz transforms. We set for $f \in L^p(\mathbf{R}^n)$, $1 \leq p < \infty$,

$$R_j(f)(x) = \lim_{\varepsilon \to 0} c_n \int_{|y| \geq \varepsilon} \frac{y_j}{|y|^{n+1}} f(x - y)\, dy, \qquad j = 1, \ldots, n, \tag{5}$$

with $c_n = \dfrac{\Gamma\left(\dfrac{n+1}{2}\right)}{\pi^{(n+1)/2}}$.

Thus $R_j$ is defined by the kernel $K_j(x) = \dfrac{\Omega_j(x)}{|x|^n}$, and $\Omega_j(x) = c_n \dfrac{x_j}{|x|}$.

We shall next derive the multipliers which correspond to the Riesz transforms, and which in fact justify their definition. Let us recall the formula (26) of §4.2 of the previous chapter. It is

$$m(x) = \int_{S^{n-1}} \Gamma(x \cdot y)\Omega(y)\, d\sigma(y), \qquad |x| = 1, \tag{6}$$

with $\Gamma(t) = \dfrac{\pi i}{2} \operatorname{sign} t + \log|1/t|$. Notice that the mapping (6) from $\Omega$ to $m$ commutes with rotations, and this is nothing but an immediate consequence of the fact that the kernel $\Lambda(x \cdot y)$ depends only on the inner product of $x$ and $y$. It is clear however that the kernels

$$(K_1(x), \ldots, K_n(x)) = c_n\left(\frac{x_1}{|x|^{n+1}}, \frac{x_2}{|x|^{n+1}}, \ldots, \frac{x_n}{|x|^{n+1}}\right)$$

satisfy the transformation law (3) (with $K_j$ in place of $m_j$). Then, in view of the commutability of the mapping $K_j \to m_j$ with rotations just alluded to, it follows that the multipliers also satisfy (3). However the $m_j$ are each homogeneous of degree 0, so the lemma shows that $m_j(x) = c\dfrac{x_j}{|x|}$. In this particular case (and because our choice of the constant $c_n$) we have $c = i$. By evaluating the $m_j$ at a fixed point in (6) this last assertion is equivalent with

$$\text{(7)} \qquad \frac{2\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+3)/2}} = \int_{S^{n-1}} |\cos\theta|\, d\sigma(y)$$

where $\theta$ denotes the angle made by the variable unit vector $y$ with a fixed direction. One may either evaluate this integral directly, or this calculation can be avoided by appealing to the general result, Theorem 5, proved below. In either case we get

$$\text{(8)} \qquad (R_jf)^\wedge(x) = i\frac{x_j}{|x|}\hat{f}(x), \qquad j = 1, \ldots, n.$$

We can express the transformation law (3) acting on the Riesz transforms in a more intrinsic manner. More precisely

$$\text{(9)} \qquad \rho R_j \rho^{-1} f = \sum_k \rho_{jk} R_k f,$$

which is the statement that under rotations in $\mathbf{R}^n$, the Riesz operators transform in the same manner as the components of a vector. The verification of (9) is immediate. It may be done using the direct definition (5) of the Riesz transforms or, because of (8), in terms of their Fourier transforms. Thus if we denote symbolically $R_j^\wedge = m_j$, then (9) becomes $\rho(m_j\rho^{-1}(f)) = \sum_k \rho_{jk} m_k f$ which is $m_j(\rho^{-1}x) = \sum_k \rho_{jk} m_k(x)$. These observations have a converse.

PROPOSITION 2. *Let $T = (T_1, T_2, \ldots, T_n)$ be an n-tuple of bounded transformations on $L^2(\mathbf{R}^n)$. Suppose*

(a) *Each $T_j$ commutes with the translation of $\mathbf{R}^n$*

(b) *Each $T_j$ commutes with the dilations of $\mathbf{R}^n$*

(c) *For every rotation $\rho = (\rho_{jk})$ of $\mathbf{R}^n$, $\rho T_j \rho^{-1} f = \sum_k \rho_{jk} T_k f$.*

*Then the $T_j$ are a constant multiple of the Riesz transforms, i.e. there exists a constant $c$, so that $T_j = cR_j$, $j = 1, \ldots, n$.*

All the elements of the proof have already been discussed. We bring them together: (i) Since the $T_j$ are bounded on $L^2(\mathbf{R}^n)$ and commute with translations they can be each realized by bounded multipliers; symbolically $T_j^\wedge = m_j$. (ii) Since the $T_j$ commute with dilations and because of the relation (1) between dilation and the Fourier transform, we see that $m_j(\delta x) = m_j(x)$, $\delta > 0$; that is, each $m_j$ is homogeneous of degree 0. (iii) Finally, assumption (c) has as a consequence the relation (3), and so by the lemma we can obtain the desired conclusion.

**1.3 An application.** One of the important applications of the Riesz transforms is that they can be used to mediate between various combinations of partial derivatives of a function. This service of the Riesz transforms will be particularly striking in Chapter V. We shall here content ourselves with two very simple illustrations, which examples have an interest on their own and have already the characteristic features of a general type of estimate which can be made in the theory of elliptic differential operators.

PROPOSITION 3. *Suppose $f$ is of class $C^2$ and has compact support. Let $\Delta f = \sum_{j=1}^{n} \dfrac{\partial^2 f}{\partial x_j^2}$. Then we have the a priori bound*

$$\left\| \frac{\partial^2 f}{\partial x_j\, \partial x_k} \right\|_p \le A_p \|\Delta f\|_p, \qquad 1 < p < \infty. \tag{10}$$

This proposition is an immediate consequence of the $L^p$ boundedness of the Riesz transforms and the identity

$$\frac{\partial^2 f}{\partial x_j\, \partial x_k} = -R_j R_k\, \Delta f. \tag{11}$$

To prove (11) we use the Fourier transform. Thus if $\hat{f}(x)$ is the Fourier transform of $f$, $\hat{f}(x) = \int_{\mathbf{R}^n} e^{2\pi i x\cdot y} f(y)\, dy$, then the Fourier transform of $\dfrac{\partial f}{\partial x_j}$ is $-2\pi i x_j\, \hat{f}(x)$, and so

$$\begin{aligned}\left(\frac{\partial^2 f}{\partial x_j\, \partial x_k}\right)^\wedge(x) &= -4\pi^2 x_j\, x_k \hat{f}(x)\\ &= -\left(\frac{i x_j}{|x|}\right)\left(\frac{i x_k}{|x|}\right)(-4\pi\, |x|^2)\hat{f}(x) = -(R_j R_k\, \Delta f)^\wedge,\end{aligned}$$

which gives (11).

Another application of interest, this time for potential theory in two-dimensions, is the following.

PROPOSITION 4. *Suppose $f$ is of class $C^1$ in $\mathbf{R}^2$ and has compact support. Then we have the a priori bound*

$$\left\|\frac{\partial f}{\partial x_1}\right\|_p + \left\|\frac{\partial f}{\partial x_2}\right\|_p \le A_p \left\|\frac{\partial f}{\partial x_1} + i\,\frac{\partial f}{\partial x_2}\right\|_p, \qquad 1 < p < \infty.$$

Needless to say, this proposition is significant only if $f$ is *complex valued.*

The proof of Proposition 4 is very much like that of the previous one, except that here the identity used is

$$\frac{\partial f}{\partial x_j} = -R_j(R_1 - iR_2)\left(\frac{\partial f}{\partial x_1} + i\,\frac{\partial f}{\partial x_2}\right), \qquad j = 1, 2.$$

A more systematic presentation of these particular facts will be given in §3.5, below.

## 2. *Poisson integrals and approximations to the identity*

**2.1** We shall now introduce a notion that will be indispensable in much of our further work. We have in mind the theory of harmonic functions. The setting for the application of this theory will be as follows. We shall think of $\mathbf{R}^n$ as the boundary hyperplane of the $(n + 1)$ dimensional upper-half space $\mathbf{R}^{n+1}_+$. In coordinate notation,

$$\mathbf{R}^{n+1}_+ = \{(x, y): x \in \mathbf{R}^n, y > 0\}.$$

We shall consider the *Poisson integral* of a function $f$ given on $\mathbf{R}^n$. This Poisson integral is effectively the solution to the Dirichlet problem for $\mathbf{R}^{n+1}_+$: find a harmonic function $u(x, y)$ on $\mathbf{R}^{n+1}_+$, whose boundary values on $\mathbf{R}^n$ (in the appropriate sense) are $f(x)$.

The formal solution of this problem can be given neatly in the context of the $L^2$ theory.

In fact, let $f \in L^2(\mathbf{R}^n)$, and let $\hat{f}$ be its Fourier transform. Consider

$$u(x, y) = \int_{t\in\mathbf{R}^n} \hat{f}(t)e^{-2\pi i t\cdot x}e^{-2\pi|t|y}\,dt, \qquad y > 0. \tag{12}$$

This integral converges absolutely, because $\hat{f} \in L^2(\mathbf{R}^n)$, and $e^{-2\pi|t|y}$ is rapidly decreasing (in $|t|$), for $y > 0$. For the same reason the integral above may be differentiated with respect to $x$ and $y$ any number of times by carrying out the operation under the sign of integration. This gives

$$\Delta u = \frac{\partial^2 u}{\partial y^2} + \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} = 0,$$

because the factor $e^{-2\pi i t\cdot x}e^{-2\pi|t|y}$ satisfies this property for each fixed $t$.

Also by Plancherel's theorem $u(x, y) \to f(x)$ in $L^2(\mathbf{R}^n)$ norm, as $y \to 0$.

This solution of the problem can also be written without explicit use of the Fourier transform. For this purpose define the *Poisson kernel* $P_y(x)$ by

$$P_y(x) = \int_{\mathbf{R}^n} e^{-2\pi i t\cdot x} e^{-2\pi |t| y}\, dt, \qquad y > 0. \tag{13}$$

Then the function $u(x, y)$ obtained above can be written as a convolution

$$u(x, y) = \int_{\mathbf{R}^n} P_y(t) f(x - t)\, dt. \tag{14}$$

We shall say that $u$ is the *Poisson integral* of $f$.

The Poisson kernel has an explicit expression.

PROPOSITION 5.

$$P_y(x) = \frac{c_n y}{(|x|^2 + y^2)^{\frac{n+1}{2}}}, \qquad c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}, \tag{15}$$

$c_n$ is the same constant that appears in the definition of the Riesz transforms ((5) above). The well-known formula (15) may be proved as follows. We use two identities:

$$(\alpha) \quad \int_{\mathbf{R}^n} e^{-\pi\delta|t|^2} e^{-2\pi i t\cdot x}\, dt = \delta^{-n/2} e^{-\pi |x|^2/\delta}, \qquad \delta > 0$$

$$(\beta) \quad e^{-\gamma} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\gamma^2/4u}\, du, \qquad \gamma > 0.$$

The first, $(\alpha)$, is immediately reducible by a change of variables to the very well-known special case $\delta = 1$. The second, $(\beta)$, expresses the exponential $e^{-\gamma}$ as a weighted average of the family of exponential $e^{-\gamma^2/4u}$, $0 < u < \infty$, and is an important instance of the principle of *subordination*.* To prove $(\beta)$, write $e^{-\gamma} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{i\gamma x}}{1 + x^2}\, dx$, and express the factor $\frac{1}{1 + x^2}$ as $\int_0^\infty e^{-(1+x^2)u}\, du$. This leads to the double integral $e^{-\gamma} = \frac{1}{\pi} \int_0^\infty e^{-u} \left( \int_{-\infty}^{\infty} e^{i\gamma x} e^{-ux^2}\, dx \right) du$ which after evaluation of the inner integral gives $(\beta)$. This being done we return to $P_y(x)$. We have by $(\beta)$ and (13):

$$P_y(x) = \frac{1}{\sqrt{\pi}} \int_{\mathbf{R}^n} \left( \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\pi^2|t|^2 y^2/u}\, du \right) e^{-2\pi i t\cdot x}\, dt.$$

* See Bochner [2], Chapter 4.

Then apply ($\alpha$) with $\delta = \dfrac{\pi y^2}{u}$. We get

$$P_y(x) = \frac{1}{\pi^{\frac{n+1}{2}}} \int_0^\infty e^{-u} e^{-\frac{|x|^2 u}{y^2}} y^{-n} u^{\frac{n-1}{2}}\, du = \frac{y}{(\pi(|x|^2 + y^2))^{\frac{n+1}{2}}} \int_0^\infty e^{-u} u^{\frac{n-1}{2}}\, du,$$

which is the desired formula for the Poisson kernel.

We list the properties of the Poisson kernel that are now more or less evident:

(i) $P_y(x) > 0$.

(ii) $\int_{\mathbf{R}^n} P_y(x)\, dx = 1$, $y > 0$; more generally, $P_y^\wedge(x) = e^{-2\pi|x|y}$ by an application of the Fourier inversion formula to (13).

(iii) $P_y(x)$ is homogeneous of degree $-n$: $P_\varepsilon(x) = P_1(x/\varepsilon)\varepsilon^{-n}$, $\varepsilon > 0$.

(iv) $P_y(x)$ is a decreasing function of $|x|$, and $P_y(x) \in L^p(\mathbf{R}^n)$, $1 \le p \le \infty$.

(v) Suppose $f \in L^p(\mathbf{R}^n)$, $1 \le p \le \infty$, then its Poisson integral $u$, given by (14), is harmonic in $\mathbf{R}_+^{n+1}$. This is a simple consequence of fact that $P_y(x)$ is harmonic in $\mathbf{R}_+^{n-1}$; the latter is immediately derivable from (13).

(vi) We have the "semi-group property" $P_{y_1} * P_{y_2} = P_{y_1+y_2}$ if $y_1 > 0$ and $y_2 > 0$. This follows immediately from the Fourier transform formula in (ii).

The boundary behavior of Poisson integrals is already described to a significant extent by the following theorem.

THEOREM 1. *Suppose $f \in L^p(\mathbf{R}^n)$, $1 \le p \le \infty$, and let $u(x, y)$ be its Poisson integral. Then:*

(a) $\sup_{y>0} |u(x, y)| \le Mf(x)$, *where $Mf$ is the maximal function of Chapter I, §1.*

(b) $\lim_{y\to 0} u(x, y) = f(x)$, *for almost every $x$.*

(c) *If $p < \infty$, $u(x, y)$ converges to $f(x)$ in $L^p(\mathbf{R}^n)$ norm, as $y \to 0$.*

The theorem will now be proved in a more general setting, valid for a large class of approximations to the identity.

**2.2** Let $\varphi$ be an integrable function on $\mathbf{R}^n$, and set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$, $\varepsilon > 0$.

THEOREM 2. *Suppose that the least decreasing radial majorant of $\varphi$ is integrable; i.e. let $\psi(x) = \sup_{|y| \ge |x|} |\varphi(y)|$, and we suppose $\int_{\mathbf{R}^n} \psi(x)\, dx = A < \infty$. Then with the same $A$,*

(a) $\sup_{\varepsilon>0} |(f * \varphi_\varepsilon)(x)| \leq AM(f)(x), f \in L^p(\mathbf{R}^n)$, $1 \leq p \leq \infty$.

(b) *If in addition* $\int_{\mathbf{R}^n}\varphi(x)\,dx = 1$, *then* $\lim_{\varepsilon\to 0} (f * \varphi_\varepsilon)(x) = f(x)$ *almost everywhere.*

(c) *If* $p < \infty$, *then* $\|f * \varphi_\varepsilon - f\|_p \to 0$, *as* $\varepsilon \to 0$.

We have already considered a special case of this situation in Chapter I, with $\varphi(x) = \dfrac{1}{m(B)}\chi_B$, the characteristic function of the unit ball $B$, divided by the measure of that ball. The point of the theorem is to reduce matters to this fundamental special case.

We begin with the proof of (c). It is to be remarked that the proof actually holds under the weaker assumption that $\varphi$ is merely integrable. (Of course the normalization $\int_{\mathbf{R}^n}\varphi\,dx = 1$ is still required.) First we point out that if $f \in L^p(\mathbf{R}^n)$, $p < \infty$, and $\|f(x - y) - f(x)\|_p = \Delta(y)$, then $\Delta(y) \to 0$, as $y \to 0$.* If $f_1$ is continuous with compact support, the assertion in that case is an immediate consequence of the uniform convergence $f_1(x - y) \to f_1(x)$, as $y \to 0$. In general write $f = f_1 + f_2$, where $f_1$ is as described and $\|f_2\|_p \leq \delta$; this is possible since such $f_1$ are dense in $L^p$, $p < \infty$. Then $\Delta(y) = \Delta_1(y) + \Delta_2(y)$, with $\Delta_1(y) \to 0$, as $y \to 0$, and $\Delta_2(y) \leq 2\delta$. This shows that $\Delta(y) \to 0$ for general $f \in L^p(\mathbf{R}^n)$, $p < \infty$. Now $f * \varphi_\varepsilon - f = \int_{\mathbf{R}^n}[f(x - y) - f(x)]\varphi_\varepsilon(y)\,dy$, because

$$\int_{\mathbf{R}^n}\varphi_\varepsilon(x)\,dx = \int_{\mathbf{R}^n}\varphi(x)\,dx = 1.$$

So

$$\|f * \varphi_\varepsilon - f\|_p \leq \int_{\mathbf{R}^n}\Delta(y)\,|\varphi_\varepsilon(y)|\,dy = \int_{\mathbf{R}^n}\Delta(\varepsilon y)\,|\varphi(y)|\,dy \to 0;$$

the latter fact is by the Lebesgue dominated convergence theorem and the fact that $\Delta(\varepsilon y) \to 0$, as $\varepsilon \to 0$. This proves assertion (c) of the theorem. We shall now prove assertion (a). With a slight abuse of notation, let us write $\psi(r) = \psi(x)$, if $|x| = r$; it should cause no confusion since $\psi(x)$ is anyway radial. Now observe that $\int_{r/2\leq|x|\leq r}\psi(x)\,dx \geq \psi(r)\int_{r/2\leq|x|\leq r}dx = \psi(r)cr^n$. Therefore the assumption $\psi \in L^1$, (and the fact that $\psi(r)$ is decreasing) proves that $r^n\psi(r) \to 0$, as $r \to 0$, or $r \to \infty$. To prove (a) we need to show that

$$(f * \psi_\varepsilon)(x) \leq A(Mf)(x) \tag{16}$$

where $f \geq 0, f \in L^p(\mathbf{R}^n)$, $1 \leq p \leq \infty$, $\varepsilon > 0$, and $A = \int_{\mathbf{R}^n}\psi(x)\,dx$.

Since the assertion (16) is clearly translation invariant (with respect to

* This statement is the continuity of the mapping $y \to f(x - y)$ of $\mathbf{R}^n$ to $L^p(\mathbf{R}^n)$.

$f$) and also dilation invariant (with respect to $\psi$), it suffices to show that

$$(f * \psi)(0) \leq A(Mf)(0). \tag{17}$$

In proving (17) we may clearly assume that $Mf(0) < \infty$. Let us write $\lambda(r) = \int_{x \in S^{n-1}} f(rx)\, d\sigma(x)$, and $\Lambda(r) = \int_{|x| \leq r} f(x)\, dx$, so

$$\Lambda(r) = \int_0^r \lambda(t) t^{n-1}\, dt.$$

We have

$$\begin{aligned}(f * \psi)(0) &= \int_{\mathbf{R}^n} f(x)\psi(x)\, dx = \int_0^\infty \lambda(r)\psi(r) r^{n-1}\, dr \\ &= \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_\varepsilon^N \lambda(r)\psi(r) r^{n-1}\, dr = \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} -\int_\varepsilon^N \Lambda(r)\, d\psi(r).\end{aligned}$$

The passage to the last equality is by integration by parts. This introduces the error $\Lambda(N)\psi(N) - \Lambda(\varepsilon)\psi(\varepsilon)$; but this term tends to zero as $\varepsilon \to 0$, and $N \to \infty$ in view of our observation regarding $\psi$ and the fact that $\Lambda(r) = \int_{|x|<r} f(x)\, dx \leq Vr^n Mf(0)$, where $V$ is the volume of the unit ball. Thus

$$f * \psi(0) = \int_0^\infty \Lambda(r)\, d(-\psi(r)) \leq VMf(0) \int_0^\infty r^n\, d(-\psi(r)).$$

So (17) and hence (16) is proved.

The almost everywhere convergence (a) is then proved in the familiar way as follows. First, one verifies that if $f_1$ is continuous and has compact support, $(f_1 * \varphi_\varepsilon)(x) \to f_1(x)$ uniformly as $\varepsilon \to 0$. Next one deals with the case $f \in L^p(\mathbf{R}^n)$, $1 \leq p < \infty$, by writing $f = f_1 + f_2$ with $f_1$ as described and with the $L^p$ norm of $f_2$ small. The argument then follows closely that given in §1.5, Chapter I, after equation (6). Thus we get that $\lim_{\varepsilon \to 0} f_\varepsilon(x)$ exists almost everywhere and equals $f(x)$. To deal with the remaining case, that of bounded $f$, we fix any ball $B$, and set ourselves the task of showing that $\lim_{\varepsilon \to 0} (f * \varphi_\varepsilon)(x) = f(x)$ for almost every $x \in B$. Let $B_1$ be any other ball which strictly contains $B$, and let $\delta$ be the distance from $B$ to the complement of $B_1$. Let $f_1(x) = \begin{cases} f(x), & x \in B \\ 0, & x \notin B \end{cases}$; $f(x) = f_1(x) + f_2(x)$. Then $f_1(x) \in L^1(\mathbf{R}^n)$, and so the appropriate conclusion holds for it. However for $x \in B$,

$$\begin{aligned}|(f * \varphi_\varepsilon)(x)| = \left| \int f(x-y)\varphi_\varepsilon(y)\, dy \right| &\leq \int_{|y| \geq \delta > 0} |f(x-y)|\, \varphi_\varepsilon(y)\, dy \\ &\leq \|f\|_\infty \int_{|y| \geq \delta/\varepsilon} |\varphi(y)|\, dy \to 0, \\ &\qquad \text{as} \quad \varepsilon \to 0.\end{aligned}$$

Theorem 2 is then completely proved. Theorem 2 then applies directly to prove theorem 1, because of properties (i)–(iv) of the Poisson kernel; in this case $\varphi(x) = \psi(x) = P_1(x)$.

There are also some variants of the result of Theorem 2 which, of course, apply equally well to Poisson integrals. The first is an easy adaptation of the argument already given, and is stated without proof.

COROLLARY. *Suppose $f$ is continuous and bounded on $\mathbf{R}^n$. Then $(f * \varphi_\varepsilon)(x) \to f(x)$ uniformly on compact subsets of $\mathbf{R}^n$.*

In particular this shows that if $f$ is a given bounded and continuous function in $\mathbf{R}^n$, we can find a function $u(x, y)$ which is continuous on the closure of $\mathbf{R}_+^{n+1}$, harmonic in the interior, and whose restriction to the boundary is the given $f$. Thus Dirichlet's problem is resolved in this case.

The second variant is somewhat more difficult. It is the analogue for finite Borel measures in place of integrable functions, and is outlined in §4.1.

**2.3 Conjugate harmonic functions.** We shall now tie together the Riesz transforms and the theory of harmonic functions, more particularly Poisson integrals. Since we are interested here mainly in the formal aspects we shall restrict ourselves to the $L^2$ case. (The $L^p$ case and related results are stated in §4.3. and §4.4.)

THEOREM 3. *Let $f$ and $f_1, \ldots, f_n$ all belong to $L^2(\mathbf{R}^n)$, and let their respective Poisson integrals be $u_0(x, y) = P_y * f$, $u_1(x, y) = P_y * f_1, \ldots$, $u_n(x, y) = P_y * f_n$. Then a necessary and sufficient condition that*

$$f_j = R_j(f), \qquad j = 1, \ldots, n, \tag{18}$$

*is that the following generalized Cauchy-Riemann equations hold:*

$$\begin{cases} \displaystyle\sum_{j=0}^{n} \frac{\partial u_j}{\partial x_j} = 0, \\ \displaystyle\frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j}, \qquad j \neq k, \quad \text{with} \quad x_0 = y. \end{cases} \tag{19}$$

It is to be noted that, at least locally, the system (19) is equivalent with the existence of a harmonic function $H$ (of the $n + 1$ variables), so that $u_j = \dfrac{\partial H}{\partial x_j}$, $j = 0, 1, 2, \ldots, n$.

The theorem is one of that class whose proof is nearly obvious but whose statement is nevertheless of some interest.

Suppose $f_j = R_j(f)$, then $\hat{f}_j(t) = \dfrac{it_j}{|t|}\hat{f}(t)$, and so by (12)

$$u_j(x, y) = \int_{\mathbf{R}^n} \hat{f}(t)\frac{it_j}{|t|}e^{-2\pi it\cdot x}e^{-2\pi|t|y}\,dt, \quad j = 1, \dots, n.$$

The equations (19) can then be immediately verified by differentiation under the integral sign, which is justified by the rapid convergence of the integrals in question.

Conversely, let $u_j(x, y) = \int_{\mathbf{R}^n}\hat{f}_j(t)e^{-2\pi it\cdot x}e^{-2\pi|t|y}\,dt$, $j = 0, 1, \dots, n$. Then the fact that $\dfrac{\partial u_0}{\partial x_j} = \dfrac{\partial u_j}{\partial x_0} = \dfrac{\partial u_j}{\partial y}$, $j = 1, \dots, n$, shows that

$$-2\pi it_j\hat{f}_0(t)e^{-2\pi|t|y} = -2\pi\,|t|f_j(t)e^{-2\pi|t|y};$$

therefore $\hat{f}_j(t) = \dfrac{it_j}{|t|}\hat{f}_0(t)$, and so

$$f_j = R_j(f_0) = R_j(f), \qquad j = 1, \dots, n.$$

The theorem indicates that it should be of interest to study harmonic functions satisfying the system (19) in analogy with complex function theory. We shall return to this point of view in Chapters V, VII, and VIII.

**2.4 A digression.** We shall now digress from our main topic in order to return to a point left open in our treatment of singular integrals in §4.6 of Chapter II. The situation there was as follows: We considered the kernel $K(x) = \dfrac{\Omega(x)}{|x|^n}$, where $\Omega$ was homogeneous of degree 0, and its restriction to the unit sphere satisfied the cancellation property (24) and the smoothness property (25) (see page 39). We were concerned with the almost everywhere existence of

$$\lim_{\varepsilon\to 0} T_\varepsilon(f) = \lim_{\varepsilon\to 0}\int_{|y|>\varepsilon}\frac{\Omega(y)}{|y|^n}f(x - y)\,dy.$$

We used, without proof, the following lemma.

LEMMA. *If* $T^*f(x) = \sup_{\varepsilon>0}|T_\varepsilon(f)|(x)$, *then*

$$\|T^*f\|_p \le A_p\,\|f\|_p, \qquad 1 < p < \infty.$$

Let $T(f)(x) = \lim_{\varepsilon\to 0} T_\varepsilon(f)(x)$, where the limit is taken in the $L^p$ norm. Its existence is guaranteed by theorem 3 in §4.2 of Chapter II. We shall prove the lemma by showing that

$$T^*(f)(x) \leq M(Tf)(x) + CM(f)(x).$$

Let $\varphi$ be a smooth non-negative function on $\mathbf{R}^n$, which is supported in the unit ball, has integral equal to one, and which is also radial and decreasing in $|x|$. Consider

$$K_\varepsilon(x) = \begin{cases} \dfrac{\Omega(x)}{|x|^n}, & \text{if } |x| \geq \varepsilon \\ 0 & \text{if } |x| < \varepsilon \end{cases}$$

This leads us to another function $\Phi$ defined by

$$\varphi * K - K_1 = \Phi, \tag{20}$$

where $\varphi * K = \lim_{\varepsilon \to 0} \varphi * K_\varepsilon = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} K(x - y)\varphi(y)\, dy$.

We shall need to prove that the smallest decreasing radial majorant of $\Phi$ is integrable (so as to apply Theorem 2). In fact if $|x| < 1$, $\Phi = \varphi * K$ which is $\int_{\mathbf{R}^n} K(y)\varphi(x - y)\, dy$ or $\int_{\mathbf{R}^n} K(y)[\varphi(x - y) - \varphi(x)]\, dx$ and hence is bounded on account of the smoothness of $\varphi$. When $1 \leq |x| \leq 2$, then $\Phi(x) = K * \varphi - K(x)$ which is again bounded by the same reason. Finally when $|x| \geq 2$,

$$\Phi(x) = \int_{\mathbf{R}^n} K(x - y)\varphi(y)\, dy - K(x) = \int_{|y| < 1} [K(x - y) - K(x)]\varphi(y)\, dy$$

so $|\Phi(x)| \leq C' \dfrac{\omega(c/|x|)}{|x|^n}$, by the estimate in §4.2 of Chapter II. Since $\omega(\delta)$ is increasing and $\displaystyle\int_0^1 \frac{\omega(\delta)\, d\delta}{\delta} < \infty$, we have proved here our assertion about $\Phi$. From (20) it follows, because the singular integral operator $\varphi \to \varphi * K$ commutes with dilations, that

$$\varphi_\varepsilon * K - K_\varepsilon = \Phi_\varepsilon, \quad \text{with} \quad \Phi_\varepsilon(x) = \varepsilon^{-n}\Phi(x/\varepsilon). \tag{21}$$

Now we claim that for any $f \in L^p(\mathbf{R}^n)$, $1 < p < \infty$,

$$(\varphi_\varepsilon * K) * f(x) = T(f) * \varphi_\varepsilon(x), \tag{22}$$

where the identity holds for every $x$. In fact we notice first that

$$(\varphi_\varepsilon * K_\delta) * f(x) = T_\delta(f) * \varphi_\varepsilon(x), \quad \text{for every} \quad \delta > 0 \tag{23}$$

because both sides of (20) are equal for each $x$ to the absolutely convergent double integral $\int_{z \in \mathbf{R}^n} \int_{|y| > \delta} K(y) f(z - y)\varphi_\varepsilon(x - z)\, dz\, dy$. Moreover $\varphi_\varepsilon \in L^q(\mathbf{R}^n)$, with $1 < q < \infty$ and $1/p + 1/q = 1$, so $\varphi_\varepsilon * K_\delta \to \varphi_\varepsilon * K$ in

$L^q$ norm, and $T_\delta(f) \to T(f)$ in $L^p$ norm, as $\delta \to 0$. This proves (22), and so by (21)

$$T_\varepsilon(f) = (Tf) * \varphi_\varepsilon - f * \Phi_\varepsilon.$$

Passing to the supremum over $\varepsilon$ and applying Theorem 2, part (a), we get our asserted majorization for $T^*f$. The $L^p$ estimates for $f \dashrightarrow Tf$ and the maximal function $M$ then prove the lemma.

## 3. *Higher Riesz transforms and spherical harmonics*

**3.1** We return to our subject proper, the consideration of special transformations of the form

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} \frac{\Omega(y)}{|y|^n} f(x - y)\, dy, \tag{24}$$

where $\Omega$ is homogeneous of degree 0 and its integral over $S^{n-1}$ vanishes. We have already considered the example $\Omega_j(y) = c\dfrac{y_j}{|y|}$, $j = 1, \ldots, n$. For $n = 1$, $\Omega(y) = c \operatorname{sign} y$, and this is the only possible case. To study the matter further for $n > 1$ we recall the expression (see (6), p. 57),

$$m(x) = \int_{S^{n-1}} \Gamma(x \cdot y)\Omega(y)\, d\sigma(y), \qquad |x| = 1$$

where $m$ is the multiplier arising from the transformation (24).

We have already remarked that the mapping $\Omega \to m$ commutes with rotations. We shall therefore consider the functions on the sphere $S^{n-1}$ (more particularly the space $L^2(S^{n-1})$) from the point of view of its decomposition under the action of rotations. As is well known, this decomposition is in terms of the spherical harmonics, and it is with a brief review of their properties that we begin.

We fix our attention, as always, on $\mathbf{R}^n$, and we shall consider polynomials in $\mathbf{R}^n$ which are also harmonic. Thus we shall define $\mathscr{H}_k$ to be the linear space of homogeneous polynomials of degree $k$ which are harmonic: the *solid spherical harmonics of degree k*. It will be convenient to restrict these polynomials to the surface of the unit sphere $S^{n-1}$, and there to define the standard inner product,

$$(P, Q) = \int_{S^{n-1}} P(x)\overline{Q(x)}\, d\sigma(x).$$

We can then affirm:

*3.1.1* The finite dimensional spaces $\{\mathcal{H}_k\}_{k=0}^{\infty}$ are mutually orthogonal. In fact if $P \in \mathcal{H}_k$, and $Q \in \mathcal{H}_j$ then

$$(k - j)\int_{S^{n-1}} P\bar{Q}\, d\sigma(x) = \int_{S^{n-1}} \left(\bar{Q}\,\frac{\partial P}{\partial \nu} - P\,\frac{\partial \bar{Q}}{\partial \nu}\right) d\sigma(x)$$

$$= \int_{|x| \leq 1} [\bar{Q}\,\Delta P - P\,\Delta \bar{Q}]\, dx = 0$$

by Green's theorem, where $\dfrac{\partial}{\partial \nu}$ denotes differentiation with respect to the outward normal, and $\Delta = \sum_{j=1}^{n} \dfrac{\partial^2}{\partial x_j^2}$ is the Laplacean.

*3.1.2* If $P$ is any homogeneous polynomial of degree $k$ (not necessarily harmonic) then $P = P_1 + |x|^2 P_2$, where $P_1$ is homogeneous of degree $k$, $P_2$ is homogeneous of degree $k - 2$, and $P_1$ is harmonic. To prove this we argue as follows. Let $\mathcal{P}_k$ denote the linear space of all homogeneous polynomials of degree $k$. We write $P(x) = \sum a_\alpha x^\alpha$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_1 + \alpha_2 \cdots + \alpha_n = k$ and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. To each such polynomial corresponds its dual object, the differential operator $P\left(\dfrac{\partial}{\partial x}\right) = \sum a_\alpha \left(\dfrac{\partial}{\partial x}\right)^\alpha$, where $\left(\dfrac{\partial}{\partial x}\right)^\alpha = \left(\dfrac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\dfrac{\partial}{\partial x_n}\right)^{\alpha_n}$. On $\mathcal{P}_k$ we define a positive inner product $\langle P, Q\rangle = P\left(\dfrac{\partial}{\partial x}\right) \bar{Q}$. Notice that two distinct monomials $x^\alpha$ and $x^{\alpha'}$ are orthogonal with respect to it, and $\langle P, P\rangle = \sum |a_\alpha|^2 \alpha!$ where $\alpha! = (\alpha_1!)(\alpha_2!) \cdots (\alpha_n!)$.

Let $|x|^2 \mathcal{P}_{k-2}$ be the subspace of $\mathcal{P}_k$ of all polynomials of the form $|x|^2 P_2$, where $P_2 \in \mathcal{P}_{k-2}$. Then its orthogonal complement (with respect to $\langle \cdot, \cdot\rangle$) is exactly $\mathcal{H}_k$. In fact $P_1$ is in this orthogonal complement if and only if $\langle |x|^2 P_2, P_1\rangle = 0$ for all $P_2$. But $\langle |x|^2 P_2, P_1\rangle = \left(P_2\left(\dfrac{\partial}{\partial x}\right)\Delta\right)\bar{P}_1 = \langle P_2, \Delta P_1\rangle$, so $\Delta P_1$ is null and thus $\mathcal{P}_k = \mathcal{H}_k \oplus |x|^2 \mathcal{P}_{k-2}$, which proves the assertion (ii).

*3.1.3* Let $H_k$ denote the linear space of restrictions of $\mathcal{H}_k$ to the unit sphere. The elements of $H_k$ are the *surface spherical harmonics* of degree $k$. Then $L^2(S^{n-1}) = \sum_{k=0}^{\infty} H_k$. The $L^2$ space is taken with respect to usual measure, and the infinite direct sum is taken in the sense of Hilbert space theory. Since we have already proved the mutual orthogonality of the subspaces $H_k$, we need only observe that every $f \in L^2(S^{n-1})$ can be approximated in the norm, by finite linear combinations of elements from $H_k$.

That this is possible can be seen as follows. Use §3.1.2 and apply it again to $P_2$, repeating this process. This shows that if $P$ is a polynomial then $P(x) = P_1(x) + |x|^2P_2(x) + |x|^4P_2(x)\ldots$, where each of the $P_j$ are harmonic polynomials. When we set $|x| = 1$, we see that the restriction of *any* polynomial on the unit sphere is a finite linear combination of spherical harmonics.

Since the restrictions of polynomials are dense in $L^2(S^{n-1})$ in the norm, the assertion is then established. It may also be restated as follows. If $f \in L^2(S^{n-1})$, then $f$ has the development

$$f(x) = \sum_{k=0}^{\infty} Y_k(x), \qquad Y_k \in H_k, \tag{25}$$

where the convergence is in the $L^2(S^{n-1})$ norm, and

$$\int_{S^{n-1}} |f(x)|^2\, d\sigma(x) = \sum_k \int_{S^{n-1}} |Y_k(x)|^2\, d\sigma(x).$$

*3.1.4* Let $\Delta_S$ denote the spherical Laplacean. If $Y_k(x) \in H_k$, then $\Delta_S Y_k(x) = -k(k+n-2)Y_k(x)$. In fact if $Y(x)$ is any function defined on the sphere, then $\Delta_S Y(x)$ equals the restriction of the ordinary Laplacean applied to $Y(x)$, but where $Y(x)$ is now defined in the neighborhood of this sphere by considering it as homogeneous of degree 0. Thus we must calculate $\Delta(|x|^{-k}P_k(x))$, where $P_k \in \mathscr{H}_k$. But this is

$$|x|^{-k}\Delta P_k + P_k\Delta(|x|^{-k}) + 2\sum_{j=1}^{n} \frac{\partial}{\partial x_j}|x|^{-k}\frac{\partial}{\partial x_j}P_k.$$

If we carry out the required differentiation, and use the fact that

$$\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} P_k = kP_k$$

(Euler's theorem for homogeneous functions), then we obtain our assertion.

*3.1.5* Suppose $f$ has the development (25). Then $f$ (after correction on a set of measure zero, if necessary) is indefinitely differentiable on $S^{n-1}$ if and only if

$$\int_{S^{n-1}} |Y_k(x)|^2\, d\sigma(x) = O(k^{-N}), \quad \text{as } k \to \infty, \text{ for each fixed } N. \tag{26}$$

To prove this write (25) as $f(x) = \sum_{k=0}^{\infty} a_k Y_0^k(x)$, where the $Y_0^k$ are normalized; our assertion is then equivalent with $a_k = O(k^{-N/2})$, as $k \to \infty$. If $f$ is of class $C^2$ then an application of Green's theorem shows that

$$\int_{S^{n-1}} \Delta_S f \overline{Y_k^0}\, d\sigma = \int_{S^{n-1}} f \Delta_S \overline{Y_k^0}\, d\sigma.$$

Thus if $f$ is indefinitely differentiable,

$$\int_{S^{n-1}} (\Delta_S^r) f \cdot \bar{Y}_k^0 \, d\sigma = \int_{S^{n-1}} f(\Delta_S^r \bar{Y}_k^0) \, d\sigma = a_k[-k(k + n - 2)]^r,$$

by §3.1.4. So $a_k = O(k^{-2r})$ for every $r$ and therefore (26) holds. To prove the converse we note that (26) implies not only that $f \in L^2(S^{n-1})$, but that for every positive integer $r$, $(\Delta_S)^r f$, (when suitably defined) also belongs to $L^2(S^{n-1})$. That this implies that $f$ can be corrected on a set of measure zero so as to become indefinitely differentiable will not be proved here; the rather technical argument will be given in Appendix C.

**3.2** Thus, after this rapid review of some of the fundamental facts concerning spherical harmonics, we return to the study of special singular integral transforms. First we deal with the interelation of spherical harmonics and the Fourier transform which is in reality the study of the decomposition of the space $L^2(\mathbf{R}^n)$ under the simultaneous action of rotations and the Fourier transform. At its source is the beautiful identity of Hecke.

THEOREM 4. *Suppose $P_k(x)$ is a homogeneous harmonic polynomial of degree $k$. Then*

$$(27) \qquad \mathscr{F}(P_k(x)e^{-\pi|x|^2}) = i^k P_k(x) e^{-\pi|x|^2}.$$

The identity to be proved can be rewritten as

$$(28) \qquad \int_{\mathbf{R}^n} P_k(x) \exp[-\pi|x|^2 + 2\pi i x \cdot y] \, dx = i^k P_k(y) e^{-\pi|y|^2}.$$

It is clear however that the left side of (28) equals $Q(y)e^{-\pi|y|^2}$ where $Q$ is a polynomial which we see when we apply the differential operator $P_k\left(\frac{\partial}{\partial y}\right)$ to both sides of the identity

$$\int_{\mathbf{R}^n} \exp[-\pi|x|^2 + 2\pi i x \cdot y] \, dx = \exp(-\pi|y|^2).$$

The problem is therefore to show that $Q(y) = P_k(iy)$. But $Q(y) = \int_{\mathbf{R}^n} P_k(x) \exp -\pi\{(x_1 - iy_1)^2 + (x_2 - iy_2)^2 \cdots (x_n - iy_n)^2\} \, dx$. However this integral is equal to $\int_{\mathbf{R}^n} P_k(x + iy)e^{-\pi|x|^2} \, dx$, after a shift of the contours of integration in $\mathbf{C}^n$, which is justified by the analyticity of $P_k(x) \exp[-\pi \sum_{j=1}^{n} x_j^2]$, and its rapid decrease. For the same reason we can replace $iy$ by $y$. This gives $Q(y/i) = \int_{\mathbf{R}^n} P_k(x + y)e^{-\pi|x|^2} \, dx$. Now $P_k$ is harmonic, so its mean value over any sphere centered at $y$ has the value

$P_k(y)$; the factor $e^{-\pi|x|^2}$ is constant on such spheres, while its total integral over $\mathbf{R}^n$ is 1. Thus $Q(y/i) = P_k(y)$, which proves the theorem.

The theorem implies the following generalization of itself, whose interest is that it links the various components of the decomposition of $L^2(\mathbf{R}^n)$, for different $n$.

If $f$ is a radial function, we write $f = f(r)$, where $|x| = r$.

COROLLARY. *Let $P_k(x)$ be a homogeneous harmonic polynomial of degree $k$ in $\mathbf{R}^n$. Suppose that $f$ is radial and $P_k(x)f(r) \in L^2(\mathbf{R}^n)$. Then the Fourier transform of $P_k(x)f(r)$ is also of the form $P_k(x)g(r)$, with $g$ a radial function. Moreover the induced transformation $f \to g$, $\mathscr{F}_{n,k}(f) = g$, depends essentially only on $n + 2k$. More precisely, we have Bochner's relation*

$$\mathscr{F}_{n,k} = i^k \mathscr{F}_{n+2k,0} \tag{29}$$

*Proof.* Consider the Hilbert space of radial functions

$$\mathscr{R} = \left\{ f(r) : \|f\|^2 = \int_0^\infty |f(r)|^2 \, r^{2k+n-1} \, dr < \infty \right\},$$

with the indicated norm. Fix now $P_k(x)$, and assume that $P_k$ is normalized, i.e., $\int_{S^{n-1}} |P_k(x)|^2 \, d\sigma(x) = 1$. Then there is an obvious unitary correspondence between the elements $f$ of $\mathscr{R}$ and the elements $f(|x|)P_k(x)$ in $L^2(\mathbf{R}^n)$, and $f(|x|)$ in $L^2(\mathbf{R}^{n+2k})$ respectively. What we have to prove is that

$$(\mathscr{F}_{n,k} f)(r) = i^k \mathscr{F}_{n+2k,0}(f)(r), \tag{30}$$

for each $f \in \mathscr{R}$. First, if $f(r) = e^{-\pi r^2}$, then (30) is an immediate consequence of the theorem (see (27)). Now consider next $e^{-\pi\delta r^2}$ for a fixed $\delta > 0$. Because of the homogeneity of $P_k$ and the interplay of dilations with the Fourier transform (see (1) in §1.2), we get successively,

$$\begin{aligned} \mathscr{F}(P_k(x)e^{-\pi\delta|x|^2}) &= \delta^{-k/2}\mathscr{F}(P_k(\delta^{\frac{1}{2}}x)e^{-\pi\delta|x|^2}) \\ &= i^k \delta^{-k/2-n/2} P_k(x/\delta^{\frac{1}{2}}) e^{-\pi|x|^2/\delta} = i^k \delta^{-k-n/2} P_k(x) e^{-\pi|x|^2/\delta}. \end{aligned}$$

This shows that $\mathscr{F}_{n,k}(e^{-\pi\delta r^2}) = i^k \delta^{-k-n/2} e^{-\pi r^2/\delta}$, and so proves (30) for $f(r) = e^{-\pi\delta r^2}$, $\delta > 0$.

To conclude the proof of the corollary it suffices to see that the linear combination of $\{e^{-\pi\delta r^2}\}_{0<\delta<\infty}$, are dense in $\mathscr{R}$. Suppose the contrary. Then there exists a non-zero $g \in \mathscr{R}$, so that $\int_0^\infty e^{-\pi\delta r^2} g(r) r^{2k+n-1} \, dr = 0$, for all $\delta > 0$. Making the change of variables $r^2 \to r$ brings us back to the Fourier-Laplace transform, and by a very well known argument we can show that $g \equiv 0$, concluding the proof of the corollary.

**3.3** We come now to what has been our main goal in our discussion of spherical harmonics.

THEOREM 5. *Let $P_k(x)$ be a homogeneous harmonic polynomial of degree $k$, $k \geq 1$. Then the multiplier corresponding to the transformation (24) with the kernel $\dfrac{P_k(x)}{|x|^{k+n}}$ is*

$$\gamma_k \frac{P_k(x)}{|x|^k}, \text{ where } \gamma_k = i^k \pi^{n/2} \frac{\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{k+n}{2}\right)},$$

Notice that if $k \geq 1$, $P_k(x)$ is orthogonal to the constants on the sphere (§3.1.1), and so its mean value over the sphere is zero.

The statement of the theorem can be interpreted as

$$\left(\frac{P_k(x)}{|x|^{k+n}}\right)^{\wedge} = \gamma_k \frac{P_k(x)}{|x|^k}. \tag{31}$$

As such it will be derived from the following closely related fact,

$$\left(\frac{P_k(x)}{|x|^{k+n-\alpha}}\right)^{\wedge} = \gamma_{k,\alpha} \frac{P_k(x)}{|x|^{k-\alpha}} \text{ with } \gamma_{k,\alpha} = i^k \pi^{n/2-\alpha} \frac{\Gamma(k/2 + \alpha/2)}{\Gamma(k/2 + n/2 - \alpha/2)}. \tag{32}$$

LEMMA. *The identity* (32) *holds in the sense that*

$$\int_{\mathbf{R}^n} \frac{P_k(x)}{|x|^{k+n-\alpha}} \hat{\varphi}(x)\, dx = \gamma_{k,\alpha} \int_{\mathbf{R}^n} \frac{P_k(x)}{|x|^{k+\alpha}} \varphi(x)\, dx \tag{33}$$

*for every $\varphi$ which is sufficiently rapidly decreasing at $\infty$, and whose Fourier transform has the same property. It is valid for all integral $k$ and for $0 < \alpha < n$.*

Observe that both the left and right side of (32) are locally integrable in the range $0 < \alpha < n$, and so the integrals in (33) both converge absolutely.

It should be pointed out that both the theorem, and the lemma from which it is deduced, are in effect special cases of the general law (29); however here the context of $L^2$ is replaced by that of "generalized functions." Still other generalizations of (29) present themselves, but we shall not pursue this point further.

**3.4** We turn to the proof of the lemma. We have already observed in §3.2 that $\mathscr{F}(P_k(x)e^{-\pi\delta|x|^2}) = i^k\delta^{-k-n/2}P_k(x)e^{-\pi|x|^2/\delta}$, so we have

$$\int_{\mathbf{R}^n} P_k(x)e^{-\pi\delta|x|^2}\hat{\varphi}(x)\, dx = i^k\, \delta^{-k-n/2} \int_{\mathbf{R}^n} P_k(x)e^{-\pi|x|^2/\delta}\varphi(x)\, dx,$$

if $\delta > 0$.

We now integrate both sides of the above with respect to $\delta$, after having multiplied the equation by a suitable power of $\delta$, ($\delta^{\beta-1}$, $\beta = \dfrac{k+n-\alpha}{2}$, to be precise).

If we use the fact that $\int_0^\infty e^{-\pi\delta|x|^2}\delta^{\beta-1}\,d\delta = (\pi|x|^2)^{-\beta}\Gamma(\beta)$, if $\beta > 0$, we get $\Gamma\left(\dfrac{k+n-\alpha}{2}\right)\pi^{-(k+n-\alpha)/2}\int_{\mathbf{R}^n}\dfrac{P_k(x)}{|x|^{k+n-\alpha}}\hat{\varphi}(x)\,dx$ for the integral on the left side. The corresponding integration for the right side gives

$$i^k\Gamma\left(\frac{k+\alpha}{2}\right)\pi^{-(k+\alpha)/2}\int_{\mathbf{R}^n}\frac{P_k(x)}{|x|^{k-\alpha}}\varphi(x)\,dx$$

which leads to identity (33). It is to be observed that when $0 < \alpha < n$ and both $\varphi$ and $\hat{\varphi}$ decrease sufficiently rapidly (the estimates $|\varphi(x)| \leq A(1+|x|)^{-n}$, and $|\hat{\varphi}(x)| \leq A(1+|x|)^{-n}$ suffice), then the double integrals that occur in the above manipulation converge absolutely. Thus the formal argument just given establishes the lemma.

To prove the theorem we make the assumption that $k \geq 1$, and we restrict $\varphi$ further by supposing that $\hat{\varphi}$ is also smooth (the differentiability of $\hat{\varphi}$ near the origin will suffice). Then,

$$\lim_{\substack{\alpha\to 0\\ \alpha>0}}\int_{\mathbf{R}^n}\frac{P_k(x)}{|x|^{k+n-\alpha}}\hat{\varphi}(x)\,dx = \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\frac{P_k(x)}{|x|^{k+n}}\hat{\varphi}(x)\,dx. \tag{34}$$

In fact, since the integral of $P_k$ over any sphere centered at the origin is zero, then

$$\int_{\mathbf{R}^n}\frac{P_k(x)}{|x|^{k+n-\alpha}}\hat{\varphi}(x)\,dx = \int_{|x|\leq 1}\frac{P_k(x)}{|x|^{k+n-\alpha}}[\hat{\varphi}(x) - \hat{\varphi}(0)]\,dx + \int_{|x|>1}\frac{P_k(x)}{|x|^{k+n-\alpha}}\hat{\varphi}(x)\,dx.$$

Passing to the limit as $\alpha \to 0$ we get for the first integral on the right side,

$$\int_{|x|\leq 1}\frac{P_k(x)}{|x|^{k+n}}[\hat{\varphi}(x) - \hat{\varphi}(0)]\,dx = \lim_{\varepsilon\to 0}\int_{\varepsilon\leq|x|\leq 1}\frac{P_k(x)}{|x|^{k+n}}\hat{\varphi}(x)\,dx,$$

which proves our assertion (34). Finally, let $f$ be any sufficiently smooth function with compact support, and for fixed $x$, set $f(x-y) = \hat{\varphi}(y)$. Then since $(\hat{\varphi})^\wedge(y) = \varphi(-y)$, we see that $\varphi(y) = \hat{f}(y)e^{-2\pi i x\cdot y}$, and so our assertion is in this case

$$\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\frac{P_k(y)}{|y|^{k+n}}f(x-y)\,dy = \gamma_k\int_{\mathbf{R}^n}\frac{P_k(y)}{|y|^k}\hat{f}(y)e^{-2\pi i x\cdot y}\,dy. \tag{35}$$

Now by the definition of the multiplier $m$, we have

$$\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\frac{P_k(y)}{|y|^{k-n}}f(x-y)\,dy=\int_{\mathbf{R}^n}m(y)\hat{f}(y)e^{-2\pi i x\cdot y}\,dy$$

where the convergence of both integrals is in the $L^2$ sense. Because the type of $f$ just described is dense in $L^2$, we get $m(y)=\gamma_k\dfrac{P_k(y)}{|y|^k}$, which establishes the theorem.

For fixed $k$, $k\geq 1$, the (finite-dimensional) linear space of operators (24), where $\Omega(y)=\dfrac{P_k(y)}{|y|^k}$ and the $P_k$ range over the homogeneous harmonic polynomials of degree $k$, form a natural generalization of the Riesz transforms; the latter arise in the special case $k=1$. Those for $k>1$, we call the higher Riesz transforms;* they can also be characterized by their invariance properties (see §4.8).

**3.5** We now consider two classes of transformations, defined on $L^2(\mathbf{R}^n)$ (which can later also be defined on $L^p(\mathbf{R}^n)$, $1<p<\infty$). The first class consists of all transformations of the form

$$T(f)=c\cdot f+\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\frac{\Omega(y)}{|y|^n}f(x-y)\,dy. \tag{36}$$

$c$ is a constant; $\Omega$ is a homogeneous function of degree 0, which is indefinitely differentiable on the sphere $S^{n-1}$, and whose mean value on that sphere is zero. The second class is given by those transformations $T$ for which

$$(Tf)\hat{}\,(y)=m(y)\hat{f}(y) \tag{37}$$

where the multiplier $m$ is homogeneous of degree 0 and is indefinitely differentiable on the sphere.

THEOREM 6. *The two classes of transformations, defined by* (36) *and* (37) *respectively, are identical.*

Suppose first that $T$ is of the form (36). Then according to the theorem in §4.2 of Chapter 2, (see also the formula (6) of the present chapter), $T$ is of the form (37) with $m$ homogeneous of degree 0 and

$$m(x)=c+\int_{S^{n-1}}\Gamma(x\cdot y)\Omega(y)\,d\sigma(y),\qquad |x|=1. \tag{38}$$

* We refer to $k$ as the *degree* of the higher Riesz transform.

Now write the spherical harmonic developments

$$\Omega(y) = \sum_{k=1}^{\infty} Y_k(y), \qquad m(x) = \sum_{k=0}^{\infty} \tilde{Y}_k(x) \quad \text{and}$$

(39)

$$\Omega_N(x) = \sum_{k=1}^{N} Y_k(x), \qquad m_N(x) = \sum_{k=0}^{N} \tilde{Y}_k(x).$$

Then by the theorem we have just proved, if $\Omega = \Omega_N$, then $m(x) = m_N(x)$, with

$$\tilde{Y}_k(x) = \gamma_k Y_k(x), \qquad 1 \leq k.$$

But $m_M(x) - m_N(x) = \int_{S^{n-1}} \Gamma(x \cdot y)[\Omega_M(y) - \Omega_N(y)]\, d\sigma(y)$. Moreover

$$\sup_{x \in S^{n-1}} |m_M(x) - m_N(x)| \leq \left(\sup_x \int_{S^{n-1}} |\Gamma(x \cdot y)|^2\, d\sigma(y)\right)^{\frac{1}{2}} \times \left(\int_{S^{n-1}} |\Omega_M - \Omega_N|^2\, d\sigma(y)\right)^{\frac{1}{2}} \to 0,$$

as $M, N \to 0$, since

$$\sup_x \int_{S^{n-1}} |\Gamma(x \cdot y)|^2 d\sigma(y) = \int_{S^{n-1}} |\Gamma(x \cdot y)|^2\, d\sigma(y) = c_1 + c_2 \int_0^{\pi} |\log|\cos\theta||^2 (\sin\theta)^{n-2}\, d\theta < \infty$$

in view of the fact that $\Gamma(t) = \dfrac{\pi i}{2} \operatorname{sign} t + \log 1/|t|$. This shows that

$$m(x) = c + \sum_{k=1}^{\infty} \gamma_k Y_k(x).$$

Now by the indefinite differentiability of $\Omega$ we have that

$$\int_{S^{n-1}} |Y_k(x)|^2\, d\sigma(x) = O(k^{-N})$$

as $k \to \infty$ for every fixed $N$. However by the explicit form of $\gamma_k$, we see that $\gamma_k \approx k^{-n/2}$, so $m(x)$ is also indefinitely differentiable on the unit sphere.

Conversely, suppose $m(x)$ is indefinitely differentiable on the unit sphere and let its spherical harmonic development be as in (39). Set $c = \tilde{Y}_0$, and $Y_k(x) = \dfrac{1}{\gamma_k} \tilde{Y}_k(x)$. Then $\Omega(y)$, given by (39), has mean value zero in the sphere, and is again indefinitely differentiable there. But as we have just seen the multiplier corresponding to this transformation is $m$; so the theorem is proved.

As an application of this theorem and a final illustration of the singular integral transforms we shall give the generalization of the estimates for partial derivatives given in §1.3.

Let $P(x)$ be a homogeneous polynomial of degree $k$ in $\mathbf{R}^n$. We shall say that $P$ is *elliptic* if $P(x)$ vanishes only at the origin. For any polynomial $P$ we consider also its corresponding differential polynomial. Thus if $P(x) = \sum a_\alpha x^\alpha$, we write $P\left(\frac{\partial}{\partial x}\right) = \sum a_\alpha \left(\frac{\partial}{\partial x}\right)^\alpha$ where $\left(\frac{\partial}{\partial x}\right)^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x}\right)^{\alpha_k}$, and with the monomials $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ (which are of degree $|\alpha| = \alpha_1 + \alpha_2 \cdots + \alpha_n$).

COROLLARY. *Suppose $P$ is a homogeneous elliptic polynomial of degree $k$. Let $\left(\frac{\partial}{\partial x}\right)^\alpha$ be any differential monomial of degree $k$. Assume $f$ is $k$ times continuously differentiable with compact support. Then we have the a priori estimate*

$$\left\| \left(\frac{\partial}{\partial x}\right)^\alpha f \right\|_p \leq A_p \left\| P\left(\frac{\partial}{\partial x}\right) f \right\|_p, \qquad 1 < p < \infty. \tag{40}$$

To prove this we note, as in §1.3, the following relation between the Fourier transform of $\left(\frac{\partial}{\partial x}\right)^\alpha f$ and $P\left(\frac{\partial}{\partial x}\right) f$,

$$P(y)\left[\left(\frac{\partial}{\partial x}\right)^\alpha f\right]^\wedge (y) = (-2\pi i y)^\alpha \left(P\left(\frac{\partial}{\partial x}\right) f\right)^\wedge (y).$$

Since $P(y)$ is non-vanishing except at the origin, $\frac{y^\alpha}{P(y)}$ is homogeneous of degree 0 and is indefinitely differentiable on the unit sphere. Thus

$$\left(\frac{\partial}{\partial x}\right)^\alpha f = T\left(P\left(\frac{\partial}{\partial x}\right) f\right),$$

where $T$ is one of the transformations of the type given by (37). By Theorem 6, $T$ is also given by (36) and hence by the results of Chapter II, we get the estimate (40). An extension of this result is indicated in §7.9 of the next chapter.

## 4. *Further results*

**4.1** Our purpose is to show that certain results for $L^1(\mathbf{R}^n)$ may be extended to the finite Borel measures on $\mathbf{R}^n$, i.e. $\mathscr{B}(\mathbf{R}^n)$:

(a) Let $d\mu \in \mathscr{B}(\mathbf{R}^n)$, and $M(d\mu)(x) = \sup \frac{1}{m(B(x, r))} \int_{B(x,r)} |d\mu|$. Then

$$m\{x : M(d\mu)(x) > \alpha\} \leq \frac{A}{\alpha} \int_{\mathbf{R}^n} |d\mu|.$$

The argument is the same as in the case of integrable functions.
(b) If $d\mu$ is purely singular, then

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} d\mu = 0 \text{ for almost every } x.$$

*Hint:* Write $d\mu = d\mu_1 + d\mu_2$ where $d\mu_1$ is supported on a closed set $F$ of measure zero and $\|d\mu_2\| \leq \delta$. Then

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} d\mu_1 = 0 \text{ for every } x \notin F.$$

A more general result of this type holds for any approximation of the identity of the type occurring in Theorem 2, in particular for Poisson integrals.
(c) Let $T_\varepsilon(d\mu)(x) = \int_{|x-y|>\varepsilon} \frac{\Omega(x-y)}{|x-y|^n}\, d\mu(y)$, where $\Omega$ satisfies the conditions of Theorems 3 and 4 of Chapter II. Then $\lim_{\varepsilon\to 0} T_\varepsilon(d\mu)(x)$ exists almost everywhere.

See e.g. Zygmund [8]; Calderón and Zygmund [1], for part (c).

**4.2** Let $u(x, y)$ be harmonic in $\mathbf{R}^{n+1}_+$.
(a) If $1 \leq p \leq \infty$, $u(x, y)$ is the Poisson integral of an $L^p(\mathbf{R}^n)$ function if and only if $\sup_{y>0} \|u(x, y)\|_p < \infty$.

(b) $u(x, y)$ is the Poisson integral of a measure in $\mathscr{B}(\mathbf{R}^n)$ if and only if

$$\sup_{y>0} \|u(x, y)\|_1 < \infty.$$

See e.g. Stein and Weiss [2]. See also §1.2.1 in Chapter VII.

**4.3** Suppose $f \in L^2(\mathbf{R}^n)$, $f_j = R_j(f)$, and $u_j(x, y)$ is the Poisson integral of $f_j$. Then

$$u_j(x, y) = \int_{\mathbf{R}^n} Q_y^{(j)}(t) f(x - t)\, dt,$$

where

$$Q_y^{(j)}(x) = c_n \frac{x_j}{(|x|^2 + y^2)^{\frac{n+1}{2}}}.$$

**4.4** This result, as well as that of Theorem 3 in §2.3, generalizes to $L^p(\mathbf{R}^n)$, $1 < p < \infty$. For details see Horváth [1]; see also §3.2 in Chapter VII. The case $n = 1$ is treated in Titchmarch [1], Chapter 5.

**4.5** It is worthwhile to observe the following easily proved facts:
(a) Let $\mathscr{A}$ be the algebra of operators on $L^2(\mathbf{R}^n)$ which is (algebraically) generated by the Riesz transforms $R_1, R_2, \ldots, R_n$. Then every higher Riesz transform belongs to $\mathscr{A}$.

(b) The closure of $\mathscr{A}$ (in the strong operator topology) is identical with the algebra of bounded transformations on $L^2(\mathbf{R}^n)$ which commute with translations and dilations.

**4.6** In §4.6 to 4.8 we shall assume that $n \geq 3$. (The cases $n = 1, 2$ would need minor modifications.) We let $SO(n)$ denote the group of proper rotations in $\mathbf{R}^n$, and $SO(n-1)$ the subgroup of those rotations leaving the direction along the $x_1$ axis fixed. Then, for every $k$, the subspace of the polynomials $P_k(x) \in \mathscr{H}_k$, which are fixed by $SO(n-1)$, i.e. for which $P_k(\rho^{-1}x) = P_k(x)$, $\rho \in SO(n-1)$, is exactly one-dimensional. See *Fourier Analysis*, Chapter IV.

**4.7** Let $V$ be a finite dimensional Hilbert space and $\rho \to R_\rho$ a continuous homomorphism from $SO(n)$ to the group of unitary transformations on $V$. The couple $(R_\rho, V)$ is called a *representation* of $SO(n)$. It is *irreducible* if there is no non-trivial subspace of $V$ invariant under the $R_\rho$, $\rho \in SO(n)$. Two representations $(R_\rho^{(1)}, V_1)$ and $(R_\rho^{(2)}, V_2)$ are *equivalent* if there is a unitary correspondence $U$, $U: V_1 \longleftrightarrow V_2$, so that $U^{-1}R_\rho^{(2)}U = R_\rho^{(1)}$.

(a) Let $V = \mathscr{H}_k$ (the linear space of homogeneous harmonic polynomials of degree $k$). Define

$$(R_\rho P(x)) = P(\rho^{-1}x), \quad \rho \in SO(n), \; P \in \mathscr{H}_k.$$

This representation is irreducible.

(b) An irreducible representation $(R_\rho, V)$ of $SO(n)$ is equivalent to one obtained from the spherical harmonics, as above, if and only if there exists a non-zero $v \in V$ so that

$$R_\rho(v) = v, \text{ all } \rho \in SO(n-1).$$

For the general theory of representations of the rotation group see Weyl [1], Boerner [1]. §4.7 can be deduced from §4.6 by using the Frobenius reciprocity theorem for compact groups. For the reciprocity theorem see Weil [1].

**4.8** Let $(R_\rho, V)$ be an irreducible representation of $SO(n)$ as in §4.7 above. Suppose $f \to T(f)$ is a bounded linear transformation from $L^2(\mathbf{R}^n)$ to $L^2(\mathbf{R}^n, V)$; thus $T$ takes complex-valued functions to functions which take their values in $V$.

(a) Suppose $T$ commutes with translations and dilations and transforms according to $(R_\rho, V)$ in the sense that

$$\rho T \rho^{-1}(f) = R_\rho Tf.$$

Then $T \equiv 0$ unless $(R_\rho, V)$ is equivalent to a representation obtained from spherical harmonics as in §4.7(a).

(b) If $(R_\rho, V)$ arises from the spherical harmonics of degree $k$, then $T$ is determined up to a constant multiple. In particular if $\beta_1, \beta_2, \ldots, \beta_N$ is a basis for the linear functionals on $V$, then each $\beta_j(Tf)$ is a higher Riesz transform of degree $k$, if $k \geq 1$ (when $k = 0$, $T$ is a constant multiple of the identity).

**4.9** The following observation will be useful later: Suppose $u(x, y)$ is the Poisson integral of $f$, $f \in L^p(\mathbf{R}^n)$. Then $\sup_{y>0} \left| y \dfrac{\partial u}{\partial x_j}(x, y) \right| \leq A(Mf)(x)$.

*Hint:* Apply Theorem 2 to the case $\varphi(x) = \dfrac{\partial}{\partial x_j}[P_1(x)]$.

A more general result of this kind is that $\sup_{y>0} |y^{|\alpha|} Du(x, y)| \leq A_\alpha Mf(x)$, where $D$ is any differential monomial in $x$ and $y$ of total degree $|\alpha|$.

## *Notes*

*Sections 1 and 2.* The connection of singular integrals (of the "Riesz transform" type) with estimates like those in §1.3 have a long history. See e.g. Friedrichs [1], for $p = 2$; also Calderón and Zygmund [1] for the case of general $p$. The identification of Riesz transforms with conjugate harmonic functions (in §2.3) goes back to Horváth [1]. For the Fourier analysis of the $n$-dimensional Poisson integral see Bochner [1], [2], and Bochner and Chandrasekharan [1]. The relation with the maximal function, which generalizes the classical result of Hardy and Littlewood, is in K. T. Smith [1]. The argument in §2.4 comes from Calderón and Zygmund [1].

*Section 3.* The main results are Theorem 4 and its corollary. The first is implicit in the work of Hecke [1] and was made explicit by Bochner, who also deduced the corollary; see Bochner [2]; also Calderón and Zygmund [5], and Calderón [3]. A more elaborate treatment of some of these topics, in particular spherical harmonics and the connection with Bessel functions, may be found in *Fourier Analysis*, Chapter IV. In comparing the present formulae with those in *Fourier Analysis* one should keep in mind that the Fourier transform defined here corresponds to the inverse Fourier transform in *Fourier Analysis*.

The germinal idea of the calculus of singular integrals in terms of their "symbols" is given in §3.5, although the latter notion is not explicitly defined there. For further details see Mihlin [1], and Calderón and Zygmund [5]. Later developments involving partial differential operators are in Calderón [3], [5], Seeley [1], Kohn and Nirenberg [1], Unterberger and Bokobza [1], and Hörmander [2]. A sketch of the history of the subject may be found in Seeley [2]. The reader is urged to consult it for the work of earlier writers; of particular note, in this connection, are the contributions of Giraud.

# CHAPTER IV

# The Littlewood-Paley Theory and Multipliers

The Littlewood-Paley theory of one-dimensional Fourier series, and its applications, represents one of the most far-reaching advances of that subject. The theory originally proceeded along three main lines, each interesting in its own right:

(i) The auxiliary $g$-function which, aside from its applications, illustrates the principle that often the most fruitful way of characterizing various analytical situations (such as finiteness of $L^p$ norms, existence of limits almost everywhere, etc.) is in terms of appropriate quadratic expressions.

(ii) The "dyadic" decomposition of a function in terms of its Fourier analysis.

(iii) The multiplier theorem of Marcinkiewicz which gives very useful sufficient conditions for $L^p$ multipliers.

This theory was developed in the main between 1930 and 1939 by Littlewood and Paley, Zygmund, and Marcinkiewicz, but it depended on complex function theory and so its full thrust was limited to the case of one-dimension. The $n$-dimensional theory is more recent and was inspired in part by the real-variable techniques presented in Chapters I and II.

We should, however, not want the reader to be left with an over-simplified picture of the above. Thus it was realized early that significant $n$-dimensional results could be deduced from the one-dimensional theory. Moreover, the $n$-dimensional theory is only partly successful in comparison with the one-dimensional case, and much remains to be done in the general context. (The latter point is taken up again §6.2.)

There are by now several possible approaches to the main results which we present, but we have purposely not chosen the shortest and most direct way; we hope, however, that the longer route we shall follow will be more instructive. In this way the reader will have a better opportunity to examine all the working parts of the complex mechanism detailed below.

## 1. *The Littlewood-Paley g-function*

**1.1** The $g$-function is a (non-linear) operator which allows one to give a useful characterization of the $L^p$ norm of a function on $\mathbf{R}^n$ in terms of the behavior of its Poisson integral. This characterization will be used not only in this chapter, but also in the succeeding chapter dealing with function spaces. The $g$-function is defined as follows. Let $f \in L^p(\mathbf{R}^n)$ and write $u(x, y)$ for its Poisson integral

$$u(x, y) = \int_{\mathbf{R}^n} P_y(t) f(x - t)\, dt$$

as defined in Chapter III, §2. We let $\Delta$ denote the Laplace operator in $\mathbf{R}^{n+1}_+$, that is $\Delta = \dfrac{\partial^2}{\partial y^2} + \sum_{j=1}^n \dfrac{\partial^2}{\partial x_j}$; $\nabla$ is the corresponding gradient, $|\nabla u(x, y)|^2 = \left|\dfrac{\partial u}{\partial y}\right|^2 + |\nabla_x u(x, y)|^2$, where $|\nabla_x u(x, y)|^2 = \sum_{j=1}^n \left|\dfrac{\partial u}{\partial x_j}\right|^2$. With these notations we define $g(f)(x)$, by

$$g(f)(x) = \left(\int_0^\infty |\nabla u(x, y)|^2\, y\, dy\right)^{1/2}. \tag{1}$$

The basic result for $g$ is the following.

THEOREM 1. *Suppose $f \in L^p(\mathbf{R}^n)$, $1 < p < \infty$. Then $g(f)(x) \in L^p(\mathbf{R}^n)$, and*

$$A'_p \|f\|_p \leq \|g(f)\|_p \leq A_p \|f\|_p. \tag{2}$$

**1.2** It is best to begin with the simple case $p = 2$. With $f \in L^2(\mathbf{R}^n)$, we have $\|g(f)\|_2^2 = \int_{y=0}^\infty \int_{\mathbf{R}^n} y |\nabla u(x, y)|^2\, dx\, dy$. The double integral may be treated either by Green's theorem (as we shall see later in §2.1) or by Plancherel's formula, when we integrate first with respect to $x$. In fact, in view of the identity

$$u(x, y) = \int_{\mathbf{R}^n} \hat{f}(t) e^{-2\pi i t\cdot x} e^{-2\pi |t| y}\, dt$$

we have

$$\frac{\partial u}{\partial y} = \int_{\mathbf{R}^n} -2\pi |t| \hat{f}(t) e^{-2\pi i t\cdot x} e^{-2\pi |t| y}\, dt,$$

and

$$\frac{\partial u}{\partial x_j} = \int_{\mathbf{R}^n} -2\pi i t_j \hat{f}(t) e^{-2\pi i t\cdot x} e^{-2\pi |t| y}\, dt.$$

Thus

$$\int_{\mathbf{R}^n} |\nabla u(x, y)|^2\, dx = \int_{\mathbf{R}^n} 8\pi^2 |t|^2 |\hat{f}(t)|^2 e^{-4\pi|t|y}\, dt, \qquad y > 0$$

and so

$$\|g(f)\|_2^2 = \int_{\mathbf{R}^n} |\hat{f}(t)|^2 \left\{8\pi^2 |t|^2 \int_0^\infty e^{-4\pi|t|y}\, y\, dt\right\} = (1/2)\, \|\hat{f}\|_2^2.$$

Hence,

$$\|g(f)\|_2 = 2^{-1/2}\, \|f\|_2 \tag{3}$$

It may be appropriate here to introduce the following two "partial" $g$-functions, one dealing with the $y$ differentiation and the other with the $x$ differentiations,

$$\begin{aligned} g_1(f)(x) &= \left(\int_0^\infty \left|\frac{\partial u}{\partial y}(x, y)\right|^2 y\, dy\right)^{1/2}, \\ g_x(f)(x) &= \left(\int_0^\infty |\nabla_x u(x, y)|^2\, y\, dy\right)^{1/2} \end{aligned} \tag{4}$$

Note that $g^2 = g_1^2 + g_x^2$, and what is more interesting, the proof of (3) also shows that

$$\|g_1(f)\|_2 = \|g_x(f)\|_2 = \tfrac{1}{2}\, \|f\|_2.$$

The whole theory could be based just as well on $g_1$ or $g_x$ instead of $g$, and anyway the three are closely related by the Riesz transforms (see §7.1).

**1.3** The $L^p$ inequalities, when $p \neq 2$, will be obtained as a corollary of the theory of singular integrals in the context of Hilbert space-valued functions, as given in §5 of Chapter II. We define the Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ which are to be considered now. $\mathscr{H}_1$ is the one-dimensional Hilbert space of complex numbers. To define $\mathscr{H}_2$ we define first $\mathscr{H}_2^0$ as the $L^2$ space on $(0, \infty)$ with measure $y\, dy$, i.e.

$$\mathscr{H}_2^0 = \left\{f : |f|^2 = \int_0^\infty |f(y)|^2\, y\, dy < \infty\right\}$$

we let $\mathscr{H}_2$ be the direct sum of $n + 1$ copies of $\mathscr{H}_2^{(0)}$; so the elements of $\mathscr{H}_2$ can be represented as $(n + 1)$ component vectors whose entries belong to $\mathscr{H}_2^0$. Since $\mathscr{H}_1$ is the same as the complex numbers, then $B(\mathscr{H}_1, \mathscr{H}_2)$ is of course identifiable with $\mathscr{H}_2$. Now let $\varepsilon > 0$, and keep it temporarily fixed.

Define

$$K_\varepsilon(x) = \left(\frac{\partial P_{y+\varepsilon}(x)}{\partial y}, \frac{\partial P_{y+\varepsilon}(x)}{\partial x_1}, \ldots, \frac{\partial P_{y+\varepsilon}(x)}{\partial x_k}\right)$$

Notice that for each fixed $x$, $K_\varepsilon(x) \in \mathscr{H}_2$. This is the same as saying that $\int_0^\infty \left|\frac{\partial P_{y+\varepsilon}(x)}{\partial y}\right|^2 y\,dy < \infty$ and $\int_0^\infty \left|\frac{\partial P_{y+\varepsilon}(x)}{\partial x_j}\right|^2 y\,dy < \infty$, for $j = 1, \ldots, n$. However it is easily seen from the explicit formula (on p. 61) for the Poisson kernel that both $\frac{\partial P_y}{\partial y}$ and $\frac{\partial P_y}{\partial x}$ are bounded by $\frac{A}{(|x|^2 + y^2)^{\frac{n+1}{2}}}$.

So* $|K_\varepsilon(x)|^2 = A^2(n+1)\int_0^\infty \frac{y\,dy}{(|x|^2 + (y+\varepsilon)^2)^{n+1}} \leq A_\varepsilon$, and $\leq A\,|x|^{-2n}$.

Thus

$$|K_\varepsilon(x)| \in L^2(\mathbf{R}^n). \tag{5}$$

Similarly

$$\left|\frac{\partial K_\varepsilon(x)}{\partial x_j}\right|^2 \leq A\int_0^\infty \frac{y\,dy}{(|x|^2 + (y+\varepsilon))^{n+2}} \leq A\int_0^\infty \frac{y\,dy}{(|x|^2 + y^2)^{n+2}} = A'\,|x|^{-2n-2}.$$

Therefore,

$$\left|\frac{\partial K_\varepsilon(x)}{\partial x_j}\right| \leq A/|x|^{n+1}, \tag{6}$$

with $A$ independent of $\varepsilon$.

Now consider the operator $T_\varepsilon$ defined by

$$T_\varepsilon(f)(x) = \int_{\mathbf{R}^n} K_\varepsilon(t) f(x-t)\,dt$$

The functions $f$ are complex-valued (take their value in $\mathscr{H}_1$), but the $T_\varepsilon f(x)$ take their values in $\mathscr{H}_2$. Observe that

$$|T_\varepsilon(f)(x)| = \left(\int_0^\infty |\nabla u(x, y+\varepsilon)|^2\, y\,dy\right)^{1/2} \leq g(f)(x). \tag{7}$$

Hence $\|T_\varepsilon f(x)\|_2 \leq 2^{-1/2}\,\|f\|_2$, if $f \in L^2(\mathbf{R}^n)$, by (3); therefore

$$|\hat{K}_\varepsilon(x)| \leq 2^{-1/2}, \tag{8}$$

(which could also be verified directly).

Because of (5), (6), and (8) and in view of Theorem 5, Chapter II, we can apply the Hilbert space version of Theorem 1, Chapter II. The conclusion is $\|T_\varepsilon(f)\|_p \leq A_p\,\|f\|_p$, $1 < p < \infty$ with $A_p$ independent of $\varepsilon$. By (7), for each $x$, $|T_\varepsilon(f)(x)|$ increases to $g(f)(x)$, as $\varepsilon \to 0$, so we obtain finally

$$\|g(f)\|_p \leq A_p\,\|f\|_p, \qquad 1 < p < \infty \tag{9}$$

* Notice that here the symbol $|K_\varepsilon(x)|$ denotes the norm in $\mathscr{H}_2$ of $K_\varepsilon(x)$, for each $x$.

**1.4** We should have liked to derive the converse inequalities,

$$A'_p \, \|f\|_p \leq \|g(f)\|_p, \qquad 1 < p < \infty, \tag{10}$$

directly from (9) and the corollary in §5.3, Chapter II. But this would have required a preliminary argument, since the operator corresponding to $g$ has been obtained as a limit, and this limit is not a principal value arising from the truncation of kernels, (see (7)); in fact, the limiting approach used here is a little more natural, since principal values are not really relevant in this context. Nevertheless, matters can be settled directly and without any difficulty. Take $g_1$ instead of $g$. Then the equality $\|g_1(f)\|_2 = (1/2)\,\|f\|_2$, for $f \in L^2(\mathbf{R}^n)$, leads by polarization to the identity

$$4\int_{\mathbf{R}^n}\int_0^\infty y\,\frac{\partial u_1}{\partial y}(x, y)\,\overline{\frac{\partial u_2}{\partial y}(x, y)}\,dy\,dx = \int_{\mathbf{R}^n} f_1(x)\bar{f_2}(x)\,dx,$$

where $f_1, f_2 \in L^2(\mathbf{R}^n)$, and where $u_j$ are the Poisson integrals of $f_j$, $j = 1, 2$. This identity, in turn, leads to the inequality

$$4\left|\int_{\mathbf{R}^n} f_1(x)\overline{f_2(x)}\,dx\right| \leq \int_{\mathbf{R}^n} g_1(f_1)(x)g_1(f_2)(x)\,dx$$

Suppose now in addition that $f_1 \in L^p(\mathbf{R}^n)$ and $f_2 \in L^q(\mathbf{R}^n)$ with $\|f_2\|_q \leq 1$, and $1/p + 1/q = 1$. Then by Hölders inequality and the result (9),

$$\left|\int_{\mathbf{R}^n} f_1(x)\bar{f_2}(x)\,dx\right| \leq \tfrac{1}{4}\,\|g_1(f_1)\|_p\,\|g_1(f_2)\|_q \leq \tfrac{1}{4}A_q\,\|g_1(f_1)\|_p \tag{11}$$

Now take the supremum in (11) as $f_2$ ranges over all function in $L^2 \cap L^q$, with $\|f_2\|_q \leq 1$. We obtain therefore the desired result (10), with $A'_p = 4/A_q$, but where $f$ is restricted to be in $L^2 \cap L^p$. The passage to the general case is provided by an easy limiting argument. Let $f_m$ be a sequence of functions in $L^2 \cap L^p$, which converge in $L^p$ norm to $f$, (a general element of $L^p$). Notice that $|g(f_m)(x) - g(f_n)(x)| \leq g(f_n - f_m)(x)$. So $g(f_n)$ converges in $L^p$ norm to $g(f)$, and we obtain the inequality (10) for $f$ as a result of the corresponding inequalities for $f_n$. We have incidentally also proved the following, which we state as a corollary.

COROLLARY. *Suppose* $f \in L^2(\mathbf{R}^n)$, *and* $g_1(f)(x) \in L^p(\mathbf{R}^n)$, $1 < p < \infty$. *Then* $f \in L^p(\mathbf{R}^n)$, *and* $A'_p\,\|f\|_p \leq \|g_1(f)\|_p$.

**1.5** There are some very simple variants of the above that should be pointed out:

(i) The results hold also with $g_x(f)$ instead of $g(f)$. The direct inequality $\|g_x(f)\|_p \leq A_p\,\|f\|_p$ is of course a consequence of the one for $g$. The converse inequality is then proved in the same way as that for $g_1$.

(ii) For any integer $k$, $k > 1$, define

$$g_k(f)(x) = \left(\int_0^\infty \left|\frac{\partial^k u}{\partial y^k}(x, y)\right|^2 y^{2k-1}\,dy\right)^{1/2}.$$

Then the $L^p$ inequalities hold for $g_k$ as well. Both (i) and (ii) are stated more systematically in §7.2, below.

(iii) For later purposes it will be useful to note that for each $x$, $g_k(f)(x) \geq A_k g_1(f)(x)$ where the bound $A_k$ depends only on $k$.

It is easily verified from the Poisson integral formula that if $f \in L^p(\mathbf{R}^n)$, $1 \leq p \leq \infty$, then

$$\frac{\partial^k u(x, y)}{\partial y^k} \to 0 \text{ for each } x, \quad \text{as} \quad y \to \infty.$$

Thus

$$\frac{\partial^k u(x, y)}{\partial y^k} = -\int_y^\infty \frac{\partial^{k-1} u(x, s)}{\partial s^{k-1}}\, s^k \frac{ds}{s^k}.$$

By Schwarz's inequality, therefore,

$$\left|\frac{\partial^k u(x, y)}{\partial y^k}\right|^2 \leq \int_y^\infty \left|\frac{\partial^{k-1} u(x, s)}{\partial s^{k-1}}\right|^2 s^{2k}\,ds \left(\int_1^\infty s^{-2k}\,ds\right).$$

Hence $(g_k(f, x))^2 \leq \dfrac{1}{2k-1}(g_{k-1}(f, x))^2$, and the assertion is proved by induction on $k$.

## 2. *The function* $g_\lambda^*$

**2.1** The proof that was given for the $L^p$ inequalities for the $g$-function did not in any essential way depend on the theory of harmonic functions, despite the fact that this function was defined in terms of the Poisson integral. In effect, all that was really used is the fact that the Poisson kernels are suitable approximations to the identity. (See the remarks at the end of the previous section, as well as §7.2.)

There is however another approach, which can be carried out without recourse to the theory of singular integrals, but which leans heavily on characteristic properties of harmonic functions. We present it here (more precisely, we present that part which deals with $1 < p \leq 2$, for the inequality (9)), because its ideas can be adapted to other situations where the methods of Chapter II are not applicable. Everything will be based on the following three observations.

LEMMA 1. *Suppose $u$ is harmonic and strictly positive. Then*

(12) $$\Delta(u)^p = p(p-1)u^{p-2}\,|\nabla u|^2.$$

LEMMA 2. *Suppose $F(x, y)$ is continuous in $\bar{\mathbf{R}}^{n+1}_+$, is of class $C^2$ in $\mathbf{R}^{n+1}_+$, and suitably small at infinity. Then*

$$\int_{\mathbf{R}^{n+1}_+} y\,\Delta F(x, y)\,dx\,dy = \int_{\mathbf{R}^n} F(x, 0)\,dx \tag{13}$$

LEMMA 3. *If $u(x, y)$ is the Poisson integral of $f$; then*

$$\sup_{y>0} |u(x, y)| \leq (Mf)(x). \tag{14}$$

The proof of Lemma 1 is a straightforward exercise in differentiation. Its main interest is that the right-hand side of (12) does not involve any of the second derivatives of $u$.

To prove Lemma 2, we use Green's theorem

$$\int_D (u\,\Delta v - v\,\Delta u)\,dx\,dy = \int_{\partial D}\left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right)d\sigma$$

where $D = B_r \cap \mathbf{R}^{n+1}_+$, with $B_r$ the ball of radius $r$ in $\mathbf{R}^{n+1}$ centered at the origin. We take $v = F$, and $u = y$. Then we will obtain our result (13) if

$$\int_D y\,\Delta F(x, y)\,dx\,dy \to \int_{\mathbf{R}^{n+1}_+} y\,\Delta F(x, y)\,dx\,dy$$

and

$$\int_{\partial D_0}\left(y\frac{\partial F}{\partial \nu} - \frac{\partial y}{\partial \nu}F\right)d\sigma \to 0, \quad \text{as} \quad r \to 0.$$

Here $\partial D_0$ is the spherical part of the boundary of $D$. This will certainly be the case, if for example $\Delta F \geq 0$, and $|F| \leq O((|x| + y)^{-n-\varepsilon})$ and $|\nabla F| = O((|x| + y)^{-n-1-\varepsilon})$, as $|x| + y \to \infty$, for some $\varepsilon > 0$.

The third lemma is of course a majorization with which the reader is by now familiar. (See part (a) of the theorem in §2.1, Chapter III).

Once these facts have been set down, the proof of the inequality $\|g(f)\|_p \leq A_p \|f\|_p$, $1 < p \leq 2$, can be accomplished in a few strokes.

Suppose first that $f \geq 0$, is indefinitely differentiable and has compact support. An examination of the Poisson kernel shows that the Poisson integral $u$ of $f$ is strictly positive in $\mathbf{R}^{n+1}_+$, and the majorizations $u(x, y) = O(|x| + y)^{-n}$ and $|\nabla u| = O(|x| + y)^{-n-1}$, as $|x| + y \to \infty$ are valid. We have

$$g(f, x)^2 = \int_0^\infty y\,|\nabla u(x, y)|^2\,dy = \frac{1}{p(p-1)}\int_0^\infty y u^{2-p}\,\Delta(u)^p\,dy$$
$$\leq \frac{1}{p(p-1)}\,[(Mf)(x)]^{2-p}\int_0^\infty y\,\Delta(u)^p\,dy$$

first using Lemma 1, then Lemma 3, and the hypothesis $1 < p \leq 2$. We can write this as

$$g(f, x) \leq C_p(Mf(x))^{(2-p)/2}(I(x))^{1/2}, \tag{15}$$

where

$$I(x) = \int_0^\infty y\, \Delta u^p\, dy.$$

However,

$$\int_{\mathbf{R}^n} I(x)\, dx = \int_{\mathbf{R}_+^{n+1}} y\, \Delta u^p\, dx\, dy = \int_{\mathbf{R}^n} u^p(x, 0)\, dx = \|f\|_p^p, \tag{16}$$

by Lemma 2. This immediately gives the desired result for $p = 2$. Suppose now $1 < p < 2$. By (15)

$$\int_{\mathbf{R}^n} (g(f, x))^p\, dx \leq C_p^p \int_{\mathbf{R}^n} (Mf)(x)^{p(2-p)/2}(I(x))^{p/2}\, dx$$
$$\leq C_p^p \left( \int_{\mathbf{R}^n} (Mf(x))^p\, dx \right)^{1/r'} \left( \int_{\mathbf{R}^n} I(x)\, dx \right)^{1/r},$$

where we have used Hölder's inequality with exponents $r$ and $r'$

$$1/r + 1/r' = 1, \qquad (1 < r < 2),$$

which is made possible by the fact that $\left(\dfrac{2-p}{2}\right) pr' = p$, and $rp/2 = 1$, if $r = 2/p$.

By (16) the last factor of the equation is $\|f\|_p^{p/r}$; the next to the last factor is majorized by $C'_p\, \|f\|_p^{p/r'}$, according to the maximal theorem of Chapter I. Inserting these two estimates gives $\|g(f)\|_p \leq A_p\, \|f\|_p$, $1 < p \leq 2$, whenever $f$ is a positive function which is indefinitely differentiable and of compact support. For general $f \in L^p(\mathbf{R}^n)$ (which we assume for simplicity to be real-valued), write $f = f^+ - f^-$ as its decomposition into positive and negative part; then we need only approximate in norm $f^+$ and $f^-$, each by a sequences of positive indefinitely differentiable functions with compact support. We omit the routine details that are needed to complete the proof.

**2.2** It is unfortunate that the elegant argument just given is not valid for $p > 2$. There is, however, a more intricate variant of the same idea which does work for the case $p > 2$, but we do not intend to reproduce it here.*

We shall, however, use the ideas above to obtain a significant generalization of the inequality for the $g$-functions. We have in mind the inequalities for the positive function $g_\lambda^*$ defined as follows,

$$(g_\lambda^*(f)(x))^2 = \int_0^\infty \int_{t \in \mathbf{R}^n} \left( \frac{y}{|t| + y} \right)^{\lambda n} |\nabla u(x - t, y)|^2\, y^{1-n}\, dt\, dy. \tag{17}$$

* See the literature cited at the end of this chapter, and also the argument in §3.3.2, Chapter VII.

**2.3** Before going any further, we shall make a few comments that will help to clarify the meaning of the complicated expression (17).

First, $g_\lambda^*(f)(x)$ will turn out to be a pointwise majorant of $g(f)(x)$. To understand this situation better we have to introduce still another quantity, which is roughly midway between $g$ and $g_\lambda^*$. It is defined as follows. Let $\Gamma$ be fixed proper cone in $\mathbf{R}_+^{n+1}$ with vertex at the origin and which contains (0, 1) in its interior. The exact form of $\Gamma$ will not really matter, but for the sake of definiteness let us choose for $\Gamma$ the right-circular cone:

$$\Gamma = \{(t, y) \in \mathbf{R}_+^{n+1} : |t| < y, y > 0\}$$

For any $x \in \mathbf{R}^n$, let $\Gamma(x)$ be the cone $\Gamma$ translated so that its vertex is at $x$. Now define the positive function $S(f)(x)$ by

$$(18) \qquad [S(f)(x)]^2 = \int_{\Gamma(x)} |\nabla u(t,y)|^2 y^{1-n}\, dy\, dt = \int_\Gamma |\nabla u(x - t, y)|^2 y^{2-n}\, dy\, dt$$

We assert, as we shall momentarily prove, that

$$(19) \qquad g(f)(x) \leq CS(f)(x) \leq C_\lambda g_\lambda^*(f)(x).$$

What interpretation can we put on the inequalities relating these three quantities? A hint is afforded by considering three corresponding approaches to the boundary for harmonic functions.

(a) With $u(x, y)$ the Poisson integral of $f(x)$, the simplest approach to the boundary point $x \in \mathbf{R}^n$ is obtained by letting $y \to 0$, (with $x$ fixed). This is the perpendicular approach, and for it the appropriate limit exists almost everywhere, as we already know.

(b) Wider scope is obtained by allowing the variable point $(t, y)$ to approach $(x, 0)$ through any cone $\Gamma(x)$, (where vertex is $x$). This is the *non-tangential* approach which will be so important for us later (in Chapters VII and VIII). As the reader may have already realized, the relation of the $S$-function to the $g$-function is in some sense analogous to the relation between the non-tangential and the perpendicular approaches; we should add that the $S$-function is of decisive significance in its own right, but we shall not pursue that matter now.*

(c) Finally the widest scope is obtained by allowing the variable point $(t, y)$ to approach $(x, 0)$ in an arbitrary manner, i.e. the unrestricted approach. The function $g_\lambda^*$ has the analogous role: it takes into account the unrestricted approach for Poisson integrals.

Notice that $g_\lambda^*(x)$ depends on $\lambda$. For each $x$, the smaller $\lambda$ the greater $g_\lambda^*(x)$, and this behavior is such that the $L^p$ boundedness of $g_\lambda^*$ depends

* See Chapter VII.

critically on the correct relation between $p$ and $\lambda$. This last point is probably the main interest in $g_\lambda^*$, and is what makes its study more difficult than $g$ (or $S$).

After these various heuristic and imprecise indications let us return to firm ground. The only thing for us to prove here is the assertion (19). The inequality $CS(f)(x) \leq C_\lambda g_\lambda^*(f)(x)$ is obvious, since the integral (17) majorizes that part of the integral taken only over $\Gamma$, and

$$\left(\frac{y}{|t| + y}\right)^{\lambda n} \geq (1/2)^{\lambda n}$$

there. The non-trivial part of the assertion is:

$$g(f)(x) \leq CS(f)(x).$$

It suffices to prove this inequality for $x = 0$. Let us denote by $B_y$ the ball in $\mathbf{R}_+^{n+1}$ centered at $(0, y)$ and tangent to the boundary of the cone $\Gamma$; the radius of $B_y$ is then proportional to $y$. Now the partial derivatives $\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial u}{\partial x_k}$ are, like $u$, harmonic functions. Thus by the mean-value theorem

$$\frac{\partial u}{\partial y}(0, y) = \frac{1}{m(B_y)} \int_{B_y} \frac{\partial u}{\partial y}(x, s)\, dx\, ds$$

(where $m(B_y)$ is the $n + 1$ dimensional measure of $B_y$, i.e. $m(B_y) = cy^{n+1}$ for an appropriate constant $c$). By Schwarz's inequality

$$\left|\frac{\partial u(0, y)}{\partial y}\right|^2 \leq \frac{1}{m(B_y)} \int_{B_y} \left|\frac{\partial u}{\partial y}(x, s)\right|^2 dx\, ds.$$

If we integrate this inequality we obtain

$$\int_0^\infty y \left|\frac{\partial u(0, y)}{\partial y}\right|^2 dy \leq \int_0^\infty c^{-1} y^{-n} \left(\int_{B_y} \left|\frac{\partial u}{\partial y}(x, s)\right|^2 dx\, ds\right) dy$$

However $(x, s) \in B_y$ clearly implies that $c_1 s \leq y \leq c_2 s$, for two positive constants $c_1$ and $c_2$. Thus, apart from a multiplicative factor, the last integral is majorized by

$$\int_\Gamma \left(\int_{c_1 s}^{c_2 s} y^{-n}\, dy\right) \left|\frac{\partial u}{\partial y}(x, s)\right|^2 dx\, ds$$

This is another way of saying that,

$$\int_0^\infty y \left|\frac{\partial u}{\partial y}(0, y)\right|^2 dy \leq c' \int_\Gamma \left|\frac{\partial u(x, y)}{\partial y}\right|^2 y^{1-n}\, dx\, dy.$$

The same is true for the derivatives $\dfrac{\partial u}{\partial x_j}$, $j = 1, \ldots n$, and adding the corresponding estimates proves our assertion.

**2.4** We are now in a position to state the result concerning $g_\lambda^*$.

THEOREM 2. *Let $\lambda$ be a parameter which is greater than* 1. *Suppose* $f \in L^p(\mathbf{R}^n)$. Then
(a) *For every* $x \in \mathbf{R}^n$, $g(f)(x) \leq C_\lambda g_\lambda^*(f)(x)$.
(b) *If* $1 < p < \infty$, *and* $p > 2/\lambda$, *then*

$$\|g_\lambda^*(f)\|_p \leq A_{p,\lambda} \|f\|_p. \tag{20}$$

Part (a) of the theorem has already been proved.

The inequalities for $p \geq 2$ will turn out to be rather easy consequences of the corresponding inequalities of the $g$-function. This we shall now see. For the case $p \geq 2$, only the assumption $\lambda > 1$ is relevant.

Let $\psi$ denote a positive function on $\mathbf{R}^n$; we claim that

$$\int_{\mathbf{R}^n} (g_\lambda^*(f)(x))^2 \psi(x)\, dx \leq A_\lambda \int_{\mathbf{R}^n} (g(f)(x))^2 (M\psi)(x)\, dx. \tag{21}$$

The left-side of (21) equals

$$\int_{y=0}^{\infty} \int_{t\in\mathbf{R}^n} y\, |\nabla u(t, y)|^2 \left[ \int_{x\in\mathbf{R}^n} \psi(x)[|t - x| + y]^{-\lambda n} y^{\lambda n}\, dx \right] dt\, dy$$

so to prove (21) we must show that

$$\sup_{y>0} \int_{\mathbf{R}^n} \psi(x)[|t - x| + y]^{-\lambda n} y^{\lambda n} y^{-n}\, dx \leq A_\lambda M(\psi)(t). \tag{22}$$

However we know by Theorem 2, §2.2, of Chapter III, that

$$\sup_{\varepsilon>0} (\psi * \varphi_\varepsilon)(t) \leq A M(\psi)(t)$$

for appropriate $\varphi$, with $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. Here we have in fact $\varphi(x) = (1 + |x|)^{-\lambda n}$, $\varepsilon = y$, and so with $\lambda > 1$ the hypotheses of that theorem are satisfied. This proves (22) and thus also (21).

The case $p = 2$ follows immediately from (21) by inserting in this inequality the function $\psi = 1$, and using the $L^2$ result for $g$. Suppose now $2 < p$; let us set $1/q + 2/p = 1$, and take the supremum of the left side over all $\psi \geq 0$, such that $\psi \in L^q(\mathbf{R}^n)$ and $\|\psi\|_q \leq 1$. The left side of (21) then gives $\|g_\lambda^*(f)\|_p^2$; Hölder's inequality yields as an estimate for the right side:

$$A_\lambda \|g(f)\|_p^2 \|M\psi\|_q.$$

However by the inequalities for the $g$-function, $\|g(f)\|_p \leq A_p' \|f\|_p$; and by the theorem of the maximal function $\|M\psi\|_q \leq A_q'' \|\psi\|_q = A_q''$, since $q > 1$, if $p < \infty$. If we substitute these in the above we get the result:

$$\|g_\lambda^*(f)\|_p \leq A_{p,\lambda} \|f\|_p, \qquad 2 \leq p < \infty, \qquad \lambda > 1.$$

**2.5** The inequalities for $p < 2$ will be proved by an adaptation of the reasoning used in §2.1 for $g$. Lemmas 1 and 2 will be equally applicable in the present situation, but we need a more general version of Lemma 3, in order to majorize the unrestricted approach to the boundary of a Poisson integral.

It is at this stage where results which depend critically on the $L^p$ class first make their appearance. Matters will depend on a variant of the maximal function which we define as follows. Let $\mu \geq 1$, and write $M_\mu(f)(x)$ for

$$(23) \qquad M_\mu(f)(x) = \left(\sup_{r>0} \frac{1}{m(B(x, r))} \int_{B(x,r)} |f(y)|^\mu \, dy\right)^{1/\mu}$$

Then $M_1(f)(x) = M(f)(x)$, and $M_\mu(f)(x) = (M\,|f|^\mu)(x))^{1/\mu}$. From the theorem of the maximal function it then immediately follows that

$$(23') \qquad \|M_\mu(f)\|_p \leq A_{p,\mu} \|f\|_p, \quad \text{for} \quad p > \mu.$$

This inequality fails for $p \leq \mu$, as in the special case $\mu = 1$.

*2.5.1* The substitute for Lemma 3 is as follows

LEMMA 4. *Let $f \in L^p(\mathbf{R}^n)$, $p \geq \mu$, $\mu \geq 1$; if $u(x, y)$ is the Poisson integral of $f$, then*

$$(24) \qquad |u(x - t, y)| \leq A\left(1 + \frac{|t|}{y}\right)^n M(f)(x),$$

*and more generally*

$$(24') \qquad |u(x - t, y)| \leq A_\mu\left(1 + \frac{|t|}{y}\right)^{n/\mu} M_\mu(f)(x).$$

We begin by deducing (24).

One notices that (24) is unchanged by the dilatation $(x, t, y) \to (x\delta, t\delta, y\delta)$; it is then clear that it suffices to prove (24) with $y = 1$.

Setting $y = 1$ in the Poisson kernel, we have $P_1(x) = \dfrac{c_n}{(1 + |x|^2)^{\frac{n+1}{2}}}$, and $u(x - t, 1) = f(x) * P_1(x - t)$, for each $t$. Theorem 2 of Chapter III (in §2.2) shows that $|u(x - t, 1)| \leq A_t(Mf)(x)$, where $A_t = \int Q_t(x)\, dx$, and

$Q_t(x)$ is the smallest decreasing radial majorant of $P_1(x - t)$, i.e.

$$Q_t(x) = c_n \cdot \sup_{|x'| \geq |x|} \left\{ \frac{1}{(1 + |x' - t|^2)^{(n+1)/2}} \right\}$$

For $Q_t(x)$ we have the easy estimates, $Q_t(x) \leq c_n$ for $|x| \leq 2\,|t|$ and $Q_t(x) \leq A'(1 + |x|^2)^{\frac{-n-1}{2}}$, for $|x| \geq 2\,|t|$, from which it is obvious that $A_t \leq A(1 + |t|)^n$ and hence (24) is proved.

Since $u(x - t, y) = \int_{s \in \mathbf{R}^n} P_y(s) f(x - t - s)\, ds$, and $\int_{\mathbf{R}^n} P_y(s)\, ds = 1$, we have $|u(x - t, y)|^\mu \leq \int_{s \in \mathbf{R}^n} P_y(s)\, |f(x - t - s)|^\mu\, ds = U(x - t, y)$, where $U$ is the Poisson integral of $|f|^\mu$. Apply (24) to $U$; this gives

$$\begin{aligned} |u(x - t, y)| &\leq A^{1/\mu}(1 + |t|/y)^{n/\mu}(M(|f|^\mu)(x))^{1/\mu} \\ &= A_\mu(1 + |t|/y)^{n/\mu} M_\mu(f)(x), \end{aligned}$$

and the Lemma is established.

*2.5.2* We shall now complete the proof of the inequality (20) for the case $1 < p < 2$, with the restriction $p > 2/\lambda$.

Let us observe that we can always find a $\mu$, $1 \leq \mu < p$, so that if we set $\lambda' = \lambda - \dfrac{2 - p}{\mu}$, then one still has $\lambda' > 1$. In fact if $\mu = p$, then $\dfrac{\lambda - 2}{\mu} - p > 1$ since $\lambda > 2/p$; this inequality can then be maintained by a small variation of $\mu$. With this choice of $\mu$ we have by Lemma 4

$$|u(x - t, y)| \left( \frac{y}{y + |t|} \right)^{n/\mu} \leq A M_\mu(f)(x) \tag{25}$$

We now proceed as in §2.1, where we treated the function $g$.

$$\begin{aligned} (g_\lambda^*(f)(x))^2 &= \frac{1}{p(p - 1)} \int_{\mathbf{R}_+^{n+1}} y^{1-n} \left( \frac{y}{y + |t|} \right)^{\lambda n} u^{2-p} |\Delta u^p|\, dt\, dy \\ &\leq A^{2-p}(M_\mu(f)(x))^{2-p} I^*(x), \end{aligned} \tag{26}$$

where

$$I^*(x) = \int_{\mathbf{R}_+^{n+1}} y^{1-n} \left( \frac{y}{y + |t|} \right)^{\lambda' n} \Delta u^p(x - t, y)\, dt\, dy$$

It is clear that

$$\begin{aligned} \int_{\mathbf{R}^n} I^*(x)\, dx &= \int_{\mathbf{R}_+^{n+1}} \int_{x \in \mathbf{R}^n} y^{1-n} \left( \frac{y}{y + |t - x|} \right)^{\lambda' n} \Delta u^p(t, y)\, dx\, dt\, dy \\ &= C_{\lambda'} \int_{\mathbf{R}_+^{n+1}} y\, \Delta u^p(t, y)\, dt\, dy. \end{aligned}$$

The last step follows from the fact that

$$y^{-n}\int_{\mathbf{R}^n}\left(\frac{y}{y+|x|}\right)^{\lambda' n} dx = \int_{\mathbf{R}^n}\frac{dx}{(1+|x|)^{\lambda' n}} = C_{\lambda'} < \infty, \quad \text{if} \quad \lambda' > 1.$$

So, by Lemma 2

$$\int_{\mathbf{R}^n} I^*(x)\,dx = C_{\lambda'}\,\|f\|_p^p. \tag{27}$$

Thus (26) takes the role of (15) and (27) that of (16).

The proof is then concluded as in §2.1 if we make use of the $L^p$ bounds for $M_\mu(f)$ in (23′), instead of those for $M(f)$.

## 3. *Multipliers* (*first version*)

**3.1** The first application of the theory of the functions $g$ and $g_\lambda^*$ will be in the study of multipliers. The theorem presented below (Theorem 3) will be a "preliminary" version of the multiplier theorem. A "final" form will be presented in §6, where a comparison of the two variants will also be made.

Let $m$ be a bounded measurable function on $\mathbf{R}^n$. One can then define a linear transformation $T_m$, whose domain is $L^2(\mathbf{R}^n) \cap L^p(\mathbf{R}^n)$, by the following relation between Fourier transforms

$$(T_m f)\hat{}\,(x) = m(x)\hat{f}(x), \qquad f \in L^2 \cap L^p. \tag{28}$$

We shall say that $m$ is *a multiplier for* $L^p$ $(1 \le p \le \infty)$ if whenever $f \in L^2 \cap L^p$ then $T_m f$ is also in $L^p$, (notice it is automatically in $L^2$), and $T_m$ is bounded, that is,

$$\|T_m(f)\|_p \le A\,\|f\|_p, \qquad f \in L^2 \cap L^p, \quad (\text{with } A \text{ independent } f). \tag{29}$$

The smallest $A$ for which (29) holds will be called the norm of the multiplier. Observe that if (29) is satisfied, and $p < \infty$, then $T_m$ has a unique bounded extension to $L^p$, which again satisfies the same inequality. We shall also write $T_m$ for this extension.

We denote by $\mathscr{M}_p$ the class of multipliers with the indicated norm. It is clearly a Banach algebra under pointwise multiplication.

We begin with some examples. The observation that the operators $T_m$ commute with translations, together with the propositions in §1.2 and §1.4 of Chapter II, lead directly to the following.

*Example* (i). $\mathscr{M}_2$ is the class of all bounded measurable functions and the multiplier norm is identical with the $L^\infty(\mathbf{R}^n)$ norm.

*Example* (ii). $\mathscr{M}_1$ is the class of Fourier transforms of elements of $\mathscr{B}(\mathbf{R}^n)$, (the finite Borel measures), and the norm of $\mathscr{M}_1$ is identical with the norm of $\mathscr{B}(\mathbf{R}^n)$.

The theory of singular integrals of Chapters II and III allows us to assert the following:

*Example* (iii). Suppose $m$ is homogeneous of degree 0. If either $m$ is indefinitely differentiable on the sphere, or more generally, if $m$ is representable in the form of equation (26) of Chapter II, (up to an additive constant), then $m \in \mathscr{M}_p$, $1 < p < \infty$.

We now return to some general considerations.

A basic duality property (which we have already used in somewhat different terms in §2.5, Chapter II) and which reflects the duality of $L^p$ spaces, is contained in the proposition that follows.

PROPOSITION. *Suppose* $1/p + 1/p'$, $1 \le p \le \infty$, *then* $\mathscr{M}_p = \mathscr{M}_{p'}$ *with an identity of norms.*

*Proof.* Let $\sigma$ denote the involution $\sigma(f)(x) = \bar{f}(-x)$. As is immediately verified $\sigma^{-1} T_m \sigma = T_{\bar{m}}$; therefore if $m$ belongs to $\mathscr{M}$, so does $\bar{m}$; moreover $\bar{m}$ has the same norm as $m$. Now by Plancherel's formula,

$$\int_{\mathbf{R}^n} T_m f \bar{g}\, dx = \int_{\mathbf{R}^n} m(x)\hat{f}(x)\overline{\hat{g}(x)}\, dx = \int_{\mathbf{R}^n} \hat{f}(x)\overline{\bar{m}\hat{g}(x)}\, dx$$

$$= \int_{\mathbf{R}^n} f\overline{T_{\bar{m}} g}\, dx, \quad \text{whenever } f,\ g \in L^2(\mathbf{R}^n)$$

Assume in addition that $f \in L^{p'}(\mathbf{R}^n)$, $g \in L^p(\mathbf{R}^n)$, and $\|g\|_p \le 1$. Then

$$\left| \int_{\mathbf{R}^n} T_m f \bar{g}\, dx \right| \le \|f\|_{p'} \|T_{\bar{m}} g\|_p \le A\, \|f\|_{p'},$$

where $A$ is the norm of the multiplier $m$ (or $\bar{m}$) in $\mathscr{M}$. Taking the supremum over all indicated $g$, gives

$$\|T_m f\|_{p'} \le A\, \|f\|_{p'}$$

Therefore $m$ belongs to $\mathscr{M}_{p'}$, and its $\mathscr{M}_{p'}$ norm is no larger than its $\mathscr{M}_p$ norm; since the situation is symmetric in $p$ and $p'$, the two norms are identical.

We have already pointed out that if $m$ is a multiplier (in $\mathscr{M}_p$), then the transformation $T_m$, which is bounded in $L^p(\mathbf{R}^n)$, commutes with translations. The converse also holds: Suppose that $T$ is a bounded linear transformation on $L^p(\mathbf{R}^n)$, $p < \infty$, which commutes with translations; then there exists an $m \in \mathscr{M}_p$ so that $T_m = T$. The proof of this fact will be outlined in §7.3 below.

After these clarifying comments about multipliers we should warn the reader that the deeper structure of the class of multipliers $\mathscr{M}_p$ (except in the "trivial" cases corresponding to $p = 1, 2$, or $\infty$) is still to a large extent unknown, even in the context of $\mathbf{R}^1$. What we shall obtain below, however, is an important sufficient condition, which incidentally contains to a large extent the results cited in example (iii).

3.2 THEOREM 3. *Suppose that $m(x)$ is of class $C^k$ in the complement of the origin of $\mathbf{R}^n$, where $k$ is an integer $> n/2$. Assume also that for every differential monomial $\left(\frac{\partial}{\partial x}\right)^\alpha$, $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_n)$, with $|\alpha| = \alpha_1 + \alpha_2 \cdots, + \alpha_n$, we have*

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha m(x)\right| \leq B\,|x|^{-|\alpha|}, \quad \textit{whenever} \quad |\alpha| \leq k. \tag{30}$$

*Then $m \in \mathscr{M}_p$, $1 < p < \infty$; that is $\|T_m f\|_p \leq A_p \|f\|_p$.*

The proof will show that the bound $A_p$ will depend only on $B, p$, and $n$.
The proof of the theorem leads to a generalization of its statement. This we formulate as a corollary.

COROLLARY. *The assumption (30) can be replaced by the weaker assumptions,*

$$|m(x)| \leq B'$$

$$\sup_{0<R<\infty} R^{2|\alpha|-n} \int_{R\leq|x|\leq 2R} \left|\left(\frac{\partial}{\partial x}\right)^\alpha m(x)\right|^2 dx \leq B', \qquad |\alpha| \leq k. \tag{31}$$

We mention now two illustrations of the relevance of the theorem:
*Example (1).* $m(x) = |x|^{it}$, where $t$ is a real number. This example has connection with the Riesz potentials of §1, Chapter V. See also §6.12 in Chapter II.
*Example (2).* $m(x)$ is homogeneous of degree 0, and is of class $C^k$ on the unit sphere. (See also §3.5 in Chapter III.)

The theorem (and corollary) will be a consequence of the following lemma. Its statement illuminates at the same time the nature of the multiplier transformations considered here, and the role played by the $g$-functions and their variants.

LEMMA. *Under the assumptions of Theorem 3 (or its corollary), let us set for each $f \in L^2(\mathbf{R}^n)$*

$$F(x) = (T_m f)(x).$$

*Then*

$$g_1(F, x) \leq B_\lambda g_\lambda^*(f, x), \quad \textit{where} \quad \lambda = 2k/n. \tag{32}$$

Thus in view of the lemma, the $g$-functions and their variants are the characterizing expressions which deal at once with all the multipliers considered. On the other hand, the fact that the relation (32) is pointwise shows that to a large extent the mapping $T_m$ is "semi-local."

The theorem is deduced from the lemma as follows. Our assumption on $k$ is such that $\lambda > 1$. Thus Theorem 2 shows us that

$$\|g_\lambda^*(f, x)\|_p \leq A_{\lambda,p} \|f\|_p, \qquad 2 \leq p < \infty \quad \text{if} \quad f \in L^2 \cap L^p.$$

However by Theorem 1, (see the corollary in §1.4), $A'_p \|F\|_p \leq \|g_1(F, x)\|_p$; therefore

$$\|F\|_p = \|T_m f\|_p \leq A_p \|f\|_p, \quad \text{if} \quad 2 \leq p < \infty \quad \text{and} \quad f \in L^2 \cap L^p.$$

That is, $m \in \mathscr{M}_p$, $2 \leq p < \infty$. By duality, the proposition in §3.1, we have also $m \in \mathscr{M}_p$, $1 < p \leq 2$, which gives the assertion of the theorem.

**3.3** We shall now prove the lemma.

Let $u(x, y)$ denote the Poisson integral of $f$, and $U(x, y)$ the Poisson integral of $F$. Then with ^ denoting the Fourier transform with respect to the $x$ variable, we have

$$\hat{u}(x, y) = e^{-2\pi|x|y}\hat{f}(x),$$

and

$$\hat{U}(x, y) = e^{-2\pi|x|y}\hat{F}(x) = e^{-2\pi|x|y}m(x)\hat{f}(x).$$

Define $M(x, y) = \int_{\mathbf{R}^n} e^{-2\pi i x \cdot t} e^{-2\pi|t|y} m(t)\, dt$. Then clearly $\hat{M}(x, y) = e^{-2\pi|x|y}m(x)$, and so

$$\hat{U}(x, y_1 + y_2) = \hat{M}(x, y_1)\hat{u}(x, y_2), \qquad y = y_1 + y_2\,, y_1 > 0.$$

This can be written as

$$U(x, y_1 + y_2) = \int_{\mathbf{R}^n} M(t, y_1)u(x - t, y_2)\, dt.$$

We differentiate this relation $k$ times with respect to $y_1$ and once with respect to $y_2$, and set $y_1 = y_2 = y/2$. This gives us the identity

(33) $$U^{(k+1)}(x, y) = \int_{\mathbf{R}^n} M^{(k)}(t, y/2)u^{(1)}(x - t, y/2)\, dt.$$

(The superscripts denote differentiation with respect to $y$.)

*3.3.1* With the aid of this identity it will not be difficult to prove the lemma. The assumptions (30) (or (31)) on $m$ need be translated in terms of $M(x, y)$. The result is:

(34) $$|M^{(k)}(t, y)| \leq B' y^{-n-k}$$

(34′) $$\int_{\mathbf{R}^n} |t|^{2k} |M^{(k)}(t, y)|^2\, dt \leq B' y^{-n}.$$

In fact, by the definition of $M$, it follows that

$$|M^{(k)}(x, y)| \leq B(2\pi)^k \int_{\mathbf{R}^n} |t|^k e^{-2\pi|t|y}\, dt = B' \int_0^\infty r^k e^{-2\pi r y} r^{n-1}\, dr = B'' y^{-n-k},$$

which is (34).

To prove (34′) let us show more particularly that

$$\int_{\mathbf{R}^n} |t^\alpha M^{(k)}(t, y)|^2\, dt \leq B' y^{-n},$$

whenever $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, so that $\alpha_1 + \alpha_2 \cdots + \alpha_n = k$, with $t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_n^{\alpha_n}$.

By Plancherel's theorem

$$\|t^\alpha M^{(k)}(t, y)\|_2 = \left\|(2\pi)^{2k}\left(\frac{\partial}{\partial x}\right)^\alpha (|x|^k m(x) e^{-2\pi|x|y})\right\|_2$$

But

$$y^{2r} \int_{\mathbf{R}^n} |x|^{2r} e^{-4\pi|x|y}\, dx \leq C y^{-n}, \qquad \text{for} \qquad 0 \leq r,$$

and by the hypothesis (30) and Leibniz's rule $\left|\left(\frac{\partial}{\partial x}\right)^\alpha (|x|^k m(x))\right| \leq B' |x|^{k-|\alpha|}$, with $|\alpha| \leq k$. So using Leibniz's rule again to evaluate

$$\left(\frac{\partial}{\partial x}\right)^\alpha (|x|^k m(x) e^{-2\pi|x|y}),$$

we get

$$\|t^\alpha M^{(k)}(t, y)\|_2^2 \leq B' y^{-n}, \qquad |\alpha| = k$$

which proves the assertion (34′).

*3.3.2* Return to the identity (33), and for each $y$ divide the range of integration into two parts, $|t| \leq y/2$ and $|t| \geq y/2$. In the first range use the estimate (34) on $M^{(k)}$ and in the second range use the estimate (34′). This together with Schwarz's inequality gives immediately

$$|U^{(k+1)}(x, y)|^2 \leq A y^{-n-2k} \int_{|t| \leq y/2} |U^{(1)}(x - t, y/2)|^2\, dt$$
$$+ A y^{-n} \int_{|t| > y/2} \frac{|U^{(1)}(x - t, y/2)|^2\, dt}{|t|^{2k}} = I_1(y) + I_2(y).$$

Now

$$(g_{k+1}(F, x))^2 = \int_0^\infty |U^{(k+1)}(x, y)|^2 y^{2k+1}\, dy = \sum_{j=1}^{2} \int_0^\infty I_j(y) y^{2k+1}\, dy.$$

However

$$\int_0^\infty I_1(y)y^{2k-1}\,dy \le B\int_{|t|<y/2} |U^{(1)}(x-t,y/2)|^2 y^{-n+1}\,dt\,dy$$
$$\le B'\int_\Gamma |\nabla u(x-t,y)|^2 y^{-n+1}\,dt\,dy$$
$$= B'(S(F,x))^2 \le B_\lambda g_\lambda^*(F,x).$$

Similarly

$$\int_0^\infty I_2(y)y^{2k-1}\,dy \le B'\int_{|t|>y} y^{-n+2k+1}|t|^{-2k}|\nabla u(x-t,y)|^2\,dt\,dy$$
$$\le B''g_\lambda^*(F,x), \qquad \text{with} \qquad n\lambda = 2k.$$

This shows that $g_{k+1}(F,x) \le B_\lambda g_\lambda^*(f,x)$. However by §1.5 (see remark (iii)), we know that $g_1(F,x) \le A_k g_{k+1}(F,x)$. Thus the proof of the lemma is concluded, and with it that of Theorem 2.

It is to be noted that the proof of the corollary is the same as that of the theorem, except for one slight change: In the lemma, the estimate $y^{2r}\int_{\mathbf{R}^n}|x|^{2r}e^{-4\pi|x|y}\,dx \le Cy^{-n}$, must be replaced by the estimate

$$y^{2r}\int_{\mathbf{R}^n}|x|^{2r}|m_0(x)|^2 e^{-4\pi|x|y}\,dx \le C'y^{-n},$$

whenever $m_0$ satisfies the inequality

$$\sup_{0<R<\infty} R^{-n}\int_{R<|x|<2R}|m_0(x)|^2\,dx \le 1.$$

## 4. *Application of the partial sums operators*

**4.1** We shall now develop the second main tool in the Littlewood-Paley theory, (the first being the usage of the functions $g$ and $g^*$). It is here already that the $n$-dimensional theory is so much more restricted than the one-dimensional case, but we postpone further discussion of this point until §4.3.

Let $\rho$ denote an arbitrary *rectangle* in $\mathbf{R}^n$. By rectangle we shall mean (in the rest of this chapter) a possibly infinite rectangle with sides parallel to the axes, i.e. the Cartesian product of $n$ intervals. For each rectangle $\rho$ denote by $S_\rho$ the "partial sum operator," that is the multiplier operator with $m = \chi_\rho$ = characteristic function of the rectangle $\rho$. So

$$(35) \qquad S_\rho(f)\hat{} = \chi_\rho\cdot\hat{f}, \qquad f\in L^2(\mathbf{R}^n)\cap L^p(\mathbf{R}^n).$$

For this operator we have the following theorem.

THEOREM 4.
$$\|S_\rho(f)\|_p \leq A_p \|f\|_p, \qquad f \in L^2 \cap L^p$$
*if* $1 < p < \infty$. *The constant* $A_p$ *does not depend on the rectangle* $\rho$.

We shall need however a more extended version of the theorem which arises when we replace complex-valued functions by functions taking their values in a Hilbert space.

Let $\mathscr{H}$ be the sequence Hilbert space, $\mathscr{H} = \{(c_j)_{j=1}^{\infty} : (\sum_j |c_j|^2)^{\frac{1}{2}} = |c| < \infty\}$. Then we can represent a function $f \in L^p(\mathbf{R}^n, \mathscr{H})$, as sequences $f(x) = (f_1(x), \ldots f_n(x), \ldots)$, where each $f_j$ is complex-valued and $|f(x)| = (\sum_{j=1}^{\infty} |f_j(x)|^2)^{\frac{1}{2}}$. Let $\mathfrak{R}$ be a sequence of rectangle, $\mathfrak{R} = \{\rho_j\}_{j=1}^{\infty}$. Then we can define the operator $S_{\mathfrak{R}}$, mapping $L^2(\mathbf{R}^n, \mathscr{H})$ to itself, by the rule

(36) $S_{\mathfrak{R}}(f) = (S_{\rho_1}(f_1), \ldots, S_{\rho_j}(f_j), \ldots)$, where $f = (f_1, f_2, \ldots, f_j, \ldots)$.

The generalization of Theorem 4 is then as follows:

THEOREM 4′. *Let* $f \in L^2(\mathbf{R}^n, \mathscr{H}) \cap L^p(\mathbf{R}^n, \mathscr{H})$. *Then*

(37)
$$\|S_{\mathfrak{R}}(f)\|_p \leq A_p \|f\|_p, \qquad 1 < p < \infty$$

*where* $A_p$ *does not depend on the family* $\mathfrak{R}$ *of rectangles.*

**4.2** The theorem will be proved in a series of steps, the first two of which already contain the essence of the matter.

*4.2.1 First stage:* Here $n = 1$, and the rectangles $\rho_1, \rho_2, \ldots \rho_j, \ldots$ are the semi-infinite intervals $(-\infty, 0)$.

We recall the Hilbert transform $f \to H(f)$, which corresponds to the multiplier $i$ sign $x$ (see Chapter III). Then clearly,

(38)
$$S_{(-\infty,0)} = \frac{I + iH}{2}$$

where $I$ is the identity, and $S_{(-\infty,0)}$ is the partial sum operator corresponding to the interval $(-\infty, 0)$. Everything will depend on the following lemma.

LEMMA. *Let* $f(x) = (f_1(x), \ldots f_j(x), \ldots) \in L^2(\mathbf{R}^n, \mathscr{H}) \cap L^p(\mathbf{R}^n, \mathscr{H})$. *Set* $\tilde{H}f(x) = (Hf_1(x), \ldots Hf_j(x), \ldots)$. *Then*

(39)
$$\|\tilde{H}f\|_p \leq A_p \|f\|_p \qquad 1 < p < \infty$$

*where* $A_p$ *is the same constant as in the scalar case, i.e. when* $\mathscr{H}$ *is one-dimensional.*

We use the vector-valued version of the Hilbert transform, as is described more generally in §5 of Chapter II. Let the Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ be both identical with $\mathscr{H}$. Take in $\mathbf{R}^1$, $K(x) = I \cdot 1/\pi x$, where $I$ is the identity mapping on $\mathscr{H}$. Then the kernel $K(x)$ satisfies all the assumptions of Theorem 5 and Theorem 3 of Chapter II. Moreover

$$\lim_{\varepsilon \to 0} \int_{|y| > \varepsilon} K(y) f(x - y)\, dy = \tilde{H}(f)(x),$$

and so our lemma is proved. (Another proof is indicated in §7.12 below.)

Now if all the rectangles are the intervals $(-\infty, 0)$, then because of (38),

$$S_{\mathfrak{R}} = \frac{I + i\tilde{H}}{2}$$

and so because of the lemma the theorem is proved in this case.

*4.2.2 Second stage:* Here $n = 1$, and the rectangles are the intervals $(-\infty, a_1), (-\infty, a_2), \ldots (-\infty, a_j), \ldots$.

Notice that $(f(x)e^{-2\pi i x\cdot a})^\wedge = \hat{f}(x + a)$, therefore $H(e^{-2\pi i x\cdot a}f)^\wedge = i \operatorname{sign} x \hat{f}(x + a)$, and hence $[e^{-2\pi i x\cdot y}H(e^{-2\pi i x\cdot a}f)]^\wedge = i \operatorname{sign}(x - a)\hat{f}(x)$. From this we see that

$$(40) \qquad (S_{(-\infty, a_j)} f_j)(x) = \frac{f_j + ie^{2\pi i x\cdot a_j} H(e^{-2\pi i x\cdot a_j} f_j)}{2}$$

If we now write symbolically $e^{-2\pi i x\cdot a}f$ for $(e^{-2\pi i x\cdot a}f_1, \ldots e^{-2\pi i x\cdot a_j}f_j, \ldots)$, where $f = (f_1, \ldots f_j, \ldots)$, then (40) may be rewritten as

$$(41) \qquad S_{\mathfrak{R}} f = \frac{f + ie^{2\pi i x\cdot a}\tilde{H}(e^{-2\pi i x\cdot a} f)}{2}$$

and so the result again follows in this case by the lemma.

*4.2.3 Third stage:* General $n$, but the rectangles $\rho_j$ are the half-spaces $x_1 < a_j$; i.e. $\rho_j = \{x : x_1 < a_j\}$.

Let $S^{(1)}_{(-\infty, a_j)}$ denote the operator defined on $L^2(\mathbf{R}^n)$, which acts only on the $x_1$ variable, by the action given by $S_{(-\infty, a_j)}$. We claim that

$$(42) \qquad S_{\rho_j} = S^{(1)}_{(-\infty, a_j)}$$

This identity is obvious for $L^2$ functions of the product form

$$f'(x_1) f''(x_2, \ldots, x_n);$$

since their linear span is dense in $L^2$ the identity (42) is established.

We now use the $L^p$ inequality, which is the result of the previous stage, for each fixed $x_2, x_3, \ldots, x_n$. We raise this inequality to the $p^{\text{th}}$ power and integrate with respect to $x_2, \ldots x_n$. This gives the desired result for the present case. Notice that the result holds as well if the half-space $\{x : x_1 < a_j\}_{j=1}^{\infty}$, is replaced by the half-space $\{x : x_1 > a_j\}_{j=1}^{\infty}$; or if the role of the $x_1$ axis is taken by the $x_2$ axis, etc.

4.2.4 *Final stage:* Observe that every general finite rectangle of the type considered is the intersection of $2n$ half-spaces, each half-space having its boundary hyperplane perpendicular to one of the axes of $\mathbf{R}^n$. Thus a $2n$-fold application of the result of the third stage proves the theorem, where the family $\mathfrak{R}$ is made up of finite rectangles. Since the bounds obtained do not depend on the family $\mathfrak{R}$, we can pass to the general case where $\mathfrak{R}$ contains possibly infinite rectangles by an obvious limiting argument.

**4.3 Some problems.** We wish to make some remarks about the limitations of Theorems 4 and 4′. When $n = 1$, the theorems deal with the partial sum operators taken with respect to intervals. Not much more can be wished for in the one-dimensional case since intervals are the only "regular" sets in $\mathbf{R}^1$: they are the only convex sets, the only connected sets, etc.

However when $n > 1$, the situation changes radically. The rectangles with sides parallel to the axes are now only *very special* sets, and the fact that we consider only those is a serious limitation of the generality of the theorems. Thus what we have proved is only an $n$-fold superposition of the one-dimensional results, and is not genuinely an $n$-dimensional result.

To clarify the situation, we wish to describe two particular test problems for an essentially $n$-dimensional theory. These problems are interesting in themselves, but the solution of each would surely have many further consequences.

PROBLEM A. *Let $B$ be the unit ball in $\mathbf{R}^n$. Can we replace the rectangle $\rho$ by the ball $B$ in Theorem 4?*

It is known that the answer can be affirmative only in the range $\dfrac{2n}{n+1} < p < \dfrac{2n}{n-1}$, but there is no positive result, except when $p = 2$. See §7.7 and §7.8 below.

PROBLEM B. *Can the rectangles of Theorem 4′ be replaced by rectangles that are each arbitrarily rotated?*

It can be shown that the positive solution of Problem A would imply the resolution of Problem B for the same $p$. It can also be shown that the answer to Problem B is in the negative for $p$ outside the interval $\frac{2n}{n+1} \leq p \leq \frac{2n}{n-1}$.*

**4.4** We state here the continuous anologue of Theorem 4′. Let $(\Gamma, d\gamma)$ be an abstract measure space, and consider the Hilbert space $\mathscr{H}$ of square integrable functions on $\Gamma$, i.e. $\mathscr{H} = L^2(\Gamma, d\gamma)$. The elements

$$f \in L^p(\mathbf{R}^n, \mathscr{H})$$

are the complex-valued functions $f(x, \gamma) = f_\gamma(x)$ on $\mathbf{R}^n \times \Gamma$, which are jointly measurable, and for which $(\int_{\mathbf{R}^n}(\int_\Gamma |f(x, \gamma)|^2 \, d\gamma)^{p/2} \, dx)^{1/p} = \|f\|_p < \infty$, (if $p < \infty$). In analogy with §4.1 let $\mathfrak{R} = \{\rho_\gamma\}_{\gamma \in \Gamma}$, and suppose that the mapping $\gamma \to \rho_\gamma$ is a measurable function from $\Gamma$ to rectangles; that is, the numerical-valued functions which assign to each $\gamma$ the components of the vertices of $\rho_\gamma$ are all measurable.

Suppose $f \in L^2(\mathbf{R}^n, \mathscr{H})$. Then we define $F = S_\mathfrak{R} f$ by the rule

$$F(x, \gamma) = S_{\rho_\gamma}(f_\gamma)(x), \qquad (f_\gamma(x) = f(x, \gamma)).$$

THEOREM 4″.

$$\|S_\mathfrak{R} f\|_p \leq A_p \|f\|_p, \qquad 1 < p < \infty \tag{42}$$

*for* $f \in L^2(\mathbf{R}^n, \mathscr{H}) \cap L^p(\mathbf{R}^n, \mathscr{H})$, *where the bound* $A_p$ *does not depend on the measure space* $(\Gamma, \gamma)$, *or on the function* $\gamma \to \rho_\gamma$.

The proof of this theorem is an exact repetition of the argument given for Theorem 4′. The reader may, if he wishes, also obtain it from Theorem 4′ by a limiting argument.

## 5. *The dyadic decomposition*

**5.1** We shall now consider a cannonical decomposition of $\mathbf{R}^n$ into rectangles. First, in the case of $\mathbf{R}^1$ we decompose it as the union of the "disjoint" intervals (that is, whose interiors are disjoint) $[2^k, 2^{k+1}]$, $-\infty < k < \infty$, and $[-2^{k+1}, -2^k]$, $-\infty < k < \infty$. This double collection of intervals, one collection for the positive half-line, the other for the negative half-line, will be the dyadic decomposition of $\mathbf{R}^1$. (Strictly speaking, the origin is left out; but for the sake of simplicity of terminology we still refer to it as the decomposition of $\mathbf{R}^1$.) Having obtained this decomposition of $\mathbf{R}^1$, we take the corresponding product decomposition for $\mathbf{R}^n$. Thus we write $\mathbf{R}^n$ as the union of "disjoint" rectangles, which rectangles are products of the intervals which occur for the dyadic decomposition of each of the axes. This is the *dyadic decomposition* of $\mathbf{R}^n$.

* Y. Meyer, personal communication.

The family of resulting rectangles will be denoted by $\Delta$. We recall the partial sum operator $S_\rho$, defined in (35) for each rectangle. Now in an obvious sense, (e.g. $L^2$ convergence)

$$\textstyle\sum_{\rho\in\Delta} S_\rho = \text{Identity}.$$

Also in the $L^2$ case the different blocks, $S_\rho(f)$, $\rho \in \Delta$, behave as if they were independent; they are of course mutually orthogonal. To put the matter precisely: The $L^2$ norm of $f$ can be given exactly in terms of the $L^2$ norms of the $S_\rho f$, that is

$$\textstyle\sum_{\rho\in\Delta} \|S_\rho f\|_2^2 = \|f\|_2^2, \tag{43}$$

(and this is true for *any* decomposition of $\mathbf{R}^n$). For the general $L^p$ case not as much can be hoped for, but the following important theorem can nevertheless be established.

THEOREM 5. *Suppose* $f \in L^p(\mathbf{R}^n)$, $1 < p < \infty$. *Then*

$(\sum_{\rho\in\Delta} |S_\rho f(x)|^2)^{\frac{1}{2}} \in L^p(\mathbf{R}^n)$, *and the ratio* $\|(\sum_{\rho\in\Delta} |S_\rho f(x)|^2)^{\frac{1}{2}}\|_p / \|f\|_p$

*is contained between two bounds* (*independent of* $f$).

**5.2** The Rademacher functions provide a very useful device in the study of $L^p$ norms in terms of quadratic expressions. These functions, $r_0(t), r_1(t), \ldots, r_m(t), \ldots$ are defined on the interval $(0, 1)$ as follows: $r_0(t) = +1$, for $0 \le t \le 1/2$, and $r_0(t) = -1$ for $1/2 < t \le 1$; $r_0$ is extended outside the unit interval by periodicity, that is $r_0(t + 1) = r_0(t)$. In general $r_m(t) = r_0(2^m t)$. The sequence of Rademacher functions are orthonormal (and in fact mutually independent) over $[0, 1]$. For our purposes their importance arises from the following fact. Suppose $\sum_0^\infty |a_m|^2 < \infty$ and set $F(t) = \sum_{m=0}^\infty a_m r_m(t)$. Then $F(t) \in L^p[0, 1]$ for every $p < \infty$, and for $p < \infty$

$$A_p \|F\|_p \le \|F\|_2 = \left(\sum_{m=0}^{\infty} |a_m|^2\right)^{\frac{1}{2}} \le B_p \|F\|_p \tag{44}$$

for two positive constants $A_p$ and $B_p$.

Thus for functions which can be expanded in terms of the Rademacher functions, all the $L^p$ norms, $p < \infty$, are comparable.

We shall also need the $n$-dimensional form of (44). We consider the unit cube $Q$ in $\mathbf{R}^n$, $Q = \{t = (t_1, t_2, \ldots t_n): 0 \le t_j \le 1\}$. Let $m$ be an $n$-tuple of non-negative integers $m = (m_1, m_2, \ldots, m_n)$. Define $r_m(t) = r_{m_1}(t_1) r_{m_2}(t_2) \cdots r_{m_n}(t_n)$. Write $F(t) = \sum a_m r_m(t)$. With

$$\|F\|_p = \left(\int_Q |F(t)|^p \, dt\right)^{1/p},$$

we also have (44), whenever $\sum |a_m|^2 < \infty$. The proof of these facts is not overly long, but it will be best not to digress at this point. For this reason we postpone it until later, and present it in an appendix.*

**5.3** We come now to the proof of the theorem itself. It will be presented in several steps.

*5.3.1* We show here that it suffices to prove the inequality

$$\left\| \left( \sum_{\rho \in \Delta} |S_\rho f(x)|^2 \right)^{\frac{1}{2}} \right\|_p \leq A_p \|f\|_p, \quad 1 < p < \infty \tag{45}$$

for $f \in L^2(\mathbf{R}^n) \cap L^p(\mathbf{R}^n)$. To see this let $g \in L^2(\mathbf{R}^n) \cap L^q(\mathbf{R}^n)$, $1/p + 1/q = 1$, and consider the identity:

$$\sum_{\rho \in \Delta} \int_{\mathbf{R}^n} S_\rho(f)\overline{S_\rho(g)}\, dx = \int_{\mathbf{R}^n} f\bar{g}\, dx$$

which follows from (43) by polarization. By Schwarz's inequality and then Hölder's inequality

$$\left| \int_{\mathbf{R}^n} f\bar{g}\, dx \right| \leq \int_{\mathbf{R}^n} \left( \sum_\rho |S_\rho f|^2 \right)^{\frac{1}{2}} \left( \sum_\rho |S_\rho g|^2 \right)^{\frac{1}{2}} dx$$

$$\leq \left\| \left( \sum_\rho |S_\rho f|^2 \right)^{\frac{1}{2}} \right\|_p \left\| \left( \sum_\rho |S_\rho g|^2 \right)^{\frac{1}{2}} \right\|_q.$$

Taking the supremum over all such $g$ with the additional restriction that $\|g\|_q \leq 1$, gives $\|f\|_p$ for the left side of the above inequality. The right side is majorized by $\|(\sum |S_\rho f|^2)^{\frac{1}{2}}\|_p A_q$, since we assume (45) for all $p$, (in particular $q$). Thus we have also

$$B_p \|f\|_p \leq \left\| \left( \sum_\rho |S_\rho f|^2 \right)^{\frac{1}{2}} \right\|_p. \tag{46}$$

To dispose of the additional assumption that $f \in L^2$, for $f \in L^p$ take $f_j \in L^2 \cap L^p$ so that $\|f_j - f\|_p \to 0$; use the inequalities (45) and (46) for $f_j$ and $f_j - f_{j'}$; after a simple limiting argument we get (45) and (46) for $f$ as well.

*5.3.2* Here we shall prove the inequality (45) for $n = 1$. We shall need first to introduce a little more notation. We let $\Delta_1$ be the family of dyadic intervals in $\mathbf{R}^1$, as explained in §5.1; we can enumerate them as $I_0, I_1, \ldots, I_m \ldots$ (the order is here immaterial). For each $I \in \Delta_1$ we consider the partial sum operator $S_I$, and a modification of it that we now define. Let

* Appendix D.

$\varphi$ be a fixed function* of class $C^1$ with the following properties:

$$\begin{cases} \varphi(x) = 1 & \text{if} \quad 1 \leq x \leq 2 \\ \varphi(x) = 0 & \text{if} \quad x \leq 1/2, \quad \text{or} \quad x \geq 4. \end{cases}$$

Suppose $I$ is any dyadic interval, and assume that it is of the form $[2^k, 2^{k+1}]$. Define $\tilde{S}_I$ by

$$(\tilde{S}_I f)\hat{}\,(x) = \varphi(2^{-k}x)\hat{f}(x) = \varphi_I(x)\hat{f}(x). \tag{47}$$

That is, $\tilde{S}_I$, like $S_I$, is a multiplier transformation where the multiplier is equal to one on the interval $I$; but unlike $S_I$, the multiplier of $\tilde{S}_I$ is smooth.

A similar definition is made for $\tilde{S}_I$ when $I = [-2^{k+1}, -2^k]$. We observe that

$$S_I\tilde{S}_I = S_I, \tag{48}$$

since $S_I$ has as multiplier the characteristic function of $I$.

Now for each $t \in [0, 1]$, consider the multiplier transformation

$$\tilde{T}_t = \sum_{m=0}^{\infty} r_m(t)\tilde{S}_{I_m}$$

That is, $\tilde{T}_t$ is for each $t$ the multiplier transformation whose multiplier is $m_t(x)$, with

$$m_t(x) = \sum_m r_m(t)\varphi_{I_m}(x). \tag{49}$$

By the definition of $\varphi_{I_m}$ it is clear that for any $x$ at most three terms in the sum (49) can be non-zero. Moreover we also see easily that

$$|m_t(x)| \leq B, \qquad \left|\frac{dm_t}{dx}(x)\right| \leq B/|x| \tag{50}$$

where $B$ is independent of $t$. Thus by the multiplier theorem (Theorem 3 in §3)

$$\|\tilde{T}_t f\|_p \leq A_p \|f\|_p, \quad \text{for} \quad f \in L^2 \cap L^p \tag{51}$$

and with $A_p$ independent of $t$. From this it follows obviously that

$$\left(\int_0^1 \|\tilde{T}_t(f)\|_p^p\, dt\right)^{1/p} \leq A_p \|f\|_p.$$

However

$$\int_0^1 \|\tilde{T}_t(f)\|_p^p\, dt = \int_0^1\int_{\mathbf{R}^1} |\sum r_m(t)(\tilde{S}_{I_m}f)(x)|^p\, dx\, dt$$
$$\geq A'_p \int_{\mathbf{R}^1}\left(\sum_m |\tilde{S}_{I_m}f(x)|^2\right)^{p/2} dx,$$

* It is kept fixed in the rest of this argument.

by the property (44) of the Rademacher functions. Thus we have

$$\left\| \left( \sum_m |\tilde{S}_{I_m}(f)|^2 \right)^{\frac{1}{2}} \right\|_p \leq B_p \, \|f\|_p. \tag{52}$$

Now apply the general theorem about partial sums, Theorem 4′, with $\mathfrak{R} = \Delta_1$ here; and using (48) we get

$$\left\| \left( \sum_m |S_{I_m}(f)|^2 \right)^{\frac{1}{2}} \right\|_p \leq C_p \, \|f\|_p, \tag{53}$$

which is the one-dimensional case of the inequality (45), and this is what we had set out to prove.

*5.3.3* We are still in the one-dimensional case, and we write $T_t$ for the operator

$$T_t = \sum_m r_m(t) S_{I_m}.$$

Our claim is that

$$\|T_t(f)\|_p \leq A_p \, \|f\|_p, \qquad 1 < p < \infty, \tag{54}$$

with $A_p$ independent of $t$, and $f \in L^2 \cap L^p$.

Write $T_t^N = \sum_{m=0}^N r_m(t) S_{I_m}$, and it suffices to show that (54) holds, with $T_t^N$ in place of $T_t$ (and with $A_p$ independent of $N$ and $t$). Since each $S_{I_m}$ is a bounded operator on $L^2$ and $L^p$, we have that $T_t^N f \in L^2 \cap L^p$ and so we can apply (46) to it, which has already been proved (in the case $n = 1$). So

$$B_p \, \|T_t^N f\|_p \leq \left\| \left( \sum^N |S_{I_m} f|^2 \right)^{\frac{1}{2}} \right\|_p \leq C_p \, \|f\|_p,$$

using also (53). Letting $N \to \infty$, we get (54).

*5.3.4* We now turn to the $n$-dimensional case and define $T_{t_1}^{(1)}$, as the operator $T_{t_1}$ acting only on the $x_1$ variable. Then by the inequality (54) we get

$$\int_{\mathbf{R}^1} |T_{t_1}^{(1)} f(x_1, x_2, \ldots x_n)|^p \, dx_1 \leq A_p^p \int_{\mathbf{R}^1} |f(x_1, \ldots x_n)|^p \, dx_1 \tag{55}$$

for almost every fixed $x_2, x_3, \ldots x_n$, since $x_1 \to f(x_1, x_2, \ldots x_n) \in L^2(\mathbf{R}^1) \cap L^p(\mathbf{R}^1)$ for almost every fixed $x_2, \ldots x_n$, if $f \in L^2(\mathbf{R}^n) \cap L^p(\mathbf{R}^n)$. If we integrate (55) with respect to $x_2, \ldots x_n$ we then obtain

$$\|T_{t_1}^{(1)} f\|_p \leq A_p \, \|f\|_p, \qquad f \in L^2 \cap L^p, \tag{56}$$

with $A_p$ independent of $t_1$. The same inequality of course holds with $x_1$ replaced by $x_2$, or $x_3$, etc.

*5.3.5* We come now to the final step of the proof. We first describe the additional notation we shall need. With $\Delta$ representing the collection of dyadic rectangles in $\mathbf{R}^n$, we write any $\rho \in \Delta$, as $\rho = I_{m_1} \times I_{m_2}, \ldots \times I_{m_n}$ where $I_0, I_1, \ldots I_m, \ldots$ represents the (arbitrary) enumeration of the dyadic intervals used above. Thus if $m = (m_1, m_2, \ldots m_n)$, (with each $m_j \geq 0$), we write $\rho_m = I_{m_1} \times I_{m_2} \cdots \times I_{m_n}$.

We now apply the operator $T_{t_1}^{(1)}$ for the $x_1$ variable, and successively its analogues for $x_2$, $x_3$, etc. The result is

(57) $$\|T_t(f)\|_p \leq A_p^n \|f\|_p.$$

Here

$$T_t = \sum_{\rho_m \in \Delta} r_m(t) S_{\rho_m}$$

with $r_m(t) = r_{m_1}(t_1) \cdots r_{m_n}(t_n)$ as described in §5.2. The inequality holds uniformly for each $(t_1, t_2, \ldots t_n)$ in the unit cube $Q$.

We raise this inequality to the $p^{\text{th}}$ power and integrate it with respect to $t$, making use of the properties of the Rademacher functions cited in (44). We then get, as in the analogous proof of (52), that

$$\left\| \left( \sum_{\rho_m \in \Delta} |S_{\rho_m} f|^2 \right)^{\frac{1}{2}} \right\|_p \leq A_p \|f\|_p$$

if $f \in L^2(\mathbf{R}^n) \cap L^p(\mathbf{R}^n)$. This together with the remarks of §5.3.1 concludes the proof of Theorem 5.

## 6. *The Marcinkiewicz multiplier theorem*

**6.1** We now present the second version of the multiplier theorem. This form is to a large extent the synthesis of the ideas developed in §4 and 5, and as such is one of the most important results of the whole theory. For the sake of clarity we state first the one-dimensional case.

THEOREM 6. *Let $m$ be a bounded function on $\mathbf{R}^1$, which is of bounded variation on every finite interval not containing the origin. Suppose*

(a) $|m(x)| \leq B$, $-\infty < x < \infty$

(b) $\int_I |dm(x)| \leq B$, *for every dyadic interval $I$.*

*Then $m \in \mathcal{M}_p$, $1 < p < \infty$; and more precisely, if $f \in L^2 \cap L^p$*

$$\|T_m(f)\|_p \leq A_p \|f\|_p,$$

*where $A_p$ depends only on $B$ and $p$.*

To present the general theorem we consider $\mathbf{R}^1$ as divided into its two half-lines, $\mathbf{R}^2$ as divided into its four quadrants, and generally $\mathbf{R}^n$ as divided into its $2^n$ "octants." Thus the first octant in $\mathbf{R}^n$ will be the open "rectangle" of those $x$ all of whose coordinates are strictly positive. We shall assume that $m(x)$ is defined on each such octant and is there continuous together with its partial derivatives up to and including order $n$. Thus $m$ may be left undefined on the set of points where one or more coordinate variables vanishes.

For every $k \leq n$, we regard $\mathbf{R}^k$ embedded in $\mathbf{R}^n$ in the following obvious way: $\mathbf{R}^k$ is the subspace of all points of the form $(x_1, x_2, \ldots, x_k, 0, \ldots 0)$.

THEOREM 6′. *Let $m$ be a bounded function on $\mathbf{R}^n$ of the type described. Suppose also*

(a) $|m(x)| \leq B$

(b) *for each* $0 < k \leq n$

$$\sup_{x_{k+1},\ldots,x_n} \int_\rho \left| \frac{\partial^k m}{\partial x_1\, \partial x_2 \cdots \partial x_k} \right| dx_1 \cdots dx_k \leq B$$

*as $\rho$ ranges over dyadic rectangles of $\mathbf{R}^k$. (If $k = n$ the "sup" sign is omitted.)*

(c) *The condition analogous to (b) is valid for every one of the $n!$ permutations of the variables $x_1, x_2, \ldots, x_n$.*

*Then $m \in \mathscr{M}_p$, $1 < p < \infty$; and more precisely, if $f \in L^2 \cap L^p$, $\|T_m f\|_p \leq A_p \|f\|_p$, where $A_p$ depends only on $B$, $p$, and $n$.*

**6.2 Comments.** Before we come to the proof of the theorem we need to clarify certain matters of a technical nature; also the relation of this theorem with the first multiplier theorem, treated in §3.

*6.2.1* Theorem 6 appears stronger than Theorem 6′ for the case $n = 1$, because the hypotheses of the second theorem require that $m(x)$ is continuously differentiable away from the origin, while Theorem 6 requires only that $m$ is of bounded variation in intervals separated from the origin. However, in reality, both results are equally strong, since whenever $m$ satisfies the hypotheses of Theorem 6, we can find a sequence $\{m_j(x)\}$ (with $m_j(x)$ in fact indefinitely differentiable away from the origin), for which the bounds of Theorem 6′ hold uniformly in $j$, and so that $m_j(x) \to m(x)$ almost everywhere. It then follows that $T_{m_j} \to T_m$, and in this way the assertion can be established. We leave the details to the interested reader.

*6.2.2* Another pedantic remark is the following. As the reader may have surmised, there is nothing indispensable about the role of the powers of 2 in the definition of the dyadic rectangles in Theorems 5, 6, and 6′. The dyadic rectangles could in fact be replaced by other rectangles; for instance, those whose vertices have coordinates $\{-\lambda_k\}_{k=-\infty}^{k=\infty}$, and $\{\lambda_k\}_{k=-\infty}^{k=\infty}$, instead of $\{-2^k\}_{k=-\infty}^{k=\infty}$ and $\{2^k\}_{k=-\infty}^{k=\infty}$, with $\lambda_{k-1}/\lambda_k \geq r > 1$, all $k$. However the conclusions obtained this way are no stronger than in the dyadic case. (See also §7.10 below.)

*6.2.3* It is more interesting to compare Theorem 6′, with our first multiplier theorem, Theorem 3 and its corollary. It is clear that for $n = 1$ Theorem 6′ is the stronger. However for $n \geq 2$, they overlap and neither includes the other. This difference for $n \geq 2$ is also illustrated by simple invariance considerations. Thus the class of multipliers treated by Theorem 3 is invariant under dilations, $m(x) \to m(\varepsilon x)$, $\varepsilon > 0$, and also under rotations, $m(x) \to m(\rho^{-1}x)$. The set of multipliers of Theorem 6′ is not invariant under rotations, but is however invariant under a larger group of dilations, $m(x) \to m(\varepsilon \circ x)$, where $(\varepsilon \circ x) = (\varepsilon_1 x_1, \varepsilon_2 x_2, \ldots, \varepsilon_n x_n)$, $x = (x_1, \ldots, x_n)$ and the $\varepsilon_j$ are *independent* non-zero quantities.

*6.2.4* Nevertheless, in various applications Theorem 6′ seems to be the more useful of the two. For example for those multipliers which arise typically in elliptic differential equations (see Chapter III, §1.3, §3.5, and also §7.9 of this chapter) both theorems apply equally well.

However the multiplier $\dfrac{x_1}{x_1 + i(x_2^2 + x_3^2 \cdots + x_n^2)}$ which appears in parabolic equations falls under the scope of Theorem 6′ only. The same can be said of the multiplier

$$\frac{|x_1|^{\alpha_1}|x_2|^{\alpha_2}\cdots|x_n|^{\alpha_n}}{(x_1^2 + x_2^2 \cdots + x_n^2)^{\alpha/2}}, \qquad \alpha = \alpha_1 + \alpha_2 \cdots + \alpha_n, \quad \text{with} \quad \alpha_j > 0,$$

which is not untypical of a class arising in connection with the study of spaces of fractional potentials. (Compare with §3.2 in Chapter V.)

*6.2.5* Finally, both theorems have very definite shortcomings. This is already so because of the matters raised in §4.3. What seems to be needed is a more far-reaching theory which gives sufficient conditions for a multiplier to belong to some $\mathscr{M}_p$, $p \neq 2$, without implying also that it belongs to all $\mathscr{M}_p$, $1 < p < \infty$. Few tools seems to be available for this difficult task. The only thing that readily suggests itself is a possible development of the ideas centering about the function $g_\lambda^*$.

*6.2.6* The limitations of Theorems 3 and 6′ may be further illustrated by the following remark. Consider the characteristic function of an arbitrary polyhedron in $\mathbf{R}^n$. By the use of the same considerations as that of Theorem 4 we may show that it is a multiplier for $L^p$, $1 < p < \infty$. But this simple example does not fall under the scope of either Theorem 3 or Theorem 6′.

**6.3 Proof.** It will be best to prove Theorem 6′ in the case $n = 2$. This case is already completely typical of the general situation, and in doing only it we can avoid some notational complications.

Let $f \in L^2(\mathbf{R}^2) \cap L^p(\mathbf{R}^2)$, and write $F = T_m f$, that is $F(x)\hat{} = m(x)\hat{f}(x)$.

Let $\Delta$ denote the dyadic rectangles, and for each $\rho \in \Delta$, write $f_\rho = S_\rho f$, $F_\rho = S_\rho F$, thus $F_\rho = T_m f_\rho$.

In view of the theorem of dyadic decompositions (Theorem 5) it suffices to show that

$$\left\| \left( \sum_{\rho \in \Delta} |F_\rho|^2 \right)^{\frac{1}{2}} \right\|_p \leq C_p \left\| \left( \sum_{\rho \in \Delta} |f_\rho|^2 \right)^{\frac{1}{2}} \right\|_p \tag{58}$$

The rectangles in $\Delta$ come in four sets, those in the first, the second, the third, and fourth quadrants respectively. In estimating the left side of (58) consider the rectangles of each quadrant separately, and assume from now on that our rectangles belong to the first quadrant.

We will express $F_\rho$ in terms of an integral involving $f_\rho$ and the partial sum operators. That this is possible is the essential idea of the proof.

Fix $\rho$ and assume $\rho = \{(x_1 x_2) : 2^k \leq x_1 \leq 2^{k+1},\ 2^l \leq x_2 \leq 2^{l+1}\}$. Then for $(x_1, x_2) \in \rho$, we have the identity

$$m(x_1, x_2) = \int_{2^k}^{x_1} \int_{2^l}^{x_2} \frac{\partial^2 m(t_1 t_2)}{\partial t_1 \, \partial t_2} \, dt_1 \, dt_2 + \int_{2^k}^{x_1} \frac{\partial}{\partial t_1} m(t_1, 2^l) \, dt_1$$
$$+ \int_{2^l}^{x_2} \frac{\partial}{\partial t_2} m(2^k, t_2) \, dt_2 + m(2^k, 2^l).$$

Now let $S_t$ denote the multiplier transformation corresponding to the rectangle $2^k < x_1 < t_1$, $2^l < x_2 < t_2$. Similarly let $S_{t_1}^{(1)}$ denote the multiplier corresponding to the rectangle $2^k < x_1 < t_1$, similarly for $S_{t_2}^{(2)}$. Thus in fact $S_t = S_{t_1}^{(1)} \cdot S_{t_2}^{(2)}$. Then the above equation is obviously

$$S_\rho T_m = \int_{2^l}^{2^{l+1}} \int_{2^k}^{2^{k+1}} S_t \frac{\partial^2 m}{\partial t_1 \, \partial t_2} \, dt_1 \, dt_2 + \int_{2^k}^{2^{k+1}} S_{t_1}^1 \frac{\partial}{\partial t_1} m(t_1, 2^l) \, dt_1 \tag{59}$$
$$+ \int_{2^l}^{2^{l+1}} \cdots + m(2^k, 2^l) S_\rho.$$

Now use the fact that $S_\rho T_m f = F_\rho$, and $S_{t_1}^{(1)} S_\rho = S_{t_1}^{(1)}$, $S_{t_2}^{(2)} S_\rho = S_{t_2}^{(2)}$ $S_t S_\rho = S_t$, together with Schwarz's inequality and the assumptions of the theorem. This gives

$$|F_\rho|^2 \le B'\left\{\iint_\rho |S_t(f_\rho)|^2 \left|\frac{\partial^2 m}{\partial t_1\,\partial t_2}\right| dt_1\,dt_2 + \int_{I_1} |S_{t_1}^{(2)}(f_\rho)|^2 \left|\frac{\partial(m(t_1, 2^l)}{\partial t_1}\right| dt_1\right.$$

$$\left. + \int_{I_2} |S_{t_2}^{(2)}(f_\rho)|^2 \left|\frac{\partial}{\partial t_2} m(2^k, t_2)\right| dt_2 + |f_\rho|\right\} \tag{60}$$

$$= \mathfrak{I}_\rho^1 + \mathfrak{I}_\rho^2 + \mathfrak{I}_\rho^3 + \mathfrak{I}_\rho^4, \quad \text{with} \quad \rho = I_1 \times I_2.$$

To estimate $\|(\sum_\rho |F_\rho|^2)^{\frac{1}{2}}\|_p$, we estimate separately the contributions of each of the four terms on the right side of (60) by the use of Theorem 4″ in §4.4. To apply that theorem in the case of $\mathfrak{I}_\rho^1$ we take for $\Gamma$ the first quadrant, and $d\gamma = \left|\dfrac{\partial^2 m(t_1, t_2)}{\partial t_1 \partial t_2}\right| dt_1\,dt_2$; the functions $\gamma \to \rho_\gamma$ are constant on the dyadic rectangles. Since for every rectangle,

$$\int_\rho d\gamma = \int_\rho \left|\frac{\partial^2 m(t_1, t_2)}{\partial t_1\,\partial t_2}\right| dt_1\,dt_2 \le B, \text{ then } \|(\Sigma_\rho |\mathfrak{I}_\rho^1|^2)^{\frac{1}{2}}\|_p \le C_p \|(\Sigma\,|f_\rho|^2)^{\frac{1}{2}}\|_p.$$

Similarly for $\mathfrak{I}^2$, $\mathfrak{I}^3$, and $\mathfrak{I}^4$, which concludes the proof.

## 7. *Further results*

**7.1** Suppose that $R_1, R_2, \ldots R_n$ are the Riesz transforms. Then

(a) $(g(f, x))^2 = (g_1(f)(x))^2 + \sum_{j=1}^n (g_1(R_j f)(x))^2$

(b) $g_1^2(f)(x) \le \sum_{j=1}^n (g_x(R_j f)(x))^2$

**7.2** (a) Suppose $\varphi$ continuously differentiable in $\mathbf{R}^n$ and

(i) $|\varphi(x)| \le A(1 + |x|)^{-n-\delta}$,

(ii) $\left|\dfrac{\partial \varphi}{\partial x_j}\right| \le A(1 + |x|)^{-n-\delta}$ for each $j = 1, \ldots, n$

(iii) $\displaystyle\int_{\mathbf{R}^n} \left|\frac{\partial \varphi}{\partial x_j}(x + t) - \frac{\partial \varphi}{\partial x_j}(x)\,dx\right| \le A\,|t|^\delta$ for some $\delta > 0$.

Define $f_\varepsilon(x) = f * \varphi_\varepsilon$, where $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. Then

$$\left\|\left(\int_0^\infty \varepsilon \left|\frac{\partial f_\varepsilon}{\partial \varepsilon}\right|^2 d\varepsilon\right)^{\frac{1}{2}}\right\|_p \le A_p \|f\|_p, \qquad f \in L^p, \qquad 1 < p < \infty.$$

If, in addition, $\left\|\left(\int_0^\infty \varepsilon \left|\dfrac{\partial f_\varepsilon}{\partial \varepsilon}\right|^2 d\varepsilon\right)^{\frac{1}{2}}\right\|_2 = C\,\|f\|_2$, $C > 0$, then also the converse inequality holds. Similarly for $\left(\int_0^\infty \varepsilon \left|\dfrac{\partial f_\varepsilon}{\partial x_k}\right|^2 d\varepsilon\right)^{\frac{1}{2}}$. (See Benedek, Calderón, and Panzone [1] for closely related results.)

(b) An example in $\mathbf{R}^1$ is given by $\left(\int_0^\infty \frac{|F(x+t) + F(x-t) - 2F(x)|^2}{t^3}\,dt\right)^{\frac{1}{2}}$,

where $F(x) = \int_0^x f(t)\,dt$. See Marcinkiewicz [2], Zygmund [1]; also Stein [3], Hörmander [1] for generalizations.

**7.3** Let $T$ be a bounded linear transformation of $L^p(\mathbf{R}^n)$ to itself, $1 \leq p \leq \infty$, which commutes with translations. Then there exists a bounded function $m$ so that $(Tf)^\wedge(x) = m(x)\hat{f}(x)$, whenever $f \in L^2 \cap L^p$.

*Outline of proof.* (i) Since $T$ commutes with translations $(Tf) * g = T(f * g)$, for appropriate $f$ and $g$. Thus $Tf * g = f * Tg$.

(ii) Let $1/p + 1/q = 1$, and suppose that $f$ and $g$ both belong to $L^p(\mathbf{R}^n) \cap L^q(\mathbf{R}^n)$. Then the convolutions $Tf * g$ and $f * Tg$ represent continuous functions, and so they are equal at every point, in particular the origin. Hence

$$\int_{\mathbf{R}^n} (Tf)(x) g(-x)\,dx = \int_{\mathbf{R}^n} (Tg)(x) f(-x)\,dx.$$

The usual duality argument then shows that $T$ is bounded on $L^q$, and by the interpolation theorem (Theorem 4, Chapter I) $T$ is also bounded on $L^2$. Finally apply the Proposition in §1.4, Chapter II. See also *Fourier Analysis*, Chapter I.

**7.4** §7.4–§7.6 give interesting illustrations of multipliers which cannot be treated by the methods of this chapter.

Let $m(x)$ be a function in $\mathbf{R}^n$ which is of the form $m(x) = \varphi_0(x) \dfrac{e^{i|x|^\alpha}}{(1 + |x|^2)^\beta}$, $\alpha > 0$, $\beta > 0$, where $\varphi_0$ is a smooth function which vanishes near the origin and is 1 for sufficiently large $\alpha$. Assume that not both $n = 1$ and $\alpha = 1$.

(a) If $|1/2 - 1/p| < \theta$, then $m(x) \in \mathcal{M}_p$. Here $\theta = 2\beta/\alpha n$.

(b) If $|1/2 - 1/p| > \theta$, then $m(x) \notin \mathcal{M}_p$.

In the exceptional case ($n = 1$, $\alpha = 1$),

(a′) With $\beta = 0$, $m \in \mathcal{M}_p \Leftrightarrow 1 < p < \infty$.

(b′) With $\beta > 0$, $m \in \mathcal{M}_p$, all $p$.

See Hirschmann [2], Wainger [1], Hörmander [3], Stein [8], and Fefferman [1].

**7.5** Suppose that $m(x) \in \mathcal{M}_p(\mathbf{R}^n)$, and is continuous at each point of $\mathbf{R}^k$, $k < n$ ($\mathbf{R}^k$ is considered as a subspace of $\mathbf{R}^n$). Then $m(x)$ restricted to $\mathbf{R}^k$ belongs to $\mathcal{M}_p(\mathbf{R}^k)$. deLeeuw [1].

**7.6** Suppose $m(x) = (m_1 * m_2)(x)$, where $m_1 \in L^r(\mathbf{R}^n)$, and $m_2 \in L^{r'}(\mathbf{R}^n)$, with $1/r + 1/r' = 1$. Then $m \in \mathcal{M}_p$, if $|1/2 - 1/p| \leq 1/r$, if $2 \leq r \leq \infty$. See Hahn [1].

**7.7** (a) Let $\chi_B$ be the characteristic function of the unit ball of $\mathbf{R}^n$. Then $\chi_B \notin \mathcal{M}_p$ if $p \leq \dfrac{2n}{n+1}$ or if $p \geq \dfrac{2n}{n-1}$. See Herz [1].

(b) More generally: suppose $m(x)$ is a radial function and suppose $m \in \mathcal{M}_p(\mathbf{R}^n)$. Then if $p < \dfrac{2n}{n+1}$, or $p > \dfrac{2n}{n-1}$, then $m$ is continuous everywhere, except possibly the origin. *Hint*: Suppose $f \in L^p(\mathbf{R}^n)$, $p < \dfrac{2n}{n+1}$ and $f$ is radial. Then $\hat{f}$ is continuous except possibly at the origin. To prove this assertion use the representation of $\hat{f}$ in terms of Bessel integrals, as in *Fourier Analysis*, Chapter IV.

**7.8** Consider the question of whether the function

$$m_\delta(x) = \begin{cases} (1 - |x|^2)^\delta, & \text{if } |x| \leq 1 \\ 0 \quad , & \text{if } |x| > 1 \end{cases}$$

is a multiplier for $L^p(\mathbf{R}^n)$, i.e. whether $m_\delta \in \mathcal{M}_p$. For $\delta = 0$, this is Problem A discussed in §4.3 above, and also in §7.7. The following positive results are known.

(a) if $\delta > \left(\dfrac{n-1}{2}\right) |1 - 2/p|$, then $m_\delta \in \mathcal{M}_p$. See Stein [1].

(b) This has recently been significantly improved in Fefferman [1]. A particular result of his is that $m_\delta \in \mathcal{M}_p$, if $n = 2$, $\delta > 2(n-1)|1/p - 3/4|$, and $1 \leq p < 6/5$. This result is in the nature of best possible, for $p$ in the range $1 \leq p < 6/5$. For $n$ dimensions, $n \geq 3$, there are similar results, in the nature of best possible, when $1 \leq p < 4n/(3n+1)$.

**7.9** Let $P(x)$ be a polynomial in $\mathbf{R}^n$ of degree $k$. Suppose $P(x)$ is elliptic in the sense that its part of homogeneous degree $k$ is non-vanishing except at the origin. Let $f$ be any $k$ times continuously differentiable function in $\mathbf{R}^n$ with compact support. Then we have the inequality

$$\left\| \left(\frac{\partial}{\partial x}\right)^\alpha f \right\|_p \leq A_p \left[ \left\| P\left(\frac{\partial}{\partial x}\right) f \right\|_p + \|f\|_p \right], \qquad 1 < p < \infty,$$

as long as $|\alpha| \leq k$.

Hint: Let $\varphi(x)$ be a smooth function which vanishes in a neighborhood of the zero set of $P$, and which is 1 outside a sufficiently large ball. Then $\varphi(x)\dfrac{x^\alpha}{P(x)}$ satisfies the conditions of the multiplier Theorems 3 or 6′; $x^\alpha(1 - \varphi(x))$ is the Fourier transform of an $L^1$ function, and finally, $x^\alpha \hat{f}(x) = x^\alpha(1 - \varphi(x))\hat{f}(x) + \varphi(x)\dfrac{x^\alpha}{P(x)} \cdot P(x)\hat{f}(x)$. See also §3.5 of Chapter III, and Agmon, Douglis and Nirenberg [1].

**7.10** (a) The conditions (a) and (b) of Theorem 6 are equivalent with the statements

$$|m(x)| \leq B'$$

$$\sup_{0 < R} \frac{1}{R} \int_{|x| \leq R} |x|\, |dm(x)| \leq B'$$

(b) What is the analogous reformulation of the conditions of Theorem 6′?

**7.11** The result of Theorem 2 in §2.4 can be strengthened as follows. Let $1 < p < 2$ and $p = 2/\lambda$. Then the mapping $f \to g_\lambda^*(f)$ is of weak-type $(p, p)$. See Fefferman [1]. An earlier result (for an analogous maximal function) is stated in §4.5 of Chapter VII.

**7.12** Let $T$ be a bounded operator of $L^p(\mathbf{R}^n)$ to itself, $1 \le p \le \infty$. Let $\mathscr{H}$ be any Hilbert space. Then $T$ has a unique "extension" to an operator $T \otimes I$, taking $L^p(\mathbf{R}^n, \mathscr{H})$ to itself, with the property that $(T \otimes I)(\varphi f(x)) = \varphi \cdot Tf(x)$, for any $\varphi \in \mathscr{H}$, and $f \in L^p(\mathbf{R}^n)$. Moreover the norm of $T \otimes I$ on $L^p(\mathbf{R}^n, \mathscr{H})$ is the same as the norm of $T$ on $L^p(\mathbf{R}^n)$. Marcinkiewicz and Zygmund [1], Zygmund [8], Chapter XV.

## *Notes*

*Section 1.* The classical theory (which used complex methods) is described in Chapters XIV and XV of Zygmund [8]; further historical references will be found there. The theorems for the $g$-function in $n$-dimensions are in Stein [3]; further generalizations were given by Hörmander [1], Schwartz [1], and Benedeck, Calderón, and Panzone [1].

*Section 2.* The function $g^*$ was studied systematically by Zygmund in [1], and the $n$-dimensional theory by Stein [6] and [10]. The particular approach, described in §2.1 is taken from Stein [10]; a related idea was independently developed by Gasper [1]. This approach is a starting point for various generalizations of the theory, as in Stein [13].

*Section 3.* The original Marcinkiewicz multiplier theorem is in Marcinkiewicz [4], where it is given in the periodic set-up. Non-periodic variants of this theorem are due to Mihlin [2], Hörmander [1], and Kree [1]. The statement of Theorem 3 is identical with Hörmander's; the present proof however, as it uses comparison with the $g$ and $g^*$ functions is different, and can be adapted in various other circumstances, as in Chapter VII below.

For a general discussion of multipliers, see also Edwards [1].

*Sections 4, 5, and 6.* The one-dimensional version of Theorem 4′ is in Zygmund [8, Chapter XV]. The more general version given here is in reality a simple consequence of this special case.

The proof given for Theorem 6′ is a simple adaptation of the original periodic argument given in Marcinkiewicz [4]. See also Lizorkin [1] and Kree [1].

CHAPTER V

# Differentiability Properties in Terms of Function Spaces

In this chapter we shall study properties of differentiability and smoothness that can best be described in the context of Banach spaces of functions. One of the motivations for this study is based on the wide scope of its applications, as a useful tool in a variety of problems in analysis, although much of what we do is in reality suggested by the ideas and methods already developed. In fact such techniques as the interpolation theorem of Marcinkiewicz, the application of harmonic functions, and the Littlewood-Paley $g$-function, are essential parts of the theory detailed below.

The function spaces we shall treat are the following:

(1) The *Sobolov spaces*, $L_k^p(\mathbf{R}^n)$. These are useful in many questions and consist of all functions on $\mathbf{R}^n$ whose derivatives up to and including order $k$ belong to $L^p(\mathbf{R}^n)$; $k$ is, of course, a non-negative integer.

The two other types of function spaces that will be considered are attempts to "generalize" the Sobolov spaces to the case when $k$ is not integral.

(2) The *potential spaces*, $\mathscr{L}_\alpha^p(\mathbf{R}^n)$, consisting of all "potentials" of order $\alpha$ of $L^p$ functions. When $\alpha$ is integral and $1 < p < \infty$, these spaces are equivalent with the Sobolov spaces.

(3) The *spaces* $\Lambda_\alpha^{p,q}$. These are function spaces defined in terms of the $L^p$ modulus of continuity. As such they represent a more easily defined "generalization" of the spaces $L_k^p(\mathbf{R}^n)$, and because of this are very useful in applications. They are, however, not a genuine generalization of the Sobolov spaces and so a comparison between them and the spaces $\Lambda_\alpha^{p,q}(\mathbf{R}^n)$ and $\mathscr{L}_\alpha^p(\mathbf{R}^n)$ is called for. This comparison may be viewed as one of the central problems treated in this chapter, and it is here where the Littlewood-Paley theory of Chapter IV is applied.

We shall begin by studying the fractional powers of the Laplacean, $(-\Delta)^{\alpha/2}$. This, together with its variant $(I - \Delta)^{\alpha/2}$, represents an important formal device that we shall use.

## 1. *Riesz potentials*

**1.1** The Fourier transform of a function $f$ which is sufficiently smooth, and small at infinity, and its Laplacean, $\Delta f = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2}$, are related by

$$(1) \qquad (-\Delta f)^\wedge(x) = 4\pi^2 \, |x|^2 \hat{f}(x).$$

From this it is only one step to replace the exponent 2 in $|x|^2$ by a general exponent $\beta$, and thus to define (at least formally) the fractional power of the Laplacean by

$$(2) \qquad ((-\Delta)^{\beta/2} f)^\wedge = (2\pi \, |x|)^\beta \hat{f}(x).$$

Of special significance will be the negative powers $\beta$ in the range, $-n < \beta < 0$. For these there will be a realization of the formal operator (2) as an integral operator. That is, with a slight change of notation we shall have

$$(3) \qquad I_\alpha(f) = (-\Delta)^{-\alpha/2}(f), \qquad 0 < \alpha < n$$

where we have defined the *Riesz potentials* by

$$(4) \qquad (I_\alpha f)(x) = \frac{1}{\gamma(\alpha)} \int_{\mathbf{R}^n} |x - y|^{-n+\alpha} f(y) \, dy,$$

with

$$\gamma(\alpha) = \pi^{n/2} 2^\alpha \Gamma(\alpha/2) \Big/ \Gamma\left(\frac{n}{2} - \frac{\alpha}{2}\right)$$

The formal manipulations have a precise meaning.

For this purpose it is convenient to use the class $\mathcal{S}$ of functions $\varphi$, which are indefinitely differentiable on $\mathbf{R}^n$ and all of whose derivatives remain bounded when multiplied by polynomials.

LEMMA 2. *Let* $0 < \alpha < n$.

(a) *The Fourier transform of the function* $|x|^{-n+\alpha}$ *is the function* $\gamma(\alpha)(2\pi)^{-\alpha} |x|^{-\alpha}$, *in the sense that*

$$(5) \qquad \int_{\mathbf{R}^n} |x|^{-n+\alpha} \varphi(x) \, dx = \int_{\mathbf{R}^n} \gamma(\alpha)(2\pi)^{-\alpha} |x|^{-\alpha} \overline{\varphi^\wedge(x)} \, dx,$$

*whenever* $\varphi \in \mathcal{S}$.

(b) *The identity* $(I_\alpha f)^\wedge = (2\pi \, |x|)^{-\alpha} \hat{f}(x)$ *holds in the sense that*

$$\int_{\mathbf{R}^n} I_\alpha(f)(x) \overline{g(x)} \, dx = \int_{\mathbf{R}^n} \hat{f}(x) (2\pi \, |x|)^{-\alpha} \overline{g^\wedge(x)} \, dx$$

*whenever* $f, g \in \mathcal{S}$.

The first part of the lemma is merely a restatement of the result in Chapter III, §3.3, since $\gamma(\alpha) = \gamma_{0,\alpha}(2\pi)^\alpha$.

Part (b) follows immediately from part (a) by writing

$$\frac{(2\pi)^\alpha}{\gamma(\alpha)} \int_{\mathbf{R}^n} f(x - y)\,|y|^{-n+\alpha}\,dy = \int_{\mathbf{R}^n} \hat{f}(-y)\,|y|^{-\alpha} e^{-2\pi i x\cdot y}\,dy,$$

(which is a rephrasing of (5)) and then integrating both sides of this identity after multiplying through by $\bar{g}(x)$.

We state now two further identities which can be obtained from Lemma 1 and which reflect essential properties of the potentials $I_\alpha$.

(6) $I_\alpha(I_\beta f) = I_{\alpha+\beta}(f), \quad f \in \mathscr{S}, \quad \alpha > 0, \quad \beta > 0, \quad \alpha + \beta < n.$

(7) $\Delta(I_\alpha f) = I_\alpha(\Delta f) = -I_{\alpha-2}(f), \quad f \in \mathscr{S}, \quad n > 3, \quad n \geq \alpha \geq 2.$

The deduction of (6) and (7) offer no real difficulties; these are best left to the interested reader to work out.

A simple consequence of (6) is the $n$-dimensional variant of the *beta integral*,

$$\int_{\mathbf{R}^n} |1 - y|^{-n+\alpha} |y|^{-n+\beta}\,dy = \frac{\gamma(\alpha)\gamma(\beta)}{\gamma(\alpha + \beta)} \tag{8}$$

with $0 < \alpha$, $0 < \beta$, $\alpha + \beta < n$.

**1.2 $L^p$ inequality for potentials.** Up to this stage we have considered the Riesz potentials only from a formal point of view; in particular, we have operated only with very smooth functions which are suitably small at infinity. But since the Riesz potentials are integral operators it is natural to inquire about their actions on the spaces $L^p(\mathbf{R}^n)$.

For this reason we formulate the following problem. Given $\alpha$, $0 < \alpha < n$, for what pairs $p$ and $q$, is the operator $f \to I_\alpha(f)$ bounded from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$? That is, when do we have the inequality

$$\|I_\alpha(f)\|_q \leq A\,\|f\|_p\,? \tag{9}$$

There is a simple necessary condition, which is merely a reflection of the homogeneity of the kernel $(\gamma(\alpha))^{-1}\,|y|^{-n+\alpha}$. In fact, consider the dilation operator $\tau_\delta$, defined by

$$\tau_\delta(f)(x) = f(\delta x), \qquad \delta > 0.$$

Then clearly

$$\tau_{\delta^{-1}} I_\alpha \tau_\delta = \delta^{-\alpha} I_\alpha, \qquad \delta > 0. \tag{10}$$

Also

$$\|\tau_\delta(f)\|_p = \delta^{-n/p}\,\|f\|_p, \qquad \|\tau_{\delta^{-1}} I_\alpha(f)\|_q = \delta^{n/q}\,\|I_\alpha(f)\|_q. \tag{11}$$

Thus (9) is possible only if

$$(12) \qquad 1/q = 1/p - \alpha/n.$$

We shall see below that this condition is also sufficient, save for two exceptional cases.

The two instances arise when $p = 1$, (then $q = n/(n - \alpha)$) and when $q = \infty$, (then $p = n/\alpha$). Let us consider the case $p = 1$. It is not hard to see that the presumed inequality

$$(13) \qquad \|I_\alpha(f)\|_{n/(n-\alpha)} \leq A \, \|f\|_1,$$

cannot hold. If in fact (13) were valid, we could put in the place of $f$ a sequence $\{f_n\}$ of positive integrable functions whose common integral is one and whose supports converge to the origin (an "approximation to the identity"). A simple limiting argument then shows that this implies that

$$\left\| \frac{1}{\gamma(\alpha)} |x|^{-n+\alpha} \right\|_{n/(n-\alpha)} \leq A < \infty,$$

which means

$$\int_{\mathbf{R}^n} |x|^{-n} \, dx < \infty,$$

and this is a contradiction.

The second atypical case occurs when $q = \infty$. Again the inequality of the type (9) cannot hold, and one immediate reason is that this case is dual to the case $p = 1$ just considered. The failure at $q = \infty$ may also be seen directly as follows: Let $f(x) = |x|^{-\alpha}(\log 1/|x|)^{-(\alpha/n)(1+\varepsilon)}$, for $|x| \leq 1/2$, and $f(x) = 0$, for $|x| > 1/2$, where $\varepsilon$ is positive but small. Then $f \in L^p(\mathbf{R}^n)$, $p = n/\alpha$, since $\int_{|x|<\frac{1}{2}} |x|^{-n}(\log 1/|x|)^{-1-\varepsilon} \, dx < \infty$. However, $I_\alpha(f)$ is essentially unbounded near the origin since

$$I_\alpha(f)(0) = \frac{1}{\gamma(\alpha)} \int_{|x| \leq \frac{1}{2}} |x|^{-n} (\log 1/|x|)^{-(\alpha/n)(1+\varepsilon)} \, dx = \infty,$$

as long as $(\alpha/n)(1 + \varepsilon) \leq 1$.

After these observations we can formulate the positive theorem: The Hardy-Littlewood-Sobolov theorem of *fractional integration.*

THEOREM 1. *Let* $0 < \alpha < n$, $1 \leq p < q < \infty$, $1/q = 1/p - \alpha/n$.

(a) *If* $f \in L^p(\mathbf{R}^n)$, *then the integral* (4), *defining* $I_\alpha(f)$, *converges absolutely for almost every* $x$.

(b) *If, in addition,* $1 < p$, *then*

$$\|I_\alpha(f)\|_q \leq A_{p,q} \, \|f\|_p.$$

(c) *If* $f \in L^1(\mathbf{R}^n)$, *then* $m\{x : |I_\alpha| > \lambda\} \leq \left(\frac{A\,\|f\|_1}{\lambda}\right)^q$, *for all* $\lambda$. *That is, the mapping* $f \to I_\alpha(f)$ *is of "weak-type"* $(1, q)$, $(1/q = 1 - \alpha/n)$.

**1.3 Proof of Theorem 1.** Let us write $K(x) = |x|^{-n+\alpha}$, and we shall consider the transformation $f \to K * f$, instead of $f \to I_\alpha(f)$ from which it differs by a constant multiple. Let us decompose $K$ as $K_1 + K_\infty$, where

$$K_1(x) = K(x) \quad \text{if} \quad |x| \leq \mu, \qquad K_1(x) = 0 \quad \text{if} \quad |x| > \mu$$
$$K_\infty(x) = K(x) \quad \text{if} \quad |x| > \mu, \qquad K_\infty(x) = 0 \quad \text{if} \quad |x| \leq \mu.$$

At this instance $\mu$ is a fixed positive constant which need not be specified. We have $K * f = K_1 * f + K_\infty * f$. The integral expressing $K_1 * f$ converges absolutely *almost everywhere* since it represents the convolution of an $L^1$ function $(K_1)$ with an $L^p$ function. Similarly the integral representing $K_\infty * f$ converges *everywhere* since it is a convolution of a function in $L^p$, $(f)$, and another in the dual space $L^{p'}$, $(K_\infty)$. In fact if $1/p + 1/p' = 1$, then $\|K_\infty\|_{p'}^{p'} = \int_{|x| > \mu} |x|^{(-n+\alpha)p'}\, dx < \infty$, since $(-n + \alpha)p' < -n$ is equivalent with $q < \infty$. Thus part (a) of the theorem is proved.

We shall show next, by a similar but more detailed reasoning, that if $1 \leq p < q < \infty$, and $1/q = 1/p - \alpha/n$, then the mapping $f \to K * f$ is of *weak-type* $(p, q)$, in the sense that

$$(14) \quad m\{x : |K * f| > \lambda\} \leq \left(A_{p,q} \frac{\|f\|_p}{\lambda}\right)^q, \quad f \in L^p(\mathbf{R}^n), \quad \text{all} \quad \lambda > 0.$$

We notice first that it suffices to prove the inequality (14) with $2\lambda$ in place of $\lambda$ in the left side of this inequality, and with $\|f\|_p = 1$. Now

$$m\{x : |K * f| > 2\lambda\} \leq m\{x : |K_1 * f| > \lambda\} + m\{x : |K_\infty * f| > \lambda\},$$

since $K * f = K_1 * f + K_\infty * f$. However

$$m\{x : |K_1 * f| > \lambda\} \leq \frac{\|K_1 * f\|_p^p}{\lambda^p} \leq \frac{\|K_1\|_1^p \|f\|_p^p}{\lambda^p} = \frac{\|K_1\|_1^p}{\lambda^p}.$$

But

$$\|K_1\|_1 = \int_{|x| \leq \mu} |x|^{-n+\alpha}\, dx = c_1 \mu^\alpha.$$

Next

$$\|K_\infty * f\|_\infty \leq \|K_\infty\|_{p'} \|f\|_p \leq \|K_\infty\|_{p'}.$$

However,

$$\|K_\infty\|_{p'} = \left(\int_{|x| > \mu} (|x|^{-n+\alpha})^{p'}\, dx\right)^{1/p'} = c_2 \mu^{-n/q},$$

and so $\|K_\infty\|_{p'} = \lambda$, if $c_2 \mu^{-n/q} = \lambda$, i.e., if $\mu = c_3 \lambda^{-q/n}$. Choose, therefore, $\mu$ to have this value. Then $\|K_\infty * f\|_\infty \leq \lambda$, and so $m\{x : |K_\infty * f| > \lambda\} = 0$.

Finally then

$$m\{x : |K * f| > 2\lambda\} \le \left(c_1 \frac{\mu^\alpha}{\lambda}\right)^p = c_4 \lambda^{-q} = c_4 \left(\frac{\|f\|_p}{\lambda}\right)^q$$

(since $\|f\|_p = 1$). This is (14), and so the mapping $f \to K * f$ is of weak type $(p, q)$. The special case for $p = 1$ then gives part (c) of the theorem, and part (b) follows by the Marcinkiewicz interpolation theorem. (See Appendix B.)

**1.4 Comment.** The following retrospective comment about the proof of the theorem is in order. In the proof of Theorem 1 for the operator $f \to K * f$ what was decisive was not the specific structure of the kernel $K$. The only thing that really mattered was the *distribution function* of $K$ (in the terminology of Chapter I.) A more detailed examination would show that we only needed the fact that $m\{x : |K(x)| \ge \lambda\} \le A\lambda^{-n/(n-\alpha)}$; that is, the kernel is of "weak type" $n/(n - \alpha)$.

If we had the stronger assumption that $K \in L^{n/(n-\alpha)}$ we would have obtained *a fortiori* the result

$$\|K * f\|_q \le A \, \|f\|_p, \quad \text{with} \quad 1/q = 1/p + 1/r - 1, \; r = n/(n - d).$$

This is essentially the more familiar Young's inequality, which is also valid when $p = 1$ or $q = \infty$. (See Appendix A.)

## 2. *The Sobolev spaces,* $L_k^p(\mathbf{R}^n)$

We come now to the study of the relation of a function and its partial derivatives. The concept of the partial derivative that will be used is the general notion given us by the theory of distributions, and the appropriate definition is stated in terms of the space $\mathcal{D}$ of all indefinitely differentiable functions on $\mathbf{R}^n$, each with compact support.

Let $\dfrac{\partial^\alpha}{\partial x^\alpha} = \dfrac{\partial^{\alpha_1 + \alpha_2 + \cdots + \alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$, be a differential monomial, whose total order is $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Suppose we are given two locally integrable functions on $\mathbf{R}^n$, $f$ and $g$. Then we say that $\dfrac{\partial^\alpha f}{\partial x^\alpha} = g$, (we add the designation "in the weak sense," whenever this is necessary to avoid ambiguities), if

$$(15) \quad \int_{\mathbf{R}^n} f(x) \frac{\partial^\alpha \varphi}{\partial x^\alpha}(x)\, dx = (-1)^{|\alpha|} \int_{\mathbf{R}^n} g(x) \varphi(x)\, dx, \quad \text{for all} \quad \varphi \in \mathcal{D}.$$

Integration by parts shows us that this is indeed the relation that we would expect if $f$ had continuous partial derivatives up to order $|\alpha|$, and $\dfrac{\partial^\alpha f}{\partial x^\alpha} = g$ had the usual meaning.

It is of course not true that every locally integrable function has partial derivatives in this sense: consider, for example, $f(x) = e^{i/|x|^n}$. However when the partial derivatives exist they are determined almost everywhere by the defining relation (15).

For any non-negative integer $k$, the Sobolov space $L_k^p(\mathbf{R}^n) = L_k^p$ is defined as the space of functions $f$, with $f \in L^p(\mathbf{R}^n)$ and where all $\dfrac{\partial^\alpha f}{\partial x^\alpha}$ exist and $\dfrac{\partial^\alpha f}{\partial x^\alpha} \in L^p(\mathbf{R}^n)$ in the above sense, whenever $|\alpha| \leq k$. This space of functions can be normed by the expression

$$\text{(16)} \qquad \|f\|_{L_k^p} = \sum_{|\alpha| \leq k} \left\| \frac{\partial^\alpha}{\partial x^\alpha} f \right\|_p, \qquad \left(\frac{\partial^0}{\partial x^0} f = f\right).$$

The resulting normed space is complete. The proof of this is as follows. If $\{f_m\}$ is a Cauchy sequence in $L_k^p$, then for each $\alpha$, $\left\{\dfrac{\partial^\alpha}{\partial x^\alpha} f_m\right\}$ is a Cauchy sequence in $L^p$, $|\alpha| \leq k$. If now $f^{(\alpha)} = \lim\limits_m \dfrac{\partial^\alpha}{\partial x^\alpha} f_m$ (the limit taken in $L^p$ norm), then clearly $\int_{\mathbf{R}^n} f \dfrac{\partial^\alpha}{\partial x^\alpha} \varphi \, dx = (-1)^{|\alpha|} \int_{\mathbf{R}^n} f^{(\alpha)} \varphi \, dx$, for each $\varphi \in \mathscr{D}$ and the assertion is proved.

It is often convenient to use an equivalent characterization of the functions of $L_k^p(\mathbf{R}^n)$, which does not explicitly involve the notion of the weak derivative given in (15).

PROPOSITION 1. *Let* $1 \leq p < \infty$. *Then* $f \in L_k^p$ *if and only if there exists a sequence* $\{f_m\}$, *so that*

(a) *each* $f_m \in \mathscr{D}$

(b) $\|f - f_m\|_p \to 0$

(c) *For each* $\alpha$, $\left\{\dfrac{\partial^\alpha f_m}{\partial x^\alpha}\right\}$ *converges in* $L^p$ *norm, for* $|\alpha| \leq k$.

That the conditions (a), (b), and (c) are sufficient is readily obvious. In fact let $f^{(\alpha)} = \lim\limits_{m \to \infty} \dfrac{\partial^\alpha f_m}{\partial x^\alpha}$, $f^{(0)} = f$; then since

$$\int f_m \frac{\partial^\alpha \varphi}{\partial x^\alpha} \, dx = (-1)^{|\alpha|} \int \frac{\partial^\alpha f_m}{\partial x^\alpha} \varphi \, dx,$$

we get

$$\int f \frac{\partial^\alpha \varphi}{\partial x^\alpha}\, dx = (-1)^{|\alpha|} \int f^{(\alpha)} \varphi \, dx;$$

this shows that $f \in L_k^p$.

The converse is more interesting. The argument that is required for its proof is typical of a great deal of similar reasoning which involves the device of *regularization*.

For this purpose let $\psi$ be a *fixed* element of $\mathscr{D}$, with the property that $\int_{\mathbf{R}^n} \psi(x)\, dx = 1$. For every $\varepsilon > 0$, consider $\psi_\varepsilon(x)$ defined by $\psi_\varepsilon(x) = \varepsilon^{-n}\psi(x/\varepsilon)$, and for each $f \in L^p$, set $f_\varepsilon = f * \psi_\varepsilon$. The family $\{f_\varepsilon\}$ is a regularization of $f$: In the present context this means

(a) $\|f_\varepsilon - f\|_p \to 0$, as $\varepsilon \to 0$

(b) each $f_\varepsilon$ is indefinitely differentiable

(c) if $f$ has a partial derivative $\dfrac{\partial^\alpha f}{\partial x^\alpha}$ (in the weak sense), then

$$\frac{\partial^\alpha f_\varepsilon}{\partial x^\alpha} = \left(\frac{\partial^\alpha f}{\partial x_\alpha}\right)_\varepsilon = \frac{\partial^\alpha f}{\partial x^\alpha} * \psi_\varepsilon.$$

(a) is valid under the more general condition that $\psi$ is integrable as we have already seen in Chapter III, §2.2.

(b) Since $f_\varepsilon(x) = \int_{\mathbf{R}^n} f(y)\psi_\varepsilon(x-y)\, dy$, it is clear by differentiation under the integral sign that $f_\varepsilon$ is indefinitely differentiable.

(c) Let us carry out this differentiation. Then

$$\begin{aligned}\frac{\partial^\alpha}{\partial x^\alpha} f_\varepsilon(x) &= \int_{\mathbf{R}^n} f(y) \frac{\partial^\alpha}{\partial x^\alpha}(\psi_\varepsilon(x-y))\, dy \\ &= (-1)^{|\alpha|}\int_{\mathbf{R}^n} f(y) \frac{\partial^\alpha}{\partial y^\alpha}(\psi_\varepsilon(x-y))\, dy.\end{aligned}$$

For each $x$, the function $y \to \psi_\varepsilon(x-y)$ is in $\mathscr{D}$, and hence an application of the definition (15) gives

$$\frac{\partial^\alpha}{\partial x^\alpha} f_\varepsilon(x) = \int \left(\frac{\partial^\alpha f}{\partial y^\alpha}\right) \psi_\varepsilon(x-y)\, dt = \left(\frac{\partial^\alpha f}{\partial x^\alpha}\right) * \psi_\varepsilon.$$

We can now apply property (a) to $\dfrac{\partial^\alpha f}{\partial x^\alpha} * \psi_\varepsilon$, and we see that $\dfrac{\partial^\alpha f_\varepsilon}{\partial x^\alpha}$ converges in $L^p$ norm as $\varepsilon \to 0$. Thus the functions $\{f_\varepsilon\}$ give the required approximation, except that they do not each have compact support, and hence a final modification is called for. Let $\eta$ be a fixed indefinitely differentiable function of compact support with $\eta(0) = 1$ and consider the two-parameter family $\{\eta(\delta x) f_\varepsilon(x)\}$, $\varepsilon > 0$, $\delta > 0$. Choose then $\varepsilon$

first so that $\frac{\partial^\alpha f_\varepsilon}{\partial x^\alpha}$ are sufficiently close to their limits. Next, with $\varepsilon$ fixed, choose $\delta$ sufficiently small so $\frac{\partial^\alpha}{\partial x^\alpha}(\eta(\delta x)f_\varepsilon(x))$ are sufficiently close to $\frac{\partial^\alpha f_\varepsilon}{\partial x^\alpha}$. Since each $\eta(\delta x)f_\varepsilon(x)$ is indefinitely differentiable with compact support the proposition is then proved.

There is a parallel proposition dealing with case $p = \infty$, but it requires the usual modification since smooth functions are not dense in the $L^\infty(\mathbf{R}^n)$ space.

Alternative characterizations in the case $n = 1$ for all $1 \leq p \leq \infty$, and in the case of general $n$ for $p = \infty$, may be found in §6.1 and §6.2 below.

*2.1.1* As far as the proof of Proposition 1 is concerned, the requirement that $\varphi$ have compact support is not absolutely necessary. We could, e.g., have carried out the proof (with a little sacrifice of elegance) by setting $\varphi(x) = \frac{c_n}{(1 + |x|^2)^{(n+1)/2}}$. Then $f_\varepsilon = f * \varphi_\varepsilon$ would equal $u(x, \varepsilon)$, where $u(x, y)$ is the Poisson integral of $f$. (See Chapter III, §2.)

It is in other problems (see for example, Chapter VI, §3.2.4.) that the regularization with a $\varphi$ of *compact* support plays a more essential role. This type of regularization has the property that $f_\varepsilon(x)$ depends only on the values of $f$ in a small neighborhood of $x$.

**2.2 Sobolov's theorem.** The importance of the Sobolov spaces just considered is that in terms of them we can account in a relatively simple way how restrictions on the "size" of partial derivatives imply corresponding restrictions on the functions in question. A general theorem may be formulated as follows.

THEOREM 2. *Suppose $k$ is a positive integer, and $1/q = 1/p - k/n$.*

(i) *If $q < \infty$ (i.e. $p < n/k$), then $L^p_k(\mathbf{R}^n) \subset L^q(\mathbf{R}^n)$ and the natural inclusion map is continuous.*

(ii) *If $q = \infty$ (i.e. $p = n/k$), then the restriction of an $f \in L^p_k(\mathbf{R}^n)$ to a compact subset of $\mathbf{R}^n$ belongs to $L^r(\mathbf{R}^n)$, for every $r < \infty$.*

(iii) *If $p > n/k$, then every $f \in L^p_k(\mathbf{R}^n)$ can be modified on a set of zero measure so that the resulting function is continuous.*

**2.3** To prove the theorem it is required that we find an appropriate way of expressing a function in terms of its partial derivatives. To do this let us proceed in a purely formal way, operating always with functions of the class $\mathscr{S}$ (or $\mathscr{D}$).

With the testing function $f$ we consider its Fourier transform $\hat{f}$. Then the Fourier transform of $\frac{\partial f}{\partial x_j}$ is $-2\pi i x_j \hat{f}(x)$. Now recall the Riesz transforms of Chapter III, §1.2. The effect of $R_j$ is multiplication on the Fourier transform side by $\frac{i x_j}{|x|}$ (see formula (8)). Thus

$$\left(R_j\left(\frac{\partial}{\partial x_j} f\right)\right)^{\wedge}(x) = 2\pi \frac{x_j^2}{|x|}\hat{f}(x).$$

In view of formula (3) we then get

$$f = I_1\left(\sum_{j=1}^{n} R_j\left(\frac{\partial}{\partial x_j} f\right)\right). \tag{17}$$

This identity, which expresses $f$ in terms of its first partial derivatives, contains two elements: the Riesz transforms and the potential of order one. The former are operators which preserve the class $L^p(\mathbf{R}^n)$, and the latter maps $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$, for appropriate $p$ and $q$ (by Theorem 1). This glimpse reveals the essential mechanism behind the present theorem.

There is, however, a simpler approach which is closely related to the identity (17), but avoids the use of the rather deep theory of the Riesz transforms. It is based on the elementary identity

$$f(x) = \frac{1}{\omega_{n-1}} \sum_{j=1}^{n} \int_{\mathbf{R}^n} \frac{\partial f}{\partial x_j}(x - y) \cdot \frac{y_j}{|y|^n}\, dy, \tag{18}$$

where $\omega_{n-1}$ is the "area" of the sphere $S^{n-1}$.

The formula (18) is proved as follows. We start with the one-dimensional formula,

$$f(x) = \int_0^\infty f'(x - t)\, dt$$

which is certainly valid whenever $f$ is a testing function. Its $n$-dimensional analogue is an immediate consequence of itself, namely

$$f(x) = \int_0^\infty (\nabla f(x - \xi t), \xi)\, dt \tag{19}$$

where $\xi$ is any unit vector, and $\nabla f$ is the vector with components

$$\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots \frac{\partial f}{\partial x_n}\right).$$

We integrate (19) over $\xi$ ranging on the unit sphere. This gives

$$f(x) = \frac{1}{\omega_{n-1}} \int_{\xi \in S^{n-1}} \int_0^\infty (\nabla f(x - \xi t), \xi)\, dt\, d\xi.$$

Changing from polar coordinates to rectangular coordinate yields (18). The formula (18) will be applied below. It may be worthwhile, however, to take this opportunity to point out some other identities of the same kind as (17), each of which is transparent on the formal level.

First, suppose we wish to express $f$ in terms of its *second* partial derivatives. Then we can do it in terms of the particular combination

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = \Delta f.$$

The identity is

$$f = -I_2(\Delta f)$$

which is just a special case of formula (7). We get in this way a classical formula of potential theory.*

Another useful observation is:

$$\text{(20)} \qquad \text{If} \quad F = I_1(f), \quad \text{then} \quad \frac{\partial F}{\partial x_j} = -R_j(f).$$

In fact according to Lemma 1, with $g = \dfrac{\partial \varphi}{\partial x_j}$, we have

$$\int I_1(f)\,\overline{\frac{\partial \varphi}{\partial x_j}}\,dx = \int \hat{f}(x)(2\pi|x|)^{-1}\overline{2\pi i x_j \hat{\varphi}(x)}\,dx$$

$$= \int \hat{f}(x) i\frac{x_j}{|x|}\,\bar{\hat{\varphi}}(x)\,dx = \int R_j(f)\bar{\varphi}\,dx.$$

So $\displaystyle\int I_1(f)\frac{\partial \bar{\varphi}}{\partial x_j}\,dx = \int R_j(f)\bar{\varphi}\,dx$, and therefore (20) is proved, at least when $f \in \mathscr{S}$.

It is also possible to extend (20) to wider classes of functions, as the need may arise (see §6.3 below).

**2.4** We prove the theorem first in the case $k = 1$, and where $1 < p$, $q < \infty$. Assume that $f \in \mathscr{D}$. Then the identity (18) shows that

$$|f(x)| \le A \sum_{j=1}^{n} \int_{\mathbf{R}^n} \left|\frac{\partial f}{\partial x_j}(x-y)\right| |y|^{-n+1}\,dy.$$

Therefore by Theorem 1, (the case $\alpha = 1$), we get

$$\text{(21)} \qquad \|f\|_q \le A' \sum_{j=1}^{n} \left\|\frac{\partial f}{\partial x_j}\right\|_p, \qquad 1/q = 1/p - 1/n.$$

* At least in the case $n \ge 3$. The case $n = 2$ must then be studied separately by a limiting argument, and gives the representation in terms of the logarithmic potential.

Let now $f$ be any function in $L_1^p(\mathbf{R}^n)$. According to Proposition 1, there exists a sequence of elements of $\mathscr{D}$, $\{f_m\}$, so that $f_m \to f$ in $L^p$ norm, and $\dfrac{\partial f_m}{\partial x_j}$ converges in $L^p$ norm. The limit $\lim\limits_{m\to\infty} \dfrac{\partial f_m}{\partial x_j}$ must equal $\dfrac{\partial f}{\partial x_j}$, since $\lim\limits_m \int \dfrac{\partial f_m}{\partial x_j} \varphi\, dx = -\lim\limits_m \int f_m \dfrac{\partial \varphi}{\partial x_j}\, dx = -\int f \dfrac{\partial \varphi}{\partial x_j}\, dx = \int \dfrac{\partial f}{\partial x_j} \varphi\, dx$, for every $\varphi \in \mathscr{D}$. Substituting in (21) we get

$$\|f_m - f_{m'}\|_q \leq A' \left\| \sum_{j=1}^n \frac{\partial f_m}{\partial x_j} - \frac{\partial f_{m'}}{\partial x_j} \right\|_p,$$

and so the sequence $f_m$ also converges in $L^q(\mathbf{R}^n)$ norm, and this limit must also equal $f$. Thus $f \in L^q(\mathbf{R}^n)$, and

$$\|f\|_q \leq A' \sum_{j=1}^n \left\| \frac{\partial f}{\partial x_j} \right\|_p \leq A' \, \|f\|_{L^p{}_1(\mathbf{R})^n} \quad , \quad f \in L_1^p(\mathbf{R}^n).$$

This shows that $f \in L^q(\mathbf{R}^n)$ and the inclusion mapping of $L_1^p(\mathbf{R}^n)$ into $L^q(\mathbf{R}^n)$ is continuous.

We consider next the situation when $k = 1$, but $q = \infty$ ($p = n/k = n$), or $p > n/k = n$. In both instances the relevant conclusions of Theorem 2 (that is, (ii) and (iii)) are local in character and so we may simplify matters by reducing to the case when $f$ (and therefore its partial derivatives) have compact support. Thus given any fixed compact set, $K$, let $\eta$ be a function in $\mathscr{D}$ which is one on that set. If $f \in L_1^p(\mathbf{R}^n)$, consider $\eta \cdot f$.

It will be enough to prove the conclusions (ii) and (iii) for $\eta f$, which incidentally also belongs to $L_1^p(\mathbf{R}^n)$. To see that $\eta f \in L_1^p(\mathbf{R}^n)$ it suffices to verify that the derivative $\dfrac{\partial}{\partial x_j}(\eta f)$, in the weak sense, equals $\dfrac{\partial \eta}{\partial x_j} f + \eta \dfrac{\partial f}{\partial x_j}$. However

$$\begin{aligned}
\int \frac{\partial}{\partial x_j}(\eta f)\varphi\, dx &= -\int \eta f \frac{\partial \varphi}{\partial x_j}\, dx \\
&= -\int f \frac{\partial(\varphi\eta)}{\partial x_j}\, dx + \int \varphi f \frac{\partial \eta}{\partial x_j}\, dx \\
&= \int \left( \frac{\partial \eta}{\partial x_j} f + \eta \frac{\partial f}{\partial x_j} \right) \varphi\, dx
\end{aligned}$$

and this assertion is proved.

We start therefore with $f \in L_1^p(\mathbf{R}^n)$ and its approximating sequence $\{f_m\}$ given by Proposition 1. Thus clearly $\eta f_m$ is an approximating sequence to $\eta f$. Now choose an $R$ so large that if $K_1$ is the (compact) support of $\eta$, then the set $K_1 - K_1$ is contained in the ball of radius $R$ about the origin.

It then follows again by the identity (18) that

$$(22)\quad |\eta(x)f_m(x)| \le A\sum_{j=1}^{n}\int_{|y|\le R}\left|\frac{\partial(\eta(x)f_m)}{\partial x_j}(x-y)\right| |y|^{-n+1}\,dy, \qquad x \in K.$$

We now use Young's inequality which states that if

$$\mathscr{C}(x) = \int_{\mathbf{R}^n} \mathscr{A}(x-y)\mathscr{B}(y)\,dy,$$

then $\|\mathscr{C}\|_r \le \|\mathscr{A}\|_p\,\|\mathscr{B}\|_s$, where $1/r = 1/p + 1/s - 1$. We set $\mathscr{A} = A\sum_{j=1}^{n}\left|\frac{\partial}{\partial x_j}(\eta f_m)\right|$; $\mathscr{B}(y) = |y|^{-n+1}$ if, $|y| \le R$, $\mathscr{B}(y) = 0$ otherwise; and we let $s$ be any exponent $< n/(n-1)$. Notice that then $\|\mathscr{B}\|_s < \infty$, since $\|\mathscr{B}\|_s^s = \int_{|y|<R} |y|^{(-n+1)s}\,dy < \infty$, because $(-n+1)s > -n$.

Hence Young's inequality shows that

$$\int_K |f_m|^r\,dx \le \int |\eta f_m|^r\,dx \le \|\mathscr{C}\|_r^r \le A' \left\|\sum_{j=1}^{n}\left|\frac{\partial(\eta f_m)}{\partial x_j}\right|\right\|_p^r.$$

Similarly,

$$\int_K |f_m - f_{m'}|^r\,dx \le A'\left\|\sum_{j=1}^{n}\left|\frac{\partial(\eta(f_m - f_{m'}))}{\partial x_j}\right|\right\|_p.$$

So we see that the sequence $\{f_m\}$, which converges to $f$ in the $L^p$ norm, also converges in the $L^r$ norm, when restricted to the set $K$. Thus $f$ is in $L^r$, when restricted to $K$.

In the present case $p = n$, and so the condition $s < n/(n-1)$, is the same as $r < \infty$, since $1/r = 1/p + 1/s - 1$. Therefore the assertion (ii) of the theorem is proved (assuming of course $k = 1$).

The argument for the proof (iii) is very similar to that just carried out, except here we use the estimate

$$\sup_x |\mathscr{C}(x)| \le \|\mathscr{A}\|_p\,\|\mathscr{B}\|_{p'}, \qquad 1/p + 1/p' = 1$$

which follows trivially from Hölder's inequality. Notice that if $p > n$, then $\|\mathscr{B}\|_{p'}^{p'} = \int_{|y|<R} |y|^{(-n+1)p'}\,dy < \infty$, so we get, in analogy with the above

$$\sup_{x\in K} |f_m(x) - f_{m'}(x)| \le A'\left\|\sum_{j=1}^{n}\left|\frac{\partial\eta(f_m - f_{m'})}{\partial x_j}\right|\right\|_p.$$

This shows that the continuous functions $\{f_m(x)\}$ converge uniformly on every compact set and hence $f$ may be taken to be continuous.

**2.5 The case $p = 1$.** With the assumption that $k = 1$, the assertions of the theorem have been completely proved save for the exceptional case

$p = 1$. The argument used so far will of course not work in this case because conclusion (b) of Theorem 1 fails for $p = 1$. A different idea is needed in this circumstance and it is contained in the following elegant inequality:

$$\|f\|_q \leq \left(\prod_{j=1}^{n} \left\|\frac{\partial f}{\partial x_j}\right\|_1\right)^{1/n}, \qquad 1/q = 1 - 1/n, \quad f \in \mathcal{S}. \tag{23}$$

We prove (23) by induction on $n$. The case $n = 1$ is trivial because it states $\|f\|_\infty \leq \|f'\|_1$, which follows immediately from $f(x) = \int_{-\infty}^{x} f'(t)\,dt$.

We assume therefore that the inequality (23) is valid for $n - 1$. For the purposes of the induction, we write $x \in \mathbf{R}^n$, as $x = (x_1, x')$, with $x' \in \mathbf{R}^{n-1}$, $x_1 \in \mathbf{R}^1$. We set

$$I_j(x_1) = \int_{\mathbf{R}^{n-1}} \left|\frac{\partial f}{\partial x_j}(x_1, x')\right| dx', \qquad j = 2, \ldots, n,$$

and

$$I_1(x') = \int_{\mathbf{R}^1} \left|\frac{\partial f}{\partial x_1}(x_1, x')\right| dx_1.$$

Suppose now that $q$ is the index that corresponds to $n$, $(q = n/(n-1))$ and $q'$ the index that corresponds to $n - 1$, $(q' = (n-1)/(n-2))$. Then by the case $n - 1$, we have

$$\left(\int_{\mathbf{R}^{n-1}} |f(x_1, x')|^{q'}\,dx'\right)^{1/q'} \leq \left(\prod_{j=2}^{n} I_j(x_1)\right)^{1/(n-1)}. \tag{24}$$

Clearly, however, $|f(x)| \leq I_1(x')$ (this is the one-dimensional case again!), so $|f|^q \leq (I_1(x'))^{1/(n-1)}|f|$, since

$$q = \frac{1}{n-1} + 1.$$

Thus

$$\begin{aligned}\int_{\mathbf{R}^{n-1}} |f|^q\,dx' &\leq \int_{\mathbf{R}^{n-1}} (I_1(x'))^{1/(n-1)}|f|\,dx' \\ &\leq \left(\int_{\mathbf{R}^{n-1}} I_1(x')\,dx'\right)^{1/(n-1)} \left(\int_{\mathbf{R}^{n-1}} |f|^{q'}\,dx'\right)^{1/q'}\end{aligned}$$

by Hölder's inequality with the conjugate exponents $n - 1$, and $q'$.

Substituting (24) in the above gives

$$\int_{\mathbf{R}^{n-1}} |f|^q\,dx' \leq \left(\int_{\mathbf{R}^{n-1}} I_1(x')\,dx'\right)^{1/(n-1)} \left(\prod_{j=2}^{n} I_j(x_1)\right)^{1/(n-1)}.$$

We integrate this with respect to $x_1$ and use Hölder's inequality again, to wit,

$$\int_{\mathbf{R}^1}\left(\prod_{j=2}^{n} I_j(x_1)\right)^{1/(n-1)} dx_1 \le \prod_{j=2}^{n}\left(\int_{\mathbf{R}^1} I_j(x_1)\,dx_1\right)^{1/(n-1)} = \prod_{j=2}^{n}\left\|\frac{\partial f}{\partial x_j}\right\|_1^{1/(n-1)}.$$

The final result is

$$\int_{\mathbf{R}^n} |f|^q\,dx \le \left(\prod_{j=1}^{n}\left\|\frac{\partial f}{\partial x_j}\right\|_1\right)^{1/(n-1)}$$

which is the desired inequality (23), since $q = n/(n-1)$.

If we use the fact that

$$\left(\prod_{j=1}^{n} a_j\right)^{1/n} \le \frac{1}{n}\sum_{j=1}^{n} a_j, \quad \text{if} \quad a_j \ge 0,$$

then as a consequence of (23) we have

$$\|f\|_q \le \frac{1}{n}\sum_{j=1}^{n}\left\|\frac{\partial f}{\partial x_j}\right\|_1, \qquad f \in \mathcal{S}, \quad 1/q = 1 - 1/n.$$

This result and the reasoning used in §2.4 above, shows that $L^q(\mathbf{R}^n) \subset L_1^1(\mathbf{R}^n)$, and that the inclusion map is continuous.

**2.6** To conclude the proof of the theorem we can argue by induction and show that the case of $k \ge 2$ can be reduced to the case $k = 1$. Let us take, for example, the assertion (i) of the theorem. The assumption $f \in L_k^p(\mathbf{R}^n)$ clearly implies that $f \in L_{k-1}^p(\mathbf{R}^n)$ and $\dfrac{\partial f}{\partial x_j} \in L_{k-1}^p(\mathbf{R}^n)$. Hence the case of the theorem for $k - 1$ implies that $f \in L^{q'}(\mathbf{R}^n)$ and $\dfrac{\partial f}{\partial x_j} \in L^{q'}(\mathbf{R}^n)$, where $1/q' = 1/p - \dfrac{(k-1)}{n}$. That is, $f \in L_1^{q'}(\mathbf{R}^n)$. The case $k = 1$ then implies that $f \in L^q(\mathbf{R}^n)$, with $1/q = 1/q' - 1/n = 1/p - \left(\dfrac{k-1}{n}\right) - 1/n = 1/p - k/n$. The corresponding inclusion mappings are also continuous. The cases (ii) and (iii) can be argued similarly.

A final remark is in order. Theorem 2 is valid for $p = 1$, unlike the closely related Theorem 1. However when $q = \infty$, (the case (ii)), it is not true that in general $f \in L^\infty$; (here the situation is again similar to that of Theorem 1). For further details see §6.3 at the end of this chapter.

## 3. *Bessel potentials*

**3.1** The Riesz potentials $I_\alpha$ lead to very elegant and useful formulae, as we have already seen. Nevertheless the present formalism suffers from a shortcoming which may be explained as follows. The importance of the potentials $I_\alpha$ lies above all in their role as "smoothing operators." While the *local* behavior, ($|x| \to 0$), of the kernels $\frac{|x|^{-n+\alpha}}{\gamma(\alpha)}$ is suited to this purpose, the *global* behavior ($|x| \to \infty$) is less favorable and leads to increasing awkwardness the greater $\alpha$ is.

A way out of this dilemma is by a modification of the Riesz potentials which maintains the essential local behavior but eliminates the irrelevant problems of infinity. There are several roughly equivalent ways of doing this, but the simplest and most natural approach consists in replacing the "non-negative" operator $-\Delta$, by the "strictly positive" operator $I - \Delta$, ($I$ = identity) and defining the *Bessel potentials* $\mathscr{J}_\alpha$ by

$$\mathscr{J}_\alpha = (I - \Delta)^{-\alpha/2}$$

in analogy with

$$I_\alpha = (-\Delta)^{-\alpha/2}.$$

To put matters in logical order, we must begin by deriving the kernel of the Bessel potential, that is the presumed function $G_\alpha(x)$, with the property that $(G_\alpha(x)) = (1 + 4\pi^2 |x|^2)^{-\alpha/2}$.

The starting point for this derivation is the idea (already used in Chapter III, §3.2) that a "general" function of $|x|$ can be expressed in terms of the $\{e^{-\pi\delta|x|^2}\}_{\delta>0}$. In this instance we have a simple identity, namely

$$(25)\quad (4\pi)^{-\alpha/2}(1 + 4\pi^2 |x|^2)^{-\alpha/2} = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-\frac{\delta}{4\pi}(1+4\pi^2|x|^2)} \delta^{\alpha/2} \frac{d\delta}{\delta}, \quad \alpha > 0$$

which is nothing but a rephrasing of the fact that

$$t^{-a} = \frac{1}{\Gamma(a)} \int_0^\infty e^{-t\delta} \delta^a \frac{d\delta}{\delta}$$

with $a = \alpha/2 > 0$.

We therefore set down the following table, where the entries on the right are the Fourier transforms of the corresponding entries on the left, with $a = \alpha/2$.

| | | |
|---|---|---|
| (i) | $e^{-\pi\|x\|^2}$ | $e^{-\pi\|x\|^2}$ |
| (ii) | $e^{-\pi\delta\|x\|^2}$ | $e^{-\pi\|x\|^2/\delta}\delta^{-n/2}$ |
| (iii) | $\int_0^\infty e^{-\pi\delta\|x\|^2} \delta^a \frac{d\delta}{\delta}$ | $\int_0^\infty e^{-\pi\|x\|^2/\delta} \delta^{-n/2} \delta^a \frac{d\delta}{\delta}$ |
| (iv) | $\Gamma(a)(\pi\|x\|^2)^{-a}$ | $\Gamma(n/2 - a)(\pi\|x\|^2)^{n/2-a}$ |

By (25) and (ii)

| | | |
|---|---|---|
| (v) | $(1 + 4\pi^2\|x\|^2)^{-\alpha/2}$ | $\frac{1}{(4\pi)^{\alpha/2}} \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-\pi\|x\|^2/\delta} e^{-\delta/4\pi} \delta^{(-n+\alpha)/2} \frac{d\delta}{\delta}$ |

Therefore we *define* $G_\alpha(x)$ by

$$(26) \qquad G_\alpha(x) = \frac{1}{(4\pi)^{\alpha/2}} \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-\pi|x|^2/\delta} e^{-\delta/4\pi} \delta^{(-n+\alpha)/2} \frac{d\delta}{\delta}.$$

PROPOSITION 2.

(1) *For each* $\alpha > 0$, $G_\alpha(x) \in L^1(\mathbf{R}^n)$.

(2) $G_\alpha\hat{}(x) = (1 + 4\pi^2 |x|^2)^{-\alpha/2}$.

Proof.

Since $\int_{\mathbf{R}^n} e^{-\pi|x|^2/\delta}\, dx = \delta^{n/2}$, Fubini's theorem applied to (26) shows

$$\int_{\mathbf{R}^n} G_\alpha(x)\, dx = \frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_0^\infty e^{-\delta/4\pi} \delta^{\alpha/2} \frac{d\delta}{\delta} = 1, \qquad \alpha > 0,$$

and so the first conclusion is demonstrated. To prove the second conclusion we use the reasoning schematized by (i)–(v) above. In fact if we let $f$ stand for one of the entries on the left and $\hat{f}$ for the corresponding entry on the right, we have whenever $\varphi \in \mathcal{S}$

$$(27) \qquad \int_{\mathbf{R}^n} f(x)\hat{\varphi}(x)\, dx = \int_{\mathbf{R}^n} \hat{f}(x)\varphi(x)\, dx.$$

First we take $f(x) = e^{-\pi\delta|x|^2}$, $\hat{f}(x) = e^{-\pi|x|^2/\delta}\delta^{-n/2}$; then $f(x) = e^{-\delta/4\pi}e^{-\pi|x|^2}$, $\hat{f}(x) = e^{-\delta/4\pi}e^{-\pi|x|^2/\delta}\delta^{-n/2}$. With this choice of $f$ and $\hat{f}$ we integrate both sides with respect to $\delta^{\alpha/2}\, d\delta/\delta$ (see (25)). An interchange of the order of integration (validated by Fubini's theorem) then shows that

$$\int_{\mathbf{R}^n} (1 + 4\pi^2 |x|^2)^{-\alpha/2}\varphi(x)\, dx = \int_{\mathbf{R}^n} G_\alpha(x)\hat{\varphi}(x)\, dx.$$

Since $G_\alpha \in L^1(\mathbf{R}^n)$ this shows that $\hat{G}_\alpha(x) = (1 + 4\pi^2 |x|^2)^{-\alpha/2}$.

A result similar to (26) which also follows from the above table is

$$(28) \qquad \frac{|x|^{-n+\alpha}}{\gamma(\alpha)} = \frac{1}{(4\pi)^{\alpha/2}} \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-\pi|x|^2/\delta} \delta^{(-n+\alpha)/2} \frac{d\delta}{\delta}.$$

If we use the fact that $e^{-\delta/4\pi} = 1 + o(e^{-\delta/4\pi})$, $\delta \to 0$, we get, upon comparing (28) with (26), that

$$(29) \qquad G_\alpha(x) = \frac{|x|^{-n+\alpha}}{\gamma(\alpha)} + o(|x|^{-n+\alpha}), \quad \text{as} \quad |x| \to 0,$$

if $0 < \alpha < n$.

A straightforward examination of the integral defining $G_\alpha(x)$ also shows that

$$(30) \qquad G_\alpha(x) = O(e^{-c|x|}) \quad \text{as} \quad |x| \to \infty, \quad \text{for some} \quad c > 0,$$

so that the kernel $G_\alpha$ is rapidly decreasing as $|x| \to \infty$.

Notice that $e^{-\pi|x|^2/\delta}e^{-\delta/4\pi}$ has as a maximum value $e^{-|x|}$ (which it attains at $\delta = 2\pi|x|$). Also if $|x| \geq 1$ then clearly $e^{-\pi|x|^2/\delta}e^{-\delta/4\pi} \leq e^{-\pi/\delta}e^{-\delta/4\pi}$. Combining the two gives, when $|x| \geq 1$, $e^{-\pi|x|^2/\delta}e^{-\delta/4\pi} \leq e^{-|x|/2}e^{-\pi/2\delta}e^{-\delta/8\pi}$. Now inserting this in the defining formula (30) yields

$$(4\pi)^{\alpha/2}\Gamma(\alpha/2)G_\alpha(x) \leq e^{-|x|/2}\int_0^\infty e^{-\pi/2\delta}e^{-\delta/8\pi}\,\delta^{(-n+\alpha)/2}\,\frac{d\delta}{\delta}$$

which is (30), with $c = 1/2$.

*3.1.1* The kernel $G_\alpha$ may also be given another integral representation which shows that it is essentially a Bessel function of the "third kind" (see §6.5 below); that was the original derivation. We shall, however, not need any of the properties of the Bessel functions and so the terminology of "Bessel potential" has for us only a vestigial significance.

**3.2 Relation between Riesz and Bessel potentials.** It may be surmised from its very definition, and also from the asymptotic relation (29), that there is an intimate connection between the Bessel potentials and the Riesz potentials. This affinity between the two is given precision in the following lemma.

LEMMA 2. *Let* $\alpha > 0$.

(i) *There exists a finite measure* $\mu_\alpha$ *on* $\mathbf{R}^n$ *so that its Fourier transform* $\hat{\mu}_\alpha$ *is given by*

$$\hat{\mu}_\alpha(x) = \frac{(2\pi|x|)^\alpha}{(1 + 4\pi^2|x|^2)^{\alpha/2}}.$$

(ii) *There exist a pair of finite measures* $\nu_\alpha$ *and* $\lambda_\alpha$ *on* $\mathbf{R}^n$ *so that*

$$(1 + 4\pi^2|x|^2)^{\alpha/2} = \hat{\nu}_\alpha(x) + (2\pi|x|)^\alpha\hat{\lambda}_\alpha(x).$$

The first part of the lemma states in effect that the following formal quotient operator is bounded on every $L^p(\mathbf{R}^n)$, $1 \leq p \leq \infty$,

$$\frac{(-\Delta)^{\alpha/2}}{(I - \Delta)^{\alpha/2}}, \qquad \alpha > 0. \tag{31}$$

The second part states also to what extent the same thing is true of the operator inverse to (31).

To prove (i) we use the expansion

$$(1 - t)^{\alpha/2} = 1 + \sum_{m=1}^{\infty} A_{m,\alpha}t^m, \tag{32}$$

which is valid when $|t| < 1$. All the $A_{m,\alpha}$ are of constant sign for $m$ sufficiently large, so $\sum |A_{m,\alpha}| < \infty$, since $(1 - t)^{\alpha/2}$ remains bounded as $t \to 1$, (if $\alpha \geq 0$). Let $t = \dfrac{1}{1 + 4\pi^2 |x|^2}$. Then

$$\left(\frac{4\pi^2 |x|^2}{1 + 4\pi^2 |x|^2}\right)^{\alpha/2} = 1 + \sum_{m=1}^{\infty} A_{m,\alpha}(1 + 4\pi^2 |x|^2)^{-m}. \tag{33}$$

However $G_{2m}(x) \geq 0$ and $\int_{\mathbf{R}^n} G_{2m}(x) e^{2\pi i x\cdot y}\, dx = (1 + 4\pi^2 |y|^2)^{-m}$.
We noticed already that $\int G_{2m}(x)\, dx = 1$ and so $\|G_{2m}\|_1 = 1$.
Thus from the convergence of $\sum |A_{m,\alpha}|$ it follows that if $\mu_\alpha$ is defined by

$$\mu_\alpha = \delta_0 + \left(\sum_{m=1}^{\infty} A_{m,\alpha} G_{2m}(x)\right) dx \tag{34}$$

with $\delta_0$ the Dirac measure at the origin, then $\mu_\alpha$ represents a finite measure. Moreover by (33),

$$\hat{\mu_\alpha}(x) = \frac{(2\pi |x|)^\alpha}{(1 + 4\pi^2 |x|^2)^{\alpha/2}}. \tag{35}$$

We now invoke the $n$-dimensional version of Weiner's theorem, to wit: If $\Phi_1 \in L^1(\mathbf{R}^n)$ and $\hat{\Phi_1}(x) + 1$ is nowhere zero, then there exists a $\Phi_2 \in L^1(\mathbf{R}^n)$ so that $(\hat{\Phi_1}(x) + 1)^{-1} = \hat{\Phi_2}(x) + 1$.
For our purposes we then write

$$\Phi_1(x) = \sum_{m=1}^{\infty} A_{m,\alpha} G_{2m}(x) + G_\alpha(x).$$

Then by (35) we see that

$$\hat{\Phi_1}(x) + 1 = \frac{(2\pi |x|)^\alpha + 1}{(1 + 4\pi^2 |x|^2)^{\alpha/2}},$$

which vanishes nowhere. Thus for an appropriate $\Phi_2 \in L^1$,

$$(1 + 4\pi^2|x|^2)^{\alpha/2} = (1 + (2\pi|x|)^\alpha)[\hat{\Phi_2}(x) + 1],$$

and so we obtain the desired conclusion with $\nu_\alpha = \lambda_\alpha = \delta_0 + \Phi_2(x)\, dx$.

**3.3 Spaces $\mathscr{L}^p_\alpha$.** It is now our intention to study more systematically the one-parameter family of operators $\{\mathscr{J}_\alpha\}_\alpha$.

For any $\alpha \geq 0$, and $f \in L^p(\mathbf{R}^n)$, $1 \leq p \leq \infty$, we can define $\mathscr{J}_\alpha(f)$, as $\mathscr{J}_\alpha(f) = G_\alpha * f$, if $\alpha > 0$, and $\mathscr{J}_0(f) = f$. In view of the fact that $\|G_\alpha\|_1 = 1$ $(= \int_{\mathbf{R}^n} G_\alpha(x)\, dx)$, we see that the convolution is in fact well-defined and

$$\|\mathscr{J}_\alpha(f)\|_p \leq \|f\|_p, \qquad 1 \leq p \leq \infty. \tag{36}$$

It is also apparent that

$$(37) \qquad \mathscr{J}_\alpha \cdot \mathscr{J}_\beta = \mathscr{J}_{\alpha+\beta}, \qquad \alpha \geq 0, \beta \geq 0$$

since $G_\alpha * G_\beta = G_{\alpha+\beta}$, as Proposition 2 shows.

The main definition that we wish to make here is that of the *potential spaces*, $\mathscr{L}^p_\alpha$. Symbolically we write

$$(38) \qquad \mathscr{L}^p_\alpha(\mathbf{R}^n) = \mathscr{J}_\alpha(L^p(\mathbf{R}^n)), \qquad 1 \leq p \leq \infty, \quad \alpha \geq 0.$$

In other words, $\mathscr{L}^p_\alpha(\mathbf{R}^n)$ is a subspace of $L^p(\mathbf{R}^n)$, consisting of all $f$ which can be written in the form $f = \mathscr{J}_\alpha(g)$, $g \in L^p(\mathbf{R}^n)$. The $\mathscr{L}^p_\alpha$ norm of $f$ is written as $\|f\|_{p,\alpha}$, and is defined to be the $L^p$ norm of $g$, i.e.,

$$(39) \qquad \|f\|_{p,\alpha} = \|g\|_p, \quad \text{if} \quad f = \mathscr{J}_\alpha(g).$$

To see that this gives a consistent definition of $\|f\|_{p,\alpha}$ we must observe that if $\mathscr{J}_\alpha(g_1) = \mathscr{J}_\alpha(g_2)$, then $g_1 = g_2$. However, if $\varphi \in \mathscr{S}$,

$$\int \mathscr{J}_\alpha(g_1)\varphi(x)\,dx = \int\int G_\alpha(x-y)g_1(y)\varphi(x)\,dx\,dy = \int g_1\mathscr{J}_\alpha(\varphi)\,dx,$$

by Fubini's theorem. So $\mathscr{J}_\alpha(g_1) = \mathscr{J}_\alpha(g_2)$ implies

$$\int_{\mathbf{R}^n} (g_1 - g_2)\mathscr{J}_\alpha(\varphi)\,dx = 0, \quad \text{all} \quad \varphi \in \mathscr{S}.$$

Now the mapping $\mathscr{J}_\alpha$ is actually an onto mapping of $\mathscr{S}$ to itself. In fact, suppose $\psi \in \mathscr{S}$ is given, and take $\hat{\varphi}(x) = \hat{\psi}(x)(1 + 4\pi^2|x|^2)^{-\alpha/2}$. Then since $\hat{\psi} \in \mathscr{S}$, so is $\hat{\varphi}$, and hence $\varphi \in \mathscr{S}$. But $\hat{\psi}(x) = \hat{\varphi}(x)(1 + 4\pi^2|x|^2)^{\alpha/2}$. Therefore $\psi = \mathscr{J}_\alpha(\varphi)$. This shows $\int (g_1 - g_2)\psi\,dx = 0$ for all $\psi \in \mathscr{S}$ and therefore $g_1 = g_2$.

It is an immediate consequence of the definition and (36) that

$$(40) \qquad \mathscr{L}^p_\beta \subset \mathscr{L}^p_\alpha, \quad \textit{and} \quad \|f\|_{p,\alpha} \leq \|f\|_{p,\beta} \quad \text{if} \quad \beta > \alpha.$$

Also

(41) $\mathscr{J}_\beta$ *is an isomorphism of* $\mathscr{L}^p_\alpha$ *to* $\mathscr{L}^p_{\alpha+\beta}$, if $\alpha \geq 0, \beta \geq 0$.

After this rehearsal of the routine related to the spaces $\mathscr{L}^p_\alpha$, we come back to matters of greater consequence. We return to the germinal idea, already exploited in §2: that of the connection between potentials and partial derivatives. In the present context it takes the form of a close connection between the scale of potential spaces $\mathscr{L}^p_\alpha(\mathbf{R}^n)$ and the scale of Sobolov spaces $L^p_k(\mathbf{R}^n)$.

THEOREM 3. *Suppose $k$ is a positive integer and $1 < p < \infty$. Then*

$$\mathscr{L}^p_k(\mathbf{R}^n) = L^p_k(\mathbf{R}^n)$$

*in the sense that* $f \in \mathscr{L}_k^p(\mathbf{R}^n)$ *if and only if* $f \in L_k^p(\mathbf{R}^n)$, *and the two norms, given respectively in (39) and (16), are equivalent.*

This identity between the spaces $\mathscr{L}_k^p$ and $L_k^p$ fails when $p = 1$ or $p = \infty$. See the discussion in §6.6 below.

**3.4 Proof of Theorem 3.** The proof of the theorem is based on the following lemma.

LEMMA 3. *Suppose* $1 < p < \infty$, *and* $\alpha \geq 1$. *Then* $f \in \mathscr{L}_\alpha^p(\mathbf{R}^n)$ *if and only if* $f \in \mathscr{L}_{\alpha-1}^p(\mathbf{R}^n)$ *and for each* $j$, $\dfrac{\partial f}{\partial x_j} \in \mathscr{L}_{\alpha-1}^p(\mathbf{R}^n)$. *Moreover, the two norms,* $\|f\|_{p,\alpha}$ *and* $\|f\|_{p,\alpha-1} + \sum_{j=1}^n \left\|\dfrac{\partial f}{\partial x_j}\right\|_{p,\alpha-1}$ *are equivalent.*

Assume first that $f \in \mathscr{L}_\alpha^p(\mathbf{R}^n)$. Then $f = \mathscr{J}_\alpha(g)$ with $g \in L^p$.
We claim that

$$(42)\qquad \frac{\partial f}{\partial x_j} = \mathscr{J}_{\alpha-1}(g^{(j)}), \quad \text{where} \quad g^{(j)} = -R_j(\mu_1 * g), \quad \text{if} \quad f = \mathscr{J}_\alpha(g).$$

This is immediately verifiable when $g$ (and then $f$) are in $\mathscr{S}$. In fact, in that case

$$\begin{aligned}\left(\frac{\partial f}{\partial x_j}\right)^{\wedge}(x) &= -2\pi i x_j \hat{f}(x)\\ &= -2\pi i x_j (1 + 4\pi^2 |x|^2)^{-\alpha/2}\hat{g}(x)\\ &= (1 + 4\pi^2|x|^2)^{-(\alpha-1)/2}\hat{g}^j(x),\end{aligned}$$

where $\hat{g}^j(x) = -\dfrac{ix_j}{|x|}\dfrac{(2\pi|x|)}{(1 + 4\pi^2|x|^2)^{1/2}}\hat{g}(x)$. This proves (42), when $g \in \mathscr{S}$. In the general case, if $g \in L^p(\mathbf{R}^n)$, there exists a sequence $g_m \in \mathscr{S}$, so that $g_m \to g$ in $L^p$ norm. The mapping $g \to \mu_1 * g$, and consequently the mapping $g \to R_j(\mu_1 * g)$ is bounded in the $L^p$ norm. The first is bounded since $\mu_1$ is a finite measure, according to Lemma 2; the second is bounded since when $1 < p < \infty$, the Riesz transforms $R_j$ are bounded (see Chapter II, and Chapter III, §1). This shows that the sequence $\left\{\dfrac{\partial f_m}{\partial x_j}\right\}$ converges in the $\mathscr{L}_{\alpha-1}^p$ norm and that (42) holds. Thus $\dfrac{\partial f}{\partial x_j} \in \mathscr{L}_{\alpha-1}^p$ and $\sum_{j=1}^n \left\|\dfrac{\partial f}{\partial x_j}\right\|_{p,\alpha-1} = \sum_{j=1}^n \|g^{(j)}\|_p \leq A_p \|g\|_p = A_p \|f\|_{p,\alpha}$. Combining this with the trivial estimate that $\|f\|_{p,\alpha-1} \leq \|f\|_{p,\alpha}$, (see (40)), we get

$$(43)\qquad \|f\|_{p,\alpha-1} + \sum_{j=1}^n \left\|\frac{\partial f}{\partial x_j}\right\|_{p,\alpha-1} \leq A_p^1 \|f\|_{p,\alpha}.$$

To prove the converse we first observe that if $f$, and the $\dfrac{\partial f}{\partial x_j}$ are all in $\mathscr{L}^p_{\alpha-1}$, then

$$f = \mathscr{J}_{\alpha-1}(g), \quad \text{and} \quad \frac{\partial f}{\partial x_j} = \mathscr{J}_{\alpha-1}\left(\frac{\partial g}{\partial x_j}\right) \tag{44}$$

where the $\dfrac{\partial g}{\partial x_j}$ exist in the weak sense and $g$ and $\dfrac{\partial g}{\partial x_j}$ belong to $L^p$.

In fact, suppose $\dfrac{\partial f}{\partial x_j} = \mathscr{J}_{\alpha-1}(g^{(j)})$. We let $\varphi'$ and $\varphi \in \mathscr{S}$. Then

$$\int_{\mathbf{R}^n} f\varphi'\,dx = \int_{\mathbf{R}^n} \mathscr{J}_{\alpha-1}(g)\varphi'\,dx = \int_{\mathbf{R}^n} g\mathscr{J}_{\alpha-1}(\varphi')\,dx.$$

Similarly $\displaystyle\int_{\mathbf{R}^n} \frac{\partial f}{\partial x_j}\varphi\,dx = \int_{\mathbf{R}^n} g^{(j)}\mathscr{J}_{\alpha-1}(\varphi)\,dx$. But

$$\int_{\mathbf{R}^n} f\frac{\partial \varphi}{\partial x_j}\,dx = -\int_{\mathbf{R}^n}\frac{\partial f}{\partial x_j}\varphi\,dx;$$

this holds whenever $\varphi \in \mathscr{D}$; and a simple limiting argument extends it to $\varphi$ which are in $\mathscr{S}$. Therefore, with $\varphi' = \dfrac{\partial \varphi}{\partial x_j}$,

$$\int_{\mathbf{R}^n} g\frac{\partial}{\partial x_j}(\mathscr{J}_{\alpha-1}\varphi)\,dx = -\int_{\mathbf{R}^n} g^{(j)}\mathscr{J}_{\alpha-1}(\varphi)\,dx$$

since $\dfrac{\partial}{\partial x_j}\mathscr{J}_{\alpha-1}(\varphi) = \mathscr{J}_{\alpha-1}\left(\dfrac{\partial \varphi}{\partial x}\right)$, for each $\varphi$ in $\mathscr{S}$, as can be verified by taking the Fourier transform. However, the mapping $\varphi \to \mathscr{J}_{\alpha-1}(\varphi)$ is onto all of $\mathscr{S}$ as we have already observed, and so we get that whenever $\psi \in \mathscr{S}$ (in particular if $\psi \in \mathscr{D}$)

$$\int_{\mathbf{R}^n} g\frac{\partial \psi}{\partial x_j}\,dx = -\int_{\mathbf{R}^n} g^{(j)}\psi\,dx,$$

which proves (44).

Since $g \in L^p_1$, we can approximate it according to the proposition in §2.1; this gives us a sequence $\{g_m\}$ in $\mathscr{D}$, (hence in $\mathscr{S}$), so that $g_m \to g$ and $\dfrac{\partial g_m}{\partial x_j} \to \dfrac{\partial g}{\partial x_j}$ in $L^p$ norm. We can write $g_m = \mathscr{J}_1(h_m)$, $h_m \in \mathscr{S}$. According to Lemma 2, part (ii), with $\alpha = 1$,

$$h_m = \nu_1 * g_m + \lambda_1 * \left(\sum_{j=1}^{n} R_j\left(\frac{\partial}{\partial x_j} g_m\right)\right)$$

and so $\|h_m\|_p \leq A_p\left[\|g_m\|_p + \sum_{j=1}^{n}\left\|\frac{\partial g_m}{\partial x_j}\right\|_p\right]$, for $1 < p < \infty$, since the $R_j$ are bounded in that range.

However $f_m = \mathscr{J}_\alpha(h_m)$, because $f_m = \mathscr{J}_{\alpha-1}(g_m)$, so $\|f_m\|_{p,\alpha} = \|h_m\|_p$. Thus

$$\|f_m\|_{p,\alpha} \leq A_p\left[\|g_m\|_p + \sum_{j=1}^{n}\left\|\frac{\partial g_m}{\partial x_j}\right\|_p\right].$$

The same inequality holds with $f_m$ replaced by $f_m - f_{m'}$, and $g_m$ replaced by $g_m - g_{m'}$. This shows that the sequence $f_m$ also converges in the $\mathscr{L}^p_\alpha$ norm; in the limit we get $f \in \mathscr{L}^p_\alpha$ and

$$\|f\|_{p,\alpha} \leq A_p\left[\|g\|_p + \sum_{j=1}^{n}\left\|\frac{\partial g}{\partial x_j}\right\|_p\right] = A_p\left[\|f\|_{p,\alpha-1} + \sum_{j=1}^{n}\left\|\frac{\partial f}{\partial x_j}\right\|_{p,\alpha-1}\right].$$

This together with (43) concludes the proof of Lemma 3.

The proof of Theorem 3 is now immediate. The identity between $L^p_k$ and $\mathscr{L}^p_\alpha$ is complete, and obvious, when $\alpha = k = 0$. However, it is clear that if $k \geq 1$, then $f \in L^p_k(\mathbf{R}^n)$ if and only if $f$ and $\dfrac{\partial f}{\partial x_j} \in L^p_{k-1}(\mathbf{R}^n)$, $j = 1, \ldots, n$. The two norms $\|f\|_{L_k{}^p(\mathbf{R}^n)}$ and

$$\|f\|_{L_{k-1}{}^p(\mathbf{R}^n)} + \sum_{j=1}^{n}\left\|\frac{\partial f}{\partial x_j}\right\|_{L_{k-1}{}^p(\mathbf{R}^n)}$$

are also obviously equivalent. Thus Lemma 3 extends the identity of $L^p_k$ and $\mathscr{L}^p_k$ from $k = 0$ to $k = 1, 2, \ldots$.

**3.5 Modulus of continuity.** Suppose that $f \in L^p(\mathbf{R}^n)$. We introduce again the $L^p$ modulus of continuity: $\omega_p(t) = \|f(x + t) - f(x)\|_p$, where the $L^p(\mathbf{R}^n)$ norm is taken with respect to the $x$ variables. We know that $\omega_p(t) \to 0$ as $|t| \to 0$ when $1 \leq p < \infty$ (see Chapter III, §2.2).

We ask ourselves the following natural question. Can the property that $f \in \mathscr{L}^p_\alpha(\mathbf{R}^n)$ be characterized in terms of the order of smallness of $\omega_p(t)$, as $|t| \to 0$? If so this would give a simple characterization of the elements of $\mathscr{L}^p_\alpha$ in terms of their smoothness.

Unfortunately this hope cannot be realized, except in certain special circumstances. The particular situations when the spaces $\mathscr{L}^p_\alpha$ can be characterized in terms of moduli of continuity are simple and worth recording. These circumstances occur only when $\alpha$ is integral (then $p$ may be arbitrary), or when $p = 2$ (then $\alpha$ may be arbitrary). We set down the details only in the cases of small $\alpha$, which cases are already entirely typical.

PROPOSITION 3. *Suppose* $1 < p < \infty$. *Then* $f \in \mathscr{L}_1^p(\mathbf{R}^n)$ *if and only if* $f \in L^p(\mathbf{R}^n)$, *and* $\omega_p(t) = O(|t|)$, *as* $|t| \to 0$.

Suppose $f \in \mathscr{D}$. If $t = |t|\,t'$, with $|t'| = 1$, then $f(x + t) - f(x) = \int_0^{|t|} (\nabla f, t')(x + st')\,ds$. Therefore by Minkowski's inequality

$$\|f(x + t) - f(x)\|_p \le |t| \sum_{j=1}^{n} \left\| \frac{\partial f}{\partial x_j} \right\|_p . \tag{45}$$

The approximation technique of §2.1 (Proposition 1) allows us to extend (45) to any $f \in L_1^p(\mathbf{R}^n)$. By Theorem 3 this holds whenever $f \in \mathscr{L}_1^p(\mathbf{R}^n)$, $1 < p < \infty$, and so $f \in \mathscr{L}_1^p(\mathbf{R}^n)$ implies $\omega_p(t) = O(|t|)$.

Conversely if $\omega_p(t) = O(|t|)$, as $|t| \to 0$, then if $e_j$ is the unit vector along the $x_j$ axis, the sequence

$$\left\{ \frac{f(x + e_j/m) - f(x)}{1/m} \right\}$$

is uniformly bounded in the $L^p(\mathbf{R}^n)$ norm. By the weak compactness of the unit sphere for $L^p$ $(1 < p)$, we can find a subsequence $\{m_k\}$, and an $f^{(j)} \in L^p(\mathbf{R})$, so that $\dfrac{f(x + e_j/m_k) - f(x)}{1/m_k} \to f^{(j)}$ weakly. In particular

$$\int_{\mathbf{R}^n} \left[ \frac{f(x + e_j/m_k) - f(x)}{1/m_k} \right] \varphi(x)\,dx = \int_{\mathbf{R}^n} f(x) \left[ \frac{\varphi(x - e_j/m_k) - \varphi(x)}{1/m_k} \right] dx$$

$$\to \int_{\mathbf{R}^n} f^{(j)} \varphi\,dx = - \int_{\mathbf{R}^n} f \frac{\partial \varphi}{\partial x_j}\,dx.$$

This shows that $\dfrac{\partial f}{\partial x_j} \in L^p(\mathbf{R}^n)$, and so $f \in L_1^p(\mathbf{R}^n) = \mathscr{L}_1^p(\mathbf{R}^n)$.

PROPOSITION 4. *Suppose* $0 < \alpha < 1$. *Then* $f \in \mathscr{L}_\alpha^2(\mathbf{R}^n)$ *if and only if* $f \in L^2(\mathbf{R}^n)$ *and* $\displaystyle\int_{\mathbf{R}^n} \frac{(\omega_2(t))^2}{|t|^{n+2\alpha}}\,dt < \infty$.

It is to be noted that because $\omega_p(t) \le 2\,\|f\|_p$, only that part of the integral $\displaystyle\int_{\mathbf{R}^n} \frac{(\omega_2(t))^2}{|t|^{n-2\alpha}}\,dt$ near the origin is critical.

In view of Plancherel's theorem, the fact that $f = \mathscr{J}_\alpha(g)$, $g \in L^2(\mathbf{R}^n)$, is equivalent with the statement that

$$\int_{\mathbf{R}^n} |\hat{f}(x)|^2 (1 + 4\pi^2 |x|^2)^\alpha dx < \infty. \tag{46}$$

Now again by Plancherel's theorem

$$(\omega_2(t))^2 = \|f(x + t) - f(x)\|_2^2 = \int_{\mathbf{R}^n} |\hat{f}(x)|^2 \left| e^{-2\pi i x \cdot t} - 1 \right|^2 dx.$$

Therefore

$$\int_{\mathbf{R}^n} \frac{(\omega_2(t))^2}{|t|^{n+2\alpha}}\,dt = \int_{\mathbf{R}^n} |\hat{f}(x)|^2 \mathscr{I}(x)\,dx,$$

with

$$\mathscr{I}(x) = \int_{\mathbf{R}^n} \frac{|e^{-2\pi i x\cdot t} - 1|^2}{|t|^{n+2\alpha}}\,dt. \tag{47}$$

An evaluation of the integral $\mathscr{I}(x)$ is easy. First $\mathscr{I}(x) = \mathscr{I}(\rho x)$, where $\rho$ is any rotation about the origin. Therefore $\mathscr{I}(x) = \mathscr{I}_0(|x|)$. Next, by homogeneity $\mathscr{I}(x) = |x|^{2\alpha}\mathscr{I}(\eta)$, where $\eta$ is any fixed unit vector. Clearly the constant $\mathscr{I}(\eta)$ satisfies the properties that $0 < \mathscr{I}(\eta) < \infty$; finiteness of $\mathscr{I}(\eta)$ follows because $|e^{-2\pi i\eta\cdot t} - 1| \leq 2$, and $|e^{-2\pi i\eta\cdot t} - 1| \leq c\,|t|$, so the integral giving $\mathscr{I}(\eta)$ converges. Therefore the conditions $f \in L^2(\mathbf{R}^n)$ and $\int_{\mathbf{R}^n} \frac{(\omega_2(t))^2}{|t|^{n+2\alpha}}\,dt < \infty$ are equivalent with the statements that $\int_{\mathbf{R}^n} |\hat{f}(x)|^2\,dx < \infty$ and $\int_{\mathbf{R}^n} |x|^{2\alpha}\,|\hat{f}(x)|^2\,dx < \infty$. These two together are clearly equivalent with (46) and so the proposition is proved.

*3.5.1* It is interesting to observe that Proposition 3, for $p = 2$, $\alpha = 1$, is not (at least in the way things are stated) a limiting case of Proposition 4, as $\alpha \to 1$. There is however a statement of the case $\alpha = 1$, in the spirit of Proposition 4, whose form anticipates some of the expressions that will interest us later.

We consider a modified modulus of continuity, $\tilde{\omega}_p(t)$, given by $\tilde{\omega}_p(t) = \|f(x + t) + f(x - t) - 2f(x)\|_p$. The point is that $\tilde{\omega}_p(t)$ is never really larger than $\omega_p(t)$, because $\tilde{\omega}_p(t) \leq \omega_p(t) + \omega_p(-t)$. On the other hand it may sometimes be effectively smaller than $\omega_p(t)$.*

PROPOSITION 5. *Suppose* $0 < \alpha < 2$. *Then* $f \in \mathscr{L}^2_\alpha(\mathbf{R}^n)$ *if and only if* $f \in L^2(\mathbf{R}^n)$ *and* $\int_{\mathbf{R}^n} \frac{(\tilde{\omega}_2(t))^2\,dt}{|t|^{n+2\alpha}} < \infty$.

The proof of this proposition is nearly identical with that of the previous one. The only difference is that the integral (47) (which converged only when $0 < \alpha < 1$) is replaced by

$$\int_{\mathbf{R}^n} \frac{|e^{-2\pi i x\cdot t} + e^{2\pi i x\cdot t} - 2|^2}{|t|^{n+2\alpha}}\,dt,$$

* For example, suppose $f \in \mathscr{S}$. Then $\omega_p(t) = O(|t|)$; but if $\frac{\omega_p(t)}{|t|} \to 0$, as $|t| \to 0$, then $f \equiv 0$. However, $\tilde{\omega}_p(t) = O(|t|^2)$, as $|t| \to 0$, always. For a deeper insight see §4.3.1 below.

which converges when $0 < \alpha < 2$. (See also the proposition in §5.2 of Chapter VIII.)

*3.5.2* While in general the space $\mathscr{L}^p_\alpha(\mathbf{R}^n)$ cannot be characterized in terms of the modulus of continuity, there are nevertheless some interesting relations, as we shall see in §5 below. Suppose that $0 < \alpha < 1$. Then if $f \in \mathscr{L}^p_\alpha$ we shall have

$$\int_{\mathbf{R}^n} \frac{(\omega_p(t))^p\,dt}{|t|^{n+\alpha p}} < \infty \quad \text{if} \quad p \geq 2,$$

and

$$\int_{\mathbf{R}^n} \frac{(\omega_p(t))^2\,dt}{|t|^{n+2\alpha}} < \infty \quad \text{if} \quad p \leq 2.$$

Conversely, suppose $f \in L^p$, then $f \in \mathscr{L}^p_\alpha$ if

$$\int_{\mathbf{R}^n} \frac{(\omega_p(t))^p\,dt}{|t|^{n-\alpha p}} < \infty \quad \text{when} \quad p \leq 2,$$

or if

$$\int_{\mathbf{R}^n} \frac{(\omega_p(t))^2\,dt}{|t|^{n-2\alpha}} < \infty \quad \text{when} \quad p \geq 2.$$

This indicates that it would be interesting to study function spaces defined in terms of their moduli of continuity. We begin by considering the simplest and most familiar example of such a function space, the space of "Lipschitz" (or "Hölder") continuous functions.*

## 4. *The spaces $\Lambda_\alpha$ of Lipschitz continuous functions*

**4.1** We start with the case $0 < \alpha < 1$. We define $\Lambda_\alpha$ as follows.

$\Lambda_\alpha = \{f : f \in L^\infty(\mathbf{R}^n), \quad \text{and} \quad \omega_\infty(t) = \|f(x+t) - f(x)\|_\infty \leq A\,|t|^\alpha\}$

The $\Lambda_\alpha$ norm is then given by

$$\|f\|_{\Lambda_\alpha} = \|f\|_\infty + \sup_{|t|>0} \frac{\|f(x+t) - f(x)\|_\infty}{|t|^\alpha}. \tag{48}$$

The first thing to observe is that the functions in $\Lambda_\alpha$ may be taken to be continuous, and so the relation $|f(x+t) - f(x)| \leq A\,|t|^\alpha$ holds for every $x$. More precisely,

PROPOSITION 6. *Every $f \in \Lambda_\alpha$ may be modified on a set of measure zero so that it becomes continuous.*

* The terminologies of functions satisfying a "Hölder condition" or a "Lipschitz condition" are equally common. Our personal preference is for the latter.

The proof can be carried out by using the device of regularization of §2.1. Any smooth regularization will do, and we shall use here that of the Poisson integral (see Chapter III, §2). Thus consider

$$u(x, y) = \int_{\mathbf{R}^n} P_y(t) f(x - t)\, dt, \qquad P_y(t) = \frac{c_n y}{(|t|^2 + y^2)^{(n+1)/2}}.$$

Then

$$u(x, y) - f(x) = \int_{\mathbf{R}^n} P_y(t)[f(x - t) - f(x)]\, dt,$$

and so

$$\|u(x, y) - f(x)\|_\infty \leq \int_{\mathbf{R}^n} P_y(t) \omega_\infty(-t)\, dt \leq A c_n y \int_{\mathbf{R}^n} \frac{|t|^\alpha}{(|t|^2 + y^2)^{(n+1)/2}}\, dt$$
$$= A' y^\alpha$$

(if $\alpha < 1$). In particular $\|u(x, y_1) - u(x, y_2)\|_\infty \to 0$, as $y_1$ and $y_2 \to 0$, and since $u(x, y)$ is continuous in $x$, then $u(x, y)$ converges uniformly as $y \to 0$. Therefore $f(x)$ may be taken to be continuous.

**4.2 A characterization.** Contrary to the situation just considered, the particular properties of the Poisson integral will play a more decisive role in the reasoning that follows. We begin by giving a characterization of $f \in \Lambda_\alpha$ in terms of their Poisson integrals $u(x, y)$.

PROPOSITION 7. *Suppose $f \in L^\infty(\mathbf{R}^n)$, and $0 < \alpha < 1$. Then $f \in \Lambda_\alpha(\mathbf{R}^n)$ if and only if*

(49)
$$\left\| \frac{\partial u(x, y)}{\partial y} \right\|_\infty \leq A y^{-1+\alpha}.$$

ADDENDUM: *If $A_1$ is the smallest constant $A$ for which* (49) *holds, then $\|f\|_\infty + A_1$ and $\|f\|_{\Lambda_\alpha}$ give equivalent norms.*

We make use of the following readily verified facts about the Poisson kernel:

(50)
$$\int_{\mathbf{R}^n} \left| \frac{\partial P_y(x)}{\partial y} \right| dx \leq c/y; \qquad \int_{\mathbf{R}^n} \frac{\partial P_y(x)}{\partial y}\, dx = 0, \qquad y > 0.$$

The first holds because of the obvious estimates for $\dfrac{\partial P_y}{\partial y}$,

$$\left| \frac{\partial P_y}{\partial y} \right| \leq c' y^{-n-1}, \qquad \left| \frac{\partial P_y}{\partial y} \right| \leq c' |x|^{-n-1}.$$

The second holds because $\int_{\mathbf{R}^n} P_y(x)\,dx \equiv 1$. Thus

$$\frac{\partial u}{\partial y}(x, y) = \int \frac{\partial P_y}{\partial y}(t) f(x - t)\,dt = \int \frac{\partial P_y}{\partial y}(t)[f(x - t) - f(x)]\,dt.$$

Hence

$$\left\| \frac{\partial u}{\partial y} \right\|_\infty \leq \frac{c c_n}{y} \|f\|_{\Lambda_\alpha} \int_{\mathbf{R}^n} \left| \frac{\partial P_y}{\partial y} \right| |t|^\alpha\,dt = c' \|f\|_{\Lambda_\alpha} y^{-1+\alpha}.$$

The converse, although not much more difficult, is far more enlightening, as it reveals an essential feature of the spaces in question. This insight is contained in the lemma below and the comments that follow.

Lemma 4. *Suppose $f \in L^\infty(\mathbf{R}^n)$ and $0 < \alpha < 1$. Then the single condition (49) is equivalent with the $n$ conditions*

$$\left\| \frac{\partial u(x, y)}{\partial x_j} \right\|_\infty \leq A' y^{-1+\alpha}, \qquad j = 1, \ldots, n. \tag{51}$$

Addendum: *The smallest $A$ in (49) is comparable to the smallest $A'$ in (51).*

We know that the relation between $\frac{\partial u}{\partial y}$, and $\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots \frac{\partial u}{\partial x_n}$ is implemented by the Riesz transforms; see Chapter III, §2.3. However the Riesz transforms do not preserve the class of bounded functions (see Chapter II, §6.1, (b)), and so there would be no identity between condition (49) and (51), when $\alpha = 1$. The meaning of the lemma is that, nevertheless, the Riesz transforms are effectively bounded on the $\Lambda_\alpha$. (The elementary proof for this stands in sharp contrast to the difficult proof for the case of the boundedness for $L^p$, $1 < p < \infty$; see also Chapter II, §6.9.)

To prove the lemma we use the following estimates

$$\left\| \frac{\partial P_y}{\partial y} \right\|_1 \leq c y^{-1}, \qquad \left\| \frac{\partial P_y}{\partial x_j} \right\|_1 \leq c y^{-1}, \qquad (y > 0). \tag{52}$$

The first already appears in (50), and the second is proved the same way.

Because $P_y = P_{y_1} * P_{y_2}$, $y = y_1 + y_2$, $y_j > 0$, we get $u(x, y) = P_{y_1} * u(x, y_2)$, and therefore with $y_1 = y_2 = y/2$,

$$\frac{\partial^2 u}{\partial y\, \partial x_j} = \left(\frac{\partial P_{y/2}}{\partial x_j}\right) * \left(\frac{\partial u}{\partial y}\right)_{y/2}.$$

Thus by (52) the assumption $\left\|\dfrac{\partial u}{\partial y}\right\| \le Ay^{-1+\alpha}$ implies the fact that

$$\left\|\frac{\partial^2 u}{\partial y\,\partial x_j}\right\|_\infty \le A_1 y^{-2+\alpha}. \tag{53}$$

However,

$$\left\|\frac{\partial}{\partial x_j}u(x, y)\right\|_\infty = \left\|\frac{\partial P_y}{\partial x_j} * f\right\|_\infty \le \left\|\frac{\partial P_y}{\partial x_j}\right\|_1 \|f\|_\infty \le cy^{-1}\|f\|_\infty$$

by (52). So

$$\frac{\partial}{\partial x_j}u(x, y) \to 0, \quad \text{as} \quad y \to \infty$$

and therefore

$$\frac{\partial}{\partial x_j}u(x, y) = -\int_y^\infty \frac{\partial^2 u(x, y')}{\partial y\,\partial x_j}\,dy'.$$

(53) the gives us that

$$\left\|\frac{\partial u}{\partial x_j}\right\|_\infty \le A_2 y^{-1+\alpha}, \quad \text{if} \quad \alpha < 1.$$

Conversely suppose that (49) is satisfied. Reasoning as before we get that $\left\|\dfrac{\partial^2 u}{\partial x_j^2}\right\|_\infty \le A_3 y^{-2+\alpha}$, $j = 1, 2, \ldots, n$. However since $u$ is harmonic, that is because

$$\frac{\partial^2 u}{\partial y^2} = -\sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2},$$

we therefore have $\left\|\dfrac{\partial^2 u}{\partial y^2}\right\|_\infty \le A_4 y^{-2+\alpha}$, and a similar integration argument then shows that $\left\|\dfrac{\partial y}{\partial y}\right\|_\infty \le A_5 y^{-1+\alpha}$.

We can now prove the converse part of Proposition 7. Suppose $\left\|\dfrac{\partial}{\partial y}u(x, y)\right\|_\infty \le Ay^{-1+\alpha}$. Then the lemma also shows that $\left\|\dfrac{\partial}{\partial x_j}u(x, y)\right\|_\infty \le A'y^{-1+\alpha}$. We write

$$\begin{aligned} f(x + t) - f(x) = {}& \{u(x + t, y) - u(x, y)\} \\ &+ \{f(x + t) - u(x + t, y)\} - \{f(x) - u(x, y)\}. \end{aligned}$$

Here $y$ does not necessarily depend on $t$ but it is best to choose $y = |t|$. Now $|u(x + t, y) - u(x, y)| \le \int_L |\nabla u(x + s, y)|\,ds$ where $L$ is the line segment (of length $|t|$) joining $x$ with $x + t$. Thus

$$|u(x + t, y) - u(x, y)| \le |t| \sum_{j=1}^n \|u_{x_j}(x, y)\|_\infty \le A_5|t|\,|t|^{-1+\alpha} = A_5|t|^\alpha.$$

Also

$$f(x+t) - u(x+t, y) = -\int_0^y \frac{\partial}{\partial y'} u(x+t, y')\, dy',$$

and so

$$|f(x+t) - u(x+t, y)| \le y \int_0^y \left\| \frac{\partial u}{\partial y'} \right\|_\infty dy' \le A_6 y^\alpha = A_6 |t|^\alpha.$$

With a similar estimate for $f(x) - u(x, y)$ the proof of the proposition is concluded.

*4.2.1* The reader should have no difficulty in verifying that the reasoning of Lemma 4 also proves the following lemma.

LEMMA 5. *Suppose* $f \in L^\infty(\mathbf{R}^n)$, *and* $0 < \alpha$. *Let* $k$ *and* $l$ *be two integers, both greater than* $\alpha$. *Then the two conditions*

$$\left\| \frac{\partial^k u(x, y)}{\partial y^k} \right\|_\infty \le A_k y^{-k+\alpha}, \quad \textit{and} \quad \left\| \frac{\partial^l u(x, y)}{\partial y^l} \right\|_\infty \le A_l y^{-l+\alpha}$$

*are equivalent. Moreover, the smallest* $A_k$ *and* $A_l$ *holding in the above inequalities are comparable.*

The utility of this lemma will be apparent soon.

**4.3** $\Lambda_\alpha$, **all** $\alpha > 0$. We can now define the space $\Lambda_\alpha(\mathbf{R}^n)$ for any $\alpha > 0$. Suppose that $k$ is the smallest integer greater than $\alpha$. We set

$$\Lambda_\alpha = \left\{ f \in L^\infty(\mathbf{R}^n) : \left\| \frac{\partial^k}{\partial y^k} u(x, y) \right\|_\infty \le A y^{-k+\alpha} \right\} \tag{54}$$

If $A_k$ denotes the smallest $A$ appearing in the inequality in (54), then we can define the $\Lambda_\alpha$ norm by

$$\|f\|_{\Lambda_\alpha} = \|f\|_\infty + A_k. \tag{55}$$

According to Proposition 7, when $0 < \alpha < 1$, this definition is equivalent with the previous one and the resulting norms are also equivalent. Lemma 5 also shows us that we could have replaced the $\dfrac{\partial^k u(x, y)}{\partial y^k}$ by the corresponding estimate for $\dfrac{\partial^l u(x, y)}{\partial y^l}$ where $l$ is *any* integer greater than $\alpha$.

A remark about the condition in (54) is in order. The estimate

$$\left\| \frac{\partial^k}{\partial y^k} u(x, y) \right\|_\infty \le A y^{-k+\alpha}$$

is of interest only for $y$ near zero, since the inequality $\left\| \frac{\partial^k}{\partial y^k} u(x, y) \right\|_\infty \leq A y^{-k}$ (which is stronger away from zero) follows already from the fact that $f \in L^\infty$, (as the argument of Lemma 4 shows). This observation allows us to assert the inclusion $\Lambda_\alpha \subset \Lambda_{\alpha'}$, if $\alpha > \alpha'$.

The definitions (54) and (55) for $\Lambda_\alpha$, when $\alpha > 0$ have a rather artificial appearance when compared with the definition given in the case $0 < \alpha < 1$ in §4.1. The reader should not be troubled about this point because the definitions just given are temporary expedients. The natural characterizations are formulated in the two propositions below.

We consider first the situation $0 < \alpha < 2$. We shall need to consider second differences, as in Proposition 5 in §3.5.1.

PROPOSITION 8. *Suppose* $0 < \alpha < 2$. *Then* $f \in \Lambda_\alpha$ *if and only if* $f \in L^\infty(\mathbf{R}^n)$ *and* $\|f(x + t) + f(x - t) - 2f(x)\|_\infty \leq A\,|t|^\alpha$. *The expression*

$$\|f\|_\infty + \sup_{|t|>0} \frac{\|f(x + t) + f(x - t) - 2f(x)\|_\infty}{|t|^\alpha}$$

*is equivalent with the* $\Lambda_\alpha$ *norm.*

We need the following observations about the differentiated Poisson kernel:

(a) $$\int_{\mathbf{R}^n} \frac{\partial^2 P_y(t)}{\partial y^2}\, dt = 0;$$

(b) $$\frac{\partial^2 P_y(t)}{\partial y^2} = \frac{\partial^2 P_y(-t)}{\partial y^2};$$

(c) $$\left| \frac{\partial^2 P_y(t)}{\partial y^2} \right| \leq c y^{-n-2};$$

(d) $$\left| \frac{\partial^2 P_y(t)}{\partial y^2} \right| \leq c\,|t|^{-n-2}.$$

The details of the calculation giving (c) and (d) are best left to the reader but it may be helpful to point out that $\frac{\partial^2 P_y(t)}{\partial y^2}$ is jointly homogeneous of degree $-n - 2$.

With these remarks we see that

$$\frac{\partial^2}{\partial y^2} u(x, y) = \frac{1}{2} \int_{\mathbf{R}^n} \frac{\partial^2}{\partial y^2} P_y(t)[f(x + t) + f(x - t) - 2f(x)]\, dt,$$

and so

$$\left\| \frac{\partial^2 u(x, y)}{\partial y^2} \right\|_\infty \leq \frac{Ac}{2} \left\{ y^{-n-2} \int_{|t| < y} |t|^\alpha\, dt + \int_{|t| \geq y} |t|^{-n-2+\alpha}\, dt \right\}.$$

Therefore

$$\left\| \frac{\partial^2 u}{\partial y^2} \right\|_\infty \leq A' y^{-2+\alpha} \quad \text{if} \quad \alpha < 2.$$

To prove the converse, write $(\Delta_t^2 F)(x) = F(x+t) + F(x-t) - 2F(x)$, and observe that if $F$ has two continuous derivatives, then

$$\Delta_t^2 F(x) = \int_0^{|t|} \left\{ \int_{-s}^{s} \frac{d^2}{d\tau^2} (F(x + t'\tau))\, d\tau \right\} ds, \quad \text{where} \quad t' = t/|t|.$$

It follows immediately that

$$\|\Delta_t^2 F\|_\infty \leq |t|^2 \left\{ \sum_{i,j} \left\| \frac{\partial^2 F}{\partial x_i\, \partial x_j} \right\|_\infty \right\}. \tag{56}$$

By the definition (54) it is clear that if $f \in \Lambda_\alpha \Rightarrow f \in \Lambda_{\alpha'}$, where $\alpha' < \alpha$. If we choose an $\alpha' < 1$, then by the results in Propositions 6 and 7 we get

$$\|u(x, y) - f(x)\|_\infty \to 0, \quad \text{and} \quad y\, \|u_y(x, y)\|_\infty \to 0 \quad \text{as} \quad y \to 0. \tag{57}$$

Now the identity

$$f(x) = u(x, 0) = \int_0^y y' \frac{\partial^2}{(\partial y')^2} u(x, y')\, dy' - y \frac{\partial u}{\partial y}(x, y) + u(x, y) \tag{58}$$

is obtained by noticing that the derivative with respect to $y$ of the extreme right-hand side vanishes, and by the use of the end-point conditions (57). However, the arguments of Lemma 4 and 5 show that the inequality $\left\| \frac{\partial^2 u(x, y)}{\partial y^2} \right\|_\infty \leq A y^{-2+\alpha}$ implies the estimates

$$\left\| \frac{\partial^2 u}{\partial x_i\, \partial x_j} \right\|_\infty \leq A' y^{-2+\alpha}, \qquad \left\| \frac{\partial^3 u}{\partial y\, \partial x_i\, \partial x_j} \right\|_\infty \leq A' y^{-3+\alpha}.$$

Thus by (56) and (58)

$$\|\Delta_t^2 f\|_\infty \leq A'' \left\{ \int_0^y y'(y')^{-2+\alpha}\, dy' + (y)^{-2+\alpha} \cdot |t|^2 \right\}.$$

Taking $y = |t|$ gives the desired result

$$\|\Delta_t^2 f\|_\infty \leq A'' |t|^\alpha, \qquad \text{if} \qquad 0 < \alpha.$$

PROPOSITION 9. *Suppose $\alpha > 1$. Then $f \in \Lambda_\alpha$ if and only if $f \in L^\infty$ and $\frac{\partial f}{\partial x_j} \in \Lambda_{\alpha-1}$, $j = 1, \ldots, n$. The norms $\|f\|_{\Lambda_\alpha}$ and $\|f\|_\infty + \sum_{j=1}^{n} \left\| \frac{\partial f}{\partial x_j} \right\|_{\Lambda_{\alpha-1}}$ are equivalent.*

Let us suppose for simplicity that $1 < \alpha \leq 2$; the other cases are argued similarly.

Observe first that $\dfrac{\partial f}{\partial x_j} \in L^\infty$. We have $\left\|\dfrac{\partial^3 u}{\partial y^3}\right\|_\infty \leq Ay^{-3+\alpha}$, which implies, as we know, $\left\|\dfrac{\partial^3 u}{\partial y^2\, \partial x_j}\right\|_\infty \leq Ay^{-3+\alpha}$. We restrict to $0 < y \leq 1$, then we see that $\left\|\dfrac{\partial^3 y}{\partial y^2\, \partial x_j}\right\|_\infty \leq Ay^{-1-\beta}$, where $\beta < 1$. An integration in $y$ then gives

$$\left\|\frac{\partial^2 u}{\partial y\, \partial x_j}\right\|_\infty \leq A\, y^{-\beta} + A\left\|\left[\frac{\partial^2 u}{\partial y\, \partial x_j}\right]_{y=1}\right\|_\infty.$$

Another integration then shows that $\left\{\dfrac{\partial}{\partial x_j} u(x, y)\right\}$ is Cauchy in the $L^\infty$ norm (as $y \to 0$) and so its limits can be taken to be $\dfrac{\partial f}{\partial x_j}$. The argument also gives the bound

$$\left\|\frac{\partial f}{\partial x_j}\right\|_\infty \leq C\, \|f\|_{\Lambda_\alpha}.$$

Since the (weak) derivative of $f$ is $\dfrac{\partial f}{\partial x_j}$, the Poisson integral of the latter is $\dfrac{\partial u}{\partial x_j}$. But $\left\|\dfrac{\partial^3 u}{\partial y^2\, \partial x_j}\right\|_\infty \leq Ay^{-3+\alpha}$. Therefore $\dfrac{\partial f}{\partial x_j} \in \Lambda_{\alpha-1}$. The converse implication is proved the same way.

The last proposition reduces the study of the spaces $\Lambda_\alpha$ to those $\alpha$ such that $0 < \alpha \leq 1$.

*4.3.1 An example.* Concerning the $\Lambda_\alpha$, $0 < \alpha \leq 1$, the following additional remarks are in order. First, when $0 < \alpha < 1$, Proposition 8 shows that if $f \in L^\infty$ the two conditions $\|f(x + t) - f(x)\|_\infty \leq A\, |t|^\alpha$ and $\|f(x + t) + f(x - t) - 2f(x)\|_\infty \leq A'\, |t|^\alpha$ are equivalent. However this is not the case when $\alpha = 1$.

EXAMPLE. *There exist $f \in L^\infty(\mathbf{R}^n)$ so that*

$$\|f(x + t) + f(x - t) - 2f(x)\|_\infty \leq A\, |t|, \qquad |t| > 0,$$

*but so that $\|f(x + t) - f(x)\|_\infty \leq A'\, |t|$ fails for all $A'$.*

One can construct such $f$ by lacunary series, and more particularly as Weierstrass-Hardy non-differentiable functions. To do this we consider the function of one variable $x$, given by $f(x) = \sum_{k=1}^{\infty} a^{-k} e^{2\pi i a^k x}$. Here $a > 1$;

for simplicity we take $a$ to be an integer and this makes $f$ periodic.* Now

$$f(x+t)+f(x-t)-2f(x) = 2\sum a^{-k}[\cos 2\pi a^k t - 1]e^{2\pi i a^k x}.$$

Therefore

$$\|f(x+t)+f(x-t)-2f(x)\|_\infty \leq 2\sum_{a^k|t|\leq 1} a^{-k}A(a^k t)^2 + 4\sum_{a^k|t|>1} a^{-k} \leq A'\,|t|.$$

We have used merely the facts that $|\cos 2\pi a^k t - 1| \leq A(a^k t)^2$, and $\leq 2$.
If however we had $\|f(x+t)-f(x)\|_\infty \leq A'\,|t|$, then by Bessel's inequality for $L^2$ periodic functions we would get

$$\begin{aligned}(A'\,|t|)^2 &\geq \int_0^1 |f(x+t)-f(x)|^2\,dx \\ &= \sum a^{-2k}|e^{2\pi i a^k t}-1|^2 \geq \sum_{a^k|t|\leq 1} |e^{2\pi i a^k t}-1|^2.\end{aligned}$$

In the range $a^k\,|t| \leq 1$ we have $|e^{2\pi i a^k t} - 1|^2 \geq c(a^k t)^2$, and so we would arrive at the contradiction

$$(A'\,|t|)^2 \geq c\,|t|^2 \sum_{a^k|t|\leq 1} 1.$$

**4.4** $\mathscr{J}_\beta\colon \Lambda_\alpha \leftrightarrow \Lambda_{\alpha+\beta}$. We shall now connect the Bessel potentials $\mathscr{J}_\beta$ with the Lipschitz spaces, $\Lambda_\alpha$.

THEOREM 4. *Suppose* $\alpha > 0$, $\beta \geq 0$. *Then* $\mathscr{J}_\beta$ *maps* $\Lambda_\alpha$ *isomorphically onto* $\Lambda_{\alpha+\beta}$.

We should explain that by "isomorphism" in this case we do not mean that the norms $\|f\|_{\Lambda_\alpha}$ and $\|\mathscr{J}_\beta f\|_{\Lambda_{\alpha+\beta}}$ are identical, but only that they are *equivalent*.

We have already noted in §3.3 that the mapping $\mathscr{J}_\beta$ is one-one. To prove that the image of $\Lambda_\alpha$ under $\mathscr{J}_\beta$ lies in $\Lambda_{\alpha+\beta}$, and that the mapping is continuous, we argue as follows. Let $u$ equal the Poisson integral of $f$, and $U$ be the Poisson integral of $\mathscr{J}_\beta(f) = G_\beta * f$. Then $u = P_y * f$, and $U = P_y * G_\beta * f$. Thus $U(x,y) = G_\beta(x,y) * f(x)$, where $G_\beta(x,y)$ is the Poisson integral of $G_\beta(x)$. The following property of $G_\beta(x,y)$ will be proved in §5.4 below.

*Suppose $l$ is an integer and $l > \beta$. Then*

$$\left\|\frac{\partial^l G_\beta(x,y)}{\partial y^l}\right\|_1 \leq A y^{-l+\beta}, \qquad y > 0. \tag{59}$$

However, we know that $P_{y_1+y_2} = P_{y_1} * P_{y_2}$, $y_1 > 0$, $y_2 > 0$; consequently,

$$U(x, y_1+y_2) = P_{y_1+y_2} * G_\beta * f = P_{y_1} * G_\beta * P_{y_2} * f = G_\beta(x,y_1) * u(x,y_2).$$

* The result also holds if $a$ is non-integral.

Let $k$ be the smallest integer larger than $\alpha$, and differentiate in the above $l$ times with respect to $y_1$ and $k$ times with respect to $y_2$. The result is

$$\frac{\partial^{k+l}U(x,y)}{\partial y^{k+l}} = \frac{\partial^l}{\partial y_1^l} G_\beta(x,y_1) * \frac{\partial^k}{\partial y_2^k} u(x,y_2), \qquad y = y_1 + y_2$$

Thus in view of (59), with $y_1 = y_2 = \dfrac{y}{2}$, we obtain

$$\left\| \frac{\partial^{k+l}U(x,y)}{\partial y^{k+l}} \right\|_1 \le A\left(\frac{y}{2}\right)^{-l-\beta} \cdot A'\left(\frac{y}{2}\right)^{-k+\alpha}.$$

Now $f \in \Lambda_\alpha$ implies that $\left\| \dfrac{\partial^k}{\partial y^k} u(x,y) \right\|_\infty \le Ay^{-k+\alpha}$; (see the definition (59)).

Moreover, clearly, $\mathscr{J}_\beta(f) \in L^\infty$, since $f \in L^\infty$. Therefore $\mathscr{J}_\beta f \in \Lambda_{\alpha+\beta}$ and the proof also shows that $\|\mathscr{J}_\beta f\|_{\Lambda_{\alpha+\beta}} \le C\,\|f\|_{\Lambda_\alpha}$.

We claim next that the image of $\Lambda_\alpha$ under $\mathscr{J}_2$ is all of $\Lambda_{\alpha+2}$. To see this let $f \in \Lambda_{\alpha+2}$. Then $f \in \Lambda_\alpha$; also $\Delta f \in \Lambda_\alpha$, the latter fact is a consequence of Proposition 9 in §4.3. Therefore $(I - \Delta)f \in \Lambda_\alpha$. However $\mathscr{J}_2[(I - \Delta)f] = f$. To prove this identity it suffices to verify that

$$\int_{\mathbf{R}^n} (\mathscr{J}_2(1 - \Delta)f)\varphi\, dx = \int_{\mathbf{R}^n} f\,\varphi\, dx$$

whenever $\varphi \in \mathscr{S}$. But

$$\begin{aligned}\int_{\mathbf{R}^n} (\mathscr{J}_2(1 - \Delta)f)\phi\, dx &= \int_{\mathbf{R}^n} (I - \Delta)f\mathscr{J}_2(\phi)\, dx \\ &= \int f(1 - \Delta)\mathscr{J}_2\phi\, dx = \int f\phi\, dx,\end{aligned}$$

since obviously $(1 - \Delta)\mathscr{J}_2\varphi = \varphi$ as the Fourier transform shows. Because $\mathscr{J}_2$ is onto, $\mathscr{J}_{2-\beta}$ is one-one, and $\mathscr{J}_2 = \mathscr{J}_{2-\beta}\mathscr{J}_\beta$, for $0 < \beta < 2$, then $\mathscr{J}_\beta$ must be onto, for that range of $\beta$. Finally by superimposing such $\mathscr{J}_\beta$ we arrive at the conclusion that $\mathscr{J}_\beta$ is onto for *any* positive $\beta$ and the theorem is then proved if we appeal to the closed graph theorem.*

## 5. *The spaces* $\Lambda_\alpha^{p,q}$

**5.1** In analogy with our definition of $\Lambda_\alpha$, and motivated by Proposition 4 in §3.5 we define the spaces $\Lambda_\alpha^{p,q}$, where $1 \le p, q \le \infty$. We begin with the case $0 < \alpha < 1$. Then $\Lambda_\alpha^{p,q}(\mathbf{R}^n)$ consists of all function $f$ in $L^p(\mathbf{R}^n)$

* We defer to the closed-graph theorem only so that we may give a quick proof of the continuity of the mappings inverse to $\mathscr{J}_\beta$. But as the reader may guess, with a little extra effort the matter could have been treated directly.

for which the norm

$$(60) \qquad \|f\|_p + \left(\int_{\mathbf{R}^n} \frac{(\|f(x+t)-f(x)\|_p)^q\, dt}{|t|^{n+\alpha q}}\right)^{1/q}$$

is finite. When $q = \infty$, the expression (60) is interpreted in the normal limiting way, namely

$$(60') \qquad \|f\|_p + \sup_{|t|>0} \frac{\|f(x+t)-f(x)\|_p}{|t|^\alpha}.$$

We see therefore that $\Lambda_\alpha^{\infty,\infty} = \Lambda_\alpha$, and that Proposition 4 states, in effect, that $\Lambda_\alpha^{2,2} = \mathscr{L}_\alpha^2$, $0 < \alpha < 1$. (The identity $\Lambda_\alpha^{2,2} = \mathscr{L}_\alpha^2$ will later be seen to be valid for all $\alpha$.)

The basic properties of the spaces, $\Lambda_\alpha$, given in Propositions 7, 8, and 9, Lemmas 4 and 5, and Theorem 4 hold with the obvious modifications for the space $\Lambda_\alpha^{p,q}$. We formulate this generally, but prove it in detail only for the direct part of the analogue of Proposition 7.

PROPOSITION 7′. *Suppose $f \in L^p(\mathbf{R}^n)$, and $0 < \alpha < 1$. Then $f \in \Lambda_\alpha^{p,q}$ if and only if*

$$(61) \qquad \left(\int_0^\infty \left(y^{\alpha-1}\left\|\frac{\partial}{\partial y} u(x,y)\right\|_p\right)^q \frac{dy}{y}\right)^{1/q} < \infty$$

*The $\Lambda_\alpha^{p,q}$ norm is equivalent with the norm*

$$\|f\|_p + \left(\int_0^\infty \left(y^{\alpha-1}\left\|\frac{\partial u}{\partial y}\right\|_p\right)^q \frac{dy}{y}\right)^{1/q}$$

We have

$$\frac{\partial}{\partial x} u(x,y) = \int_{\mathbf{R}^n} \frac{\partial P_y(t)}{\partial t}[f(x-t)-f(x)]\, dt$$

and therefore by the elementary estimate $\left|\frac{\partial P_y(t)}{\partial y}\right| \le c' y^{-n-1}$, $\left|\frac{\partial P_y(t)}{\partial y}\right| \le c'\,|t|^{-n-1}$, already used, we see that

$$\left\|\frac{\partial}{\partial y} u(x,y)\right\|_p \le c' y^{-n-1}\int_{|t|\le y} \|f(x+t)-f(x)\|_p\, dt$$
$$+\, c'\int_{|t|>y} \|f(x+t)-f(x)\|_p \frac{dt}{|t|^{n+1}}.$$

Next set $t = r\xi \in \mathbf{R}^n$, with $r = |t|$, and $|\xi| = 1$. Then with

$$\|f(x+t)-f(x)\|_p = \omega_p(t) = \omega_p(r\xi),$$

we write $\Omega(r) = \int_{S^{n-1}} \omega_p(r\xi)\, d\xi$. The inequality above becomes

$$\left\|\frac{\partial u}{\partial y}\right\|_p \le c' y^{-n-1} \int_0^y \Omega(r) r^{n-1}\, dr + c' \int_y^\infty \Omega(r) r^{-2}\, dr.$$

(because $dt = d\xi r^{n-1}\, dr$)

Therefore by the Hardy inequalities (see Appendix A)

$$\left(\int_0^\infty \left(y^{\alpha-1}\left\|\frac{\partial u}{\partial y}\right\|_p\right)^q \frac{dy}{y}\right)^{1/q} \le c\left(\int_0^\infty [\Omega(r) r^{-\alpha}]^q \frac{dr}{r}\right)^{1/q}.$$

However, $\Omega(r)^q \le c \int_{S^{n-1}} (\omega(r\xi))^q\, d\xi$, by Hölder's inequality.* Substituting this in the above leads to the bound

$$c''\left(\int_{S^{n-1}} \int_0^\infty (\omega(r\xi))^q r^{-\alpha q} \frac{dr}{r}\, d\xi\right)^{1/q} = c''\left(\int_{\mathbf{R}^n} \frac{(\|f(x+t) - f(x)\|_p^q}{|t|^{n+\alpha q}}\, dt\right)^{1/q}.$$

In the same way we can prove

Lemma 4′. *Suppose $f \in L^p(\mathbf{R}^n)$, $0 < \alpha < 1$. Then the single condition* (61) *is equivalent with the n conditions*

$$\left(\int_0^\infty \left(y^{\alpha-1}\left\|\frac{\partial u}{\partial x_j}\right\|_p\right)^q \frac{dy}{y}\right)^{1/q} < \infty, \qquad j = 1, 2, \ldots, n. \tag{62}$$

*The norm which results if we replace the quantity in* (61) *by the sum of the n quantities in* (62) *is equivalent with the original $\Lambda_\alpha^{p,q}$ norm.*

Before going further it will be well to record the more general assertion which is in back of Lemma 4′. It is the inequality

$$\left(\int_0^\infty (y^{\alpha-k} \|D^k u\|_p)^q \frac{dy}{y}\right)^{1/q} \le A\left(\int_0^\infty \left(y^{\alpha-k}\left\|\frac{\partial^k u}{\partial y^k}\right\|_p\right)^q \frac{dy}{y}\right)^{1/q}. \tag{62′}$$

Here $k$ is a positive integer, $0 < \alpha < k$, and $D^k$ is any differential monomial in $x_1, x_2, \ldots, x_n, y$, of total order $k$. The proof follows by the same arguments.

Proceeding as before we define next the spaces $\Lambda_\alpha^{p,q}(\mathbf{R}^n)$ for any $\alpha > 0$. Let $k$ be the smallest integer greater than $\alpha$. We set

$$\Lambda_\alpha^{p,q}(\mathbf{R}^n) = \left\{f \in L^p(\mathbf{R}^n) : \left(\int_0^\infty \left(y^{\alpha-k}\left\|\frac{\partial^k u}{\partial y^k}\right\|_p\right)^q \frac{dy}{y}\right)^{1/q} < \infty\right\} \tag{63}$$

Then $\Lambda_\alpha^{p,q}$ norm is defined by

$$\|f\|_{\Lambda_\alpha^{p,q}} = \|f\|_p + \left(\int_0^\infty \left(\left\|y^{\alpha-k} \frac{\partial^k u}{\partial y^k}\right\|_p\right)^q \frac{dy}{y}\right)^{1/q}.$$

* Use $\Omega(r) \le \sup_\xi \omega(r, \xi)$, when $q = \infty$.

We should remark, as we did in the special case of $\Lambda_\alpha$, than an equivalent definition and equivalent norm would have been obtained had we replaced the integer $k$ by any other integer $l$, with $l > \alpha$. (This is already implicit in (62').) In complete analogy with Propositions 8 and 9 and Theorem 4 we state the following.

PROPOSITION 8'. *Suppose* $0 < \alpha < 2$. *Then* $f \in \Lambda_\alpha^{q,p}$ *if and only if* $f \in L^p(\mathbf{R}^n)$ *and*

$$\left(\int_{\mathbf{R}^n} \frac{(\|f(x+t) + f(x-t) - 2f(x)\|_p)^q}{|t|^{n+\alpha q}}\, dt\right)^{1/q} < \infty.$$

*The expression*

$$\|f\|_p + \left(\int_{\mathbf{R}^n} \frac{(\|f(x+t) + f(x-t) - 2f(x)\|_p)^q}{|t|^{n+\alpha q}}\, dt\right)^{1/q}$$

*is equivalent with the* $\Lambda_\alpha^{p,q}$ *norm.**

PROPOSITION 9'. *Suppose* $\alpha > 1$. *Then* $f \in \Lambda_\alpha^{p,q}$ *if and only if* $f \in L^p(\mathbf{R}^n)$ *and* $\dfrac{\partial f}{\partial x_j} \in \Lambda_{\alpha-1}^{p,q}$. *The norm* $\|f\|_{\Lambda_{\alpha-1}^{p,q}}$ *and* $\|f\|_p + \sum_{j=1}^n \left\|\dfrac{\partial}{\partial x_j} f\right\|_{\Lambda_{\alpha-1}^{p,q}}$ *are equivalent.*

THEOREM 4'. *Suppose* $\alpha > 0$, $\beta \geq 0$. *Then* $\mathscr{J}_\beta$ *maps* $\Lambda_\alpha^{p,q}$ *isomorphically onto* $\Lambda_{\alpha+\beta}^{p,q}$.

**5.2 A further look at $\Lambda_\alpha^{p,q}$.** After these rather mechanical preliminaries we intend to make some more interesting observations about the space $\Lambda_\alpha^{p,q}$. The first question is, what are the roles of the indices $\alpha$, $p$, and $q$? The answer is roughly as follows. First the index $p$ indicates the basic norm that is used; next $\alpha$ gives the order of smoothness involved, and the index $q$ represents a second-order (and rather subtle) correction to this order of smoothness. A precise result is as follows.

PROPOSITION 10. *The inclusion* $\Lambda_{\alpha_1}^{p,q_1}(\mathbf{R}^n) \subset \Lambda_{\alpha_2}^{p,q_2}(\mathbf{R}^n)$ *holds if either* (a) *if* $\alpha_1 > \alpha_2$ *(then* $q_1$ *and* $q_2$ *need not be related), or* (b) *if* $\alpha_1 = \alpha_2$ *and* $q_1 \leq q_2$.

* For $q = \infty$, we interpret

$$\left(\int_{\mathbf{R}^n} \frac{(\|f(x+t) + f(x-t) - 2f(x)\|_p)^q}{t^{n+\alpha q}}\, dt\right)^{1/q}$$

as

$$\sup_{|t|>0} \frac{\|f(x+t) + f(x-t) - 2f(x)\|_p}{|t|^\alpha}$$

The proof is based on the following lemma, which in reality is nothing but a variant of the usual maximum principle for harmonic functions.

LEMMA 6. *Suppose $f \in L^p(\mathbf{R}^n)$; then for any integer $k$ the function* $\left\| \frac{\partial^k}{\partial y^k} u(x, y) \right\|_p$ *is a non-increasing function of $y$ for $0 < y < \infty$.*

Consider first the case $k = 0$. Since $P_{y_1} * P_{y_2} = P_{y_1+y_2}$ we have

$$u(x, y_1 + y_2) = P_{y_1} * u(x, y_2), \tag{64}$$

and so $\|u(x, y_1 + y_2)\|_p \leq \|P_{y_1}\|_1 \|u(x, y_2)\|_p$. Because $\|P_{y_1}\|_1 = 1$ we obtain $\|u(x, y_1 + y_2)\|_p \leq \|u(x, y_2)\|_p$ and the assertion is proved in this case. To prove the general case of the lemma, differentiate the identity (64) $k$ times with respect to $y_2$ and argue similarly.

Let us now prove part (b) of the Proposition (part (a) is argued similarly, and anyway that conclusion is even less delicate). Assume $q_1 < \infty$, and

$$\left(\int_0^\infty \left(y^{\alpha-k} \left\| \frac{\partial^k u}{\partial y^k} \right\|_p\right)^{q_1} \frac{dy}{y}\right)^{1/q_1} = A \tag{65}$$

Then

$$\int_{y_0/2}^{y_0} y^{(\alpha-k)q_1} \left\| \frac{\partial^k u}{\partial y^k} \right\|_p^{q_1} \frac{dy}{y} \leq A^{q_1}.$$

However, by the lemma $\left\| \frac{\partial^k u}{\partial y^k} \right\|_p$ takes its minimum value at the end point $(y = y_0)$ of the above integral. So we get

$$\left\| \frac{\partial^k}{\partial y^k} u(x, y_0) \right\|_p^{q_1} \int_{y_0/2}^{y_0} y^{(\alpha-k)q_1} \frac{dy}{y} \leq A^{q_1},$$

that is

$$\left\| \frac{\partial^k u}{\partial y^k} \right\|_p \leq cAy^{-k+\alpha} \tag{66}$$

In other words $f \in \Lambda_\alpha^{p,q_1}$ implies also that $f \in \Lambda_\alpha^{p,\infty}$. Combining (66) with (65) shows easily that

$$\left(\int_0^\infty \left(y^{\alpha-k} \left\| \frac{\partial^k u}{\partial y} \right\|_p\right)^{q_2} \frac{dy}{y}\right)^{1/q_2} < \infty, \qquad q_2 \geq q$$

and so $f \in \Lambda_\alpha^{p,q_2}$.

For other inclusion relations of this type see also §6.7 below.

**5.3 Comparison of $\mathscr{L}_\alpha^p$ with $\Lambda_\alpha^{p,q}$** We come now to one of our main goals whose interest justifies much of the preparatory work described in

§4 and §5. The comparison between the potential spaces $\mathscr{L}_\alpha^p$ and the Lipschitz spaces $\Lambda_\alpha^{p,q}$ achieves the deepest insight in this chapter and incidentally the only one which uses the Littlewood-Paley theory of Chapter IV. The result is as follows

THEOREM 5. *Suppose $1 < p < \infty$ and $\alpha > 0$. Then*

$$\text{(A)} \qquad \mathscr{L}_\alpha^p \subset \Lambda_\alpha^{p,p} \quad \textit{if} \quad p \geq 2$$

$$\text{(B)} \qquad \mathscr{L}_\alpha^p \subset \Lambda_\alpha^{p,2} \quad \textit{if} \quad p \leq 2$$

$$\text{(C)} \qquad \Lambda^{p,p} \subset \mathscr{L}_\alpha^p \quad \textit{if} \quad p \leq 2$$

$$\text{(D)} \qquad \Lambda_\alpha^{p,p} \subset \mathscr{L}_\alpha^p \quad \textit{if} \quad p \geq 2.$$

The fact that sharper inclusion relations of this type are not possible is contained in §6.8 and §6.9 below.

Because of the isomorphisms given by the operators $\mathscr{J}_\beta$ (see Theorem 4, and (41) in §3.3), it suffices to prove the inclusion relations for any particular value of $\alpha$. It will be convenient for us to take $\alpha = 1$. In view of Theorem 3 of §3.3 the space $\mathscr{L}_1^p$ is equivalent with $L_1^p$, (when $1 < p < \infty$) and we have therefore reduced considerations to the proof of the inclusion relations for $\alpha = 1$, and with $\mathscr{L}_1^p$ replaced by $L_1^p$.

The norms in $\Lambda_1^{p,q}$ are expressed in terms of quantities which involve the second derivatives of $u$, with $u$ the Poisson integral of $f$. It is for this reason we consider the following variants of the Littlewood-Paley functions:

$$\text{(67)} \qquad \begin{cases} \mathscr{G}_p(x) = \left(\displaystyle\int (y\,|\nabla^2 u(x,y)|)^p \frac{dy}{y}\right)^{1/p}, & \text{if} \quad p < \infty \\ \mathscr{G}_\infty(x) = \displaystyle\sup_{y>0} y\,|\nabla^2 u(x,y)| \end{cases}$$

Here

$$|\nabla^2 u(x,y)|) = \sum_{k=0}^n \sum_{j=0}^n \left| \frac{\partial^2}{\partial x_j\,\partial x_k} u(x,y) \right|^2, \quad \text{with} \quad x_0 = y.$$

Assume that $\dfrac{\partial f}{\partial x_j} \in L^p(\mathbf{R}^n)$, $j = 1, \ldots, n$. Then since $u$ is the Poisson integral of $f$, the Poisson integral of $\dfrac{\partial f}{\partial x_j}$ is $\dfrac{\partial u}{\partial x_j}$, as we have already pointed out in §4.3. Recalling the definition of the $g$-function, (see Chapter IV, §1.1), we see that

$$\left[g\left(\frac{\partial f}{\partial x_j}\right)(x)\right]^2 = \sum_{k=0}^n \int_0^\infty y \left| \frac{\partial^2}{\partial x_j\,\partial x_k} u(x,y) \right|^2 dy, \qquad x_0 = y.$$

However, $\dfrac{\partial^2 u}{\partial y^2} = -\sum_{j=1}^{n} \dfrac{\partial^2 u}{\partial x_j^2}$, and therefore

$$\mathscr{G}_2(x) \leq c \sum_{j=1}^{n} g\left(\frac{\partial f}{\partial x_j}\right)(x).$$

By §4.9 of Chapter III

$$\sup_{y>0} \left| y \frac{\partial^2}{\partial y^2} u(x, y) \right| \leq A \sum_{j=1}^{n} M\left(\frac{\partial f}{\partial x_j}\right)(x)$$

Thus

$$\| \mathscr{G}_2(x)\|_p \leq A_p \sum_{j=1}^{n} \left\| \frac{\partial f}{\partial x_j} \right\|_p \tag{68}$$

and

$$\| \mathscr{G}_\infty(x)\|_p \leq A_p \sum_{j=1}^{n} \left\| \frac{\partial f}{\partial x_j} \right\|_p . \tag{69}$$

The inequality for $\mathscr{G}_2$ holds because of Theorem 1 in Chapter VI, and that for $\mathscr{G}_\infty$ is a consequence of the maximal theorem in Chapter I. Now it is clear that $\mathscr{G}_p^p(x) \leq \mathscr{G}_2^2(x)\mathscr{G}_\infty^{p-2}(x)$, if $p \geq 2$. Hence

$$\mathscr{G}_p(x) \leq \mathscr{G}^{2/p}(x)\, \mathscr{G}_\infty^{(p-2)/p}(x) = \mathscr{G}_2^\theta\, (x)\mathscr{G}_\infty^{1-\theta}(x),$$

(with $\theta = 2/p$).

Therefore by Hölder's inequality (68), and (69),

$$\|\mathscr{G}_p(x)\|_p \leq \| \mathscr{G}_2(x)\|_p^\theta\, \| \mathscr{G}_\infty(x)\|_p^{1-\theta} \leq A_p \sum_{j=1}^{\infty} \left\| \frac{\partial f}{\partial x_j} \right\|_p$$

We have then that in particular

$$\int_0^\infty \left( y \left\| \frac{\partial^2 u}{\partial y} \right\|_p \right)^p \frac{dy}{y} < \infty, \quad \text{if} \quad 2 \leq p < \infty.$$

This shows that if $2 \leq p < \infty$, then $L_1^p(\mathbf{R}^n) \subset \Lambda_1^{p,p}(\mathbf{R}^n)$, and conclusion (A) is proved.

To prove (B) apply Minkowski's inequality for integrals in the form that if $F(x, y) \geq 0$, and $r \geq 1$

$$\left(\int_0^\infty \left\{\int_{\mathbf{R}^n} F(x, y)\, dx\right\}^r y\, dy\right)^{1/r} \leq \int_{\mathbf{R}^n} \left(\int_0^\infty F^r(x, y) y\, dy\right)^{1/r} dx, \tag{70}$$

to the effect that the norm of an integral is not greater than the integral of the norms. Take $r = 2/p$ (here $p \leq 2$), and $F(x, y) = |\nabla u(x, y)|^p$. Then (70) can be rewritten as

$$\int_0^\infty y \, \| \, |\nabla^2 u| \, \|_p^2 \, dy \leq \left(\int_{\mathbf{R}^n} (\mathscr{G}_2(x))^p \, dx\right)^{2/p},$$

and the latter is finite by (68), if $f \in L_1^p$. Therefore $f \in \mathscr{L}_1^p \Rightarrow f \in \Lambda_1^{p,2}$, if $p \leq 2$ and (B) is proved.

The arguments above also show that $\|f\|_{\Lambda_1^{p,q}} \leq A_p \|f\|_{L_1^p}$, with $q = p$, if $2 \leq p < \infty$, and $q = 2$, if $1 < p \leq 2$.

We shall prove the converse inclusions, (C) and (D) by establishing the *a priori* inequalities

$$\|f\|_{L_1^p} \leq A_p \|f\|_{\Lambda_1^{p,q}}, \tag{71}$$

with $q = p$ if $1 < p \leq 2$, and $q = 2$, if $2 \leq p < \infty$, under the assumption that $f$ belongs to $L_1^p$.

This turns out to be merely an inversion of the arguments just given. In fact when $r \leq 1$, then Minkowski's inequality for integrals shows that (70) holds with a reversal of the sign of inequality. Thus we get

$$\left(\int_{\mathbf{R}^n} (\mathscr{G}_p(x))^p \, dx\right)^{2/p} \leq \int_0^\infty y \, \|\nabla^2 u\|_p^2 \, dy, \quad \text{if} \quad p \geq 2.$$

But since $A_p' \left\| \dfrac{\partial f}{\partial x_j} \right\|_p \leq \|\mathscr{G}_2(x)\|_p$, according to the converse of Theorem 1 of Chapter IV, we obtain (71) for $2 \leq p < \infty$.

Similarly if $1 < p \leq 2$, then $\mathscr{G}_2(x) \leq \mathscr{G}_p^\theta(x)\mathscr{G}_\infty^{1-\theta}(x)$, with $\theta = p/2$, and therefore by Hölder's inequality

$$\|\mathscr{G}_2(x)\|_p \leq \|\mathscr{G}_p(x)\|_p^\theta \, \|\mathscr{G}_\infty(x)\|_p^{1-\theta}.$$

Again by the Littlewood-Paley Theorem, $\|\mathscr{G}_2\|_p$ exceeds $A_p' \left\| \dfrac{\partial f}{\partial x_j} \right\|_p$ and then by (69)

$$\left(A'' \sum \left\| \frac{\partial f}{\partial x_j} \right\|_p\right)^\theta \leq \|\mathscr{G}_p(x)\|_p^\theta$$

However,

$$\begin{aligned} \|\mathscr{G}_p(x)\|_p &= \left(\int_0^\infty y^p \, \|\nabla^2 u\|_p^p \frac{dy}{y}\right)^{1/p} \\ &\leq c\left(\int_0^\infty y^p \left\|\frac{\partial^2 u}{\partial y^2}\right\|_p^p \frac{dy}{y}\right)^{1/p} \leq c\,\|f\|_{\Lambda_1^{p,p}} \end{aligned}$$

This is because of (62′) and the definition of the $\Lambda^{p,}{}_p$ norm. Therefore (71) is also proved for $1 < p \leq 2$.

Finally, to lift the restriction $f \in L_1^p$, we consider $u(x, \varepsilon)$ instead of $f$, with $\varepsilon > 0$. Then clearly $u(x, \varepsilon) \in L_1^p(\mathbf{R}^n)$, if $f \in \Lambda_1^{p,q}$ (since then $f \in L^p(\mathbf{R}^n)$). Therefore by (71)

$$\|u(x, \varepsilon)\|_{L_1^p} \leq A_p \|u(x, \varepsilon)\|_{\Lambda_1^{p,q}} \leq A_p \|f\|_{\Lambda_1^{p,q}}$$

So the family $u(x, \varepsilon)$ converges in $L^p$ norm to $f$, and its $L_1^p$ norm remains uniformly bounded. From this, we see that for each $j$,

$$\int_{\mathbf{R}^n} \frac{\partial}{\partial x_j} u(x, \varepsilon)\varphi \, dx \to -\int_{\mathbf{R}^n} f(x) \frac{\partial \varphi}{\partial x_j} \, dx,$$

whenever $\varphi \in \mathscr{D}$, and that the linear functional $\varphi \to \int_{\mathbf{R}^n} f \frac{\partial \varphi}{\partial x_j} \, dx$ is bounded in the norm dual to that of $L^p(\mathbf{R}^n)$. Therefore by the Riesz representation theorem there exists a $g_j$ so that

$$\int_{\mathbf{R}^n} f \frac{\partial \varphi}{\partial x_j} = -\int_{\mathbf{R}^n} g_j \varphi \, dx,$$

with $g_j \in L^p$. This shows that $f \in L_1^p$ and the theorem is completely proved.

**5.4 A point left open.** We return to the proof of (59) we had postponed until now. If we look back to the definition of $\Lambda_\beta^{p,q}$ given in §5.1 we see that what we need to show has the following interpretation:

$$G_\beta(x) \in \Lambda_\beta^{1,\infty}, \quad \text{if} \quad \beta > 0. \tag{72}$$

Let us first consider the case $0 < \beta < 1$. Since $G_\beta \in L^1(\mathbf{R}^n)$, we must see, according to Proposition 7′, that

$$\int_{\mathbf{R}^n} |G_\beta(x + t) - G_\beta(x)| \, dx \leq A \, |t|^\beta.$$

We write

$$\int_{\mathbf{R}^n} |G_\beta(x + t) - G_\beta(x)| \, dt = \int_{|x| \leq 2|t|} | \cdot | \, dt + \int_{|x| > 2|t|} | \cdot | \, dt.$$

The first integral can be estimated by

$$\int_{|x| \leq 2|t|} [|G_\beta(x + t)| + |G_\beta(x)|] \, dx \leq 2 \int_{|x| \leq 3|t|} |G_\beta(x)| \, dx$$

Because $G_\beta(x) \leq c \, |x|^{-n+\beta}$ (see (29) and (30) in §3.1) we see that

$$2 \int_{|x| \leq 3|t|} |G_\beta(x)| \, dx \leq A \, |t|^\beta.$$

Next by differentiating the formula (26) in §3.1 we are led quickly to the bound

$$\begin{aligned}
\left| \frac{\partial G_\beta}{\partial x_j} \right| &= c \, |x_j| \int_0^\infty e^{-\pi|x|^2/\delta} e^{-\delta/4\pi} \, \delta^{(\beta-n-2)/2} \frac{d\delta}{\delta} \\
&\leq c \, |x_j| \int_0^\infty e^{-\pi|x|^2/\delta} \, \delta^{(\beta-n-2)/2} \frac{d\delta}{\delta} = c' \, |x_j| \, |x|^{-n+\beta-2} \\
&\leq c' \, |x|^{-n+\beta-1}
\end{aligned}$$

Therefore $|G_\beta(x+t) - G_\beta(x)| \le c'' \, |t| \, |x|^{-n+\beta-1}$, if $|x| \ge 2\,|t|$ and hence $\int_{|x|\ge 2|t|} |G_\beta(x+t) - G_\beta(x)| \, dx \le A\,|t|^\beta$, if $0 < \beta < 1$. Thus (59) and also (72) are proved when $0 < \beta < 1$.

To pass to the general case we observe that wherever $r$ is a positive integer, $G_{\beta r} = G_\beta * G_\beta * \cdots * G_\beta$, ($r$ factors), and $P_y = P_{y_1} * P_{y_2} \cdots * P_{y_r}$, if $y = y_1 + y_2 \cdots + y_r$, and $y_k > 0$. Consequently

$$G_{\beta r}(\cdot, y) = G_\beta(\cdot, y_1) * G_\beta(\cdot, y_2) \cdots * G_\beta(\cdot, y_r)$$

Now differentiate this relation once with respect to each of the variables $y_1, y_2, \ldots, y_r$, and then set $y_1 = y_2 \cdots y_2 = y/r$. The result is

$$\left\| \frac{\partial^r G_{\beta r}}{\partial y^r}(x, y) \right\| \le A y^{-\beta r}$$

Since $0 < \beta < 1$, and is otherwise arbitrary we get the required estimate (for $\beta r$ in place of $\beta$) in (59); this also implies (72).

## 6. *Further results*

**6.1** $f$ belongs to $L_1^p(\mathbf{R}^n)$ if and only if $f \in L^p(\mathbf{R}^n)$ and (i) $f$ can be modified on a set of measure zero so that it is absolutely continuous in the sense of Tonelli; (ii) $\dfrac{\partial f}{\partial x_j} \in L^p(\mathbf{R}^n)$, $j = 1, \ldots, n$ (the derivatives exist almost everywhere).

**6.2** $f$ belongs to $L_k^\infty(\mathbf{R}^n)$, $k \ge 1$ if and only if $f$ can be modified on a set of measure zero so that either of the following two equivalent conditions are satisfied.

(a) $f$ has continuous partial derivatives of total order $\le k - 1$. Moreover, whenever $g = \dfrac{\partial^\alpha f}{\partial x^\alpha}$, $|\alpha| \le k - 1$, then

$$\sup_x |g(x)| < \infty \quad \text{and} \quad \sup_{x,x'} \frac{|g(x) - g(x')|}{|x - x'|} < \infty.$$

(b) There exists a sequence $\{\varphi_n\}$, $\varphi_n \in \mathscr{D}$, so that $\varphi_n \to f$ uniformly on every compact set and

$$\sup_{|\alpha|\le k} \sup_n \left\| \frac{\partial^\alpha \varphi_n}{\partial x^\alpha} \right\|_\infty < \infty.$$

(Hint: See the proof of Proposition 3 in §3.5, and Proposition 1 in §2.1 respectively.)

**6.3** Suppose $1 < p < \infty$, and $1/p = k/n$. Then there exists an $f \in L_k^p(\mathbf{R}^n)$ which is essentially unbounded in the neighborhood of every point.

*Hint:* Consider, for example, the case $n = 2$, $k = 1$, (then $p = 2$). Let $\varphi(x) = |x|^{-1}(\log 1/|x|)^{-1}$, if $|x| \leq 1/2$, $\varphi = 0$, otherwise. Set $f_0 = I_1(\varphi)$. Then $\dfrac{\partial f_0}{\partial x_j} = R_j(\varphi) \in L^2$. However, $f_0$ is not bounded near the origin. One may also construct a similar $f_0$ more directly by taking $f_0(x) = \log\log 1/|x|$, for small $x$, and $f_0$ positive, smooth, and with compact support, away from the origin. Finally set $f(x) = \sum_{k=1}^{\infty} 2^{-k} f_0(x - r_k)$, where $\{r_k\}$ is dense set in $\mathbf{R}^n$.

**6.4** The following is the generalization of inequality (23) in §2.5. Suppose $1 < k \leq n$.

Then with $1/q = 1 - k/n$, $f \in \mathscr{D}$,

$$\|f\|_q \leq \left(\prod_{i_1, \cdots i_k} \left\| \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}} \right\|_1\right)^{1/\binom{n}{k}}$$

where the product ranges over the $\dbinom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$ ways of picking distinct $i_1, i_2, \ldots, i_k$ from $1, 2, \ldots, n$.

*Hint:* Consider, for example, $k = n - 1$. Write $I_j(x_j) = \displaystyle\int_{\mathbf{R}^k} \left| \frac{\partial^{n-1} f}{\partial \hat{x}_j^{n-1}} \right| d\hat{x}_j$, where the symbol $\hat{x}_j$ indicates that the variable has been omitted. Clearly $|f(x)| \leq I_j(x_j)$, and so $|f(x)|^n \leq \prod_{j=1}^{n} I_j(x_j)$. Integrate. (If we start with the identity $f(x) = \frac{1}{2}\displaystyle\int_{-\infty}^{\infty} \operatorname{sign}(x - t) f'(t)\,dt$, instead of $f(x) = \displaystyle\int_{-\infty}^{x} f'(t)\,dt$, then the above inequality is improved by a factor of $2^{-k}$.)

**6.5** An alternative formula to (26) for the Bessel kernel is

$$G_\alpha(x) = c_\alpha e^{-|x|} \int_0^\infty e^{-|x|t}\left(t + \frac{t^2}{2}\right)^{(n-\alpha-1)/2} dt, \quad \text{for} \quad 0 < \alpha < n + 1$$

$$c_\alpha^{-1} = (2\pi)^{(n-1)/2} 2^{\alpha/2}\, \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{n - \alpha + 1}{2}\right).$$

See Aronszajn and Smith [1].

**6.6** The following describe the possible inclusion relations between $L_k^p(\mathbf{R}^n)$ and $\mathscr{L}_k^p(\mathbf{R}^n)$, in the extreme cases $p = 1$, and $p = \infty$.

(a) When $n = 1$, then $L_k^p(\mathbf{R}^1) = \mathscr{L}_k^p(\mathbf{R}^1)$, when $k$ is even and $p = 1$, or $\infty$.

(b) When $n > 1$, then $L_k^p(\mathbf{R}^n) \subset \mathscr{L}_k^p(\mathbf{R}^n)$, when $k$ is even, and $p = 1$ or $\infty$; the reverse inclusion fails for both $p = 1$ and $p = \infty$.

(c) For all $n$, if $k$ is odd then neither $L_k^p(\mathbf{R}^n) \subset \mathscr{L}_k^p(\mathbf{R}^n)$ nor $\mathscr{L}_k^p(\mathbf{R}^n) \subset L_k^p(\mathbf{R}^n)$.

*Hints:* For (a) use the fact that $f \in L^p(\mathbf{R}^1)$ and $\dfrac{d^2 f}{dx^2} \in L^p(\mathbf{R}^1)$ implies $\dfrac{df}{dx} \in L^p(\mathbf{R}^1)$. To show that $\mathscr{L}_k^p(\mathbf{R}^n) \not\subset L_k^p(\mathbf{R}^n)$, use the unboundedness of the higher Riesz transforms for $L^1$ and $L^\infty$ (see §6.1 in Chapter II). To see, e.g. that

$L_1^\infty(\mathbf{R}^n) \not\subset \mathscr{L}_1^\infty(\mathbf{R}^n)$, use the function $G_{n+1}(x)$. From formula (26) it follows easily that $G_{n+1}$ and $\dfrac{\partial G_{n+1}}{\partial x_j} \in L^\infty$, thus $G_{n+1} \in L_1^\infty$. However, $G_{n+1} \notin \mathscr{L}_1^\infty(\mathbf{R}^n)$, since $G_n(x) \approx \log \dfrac{1}{|x|}$, as $|x| \to 0$, and so $G_n \notin L^\infty$. The fact that $G_n \approx \log \dfrac{1}{|x|}$ as $|x| \to 0$ also follows from (26), in the same way as (29) is proved. Special functions useful in this connection are studied in Wainger [1].

**6.7**

(a) $\Lambda_{\alpha_1}^{p_1,q}(\mathbf{R}^n) \subset \Lambda_{\alpha_2}^{p_2,q}(\mathbf{R}^n)$ if $\alpha_1 \geq \alpha_2$ and $\alpha_1 - \dfrac{n}{p_1} = \alpha_2 - \dfrac{n}{p_2}$.

(b) If $f \in \Lambda_{\alpha_j}^{p_j,q_j}(\mathbf{R}^n)$ where $j = 0, 1$, then $f \in \Lambda_\alpha^{p,q}(\mathbf{R}^n)$, where

$$\alpha = \alpha_0(1 - \theta) + \alpha_1\theta,$$

$$\frac{1}{p} = \frac{1}{p_0}(1 - \theta) + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1}{q_0}(1 - \theta) + \frac{\theta}{q_1},$$

for each $0 < \theta < 1$. See Hardy-Littlewood [2], Taibleson [2].

**6.8** Suppose that $f_{\alpha,\sigma}(x) = e^{-\pi x^2} \cdot \sum_{k=1}^{\infty} a^{-k\alpha} k^{-\sigma} e^{2\pi i a^k x}$, $x \in \mathbf{R}^1$, where $a$ is an integer $> 1$.

(i) $f_{\alpha,\sigma} \in \mathscr{L}_\alpha^p(\mathbf{R}^1) \Leftrightarrow \sigma > \dfrac{1}{2}$, for $1 \leq p < \infty$.

(ii) $f_{\alpha,\sigma} \in \Lambda_\alpha^{p,q}(\mathbf{R}^1) \Leftrightarrow \sigma > \dfrac{1}{q}$, (for $1 \leq p \leq \infty$).

Thus $\mathscr{L}_\alpha^p(\mathbf{R}^1) \not\subset \Lambda_\alpha^{p,q}(\mathbf{R}^1)$ if $q < 2$, and $\Lambda_\alpha^{p,q}(\mathbf{R}^1) \not\subset \mathscr{L}_\alpha^p(\mathbf{R}^1)$ if $q > 2$.

**6.9** Let $g_{\alpha,\delta,p}(x) = |x|^{\alpha - n/p}(\log 1/|x|)^{-\delta}$ for $|x| < \dfrac{1}{2}$, and assume that $g_{\alpha,\delta,p}$ is smooth away from the origin and has compact support. Assume $\alpha < n/p$.

(i) $g_{\alpha,\delta,p} \in \mathscr{L}_\alpha^p(\mathbf{R}^n) \Leftrightarrow \delta p > 1$

(ii) $g_{\alpha,\delta,p} \in \Lambda_\alpha^{p,q}(\mathbf{R}^n) \Leftrightarrow \delta q > 1$

Thus $\mathscr{L}_\alpha^p(\mathbf{R}^n) \not\subset \Lambda_\alpha^{p,q}(\mathbf{R}^n)$ if $q < p$, and $\Lambda_\alpha^{p,q}(\mathbf{R}^n) \not\subset \mathscr{L}_\alpha^p(\mathbf{R}^n)$ if $q > p$. For examples closely related to §6.8 and §6.9 see Taibleson [2].

**6.10** Suppose $0 < \alpha < 2$. Then $f \in \mathscr{L}_\alpha^p(\mathbf{R}^n) \Leftrightarrow f \in L^p$ and

(i) $\lim\limits_{\varepsilon \to 0} I_\varepsilon \equiv \lim\limits_{\varepsilon \to 0} \displaystyle\int_{|t| > \varepsilon} \frac{[f(x + t) - f(x)]}{|t|^{n+\alpha}}\, dt$ converges in $L^p$ norm, if $1 \leq p < \infty$.

(ii) $I_\varepsilon$ remains bounded in $L^\infty$ norm, when $p = \infty$. See Stein [7]; also Wheeden [2].

*Hint:* Verify that if $f \in \mathscr{D}$, $\lim_{\varepsilon \to 0} \int_{|t| \geq \varepsilon} \frac{f(x+t) - f(x)}{|t|^{n+\alpha}}\, dt = c_\alpha(-\Delta)^{\alpha/2} f$. Conversely, suppose $f = \mathscr{I}_\alpha(g)$, with $g \in L^p$. Then $f = I_\alpha(\gamma)$, $\gamma \in L^p$. (See §3.2). Also

$$\int_{|t| > \varepsilon} \frac{f(x+t) - f(x)}{|t|^{n+\alpha}}\, dt = \int K_\varepsilon(t)\gamma(x+t)\, dt,$$

where $K_\varepsilon(t) = \varepsilon^{-n} K(t/\varepsilon)$; it can be shown that $|K(x)| \leq A\,|x|^{-n+\alpha}$ and $|K(x)| \leq A\,|x|^{-n+\alpha-2}$, thus $K \in L^1(\mathbf{R}^n)$.

**6.11** (a) The space $\mathscr{L}^p_\alpha(\mathbf{R}^n)$ is an algebra under pointwise multiplication if and only if every element of $\mathscr{L}^p_\alpha(\mathbf{R}^n)$ is continuous. This holds if and only if $p > n/\alpha$.

(b) Let $\chi_K$ be the characteristic function of an arbitrary convex set $K \subset \mathbf{R}^n$. Then the mapping $f \to \chi_K \cdot f$ is continuous in $\mathscr{L}^p_\alpha(\mathbf{R}^n)$ when $0 \leq \alpha < 1/p$. For these and related results see Strichartz [1]. For $p = 2$, see also Hirschmann [2].

**6.12** Suppose $F = I_\alpha(f)$, and $0 < \alpha < 1$.

$$(\mathscr{D}_\alpha F)(x) = \left( \int_{\mathbf{R}^n} \frac{|F(x+t) - F(x)|^2}{|t|^{n+2\alpha}}\, dt \right)^{1/2}$$

Then

$$A_\alpha g_1(f)(x) \leq \mathscr{D}_\alpha(F)(x) \leq B_\alpha g^*_\lambda(f)(x)$$

where $\lambda < 1 + 2\alpha/n$, and the functions $g_1$ and $g_*$ have been defined in Chapter IV, (§1.2 and §2.2 respectively). $A_\alpha$ and $B_\alpha$ are appropriate constants.

*Hint:* Write $U(x, y)$ and $u(x, y)$ for the Poisson integrals of $F$ and $f$ respectively. Since

$$\frac{\partial^2 U}{\partial y^2} = \int \frac{\partial^2 P_y}{\partial y^2}(t)\,[F(x+t) - F(x)]\, dt,$$

simple estimates show that $\int_0^\infty y^{3+2\alpha} \left| \frac{\partial^2 U}{\partial y^2} \right|^2 dy \leq c_1(\mathscr{D}_\alpha(F))^2$. Next

$$\begin{aligned} \frac{\partial u}{\partial y}(x, y) &= \frac{1}{\Gamma(1-\alpha)} \int_0^\infty \frac{\partial^2 U}{\partial y^2}(x, y+s) s^\alpha\, ds \\ &= \frac{1}{\Gamma(1-\alpha)} \int_y^\infty \frac{\partial^2 U}{\partial y^2}(x, s)(s-y)^\alpha\, ds \end{aligned}$$

Therefore $\int_0^\infty y \left| \frac{\partial u}{\partial y} \right|^2 dy \leq c_2 \int_0^\infty y^{3+2\alpha} \left| \frac{\partial^2 U}{\partial y^2} \right|^2 dy$, and so $A_\alpha g_1(f)(x) \leq \mathscr{D}_\alpha(F)(x)$. Conversely, we have

$$|F(x+t) - F(x)| \leq \int_{L_1} |\nabla U|\, ds + \int_{L_2} |\nabla U|\, ds + \int_{L_3} |\nabla U|\, ds$$

where $L_1$, $L_2$, and $L_3$ are respectively the line segments joining $(x, 0)$ with $(x, y)$; $(x + t, 0)$ with $(x + t, y)$; and $(x + t, y)$ with $(x, y)$. However $U(x, y) = \frac{1}{\Gamma(\alpha)} \int_0^\infty u(x, y + s) s^{-1+\alpha}\, ds$, therefore

$$|\nabla U(x, y)| \leq \frac{1}{\Gamma(\alpha)} \int_0^\infty |\nabla u(x, y + s)|\, s^{-1+\alpha}\, ds.$$

If we substitute this estimate in the above, set $y = |t|$, and carry out the indicated integrations, we get after some further reduction the result $\mathscr{D}_\alpha(F)(x) \leq B_\alpha g_\lambda^*(f)(x)$, with $\lambda < 1 + \frac{2\alpha}{n}$. See also the bibliographical references in §6.13 below.

**6.13** (a) Suppose that $0 < \alpha < 1$, and $\frac{2n}{n + \alpha} < p < \infty$ (the latter holds in particular if $2 \leq p < \infty$). Then $f \in \mathscr{L}_\alpha^p(\mathbf{R}^n)$ if and only if $f \in L^p(\mathbf{R}^n)$ and $\mathscr{D}_\alpha(f) \in L^p(\mathbf{R}^n)$. Also $\|f\|_{p,\alpha}$ is comparable with $\|f\|_p + \|\mathscr{D}_\alpha(f)\|_p$.

(b) The similar result holds in the larger range $0 < \alpha < 2$ if $\mathscr{D}_\alpha(f)(x)$ is replaced by

$$\left(\int_{\mathbf{R}^n} \frac{|f(x + t) + f(x - t) - 2f(x)|^2}{|t|^{n+2\alpha}}\, dt\right)^{1/2}$$

The results of §6.12 and §6.13 (a) and (b) are stated in Stein [7]. For the earlier (one-dimensional) theory see Marcinkiewicz [2], Zygmund [1], and Hirschmann [1]. For a recent stronger result, dealing with the critical case $p = \frac{2n}{n + \alpha}$, see Fefferman [1].

(c) A variant of (a) above holds for all $p$ in $1 < p < \infty$ if $\mathscr{D}_\alpha(f)$ is replaced by

$$\left(\int_0^\infty \left\{\int_B |f(x + rt) - f(x)|\, dt\right\}^2 \frac{dr}{r^{1+2\alpha}}\right)^{1/2}$$

where $B$ is the unit ball. See Strichartz [1].

**6.14** Let $\beta > \alpha$. Then $T$ is a bounded linear transformation from $\Lambda_\alpha(\mathbf{R}^n)$ to $\Lambda_\beta(\mathbf{R}^n)$ which commutes with translations if and only if $T$ is of the form $Tf = K * f$, with $K \in \Lambda_{\beta-\alpha}^{1,\infty}(\mathbf{R}^n)$. See Zygmund [6] for the case $n = 1$, and Taibleson [2] for the general case.

**6.15** (a) Suppose $T$ is of the type discussed in §6.14 above. Then $T$ maps $L^p(\mathbf{R}^n)$ boundedly into $L^q(\mathbf{R}^n)$ if $1 < p$, $q < \infty$ and $1/q = 1/p - \frac{(\beta - \alpha)}{n}$.

(b) However, there exists a $T$ commuting with translations and mapping $\Lambda_\alpha(\mathbf{R}^n)$ to $\Lambda_\alpha(\mathbf{R}^n)$ boundedly, but which is not bounded on $L^p(\mathbf{R}^n)$, for $p \neq 2$. See Stein and Zygmund [2]; earlier results in this direction are in Hardy and Littlewood [3].

**6.16** The last set of results deal with the space of functions of *bounded mean oscillation*; it illustrates the fact that this class arises often as a substitute for the space $L^\infty$ in the usual limiting cases where results break down for $L^\infty$.

(a) Suppose $f$ is defined on $\mathbf{R}^n$. Then it is said to be of bounded mean oscillation (on $\mathbf{R}^n$), (abbreviated as BMO), if there exists a constant $M$, so that $\frac{1}{m(Q)}\int_Q |f(x) - a_Q|\,dx \le M$, for every cube $Q$ in $\mathbf{R}^n$; $a_Q$ is the mean-value of $f$ over $Q$. Notice that every bounded function is BMO; however, the function $\log|x|$ can be seen to be BMO, so the converse does not hold. That this example is to a certain extent typical can be seen by the fact that it is possible to make the estimate

$$m\{x \in Q : |f(x) - a_Q| > \alpha\} \le e^{-c\alpha/M} m(Q), \quad \text{every} \quad \alpha > 0.$$

In particular if $f$ is BMO, then $\int_Q e^{a|f|}\,dx < \infty$, for every cube $Q$, for appropriate positive $a$. See John and Nirenberg [1].

(b) Let $T$ be one of the singular integrals transforms dealt with by Theorem 1, its corollary, or Theorems 2 and 3 of Chapter II. Suppose $f$ is bounded. Then $Tf$ is BMO. See Stein [8].

**6.17** (a) Suppose $f$ is in BMO. Then $\mathscr{I}_\alpha f \in \Lambda_\alpha(\mathbf{R}^n)$ for $\alpha > 0$.

(b) Suppose $f$ is of weak type $p$; that is $m\{x : |f(x)| > \lambda\} \le A\lambda^{-p}$, $0 < \lambda < \infty$, with $1 < p < \infty$. Then $\mathscr{I}_\alpha(f) \in \text{BMO}$, if $\alpha = n/p$. (Compare with Theorem 1 in this chapter.)

See Stein and Zygmund [2].

**6.18** Suppose $f \in \Lambda_1^{\infty,2}$. That is, suppose $f \in L^\infty(\mathbf{R}^n)$ and

$$\int_{\mathbf{R}^n} \frac{\|f(x+t) + f(x-t) - 2f(x)\|_\infty^2}{|t|^{n+2}}\,dt < \infty.$$

Then $\dfrac{\partial f}{\partial x_j} \in \text{BMO}$, $j = 1, \ldots, n$.

*Hint:* Using the reasoning of §4 and §5, the assumptions can be shown to imply the existence of a function $\delta(s)$ on $0 < s < \infty$, so that

(i) $\delta(s)$ is non-decreasing on $0 < s < \infty$

(ii) $\displaystyle\int_0^1 \frac{\delta^2(s)}{s}\,ds < \infty$

(iii) $\|f(x+t) + f(x-t) - 2f(x)\|_\infty \le |t|\,\delta(|t|)$.

With this done one can then adapt the reasoning in John and Nirenberg [1] given for the case $\delta(s) = \left(\log \frac{1}{s}\right)^{-1/2-\varepsilon}$, $\varepsilon > 0$ $(s < 1/2)$. An earlier related result is in M. Weiss and Zygmund [1].

## Notes

*Section 1.* For the Riesz potentials see M. Riesz [2], and an earlier one-dimensional version in Weyl [1]. The $L^p$ inequalities of Theorem 1 are due to Hardy and Littlewood [2], when $n = 1$, and to Sobolov [1] for general $n$. The fact that the fractional integration mapping is of weak-type $(1, n/(n - \alpha)$ appears first in Zygmund [4]. For a general treatment in terms of Lorentz spaces see O'Neil [1]; the simple proof given in the text is taken from Muckenhoupt and Stein [1].

*Section 2.* Theorem 2 goes back to Sobolov [1]. The case $p = 1$, was however not dealt with until later by Gagliardo [2] and Nirenberg [1].

*Section 3.* The Bessel potentials and the corresponding spaces $\mathscr{L}^p_\alpha$ were introduced by Aronszajn and Smith [1], and Calderón [4]. Lemma 2 connecting the Bessel and Riesz potentials is stated in Stein [7]. The identification of $\mathscr{L}^p_k$ with $L^p_k$, $1 < p < \infty$, is proved in Calderón [4], and the characterization of $\mathscr{L}^2_\alpha$ given by Proposition 4 is taken from Aronszajn and Smith [1].

*Sections 4 and 5.* For the case $n = 1$ most of the results given here were formulated and proved in one form or another (and sometimes only implicitly) by Hardy and Littlewood [2], and [3], Zygmund [6], and Hirschmann [1]. For the first explicit treatment of the spaces $\Lambda^{p,q}_\alpha$ (in $n$-dimensions) see Besov [1]; this was preceded however by a significant paper of Gagliardo [1]. The presentation given in these two sections leans heavily on the systematic treatment of Taibleson [2]; Theorems 4′ and 5 in particular are due to him. The reader may also consult the earlier survey paper of Nikolskĭi [1].

CHAPTER VI

# Extensions and Restrictions

If we want to apply the results of harmonic analysis of $\mathbf{R}^n$ to a variety of other problems we are often faced with the following situation. Let $S$ be a subset of $\mathbf{R}^n$, (the nature of $S$ will be specified later), and consider one of the Banach spaces of functions on $\mathbf{R}^n$ we have already studied. Two problems then arise. The *restriction* problem: What is the space of functions that arise by restriction to $S$ of the functions in the given Banach space? There is also the closely related *extension* problem: Given an appropriate space of functions defined on $S$, how can these functions be extended to $\mathbf{R}^n$?

The techniques and results differ depending on the nature of the set $S$, although there is some overlapping. We shall single out three cases which seem to be of genuine interest.

(i) The set $S$ is an arbitrary closed set $F$. The appropriate function spaces are those composed of functions which have continuous partial derivatives up to a certain order, together with bounds on their moduli of continuity. The type of extension considered goes back to Whitney and we follow his construction except for details.

(ii) The set $S$ is a domain (open subset of $\mathbf{R}^n$) whose boundary satisfies a certain minimal smoothness condition. If the domain had a smooth (say $C^\infty$) boundary the extension result would be much easier and a simpler construction would do the job. The main point of the given extension is that one needs to assume what amounts roughly to only one order of differentiability of the boundary, and obtain extensions for all orders of differentiability. The presentation we shall give (in §3) will be based on ideas different from the one initially introduced by Calderón in this context. The gist of his method is outlined in §4.8 below.

(iii) The set $S$ is a linear sub-variety $\mathbf{R}^m$ of $\mathbf{R}^n$. Looked at from the point of view of the restriction problem, there is in general a loss of smoothness in going from appropriate functions on $\mathbf{R}^n$ to functions on $\mathbf{R}^m$. Since $\mathbf{R}^m$ has Lebesgue measure zero in $\mathbf{R}^n$, there is also the problem of giving the functions in $\mathbf{R}^n$ their *natural* definition on $\mathbf{R}^m$, so that the restriction may be well defined. This kind of difficulty did not arise in case (i) because there continuous functions are dealt with exclusively. For the present problem the functions considered may be discontinuous at every point, but they do have certain average continuity. A striking aspect turns out to be the fact that the function spaces appropriate for $\mathbf{R}^m$ (for the restrictions), may in character be quite different from those appropriate for $\mathbf{R}^n$.

We begin this chapter by giving the details of the decomposition of an arbitrary open set in $\mathbf{R}^n$ into a suitable "disjoint" union of cubes. The usefulness of this decomposition has already been indicated in Chapter I. Here we apply it again, and the partition of unity based on it, as the main tool in the extension of the type (i). It arises again, if only implicitly, in the extension of type (ii).

## 1. *Decomposition of open sets into cubes*

In what follows, $F$ will denote an arbitrary non-empty closed set in $\mathbf{R}^n$, $\Omega$ its complement. By a *cube* we mean a closed cube in $\mathbf{R}^n$, with sides parallel to the axes, and two such cubes will be said to be *disjoint* if their interiors are disjoint. For such a cube $Q$, diam $(Q)$ denotes its diameter, and dist $(Q, F)$ its distance from $F$.

**1.1 Theorem 1.** *Let $F$ be given. Then there exists a collection of cubes $\mathscr{F}$, $\mathscr{F} = \{Q_1, Q_2, \ldots Q_k, \ldots\}$ so that*

(1) $\bigcup_k Q_k = \Omega = ({}^cF)$,

(2) *The $Q_k$ are mutually disjoint,*

(3) $c_1 \operatorname{diam}(Q_k) \leq \operatorname{dist}(Q_k, F) \leq c_2 \operatorname{diam}(Q_k)$.

*The constants $c_1$ and $c_2$ are independent of $F$. In fact we may take $c_1 = 1$ and $c_2 = 4$.*

**1.2** Consider the lattice of points in $\mathbf{R}^n$ whose coordinates are integral. This lattice determines a *mesh* $\mathscr{M}_0$, which is a collection of cubes: namely all cubes of unit length, whose vertices are points of the above lattice. The mesh $\mathscr{M}_0$ leads to a two-way infinite chain of such meshes $\{\mathscr{M}_k\}_{-\infty}^{\infty}$, with $\mathscr{M}_k = 2^{-k}\mathscr{M}_0$.

Thus each cube in the mesh $\mathscr{M}_k$ gives rise to $2^n$ cubes in the mesh $\mathscr{M}_{k+1}$ by bisecting the sides. The cubes in the mesh $\mathscr{M}_k$ each have sides of length $2^{-k}$ and are thus of diameter $\sqrt{n}\, 2^{-k}$.

In addition to the meshes $\mathscr{M}_k$ we consider the layers $\Omega_k$, defined by $\Omega_k = \{x : c2^{-k} < \operatorname{dist}(x, F) \leq c2^{-k+1}\}$; $c$ is a positive constant we shall fix momentarily. Obviously $\Omega = \bigcup_{k=-\infty}^{\infty} \Omega_k$.

We now make an initial choice of cubes, and denote the resulting collection by $\mathscr{F}_0$. Our choice is made as follows. We consider the cubes of the mesh $\mathscr{M}_k$, (each such cube is of size approximately $2^{-k}$), and include a cube of this mesh in $\mathscr{F}_0$ if it intersects $\Omega_k$, (the points of the latter are all approximately at a distance $2^{-k}$ from $F$). That is we take

$$\mathscr{F}_0 = \bigcup_k \{Q \in \mathscr{M}_k : Q \cap \Omega_k \neq 0\}.$$

We then have

$$\bigcup_{Q \in \mathscr{F}_0} Q = \Omega.$$

For appropriate choice of $c$,

(3) $$\operatorname{diam}(Q) \leq \operatorname{dist}(Q, F) \leq 4 \operatorname{diam}(Q), \qquad Q \in \mathscr{F}_0.$$

Let us prove (3) first. Suppose $Q \in \mathscr{M}_k$; then the diameter of $Q = \sqrt{n}\, 2^{-k}$. Since $Q \in \mathscr{F}_0$, there exists $x \in Q \cap \Omega_k$. Thus $\operatorname{dist}(Q, F) \leq \operatorname{dist}(x, F) \leq c2^{-k-1}$, and $\operatorname{dist}(Q, F) \geq \operatorname{dist}(x, F) - \operatorname{diam}(Q) > c2^{-k} - \sqrt{n}\, 2^{-k}$. If we choose $c = 2\sqrt{n}$ we get (3).

Then by (3) the cubes $Q \in \mathscr{F}_0$ are disjoint from $F$ and clearly cover $\Omega$. Therefore (1) is also proved. Notice that the collection $\mathscr{F}_0$ has all our required properties, except that the cubes in it are not necessarily disjoint. To finish the proof of the theorem we need to refine our choice leading to $\mathscr{F}_0$, eliminating those cubes which were really unnecessary.

We require the following simple observation. Suppose $Q_1$ and $Q_2$ are two cubes (taken respectively from the mesh $\mathscr{M}_{k_1}$ and $\mathscr{M}_{k_2}$). Then if $Q_1$ and $Q_2$ are not disjoint, one of the two must be contained in the other. (In particular $Q_1 \subset Q_2$, if $k_1 \geq k_2$.)

Start now with any cube $Q \in \mathscr{F}_0$, and consider *the maximal* cube in $\mathscr{F}_0$ which contains it. In view of the inequality (3) for any cube $Q'$ in $\mathscr{F}_0$, which contains $Q$ in $\mathscr{F}_0$ we have $\operatorname{diam}(Q') \leq 4 \operatorname{diam}(Q)$. Moreover any two cubes $Q'$ and $Q''$ which contain $Q$ have obviously a non-trivial intersection. Thus by the observation made above each cube $Q \in \mathscr{F}_0$ has a *unique* maximal cube in $\mathscr{F}_0$ which contains it. By the same token these maximal cubes are also disjoint. We let $\mathscr{F}$ denote the collection of maximal cubes of $\mathscr{F}_0$. Then obviously

(1) $\bigcup_{Q \in \mathscr{F}} Q = \Omega$,

(2) The cubes of $\mathscr{F}$ are disjoint,

(3) $\operatorname{diam}(Q) \leq \operatorname{dist}(Q, F) \leq 4 \operatorname{diam}(Q), \qquad Q \in \mathscr{F}$.

Theorem 1 is therefore proved.

**1.3 A partition of unity.** We shall now make a few observations about the family $\mathscr{F}$ of cubes whose existence is guaranteed by Theorem 1. Let us say that two distinct cubes of $\mathscr{F}$, $Q_1$ and $Q_2$, *touch* if their boundaries have a common point. (We remind the reader that two distinct cubes of $\mathscr{F}$ always have disjoint interiors.)

PROPOSITION 1. *Suppose $Q_1$ and $Q_2$ touch. Then*

$$(1/4) \text{ diam } (Q_2) \leq \text{diam } (Q_1) \leq 4 \text{ diam } (Q_2).$$

We know that $\text{dist}\,(Q_1, F) \leq 4 \text{ diam } (Q_1)$. Then $\text{dist}\,(Q_2, F) \leq 4 \text{ diam } (Q_1) + \text{diam } (Q_1) = 5 \text{ diam } (Q_1)$, since $Q_1$ and $Q_2$ touch. But diam $(Q_2) \leq$ dist $(Q_2, F)$, therefore diam $(Q_1) \leq 5$ diam $(Q_2)$. However diam $(Q_2) = 2^k$ diam $(Q_1)$ for some integral $k$, thus diam $(Q_1) \leq 4$ diam $(Q_2)$, and the symmetrical implication proves the proposition.

We now set $N = (12)^n$. The exact size of $N$ needed in what follows is of no importance; what matters is that it can be chosen to depend only on the dimension $n$, and in particular to be independent of the closed set $F$.

PROPOSITION 2. *Suppose $Q \in \mathscr{F}$. Then there are at most $N$ cubes in $\mathscr{F}$ which touch $Q$.*

If the cube $Q$ belongs to the mesh $\mathscr{M}_k$, then as is easily seen, there are $3^n$ cubes (including $Q$) which belong to the mesh $\mathscr{M}_k$ and touch $Q$. Next, each cube in the mesh $\mathscr{M}_k$ can contain at most $4^n$ cubes of $\mathscr{F}$, of diameter $\geq (1/4)$ diameter of $Q$. If we combine this with Proposition 1 we get the proof of Proposition 2.

Let now $Q_k$ denote any cube in $\mathscr{F}$. Write $x^k$ as the center of this cube and $l_k$ the common length of its sides. Then of course diam $(Q_k) = \sqrt{n}\, l_k$. For any $\varepsilon$, $0 < \varepsilon < 1/4$, which is arbitrary but will be kept fixed in what follows, denote by $Q_k^*$ the cube which has the same center as $Q_k$ but is expanded by the factor $1 + \varepsilon$; that is, $Q_k^* = (1 + \varepsilon)[Q_k - x^k] + x^k$. Clearly $Q_k \subset Q_k^*$, and the cubes $Q_k^*$ no longer have disjoint interiors. However the following holds:

PROPOSITION 3. *Each point of $\Omega$ is contained in at most $N$ of the cubes $Q_k^*$.*

Let $Q$ and $Q_k$ be two cubes of $\mathscr{F}$. We claim that $Q_k^*$ intersects $Q$ only if $Q_k$ touches $Q$. In fact consider the union of $Q_k$ with all the cubes in $\mathscr{F}$ which touch $Q_k$; since the diameters of these cubes are all $\geq (1/4)$ diameter of $Q_k$, it is clear that this union contains $Q_k^*$. Therefore $Q$ intersects $Q_k^*$ only if $Q$ touches $Q_k$. However any point $x \in \Omega$ belongs to some cube $Q$, and therefore by Proposition 2 there are at most $N$ cubes $Q_k^*$ which contain $x$.

The proof also shows that every point of $\Omega$ is contained in a small neighborhood intersecting at most $N$ cubes $Q_k^*$.

Now let $Q_0$ denote the cube of unit length centered at the origin. Fix a $C^\infty$ function $\varphi$ with the properties: $0 \leq \varphi \leq 1$; $\varphi(x) = 1$, $x \in Q_0$; and $\varphi(x) = 0$, $x \notin (1 + \varepsilon)Q_0$.

Let $\varphi_k$ denote the function $\varphi$ adjusted to the cube $Q_k$; that is

$$\varphi_k(x) = \varphi\left(\frac{x - x^k}{l_k}\right).$$

Recall that $x^k$ is the center of $Q_k$ and $l_k$ is the common length of its sides. Notice that therefore $\varphi_k(x) = 1$ if $x \in Q_k$, and $\varphi_k(x) = 0$ if $x \notin Q_k^*$. It is to be observed that

$$\left| \left(\frac{\partial}{\partial x}\right)^\alpha \varphi_k(x) \right| \leq A_\alpha (\operatorname{diam} Q_k)^{-|\alpha|}. \tag{4}$$

We now define $\varphi_k^*(x)$ for $x \in {}^cF$ by

$$\varphi_k^*(x) = \frac{\varphi_k(x)}{\Phi(x)}, \quad \text{where} \quad \Phi(x) = \sum_k \varphi_k(x).$$

The obvious identity

$$\sum_k \varphi_k^*(x) \equiv 1, \qquad x \in {}^cF \tag{5}$$

then defines our required partition of unity.

## 2. *Extension theorems of Whitney type*

**2.1 The regularized distance.** The ideas of the extension theorem of Whitney are implicitly contained in the partition of unity (5) just developed, and are further suggested by the construction of the regularized distance function which we shall now describe.

Let $F$ be an arbitrary closed set in $\mathbf{R}^n$, and following the notation of Chapter I, let $\delta(x)$ denote the distance of $x$ from $F$. While this function is smooth on $F$ (it vanishes there) it is in general not more differentiable on ${}^cF$ than the obvious Lipschitz-condition-inequality $|\delta(x) - \delta(y)| \leq |x - y|$ would indicate.

For several applications it is desirable to replace $\delta(x)$ by a regularized distance which is smooth for $x \in {}^cF$, as $x$ stays away from $F$. In addition this regularized distance is to have essentially the same profile as $\delta(x)$.

Its existence is guaranteed by the following theorem.

THEOREM 2. *There exists a function $\Delta(x) = \Delta(x, F)$ defined in ${}^cF$ such that*

(a) $c_1\,\delta(x) \leq \Delta(x) \leq c_2\,\delta(x), \qquad x \in {}^cF$

(b) $\Delta(x)$ *is* $C^\infty$ *in* ${}^cF$ *and*

$$\left| \frac{\partial^\alpha}{\partial x^\alpha} \Delta(x) \right| \leq B_\alpha(\delta(x))^{1-|\alpha|}.$$

$B_\alpha$, $c_1$ *and* $c_2$ *are independent of* $F$.

The construction of $\Delta(x)$ is given in one stroke. In fact we can set

$$\Delta(x) = \sum_k \operatorname{diam}(Q_k)\varphi_k(x). \tag{6}$$

Observe that if $x \in Q_k$, then $\delta(x) = \operatorname{dist}(x, F) \leq \operatorname{dist}(Q_k, F) + \operatorname{diam}(Q_k) \leq 5 \operatorname{diam}(Q_k)$ by inequality (3). Also if $x \in Q_k^*$, then $\delta(x) \geq \operatorname{dist}(Q_k, F) - 1/4 \operatorname{diam}(Q_k) \geq (3/4) \operatorname{diam}(Q_k)$, because of (3). To summarize:

(7) *If* $x \in Q_k$ *then* $\delta(x) \leq 5 \operatorname{diam}(Q_k)$. *If* $x \in Q_k^*$, *then* $\delta(x) \geq (3/4) \operatorname{diam}(Q_k)$.

However if $x \in Q_k$, then $\varphi_k(x) = 1$, so $\Delta(x) \geq \operatorname{diam}(Q_k) \geq \dfrac{\delta(x)}{5}$. On the other hand, any given $x$ lies in at most $N$ of the $Q_k^*$, and thus $\Delta(x) \leq \sum_{x \in Q_k^*} \operatorname{diam}(Q_k) \leq (4/3)\, N\, \delta(x)$.

We have therefore proved conclusion (a) with $c_1 = \frac{1}{5}$ and $c_2 = (4/3)N$.

To prove conclusion (b) we argue similarly but invoke inequality (4), and the observation (analogous to (7)) that if $x \in Q_k^*$, then $\delta(x) \leq 6 \operatorname{diam} Q_k$. This gives the desired result with $B_\alpha = A_\alpha N\, 6^{|\alpha|-1}$.

We shall not follow up this construction now, and defer its application until §3. We wish here to remark that the bounds given by (b) on the derivatives of $\Delta(x)$, although they blow up as $x$ approaches $F$, are in general the best possible under the circumstances. This can already be seen in the case of $\mathbf{R}^1$ if we take for $F$ the complement of the open set $\bigcup_{j=-\infty}^{\infty} (2^{-j}, 2^{-j-1})$.

On each interval the regularized distance function must rise from zero to at least $c_1 2^{-j-1}$, in passing over a distance of length $2^{-j-1}$ and so the first derivative must attain a size at least as large as $c_1$; by the same token the first derivative must attain a size smaller than $-c_1$, and so the second derivative must somewhere in that interval be at least as large as $c_1 2^{j+1}$, etc.

**2.2 The first extension operator, $\mathscr{E}_0$.** Let $F$ be a closed set in $\mathbf{R}^n$. Our purpose here will be to describe an operator $\mathscr{E}_0$ which extends functions defined on $F$ to functions defined on $\mathbf{R}^n$. Its main properties are expressible in terms of function spaces involving one order or less of differentiability. As such $\mathscr{E}_0$ is the simplest of a hierarchy of extension operators required for differentiability of higher order.

The definition of $\mathscr{E}_0$ is as follows. Consider the set $F$ and the family of cubes $\{Q_k\}$ given in Theorem 1. For each cube $Q_k$ fix a point $p_k$ in $F$ with the property that dist $(Q_k, F) =$ dist $(Q_k, p_k)$.

Such a point $p_k$ of course exists since $F$ is closed. While it is not unique any fixed choice of a point of minimum distance from $F$ will do. In fact any choice of a point $p_k \in F$ with the property that the distance of $p_k$ to $Q_k$ is comparable to the distance of $Q_k$ from $F$ would do just as well, but it is somewhat simpler if we specify $p_k$ as above.

Let now $f$ be given on $F$. Consider the function $\mathscr{E}_0(f)$ defined by $\mathscr{E}_0(f)(x) = f(x)$, $x \in F$, and

$$\mathscr{E}_0(f)(x) = \sum_k f(p_k)\varphi_k^*(x), \qquad x \in {}^cF, \tag{8}$$

where $\{\varphi_k^*(x)\}$ is the partition of unity described at the end of §1.3.

It is to be observed that if $x \in {}^cF$, then it belongs to at most $N$ cubes $Q_k^*$, and since the $\varphi_k^*$ are supported in $Q_k^*$, the sum in (8) is really a finite sum and thus $\mathscr{E}_0(f)(x)$ is well-defined. The first properties of $\mathscr{E}_0$ will now be given.

PROPOSITION. *Suppose $f$ is a given function on $F$. Then $\mathscr{E}_0(f)$ is an extension of $f$ to $\mathbf{R}^n$. Assume, in addition, that $f$ is continuous on $F$. Then $\mathscr{E}_0(f)$ is continuous on $\mathbf{R}^n$, and in fact is $C^\infty$ in ${}^cF$.*

That $\mathscr{E}_0(f)$ is an extension of $f$ is by definition. To prove the continuity of $\mathscr{E}_0(f)$, and to make later estimates it is convenient to use the following notational convention: Suppose $A$ and $B$ are two positive quantities; then we write $A \approx B$ to mean that $A$ and $B$ are *comparable*. In the context of this chapter this means that there exist two constants $c_1$ and $c_2$ so that $c_1 A \leq B \leq c_2 A$; it will be understood that these constants, $c_1$ and $c_2$, may depend on the dimension $n$, but are otherwise independent of the set $F$, the cubes $Q_k$, the function $f$, etc.

With this notation we observe first that

$$\text{if} \quad x \in Q_k^*, \quad \text{then} \quad |x - p_k| \approx \text{diam}\,(Q_k). \tag{9}$$

Also

$$\text{dist}\,(Q_k^*, F) \approx \text{diam}\,(Q_k), \qquad \text{(see (7))}. \tag{10}$$

Now if $y \in F$, $x \in Q_k^*$, then $|y - p_k| \leq |y - x| + |p_k - x|$. But clearly

$|y - x| \geq \operatorname{dist}(Q_k^*, F)$, and therefore by (9) and (10):

(11) *if* $y \in F$ *and* $x \in Q_k^*$, *then* $|y - p_k| \leq c\,|y - x|$.

We are now ready to prove the continuity of $\mathscr{E}_0(f)$. We have already observed that each point of $x \in {}^cF$ belongs to a neighborhood which intersects at most $N$ of the cubes $Q_k^*$. Since each of the functions $\varphi_k^*$ are $C^\infty$ in ${}^cF$, this shows that $\mathscr{E}_0(f)(x)$ is $C^\infty$ in ${}^cF$, and hence certainly continuous in ${}^cF$.

Now let $y$ be a fixed point of $F$. We want to prove the continuity of $\mathscr{E}_0(f)(x)$ at $x = y$. Consider therefore $\mathscr{E}_0(f)(y) - \mathscr{E}_0(f)(x) = f(y) - \mathscr{E}_0(f)(x)$, with $x \to y$. For those $x$ which belong to $F$ this difference is $f(y) - f(x)$ and matters are reduced to the given continuity of $f$ on $F$. Suppose therefore that $x \to y$, with $x \in {}^cF$. Then

$$f(y) - \mathscr{E}_0(f)(x) = f(y) - \sum f(p_k)\varphi_k^*(x) = \sum (f(y) - f(p_k))\varphi_k^*(x),$$

because $\sum \varphi_k^*(x) \equiv 1$, $x \in {}^cF$.

We now use the observation (11) together with the fact that $\varphi_k^*(x)$ is supported in $Q_k^*$ to see that

$$|f(y) - \mathscr{E}_0(f)(x)| \leq \sup_{y' \in F} |f(y) - f(y')| \to 0,$$

$$\text{as} \quad x \to y, \quad \text{with} \quad |y - y'| \leq c\,|y - x|.$$

*2.2.1 Theorem 3.* It is desirable to go further and express the continuity properties of the linear operator $f \to \mathscr{E}_0(f)$ in terms of Banach spaces. The most appropriate function spaces are those given in terms of the modulus of continuity, and in particular the Lipschitz spaces. For this purpose let $0 < \gamma \leq 1$, and define

$$\operatorname{Lip}(\gamma, \mathbf{R}^n) = \{f : |f(x)| \leq M,\ |f(x) - f(y)| \leq M\,|x - y|^\gamma,\ x, y \in \mathbf{R}^n\}.$$

Lip $(\gamma, \mathbf{R}^n)$ becomes a Banach space if we take for norm the smallest $M$ in the above definition.*

It is to be noted that when $0 < \gamma < 1$, then Lip $(\gamma, \mathbf{R}^n)$ is equivalent with the space $\Lambda_\gamma = \Lambda_\gamma^{\infty,\infty}$ studied in Chapter V, §4. However it is important to point out that in the present context we have a different transition as $\gamma \to 1$. Namely Lip $(1, \mathbf{R}^n)$ is isomorphic to $L_1^\infty(\mathbf{R}^n)$ the space of bounded function on $\mathbf{R}^n$ whose first derivatives are bounded, and *not* to $\Lambda_1 (= \Lambda_1^{\infty,\infty})$; see §4.3.1 and §6.2 of Chapter V.

If $F$ is any closed set we define Lip $(\gamma, F)$ similarly as consisting of those $f$ defined on $F$ for which

(12) $|f(x)| \leq M$ and $|f(x) - f(y)| \leq M\,|x - y|^\gamma$, $\quad x, y \in F$.

Again Lip $(\gamma, F)$ is a Banach space, with the smallest $M$ as norm.

* When $\gamma > 1$, the space defined above consists of constants only.

THEOREM 3. *The linear extension operator* $\mathscr{E}_0$ *maps* $\text{Lip}(\gamma, F)$ *continuously into* $\text{Lip}(\gamma, \mathbf{R}^n)$, *if* $0 < \gamma \leq 1$. *The norm of this mapping has a bound independent of the closed set* $F$.

*2.2.2* In order to prove the theorem we begin by recording the inequality

$$\left| \frac{\partial^\alpha}{\partial x^\alpha} \varphi_k^*(x) \right| \leq A'_\alpha (\operatorname{diam} Q_k)^{-|\alpha|}. \tag{13}$$

It can be derived as an easy consequence of the analogous inequality (4) for $\varphi_k$ in §1.3; we leave the straightforward details to the reader.

Now let us assume that $f$ satisfies the inequality (12) with $M = 1$. We have already observed that whatever $f$ is, $\mathscr{E}_0(f)$ is $C^\infty$ in ${}^cF$. Here we shall need the quantitative estimate

$$\left| \frac{\partial}{\partial x_i} \mathscr{E}_0(f)(x) \right| \leq c(\delta(x))^{\gamma-1} \qquad i = 1, \ldots, n, \quad x \in {}^cF; \tag{14}$$

and $\delta(x)$ denotes the distance of $x$ from $F$.

In fact

$$\frac{\partial}{\partial x_i} \mathscr{E}_0(f)(x) = \sum_k f(p_k) \frac{\partial \varphi_k^*(x)}{\partial x_i} = \sum_k (f(p_k) - f(y)) \frac{\partial \varphi_k^*(x)}{\partial x_i},$$

in view of the fact that $\sum \dfrac{\partial \varphi_k^*}{\partial x_i}(x) \equiv 0$, by (5). For any $x \in {}^cF$ choose $y$ to be a point in $F$ closest to $x$, that is $|x - y| = \delta(x)$.

Consider next those cubes $Q_k^*$ so that $x \in Q_k^*$. There are at most $N$ of these and we always have $|y - p_k| \leq c\,|x - y| = c\delta(x)$ for these cubes, as was observed in (11) (see §2.2). Therefore

$$\left| \frac{\partial}{\partial x_i} \mathscr{E}_0(f)(x) \right| \leq A_1 \sum_{x \in Q_k^*} |f(p_k) - f(y)|\,(\operatorname{diam} Q_k)^{-1}.$$

Clearly, however, if $x \in Q_k^*$ then $\delta(x) \approx \operatorname{diam}(Q_k)$. Thus

$$\left| \frac{\partial}{\partial x_i} \mathscr{E}_0(f)(x) \right| \leq A' \left( \sum_{x \in Q_k^*} |p_k - y|^\gamma \right) \delta^{-1}(x) \leq c'\, \delta(x)^{\gamma - 1},$$

which proves (14).

The estimate (14) is the appropriate one for points away from $F$. For points near $F$ we observe that if $y \in F$, $x \in {}^cF$, then

$$\mathscr{E}_0(f)(y) - \mathscr{E}_0(f)(x) = f(y) - \mathscr{E}_0 f(x) = \sum_k (f(y) - f(p_k))\varphi_k^*(x);$$

therefore by (11)

$$(15)\quad |\mathscr{E}_0(f)(y) - \mathscr{E}_0(f)(x)| \le \sup_{|y-p_k| \le c|y-x|} |f(y) - f(p_k)| \le c\,|y - x|^\gamma,$$
$$\text{if} \quad y \in F, \quad x \in {}^cF.$$

Suppose now that both $y$ and $x$ are in ${}^cF$. Let $L$ be the line segment joining them and we consider the two cases: (i) the distance of $L$ from $F$ exceeds the length of $L$ $(= |x - y|)$, (ii) the distance of $L$ from $F$ is not larger than the length of $L$. In the first case we have simply

$$|\mathscr{E}_0(f)(y) - \mathscr{E}_0(f)(x)| \le |y - x| \sup_{x' \in L} |\nabla \mathscr{E}_0(f)(x')|$$
$$\le c\,|y - x| \sup_{x' \in L} (\delta(x'))^{\gamma - 1},$$

by (14) since in this case $\delta(x') > |y - x|$, $x' \in L$. We obtain as a result $|\mathscr{E}_0(f)(y) - \mathscr{E}_0(f)(x)| \le c\,|y - x|^\gamma$. In the second case we can find a point $x' \in L$, and a point $y' \in F$ so that $|x' - y'| \le |y - x|$. Therefore $|y' - x| \le 2\,|y - x|$ and $|y' - y| \le 2\,|y - x|$. If we apply (15) to $\mathscr{E}_0(f)(y') - \mathscr{E}_0(f)(x)$ and $\mathscr{E}_0(f)(y') - \mathscr{E}_0(f)(y)$ we get again

$$|\mathscr{E}_0(f)(y) - \mathscr{E}_0(f)(x)| \le c'\,|y - x|^\gamma.$$

Finally if $x$ and $y \in F$ we have trivially $|\mathscr{E}_0(f)(y) - \mathscr{E}_0(f)(x)| \le |y - x|^\gamma$.

Observe also that if the absolute value of $f$ is bounded by 1, then the same is true for the absolute value of $\mathscr{E}_0(f)$. Theorem 3 is therefore completely proved if we note that all our bounds are independent of the closed set $F$.

*2.2.3 A corollary.* The proof of Theorem 3 leads to a simple generalization of itself. Let $\omega(\delta)$, $0 < \delta < \infty$, be a *modulus of continuity*, that is, a positive increasing function of $\delta$; assume it is *regular* in the sense that: (1) $\dfrac{\omega(\delta)}{\delta}$ is increasing as $\delta \to 0$, and (2) $\omega(2\delta) \le c\omega(\delta)$. (The first condition among other things excludes $\omega(\delta) = \delta^\gamma$ for $\gamma > 1$. The second condition makes the statement of the result neater.) Define Lip $(\omega, F) = \{f : |f(x)| \le M, |f(x) - f(y)| \le M\omega(|x - y|),\ x, y \in F\}$ with norm the smallest $M$. Then:

COROLLARY. *$\mathscr{E}_0$ is a continuous mapping of* Lip $(\omega, F)$ *into* Lip $(\omega, \mathbf{R}^n)$.

The proof is merely a repetition of that of Theorem 3. Notice that the condition $\omega(2\delta) \le c\omega(\delta)$ and its non-decreasing character implies that for every positive $c_1$ there exists a positive $c_2$ so that $\omega(c_1\delta) \le c_2\omega(\delta)$, $0 < \delta < \infty$.

**2.3 The extension operators $\mathscr{E}_k$.** In generalizing the results of §2.2 to higher derivatives the first requirement is the corresponding definition of Lip $(\gamma, F)$ when $\gamma > 1$. For this purpose let $k$ be a non-negative integer and assume that $k < \gamma \leq k + 1$.

We shall say that a function $f$, defined on $F$ belongs to Lip $(\gamma, F)$ if there exists functions $f^{(j)}$, $0 \leq |j| \leq k$ defined on $F$, with $f^{(0)} = f$, and so that if

$$f^{(j)}(x) = \sum_{|j+l| \leq k} \frac{f^{(j+l)}(y)}{l!}(x - y)^l + R_j(x, y) \tag{16}$$

then

$$|f^{(j)}(x)| \leq M \quad \text{and} \quad |R_j(x, y)| \leq M\,|x - y|^{\gamma - |j|}, \quad \text{all } x, y \in F, \quad |j| \leq k. \tag{17}$$

Several explanations are in order concerning this definition. $j$ and $l$ denote multi-indices $j = (j_1, j_2, \ldots, j_n)$, $l = (l_1, l_2, \ldots, l_n)$ with $j! = j_1!\,j_2! \cdots j_n!$, and $|j| = j_1 + j_2 \cdots + j_n$; $x^l = x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}$.

It is to be noted that the function $f = f^{(0)}$ does not necessarily determine the $f^{(j)}$, $(0 < |j| \leq k)$, uniquely; (consider, for example, the case of an $f$ defined on $F$, where $F$ is a finite set). Thus in order to avoid this ambiguity, when speaking of an element of Lip $(\gamma, F)$ we shall mean in fact the collection $\{f^{(j)}(x)\}_{|j| \leq k}$. The norm of an element in Lip $(\gamma, F)$ will then be taken to be the smallest $M$ for which the inequality (17) holds. In the definition just adopted we make an exception if $F = \mathbf{R}^n$. By Lip $(\gamma, \mathbf{R}^n)$ we shall then mean the linear space of the $f = f^{(0)}$ only; for which, of course there exists $f^{(j)}$ satisfying (16) and (17).

Again the norm is taken to be the smallest $M$ satisfying (17). This convention, which is adopted purely for a notational ease is consistent with the general definition of Lip $(\gamma, F)$, since it can easily be seen that the $f^{(j)}$, $1 \leq |j|$, are uniquely determined by $f$, if $F = \mathbf{R}^n$.

More particularly if $f \in$ Lip $(\gamma, \mathbf{R}^n)$ according to the definition just given, then $f$ is continuous and bounded and has continuous bounded partial derivatives of order not greater than $k$; furthermore $\dfrac{\partial^j f}{\partial x^j} = f^{(j)}$, $|j| \leq k$, and the functions $f^{(j)}$, for $|j| = k$ belong to the space Lip $(\gamma - k, \mathbf{R}^n)$ considered in §2.2.1. The converse is also true and easily established. Therefore if $\gamma$ is not integral Lip $(\gamma, \mathbf{R}^n)$ is equivalent with $\Lambda_\gamma$; see Chapter V, §4, and Proposition 9, page 147, in particular. When $\gamma$ is integral, $\gamma = k + 1$, then Lip $(k + 1, \mathbf{R}^n)$ is equivalent with $L^\infty_{k-1}(\mathbf{R}^n)$; see §6.2 in Chapter V.

Let now $\{f^{(j)}\}_{|j| \leq k}$ be a collection of functions defined on $F$. The linear mapping $\mathscr{E}_k$ will assign to any such collection a function $\mathscr{E}_k(f^{(j)})$

defined on $\mathbf{R}^n$, which will visibly be an extension of $f^{(0)} = f$ to $\mathbf{R}^n$. For simplicity of notation we shall *denote this extension by $f$ also*. Our definition of $\mathscr{E}_k$ is as follows:

$$(18)\qquad \begin{cases} \mathscr{E}_k(f^{(j)}) = f^{(0)}(x), & x \in F \\ \mathscr{E}_k(f^{(j)}) = \sum_i' P(x, p_i)\varphi_i^*(x), & x \in {}^cF. \end{cases}$$

$P(x, y)$ denotes the polynomial in $x$ giving the Taylor expansion of $f$ about the point $y \in F$, that is

$$P(x, y) = \sum_{|l| \leq k} \frac{f^{(l)}(y)(x - y)^l}{l!}, \qquad x \in \mathbf{R}^n, \quad y \in F.$$

$p_i$ is as in §2.2, a point in $F$ of minimum difference from the cube $Q_i$. Finally the symbol $\sum'$ indicates that the summation is taken only over those cubes $Q_i$ near $F$; more precisely, those whose distance from $F$ is not greater than one.

THEOREM 4. *Suppose $k$ is a non-negative integer, $k < \gamma \leq k + 1$, and $F$ a closed set in $\mathbf{R}^n$.*

*Then the mapping $\mathscr{E}_k$ is a continuous mapping of* Lip $(\gamma, F)$ *to* Lip $(\gamma, \mathbf{R}^n)$ *which gives an extension of $f^{(0)}$ to all of $\mathbf{R}^n$. The norm of this mapping has a bound independent of $F$.*

*2.3.1* In addition to the notation $P(x, y)$ just introduced it is convenient to write $P_j(x, y)$ for $\sum_{|j+l| \leq k} \frac{f^{(j+l)}(y)}{l!}(x - y)^l$, $|j| \leq k$. Then of course $f^{(j)}(x) = P_j(x, y) + R_j(x, y)$, $x, y \in F$. We have $P(x, y) = P_0(x, y)$, and so consistent with this we set $R(x, y) = R_0(x, y)$.

LEMMA. *Suppose $b, a \in F$, $x \in \mathbf{R}^n$, then*

$$P(x, b) - P(x, a) = \sum_{|l| \leq k} R_l(b, a)\frac{(x - b)^l}{l!},$$

*and more generally*

$$P_j(x, b) - P_j(x, a) = \sum_{|j+l| \leq k} R_{j+l}(b, a)\frac{(x - b)^l}{l!}.$$

To prove the lemma we expand the polynomial in $x$, $P(x, b) - P(x, a)$, in its Taylor expansion about the point $b$. The coefficient of $\frac{(x - b)^l}{l!}$ is then $\frac{\partial^l}{\partial x^l}(P(x, b) - P(x, a)]_{x=b}$. However $\frac{\partial^l}{\partial x^l}(P(x, y)) = P_l(x, y)$, and $P_l(b, b) = f^{(l)}(b)$ which proves the lemma for $j = 0$. The case for $j \neq 0$ can of course be considered as a special instance of the case already proved.

Let us now turn our attention to the sum $\sum'$ and see how it differs from $\sum$. The observations we have already made, see (7), (10), and property (3) show that

(19) *if* $x \in Q_k^*$, *then* $\delta(x) = \operatorname{dist}(x, F) \approx \operatorname{dist}(Q_k, F) \approx \operatorname{dist}(Q_k^*, F)$.

Therefore if $\delta(x) \leq c_1$, for some appropriate positive $c_1$, which is sufficiently small, then the sum $\sum'$ is the full sum taken over all cubes. If $\delta(x) \geq c_2$, for another positive constant which is sufficiently large, then there are no terms in the sum $\sum'$, and thus $f$ vanishes for $\delta(x) \geq c_2$. Finally if $c_1 \leq \delta(x) \leq c_2$, then only a bounded number of terms occur, and in view of the bounds of the derivatives of $\varphi_i^*(x)$ given by (13) and the bounds on $f^{(j)}$ given by (17), we see that there $\left|\dfrac{\partial^\alpha}{\partial x^\alpha}(f(x))\right| \leq A_\alpha' M$, all $\alpha$. Because of this we shall be able to limit our considerations to $x$ close to $F$, namely $\delta(x) \leq c_1$. We shall also assume the normalization $M = 1$.

*2.3.2* We now claim that the following hold:

(a) $|f(x) - P(x, a)| \leq A\,|x - a|^\gamma$, for $x \in \mathbf{R}^n$, $a \in F$

(a′) $|f^{(j)}(x) - P_j(x, a)| \leq A\,|x - a|^{\gamma - |j|}$, for $x \in \mathbf{R}^n$, $a \in F$, $|j| \leq k$

(b) $|f^{(j)}(x)| \leq A$, for $|j| \leq k$,

(b′) $|f^{(j)}(x)| \leq A(\delta(x))^{\gamma-k-1}$, for $x \in {}^cF$, $|j| = k + 1$.

To prove (a) notice first that it holds for $A = 1$, when $x \in F$, as a result of our assumptions. Suppose that $x \in {}^cF$, and $\delta(x) \leq c_1$ (otherwise matters become obvious in view of the remarks made earlier). Then $f(x) - P(x, a) = \sum_i \{P(x, p_i) - P(x, a)\}\varphi_i^*(x)$. We invoke now the lemma and get in view of our assumptions

$$|f(x) - P(x, a)| \leq \sum_{|l| \leq k} \sum |p_i - a|^{\gamma - |l|}|x - a|^{|l|}$$

where the inner (un-indexed) sum is taken over those cubes $Q_i$, (there are at most $N$), so that $x \in Q_i^*$. By (11), $|p_i - a| \leq c_1\,|x - a|$, and therefore (a) is proved.

In proving (a′) we may restrict ourselves again to points $x \in {}^cF$ with $\delta(x) \leq c_1$. Here $f^{(j)}(x) = \left(\dfrac{\partial^j f}{\partial x^j}\right)(x)$. Thus

$$f^{(j)}(x) = \sum_i \left(\frac{\partial}{\partial x}\right)^j (P(x, p_i))\varphi_i^*(x) + \text{other terms}.$$

If we disregard the "other terms" and notice that $\left(\frac{\partial}{\partial x}\right)^j P(x, p_i) = P_j(x, p_i)$ then we get (a′) just as (a). The other terms are themselves sums of expressions like

$$\sum_i P_{j-l}(x, p_i)\left(\frac{\partial}{\partial x}\right)^l \varphi_i^*(x) \tag{20}$$

where $0 < |l|$, and $l_i \le j_i$, $i = 1, \ldots, n$. Since $\sum_i \left(\frac{\partial}{\partial x}\right)^l \varphi_i^*(x) = 0$, then these sums are in turn equal to

$$\sum_i \{P_{j-l}(x, p_i) - P_{j-l}(x, a)\}\left(\frac{\partial}{\partial x}\right)^l \varphi_i^*(x). \tag{21}$$

The same argument as before together with the estimate (13) for $\left(\frac{\partial}{\partial x}\right)^l \varphi_i^*$, and the inequalities (19) then prove (a′).

Inequality (b) (again for $\delta(x) \le c_1$) is an immediate consequence of (a′) if for the point $a$ we take a point in $F$ of bounded distance from $x$. (Incidentally it was at the stage of the proof of inequality (b) that it really mattered that we defined $f$ in terms of $\sum_i' P(x, p_i)\varphi_i^*(x)$ instead of the full sum over all cubes.)

Finally we come to the proof of (b′). If we carry out the differentiation for $\delta(x) \le c_1$ then we get that $f^{(j)}(x)$ equals a sum of expressions of the form (20). Since $|j| = k + 1$, it must follow that $|l| > 0$, otherwise $P_j(x, p_i) \equiv 0$. Thus each sum can be rewritten as one of the form (21), where we choose $a$ to be a point in $F$ of minimum distance from $x$. In view of the lemma, and inequalities (11), (13) and (19) we get that each sum (21) is dominated by a finite sum of terms of the form $A\,|p_i - a|^{\gamma-|j|+|l|}(\delta(x))^{-|l|} \le A'\delta(x)^{\gamma-k-1}$, and (b′) is also proved.

*2.3.3* Having disposed the inequalities (a), (a′), (b), and (b′) we can now finish the proof of Theorem 4. The case $k = 0$ is of course Theorem 3 (§2.2.1). We shall consider in detail the case $k = 1$, which is already entirely typical; we have then $1 < \gamma \le 2$.

The inequality (a) shows that the function $f$ has first partial derivatives at every point of $F$, and these are the $f^{(j)}(x)$, with $|j| = 1$. However $f$ is $C^\infty$ in ${}^cF$, and thus $\left(\frac{\partial}{\partial x}\right)^j f = f^{(j)}$ exists for each point in ${}^cF$. Inequality (a′) shows that the resulting $f^{(j)}$, $|j| = 1$ are continuous through $\mathbf{R}^n$. Now let $g$ denote one of these first partial derivatives, then (a′) and (b′) respectively imply that

$$|g(x) - g(a)| \le A\,|x - a|^{\gamma-1}, \qquad x \in \mathbf{R}^n, a \in F$$

and

$$\left|\frac{\partial}{\partial x_i} g(x)\right| \leq A(\delta(x))^{\gamma-2}, \qquad i = 1, \ldots, n, \ x \in {}^cF.$$

These two inequalities are of the same form as (15) and (14) in the proof of the theorem for $k = 0$, (but with $\gamma$ instead of $\gamma - 1$). Following the argument given there it follows that each $g(x) \in \mathrm{Lip}\,(\gamma - 1, \mathbf{R}^n)$ and this is the desired result when $k = 1$. The proof for $k \geq 2$ can then be carried out by induction, the inductive step being very similar to the case $k = 1$ just given.

For the variant of the theorem analogous to the corollary in §2.2.3 see §4.6 below; see also §4.7 where another version is given.

## 3. *Extension theorem for a domain with minimally smooth boundary*

Let $D$ be an open set in $\mathbf{R}^n$. Our purpose will be to describe an operator $\mathfrak{E}$ which extends functions defined in $D$ to $\mathbf{R}^n$. The operator that will be given will be universal in the sense that it will simultaneously extend all orders of differentiability. This is to be contrasted with the hierarchy of operators $\mathscr{E}_k$, of increasing complexity in $k$, that we needed to perform the job for an arbitrary closed set $F$. The construction of $\mathfrak{E}$ will be possible if the boundary of $D$ satisfies some minimal smoothness property, which is approximately equivalent to saying that it is of class Lip 1. It will be seen momentarily that this condition cannot really be relaxed. What is striking therefore is that one order of differentiability of the boundary is roughly speaking just the right requirement to allow extension of all orders of differentiability.

**3.1 Statement of the theorem.** The appropriate function spaces to be used here are the Sobolov spaces $L_k^p(D)$, defined in analogy with the special case $D = \mathbf{R}^n$ as follows. Let $C_0^\infty(D)$ denote the class of $C^\infty$ functions with compact support, lying in $D$. Then a locally integrable function $f$ defined in $D$ has a (weak) derivative $\dfrac{\partial^\alpha f}{\partial x^\alpha} = g$ which is locally integrable, if

$$\int_D f \frac{\partial^\alpha}{\partial x^\alpha} \varphi \, dx = (-1)^{|\alpha|} \int_D g\varphi \, dx, \quad \text{for all} \quad \varphi \in C_0^\infty(D).$$

If $g \in L^p(D)$, then we say $\dfrac{\partial^\alpha f}{\partial x^\alpha} \in L^p(D)$. Now if $k$ is an integer then

$$L_k^p(D) = \left\{ f \in L^p(D) : \frac{\partial^\alpha f}{\partial x^\alpha} \in L^p(D), \quad \text{all} \quad |\alpha| \leq k \right\}.$$

The norm, on the resulting equivalence classes is given by

$$\|f\|_{L_k^p(D)} = \sum_{|\alpha| \leq k} \left\| \frac{\partial^\alpha f}{\partial x^\alpha} \right\|_{L^p(D)}.$$

Our objective will be to prove the following theorem.

THEOREM 5. *Let $D$ be a domain whose boundary satisfies the minimal smoothness condition given by* (i), (ii), *and* (iii) *in §3.3. Then there exists a linear operator $\mathfrak{E}$ mapping functions on $D$ to functions on $\mathbf{R}^n$ with the properties*

(a) $\mathfrak{E}(f)\,|_D = f$, *that is, $\mathfrak{E}$ is an extension operator.*

(b) $\mathfrak{E}$ *maps $L_k^p(D)$ continuously into $L_k^p(\mathbf{R}^n)$ for all $p$, $1 \leq p \leq \infty$, and all non-negative integral $k$.*

Notice that for these domains the theorem also solves the restriction problem for $L_k^p(\mathbf{R}^n)$. In fact if $D$ is any domain in $\mathbf{R}^n$ it is obvious that the restriction to $D$ of any element of $L_k^p(\mathbf{R}^n)$ belongs to $L_k^p(D)$.

**3.2 A basic special case.** The main element of the proof of the theorem is contained in a substantial special case which we formulate and discuss separately.

For this purpose it is convenient for the sake of notation to change our setting from $\mathbf{R}^n$ to $\mathbf{R}^{n+1}$. We consider the points in $\mathbf{R}^{n+1}$ as pairs $(x, y)$, where $x \in \mathbf{R}^n$, and $y \in \mathbf{R}^1$. The domains $D$ (now open sets of $\mathbf{R}^{n+1}$) we shall consider are the *special Lipschitz domains* defined as follows.

Let $\varphi : \mathbf{R}^n \to \mathbf{R}^1$ be a function which satisfies the Lipschitz condition

$$|\varphi(x) - \varphi(x')| \leq M\,|x - x'|, \quad \text{all} \quad x, x \in \mathbf{R}^n. \tag{22}$$

In terms of this function we can define the special Lipschitz domain it determines to be the set of points lying above the hypersurface $y = \varphi(x)$ in $\mathbf{R}^{n+1}$, i.e.,

$$D = \{(x, y) \in \mathbf{R}^{n+1} : y > \varphi(x)\}. \tag{23}$$

The smallest $M$ for which (22) holds will be called the *bound* of the special Lipschitz domain $D$.

The special case we have in mind can be formulated as follows.

THEOREM 5′. *Let $D$ be a special Lipschitz domain in $\mathbf{R}^{n+1}$. Then there exists a linear extension operator $\mathfrak{E}$ taking appropriate functions on D to functions on $\mathbf{R}^{n+1}$ with the property that $\mathfrak{E}$ maps $L_k^p(D)$ continuously into $L_k^p(\mathbf{R}^{n+1})$, $1 \leq p \leq \infty$, $k$ integral. Moreover the norms of these mappings have bounds which depend only on the number $n$, the order of differentiability $k$, and the bound of the special Lipschitz domain.*

We shall first point out that the Lipschitz condition (22) for the boundary of the domain is in the nature of the best possible. Suppose we consider in $\mathbf{R}^2$ the domain where $\varphi(x) = |x|^\gamma$, $\gamma < 1$, that is $D = \{(x, y): y > |x|^\gamma\}$. Here $\varphi$ satisfies a Lipschitz condition of order $\gamma$, and violates condition (22) near the origin only. Let us set $f(x, y) = y^{-\beta}$ in $D$ near the origin, $f \in C^\infty$ away from the origin and suppose $f$ has bounded support. Notice that $f \in L_1^{2+\varepsilon}(D)$ for some $\varepsilon > 0$ as soon as $\dfrac{1}{\gamma} + 2(\beta - 1) > -1$, which can always be achieved with *negative* $\beta$, no matter how close $\gamma$ is to 1. However if the extension theorem were valid for this type $D$ then the extended $f$ would have to be in $L_1^{2+\varepsilon}(\mathbf{R}^2)$, and so by Sobolov's theorem (Theorem 2 in §2.2 of Chapter V) it would have to be continuous, which is a contradiction.

*3.2.1 Outline.* Let us consider the domain $D$, and the points $(x, y)$ which lie outside its closure. Our problem is to define $\mathfrak{E}(f)(x, y)$, (here $y < \varphi(x)$), where $f$ is given in $D$. For fixed $x$ we shall define $\mathfrak{E}(f)(x, y)$ for $\varphi(x) > y$ in terms of a suitable average of the values of $f$ on the segment where $\varphi(x) < y$. Two things are needed so that we can implement this idea. First, an appropriate weighting function in terms of which the averages will be defined. Secondly, a device to get around the difficulty that the difference $\varphi(x) - y$ allows for at most one order of differentiation (in $x$). Matters are taken care of by the following two lemmas.

LEMMA 1. *There exists a continuous function $\psi$ defined on* $[1, \infty)$ *which is rapidly decreasing at $\infty$, that is $\psi(\lambda) = O(\lambda^{-N})$, as $\lambda \to \infty$ for every $N$, and which satisfies in addition the properties*

$$\int_1^\infty \psi(\lambda)\, d\lambda = 1, \qquad \int_1^\infty \lambda^k \psi(\lambda)\, d\lambda = 0, \quad \textit{for} \quad k = 1, 2, \ldots.$$

LEMMA 2. *Let $F = \bar{D}$. Suppose $\Delta(x, y)$ is the regularized distance from $F$, as given in Theorem 2, in §2.1. Then there exists a constant $c$, (which depends only on the Lipschitz bound of $D$), so that if $(x, y) \in {}^cF$, then $c\Delta(x, y) \geq \varphi(x) - y$.*

For simplicity of notation we shall write $\delta^* = 2c\Delta$, so then $\delta^*(x, y) \geq 2(\varphi(x) - y)$.

We now write down what will turn out to be the extension operator for $D$. If $(x, y) \in {}^c\bar{D}$, we will set

$$\mathfrak{E}(f)(x, y) = \int_1^\infty f(x, y + \lambda\, \delta^*(x, y))\psi(\lambda)\, d\lambda, \tag{24}$$

where the integral will be defined in an appropriate limiting sense.

The plan of how we are to proceed is as follows. First in §3.2.2 we give the proof of the two lemmas. Next we show that the operator $\mathfrak{E}$ defined by (24) accomplishes the goal of Theorem 5′, namely the extension when $D$ is a special Lipschitz domain. Finally the $\mathfrak{E}$ corresponding to the more general domain considered will be constructed in terms of the operators corresponding to the special domains.

*3.2.2 Proof of Lemmas 1 and 2.* An elementary function can be given which satisfies the conclusion of Lemma 1, namely

$$\psi(\lambda) = \frac{e}{\pi\lambda} \cdot \operatorname{Im}\,(e^{-\omega(\lambda-1)^{1/4}})$$

where $\omega = e^{-i\pi/4}$.

In fact we consider the single-valued analytic function $e^{-\omega(z-1)^{1/4}}$ in the complex plane which is slit along the real axis from 1 to $+\infty$. In this connection we take the contour $\gamma$ which goes from $+\infty$ to 1 above the slit, makes an infinitesimal half-loop about 1 and returns to $+\infty$ below the slit. Then since $e^{-\omega(z-1)^{1/4}}$ decreases rapidly as $z \to \infty$ we get by Cauchy's theorem that

$$-\frac{1}{2\pi i}\int_\gamma \frac{1}{z}\, e^{-\omega(z-1)^{1/4}}\, dz = e^{-\omega(z-1)^{1/4}}\Big]_{z=0} = e^{-1}$$

while

$$\frac{1}{2\pi i}\int_\gamma \frac{z^k}{z}\, e^{-\omega(z-1)^{1/4}}\, dz = 0, \quad \text{if} \quad k = 1, 2, \ldots .$$

To prove Lemma 2 we use a simple geometrical interpretation of the Lipschitz condition for the boundary of $D$. Let $\Gamma_-$ be the (lower) cone with vertex at the origin given by $\Gamma_- = \{(x, y): M\,|x| < |y|, y < 0\}$. For any point $p \in \mathbf{R}^{n-1}$, denote by $\Gamma_-(p)$ the cone $\Gamma_-$ translated so that its vertex is $p$.

Now it is immediate that the Lipschitz condition (22) implies that if $p$ is any point on the boundary of $D$, i.e. $p = (x_1, y_1)$, with $y_1 = \varphi(x_1)$, then $\Gamma_-(p) \subset {}^c\bar{D} = {}^cF$. Next, let $(x, y)$ denote any point in ${}^c\bar{D}$, and let $p = (x, \varphi(x))$ be the point on the boundary of $D$ lying above it. Then of course $(x, y) \in \Gamma_-(p)$, and no point of $\bar{D}$ is closer to $(x, y)$ than the boundary of $\Gamma_-(p)$. Clearly $(x, y)$ lies along the central axis of the circular cone $\Gamma_-(p)$, and a simple geometrical argument shows that this minimum distance must be at least $\dfrac{\varphi(x) - y}{\sqrt{1 + M^{-2}}}$. Therefore

$$\delta(x, y) \geq (1 + M^{-2})^{-1/2}(\varphi(x) - y),$$

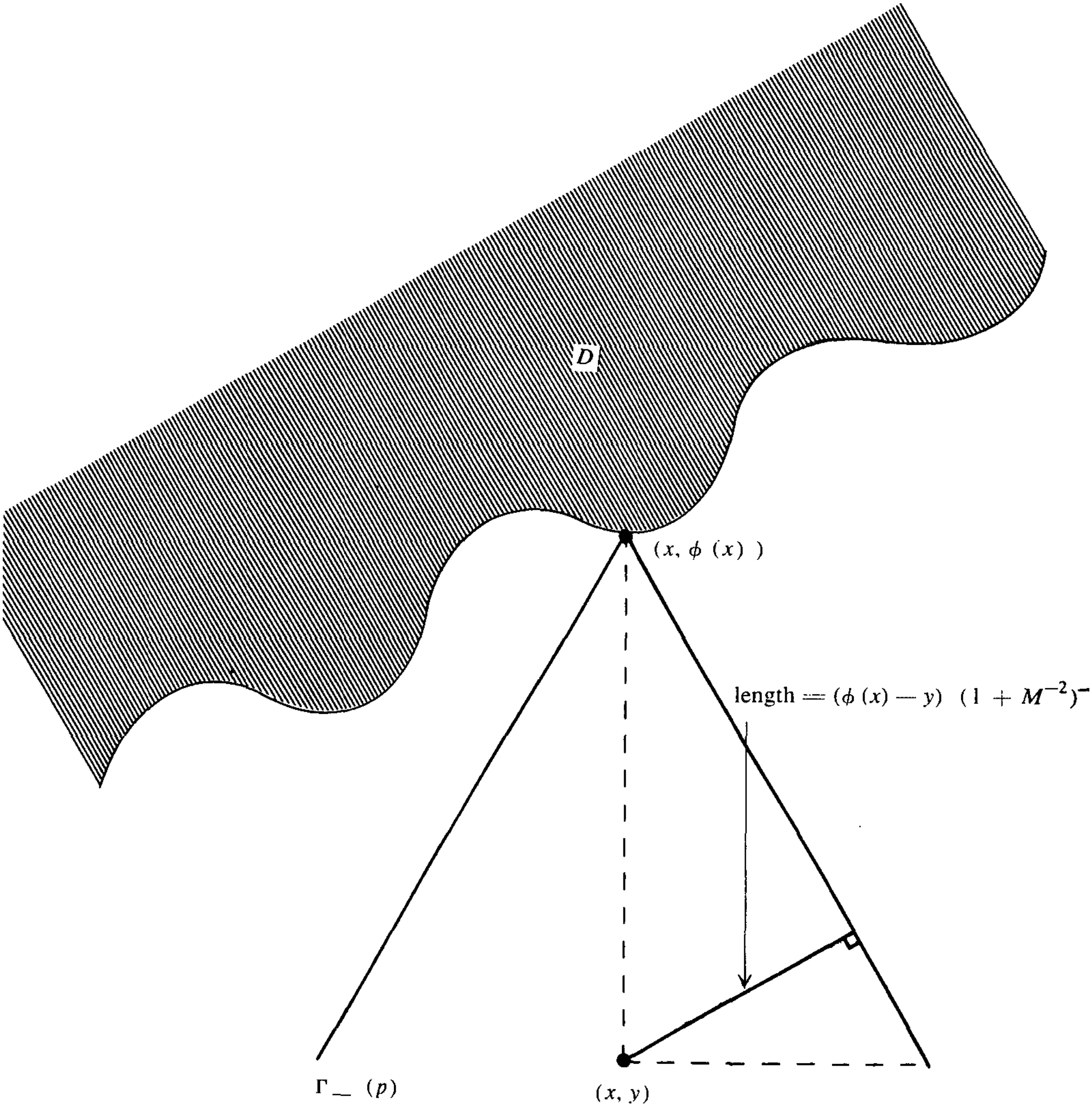



*Figure 3.* The Lipschitz domain $D$, and the exterior cone $\Gamma_{-}(p)$.

and hence $c\Delta(x, y) \geq \varphi(x) - y$, with $c = 5(1 + M^{-2})^{1/2}$, because by Theorem 2 we have $c_1\delta \leq \Delta$, and the proof of theorem shows we may take $c_1 = 1/5$; (see §2.1).

*3.2.3* Armed with Lemmas 1 and 2 and the resulting definition (24) for $\mathfrak{E}$ we come now to the proof of Theorem 5′. We assume that $f \in L_k^p(D)$;

we suppose also that $f$ is $C^\infty$ in $D$ and it, together with all its partial derivatives are continuous and bounded in $\bar{D}$. The last set of conditions are of course not satisfied for all $f \in L_k^p(D)$. But it will be our intention to show that for such $f$ we have the inequality

$$\|\mathfrak{E}(f)\|_{L_k^p(\mathbf{R}^{n+1})} \leq A_{k,n}(M) \|f\|_{L_k^p(D)}. \tag{25}$$

With this *a priori* inequality we shall be able to treat the general $f$ in $L_k^p(D)$ by a passage to the limit.

Now define $\mathfrak{E}(f)(x, y) = f(x, y)$, if $(x, y) \in \bar{D}$ and $\mathfrak{E}(f)(x, y) = \int_1^\infty \psi(\lambda) f(x, y + \lambda\delta^*(x, y))\, d\lambda$, for $(x, y) \in {}^c\bar{D}$. Notice that in view of the fact that $\delta^*(x, y) \geq 2(\varphi(x) - y)$, we get that

$$y + \lambda\delta^*(x, y) \geq y + 2(\varphi(x) - y) = \varphi(x) + \varphi(x) - y > \varphi(x), \text{ if } \lambda \geq 1.$$

This, and the assumed boundedness of $f$, shows that the integral giving $\mathfrak{E}$ is well defined.

We are now faced by the following situation: let $D$ be as before and $D_- = \{(x, y): \varphi(x) > y\}$, that is those points lying strictly below our Lipschitz hypersurface. Then of course $\bar{D} \cup \bar{D}_- = \mathbf{R}^n$, but $\bar{D}$ and $\bar{D}_-$ intersect. The properties of $f$ that we are momentarily taking for granted assure that $\mathfrak{E}(f)$ is continuous with all its partial derivatives in $\bar{D}$. Next observe that for such $f$, $\mathfrak{E}(f)$ is $C^\infty$ in $D_-$ and all its partial derivatives are continuous (and bounded in $\bar{D}_-$). The argument is entirely typical for say, $\dfrac{\partial^2\mathfrak{E}(f)}{\partial x_j^2}$. Carrying out the differentiation gives

$$\begin{aligned}\frac{\partial^2\mathfrak{E}(f)}{\partial x_j^2} &= \int_1^\infty f_{jj}(\cdot)\psi(\lambda)\, d\lambda + \int_1^\infty f_{jy}(\cdot)\lambda\, \delta_j^*\psi(\lambda)\, d\lambda \\ &= \int_1^\infty f_{yy}(\cdot)(\lambda\, \delta_j^*)^2\psi(\lambda)\, d\lambda + \int_1^\infty f_y(\cdot)\lambda\, \delta_{jj}^*\psi(\lambda)\, d\lambda.\end{aligned} \tag{26}$$

We have used the following abbreviation: the notation $f_{jj}$ means $\dfrac{\partial^2 f}{\partial x_j^2}$; similarly $\dfrac{\partial f}{\partial y}$ is designated by $f_y$. The $(\cdot)$ stands for $(x, y + \lambda\delta^*)$.

In view of the assumed differentiability of $f$ in $D$ the above shows $\dfrac{\partial^2}{\partial x_j^2}\mathfrak{E}(f)(x, y)$ is well defined for $(x, y) \in D_-$. (In fact it is just as obvious that $\mathfrak{E}f(x, y)$ is $C^\infty$ in $D_-$.) Next let $(x, y)$ tend to a boundary point of $D_-$, namely $(x^0, y^0) \in \bar{D}_- \cap \bar{D}$. Then $\delta^*(x, y) \to 0$ and since $\delta_j^*$ remains bounded (see conclusion (b) of Theorem 2) while

$$\int_1^\infty \psi(\lambda)\, d\lambda = 1, \qquad \int_1^\infty \lambda\psi(\lambda)\, d\lambda = 0, \qquad \int_1^\infty \lambda^2\psi(\lambda)\, d\lambda = 0,$$

the first three terms on the right side converge to a total which equals

$$\lim_{\substack{(x,y)\in D \\ (x,y)\to(x^0,y^0)}} \frac{\partial^2 f}{\partial x_j^2}(x, y).$$

The difficulty with the last term is that it involves more than one derivative of the regularized distance $\delta^*$, which is then no longer bounded. However we can write

$$f_y(x, y + \lambda\delta^*) = f_y(x, y + \delta^*) + (\lambda - 1)\delta^* f_{yy}(x, y + \delta^*) \\ + O([(\lambda - 1)\delta^*]^2).$$

Substituting in the last term gives two further integrals which vanish identically together with a remainder, which is

$$O\left((\delta^*)^2\, \delta^*_{x_j x_j} \int_1^\infty (\lambda - 1)^2 \lambda \psi(\lambda)\, d\lambda\right).$$

Now $\psi$ is rapidly decreasing, therefore by Theorem 2 we have that the whole quantity is $O(\delta) \to 0$, as $(x, y) \to (x^0, y^0)$.

To summarize, since $f$ is continuous (and bounded) with all its partial derivatives in $\bar{D}$, then the same can be said about $\mathfrak{E}f$ in $\bar{D}_-$, and $f$ and $\mathfrak{E}f$ agree on $\bar{D} \cap \bar{D}_-$, together with all their partial derivatives. This shows also that $\mathfrak{E}f \in C^\infty(\mathbf{R}^{n-1})$, which in effect means that the two pieces of $\mathfrak{E}f$, one coming from $D$, the other from $D_-$, have in reality been joined properly. Let us prove this by first showing that $\mathfrak{E}f$ is of class $C^1$. (The continuity of $\mathfrak{E}f$ has already been demonstrated.)

It will be necessary to show that

$$\mathfrak{E}(f)(u) - \mathfrak{E}(f)(v) = (u - v) \cdot (\nabla\mathfrak{E}f)(u) + o(|u - v|),$$

for any point $v \in \mathbf{R}^{n-1}$, as $u \to v$. There is nothing further to prove if $v$ is in either $D$ or $D_-$. Assume therefore that $v$ is on the (common) boundary of these two domains. Suppose $u \in \bar{D}_-$; the argument if $u \in \bar{D}$ is entirely similar. We claim that $v$ and $u$ can be joined by a broken line segment which except for $v$ and $u$ is entirely in $D_-$, and its total length is not greater than $c\,|v - u|$. In fact there exists a $w \in D_-$, so that the segments joining $v$ to $w$, and $w$ to $u$ have the required property. To find this $w$, notice that either $u \in \Gamma_-(v)$, (then we can pick $w = u$) or else the cones $\Gamma_-(v)$ and $\Gamma_-(u)$ must intersect. The nearest point of the intersection will do for $w$. Now $\mathfrak{E}(f)(u) - \mathfrak{E}(f)(w) = (u - w)(\nabla\mathfrak{E}f)(u) + o(|u - w|)$, and $\mathfrak{E}(f)(w) - \mathfrak{E}(f)(v) = (w - v)(\nabla\mathfrak{E}f)(w) + o(|w - v|)$. Adding the two and using the fact that $(\nabla\mathfrak{E}f)(w) - (\nabla\mathfrak{E}f)(u) = o(1)$, as $|u - v| \to 0$ in $\bar{D}$ gives the required result. Similarly it is seen that $\mathfrak{E}f \in C^k(\mathbf{R}^{n+1})$ for every $k$.

The next step is to prove the inequality (25), which we will do by proving a corresponding inequality for each fixed $x$, and then integrating the result in $x$.

Consider first the case $k = 0$. Let us fix on $x^0$ and assume (for notational convenience) that $\varphi(x^0) = 0$. Then

$$|\mathfrak{E}f(x^0, y)| \le A\int_1^\infty |f(x^0, y + \lambda\,\delta^*)|\,\frac{d\lambda}{\lambda^2}, \qquad y < 0.$$

We are of course using the fact that $|\psi(\lambda)| \le A/\lambda^2$. Since $\delta^* \ge 2(\varphi(x) - y)$, we have $\delta^* \ge 2\,|y|$ in this case. Also of course in general $\varphi(x) - y \ge$ distance $(x, y)$ from $\bar{D}$, therefore in this case $\delta^* \le a\,|y|$. Put $s = y + \lambda\delta^*$, with $y$ fixed, then $ds = d\lambda$ and the above inequality becomes

$$|\mathfrak{E}f(x, y)| \le A\,\delta^*\int_{|y|}^\infty f(x^0, s)\,(s - y)^{-2}\,ds, \qquad y < 0.$$

Therefore

$$(27) \qquad |\mathfrak{E}(f)(x^0, y)| \le Aa\,|y|\int_{|y|}^\infty |f(x^0, s)|\,\frac{ds}{s^2}, \qquad y < 0.$$

Hardy's inequality (see Appendix A, p. 272) then shows that

$$\left(\int_{-\infty}^0 |\mathfrak{E}(f)(x^0, y)|^p\,dy\right)^{1/p} \le A'\left(\int_0^\infty |f(x^0, y)|^p\,dy\right)^{1/p}.$$

If we drop the condition that $\varphi(x^0) = 0$, which we may after a suitable translation in $y$, we get

$$\left(\int_{\infty}^\infty |\mathfrak{E}(f)(x^0, y)|^p\,dy\right)^{1/p} \le A\left(\int_{\varphi(x^0)}^\infty |f(x^0, y)|^p\,dy\right)^{1/p}.$$

Raising both sides to the $p^{\text{th}}$ power and integrating for $x^0 \in \mathbf{R}^n$ then gives the inequality (25), for $k = 0$. The proof for $k > 0$ is similar. Consider for example the case $k = 2$. Here the consideration of $\dfrac{\partial^2\mathfrak{E}}{\partial x_j^2}$ is again typical. The first three terms in the right side of (26) are handled in the same way, except that now we use $|\psi(\lambda)| \le A/\lambda^4$, $\lambda \ge 1$. Only the last term needs to be dealt with separately. We write

$$(28) \qquad f_y = f_y(x^0, y + \delta^*) + \int_{y+\delta^*}^{y+\lambda\delta^*} f_{yy}(x^0, t)\,dt$$

and substitute this in (26). The contribution of the integral involving $f_y(x^0, y + \delta^*)$ vanishes identically by the orthogonality condition on $\psi$, and we are reduced to estimating

$$|y|^{-1}\int_1^\infty \left\{\int_{y+\delta^*}^{y+\lambda\delta^*} |f_{yy}(x^0, t)|\,dt\right\}\lambda^{-3}\,d\lambda.$$

An interchange of the order of integration reduces this to the earlier case (analogous to the right-side of (27)), and disposes of the case $k = 2$.

For general $k$, if we carry out the differentiation under the integral sign in (24), then we will get various orders derivatives of $f$. Whenever the total order appearing in $f$ is less than $k$, (it may be as low as 1), then we write the Taylor expansion of this derivative about the point $(x^0, y + \delta^*)$; carry it up to order $k$ with integral remainder, and then proceed as above.

In fact assume the order of differentiation appearing in $f$ is $k_0$, with $k_0 < k$. Let $g$ be that partial derivative of $f$ of order $k_0$. Then we write

$$g(x^0, y + \lambda\,\delta^*) = \sum_{j=0}^{l-1} \frac{((\lambda - 1)\,\delta^*)^j}{j!} \frac{\partial^j}{\partial y^j} g\Big]_{(x^0, y+\delta^*)} + \frac{1}{l!}\int_{\delta^*}^{\lambda\delta^*} (\lambda\,\delta^* - t)^{l-1} \frac{\partial^l}{\partial t^l} g(x^0, y + t)\, dt$$

$$\text{with} \quad k_0 + l = k$$

Only the integral gives a non-zero contribution, but it is dominated by

$$A(\lambda\,\delta^*)^{l-1}\int_{\delta^*}^{\lambda\delta^*} \left| \frac{\partial^l}{\partial t^l} g(x^0, y + t) \right| dt,$$

and the argument is then as before.

We should notice that in all these calculations, leading to the proof of (25), the only effect the region $D$ has on the bounds that appear is via its Lipschitz bound $M$ of (22).

*3.2.4* The last step of the proof of Theorem 5′ will be to remove the restriction that $f$ is $C^\infty$ in $D$ and it and all its partial derivatives are continuous in $\bar{D}$ and bounded.

For this purpose suppose $\eta \in C^\infty(\mathbf{R}^{n+1})$ is non-negative, has total integral 1, and that the support of $\eta$ lies strictly in the interior of the cone $\Gamma_-$. For any $\varepsilon \geq 0$ write $\eta_\varepsilon(u) = \varepsilon^{-n-1}\eta(u/\varepsilon)$, $(u \in \mathbf{R}^{n+1})$, and $f_\varepsilon(u) = \int f(u - v)\eta_\varepsilon(v)\, dv$, where $f$ is given in $L_k^p(D)$. Notice that if $u \in \bar{D}$, the integral involves only $u - v \in D$, and so is well-defined. Moreover since the support of $\eta_\varepsilon$ is strictly inside $\Gamma_-$, we see that the integral defines $f_\varepsilon(u)$ in a neighborhood of $\bar{D}$, and $f_\varepsilon$ is $C^\infty$ there. It is also clear that

$$\left\| \frac{\partial^\alpha f_\varepsilon}{\partial x^\alpha} \right\|_{L^p(D)} \leq \left\| \frac{\partial^\alpha f}{\partial x^\alpha} \right\|_{L^p(D)}, \qquad |\alpha| \leq k,$$

therefore

$$\|f_\varepsilon\|_{L_k^p(D)} \leq \|f\|_{L_k^p(D)}. \tag{29}$$

Now if $p < \infty$, we know that $f_\varepsilon \to f$ in the norm of $L_k^p(D)$. For $p = \infty$ it suffices to content ourselves with the weaker statement that if $k \geq 1$, then $f_\varepsilon \to f$ in the $L^\infty_{k-1}$ norm. (See the closely related facts, §2.1 and §6.2 of Chapter V.)

Now let $\mathfrak{E}_\varepsilon(f) = \mathfrak{E}(f_\varepsilon)$, then we have by (25) and (29) that

$$\|\mathfrak{E}_\varepsilon(f)\|_{L^p_k(\mathbf{R}^{n+1})} \le A_{k,n}(M)\, \|f\|_{L^p_k(D)}. \tag{30}$$

Now clearly $\|f_\varepsilon - f_{\varepsilon'}\|_{L^p_{k-1}(D)} \to 0$, as $\varepsilon, \varepsilon' \to 0$. Therefore $\mathfrak{E}_\varepsilon(f)$ is Cauchy in $L^p_{k-1}(\mathbf{R}^{n+1})$, and its limit also satisfies (30), which means we have proved the existence of an operator $\mathfrak{E} = \lim \mathfrak{E}_\varepsilon$, extending our $\mathfrak{E}$ defined originally on $C^\infty$ functions on $\bar{D}$. This is obviously the required extension operator satisfying (25). Theorem 5′ is therefore completely proved.

**3.3 The general case.** Having disposed of the case of the special Lipschitz domain, we leave the setting of $\mathbf{R}^{n+1}$ and return to $\mathbf{R}^n$. It will be convenient to modify our terminology slightly by referring to rotations of these domains also as special Lipschitz domains. The notion of the Lipschitz bound of such domain is then defined in the obvious manner, and is of course rotation invariant.

Now let $D$ be an open set in $\mathbf{R}^n$, and let $\partial D$ be its boundary. We shall say that $\partial D$ is *minimally smooth*, if there exists an $\varepsilon > 0$, an integer $N$, an $M > 0$, and a sequence $U_1, U_2, \ldots U_n \ldots$ of open sets so that:

(i) If $x \in \partial D$, then $B(x, \varepsilon) \subset U_i$, for some $i$; $B(x, \varepsilon)$ is the ball of center $x$ and radius $\varepsilon$.

(ii) No point of $\mathbf{R}^n$ is contained in more than $N$ of the $U_i$'s.

(iii) For each $i$ there exists a special Lipschitz domain $D_i$ whose bound does not exceed $M$ so that

$$U_i \cap D = U_i \cap D_i.$$

Some examples of the above are:

*Example 1.* Suppose $D$ is a bounded domain in $\mathbf{R}^n$ whose boundary is $C^1$ embedded in $\mathbf{R}^n$. In this case only finitely many $U_i$'s are needed.

*Example 2.* $D$ is any open bounded convex set. Again only finitely many $U_i$'s are required.

*Example 3.* $D \subset \mathbf{R}^1$, and $D = \bigcup_j I_j$, where $I_j$ are disjoint open intervals. The conditions (i)–(iii) are satisfied if there exists a $\delta > 0$, so that length $I_j \ge \delta$, and dist $(I_j, I_k) \ge \delta$, if $j \ne k$. In this example an infinite number of $U_i$'s are required if there are infinitely many $I_j$. The conditions length $I_j \ge \delta$, dist $(I_j, I_k) \ge \delta$ are also necessary. The reader can easily verify that the condition length $I_j \ge \delta$ is required if we are to have a bounded extension for $L_1^1$, while the condition dist $(I_j, I_k) \geq \delta$ is needed in order to have a bounded extension for $L_1^\infty$.

*3.3.1* We shall prove Theorem 5 by reducing it to Theorem 5′ for special Lipschitz domains. The argument however is somewhat tricky.

For any set $U \subset \mathbf{R}^n$ and any $\varepsilon > 0$, denote by $U^\varepsilon = \{x : B(x, \varepsilon) \subset U\}$. Notice that $U^\varepsilon \subset U$ and the condition (i) can be restated by saying $\bigcup_i U_i^\varepsilon \supset \partial D$. Let $\eta(x)$ denote a fixed $C^\infty$ function in $\mathbf{R}^n$ of total integral one, whose support is contained in the unit ball, and denote $\eta_\varepsilon(x) = \varepsilon^{-n}\eta(x/\varepsilon)$. Suppose that $\chi_i$ is the characteristic function of $U_i^{3\varepsilon/4}$, and let

$$\lambda_i(x) = (\chi_i * \eta_{\varepsilon/4})(x).$$

The following properties of the $\lambda_i$ are now evident:

(a) each $\lambda_i$ is supported in $U_i$;

(b) $\lambda_i(x) = 1$, if $x \in U_i^{\varepsilon/2}$, and so in particular if $x \in U_i^\varepsilon$;

(c) each $\lambda_i \in C^\infty$ has bounded derivatives of all orders and the bounds of the derivatives of $\lambda_i$ can be taken to be independent of $i$. (These depend only on the $L^1$ norms of the corresponding derivatives of $\eta_{\varepsilon/4}$.)

Consider in addition three other open sets, covering respectively a neighborhood of $D$, the boundary of $D$, and that part of the interior of $D$ away from the boundary, namely

$$U_0 = \{x : \text{dist}\,(x, D) < \varepsilon/4\}$$
$$U_+ = \{x : \text{dist}\,(x, \partial D) < (3/4)\varepsilon\}$$
$$U_- = \{x \in D : \text{dist}\,(x, \partial D) > \varepsilon/4\}.$$

Write $\chi_0$, $\chi_-$, and $\chi_-$ for the characteristic functions of these sets and regularize the functions as above, to wit $\lambda_0 = \chi_0 * \eta_{\varepsilon/4}$, $\lambda_+ = \chi_+ * \eta_{\varepsilon/4}$, and $\lambda_- = \chi_- * \eta_{\varepsilon/4}$. Then clearly $\lambda_0(x) = 1$, if $x \in \bar{D}$; $\lambda_+(x) = 1$ if dist $(x, \partial D) \leq \varepsilon/2$; and $\lambda_-(x) = 1$ if $x \in D$ and dist $(x, \partial D) \geq \varepsilon/2$. Moreover the supports of $\lambda_0$, $\lambda_+$, and $\lambda_-$ lie respectively in the $\varepsilon/2$-neighborhood of $D$, the $\varepsilon$-neighborhood of $\partial D$, and in $D$. Also the functions are bounded in $\mathbf{R}^n$ together with all their partial derivatives. Now set

$$\Lambda_+ = \lambda_0\left(\frac{\lambda_+}{\lambda_+ + \lambda_-}\right) \quad \text{and} \quad \Lambda_- = \lambda_0\left(\frac{\lambda_-}{\lambda_+ + \lambda_-}\right)$$

We see that $\lambda_0$ is supported in the set where $\lambda_+ + \lambda_- \geq 1$. Therefore $\Lambda_+$ and $\Lambda_-$ also have all their derivatives bounded in $\mathbf{R}^n$, and $\Lambda_+ + \Lambda_- = 1$ if $x \in \bar{D}$, while $\Lambda_+ + \Lambda_- = 0$ outside the $\varepsilon/2$-neighborhood of $D$.

Recall that the open sets $U_1, U_2, \ldots U_i, \ldots$ covering the boundary of $D$ had special Lipschitz domains $D_1, D_2, \ldots, D_i, \ldots$ associated with them.

Let $\mathfrak{E}^i$ be the extension operator for $L^p_k(D_i)$ whose properties are given by Theorem 5′.

After all these preliminaries we can finally write down the required extension operator $\mathfrak{E}$ for $D$. In fact for $f \in L^p(D)$ define

$$(31)\qquad (\mathfrak{E}f)(x) = \Lambda_+(x)\left\{\frac{\sum_{i=1}^{\infty} \lambda_i(x)\mathfrak{E}^i(\lambda_i f)}{\sum_{i=1}^{\infty} \lambda_i^2(x)}\right\} + \Lambda_-(x)f(x).$$

Observe the following facts:

(d) For $x$ in the support of $\Lambda_+$, (more generally if dist $(x, \partial D) \leq \varepsilon/2$), then $x \in U_i^{\varepsilon/2}$ for at least one $i$, and thus $\sum \lambda_i^2(x) \geq 1$ there. (See (b) above.)

(e) For each $x$ the sum (30) involves at most $N + 1$ non-vanishing terms (because of the condition (ii) on the covering $\{U_i\}$);

(f) The term $\Lambda_-(x)f(x)$ is well-defined since the support of $\Lambda_-$ is contained in $D$;

(g) The terms $\mathfrak{E}^i(\lambda_i f)$ are well-defined since the $\lambda_i f$ are given in the special Lipschitz domains $D_i$;

(h) It is evident that $(\mathfrak{E}f)(x) = f(x)$ for $x \in D$.

In order to prove the basic inequality

$$(32)\qquad \|\mathfrak{E}(f)\|_{L^p_k(\mathbf{R}^n)} \leq A_{kn}(D)\,\|f\|_{L^p_k(D)}, \quad \text{if} \quad f \in L^p_k(D)$$

we require the following remark.

PROPOSITION. *Suppose* $A(x) = \sum_{i=1}^{\infty} a_i(x)$, *and for each* $x$ *at most* $N$ *of the terms* $\{a_i(x)\}$ *are non-vanishing. Then*

$$\|A(x)\|_p \leq N^{1-1/p}\left(\sum_i \|a_i(x)\|_p^p\right)^{1/p} \quad \text{if} \quad p < \infty$$

and

$$\|A(x)\|_\infty \leq N \sup_i \|a_i(x)\|_\infty, \qquad \text{if} \quad p = \infty.$$

The symbol $\|\cdot\|_p$ denotes the standard $L^p$ norm. The case $L^\infty$ of this is trivial as well as the $L^1$ case. The general case is not much more difficult, and follows from the observation that

$$|A(x)|^p \leq N^{p-1} \sum_{i=0}^{\infty} |a_i(x)|^p$$

which is in turn an obvious consequence of Hölder's inequality.

We prove first (32) when $k = 0$. Using properties (a)–(c) of the $\lambda_i$, and then (d) to (h), together with the Proposition and the case ($k = 0$) of

Theorem 5′ we get, if $p < \infty$,

$$\begin{aligned}
\|\mathfrak{E}f\|_p &\le N^{1-1/p}\Big(\sum_i\int_{U_i}|\mathfrak{E}^i(\lambda_i f)|^p\,dx\Big)^{1/p} + \Big(\int_D |f(x)|^p\,dx\Big)^{1/p}\\
&\le AN^{1-1/p}\Big(\sum_i\int_D |\lambda_i f|^p\,dx\Big)^{1/p} + \Big(\int_D |f|^p\,dx\Big)^{1/p}\\
&\le AN\Big(\int_D |f|^p\,dx\Big)^{1/p} + \Big(\int_D |f|^p\,dx\Big)^{1/p}, \quad \text{since} \quad \Big(\sum_i \lambda_i\Big)^{1/p} \le N^{1/p}.
\end{aligned}$$

A similar reasoning holds for $p = \infty$. This proves (32) for $k = 0$. A very similar argument works for all $k$ since every fixed partial derivative of the $\lambda_i$, $i = 1, 2, \ldots, \Lambda_+$ and $\Lambda_-$ are all uniformly bounded. The proof of Theorem 5 is therefore concluded.

## 4. *Further results*

The following paragraphs §4.1 to 4.5, deal with the restriction of functions in $\mathscr{L}^p_\alpha(\mathbf{R}^n)$ to linear sub-varieties.

**4.1** Let $f$ be a locally integrable function on $\mathbf{R}^n$. We shall say that *$f$ can be strictly defined at $x^0$*, if $\lim_{\varepsilon\to 0}\dfrac{1}{m(B_\varepsilon)}\displaystyle\int_{B_\varepsilon} f(x^0 - t)\,dt$ exists, where $B_\varepsilon$ denotes the ball of radius $\varepsilon$ centered at the origin. In this case we redefine (if necessary) $f$ to have this value at $x^0$. If this is done at each point $x^0$ where this is possible we say that $f$ is *strictly defined.* Thus by the fundamental theorem of Chapter I every locally integrable function can be strictly defined, and agrees with the original function, almost everywhere. For the study of the restrictions of $\mathscr{L}^p_\alpha$ the following lemma is important.

LEMMA. *Suppose $f \in \mathscr{L}^p_\alpha(\mathbf{R}^n)$, $f = G_\alpha * \varphi$, with $\alpha > 0$, $1 \le p \le \infty$. Suppose $x^0$ is a point where the integral $\int_{\mathbf{R}^n} G_\alpha(x - t)\varphi(t)\,dt = f(x)$ representing $f$ converges absolutely. Then*

$$\frac{1}{m(B_\varepsilon)}\int_{B_\varepsilon} |f(x^0 - t) - f(x^0)|\,dt \to 0, \quad \text{as} \quad \varepsilon \to 0.$$

Thus $f$ can be strictly defined at $x^0$.

**4.2** We consider a linear $m$-dimensional sub-variety of $\mathbf{R}^n$. It will be no loss of generality to assume that it is the subspace $\mathbf{R}^m$ of $\mathbf{R}^n$ of points whose first $m$ coordinates are arbitrary, and whose last $n - m$ coordinates vanish. Now assume that $\alpha > (n - m)/p$, $1 \le p \le \infty$, and $f \in \mathscr{L}^p_\alpha(\mathbf{R}^n)$. Then $f$ can be strictly defined at all points of $\mathbf{R}^m$, except for a subset of $\mathbf{R}^m$ of $m$-dimensional Lebesgue measure zero. If we denote this restriction by $\mathscr{R}(f)$, then $\mathscr{R}(f) \in L^p(\mathbf{R}^m)$, and the mapping $f \to \mathscr{R}(f)$ of $\mathscr{L}^p_\alpha(\mathbf{R}^n)$ to $L^p(\mathbf{R}^m)$ is continuous, i.e.

$$\|\mathscr{R}f\|_{L^p(\mathbf{R}^m)} \le A\,\|f\|_{\mathscr{L}_\alpha{}^p(\mathbf{R}^n)}.$$

**4.3** The $\mathscr{R}(f)$ described above belongs not only to $L^p(\mathbf{R}^m)$, but also to an appropriate $\Lambda$ space. More precisely, let $\beta = \alpha - (n - m)/p > 0$, and $1 < p < \infty$. If $f \in \mathscr{L}^p_\alpha(\mathbf{R}^n)$, then $\mathscr{R}(f) \in \Lambda^{p,p}_\beta(\mathbf{R}^m)$, and the mapping $f \to \mathscr{R}(f)$ is continuous, that is

$$\|\mathscr{R}(f)\|_{\Lambda_\beta^{p,p}(\mathbf{R}^m)} \leq A\,\|f\|_{\mathscr{L}_\alpha^p(\mathbf{R}^n)}.$$

**4.4** The converse to §4.3 also holds. Thus the restriction of elements of $\mathscr{L}^p_\alpha(\mathbf{R}^n)$ to $\mathbf{R}^m$ consists of exactly $\Lambda^{p,p}_\beta(\mathbf{R}^m)$, with $\beta = \alpha - \dfrac{(n-m)}{p}$. We give an explicit description of this converse for the case $m = n - 1$. Let $\varphi$ be a fixed $C^\infty$ function on $\mathbf{R}^{n-1}$ with compact support, such that $\int_{\mathbf{R}^{n-1}} \varphi(x)\,dx = 1$. Let $\eta$ be a fixed $C^\infty$ function on $\mathbf{R}^1$ with compact support, so that $\eta(0) = 1$. For each locally integrable function $f$ on $\mathbf{R}^{n-1}$ consider its extension to $\mathbf{R}^n$ given by

$$(\mathbf{E}f)(x, y) = \eta(y) \int_{\mathbf{R}^{n-1}} f(x - yt)\varphi(t)\,dt, \qquad (x, y) \in \mathbf{R}^{n-1} \times \mathbf{R}^1 = \mathbf{R}^n.$$

It can then be shown that $f \in \Lambda^{p,p}_\beta(\mathbf{R}^{n-1})$ implies that $\mathbf{E}f \in \mathscr{L}^p_\alpha(\mathbf{R}^n)$, and

$$\|\mathbf{E}(f)\|_{\mathscr{L}_\alpha^p(\mathbf{R}^n)} \leq A\,\|f\|_{\Lambda_\beta^{p,p}(\mathbf{R}^{n-1})}.$$

Here it is assumed that $\beta = \alpha - 1/p > 0$.

A more precise form of this extension is as follows. Let $k$ be the largest integer not greater than $\beta$. Let $\eta_0(y), \ldots, \eta_k(y)$, be $C^\infty$ functions on $\mathbf{R}^1$ with compact support, with the property that $\left.\dfrac{d^l}{dy^l}\eta_j(y)\right|_{y=0} = \delta_{jl}$, $0 \leq j,\ l \leq k$. Assume that $f_j \in \Lambda^{p,p}_{\beta-j}(\mathbf{R}^{n-1})$, $0 \leq j \leq k$. Set

$$F(x, y) = \sum_{j=0}^{k} \eta_j(y) \int_{\mathbf{R}^{n-1}} f_j(x - yt)\varphi(t)\,dt.$$

Then $F \in \mathscr{L}^p_\alpha(\mathbf{R}^n)$,

$$\|F\|_{\mathscr{L}_\alpha^p(\mathbf{R}^n)} \leq A\left\{\sum_{j=0}^{k} \|f_j\|_{\Lambda_{\beta-j}^{p,p}(\mathbf{R}^{n-1})}\right\},$$

and $\mathscr{R}\left(\dfrac{\partial^j F(x, y)}{\partial y^j}\right) = f_j(x), j = 0, \ldots, k$, where $\mathscr{R}$ denotes the restriction to the hyperplane $y = 0$. For the above results see Stein [7], and the later treatments in Aronszajn, Mulla, and Szeptycki [1] and Lizorkin [2]. For the background see Gagliardo [1] and Aronszajn and Smith [1].

**4.5** Analogous results hold for the space $\Lambda^{p,q}_\alpha(\mathbf{R}^n)$. If $\alpha > \dfrac{(n-m)}{p}$, then the restriction of an element in $\Lambda^{p,q}_\beta(\mathbf{R}^n)$ to $\mathbf{R}^m$ belongs to $\Lambda^{p,q}_\alpha(\mathbf{R}^m)$ where $\beta = \alpha - \dfrac{(n-m)}{p}$. Conversely every function in $\Lambda^{p,q}_\beta(\mathbf{R}^m)$ can be extended to $\mathbf{R}^n$ so that it is an element in $\Lambda^{p,q}_\alpha(\mathbf{R}^n)$. For details see Besov [1], and Taibleson [2].

**4.6** Let $\omega(\delta)$, $0 < \delta$, be a regular modulus of continuity as defined in §2.2.3 above. For any closed set $F$ and every non-negative integer $k$ define the space Lip $(k + \omega, F)$ as in §2.3 (equations (16) and (17)), except that the assumption on $R_j$ should read

$$|R_j(x, y)| \leq M\,|x - y|^{k-|j|}\omega(|x - y|).$$

Then the operator $\mathscr{E}_k$ gives an extension from Lip $(k + \omega, F)$ to Lip $(k + \omega, \mathbf{R}^n)$.

**4.7** In the extension theorem of §2.3, assume $\gamma = k$ and in addition

$$R_j(x, y) = o(|x - y|^{k+|j|}),$$

in the sense that for any $\bar{x} \in F$, and any $\varepsilon > 0$, there exists a $\delta > 0$, so that $|\bar{x} - x| < \delta$, $|\bar{x} - y| < \delta$, and $x, y \in F$, implies that $|R_j(x, y)| \leq \varepsilon\,|x - y|^{k-|j|}$. Then $\mathscr{E}_k(f) \in C^k(\mathbf{R}^n)$. This is essentially the original extension theorem of the type in Whitney [1].

**4.8** The first extension theorem for functions in $L_k^p(D)$, where $D$ is a region of the kind treated in §3, was proved by the following argument.

In giving an outline of this argument we shall limit ourselves to a *special Lipschitz domain* $D \subset \mathbf{R}^n$ (as defined in §3.2 for $\mathbf{R}^{n+1}$). For our purpose all we need as a consequence of this definition is the existence of a fixed cone $\Gamma$, so that wherever $x \in D$, then $x + \Gamma \subset D$. Now fix $k \geq 1$ in the rest of this discussion. We choose a fixed function $\varphi$ supported in $-\Gamma$, with the following additional properties: (1) $\varphi$ has bounded support. (2) Near the origin $\varphi$ equals a function which is homogeneous of degree $-n + k$. (3) $\varphi$ is $C^\infty$ in $\mathbf{R}^n - \{0\}$. (4) $\lim_{\varepsilon\to 0} \int_{S^{n-1}} \varepsilon^{n-k}\varphi(\varepsilon x')\,d\sigma(x') = 1/(k-1)!$. (Observe that in view of the assumed homogeneity of $\varphi$ near the origin, the integral is actually constant for small $\varepsilon$.)

Let $\psi(\rho y') = \left(\dfrac{\partial}{\partial\rho}\right)^k(\rho^{n-1}\varphi(\rho y'))$, $(y = \rho y', |y| = \rho)$. Then $\psi$ vanishes near the origin, and so is $C^\infty$ everywhere (and of course has bounded support). Let

$$(\mathfrak{E}f)(x) = \int \varphi(\rho y')\rho^{n-1}\left(\frac{\partial}{\partial\rho}\right)^k f(x - \rho y')\,d\rho\,dy'$$
$$-\int \psi(\rho y')f(x - \rho y')\rho^{n-1}\,d\rho\,dy'.$$

The first integral may be rewritten as

$$(-1)^k\int_{\mathbf{R}^n}\varphi(y)\sum_{|\alpha|=k}\frac{k!}{\alpha!}y^\alpha\left(\frac{\partial}{\partial x}\right)^\alpha f(x - y)\,|y|^{-k}\,dy$$

while the second integral clearly equals

$$\int_{\mathbf{R}^n}\psi(y)f(x - y)\,dy.$$

We remark that $f$, and $\left(\frac{\partial}{\partial x}\right)^\alpha f$, $|\alpha| = k$, are given to us only in $D$. For the purposes of the integral formula defining $\mathfrak{E}f$, we set $f$ and $\left(\frac{\partial}{\partial x}\right)^\alpha f$ equal to zero outside $D$.

It can then be verified that $\mathfrak{E}f$ indeed extends $f$, and

$$\|\mathfrak{E}f\|_{L_k^p(\mathbf{R}^n)} \le A_{p,k} \|f\|_{L_k^p(D)}, \quad 1 < p < \infty.$$

The latter fact is proved by observing that a differentiation of order $k$ applied to the first integral defining $\mathfrak{E}f$ leads essentially to a singular integral operator of the kind treated in §4 of Chapter II. This explains the occurrence of the limitation $1 < p < \infty$. The reader should consult Calderón [4] for further details.

We make two further remarks about this extension. The operator $\mathfrak{E}$ depends on the particular $k$, and so is not universal, in the sense that the extension given in §3 is. On the other hand it has the interesting property that away from $D$, $\mathfrak{E}f$ depends only on the part of $f$ near the boundary of $D$. This means that if $f \in L_k^p(D)$, and $f$ vanishes near the boundary of $D$, then $\mathfrak{E}f(x) = 0$, if $x \notin D$.

## Notes

*Sections 1 and 2.* The fundamental paper is Whitney [1]. The particular form of Theorem 1, is in Stein [10]; Theorem 2 is in Calderón and Zygmund [7]. See also Glaeser [1].

*Section 3.* The extension theorem for functions defined in domains with Lipschitz boundaries originates in Calderón [4]. Some of the ideas used were implicit in Sobolov [2]. Calderón's extension does not apply to the limiting cases $p = 1$, or $p = \infty$, since it relies on the $L^p$ boundedness of the singular integrals treated in Chapter II. For the present version, see Stein [10]. Ideas related to the present extension and applied to the case $p = 2$ can be found in Adams, Aronszajn and Smith [1].

# CHAPTER VII

# Return to the Theory of Harmonic Functions

We return to the theory of harmonic functions to make a deeper study of some of its aspects and in particular with that part dealing with the notion of conjugate harmonic functions first discussed in Chapter III. The main threads of our development will be as follows:

(A) The notion of *non-tangential convergence.* While Theorem 1 in Chapter III guarantees the existence almost everywhere of boundary values for the perpendicular approach for Poisson integrals, the more general non-tangential limits also exist, and this is fundamental in what follows. It allows for a wide extension of the classical Fatou theorem to a purely "local" setup. These matters are dealt with in §1.

(B) *The "area integral" of Lusin.* This object was already pointed out in §2 of Chapter IV, but our attention there was focused on the closely related $g$ and $g^*$ functions.

The basic role of the area integral is due to the fact that it serves to characterize at once both non-tangential convergence and also $L^p$ and $H^p$ norms. The characterization of non-tangential convergence for harmonic functions will be taken up in §2 and will find important application to the study of differentiability properties of functions in Chapter VIII.

(C) *$H^p$ theory for conjugate harmonic functions.* The notion of conjugacy was previously introduced in the generalized Cauchy-Riemann equations in §2.3 of Chapter III, in connection with the Riesz transforms. Here we shall see that by using it, a certain subharmonicity, and the area integral of Lusin we can study the analogue for $p = 1$ of various results for singular integrals operators and multiplier transformations. One of the ideas (characterizing the $L^p$ boundedness of these operators in terms of $g$, or $g^*$) was already used in §3 of Chapter IV.

## 1. *Non-tangential convergence and Fatou's theorem*

**1.1** We shall use the following notation: $\mathbf{R}^{n+1}_+$ is the $n + 1$ dimensional half-space of points $(x, y)$, with $y > 0$, $x \in \mathbf{R}^n$. Its boundary $\{(x, 0)\}$ is identical with $\mathbf{R}^n$. For any $x^0 \in \mathbf{R}^n$, and $\alpha > 0$, $\Gamma_\alpha(x^0)$ will denote the (infinite) cone, $\Gamma_\alpha(x^0) = \{(x, y) \in \mathbf{R}^{n+1}_+ : |x - x^0| < \alpha y\}$, whose vertex is at $x^0$. If $u(x, y)$ is defined at those points in $\mathbf{R}^{n-1}_+$ near a boundary point $(x^0, 0)$, then $u$ has a non-tangential limit (which equals $l$) at $(x^0, 0)$ if for every $\alpha > 0$ the conditions $(x, y) \in \Gamma_\alpha(x^0)$ and $(x, y) \to (x^0, 0)$ imply that $u(x, y) \to l$.

The basic result about non-tangential convergence for Poisson integrals is contained in the following extension of Theorem 1 of Chapter III (see p. 62).

THEOREM 1. *Suppose $f \in L^p(\mathbf{R}^n)$, $1 \le p \le \infty$, and let $u(x, y)$ be its Poisson integral. Assume that $\alpha$ is a fixed positive number. Then*

(a)
$$\sup_{(x,y)\in\Gamma_\alpha(x^0)} |u(x, y)| \le A_\alpha M(f)(x^0),$$

*where $Mf$ is the maximal function of Chapter I, §1. $A_\alpha$ is independent of $f$.*

(b)
$$\lim_{\substack{(x,y)\to(x^0,0)\\(x,y)\in\Gamma_\alpha(x^0)}} u(x, y) = f(x^0),$$

*for almost every $x^0$, and in particular for every point $x^0$ in the Lebesgue set of $f$.*

This theorem is actually a relatively easy consequence of the corresponding theorem in Chapter III.

To prove (a), recall that

$$P_y(x) = \frac{c_n y}{(|x|^2 + y^2)^{(n+1)/2}},$$

(Proposition 1, Chapter III). Therefore as is easily seen,

(1)
$$P_y(x - t) \le A_\alpha P_y(x) \quad \text{if} \quad |t| < \alpha y.$$

However,

$$u(x^0 - t, y) = \int_{\mathbf{R}^n} P_y(x^0 - t - z, y) f(z)\, dx.$$

Therefore

$$\begin{aligned}\sup_{|t|<\alpha y} |u(x^0 - t, y)| &= \sup_{(x,y)\in\Gamma_\alpha(x^0)} |u(x, y)| \\ &\le \sup_{|t|<\alpha y} \int_{\mathbf{R}^n} P_y(x^0 - t - z, y)\ |f(z)|\, dz \\ &\le A_\alpha \sup_y \int_{\mathbf{R}^n} P_y(x^0 - z)\, |f(z)|\, dz \\ &\le A_\alpha (Mf)(x^0).\end{aligned}$$

Thus (a) is proved.

Incidentally, the simple property (1) can in effect be re-expressed in the fact that if $P_1(x) = \varphi(x)$, then $\sup_{|t|<1} \varphi(x - t) \leq A\varphi(x)$ for some constant $A$ independent of $x$. (With this remark the reader should have no difficulty in formulating and proving a general non-tangential version for the approximations $f * \varphi_\varepsilon$, analogous to Theorem 2 of §2.2, Chapter III.)

To prove (b) assume that $x^0$ is a point of the Lebesgue set of $f$. Thus for any $\varepsilon > 0$, there exists a $\delta > 0$ so that

$$\frac{1}{m(B(r))} \int_{B(r)} |f(x^0 - z) - f(x^0)| \, dz < \varepsilon,$$

whenever $r > \delta$; $B(r)$ denotes the ball of radius $r$ about the origin. Now set $g(x) = f(x) - f(x^0)$, if $|x - x^0| \leq \delta$ and $g(x) = 0$ if $|x - x^0| > \delta$. Notice, therefore, that $M(g)(x^0) < \varepsilon$. We have, however,

$$u(x^0 - t, y) - f(x^0) = \int_{\mathbf{R}^n} P_y(z - t)[f(x^0 - z) - f(x^0)] \, dz.$$

So by (1), if $|t| < \alpha y$,

$$\begin{aligned} |u(x^0 - t, y) - f(x^0)| &\leq A_\alpha \int_{\mathbf{R}^n} P_y(z) \, |f(x^0 - z) - f(x^0)| \, dz \\ &= A_\alpha \left\{ \int_{|z|<\delta} + \int_{|z|\geq\delta} \right\}. \end{aligned}$$

Now

$$\int_{|z|\leq\delta} \leq \int_{\mathbf{R}^n} P_y(t)|g(x^0 - z)| dz \leq M(g)(x^0) < \varepsilon,$$

by Theorem 1 (part (1)) in Chapter III. However,

$$\int_{|z|\geq\delta} P_y(z) \, |f(x^0 - z) - f(x^0)| \, dz \to 0, \quad \text{as} \quad y \to 0$$

which follows immediately from the fact that

$$\left( \int_{|z|>\delta} [P_y(z)]^q \, dz \right)^{1/q} \leq A_\delta y \to 0,$$

for $1 \leq q \leq \infty$, where $q$ is chosen to be successively the exponent dual to $p$, and then 1. Therefore $\limsup_{|t|<\alpha y, y\to 0} |u(x^0 - t, y) - f(x^0)| \leq A_\alpha \varepsilon$, which proves the required non-tangential convergence at every point of the Lebesgue set of $f$. Since almost every point is a point of the Lebesgue set (see §1.8 in Chapter I) we have therefore concluded the proof of the theorem.

**1.2 Fatou's theorem.** We characterize first bounded harmonic functions in $\mathbf{R}_+^{n+1}$.

PROPOSITION 1. *Suppose $u$ is given in $\mathbf{R}_+^{n+1}$. Then $u$ is the Poisson integral of a function in $L^\infty(\mathbf{R}^n)$ if and only if $u$ is harmonic and bounded.*

That the Poisson integral of a bounded function is (harmonic) and bounded follows from the considerations in §2 of Chapter III.

To prove the converse assume that $u$ is harmonic and $|u| \le M$ in $\mathbf{R}_+^{n+1}$. For each integer $k$ set $f_k(x) = u\left(x, \frac{1}{k}\right)$, and let $u_k(x, y)$ be the Poisson integral of $f_k$. Finally let $\Delta_k(x, y) = u\left(x, y + \frac{1}{k}\right) - u_k(x, y)$. Notice the following obvious properties of $\Delta_k$. First, $\Delta_k$ is harmonic in $\mathbf{R}_+^{n+1}$; also $|\Delta_k| \le \left|u\left(x, y + \frac{1}{k}\right)\right| + |u_k(x, y)| \le 2M$, so $\Delta_k$ is bounded. Finally by the corollary (on p. 65) in §2.2 of Chapter III, since $f_k$ is continuous and bounded, $u_k(x, y)$ is continuous in $\overline{\mathbf{R}}_+^{n+1}$ and $u_k(x, 0) = f_k(x)$; thus $\Delta_k$ is continuous in $\overline{\mathbf{R}}_+^{n+1}$ and $\Delta_k(x, 0) = 0$. We want to conclude on the basis of these properties that $\Delta_k \equiv 0$. To do this it suffices to show that $\Delta_k(0, 1) = 0$, since our assumptions are invariant under translation in the $x$ variables and dilations jointly in the $x$ and $y$ variables and any point in $\mathbf{R}_+^{n+1}$ can be mapped to $(0, 1)$ by a product of those transformations. For fixed $\varepsilon > 0$ consider the function

$$U(x, y) = \Delta_k(x, y) + 2M\varepsilon y + \varepsilon\left[\prod_{j=1}^{n} \cosh\left(\frac{\varepsilon\pi}{4n^{1/2}} x_j\right)\right] \cos\left(\frac{\varepsilon\pi}{4} y\right)$$

which is clearly harmonic in $\mathbf{R}_+^{n+1}$, and continuous in $\overline{\mathbf{R}}_+^{n+1}$. We restrict our attention to the cylindrical section $\Sigma = \left\{(x, y) : 0 \le y \le \frac{1}{\varepsilon}, |x| \le R\right\}$, where $R$ is sufficiently large. On that part of the boundary of $\Sigma$ where $y = 0$, we have that $U(x, y) \ge 0$ since $\Delta_k = 0$ there; where $y = \frac{1}{\varepsilon}$, we have $U(x, y) \ge 0$ since $|\Delta_k| \le 2M$. Finally if $R$ is sufficiently large, then again since $\Delta_k$ is bounded, we have $U(x, y) \ge 0$ on that part of the boundary. Thus by the maximum principle (see Appendix C) we have that $U(0, 1) \ge 0$, which means that $\Delta_k(0, 1) \ge -(2M + 1)\varepsilon$; a similar conclusion holds with $-\Delta_k$ in place of $\Delta_k$, and as a result $\Delta_k \equiv 0$ as desired. This can be rewritten as

$$u\left(x, y + \frac{1}{k}\right) = \int_{\mathbf{R}^n} P_y(x - t) f_k(t)\, dt. \tag{2}$$

Now recall that

$$\|f_k\|_\infty = \left\|u\left(x, \frac{1}{k}\right)\right\|_\infty \le M.$$

Therefore by a familiar weak-compactness argument, there is an $f \in L^\infty$, $\|f\|_\infty \le M$, and a subsequence $\{f_{k'}\}$, so that $f_{k'} \to f$ weakly in the sense that $\int_{\mathbf{R}^n} f_{k'}\varphi\, dt \to \int_{\mathbf{R}^n} f\varphi\, dt$, for any $\varphi \in L^1(\mathbf{R}^n)$. For fixed $(x, y) \in \mathbf{R}^{n+1}_+$, choose $\varphi(t) = P_y(x - t)$. Then (2) becomes in the limit

$$u(x, y) = \int_{\mathbf{R}^n} P_y(x - t) f(t)\, dt$$

which means that $u$ is the Poisson integral of the bounded function $f$. The proposition is therefore completely proved.

When we combine the proposition and Theorem 1 we immediately get the $n$-dimensional version of a classical theorem of Fatou.

THEOREM 2. *Suppose $u$ is harmonic and bounded in $\mathbf{R}^{n+1}_+$. Then $u$ has non-tangential limits at almost every point of the boundary $(\mathbf{R}^n)$ of $\mathbf{R}^{n+1}_+$.*

*1.2.1* Proposition 1 quickly leads to a generalization of itself.

COROLLARY. *Suppose $u$ is harmonic in $\mathbf{R}^{n+1}_+$, and $1 \le p \le \infty$. If $\sup_{y>0} \|u(\cdot, y)\|_{L^p(\mathbf{R}^n)} < \infty$, then $u$ is the Poisson integral of an $f \in L^p(\mathbf{R}^n)$, if $1 < p$. If $p = 1$, $u$ is the Poisson integral of a finite measure.*

This result was stated without proof in §4.2, Chapter III. The easy converse is contained in §2.2 of that chapter. To prove the corollary, assume $p < \infty$. For each $(x, y) \in \mathbf{R}^{n+1}_+$ let $B$ be the ball whose center is $(x, y)$ and radius is $y$. By the mean value theorem

$$|u(x, y)|^p \le \frac{1}{m(B)} \iint_B |u(x', y')|^p\, dx'\, dy'.$$

However, $B \subset \{(x', y'):\ 0 < y' < 2y\}$, and $m(B) = cy^{n+1}$. Therefore

$$|u(x, y)|^p \le c' y^{-n-1} \int_0^{2y} \int_{\mathbf{R}^n} |u(x', y')|^p\, dx'\, dy',$$

and

$$|u(x, y)| \le c'' y^{-n/p}.$$

For each positive integer $k$ we can therefore apply Proposition 1 (and Theorem 2) and obtain $u\left(x, y + \frac{1}{k}\right) = P_y * f_k$, with $f_k(x) = u\left(x, \frac{1}{k}\right)$.

However, by assumption $\sup_k \|f_k\|_p < \infty$, and thus the familiar weak compactness arguments apply. More specifically, if $p > 1$, there exists an $f \in L^p(\mathbf{R}^n)$ and a subsequence $\{f_{k'}\}$, so that $f_{k'} \to f$ weakly. For $p = 1$ there exists a finite measure $d\mu$ and a subsequence $\{f_{k'}\}$ so that $f_{k'} \to d\mu$ weakly. In either case it follows that $u(x, y) = P_y * f$ or $u(x, y) = P_y * d\mu$ respectively. Observe that $\|f\|_p = \sup_{y>0} \|u(\cdot, y)\|_p$, if $p > 1$, and $\|d\mu\| = \sup_{y>0} \|u(\cdot, y)\|_1$.

**1.3 A local version.** We define first the appropriate notion of non-tangential boundedness. For any $\alpha > 0$, and $h > 0$, we let $\Gamma_\alpha^h(x^0)$ denote the *truncated cone* $\Gamma_\alpha^h(x_0) = \{(x, y) \in \mathbf{R}_+^{n+1} : |x - x^0| < \alpha y,\ 0 < y < h\}$. If $u$ is defined in $\mathbf{R}_+^{n+1}$, we say that $u$ is *non-tangentially bounded* at $x^0$ if for some $\alpha$ and $h$,

$$\sup |u(x, y)| < \infty,\ (x, y) \in \Gamma_\alpha^h(x^0).$$

Notice the non-tangential boundedness at $x^0$ requires a condition with respect to only one truncated cone whose vertex is $x^0$, while the existence of the non-tangential limit at $x^0$ (as described in §1.1 above) requires a condition for *all* cones at $x^0$.

The main result of this section is the following local analogue of Fatou's theorem (Theorem 2).

THEOREM 3. *Suppose u is harmonic in* $\mathbf{R}_+^{n+1}$. *Let E be a subset of* $\mathbf{R}^n$ *and suppose that u is non-tangentially bounded at every* $x^0 \in E$. *Then u has a non-tangential limit at almost every* $x^0 \in E$.

*1.3.1* A preliminary argument allows us to uniformize the situation.

LEMMA. *Let u be continuous in* $\mathbf{R}_+^{n+1}$, *and suppose it is non-tangentially bounded at every point of the set* $E$, $E \subset \mathbf{R}^n$. *Then for* $\varepsilon > 0$, *there exists a compact set* $E_1$ *with:*

(1) $E_1 \subset E$, $m(E - E_1) < \varepsilon$.

(2) *For any* $\alpha > 0$, $h > 0$, *there is a bound* $M = M(\alpha, h, \varepsilon)$, *so that* $|u(x, y)| \le M$, $(x, y) \in \Gamma_\alpha^h(x^0)$, $x^0 \in E_1$.

*Proof. 1st step.* By considering only $\alpha$ and $h$ with rational values, we can find an $E_0$, with $m(E - E_0)$ small (say $m(E - E_0) < \varepsilon/3$), so that $|u(x, y)| \le M$, $(x, y) \in \Gamma_\alpha^h(x^0)$, $x^0 \in E_0$, with some fixed $\alpha$ and $h$. We may assume that $E_0$ is also compact.

*2nd step.* With this $E_0$ fixed and $k$ a large integer we shall choose a further subset $E_{00}$, $E_{00} \subset E_0$, with $m(E_0 - E_{00}) < \varepsilon/3$ so that

$$|u(x, y)| \leq M', \ (x, y) \in \Gamma_k^{\bar{\delta}}(x^0), \ x^0 \in E_{00}.$$

To do so we proceed as follows. If $\eta < 1$ and $\varepsilon/3$ are given, there exists a $\delta > 0$, and a subset $E_{00}$ so that $m(E_0 - E_{00}) < \varepsilon/3$ and

$$\frac{m(B(x, r) \cap E_0)}{m(B(x, r))} \geq \eta,$$

for $x \in E_{00}$, and $r \leq \delta$; this is because almost every point of $E_0$ is a point of density. Our required boundedness on $E_{00}$ will hold if we show that for some $\bar{\delta}$ sufficiently small we have

$$\text{(3)} \qquad \Gamma_k^{\bar{\delta}}(x^0) \subset \bigcup_{x' \in E_0} \Gamma_\alpha^h(x'), \quad \text{all} \quad x^0 \in E_{00}.$$

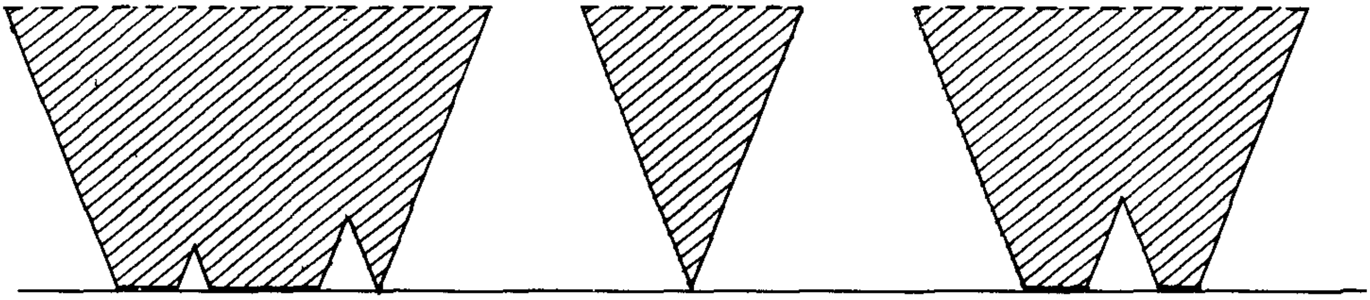
To prove (3) assume for simplicity that $x^0 = 0$. We must therefore consider pairs $(x, y)$ in the cone $|x| < ky$ (with $y < \bar{\delta}$). Fix such a pair $(x, y)$. We want to show that there exists an $x' \in E_0$, so that $|x - x'| < \alpha y$. Suppose, on the contrary, that for this $x$ (and $y$) there is no such $x'$. Then the ball $|x - x'| < \alpha y$ is in ${}^cE_0$. This is a ball of radius $\alpha y$ to be compared with the (large) ball of radius $ky$ (centered at the origin). Since the origin is in $E_{00}$ by assumption, the relative measure in the large ball is at most $1 - \alpha/k$, which would be a contradiction if we had chosen $\eta > 1 - \alpha/k$. Thus for sufficiently small $y$ we can find a required $x'$, and (3) is proved, with $\bar{\delta} = \delta/k$. In view of the continuity of $u$ it is clear that $u$ is bounded for points at a positive distance from $\mathbf{R}^n$. Notice that we can also choose $E_{00}$ to be compact.

*Step 3.* For each integer $k$ we construct a subset $E_{00}$ of this type. An appropriate intersection of a countable collection of such $E_{00}$ gives the required set $E_1$.

*1.3.2* With $E$ any compact subset of $\mathbf{R}^n$, we define the open region $\mathscr{R}$ by

$$\text{(4)} \qquad \mathscr{R} = \bigcup_{x^0 \in E} \Gamma_\alpha^h(x^0).$$

A moment's reflection shows that the above lemma allows one to reduce Theorem 3 to its essential core: *Assume $u$ is harmonic in $\mathbf{R}^{n+1}_+$ and $|u| \leq 1$ for $(x, y) \in \mathscr{R}$. Then for almost every $x^0 \in E$ the limit*, $\lim u(x, y)$ *exists as* $(x, y) \to (x^0, 0)$, *and* $(x, y) \in \mathscr{R}$. This we now proceed to prove.



$x^0$

*Figure 4.* The region $\mathscr{R}$, which is the union of cones $\Gamma_\alpha^h(x^0)$, $x^0 \in E$.

*1.3.3* For each $m > 0$, we define the function $\varphi_m$ on $\mathbf{R}^n$ as follows: $\varphi_m(x) = u\left(x, \frac{1}{m}\right)$, if $\left(x, \frac{1}{m}\right) \in \mathscr{R}$; $\varphi_m = 0$ otherwise. Let $\varphi_m(x, y)$ be the Poisson integral of $\varphi_m$, that is $\varphi_m(x, y) = (P_y * \varphi_m)(x)$. Define $\psi_m(x, y)$ by

$$u\left(x, y + \frac{1}{m}\right) = \varphi_m(x, y) + \psi_m(x, y). \tag{5}$$

Now the sequence of functions $\{\varphi_m(x)\}$ are uniformly bounded in the $L^\infty(\mathbf{R}^n)$ norm, and thus we can find a $\varphi(x) \in L^\infty$ (in fact $|\varphi(x)| \leq 1$), and a subsequence $\{\varphi_{m'}\}$ so that $\varphi_{m'} \to \varphi$ weakly. Let $\varphi(x, y) =$ Poisson integral of $\varphi$. Then clearly $\varphi_{m'}(x, y) \to \varphi(x, y)$ at each point $(x, y) \in \mathbf{R}^{n+1}_+$; also $u\left(x, y + \frac{1}{m}\right) \to u(x, y)$ and so $\psi_{m'}(x, y)$ converges pointwise to $\psi(x, y)$ and we have

$$u(x, y) = \varphi(x, y) + \psi(x, y). \tag{5'}$$

The function $\varphi(x, y)$ is the Poisson integral of a bounded function and so non-tangential convergence almost everywhere holds for it. The function $\psi$ is constructed to have zero boundary values on $E$, and this is what we in fact intend to prove about it in terms of a simple inequality. To do this we consider an auxiliary harmonic function $H(x, y)$ with the following properties in the region $\mathscr{R}$. We divide the boundary of $\mathscr{R}$ into two parts: we write $\partial\mathscr{R} = \mathscr{B} = \mathscr{B}_0 \cup \mathscr{B}_-$, where $\mathscr{B}_0$ is the part of the region which intersects the boundary hyperplane $\mathbf{R}^n$, and $\mathscr{B}_+$ is that part of the boundary lying above $\mathbf{R}^n$. That is $\mathscr{B}_0 = \{(x, 0) \in \bar{\mathscr{R}} - \mathscr{R}\}$, and $\mathscr{B}_+ = \{(x, y) \in \bar{\mathscr{R}} - \mathscr{R}, y > 0\}$.

$H$ will have the following properties:

(i) $H$ is harmonic in $\mathbf{R}^{n+1}_+$

(ii) $H \geq 0$ in $\mathbf{R}^{n+1}_+$, (this is appropriate for $\mathscr{B}_0$)

(iii) $H \geq 2$ on $\mathscr{B}_+$

(iv) as $(x, y) \to (x^0, 0)$ non-tangentially, $H(x, y) \to 0$, for almost all $x^0 \in E$.

The construction of $H$ is simple. Let $\chi$ denote the characteristic function of the complement of $E$. For a constant $c$ to be determined momentarily we write $H(x, y) = c\{(P_y * \chi)(x) + y\}$. The properties (i) and (ii) are obvious; (iv) is an immediate consequence of Theorem 1 of this chapter; the only one that requires further examination is (iii).

Now the boundary $\mathscr{B}_+$ can be further decomposed into its "trivial" part namely that part for which $y = h$, and its non-trivial part, that for which $0 < y < h$. For the trivial part we can always assure (iii) by taking $c$ large enough $\left(c \geq \frac{2}{h}\right)$. It remains to consider the non-trivial part of the boundary $\mathscr{B}_+$. It should be observed, and this is also important for later considerations, that the part of the boundary in question is a part of the Lipschitz hypersurface given by $y = \frac{1}{\alpha}\operatorname{dist}(x, E)$. We can therefore conclude that for such points $(x, y)$, the ball $B$ in $\mathbf{R}^n$ whose center is $x$ and which has radius $\alpha y$ lies outside $E$. Hence

$$(P_y * \chi)(x) = c_n y \int_{\mathbf{R}^n} \frac{\chi(t)}{(|\chi - t|^2 + y^2)^{(n+1)/2}}\, dt \geq c_n y \int_B \cdots dt$$
$$= c_n y \int_{|t| < \alpha y} \frac{1 \cdot dt}{(|t|^2 + y^2)^{(n+1)/2}} = \text{constant}.$$

The fact that the last integral is a constant can be seen by an obvious change of variables. Again taking $c$ large enough we see that all the properties of $H$ have been verified. We shall now prove that for fixed $m$

$$(6) \qquad |\psi_m(x, y)| \leq H(x, y) \quad \text{when} \quad (x, y) \in \mathscr{R}.$$

In view of the harmonic character of $\psi_n$ and $H$ and the maximum principle if this were not so there would exist an $\varepsilon > 0$, and a sequence of points $(x_k, y_k)$ converging to a point on the boundary of $\mathscr{R}$ so that $|\psi_m(x_k, y_k)| \geq H(x_k, y_k) + \varepsilon$. First $u$ and $\varphi_n$ are both bounded by 1 in absolute value, and therefore $|\psi_m(x, y)| \leq 2$ and so by property (iii) the limit of $\{(x_k, y_k)\}$ cannot be on $\mathscr{B}_+$. However, $\varphi_m(x, y)$ is the Poisson integral of a function which is continuous on an open set containing $E$, and its values there are $u\left(x, \frac{1}{m}\right)$. Thus by Theorem 1, part (b)

$$\lim_k \left\{u\left(x_k, y_k + \frac{1}{m}\right) - \varphi_m(x_k, y_k)\right\} = \lim_k \psi_m(x_k, y_k) = 0,$$

and again we obtain a contradiction, this time by (ii). We have therefore proved (6), and we can now let $m \to \infty$ through the subsequence $\{m'\}$.

The result is

$$(6') \qquad |\psi(x, y)| \le H(x, y)$$

Property (iv) then gives us the required convergence of $\psi$ (to zero) as $(x, y) \to (x^0, 0)$, $(x, y) \in \mathscr{R}$, for almost every $(x^0, 0) \in E$. This concludes the proof of Theorem 3.

*1.3.4* It is interesting to point out that this theorem and many other results of this chapter (in particular Theorem 5 below) do not have valid analogues if non-tangential convergence is replaced by perpendicular convergence to the boundary. See §4.12 below.

## 2. *The area integral*

**2.1** The theorem we have just proved shows that for a harmonic function in $\mathbf{R}_+^{n+1}$ the properties of non-tangential boundedness and existence of non-tangential limits are almost everywhere equivalent. Looked at from a broad point of view the two properties (boundedness and existence of limits) are not so very different. There is, however, another condition, almost everywhere equivalent with the first two, which has a different character, and finds its expression in a certain quadratic integral introduced by Lusin.

We fix the shape of our typical truncated cone $\Gamma_\alpha^h$; (that is we fix $\alpha$ and $h$). Whenever $u$ is given in $\mathbf{R}_+^{n+1}$ (or sub-region of $\mathbf{R}_+^{n+1}$ which contains the part near the boundary $\mathbf{R}^n$) we define

$$(7) \qquad S(u)(x^0) = \left(\iint_{\Gamma_{\alpha(x^0)}^h} |\nabla u|^2 \, y^{1-n} \, dy \, dx\right)^{1/2}.$$

(This is to be compared with the variant occurring in §2.3 of Chapter IV; see p. 89.)

Here

$$|\nabla u|^2 = \left|\frac{\partial u}{\partial y}\right|^2 + \sum_{j=1}^{n} \left|\frac{\partial u}{\partial x_j}\right|^2,$$

and we have written a double integral in (7) to emphasize that we are dealing with an $n + 1$ dimensional integration in distinction to certain $n$-dimensional integrations to be carried out below. The reason for the name "area integral" arose when $n = 1$. Then $(S(u)(x^0))^2$ represents the area (points counted with their multiplicity) of the image in $\mathbf{R}^2$ of the triangle $\Gamma_\alpha^h(x^0)$ under the analytic mapping $z \to F(z)$, where the real part of $F$ is $u$. When $n > 1$, $S$ loses this simple interpretation, but we shall nevertheless continue to use the terminology of the case $n = 1$. The characterization of non-tangential convergence in terms of the area integral is as follows.

THEOREM 4. *Let $u$ be harmonic in $\mathbf{R}^{n+1}_+$. Then except for a set of points $x^0$ of zero measure in $\mathbf{R}^n$, the following two conditions are equivalent:*
(1) *$u$ has a non-tangential limit at $x^0$*
(2) $S(u)(x^0) < \infty$

*2.1.1* A word of explanation is needed about the statement of the theorem. The proof that (1) $\Rightarrow$ (2) will show in fact that except for a subset of measure zero, at the points $x^0$ where the non-tangential limit exists we have $S(u)(x^0) < \infty$, no matter what choice of the parameters $\alpha$ and $h$ we make that determine the shape of the truncated cone.

Conversely suppose that we have $S(u)(x^0) < \infty$ for $x^0$ in a given set, where the truncated cones may vary from point to point. Then the proof of the theorem will show that $u$ has non-tangential limits at almost every point of the set.

In particular it follows that assuming the finiteness of $S$ for variable cones is almost everywhere equivalent with assuming the finiteness of $S$ for all truncated cones. (This last point, however, is quite elementary and could be proved directly if one wished.)

*2.2.1* The proof of the theorem will depend on an application of Green's theorem to the region $\mathscr{R}$ defined in §1.3.2, (p. 202). The boundary of this region has just barely the required regularity to utilize Green's theorem and so it is convenient to approximate $\mathscr{R}$ by a family of regions $\{\mathscr{R}_\varepsilon\}$ with very smooth boundaries. The family $\partial\mathscr{R}_\varepsilon$ have a certain uniform smoothness which reflects the minimal smoothness of $\partial\mathscr{R}$.

LEMMA. *There exists a family of regions $\{\mathscr{R}_\varepsilon\}$, $\varepsilon > 0$, with the following properties:*
($\alpha$) $\overline{\mathscr{R}}_\varepsilon \subset \mathscr{R}$, $\mathscr{R}_{\varepsilon_1} \subset \mathscr{R}_{\varepsilon_2}$ *if* $\varepsilon_2 < \varepsilon_1$.
($\beta$) $\mathscr{R}_\varepsilon \to \mathscr{R}$ *as* $\varepsilon \to 0$ (i.e. $\cup\mathscr{R}_\varepsilon = \mathscr{R}$).
($\gamma$) *the boundary $\partial\mathscr{R}_\varepsilon = \mathscr{B}_\varepsilon$ of the region is the union of two parts $\mathscr{B}^1_\varepsilon \cup \mathscr{B}^2_\varepsilon$, so that $\mathscr{B}^2_\varepsilon$ is a portion of the hyperplane $y = h - \varepsilon$ and*
($\delta$) *$\mathscr{B}^1_\varepsilon$ is a portion of the hypersurface $y = \alpha^{-1}\delta_\varepsilon(x)$ where $\delta_\varepsilon \in C^\infty$, and*

$$\left|\frac{\partial\delta_\varepsilon}{\partial x_j}\right| \leq 1, \qquad j = i, \ldots, n.$$

Set $\delta(x) = \text{dist}\,(x, E)$. Then clearly $|\delta(x) - \delta(x')| \leq |x - x'|$, if $x, x' \in \mathbf{R}^n$. Suppose $\varphi \in C^\infty(\mathbf{R}^n)$, $\varphi$ has compact support, is positive, and

$$\int_{\mathbf{R}^n} \varphi(x)\, dx = 1.$$

Write $\tilde{\delta}_\eta(x) = \varphi_\eta * \delta$, with $\varphi_\eta(x) = \eta^{-n}\varphi(x/\eta)$. Thus it is clear that $\tilde{\delta}_\eta \in C^\infty$ and $\left|\frac{\partial \tilde{\delta}_\eta}{\partial x_j}\right| \leq 1$; also $\tilde{\delta}_\eta \to \delta$ uniformly as $\eta \to$. It is then easy to see that with an appropriate choice of $\eta$ and $\eta'$ ($\eta'$ is a constant $> 0$) the functions $\delta_\varepsilon(x) = \tilde{\delta}_\eta(x) + \eta'$ and the corresponding regions $\mathscr{R}_\varepsilon = \{(x, y): \delta_\varepsilon(x) < \alpha y,\ 0 < y < h - \varepsilon\}$, satisfy all the conclusions of our lemma.

*2.2.2* For fixed $\alpha$ and $h$ we have our typical truncated cone $\Gamma_\alpha^h = \Gamma_\alpha^h(0)$. It will be necessary in what is done below to consider another truncated cone strictly containing it. We fix it by fixing $\beta$ and $k$, with $\beta > \alpha$, and $k > h$. Then clearly $\Gamma_\beta^k \supset \Gamma_\alpha^h$, and the only point that the boundaries of these cones have in common is their common vertex.

LEMMA. *Suppose $u$ is harmonic in $\Gamma_\beta^k$.*

(i) *If $|u| \leq 1$ in $\Gamma_\beta^k$, then $|y\nabla u| \leq c$ in $\Gamma_\alpha^h$.*

(ii) *If $\iint_{\Gamma_\beta^k} |\nabla u|^2 y^{1-n}\, dx\, dy \leq 1$, then $|y\nabla u| \leq c$ in $\Gamma_\alpha^h$. $c$ is a constant which depends only on $\alpha$, $\beta$, $h$, $k$ and the dimension $n$.*

Observe that there exists a constant $c_1$, $c_1 > 0$, with the property that if $(x, y)$ is any point of the smaller cone, $\Gamma_\alpha^h$, the ball $B$ whose center is $(x, y)$ and has radius $c_1 y$ lies in $\Gamma_\beta^k$. Now by the mean-value property of harmonic functions we can assert the following inequalities (see Appendix C):

$$|(\nabla u)(x, y)| \leq c_2 r^{-1} \sup_{(x,y)\in B} |u(x, y)|$$

and

$$|(\nabla u)(x, y)|^2 \leq c_2 r^{-n-1} \iint_B |\nabla u|^2\, dx'\, dy'$$

where $r$ = radius of $B$ ($= c_1 y$).

The first of these two inequalities gives conclusion (i) immediately. To prove the second conclusion observe that for $(x', y') \in B$, the value $y'$ remains comparable to the radius of $B$ and more particularly $r \geq c_3 y'$, and so $y \geq c_4 y'$. Thus

$$y^2 |\nabla u|^2 \leq c_4 \iint_B |\nabla u|^2\, y^{1-n}\, dx'\, dy \leq c_4 \iint_{\Gamma_\beta^k} |\nabla u|^2 y^{1-n}\, dx'\, dy.$$

The lemma is therefore proved.

**2.3 Proof of the direct part,** (1) $\Rightarrow$ (2). Suppose $\alpha$ and $h$ are given. It suffices to show that at almost every point $x^0$ of a given compact set $E$, we have $\int\int_{\Gamma^h(x^0)} |\nabla u|^2 y^{1-n}\, dx\, dy < \infty$, if we assume that

$$\sup_{x^0 \in E} \sup_{(x,y)\in \Gamma_\beta{}^k(x^0)} |u(x, y)| \leq 1 \tag{8}$$

for some fixed $\beta > \alpha$, $k > h$. Indeed assuming that $u$ has non-tangential limits at a given set, we can always find compact subsets whose measure is arbitrarily close to the given set, and where the uniformity leading to (8) would hold, (after multiplication by a suitable non-zero constant).

We therefore write $\tilde{\mathscr{R}} = \bigcup_{x^0 \in E} \Gamma_\beta^k(x^0)$, (in the same way as

$$\mathscr{R} = \bigcup_{x^0 \in E} \Gamma_\alpha^h(x^0)).$$

Our assumption becomes, therefore, that $|u| \leq 1$ in $\tilde{\mathscr{R}}$.

In order to show that $S(u)(x^0) < \infty$ almost everywhere in $E$ it suffices to show that $\int_E S^2(u)(x^0)\, dx^0 < \infty$. This integral equals

$$\iint \left(\int_E \psi(x^0, x, y)\, dx^0\right) y^{1-n} |\nabla u(x, y)|^2\, dx\, dy$$

where $\psi$ is the characteristic function of the set $\{|(x - x^0| < \alpha y, 0 < y < h\}$. However

$$\int_E \psi(x^0, x, y)\, dx^0 \leq \int_{|x^0 - x| < \alpha y} dx^0 = c y^n,$$

thus we want to see why

$$\iint_{\mathscr{R}} y\, |\nabla u(x, y)|^2\, dx\, dy < \infty. \tag{9}$$

We replace now $\mathscr{R}$ by the approximating regions $\mathscr{R}_\varepsilon$ and then (9) is equivalent with

$$\iint_{\mathscr{R}_\varepsilon} y\, |\nabla u|^2\, dx\, dy \leq A < \infty \tag{9'}$$

with $A$ independent of $\varepsilon$. To evaluate (9′) we use Green's theorem in the form

$$\iint_{\mathscr{R}_\varepsilon} (A\, \Delta B - B\, \Delta A)\, dx\, dy = \int_{\mathscr{B}_\varepsilon} \left(A \frac{\partial B}{\partial n_\varepsilon} - B \frac{\partial A}{\partial n_\varepsilon}\right) d\tau_\varepsilon \tag{10}$$

for the region with smooth boundary $\partial \mathscr{R}_\varepsilon = \mathscr{B}_\varepsilon$, where $\dfrac{\partial}{\partial n_\varepsilon}$ indicates the directional derivative along the outward normal and $d\tau_\varepsilon$ is the element of "area" of $\partial\mathscr{R}_\varepsilon$. Now let $B = \dfrac{u^2}{2}$, and $A = y$, then $\Delta B = |\nabla u|^2$ and $\Delta A = 0$. Since $\partial \mathscr{R}_\varepsilon \subset \tilde{\mathscr{R}}$ the estimate (i) of the lemma holds there. That is, $\left|\frac{\partial B}{\partial n_\varepsilon}\right| \le |u|\,|\nabla u|$, and so $\left|\frac{A\partial B}{\partial n_\varepsilon}\right| \le c$ on $\mathscr{B}_\varepsilon$. Similarly, since $\left|\frac{\partial y}{\partial n_\varepsilon}\right| \le 1$ we have $\left|\frac{B\partial y}{\partial n_\varepsilon}\right| \le \frac{1}{2}$. Altogether then the integrand of the integral on $\mathscr{B}_\varepsilon$ in (10) is uniformly bounded, and so

$$\left|\int_{\mathscr{B}_\varepsilon}\left(\frac{A\partial B}{\partial n_\varepsilon} - \frac{B\partial A}{\partial n_\varepsilon}\right) d\tau_\varepsilon\right| \le c\int_{\mathscr{B}_\varepsilon} d\tau_\varepsilon.$$

However,

$$\int_{\mathscr{B}_\varepsilon} d\tau_\varepsilon = \int_{\mathscr{B}_\varepsilon{}^1} d\tau_\varepsilon + \int_{\mathscr{B}_\varepsilon{}^2} d\tau_\varepsilon$$

where $\mathscr{B}_\varepsilon^2$ is a portion of the hyperplane $y = h - \varepsilon$, and $\mathscr{B}_\varepsilon^1$ is a portion of the surface $y = \alpha^{-1}\delta_\varepsilon(x)$. On $\mathscr{B}_\varepsilon^2$ we have $d\tau_\varepsilon = dx$, while on $\mathscr{B}_\varepsilon^1$

$$d\tau_\varepsilon = \left(\sqrt{1 + \alpha^{-2}\sum_{j=1}^{n}\left(\frac{\partial \delta_\varepsilon}{\partial x_j}\right)^2}\right) dx \le \sqrt{1 + \alpha^{-2}n}\, dx$$

by property ($\delta$) of the lemma in §2.2.1. Since in any case $\mathscr{R}$ and thus $\mathscr{B}_\varepsilon$ is contained in a fixed compact set, it then follows that $\int_{\mathscr{B}_\varepsilon} d\tau_\varepsilon \le$ constant $< \infty$, and so (9) is proved.

**2.4 Proof of the converse part,** (2) $\Rightarrow$ (1). We may reduce our assumptions to the following. We can assume that for certain $\beta$ and $k$ (which may be small), and a given bounded set $E_0$ we have

$$\iint_{\Gamma_\beta^k(x^0)} |\nabla u|^2 y^{1-n} dx\, dy \le 1, \quad \text{for} \quad x^0 \in E_0.$$

Now let $E$ be chosen as follows: $E \subset E_0$, $E$ is compact and $m(E_0 - E)$ is small; also there exists an $\eta > 0$, so that if $x^0 \in E_0$ then

$$m\{\{|x - x^0| < r\} \cap E_0\} \ge \tfrac{1}{2} m\{|x - x^0| < r\},$$

for $0 < r < \eta$. At the end of the proof we let $m(E_0 - E) \to 0$.

Such choices of $E$ are possible since almost every point of $E_0$ is a point of density of $E_0$.

Now suppose $\alpha$ and $h$ are fixed with $\alpha < \beta$, $h < k$. We shall then study $u$ on $\mathscr{R} = \bigcup_{x^0 \in E} \Gamma_\alpha^h(x^0)$. We will now try to reverse the logical chain used in the direct part, and our first objective is the proof of (9).

Since we assumed that $E_0$ was bounded, we get

$$\text{(11)} \qquad \int_{E_0}\left\{\iint_{\Gamma_\beta^k} |\nabla u|^2\, y^{1-n}\, dx\, dy\right\} dx^0 < \infty.$$

It is now necessary to estimate $\int_{E_0} \tilde{\psi}(x_0, x, y)\, dx^0$ from below for $(x, y) \in \mathscr{R}$. Here $\tilde{\psi}$ is the characteristic function of $\{|x - x_0| < \beta y, 0 < y < k\}$. Now $(x, y) \in \mathscr{R}$ means there exists a $z \in E$, so that $|x - z| < \alpha y$, $0 < y < h$. Hence we see that

$$\int_{E_0} \tilde{\psi}(x_0, x, y)\, dx^0 \geq \int_{E_0 \cap \{|x^0 - z| < (\beta - \alpha) y\}} dx^0.$$

By the property of relative density not less than $\frac{1}{2}$, which was assumed for $E$, we get that the second integral exceeds $cy^n$, for some constant $c$. Substituting this in (11) then gives (9). But (9) is equivalent with (9′) and in turn this is the same as

$$\left| \int_{\mathscr{B}_\varepsilon} \left( y \frac{\partial u^2}{\partial n_\varepsilon} - u^2 \frac{dy}{\partial n_\varepsilon} \right) d\tau_\varepsilon \right| \leq c < \infty. \tag{12}$$

However, $\mathscr{B}_\varepsilon = \mathscr{B}_\varepsilon^1 \cup \mathscr{B}_\varepsilon^2$, and $\mathscr{B}_\varepsilon^2$ is at a strictly positive distance from the set $\{y = 0\}$, and is also compact. Thus the contribution in (12) coming from $\mathscr{B}_\varepsilon^2$ is bounded and hence we have

$$\left| \int_{\mathscr{B}_\varepsilon{}^1} \left( y \frac{\partial u^2}{\partial n_\varepsilon} - u^2 \frac{\partial y}{\partial n_\varepsilon} \right) d\tau_\varepsilon \right| \leq c < \infty. \tag{12′}$$

Now on $\mathscr{B}_\varepsilon^1$, $\dfrac{\partial y}{\partial n_\varepsilon} \leq -\alpha(\alpha^2 + n^2)^{-1/2}$. In fact $\dfrac{\partial}{\partial n_\varepsilon}$ is the outward normal derivative to the surface whose equation is $F_\varepsilon(x, y) \equiv \alpha y - \delta_\varepsilon(x) = 0$. The direction of $\dfrac{\partial}{\partial n_\varepsilon}$ is then given by the unit vector with the same direction as

$$\left( -\frac{\partial F_\varepsilon}{\partial y}, \frac{\partial F_\varepsilon}{\partial x_1}, \ldots, \frac{\partial F_\varepsilon}{\partial x_n} \right) = \left( -\alpha, \frac{\partial \delta_\varepsilon}{\partial x_j}, \ldots, \frac{\partial \delta_\varepsilon}{\partial x_n} \right).$$

Since $\left| \dfrac{\partial \delta_\varepsilon}{\partial x_j} \right| \leq 1$ we see that $\dfrac{\partial y}{\partial n_\varepsilon} \leq -\alpha(\alpha^2 + n^2)^{-1/2}$.

Next let $\mathscr{I}_\varepsilon^2 = \int_{\mathscr{B}_\varepsilon{}^1} u^2\, d\tau_\varepsilon$. According to what we have just said

$$\mathscr{I}_\varepsilon^2 \leq c_1 \int_{\mathscr{B}_\varepsilon{}^1} |u|\, y \left| \frac{\partial u}{\partial n_\varepsilon} \right| d\tau_\varepsilon + c_2, \quad \text{by (12′)}.$$

Now $\mathscr{B}_\varepsilon^1 \subset \tilde{\mathscr{R}} = \bigcup_{x^0 \in E} \Gamma_\beta^k(x^0)$, thus by the conclusion (ii) in the lemma of §2.2.2 we see that $y \left| \dfrac{\partial u}{\partial n_\varepsilon} \right| \leq y\, |\nabla u| \leq c$ there. Hence

$$\int_{\mathscr{B}_\varepsilon{}^1} |u| y \left| \frac{\partial u}{\partial n_\varepsilon} \right| d\tau_\varepsilon \leq c \int_{\mathscr{B}_\varepsilon{}^1} |u|\, d\tau_\varepsilon \leq c \mathscr{I}_\varepsilon \left( \int_{\mathscr{B}_\varepsilon{}^1} d\tau_\varepsilon \right)^{1/2} \leq c_3 \mathscr{I}_\varepsilon.$$

Altogether this gives $\mathscr{I}_\varepsilon^2 \le c_3 \mathscr{I}_\varepsilon + c_2$, and hence $\mathscr{I}_\varepsilon$ is bounded in $\varepsilon$. We have therefore

$$\int_{\mathscr{B}_\varepsilon{}^2} u^2 \, d\tau_\varepsilon \le \text{constant}. \tag{13}$$

*2.4.1* We continue with the proof of the converse direction (2) $\Rightarrow$ (1). We seek to majorize the function $u$ by another, $v$, whose non-tangential behavior is known to us. We proceed as follows. The surface $\mathscr{B}_\varepsilon^1$ is a portion of the surface $y = \alpha^{-1}\delta_\varepsilon(x)$. Let $f_\varepsilon(x)$ be the function defined on $y = 0$ which is the projection on $y = 0$ of the restriction of $u$ to $\mathscr{B}_\varepsilon^1$, and zero otherwise. That is

$$f_\varepsilon(x) = u(x, \alpha^{-1}\, \delta_\varepsilon(x))$$

for those $(x, 0)$ lying below $\mathscr{B}_\varepsilon^1$, and $f_\varepsilon(x) = 0$ otherwise. Since clearly $d\tau_\varepsilon \ge dx$, we have that

$$\int_{\mathbf{R}^n} |f_\varepsilon(x)|^2 \, dx \le \int_{\mathscr{B}_\varepsilon{}^1} |u(x)|^2 \, d\tau_\varepsilon \le c.$$

Let now $v_\varepsilon(x, y)$ be the Poisson integral of $|f_\varepsilon|$. We shall show that for two appropriate constants $c_1$ and $c_2$

$$|u(x, y)| \le c_1 v_\varepsilon(x, y) + c_2, \quad \text{with} \quad (x, y) \in \mathscr{R}_\varepsilon \tag{14}$$

By the maximum principle for harmonic functions it suffices to show that the above inequality holds for points on the boundary $\mathscr{B}_\varepsilon$ of $\mathscr{R}_\varepsilon$. We have $\mathscr{B}_\varepsilon = \mathscr{B}_\varepsilon^1 \cup \mathscr{B}_\varepsilon^2$, where $\mathscr{B}_\varepsilon^2$ is a portion of the hyperplane $y = h - \varepsilon$ lying in a fixed sphere. Since $v_\varepsilon \ge 0$ we can satisfy (14) on $\mathscr{B}_\varepsilon^2$ by choosing $c_2$ large enough (and independent of $\varepsilon$).

It remains to consider points $(x, y) \in \mathscr{B}_\varepsilon^1$. Since $\mathscr{B}_\varepsilon^1 \subset \mathscr{R} = \bigcup_{x^0 \in E} \Gamma_\alpha^h(x^0)$ we can find a constant $c > 0$ so the ball $B$ of center $(x, y)$ and radius $cy$ lies entirely in $\bigcup_{x^0 \in E} \Gamma_{\alpha^*}^{h^*}(x^0)$, where $\alpha < \alpha^* < \beta$ and $h < h^* < k$. Recall that for the cones $\Gamma_\beta^k(x^0)$ we have the property

$$\iint_{\Gamma_\beta{}^k(x^0)} |\nabla u|^2 \, y^{1-n} \, dx \, dy \le 1.$$

Therefore by the lemma in §2.2.2 we have that $y\,|\nabla u| \le c$, for all points in the ball $B$. Now $|u(p_1) - u(p_2)| \le |p_1 - p_2| \sup_l |\nabla u|$ where $l$ is the line segment joining $(p_1, p_2)$; thus if $p_1 = (x, y)$, and $p_2$ is any other point of the ball $B$, $|p_1 - p_2| \le$ radius of $B = cy$ and hence

$$|u(p_1) - u(p_2)| \le c_2. \tag{15}$$

Next suppose $S_\varepsilon$ is that portion of the surface $\mathscr{B}_\varepsilon^1$ which is contained in the ball $B$. Write $|S_\varepsilon| = \int_{\mathscr{B}_\varepsilon{}^1 \cap B} d\tau_\varepsilon$. Then because of (15) we clearly have

$$|u(p_1)| \le \frac{1}{|S_\varepsilon|} \int_{S_\varepsilon} |u_\varepsilon(p_2)| \, d\tau_\varepsilon(p_2) + c_2$$

Because $d\tau_\varepsilon \ge dx$ and $B$ has radius $cy$ it is clear that $|S_\varepsilon| \ge ay^n$, where $a$ is an appropriate constant, $a > 0$. Recalling the definition of $f_\varepsilon(x)$ and the fact that $d\tau_\varepsilon \le c\, dx$, we then get

$$|u(p_1)| = |u(x, y)| \le by^{-n} \int_{|z-x| < cy} |f_\varepsilon(z)| \, dz + c_2.$$

The Poisson kernel $P_y$ has the property that

$$P_y(z) \ge c_1^{-1} b y^{-n}, \quad \text{for} \quad |z| < cy$$

(for an appropriate constant $c_1$).

If we substitute in the above, since $v_\varepsilon$ is the Poisson integral of $|f_\varepsilon|$ we get

$$|u(x, y)| \le c_1 v_\varepsilon(x, y) + c_2 \quad \text{for} \quad (x, y) \in \mathscr{R}_\varepsilon.$$

In view of the uniform boundedness in the $L^2$ norm of the $|f_\varepsilon|$ we can select a subsequence $|f_{\varepsilon'}|$ which converges weakly to an $f$ in $L^2(\mathbf{R}^n)$. Let $v$ denote the Poisson integral of $f$; thus $v_{\varepsilon'}(x, y)$ converges pointwise to $v(x, y)$, for $(x, y) \in \mathbf{R}_+^{n+1}$. Finally since $\mathscr{R}_\varepsilon \to \mathscr{R}$ we get from (14) that

$$(14') \qquad |u(x, y)| \le c_1 v(x, y) + c_2, \qquad (x, y) \in \mathscr{R}.$$

In that $v$ is the Poisson integral of an $L^2$ function, it is by Theorem 1 non-tangentially bounded at almost every point of $\mathbf{R}^n$ and hence almost everywhere at $E$. Because of (14′) the same is true for $u$, and therefore in view of Theorem 3, $u$ has non-tangential limits at almost every point of $E$.

**2.5 Application to non-tangential convergence.** Let $u_0, u_1, \ldots, u_n$ be a system of conjugate harmonic functions in the sense of Chapter III, §2.3. For simplicity of notation we write $y = x_0$. Then these $n + 1$ functions satisfy the equations

$$\sum_{j=0}^{n} \frac{\partial u_j}{\partial x_j} = 0,$$

and

$$\frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j}, \qquad 0 \le k, j \le n.$$

Our purpose now will be to prove the following theorem which is itself implied by the theorem for the area integral just proved.

THEOREM 5. *The following two statements are equivalent for any set* $E \subset \mathbf{R}^n$:

(a) $u_0$ *has a non-tangential limit at almost every point of* $E$.

(b) $u_1, u_2, \ldots, u_n$ *all have non-tangential limits at almost every point of* $E$.

The reader will have no difficulty in deducing the following consequence of Theorem 5.

COROLLARY. *Suppose* $H$ *is harmonic in* $\mathbf{R}^{n+1}_+$, *and* $\dfrac{\partial^k}{\partial y^k} H$ *has non-tangential limits at almost every point of a set* $E$, $E \subset \mathbf{R}^n$. *Then the same is true of* $P\left(\dfrac{\partial}{\partial x}\right) H$, *where* $P\left(\dfrac{\partial}{\partial x}\right)$ *is a homogeneous polynomial in*

$$\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n},$$

*of degree* $k$.

Because of the interpretation of conjugate harmonic functions in terms of Riesz transforms (as described in Chapter III, §2.2), Theorem 5 can be viewed in the following light. Namely, it is a local version, dealing with behavior almost everywhere in arbitrary sets for Riesz transforms; the global version, the one giving $L^p(\mathbf{R}^n)$ inequalities, was of course treated in Chapters II and III.

The case $n = 1$ is exceptional in the proof of Theorem 5, since then clearly $|\nabla u_0|^2 = |\nabla u_1|^2$, and thus $S(u_0)(x) = S(u_1)(x)$. In that case, therefore, the theorem is an immediate consequence of Theorem 4. However, in the general case the fact that $S(u_0)(x)$ dominates the $S(u_j)(x)$, $j = 1, \ldots, n$, requires an extra argument, which we shall now give.

*2.5.1 A lemma.*

LEMMA. *Let* $\Gamma^h_\alpha$ *and* $\Gamma^k_\beta$ *be a pair of truncated cones (for simplicity we assume that their vertices are at the origin). Suppose* $\alpha < \beta$, $h < k$, *and so* $\Gamma^h_\alpha$ *is strictly contained in* $\Gamma^k_\beta$. *Assume that*

$$\iint_{\Gamma^k_\beta} \left|\frac{\partial u}{\partial y}\right|^2 y^{1-n}\, dx\, dy < \infty.$$

*Then*

$$\iint_{\Gamma^h_\alpha} \left|\frac{\partial u}{\partial x_j}\right|^2 y^{1-n}\, dx\, dy < \infty, \qquad j = 1, \ldots, n.$$

We proceed as follows. We let $\rho$ denote any unit vector of direction determined by the smaller cone $\Gamma_\alpha^h$; that is $\rho$ is any unit vector so that $\rho s \in \Gamma_\alpha^h$ for some $s > 0$. Let $s_0$ denote the least upper bound of $s$ so that $\rho s \in \Gamma_\alpha^h$. Then clearly we always have $h \leq s_0 \leq h^*$ where $h^*$ is a constant which depends on the cone $\Gamma_\alpha^h$ only. For any function $U$ defined in $\Gamma_\alpha^h$ we denote its restriction to the ray determined by $\rho$ as $U_\rho$; more precisely $U_\rho(s) = U(\rho s)$. We shall show

$$\int_0^h s \left| \frac{\partial u_\rho(s)}{\partial x_j} \right|^2 ds \leq A < \infty$$

where the constant $A$ is independent of the direction $\rho$ in question. An integration over all relevant $\rho$ will then give our conclusion.

We write

$$\frac{\partial u}{\partial x_j} = -\int_y^h \frac{\partial^2 u}{\partial x_j \, \partial y} (x, \tau) \, d\tau + \frac{\partial u}{\partial x_j} (x, h) \tag{16}$$

The second term on the right side is harmless since it is uniformly bounded; (it represents values of $\dfrac{\partial u}{\partial x_j}$ taken at points strictly away from the boundary of the larger cone $\Gamma_\alpha^k$). We therefore need to deal with only the integral on the right side. For this we use the following estimate which is a consequence of the mean-value theorem (see Appendix C).

$$\left| \frac{\partial^2 u}{\partial x_j \, \partial y} (x, \tau) \right|^2 \leq \frac{c r^{-2}}{m(B)} \iint_B \left| \frac{\partial u}{\partial y} \right|^2 dx' \, dy'. \tag{17}$$

Here $B$ is a ball centered at the point $(x, \tau)$ whose radius is $r$. We now choose a constant $c_1$ with the property that if $(x, \tau) \in \Gamma_\alpha^h$ then the ball $B$ of radius $r = c_1 \tau$ lies entirely in the larger cone $\Gamma_\beta^k$. With this choice (17) becomes

$$\left| \frac{\partial^2 u}{\partial x_j \, \partial y} (x, \tau) \right| \leq c_2 \tau^{-(n+3)/2} \left( \iint_B \left| \frac{\partial u}{\partial y} \right|^2 dx' \, dy' \right)^{1/2} \tag{17'}$$

Now call $S_\tau$ the "layer" in the larger cone $\Gamma_\beta^k$ given by

$$S_\tau = \{(x, y) : |x| < \beta y, \ \tau - c_1 \tau < y < \tau + c_1 \tau\}.$$

Then clearly since $B \subset \Gamma_\beta^k$, we have $B \subset S_\tau$ and hence if we write

$$\mathscr{I}_\tau = \iint_{S_\tau} \left| \frac{\partial u}{\partial y} \right|^2 dx' \, dy'$$

the inequality (17′) becomes

$$\left| \frac{\partial^2 u}{\partial x_j \, \partial y} (x, \tau) \right| \leq c_2 \tau^{-(n+3)/2} \mathscr{I}_\tau^{1/2}. \tag{17''}$$

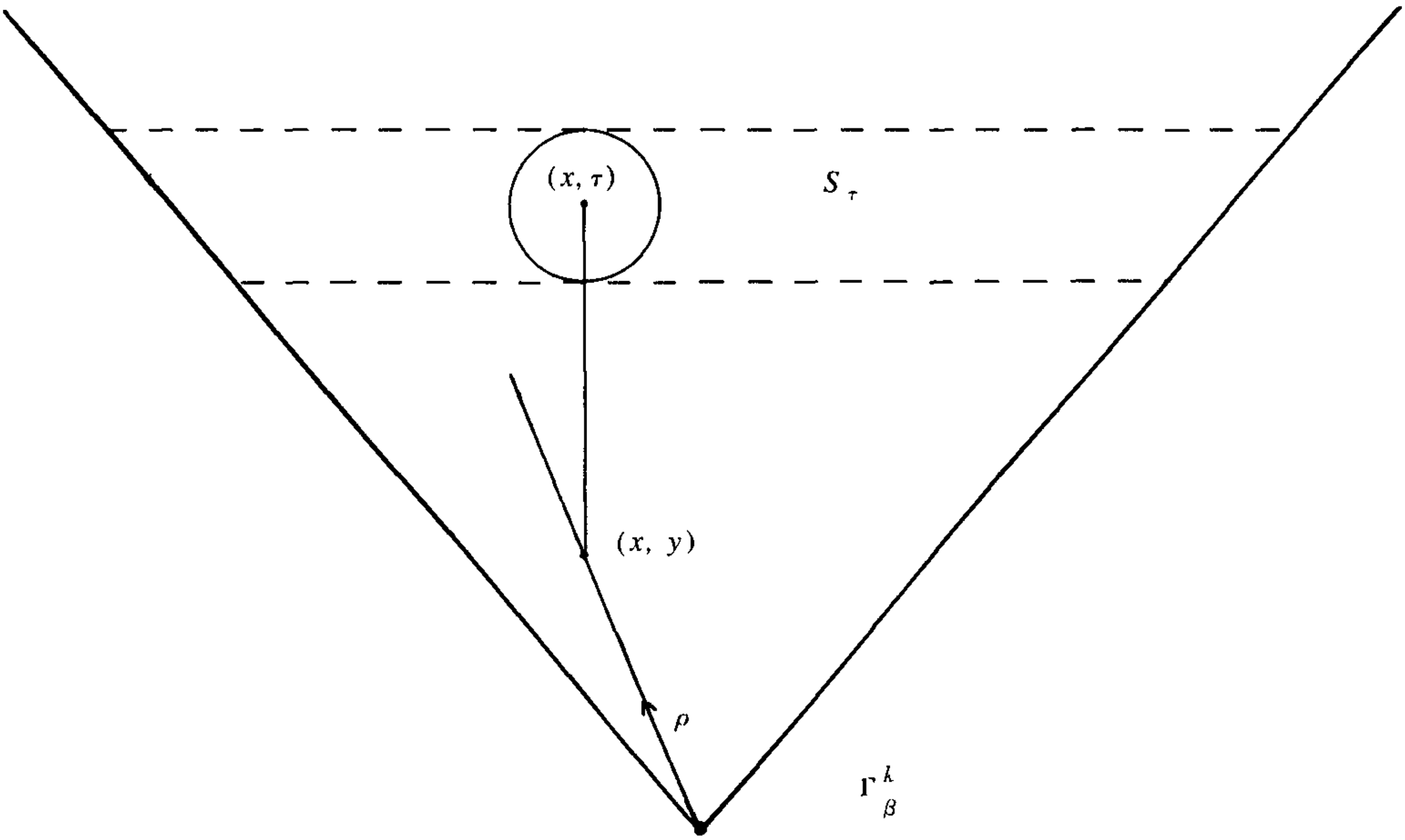



*Figure 5.* The situation in the proof of lemma §2.5.1.

We call $\theta$ the angle that $\rho$ makes with the $y$-axis. Since this direction is contained in the cone $|x| < \alpha y$, $y > 0$, it follows that $\cos\theta \geq a_0 = (1 + \alpha^2)^{-1/2} > 0$. With this observation we can insert (17″) in (16). The result is

$$\left| \frac{\partial u_\rho(s)}{\partial x_j} \right| \leq c_3 \int_{sa_0}^{h} \tau^{-(n+3)/2} \mathscr{I}_\tau^{1/2}\, d\tau + c_4.$$

Now use the Hardy inequality (see Appendix A) and we get

$$\int_0^h s \left| \frac{\partial u_\rho(s)}{\partial x_j} \right|^2 ds \leq c_5 \int_0^h \tau^{-n} \mathscr{I}_\tau\, d\tau + c'.$$

However,

$$\int_0^h \tau^{-n} \mathscr{I}_\tau\, d\tau = \int_0^h \tau^{-n} \left\{ \int\int_{S_\tau} \left| \frac{\partial u}{\partial y} \right|^2 dx'\, dy' \right\} d\tau$$

$$\leq \iint_{\Gamma_\beta^k} \left| \frac{\partial u}{\partial y} \right|^2 \left\{ \int \tau^{-n} \chi(\tau, x', y')\, d\tau \right\} dx'\, dy'.$$

where $\chi$ is the characteristic function of the layer $S_\tau$ in $\Gamma_\beta^k$. But

$$\int \tau^{-n} \chi(\tau, x', y)\, d\tau \leq \int_{\tau - c_1\tau < y < \tau + c_1\tau} \tau^{-n}\, d\tau = \int_{y/(1+c_1)}^{y/(1-c_1)} \tau^{-n}\, d\tau = c' y^{-n+1}.$$

Inserting this in the above gives

$$\int_0^h s \left| \frac{\partial u_\rho(s)}{\partial x_j} \right|^2 ds \le c \iint_{\Gamma_\beta^k} \left| \frac{\partial u}{\partial y} \right|^2 y^{1-n} \, dx \, dy + c$$

and a final integration over $\rho$ concludes the proof of the lemma.

*2.5.2* For further reference we state two closely related inequalities. In this connection we let $\Gamma_\alpha$ denote the infinite cone $\Gamma_\alpha = \{(x, y) : |x| < \alpha y\}$. We consider also $\Gamma_\beta$, with $\beta > \alpha$. For each positive $k$ we denote by $|\nabla^k u|^2$ the square of the $k^{\text{th}}$ gradient of $u$, namely that positive definite quadratic expression in the partial derivatives of order $k$ which is recursively defined by

$$|\nabla^k u|^2 = \sum_{j=0}^{n} \left| \nabla^{k-1} \frac{\partial u}{\partial x_j} \right|^2, \qquad (x_0 = y).$$

LEMMA 1. *Suppose $u$ is harmonic in $\Gamma_\beta$, then for each $k \ge 1$*

$$\iint_{\Gamma_\alpha} |\nabla^k u|^2 \, y^{2k-n-1} dx \, dy \le c_{\alpha,\beta,k} \iint_{\Gamma_\beta} |\nabla u|^2 \, y^{1-n} \, dx \, dy.$$

LEMMA 2. *Suppose $u$ is harmonic in $\Gamma_\beta$, and $|\nabla u| \to 0$ as $y \to \infty$, for $(x, y) \in \Gamma_\beta$. Then for each $k \ge 1$*

$$\iint_{\Gamma_\beta} |\nabla u|^2 \, y^{1-n} \, dx \, dy \le c_{\beta,k} \iint_{\Gamma_\beta} |\nabla^k u|^2 \, y^{2k-n-1} \, dx \, dy.$$

To prove Lemma 1 we use the inequality

$$|\nabla^k u(x, \tau)|^2 \le c_k \frac{r^{-2(k-1)}}{m(B)} \iint_B |\nabla u|^2 \, dx' \, dy' \tag{18}$$

where $B$ is a ball of radius $r = c_1 \tau$ centered at $(x, \tau)$; the rest is then as in §2.5.1.

To prove Lemma 2 we observe that $|\nabla u| \to 0$ implies by (18) that $|\nabla^k u| \to 0$ as $y \to \infty$, for any proper sub-cone of $\Gamma_\beta$. Now if $k > 1$, then

$$\frac{\partial u}{\partial x_j}(x, y) = \frac{-1}{(k-2)!} \int_y^\infty \frac{\partial^k}{\partial y^k \, \partial x_j} u(x, \tau)(y - \tau)^{k-2} \, d\tau \tag{19}$$

By the inequalities for integral operators with homogeneous kernels (see Appendix A) we get from (19) that

$$\int_{|x|/\beta}^\infty \left| \frac{\partial u}{\partial x_j}(x, y) \right|^2 y^{1-n} \, dy \le c_k \int_{|x|/\beta}^\infty \left| \frac{\partial^k u}{\partial y^{k-1} \, \partial x_j}(x, y) \right|^2 y^{2k-n-1} \, dy$$

and a final integration in $x$ gives the desired result, for $\dfrac{\partial u}{\partial x_j}$. A similar argument works for $\dfrac{\partial u}{\partial y}$.

The lemma in §2.5.1 as well as those in §2.5.2 tell, among other things, that as far as the basic properties of the area integral are concerned we could have used higher derivatives as well as first derivatives. However, it is important to note that we could not have used the zero derivative. In fact, observe that the analogue for the area integral for $k = 0$ is infinite even when $u$ is a constant!

### 3. *Application of the theory of $H^p$ spaces*

We intend to study further the notion of conjugacy given by the generalized Cauchy-Riemann equations

$$(18) \qquad \sum_{j=0}^{n} \frac{\partial u_j}{\partial x_j} = 0, \qquad \text{and} \qquad \frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j}, \qquad x_0 = y.$$

One of the tools will be a certain result of subharmonicity which is subsumed in the following lemma.

**3.1 A subharmonic property of the gradient.** We write $F = (u_0, u_1, \ldots, u_n)$, where the $u_j$ satisfy the Cauchy-Riemann equations (18) in some open set. (Observe that this implies that in some neighborhood of each point of this set the vector $F$ is the gradient of a harmonic function $H$ in that neighborhood.) For certain technical reasons it will be convenient to assume that the $u_j$ take their values in a fixed finite-dimensional inner product space (i.e., Hilbert space) over the real numbers. We write $|\cdot|$ for the norm in that space and $u_i \cdot u_i'$ for the inner product of the $u_i$ with $u_i'$. We write also

$$|F| = \left(\sum_{j=0}^{n} |u_j|^2\right)^{1/2}, \qquad \text{and} \qquad \Delta = \sum_{j=0}^{n} \frac{\partial^2}{\partial x_j}.$$

LEMMA. *Suppose $|F| > 0$ at a point. If $q \geq \dfrac{n-1}{n}$, then $\Delta(|F|^q) \geq 0$ there. More precisely, if $q > \dfrac{n-1}{n}$, then*

$$(19) \qquad C_q\,|F|^{q-2}\,|\nabla F|^2 \geq \Delta(|F|^q) \geq c_q\,|F|^{q-2}\,|\nabla F|^2$$

*for two positive constants $C_q$ and $c_q$.**

* The proof will show that we may take $C_q = q(q-1)$, $c_q = q$, if $q \geq 2$ and $C_q = q$, $c_q = q\left[1 + (q-2)\left(\dfrac{n}{n+1}\right)\right]$ if $q \leq 2$. The condition $q \geq \dfrac{n-1}{n}$ is exactly the condition $c_q \geq 0$. Observe also that $C_2 = c_2 = 2$.

To understand the thrust of this lemma we remark that the assertion that $\Delta |F|^q \geq 0$, if $q \geq 1$, is a consequence of the fact that each $u_j$ is harmonic, and does not otherwise depend on the differential relations (18) for the $u_j$. The particular usefulness of this lemma is that it holds for some $q < 1$. The actual range of $q$ given, $q \geq \frac{n-1}{n}$ is best possible as simple examples show.

The inequality (19) should also be compared to the following two simpler identities. First

$$\Delta u^q = q(q-1)u^{q-2}|\nabla u|^2, \tag{20}$$

whenever $u$ is harmonic and $u > 0$. We have already used this identity in Chapter IV, §2.

The second identity is

$$\Delta |F|^q = \frac{q^2}{2}|F|^{q-2}|\nabla F|^2 = q^2|F|^{q-2}|F'|^2, \tag{21}$$

in the case when $n = 1$ and $F = u_0 - iu_1$ is a holomorphic function of $x_0 + ix_1$.

*3.1.3* To prove the lemma we use the notation $F \cdot G = u_0 \cdot v_0 + u_1 \cdot v_1 + \cdots + u_n \cdot v_n$, where $F = (u_0, u_1, \ldots, u_n)$, $G = (v_0, v_1, \ldots, v_n)$; also $F_{x_j} = \frac{\partial F}{\partial x_j}$; then $|\nabla F|^2 = \sum_{j=0}^{n} |F_{x_j}|^2$. Now $\frac{\partial}{\partial x_j}(F \cdot G) = F_{x_j} \cdot G + F \cdot G_{x_j}$; therefore

$$\frac{\partial}{\partial x_j}|F|^q = \frac{\partial}{\partial x_j}(F \cdot F)^{q/2} = q|F|^{q-2}(F_{x_j} \cdot F).$$

Hence,

$$\frac{\partial^2}{\partial x_j^2}|F|^q = q(q-2)|F|^{q-4}(F_{x_j} \cdot F)^2 + q|F|^{q-2}\{|F_{x_j}|^2 + F_{x_j x_j} \cdot F\}.$$

Adding with respect to $j$ and taking into account the fact that $F_{x_0 x_0} + F_{x_1 x_1} \cdots + F_{x_n x_n} = 0$ we obtain

$$\Delta |F|^q = q|F|^{q-4}\left\{(q-2)\sum_j (F_{x_j} \cdot F)^2 + |F|^2 \sum_j |F_{x_j}|^2\right\}.$$

It is therefore only a matter of comparing

$$(q-2)\sum_j (F_{x_j} \cdot F)^2 + |F|^2 \sum_j |F_{x_j}|^2 \tag{22}$$

with $|F|^2 |\nabla F|^2 = |F|^2 \sum_j |F_{x_j}|^2$.

By Schwarz's inequality $\sum_j (F_{x_j} \cdot F)^2 \leq |F|^2 \sum_j |F_{x_j}|^2$ and so (22) is certainly dominated by $|F|^2 |\nabla F|^2$, if $q \leq 2$, and $(q - 1) |F|^2 |F|^2$ if $q \geq 2$. This shows that $\Delta |F|^q \leq C_q |F|^{q-2} |\nabla F|^2$. Similarly the bound of $\Delta |F|^q$ from below is simple when $q \geq 2$, and does not use the full thrust of the Cauchy-Riemann equations (18). We now assume that $q < 2$. Then the heart of the matter is contained in the following improvement over Schwarz's inequality

$$(23) \qquad \sum_j (F_{x_j} \cdot F)^2 \leq \left(\frac{n}{n+1}\right) |F|^2 \sum_j |F_{x_j}|^2$$

which we shall prove momentarily. Assuming (23) we see that (22) is bounded from below by

$$|F|^2 \sum_j |F_{x_j}|^2 \left\{1 + (q-2)\left(\frac{n}{n+1}\right)\right\}.$$

Inserting this in the above gives $\Delta |F|^q \geq c_q |F|^{q-2} |\nabla F|^2$, with $c_q = q\left\{1 + (q-2)\left(\frac{n}{n+1}\right)\right\}$, if $q \leq 2$. The lemma will then be proved once we have established the inequality (23), which makes full use of the generalized Cauchy-Riemann equations.

*3.1.2* We observe first that it suffices to prove (23) in the special case when the functions $u_0, u_1, \ldots, u_n$ are real-valued, that is when the Hilbert spaces in question are one-dimensional. The passage to the general case considered is then effected by introducing any orthonormal basis.

Let now $\mathcal{M} = (m_{jk})$ be an $(n+1) \times (n+1)$ matrix with (real) entries $m_{jk}$. We will consider two norms on such matrices. First the ordinary norm $\|\mathcal{M}\| = \sup |\mathcal{M}(F)|$, taken over all vectors $F$ of length 1. Also the Hilbert-Schmidt norm, $\|\|\mathcal{M}\|\|$, given by $\|\|\mathcal{M}\|\|^2 = \sum_{j,k} |m_{jk}|^2$. By Schwarz's inequality it is clear that $\|\mathcal{M}\| \leq \|\|\mathcal{M}\|\|$. Suppose now that we assume that the matrix $\mathcal{M}$ is symmetric and has trace zero. Then we can strengthen the above inequality to read

$$(23') \qquad \|\mathcal{M}\|^2 \leq \left(\frac{n}{n+1}\right) \|\|\mathcal{M}\|\|^2.$$

To prove (23′) we observe that both norms are orthogonal invariants and after a proper choice of basis in the $n + 1$ dimensional Euclidean space we can assume that the symmetric matrix $\mathcal{M}$ is diagonal, with diagonal entries $\lambda_0, \lambda_1, \ldots, \lambda_n$. Since the trace of $\mathcal{M}$ vanishes we get $\sum_{j=0}^{n} \lambda_j = 0$.

Therefore $\lambda_{j_0} = -\sum_{j \neq j_0} \lambda_j$; so by Schwarz's inequality $\lambda_{j_0}^2 \leq n \sum_{j \neq j_0} \lambda_j^2$, and consequently

$$\sup_j \lambda_j^2 = \lambda_{j_0}^2 \leq \left(\frac{n}{n+1}\right) \sum_j \lambda_j^2.$$

This is (23′) in case of a diagonal matrix, and thus by what has been said already (23′) is proved for any symmetric matrix of trace zero. With $F = (u_0, \ldots, u_n)$, let $m_{jk} = \dfrac{\partial u_j}{\partial x_j}$. Then the generalized Cauchy-Riemann equations (18) state exactly that the matrix $\mathcal{M} = (m_{jk})$ is symmetric and has trace zero. It is then easy to see that (23′) implies (23) and the lemma is completely proved.

**3.2 $H^p$ spaces; in particular $H^1$.** In analogy with the classical theory we define the $n$-dimensional form of the Hardy space $H^p$ as follows. Suppose $F = (u_0, u_1, \ldots, u_n)$ satisfies the generalized Cauchy-Riemann equations (18) in $\mathbf{R}_+^{n+1}$. For $p > 0$ we say that $F \in H^p$ if

$$\sup_{y>0} \left(\int_{\mathbf{R}^n} |F(x, y)|^p \, dx\right)^{1/p} < \infty \tag{24}$$

We shall write $\|F\|_p$ for the quantity appearing above. It is a norm when $p \geq 1$.

Because of the lemma in §3.1 it will be shown that some of the classical theory ($n = 1$) of $H^p$ spaces extends to $n$-dimensions when $p > \dfrac{n-1}{n}$, and what is most interesting for our purposes, is that this contains the case of $p = 1$. To understand better the implications of the latter point we consider first the $H^p$ spaces when $1 < p < \infty$.

Suppose therefore that $F \in H^p$. Then by the corollary in §1.2.1 there exist $f_0, f_1, f_2, \ldots, f_n$, each in $L^p(\mathbf{R}^n)$, so that $u_j(x, y)$ is the Poisson integral of $f_j$, $j = 0, \ldots, n$. Also by §4.4 of Chapter III $f_j = R_j(f_0)$, where $R_1, R_2, \ldots, R_n$ are the Riesz transforms. Conversely, suppose $f_0 \in L^p(\mathbf{R}^n)$, and let $f_j = R_j(f_0)$, and $u_j(x, y)$ be the Poisson integrals of $f_j$, $j = 0, \ldots, n$. Then $F = (u_0, u_1, \ldots, u_n) \in H^p$; moreover $\|f_0\|_p \leq \|F\|_p \leq A_p \|f_0\|_p$.

To summarize: for $1 < p < \infty$ the space $H^p$ is naturally equivalent with the space $L^p(\mathbf{R}^n)$.

For $p = 1$ this identity no longer holds and we may view the space $H^1$ as a substitute for $L^1(\mathbf{R}^n)$. Our purpose will be to show that in this context various results which break down for $L^1$ have correct versions for $H^1$. In this way the $H^1$ theorems may be thought to be in some sense analogous and complementary to the results involving weak-type (1, 1) which occur in Chapters I and II. Our first theorem of this kind deals in effect with the maximal function.

THEOREM 6. *Let $F \in H^1$. Then $\lim_{y\to 0} F(x, y) = F(x)$ exists almost everywhere and in $L^1(\mathbf{R}^n)$ norm. Also*

$$\int_{\mathbf{R}^n} \sup_{y>0} |F(x, y)|\, dx \leq A \sup_{y>0} \int_{\mathbf{R}^n} |F(x, y)|\, dx = \|F\|_1.$$

Before we come to the proof of this theorem we shall formulate several corollaries which will indicate more clearly what is actually involved.

Let $d\mu_0$ be a finite measure on $\mathbf{R}^n$. We shall say that the Riesz transform $R_j(d\mu_0)$ is also a measure, say $d\mu_j$, if the identity

$$\hat{\mu}_j(x) = i\frac{x_j}{|x|}\hat{\mu}_0(x) \tag{25}$$

holds, where $\hat{\mu}_0$ and $\hat{\mu}_j$ denote respectively the Fourier transforms of the measure $d\mu_0$ and $d\mu_j$. This is of course consistent with the usual definition in view of identity (8) in Chapter III, (see p. 58).

As a special case of this definition we can assume that $d\mu_0 = f_0\, dx$ where $f_0 \in L^1(\mathbf{R}^n)$. We say then analogously that $R_j(f_0) \in L^1(\mathbf{R}^n)$ if there exists an $f_j \in L^1(\mathbf{R}^n)$ so that $\hat{f}_j(x) = i\dfrac{x_j}{|x|}\hat{f}_0(x)$.

COROLLARY 1. *Suppose $d\mu_0$ is a finite measure and all its Riesz transforms $R_j(d\mu)$ are also finite measures, $R_j(d\mu_0) = d\mu_j$, $j = 1, \ldots, n$. Then there exist $L^1(\mathbf{R}^n)$ functions $f_0, f_1, \ldots, f_n$, so that $d\mu_j = f_j\, dx$, $j = 0, \ldots, n$.*

*The space $H^1$ is naturally isomorphic with the space of $L^1(\mathbf{R}^n)$ functions $f_0$ which have the property that $R_j(f_0) \in L^1(\mathbf{R}^n)$, $j = 1, \ldots, n$. The $H^1$ norm is then equivalent with $\|f_0\|_1 + \sum_{j=1}^{n} \|R_j(f_0)\|_1$.*

This space of $f_0$ consists in effect of the "real parts" of the boundary values of $F$ in $H^1$. It is tempting to refer to this Banach space as $H^1$ also, whenever this does not cause confusion with the parent space of $F$ initially defined.

COROLLARY 2. *Suppose $f_0 \in L^1(\mathbf{R}^n)$ and $R_j(f_0) \in L^1(\mathbf{R}^n)$, $R_j(f_0) = f_j$, $j = 1, \ldots, n$. Then*

$$\sum_{j=0}^{n} \int_{\mathbf{R}^n} \sup_{y>0} |u_j(x, y)|\, dx \leq A \sum_{j=0}^{n} \|f_j\|_1.$$

*3.2.1* We prove Theorem 6 and its corollaries. To begin with let us suppose, as we did earlier, that if $F = (u_0, \ldots, u_n)$, each of the $u_j$'s take their values in a fixed finite-dimensional Hilbert space; we call this inner-product space $V_1$. We shall also need another finite dimensional Hilbert space $V_2$ and we consider $V = V_1 \oplus V_2$, their direct sum; $V_1$ and $V_2$ are orthogonal complements in $V$.

We next find a fixed function in $\mathbf{R}^{n+1}_+$, $\Phi(x, y) = (v_0(x, y), v_1(x, y), \ldots, v_n(x, y))$, so that the $v_j$'s take their values in $V_2$ and

(i) The $v_j$ satisfy the Cauchy-Riemann equations (18).

(ii) $|\Phi(x, y)| = c\,|(x, y + 1)|^{-n-1} = c(|x|^2 + (y + 1)^2)^{-(n+1)/2}$.

To do this let $V_2$ denote the standard coordinate space of $(n + 1)$ dimensions; let $H(x, y)$ be the harmonic function $|(x, y + 1)|^{-n+1} = (|x|^2 + (y + 1)^2)^{-(n-1)/2}$.* For every $j$, $0 \leq j \leq n$ set

$$v_j(x, y) = \left(\frac{\partial^2 H}{\partial x_j\, \partial x_k}\right)_{k=0}^{n}.$$

Then

$$|\Phi(x, y)|^2 = \sum_{j=0}^{n} \sum_{k=0}^{n} \left|\frac{\partial^2 H}{\partial x_j\, \partial x_k}\right|^2.$$

It is therefore easy to verify (i) and (ii) with $c^2 = (n^2 - 1)(n^2 - n)$.

Next, for every $\varepsilon > 0$ we define $F_\varepsilon$ by

$$F_\varepsilon(x, y) = F(x, y + \varepsilon) + \varepsilon\Phi(x, y). \tag{26}$$

If we write $F_\varepsilon(x, y) = (u_0^\varepsilon(x, y), \ldots, u_n^\varepsilon(x, y))$, then

$$u_j^\varepsilon(x, y) = u_j(x, y + \varepsilon) + \varepsilon v_j(x, y),$$

and so the $u_j^\varepsilon$ take their values in $V_1 \oplus V_2 = V$. Observe also that the components $u_j^\varepsilon$ of $F_\varepsilon$ satisfy the Cauchy-Riemann equations, and that $F_\varepsilon$ is continuous in $\overline{\mathbf{R}}^{n+1}_+$.

Now because of our assumptions on $F$, and in view of the corollary in §1.2.1 of this chapter, we can assert that each $u_j(x, y)$ is the Poisson integral of a finite measure. Thus it is easy to see that for every fixed $\varepsilon$ each $u_j(x, y + \varepsilon)$ tends to zero as $|(x, y)| \to \infty$, for $(x, y) \in \overline{\mathbf{R}}^{n+1}_+$. The same is true for the components of $\Phi$ (by property (ii)) and so $|F_\varepsilon(x, y)| \to 0$ as $|(x, y)| \to \infty$ in $\overline{\mathbf{R}}^{n+1}_+$. In addition to this we have $|F_\varepsilon(x, y)|^2 = |F(x, y + \varepsilon)|^2 + \varepsilon^2\,|\Phi(x, y)|^2 > 0$, because of the orthogonality of $V_1$ and $V_2$ in $V$. Thus $|F_\varepsilon|^q$ is smooth, and we can apply the lemma in §3.1 with $q = \dfrac{n-1}{n}$, obtaining $\Delta(|F_\varepsilon|^q) \geq 0$ everywhere. Set

$$g_\varepsilon(x) = |F_\varepsilon(x, 0)|^q = (|F(x, \varepsilon)|^2 + \varepsilon^2\,|\Phi(x, 0)|^2)^{q/2}.$$

* We assume here and in the rest of this chapter that $n > 1$. The argument for $n = 1$ needs certain slight modifications.

Then

$$\int_{\mathbf{R}^n} (g_\varepsilon(x))^p \, dx = \int_{\mathbf{R}^n} |F_\varepsilon(x, 0)| \, dx \leq \|F\|_1 + \varepsilon \|\Phi\|_1 \tag{27}$$

where $p = 1/q$, and it is important that as a result $p > 1$. Let $g_\varepsilon(x, y)$ be the Poisson integral of $g_\varepsilon$. We claim that

$$|F_\varepsilon(x, y)|^q \leq g_\varepsilon(x, y), (x, y) \in \overline{\mathbf{R}}^{n+1}_+. \tag{28}$$

To verify (28) observe that both $F_\varepsilon$ and $g_\varepsilon$ are continuous in $\overline{\mathbf{R}}^{n+1}_+$ and $F_\varepsilon$ vanishes at infinity. We have $\Delta(|F_\varepsilon|^q - g_\varepsilon) \geq 0$, and therefore in view of the maximum principle of Appendix C it suffices to verify (27) on the boundary, $y = 0$, for which the equality sign holds. Therefore (28) is proved.

We now select a subsequence of the family $\{g_\varepsilon(x)\}$, $\varepsilon \to 0$, which converges weakly to a function $g$ in $L^p(\mathbf{R}^n)$. Because of (27) we have $\|g\|_p^p \leq \|F\|_1$. If $g(x, y)$ denotes the Poisson integral of $g$ then (28) leads to

$$|F(x, y)|^q \leq g(x, y) \tag{28$'$}$$

However, by the maximal theorem for Poisson integrals (see Chapter III, p. 62) we have $\sup_{y>0} |F(x, y)|^q \leq \sup_{y>0} g(x, y) \leq (Mg)(x)$, and therefore

$$\int_{\mathbf{R}^n} \sup_{y>0} |F(x, y)| \, dx \leq \int_{\mathbf{R}^n} (Mg(x))^p \, dx \leq A_p^p \int_{\mathbf{R}^n} (g(x))^p \, dx \leq A_p^p \|F\|_1.$$

This proves the main conclusion of Theorem 6. That $\lim_{y\to 0} F(x, y)$ exists almost everywhere follows because the Poisson integral of a finite measure has this property (see §4.1 of Chapter III). The almost everywhere convergence can also be proved by appealing to (28$'$) which shows that the $F(x, y)$ is almost everywhere non-tangentially bounded, and then by using Theorem 3 of this chapter.

Finally the convergence in the $L^1$ norm is a consequence of the almost everywhere convergence and the maximal inequality just proved which shows that $|F(x, y)|$ is majorized by a fixed integrable function.

To prove Corollary 1 we merely notice that if $u_j(x, y)$ are the Poisson integrals of $d\mu_j$ and $d\mu_j = R_j(d\mu_0)$, then $F = (u_0, \ldots, u_n)$ satisfies the Cauchy-Riemann equations. Also $\sup_{y>0} \|u_j(x, y)\|_1 = \|d\mu_j\| < \infty$, and so $F \in H^1$. Let $f_j(x) = \lim_{y\to 0} u_j(x, y)$, where the existence of this limit (in the $L^1$ norm) is guaranteed by the theorem. If $\hat{\mu}_j$ and $\hat{f}_j$ are the Fourier transforms of $d\mu_j$ and $f_j$ respectively, we have $(\mu(x, y))\hat{} = \hat{\mu}_j(x)e^{-2\pi|x|y}$. Thus $\hat{\mu}_j(x)e^{-2\pi|x|y} \to \hat{f}_j(x)$, as $y \to 0$; therefore $\hat{\mu}_j(x) = \hat{f}_j(x)$ and $d\mu_j = f_j \, dx$.

Corollary 2 follows from the above once we notice that if $d\mu_j = f_j(x)\,dx$, then $\|d\mu_j\| = \|f_j\|_1$.

**3.3 The area integral and $H^1$.** If $f \in L^p(\mathbf{R}^n)$, we have studied in Chapter IV, and also in part in the present chapter, three related auxiliary expressions, namely

$$g(f)(x), \qquad S(f)(x), \qquad \text{and} \qquad g_\lambda^*(f)(x).$$

These were defined as follows. If $u(x, y)$ is the Poisson integral of $f$, then

$$g(f)(x) = \left(\int_0^\infty y\,|\nabla u(x, y)|^2\,dy\right)^{1/2}$$

and

$$g_\lambda^*(f)(x) = \left(\int_0^\infty \int_{\mathbf{R}^n} |\nabla u(x - t, y)|^2 \left(\frac{y}{|t| + y}\right)^{n\lambda} y^{1-n}\,dt\,dy\right)^{1/2}$$

To define the area integral $S$ we need to fix the cone entering in its definition. In what follows it will be convenient to fix this cone as the usual right circular cone. Thus we take $S$ to be given by

$$S(f)(x) = \left(\iint_{|t| < y} |\nabla u(x - t, y)|^2\,y^{1-n}\,dt\,dy\right)^{1/2}.$$

We observed in Chapter IV that $g(f)(x) \leq cS(f)(x) \leq c_\lambda g_\lambda^*(f)(x)$, and proved in Theorems 1 and 2 that

$$B_p\,\|f\|_p \leq \|g(f)\|_p \leq c_\lambda\,\|g_\lambda^*(f)\|_p \leq A_{p,\lambda}, \|f\|_p$$

if $1 < p < \infty$, when $\lambda \geq 2$.

Our purpose will be to formulate the extension of these facts for $p = 1$ in the context of the $H^1$ spaces, and then to apply this in §3.4 below.

Let $F \in H^p$, $1 \leq p$, and define $S(F)(x)$ and $g_\lambda^*(F)(x)$ in analogy to the above, namely

$$S(F)(x) = \left(\iint_{|t| \leq y} |\nabla F(x - t, y)|^2\,y^{1-n}\,dt\,dy\right)^{1/2}$$

$$g_\lambda^*(F)(x) = \left(\iint |\nabla F(x - t, y)|^2 \left(\frac{y}{|t| + y}\right)^{n\lambda} y^{1-n}\,dt\,dy\right)^{1/2}.$$

Again, of course, $S(F)(x) \leq c_\lambda g_\lambda^*(F)(x)$. Our results are as follows:

THEOREM 7. *If $F \in H^p$, $1 \leq p < \infty$, then $\|g_\lambda^*(F)\|_p \leq A_{p,\lambda}\|F\|_p$, as long as $p > \dfrac{2}{\lambda}$. In particular $\|S(F)\|_p \leq A_p\,\|F\|_p$.*

THEOREM 8. *Suppose $F \in H^p$, $1 \leq p < \infty$. Then $B_p\,\|F\|_p \leq \|S(F)\|_p$.*

It can be shown that results of this kind are also valid for $p > \dfrac{n-1}{n}$, (see §4.9 below), but in its present formulation the theorem and proof are already fairly typical of the general case. The interest in these theorems is for us the case $p = 1$, since the case in that part of the theorem when $p > 1$ is contained in the results of Chapter IV. In the arguments that follow we therefore limit ourselves to $p = 1$.

*3.3.1 Proof of Theorem 7.* For the proof of this and the succeeding theorems it will be very convenient to restrict our attention to an appropriate sub-class of $H^p$ which we now define.

Let $H_0^p$ consist of all $F$ such that

(i) $F \in H^p$

(ii) $F$ is continuous in $\overline{\mathbf{R}}_+^{n+1}$; it is also rapidly decreasing, that is $pF$ is bounded in $\overline{\mathbf{R}}_+^{n+1}$ for every polynomial $p$ in the $x_1, \ldots, x_n$ and $y$

(iii) Property (ii) holds also for every order partial derivative of $F$ in the $x_1, \ldots, x_n$ and $y$.

What will be important for our purposes is the following fact.

LEMMA. *$H_0^p$ is dense in $H_p$, $1 \leq p < \infty$.*

When $1 < p < \infty$ the proof of the lemma can be given by a rather straightforward limiting argument since the Riesz transforms $R_j$ are continuous in $L^p$. The lack of continuity for $p = 1$ complicates the situation for that basic special case, and it is just this case which concerns us here. However, to come directly to the presentation of both Theorems 7 and 8 we shall postpone the proof of the lemma to §3.3.3 below.

Assuming the truth of this lemma, we are therefore faced with the task of showing that

$$\|g_\lambda^*(F)\|_1 \leq A_\lambda \|F\|_1,\ \lambda > 2,\ F \in H_0^1. \tag{29}$$

It will be useful at this stage to make a further reduction of our problem by replacing $F$ by $F + \varepsilon\Phi$, in analogy to the proof of §3.2.1. It is to be recalled that the components of the $\Phi$ are orthogonal to those of $F$, that they also satisfy the Cauchy-Riemann equations, and that $|\Phi(x, y)| = c\,|(x, y + 1)|^{-n-1}$. (For later purposes it is to be observed that $|\nabla\Phi| = c'\,|(x, y + 1)|^{-n-2}$.)

The main point of introducing the perturbation $\varepsilon\Phi$ is of course to eliminate the zeroes of $F$; in fact we have $|F + \varepsilon\Phi|^2 = |F|^2 + \varepsilon^2\,|\Phi|^2 > 0$. Our objective will therefore be (29) with $F$ replaced by $F + \varepsilon\Phi$.

Our argument will be a modification of the presentation for $p > 1$ given in §2 of Chapter IV, and in particular in §2.5.2 of that chapter. Let us draw the parallel in detail.

Lemma 1 (p. 86) will be replaced by the inequality

$$C_1 |F|^{-1} |\nabla F|^2 \geq \Delta(|F|) \geq c_1 |F|^{-1} |\nabla F|^2 \tag{30}$$

which is merely a special case of the lemma in §3.1 of the present chapter.

Lemma 2 (p. 87) will be restated with a slight modification as follows. Suppose $G$ is continuous in $\overline{\mathbf{R}}^{n+1}_{+}$, $G$ is of class $C^2$ in $\mathbf{R}^{n+1}_{+}$, $y\,\Delta G \in L^1(\mathbf{R}^{n+1}_{+}, dy\,dx)$, $|G(x, y)| \leq |(x, y)|^{-n-\varepsilon}$, and

$$|\nabla G(x, y)| \leq A\,|(x, y)|^{-n-1-\varepsilon},$$

with $\varepsilon > 0$. Then

$$\iint_{\mathbf{R}^{n+1}_{+}} y\,\Delta G\,dx\,dy = \int_{\mathbf{R}^n} G(x, 0)\,dx. \tag{31}$$

Finally for the maximal Lemma 3, and its variant inequality (24), (see p. 92), we shall have

*For any fixed $\mu$, $\mu$ sufficiently close to 1, $\mu < 1$,*

$$|F(x - t, y)| \leq (1 + |t|/y)^{n/\mu} F^*_\mu(x), \quad \textit{with} \int_{\mathbf{R}^n} F^*_\mu(x)\,dx \leq A_\mu \,\|F\|_1. \tag{32}$$

In fact by the inequality (24) of Chapter IV we have

$$|g(x - t, y)| \leq A\left(1 + \frac{|t|}{y}\right)^n M(g)(x),$$

where $g(x, y)$ is the Poisson integral of an arbitrary function $g(x)$. We now invoke the majorization (28′), namely $|F(x, y)|^q \leq g(x, y)$ with $g(x) = |F(x, 0)|^q$ where we choose $q = \mu < 1$. Then

$$|F(x - t, y)| = A^{1/\mu}(1 + |t|/y)^{n/\mu} M^{1/\mu}(g),$$

and (32) is proved with

$$F^*(x) = A^{1/\mu} M^{1/\mu}(g)(x),$$

when we recall that if $p = 1/q$, then $\|g\|_p^p \leq \|F\|_1$.

Now whatever fixed $\lambda$, $\lambda > 2$ we start with, we can find a $\lambda' > 1$, and a $\mu < 1$, $\mu$ sufficiently close to 1, so that $\lambda' = \lambda - 1/\mu$. We set

$$I^*(x) = \int_{\mathbf{R}^{n+1}_{+}} y^{1-n}\left(\frac{y}{y + |t|}\right)^{\lambda' n} \Delta G(x - t, y)\,dt\,dy,$$

where $G(x, y) = |F(x, y) + \varepsilon\Phi(x, y)|$. Observe (as in §2.5.2 of Chapter

IV) that

$$\int_{\mathbf{R}^n} I^*(x)\,dx = c_{\lambda'} \iint_{\mathbf{R}_+^{n+1}} y\,\Delta G(t, y)\,dt\,dy = c_{\lambda'} \int_{\mathbf{R}^n} G(t, 0)\,dt$$

$$= c_{\lambda'} \int_{\mathbf{R}^n} |F(x, 0) + \varepsilon\Phi(x, 0)|\,dx,$$

as soon as we verify that $G$ satisfies the requirements given for (31). Now since $F \in H_0^1$, it is rapidly decreasing together with all its partial derivatives; also $|\Phi(x, y)| = c\,|(x, y + 1)|^{-n-1}$ and $|\nabla\Phi| = c'\,|(x, y + 1)|^{-n-2}$. Therefore by (30) for $F + \varepsilon\Phi$ in place of $F$ we have

$$\Delta\,|F + \varepsilon\Phi| \leq A\,|(x, y + 1)|^{-n-3}.$$

This shows $y\,\Delta G \in L^1(\mathbf{R}_+^{n+1}, dy\,dx)$. Similarly

$$|G(x, y)| \leq A\,|(x, y + 1)|^{-n-1}$$

and $|\nabla G(x, y)| \leq A\,|(x, y + 1)|^{-n-2}$.

Next by (30) again

$$g_\lambda(F + \varepsilon\Phi)^2(x) \leq c_1^{-1} \iint_{\mathbf{R}_+^{n+1}} y^{1-n}\left(\frac{y}{y + |t|}\right)^{\lambda n} G\,\Delta G\,dt\,dy.$$

The latter is majorized by $c_1^{-1} I^*(x) F^*_{\mu,\varepsilon}(x)$, because of (32); here

$$\sup_{t,y}\left(\frac{y}{y + |t|}\right)^{n/\mu} G(x - t, y) = F^*_{\mu,\varepsilon}(x)$$

and

$$\|F^*_{\mu,\varepsilon}\|_1 \leq A_\mu\,\|F + \varepsilon\Phi\|_1.$$

Thus by Schwarz's inequality

$$\int_{\mathbf{R}^n} g^*_\lambda(F + \varepsilon\Phi)(x)\,dx \leq c_1^{-1/2} c_{\lambda'}^{1/2}\,\|F + \varepsilon\Phi\|_1.$$

Letting $\varepsilon \to 0$ proves our desired inequality (29) and thus Theorem 7.

*3.3.2 Proof of Theorem 8.* In addition to the operator $S$ we want to consider a whole family of variants, defined for any $q$, with $q > \dfrac{n-1}{n}$. We set

$$\mathfrak{S}_q(F)(x) = \left(\iint_{\Gamma(x)} y^{1-n}\,\Delta(|F|^q)(y, t)\,dy\,dt\right)^{1/q}$$

where $\Gamma(x)$ is our basic cone,

$$\Gamma(x) = \{(t, y) : |x - t| < y\}.$$

$\mathfrak{S}_q(F)$ is well-defined, since

$$\Delta(|F|)^q \geq 0 \qquad \text{for} \qquad q \geq \frac{n-1}{n}$$

by the lemma in §3.1. We make the following observations about $\mathfrak{S}_q$.

$$(\alpha) \qquad \mathfrak{S}_2(F) = \sqrt{2}\, S(F)$$

because $\Delta\,|F|^2 = 2\,|\nabla F|^2$, by the lemma in §3.1. In addition $\mathfrak{S}_q$ satisfies a certain convexity property in $q$, namely

$$(\beta) \qquad \mathfrak{S}_q(F) \leq c(\mathfrak{S}_{q_0}(F))^{1-\theta}(\mathfrak{S}_{q_1}(F))^{\theta}$$

whenever $q_j > (n-1)/n$, and $1/q = (1-\theta)/q_0 + \theta/q_1$ where $0 \leq \theta \leq 1$. $c$ is a constant which depends on $q_0, q_1$ and $\theta$, but is independent of $F$. Inequality $(\beta)$ is a consequence of Hölder's inequality and of the lemma in §3.1. This lemma states, in effect, that $\Delta\,|F|^q$ is comparable with

$$|F|^{q-2}\,|\nabla F|^2$$

and so, in effect, $\mathfrak{S}_q(F)$ behaves like a $q^{\text{th}}$ norm of $|F|$.

After these preliminaries we come to the main point. In proving the inequality

$$(33) \qquad B\,\|F\|_1 \leq \|S(F)\|_1, \qquad F \in H^1$$

we shall make the same reduction we did in the proof of Theorem 7. That is, we assume that in place of $F$ we have $F + \varepsilon\Phi$, where $F \in H_0^1$, and $\Phi$ is as above. In order to simplify the notation we shall assume that $F$ already has the required form. Write now $G(x, y) = |F(x, y)|$ and we apply the identity $\iint_{\mathbf{R}_+^{n+1}} y\,\Delta G\,dx\,dy = \int_{\mathbf{R}^n} G(x, 0)\,dx$ again, which we already saw was perfectly legitimate. In fact

$$\int_{\mathbf{R}^n} \mathfrak{S}_1(F)\,dx = \int_{\mathbf{R}^n}\left(\iint_{\Gamma(x)} \Delta\,|F(t, y)|\,y^{1-n}\,dy\,dt\right)dx$$
$$= c\iint_{\mathbf{R}_+^{n+1}} y\,\Delta\,|F(x, y)|\,dx\,dy = c\int_{\mathbf{R}^n} |F(x, 0)|\,dx.$$

Thus, $\|F\|_1 = c^{-1}\,\|\mathfrak{S}_1(F)\|_1$ and hence by $(\beta)$ we have using Hölder's inequality

$$(34) \quad \|F\|_1 \leq c'\,\|(\mathfrak{S}_2(F))^{\theta'}(\mathfrak{S}_\eta(F))^{1-\theta'}\|_1 \leq c'\,\|\mathfrak{S}_2(F)\|_1^{\theta'}\,\|\mathfrak{S}_\eta(F)\|_1^{1-\theta'}$$

where $\eta$ is an arbitrary, but fixed exponent in the range $\dfrac{n-1}{n} < \eta < 1$, and $\theta'$ is chosen accordingly.

If we can prove the inequality

$$(35) \qquad \|\mathfrak{S}_\eta(F)\|_1 \leq c\,\|F\|_1$$

then substituting this in (34) gives $\|F\|_1 \leq c' \|\mathfrak{S}_2(F)\|_1$, which is (33). We turn our attention therefore to (35).

The question whether $\mathfrak{S}_\eta(F) \in L^1(\mathbf{R}^n)$ is the same as that whether $\mathfrak{S}_\eta(F)^\eta \in L^{1/\eta}(\mathbf{R}^n)$, and since $\eta < 1$, the exponent $1/\eta$ is greater than one. Let $r$ be the index conjugate to $1/\eta$, namely $1/r + \eta = 1$. Then what we need to do is estimate

$$\sup_\varphi \int_{\mathbf{R}^n} \mathfrak{S}_\eta(F)^\eta(x)\varphi(x)\, dx$$

where $\varphi$ ranges over an (appropriate) dense subset of elements in $L^r(\mathbf{R}^n)$, with $\|\varphi\|_r \leq 1$.

For our $\varphi$ we choose the non-negative $C^\infty$ functions on $\mathbf{R}^n$, each with compact support, and so that $\|\varphi\|_r \leq 1$. Now

$$\begin{aligned} \mathfrak{S}_\eta(F)^\eta(x) &= \iint_{\Gamma(x)} \Delta(|F|^\eta(t, y))y^{1-n}\, dt\, dy \\ &= \iint_{\mathbf{R}_+^{n+1}} \psi(x, t, y)\, \Delta(|F|^\eta(t, y))y^{1-n}\, dt\, dy \end{aligned}$$

with $\psi(x, t, y)$ the characteristic function of the cone $\Gamma(x) = \{|x - t| < y\}$. If $P_y(x)$ is the Poisson kernel then $\psi(x, t, y)y^{-n} \leq cP_y(x)$, therefore

$$\int_{\mathbf{R}^n} \mathfrak{S}_\eta(F)^\eta(x)\varphi(x)\, dx \leq c\iint_{\mathbf{R}_+^{n+1}} \varphi(x, y)\, \Delta\, |F(x, y)|^\eta\, y\, dx\, dy \tag{36}$$

with $\varphi(x, y)$ the Poisson integral of $\varphi$, that is $\varphi(x, y) = (P_y * \varphi)(x)$. We next observe the following differential identity: if $\Delta(A) = 0$

$$A\, \Delta(B) = \Delta(AB) - 2\sum_{j=0}^{n} \frac{\partial A}{\partial x_j}\frac{\partial B}{\partial x_j}$$

Set $A = \varphi(x, y)$, $B = |F(x, y)|^\eta$, $(y = x_0)$, then the right side of (36) is majorized by a constant multiple of

$$\iint_{\mathbf{R}_+^{n+1}} \Delta(\varphi\, |F|^\eta)y\, dy\, dx + 2\iint_{\mathbf{R}_+^{n+1}} |\nabla\, |F|^\eta\,| \cdot |\,\nabla\varphi|\; y\, dy\, dx.$$

The integral on the left of the above can be evaluated by the identity (31) whose application is again legitimate in view of the assumed form of $F$. Its value is then $\int_{\mathbf{R}^n} |F(x, 0)|^\eta\, \varphi(x, 0)\, dx$, and this is majorized by

$$\|F\|_1^\eta\, \|\varphi\|_r \leq \|F\|_1^\eta.$$

The integral on the right equals a constant multiple of

$$\int_{\mathbf{R}^n} \left\{ \iint_{\Gamma(x)} |\nabla\, |F|^\eta(t, y)\,| \cdot |\,\nabla\varphi(t, y)|\; y^{1-n}\, dt\, dy \right\} dx. \tag{37}$$

Now $|\nabla |F|^\eta| \leq \text{constant} \, |F|^{\eta-1} \, |\nabla F|$. Also by the maximal inequality ((32), p. 226) we have $\sup_{(t,y)\in\Gamma(x)} |F(t, y)| \leq cF^*(x)$, with $\int_{\mathbf{R}^n} F^*(x) \, dx \leq A \, \|F\|_1$. Finally $\Delta \, |F|^\eta \geq \text{constant} \, |F|^{\eta-2} \, |\nabla F|^2$. Altogether then, using Schwarz's inequality, we see that (37) is majorized by a constant multiple of

$$\int_{\mathbf{R}^n} (F^*(x))^{\eta/2} (\mathfrak{S}_\eta(F))^{\eta/2} S(\varphi) \, dx$$

To this integral we apply Hölder's inequality in the form

$$\int_{\mathbf{R}^n} A_1 A_2 A_3 \, dx \leq \|A_1\|_{p_1} \, \|A_2\|_{p_2} \, \|A_3\|_{p_3},$$

where $1/p_1 + 1/p_2 + 1/p_3 = 1$, with $p_1 = p_2 = 2/\eta$, and $p_3 = r$ (recall $1/r + \eta = 1$). The majorant is then $\|F^*\|_1^{\eta/2} \, \|\mathfrak{S}_\eta(F)\|_1^{\eta/2} \, \|S(\varphi)\|_r$ which in turn is dominated by a constant multiple of $\|F\|_1^{\eta/2} \, \|\mathfrak{S}_\eta(F)\|_1^{\eta/2}$; this is true on two counts, first since $\|S(\varphi)\|_r \leq A \, \|\varphi\|_r \leq A$ by the results of Chapter III (here $r > 1$), and also $\|F^*\|_1 \leq A \, \|F\|_1$ as already pointed out.

In summary we have obtained the following:

$$\begin{aligned} \|\mathfrak{S}_\eta(F)\|_1^\eta &= \sup_\varphi \int_{\mathbf{R}^n} \mathfrak{S}_\eta(F)^\eta \varphi \, dx \\ &\leq \|F\|_1^\eta + A \, \|F\|^{\eta/2} \, \|\mathfrak{S}_\eta(F)\|_1^{\eta/2} \end{aligned}$$

This implies inequality (35) and therefore inequality (33), giving the desired conclusion for our $F$ of the special form $F + \varepsilon\Phi$, with $F \in H_0^1$. The limiting passage to general $F$ in $H^1$ is then routine.

*3.3.3 Proof of the density lemma.* We dispose here of the lemma which was stated without proof in §3.3.1 above. We are going to prove that $H_0^1$ is dense in $H^1$.

We suppose $f \in L^1(\mathbf{R}^n)$, and $f$ has the property that the $R_j(f) = f_j$ all belong to $L^1(\mathbf{R}^n)$. (The assumption means of course that $i \dfrac{x_j}{|x|} \hat{f}(x)$ each are Fourier transforms of $L^1$ functions; observe that as a consequence $\hat{f}(0) = \hat{f}_j(0) = 0, j = 1, \ldots, n$.)

Our proof of the lemma will be in two steps. First we shall see that for each $f$ of the above kind we can find a sequence $\{f^{(k)}\}_k \in L^1$, so that $\hat{f}^{(k)}$ has support which is compact and at a positive distance from the origin, and so that $f^{(k)} \to f$ and $R_j(f^{(k)}) \to R_j(f)$ in the $L^1$ norm, as $k \to \infty$.

Let us choose a fixed $C^\infty$ function in $\mathbf{R}^n$ with compact support, $\Phi$, with the additional property that $\Phi(x) = 1$, for $|x| \leq 1$. For each $\delta > 0$, define the transformation $T_\delta$ on $L^1(\mathbf{R}^n)$ by $(T_\delta f)\hat{}\,(x) = \Phi(x/\delta)\hat{f}(x)$.

Clearly
$$T_\delta(f)(x) = \delta^n \int_{\mathbf{R}^n} f(x-y)\varphi(\delta y)\,dy$$
with $\hat{\varphi} = \Phi$. It is also to be observed that $\|T_\delta(f)\|_1 \le A\,\|f\|_1$ with $A$ independent of $\delta$ or $f$. Consider $T_N(I - T_\varepsilon)f$. As $N \to \infty$ and $\varepsilon \to 0$ it converges to $f$ in $L^1(\mathbf{R}^n)$ norm, if $f$ belongs to the closed subspace $L^1_0$ of $L^1(\mathbf{R}^n)$ consisting of functions whose Fourier transform vanishes at the origin. To prove this assertion it suffices, in view of the uniform boundedness of the operators $T_N(I - T_\varepsilon)$, to verify it for a dense subset of this subspace. An appropriate such subset consists of $L^1(\mathbf{R}^n)$ functions whose Fourier transform has support which is compact and is at a positive distance from the origin. For those $f$, obviously $T_N(I - T_\varepsilon)f = f$ for $N$ sufficiently large and $\varepsilon$ sufficiently small. That such $f$ are dense in the closed subspace $L^1_0$ can be proved directly by elementary computation, or one can appeal to Wiener's theorem characterizing the maximal ideals of $L^1(\mathbf{R}^n)$. In any case then we take $T_k(I - T_{1/k})f = f^{(k)}$, so $f^{(k)} \to f$ in $L^1$ norm. Also $R_j(f^{(k)}) = T_k(I - T_{1/k})R_j(f)$, so $R_j(f^{(k)}) \to R_j(f)$ and we have achieved the first step.

We may assume now that $f \in L^1(\mathbf{R}^n)$ and that the support of $\hat{f}$ is a compact set $K$ disjoint from the origin. Consider a standard regularization of $\hat{f}$ given by $\hat{f} * k^n\psi(kx)$ where $\psi$ is $C^\infty$ with compact support, $\int_{\mathbf{R}^n} \psi\,dx = 1$. Observe that the $\hat{f} * k^n\psi(kx)$ belong to $C^\infty$, and if $k$ is sufficiently large they have a common support contained in a compact set $K'$ which is at a positive distance from the origin. If $\Psi\hat{\ }(x) = \psi(x)$ then $\Psi(0) = 1$ and $\hat{f} * k^n\psi(kx)$ is the Fourier transform of $f(x)\Psi(x/k) = f_k$ which clearly converges to $f$ in $L^1(\mathbf{R}^n)$ norm. We claim that $R_j(f_k) \to R_j(f)$ also. In fact for each compact set $K'$ of the type described there is a $C^\infty$ function $m_j(x)$ so that $m_j(x) = i\dfrac{x_j}{|x|}$ for $x \in K'$. Let $M_j$ be the $L^1$ function determined by $\hat{M}_j(x) = m_j(x)$. Then
$$M_j * f_k = R_j(f_k).$$
The convergence of $R_j(f_k)$ to $R_j(f)$ in the $L^1(\mathbf{R}^n)$ norm is then obvious. It is also apparent that the element of $H^1$ whose boundary values are $(f_k, R_1(f_k), \ldots, R_n(f_k))$ is in fact in $H^1_0$. This proves the lemma.

It will be useful to record the essence of the lemma as follows. Consider the Banach space $\{f \in L^1(\mathbf{R}^n);\ R_j(f) \in L^1(\mathbf{R}^n), j = 1, \ldots, n\}$, with norm $\|f\| = \|f\|_1 + \sum_{j=1}^n \|R_j(f)\|_1$. Then we have proved above:

COROLLARY. *The collection of $f$ whose Fourier transforms are $C^\infty$ and which have compact support strictly disjoint from the origin is dense in the whole Banach space.*

Let us call by $H^1_{00}$ the corresponding subspace of $F$ in $H^1$. We have of course $H^1_{00} \subset H^1_0 \subset H^1$, and we have shown that $H^1_{00}$ is dense in $H^1$.

**3.4 Multiplier transformations in** $H^1$. After all the exertions of the previous pages in dealing with the functions $S(F)$ and $g^*_\lambda(F)$ we come now to some results which indicate that these efforts were justified. We intend to show that many of the singular integral operators (more broadly: multiplier transformations) studied in Chapters II, III, and IV extend to bounded operators on $H^1$.

We require a definition. Let $m(x)$ be a function defined on $\mathbf{R}^n$. Suppose that whenever $F \in H^1$ we can find another element of $H^1$, namely $\tilde{F}$, with the property that if $\lim_{y \to 0} F(x, y) = F(x, 0) = (f_0(x), f_1(x), \ldots, f_n(x))$, and $\tilde{F}(x, 0) = (\tilde{f}_0(x), \ldots, \tilde{f}_n(x))$, we have

$$(38) \qquad (\tilde{f}_j)\hat{}\,(x) = m(x)\hat{f}_j(x), \qquad j = 0, 1, \ldots, n.$$

The function $m$ will then define a mapping $T_m$ of $H^1$ to itself, given by $\tilde{F} = T_m(F)$. If $T_m$ is bounded on $H^1$ (*) we shall say that $m$ is a multiplier for $H^1$. The matter can be put in another way. We can say that $m$ is a multiplier for $H^1$ if there exists a constant $A$ with the following property: whenever $f_0, f_1, \ldots, f_n$ are $L^1(\mathbf{R}^n)$ and $f_j = R_j(f_0)$, then there exists $L^1(\mathbf{R}^n)$ functions $\tilde{f}_0, \tilde{f}_1, \ldots, \tilde{f}_n$ defined by (38) so that

$$\sum_{j=0}^{n} \|\tilde{f}_j\|_1 \leq A \sum_{j=0}^{n} \|f_j\|_1.$$

Our theorem is as follows:

THEOREM 9. *Suppose that $m(x)$ is of class $C^{(n+1)}$ in the complement of the origin of $\mathbf{R}^n$. Assume that*

$$(39) \qquad \sup_{0<R<\infty} R^{2|\alpha|-n} \int_{R \leq |x| \leq 2R} \left|\left(\frac{\partial}{\partial x}\right)^\alpha m(x)\right|^2 dx \leq B$$

*for every differential monomial* $\left(\frac{\partial}{\partial x}\right)^\alpha$, *where* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ *and* $|\alpha| = \alpha_1 + \alpha_2 \cdots + \alpha_n \leq n + 1$. *Then $m$ is a multiplier for $H^1$; that is,*

$$(40) \qquad \|T_m(F)\|_1 \leq A\, \|F\|_1.$$

* The assumption that $T_m$ is bounded on $H^1$ is strictly speaking unnecessary since it follows automatically from the closed graph theorem and the assumption that $T_m$ is defined on all of $H^1$.

Among the operators that are covered by this theorem are the following:

(i) The operators of the kind

$$f \to \lim_{\varepsilon \to 0} \int_{|y| > \varepsilon} \frac{\Omega(y)}{|y|^n} f(x - y)\, dy$$

if $\Omega$ is a sufficiently smooth homogeneous function of degree 0 with vanishing mean-value on the unit sphere. Operators of this type arose in §4 of Chapter II, and contain the class of operators described in Theorem 6 (p. 75) of Chapter III. The latter class includes of course the Riesz transforms and the algebra they generate.

(ii) A substantial sub-class of the multipliers arising in the $L^p$ multiplier theorem (Theorem 3 and its corollary) in Chapter IV (see p. 96).

Incidentally the proof will show that $A \leq CB$, where $C$ is some absolute constant; thus the theorem leads directly to a slight extension of itself where the condition of $C^{(n+1)}$ is relaxed to some condition involving differentiality in the $L^2$ context. We shall however, not pursue this refinement.

*3.4.1* The theorem will be a direct consequence of the following lemma:

LEMMA. *Suppose $F$ belongs to the dense subspace $H^1_{00}$. Then $T_m(F) \in H^1$ and*

$$S(T_m(F))(x) \leq A' g^*_\lambda(F)(x), \quad \text{with} \quad \lambda = \frac{2n+2}{n} \tag{41}$$

The lemma is proved in much the same way as the corresponding lemma in §3.2 of Chapter IV.

First since $F \in H^1_{00}$, each $\hat{f}_j$ is $C^\infty$ and has compact support away from the origin. Hence in view of the fact that $m$ is $C^{(n+1)}$ away from the origin it follows that $m(x)\hat{f}_j(x)$ is of class $C^{(n+1)}$ and has compact support, and therefore is the Fourier transform of an $L^1$ function. Thus by definition (38) $T_m(F) \in H^1$.

Let us define the harmonic function $M(x, y)\, y > 0$, by

$$M(x, y) = \int_{\mathbf{R}^n} e^{-2\pi i x \cdot t} e^{-2\pi |t| y} m(t)\, dt.$$

Then following the argument in §3.3 of Chapter IV we have

$$\tilde{F}(x, y) = (T_m F)(x, y) = \int_{\mathbf{R}^n} M(t, y_1) F(x - t, y_2)\, dt, \qquad y = y_1 + y_2,$$

and therefore

$$|\nabla^{(k+1)} \tilde{F}(x, y)| \leq \int_{\mathbf{R}^n} |\nabla^k M(t, y/2)|\, |\nabla F(x - t, y/2)|\, dt \tag{42}$$

where $\nabla^k$ denotes the $k^{\text{th}}$ gradient.

The conditions (39) may be reinterpreted in terms of $M$. Following the same reasoning used in Chapter IV we get

$$|\nabla^k M(t, y)| \leq B' y^{-n-k} \tag{43}$$

$$\int_{\mathbf{R}^n} |t|^{2k} |\nabla^k M(t, y)|^2 \, dt \leq B' y^{-n}, \quad \text{for} \quad k = n + 1. \tag{43'}$$

In proving the inequality (41) it suffices by translation invariance to consider the origin only. Inserting (43) and (43′) in (42) and using Schwarz's inequality gives (with $k = n + 1$)

$$\begin{aligned} |\nabla^{(k-1)} \tilde{F}(x, y)|^2 &\leq A y^{-n-2k} \int_{|t| \leq 2y} |\nabla F(x - t, y/2)|^2 \, dt \\ &\quad + A y^{-n} \int_{|t| > 2y} |\nabla F(x - t, y/2)|^2 \, |t|^{-2k} \, dt \\ &= I_1(x, y) + I_2(x, y) \end{aligned}$$

Thus

$$\iint_{|x| \leq y} |\nabla^{k-1} \tilde{F}(x, y)|^2 \, y^{2k-n+1} \, dx \, dy \leq \sum_{j=1}^{2} \iint_{|x| \leq y} I_j(x, y) y^{2k-n+1} \, dx \, dy$$

To evaluate

$$\iint_{|x| \leq y} I_1(x, y) y^{2k-n+1} \, dx \, dy$$

observe that $|t| \leq 2y$ and $|x| \leq y$ implies that $|x - t| \leq 3y$. Thus a simple computation shows that this integral is dominated by a constant multiple of

$$\iint_{|x'| \leq 6y} |\nabla F(x', y)|^2 \, y^{1-n} \, dx' \, dy.$$

Similarly the integral

$$\iint_{|x| \leq y} I_2(x, y) y^{2k-n+1} \, dx \, dy$$

is dominated by a constant multiple of

$$\iint_{|x'| > 2y} |\nabla F(x', y)|^2 \, y^{1-n} \left( \frac{y}{|x'|} \right)^{2k} dx' \, dy$$

Both of these results are of course majorized by a constant multiple of $(g_\lambda^*(F)(0))^2$, where $\lambda = \dfrac{2k}{n}$. Since $k = n + 1$, we get $\lambda = \dfrac{2n + 2}{n}$. Altogether we have verified that

$$\iint_{|x| \leq y} |\nabla^{(k-1)} \tilde{F}(x, y)|^2 \, y^{2k-n+1} \, dx \, dy \leq A' [g_\lambda^*(F)(0)]^2.$$

We now invoke Lemma 2 in §2.5.2 of the present chapter (see p. 216). As a result we get

$$S(\tilde{F})(0) = S(T_m F)(0) \leq A g_\lambda^*(F)(0)$$

and after a translation by an arbitrary $x$ this is (41), and the lemma is proved.

The proof of Theorem 9 is then concluded as follows. Theorems 7 and 8 now show immediately that $\|T_m(F)\|_1 \leq A \|F\|_1$, whenever $F \in H^1_{00}$, with $A$ independent of $F$.

The bounded operator $T_m$ defined on $H^1_{00}$ then has an abstract extension as a bounded operator to all of $H^1$. It is then a trivial matter to see by a limiting argument that this extension satisfies the defining property (38). Theorem 9 is therefore completely proved.

## 4. *Further results*

**4.1** The results of §1 have analogues when the upper half-plane $\mathbf{R}^{n+1}_+$ is replaced by the unit ball $B^{n+1}$ in $\mathbf{R}^{n+1}$ with its boundary the unit sphere $S^n$. Let $\mathscr{P}(x, y)$ be the spherical Poisson kernel $\mathscr{P}(x, y) = c_n \dfrac{1 - |x|^2}{|x - y|^{n+1}}$, $(|x| < 1, |y| = 1)$, and $d\sigma(y)$ be the induced Lebesgue measure on $S^n$. For every $f \in L^p(S^n, d\sigma)$, its Poisson integral is $u(x) = \int_{S^n} \mathscr{P}(x, y) f(y)\, d\sigma(y)$. One then has the following. Suppose $u$ is harmonic in $B^{n+1}$.

(a) $u$ is the Poisson integral of an $L^p$, $1 < p \leq \infty$ function if and only if $\sup\limits_{0<r<1} \left(\int_{S_n} |u(ry)|^p\, d\sigma(y)\right)^{1/p} < \infty$

(b) $u$ is the Poisson integral of a finite measure on $S$ if and only if

$$\sup_{0<r<1} \int_{S^n} |u(ry)|\, d\sigma(y) < \infty$$

(c) $u$ is the Poisson integral of a finite positive measure on $S^n$ if and only if $u \geq 0$ in $B^{n+1}$.

Under any of the three above conditions the (appropriately defined) non-tangential limits of $u$ exist almost everywhere on $S^n$.

**4.2** Suppose $u(x, y)$ is harmonic in $\mathbf{R}^{n+1}_+$. Then $u(x, y) \geq 0$ if and only if it is of the form

$$u(x, y) = \int_{\mathbf{R}^n} P_y(x - t)\, d\mu(t) + ay, \qquad a \geq 0$$

where $d\mu$ is non-negative Borel measure for which

$$\int_{\mathbf{R}^n} \frac{d\mu(t)}{(1 + |t|^2)^{(n+1)/2}} < \infty. \quad \text{(Hint: use §4.1(c)).}$$

**4.3** The fact that non-tangential boundedness implies at almost every point the existence of non-tangential limits (Theorem 3) has been generalized in several directions.

(a) It suffices to assume that at the points in question, the given harmonic function is non-tangentially bounded from below. (Carleson [1])

(b) These results can be extended to domains which have a Lipschitz boundary. (R. Hunt and Wheeden [1])

**4.4** Let $d\mu$ be any non-negative measure on $\mathbf{R}^{n+1}_+$ with the property that $\mu(Q) \leq c(\text{diam } Q)^n$ for any cube $Q$ in $\mathbf{R}^{n+1}_+$ which touches the boundary, $\mathbf{R}^n$. Let $u(x, y)$ be the Poisson integral of an $L^p(\mathbf{R}^n)$ function. Then

$$\left(\iint_{\mathbf{R}^{n+1}_+} |u(x, y)|^p \, d\mu\right)^{1/p} \leq cA_p\left(\int_{\mathbf{R}^n} |f(x)|^p \, dx\right)^{1/p}, \qquad 1 < p \leq \infty.$$

See Carleson [2]; also Hörmander [4].

That this result is in reality a consequence of the usual maximal theorem (Theorem 1, Chapter I) can be seen as follows. Let $\Phi(x, y)$ and $\varphi(x)$ be non-negative functions in $\mathbf{R}^{n+1}_+$ and $\mathbf{R}^n$ related by the non-tangential inequality $\sup\limits_{|x-x'|<y} \Phi(x', y) \leq \varphi(x)$. Then $\mu\{(x, y) : \Phi > \alpha\} \leq cm\{x : \varphi > \alpha\}$ for each $\alpha$, and as a result

$$\iint_{\mathbf{R}^{n+1}_+} \Phi^p \, d\mu \leq c \int_{\mathbf{R}^n} \varphi^p(x) \, dx$$

Once this is observed, we need only take $\Phi(x, y) = |u(x, y)|$, $\varphi(x) = AM(f)(x)$. The non-tangential inequality $\sup\limits_{|x-x'|<y} \Phi(x', y) \leq \varphi(x)$ is then contained in Theorem 1, and the result follows from the $L^p$ inequality for $M(f)$.

**4.5** There is another type of maximal inequality for Poisson integrals. It has some of the features of the $g_\lambda^*$ function in that there is a critical $L^p$ class depending on $\lambda$. Define $\mathscr{M}_\lambda$ by

$$\mathscr{M}_\lambda(f)(x) = \sup_{y>0} \left(\int_{\mathbf{R}^n} |u(x - t, y)|^2 \, y^{-n} \left(\frac{y}{|t| + y}\right)^{n\lambda} dt\right)^{1/2}$$

Notice that $\mathscr{M}_\lambda(f)(x) \geq c_\lambda M(f)(x)$, if $f \geq 0$. Let $1 < \lambda \leq 2$.

(a) If $p = 2/\lambda$ the mapping $f \to \mathscr{M}_\lambda(f)$ is of weak-type $(p, p)$

(b) If $p > 2/\lambda$

$$\|\mathscr{M}_\lambda(f)\|_p \leq A_{p,\lambda} \|f\|_p$$

(c) If $p < 2/\lambda$, there exists an $f \in L^p(\mathbf{R}^n)$ so that $\mathscr{M}_\lambda(f)(x) = \infty$ everywhere. See Stein [4].

**4.6** Let $H$ be the harmonic function $\dfrac{|(x, y)|^{-n+1}}{-n + 1}$, $n > 1$, (with $|(x, y)| = (y^2 + x_1^2 \cdots + x_n^2)^{1/2}$).

Set $F = \nabla H = \left(\frac{\partial H}{\partial y}, \frac{\partial H}{\partial x_1}, \cdots \frac{\partial H}{\partial x_n}\right)$. Then $|F| = |(x, y)|^{-n}$, and $\Delta |F|^q = nq[nq - n + 1]\,|(x, y)|^{-nq-2}$. Thus if $q > 0$, $\Delta |F|^q \geq 0$ only if $q \geq \left(\frac{n-1}{n}\right)$.

**4.7** The $L^p$ inequalities for fractional integration (see §1.2 in Chapter VI) are valid for $p = 1$ in the context of $H^1$. If $f \in L^1(\mathbf{R}^n)$ and $R_j(f) \in L^1(\mathbf{R}^n)$, $j = 1, \cdots, n$. Then

$$I_\alpha(f) = \frac{1}{\gamma(\alpha)} \int_{\mathbf{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}}\, dy \in L^q(\mathbf{R}^n)$$

and $\|I_\alpha(f)\|_q \leq A_\alpha(\|f\|_1 + \sum_{j=1}^n \|R_j(f)\|_1)$ with $1/q = 1 - \alpha/n$, and $0 < \alpha < n$. See Stein and Weiss [2].

**4.8** Suppose $f \geq 0$, $f \in L^1(\mathbf{R}^n)$. Define $R_j(f)$ by

$$\lim_{\varepsilon \to 0} c_n \int_{|y| \geq \varepsilon} \frac{y_j}{|y|^{n+1}} f(x - y)\, dy,$$

which we know exists almost everywhere (see Theorem 4, Chapter I).

Suppose each $R_j(f)$ is integrable on each compact set. Then $|f| \log (2 + |f|)$ is also integrable on each compact set. (*Hint:* use Corollary 2 of Theorem 6 in §3.2 of this chapter, together with §5.2(c) in Chapter I. For details see Stein [12].)

**4.9** Theorems 6 to 9 have extensions to $H^p$, for $p > \frac{n-1}{n}$ by much the same methods as those given here for $p = 1$.

(a) If $F \in H^p$, then $\lim_{y \to 0} F(x + iy) = F(x)$ exists almost everywhere,

$$\int_{\mathbf{R}^n} |F(x + iy) - F(x)|^p\, dx \to 0,$$

as $y \to 0$, and $\int_{\mathbf{R}^n} \sup_{y>0} |F(x + iy)|^p\, dx \leq A_p^p \|F\|_p^p$. Stein and Weiss [2].

(b) $B_p \|F\|_p \leq \|S(F)\|_p \leq A_p \|F\|_p$. Calderón [6], Segovia [1] and also Gasper [1]

(c) $$\|g_\lambda^*(F)\|_p \leq A_{p,\lambda} \|F\|_p, \quad \text{if} \quad p > 2/\lambda$$

(d) $m$ is a multiplier of $H^p$ if $|m(x)| \leq B$ and

$$\sup_{0<R<\infty} R^{2|\alpha|-n} \int_{R<|x|\leq 2R} \left|\left(\frac{\partial}{\partial x}\right)^\alpha m(x)\right|^2 dx \leq B$$

where $|\alpha| \leq k$, and $k > n/p$, $(p \leq 2)$. For (c) and (d) see Stein [9].

**4.10** Suppose $a(x)$ is defined on $\mathbf{R}^1$ and $|a(x) - a(y)| \le M|x - y|$, $x, y \in \mathbf{R}^1$. Let $T_\varepsilon(f)(x) = \int_{|x-y|\ge\varepsilon} \left\{\dfrac{a(x) - a(y)}{(x - y)^2}\right\} f(y)\, dy$. Then $\|T_\varepsilon(f)\|_p \le A_p \|f\|_p$, $1 < p < \infty$, with $A_p$ independent of $\varepsilon$. This result can be proved as an application of Theorem 8. For details and further results of this kind see Calderón [6].

**4.11** Let $u(x, y)$ be harmonic in $\mathbf{R}^{n+1}_+$. Then at *an individual* point $x^0$ the three conditions

(1) $u$ is non-tangentially bounded at $x^0$;

(2) $u$ has a non-tangential limit at $x^0$; and

(3) $\iint_{\Gamma(x^0)} |\nabla u|^2 y^{1-n}\, dx\, dy < \infty$

may all be independent, except that of course (2) implies (1).

To see this take the case $n = 1$. Consider the three functions $u(x, y)$ given by $e^{-i/z}$, $e^{i\gamma \log z}$, and $(\log z)^{1-\delta}$ where $\gamma$ is real, $1/2 < \delta < 1$; each are single valued and holomorphic in $z = x + iy$ for $(x, y) \in \mathbf{R}^2_+$. Let $x^0 = 0$. For the first we have (2) (and (1)) but not (3); for the second we have (1), but not (2) and (3); and for the third function we have (3), but not (2) or (1).

**4.12** That many of the results of this chapter do not hold when the non-tangential approach is replaced by perpendicular convergence to the boundary can be seen by the use of the following theorem.

THEOREM. *Let $E$ be a set of the first category (and possibly of full measure) in $\mathbf{R}^1$. Suppose $\Phi(x, y)$ is any continuous function in $\mathbf{R}^2_+$. Then there exists an analytic function $F(z)$, $z = x + iy$ in $\mathbf{R}^2_+$ so that* $\lim_{y\to 0} \{F(x + iy) - \Phi(x, y)\} = 0$ *for each $x \in E$.* (See Bagemhil and Seidel [1]).

(a) By the aid of this theorem, choosing an appropriate $\Phi$ which is bounded but oscillates as $y \to 0$, we see that the analogue of Theorem 3 in §1.3 for perpendicular approach is not valid. Similarly by choosing a $\Phi$ which in addition is purely real-valued, we see that the analogue of Theorem 5 in §2.5 also fails.

(b) For fixed $\Psi(y)$, let $\Phi(x, y) = \Psi'(y)$, and suppose that $f(z)$ is analytic in $\mathbf{R}^2_+$, with the property that $f'(x + iy) - \Psi'(y) \to 0$, as $y \to 0$, $x \in E$. Then clearly $f(x + iy) - \Psi(y) \to$ limit, as $y \to 0$, $x \in E$. If we choose $\Psi(y) = ye^{i/y}$, we see that $f(x + iy)$ can tend to a limit almost everywhere, as $y \to 0$, but $\int_0^\varepsilon y\, |f'(x + iy)|^2\, dy = \infty$ almost everywhere. Conversely, if we choose $\Psi(y) = (\log 1/y)^\delta$, $(0 < \delta < 1/2)$ for $y \le 1$, then we see that $\int_0^\varepsilon y\, |f'(x + iy)|^2\, dy$ can be finite almost everywhere, without $\lim_{y\to 0} f(x + iy)$ existing almost everywhere. This shows that the perpendicular analogue of the area theorem (in §2) is also not valid.

## Notes

*Section 1.* The local version of Fatou's theorem (when $n = 1$) was proved by Privalov using complex methods. See Zygmund [8, Chapter XIV]. The general version, and proof, given here are taken from Calderón [1].

*Section 2.* The area integral introduced by Lusin when ($n = 1$), and in this case Theorem 4 is due to Marcinkiewicz and Zygmund and Spencer (see Zygmund [8, Chapter XIV]). For the case of general $n$ see Stein [5], but one direction of the implication had been proved earlier by Calderón [2]. Theorem 5, dealing with non-tangential convergence of conjugate harmonic functions (which will be a key tool in Chapter VIII) goes back to Plessner, when $n = 1$. The argument using complex methods can be found in Zygmund [8, Chapter XIV]. The general case is in Stein [5], as well as the lemma in §2.5.1 used to prove it.

*Section 3.* The classical theory of $H^p$ spaces may be found in, for example, Zygmund [8, Chapter VII], and K. Hoffman [1]. The $n$-dimensional real theory was begun in Stein and G. Weiss [2], and is based on a variant of the lemma in §3.1. Theorem 7 was announced in Stein [9]. The argument of Theorem 8 is a simplification of the original proof given by Segovia [1]. For the chain of ideas leading up to Theorem 8 and the reasoning presented here see also Zygmund [8, Chapter XIV], and the significant refinement in Calderón [6].

Theorem 9, which extends to $H^1$ many of the singular integral theorems, is in Stein [9]. For the extension of the theory of $H^p$ spaces to systems generalizing (18) see Calderón and Zygmund [8], Stein and G. Weiss [2], and *Fourier Analysis*, Chapter VI.

CHAPTER VIII

# Differentiation of Functions

In the present chapter we want to bring together various techniques developed in this monograph to study differentiability properties of functions of several variables. In keeping with our approach we shall not aim at the greatest generality, but we will instead pick out certain salient features of a theory which has not yet reached maturity. We shall be concerned with the following problems.

(A) What are conditions that guarantee that a function has derivatives almost everywhere? This is a special case of a further problem which is central in our considerations here.

(B) What are the conditions on a function relative to a given measurable subset $E$ of $\mathbf{R}^n$, that guarantee that this function is differentiable at almost every point of $E$?

We adopt the following viewpoint with respect to the second problem. We look first for the appropriate *global* analogue of our problem. Here the result is usually stated in terms of identities or inequalities that involve norms of function spaces and are valid everywhere (or almost everywhere) for all of $\mathbf{R}^n$. The derived *local* analogue, which is often deeper, is then given by similar conditions and conclusions, but relative to an arbitrary measurable set $E$.

We have already dealt with one example of this global and local pairing in Chapter VII (in §1.2), where we considered two versions of Fatou's theorem on boundary values of harmonic functions.

We have made it our goal in this chapter to present three local theorems giving conditions for differentiability. Their global analogues are the following: the differentiability almost everywhere of Lipschitz functions; the boundedness of singular integral transformations on $L^q(\mathbf{R}^n)$, $1 < q < \infty$; and the characterization of the $L^q$ class of a function in terms of the $L^q$ class of the $g$-function or Lusin integral.

In addition to the global results already alluded to, the main technical ingredients turn out to be as follows.

(i) Generalizations of the ordinary definition of differentiability.

(ii) A basic splitting lemma into "good" and "bad" parts which in some sense may be viewed as the local analogue of the corresponding global splitting used in the study of singular integrals in Chapter II.

(iii) An argument of *desymmetrization* which allows one to go from symmetric assumptions to non-symmetric conclusions.

The ideas (i) and (ii) are presented in §1 and §2 and (iii) is detailed in §4.

We add a pedantic comment. It is in the nature of the subject matter considered here that in principle we cannot exclude the possibility that non-measurable sets enter at certain stages of the argument. While this is an awkward point it does not impede the development that follows. We shall return to this in greater detail in §3.1.1 later, but let us at this point already make the following convention. When the words "function" or "set" are written below we will mean Lebesgue measurable function or Lebesgue measurable set, unless there is an explicit qualification to the contrary.

## 1. *Several notions of pointwise differentiability*

**1.1** Suppose that $f$ is defined in an open neighborhood of a set $E$ in $\mathbf{R}^n$. Let $x^0 \in E$. We shall say that $f$ has an *ordinary derivative* (or *is differentiable*) at $x^0$, if there exists a linear function $\Lambda = \Lambda_{x^0}$ so that

$$f(x^0 + y) = f(x^0) + \Lambda(y) + o(|y|) \tag{1}$$

as $|y| \to 0$, or what is the same that

$$\sup_{|y| < r} \frac{|f(x^0 + y) - f(x^0) - \Lambda(y)|}{|y|}$$

tends to zero with $r$.

On the other hand if $f$ is (say) locally integrable we can define its first partial derivatives in the *weak sense* (the distribution sense) as in Chapter V, §2.1, by saying that $\dfrac{\partial f}{\partial x_k} = f_k$, $k = 1, \ldots, n$, with the $f_k$ locally integrable if

$$\int_{\mathbf{R}^n} f \frac{\partial \varphi}{\partial x_k}\, dx = -\int_{\mathbf{R}^n} f_k \varphi\, dx \tag{2}$$

for every smooth function $\varphi$ whose support is compact and is strictly inside the domain of definition of $f$.

The first important question that arises is whether as a consequence of the second definition $f$ has a derivative in the pointwise sense (1) for almost every $x^0$.

In the case of one dimension the answer is of course yes since such $f$ are locally absolutely continuous on $\mathbf{R}^1$ (see §6.1 of Chapter V). The situation however differs when the dimension is greater than one. The example in §6.3 of Chapter V shows that the $\dfrac{\partial f}{\partial x_k}$ may exist in the weak sense and be in $L^p(\mathbf{R}^n)$, with $p \le n$; at the same time $f$ may fail to have a derivative in the sense of (1) at *every* point $x^0$, since $f$ may be unbounded near every $x^0$. We are therefore motivated to relax the requirement in the definition of the derivative so as to conform with the type of behavior we can expect of $f$. We shall say that $f$ has a derivative at $x^0$ *in the $L^q$ sense*, with $1 \le q < \infty$, if

$$(3) \quad \left(h^{-n}\int_{|y|<h} |f(x^0 + y) - f(x^0) - \Lambda(y)|^q \, dy\right)^{1/q} = o(h), \quad \text{as} \quad h \to 0.$$

This is clearly a generalization of the initial definition (1).

The following result clarifies the significance of the notion just introduced. We assume that $n > 1$.

THEOREM 1. *Suppose $f$ is a locally integrable function given in an open set $\Omega$, with the property that the $\dfrac{\partial f}{\partial x_j}$, $j = 1, \ldots, n$ exist weakly there, and $f$ and the $\dfrac{\partial f}{\partial x_j}$ belong locally to $L^p(\mathbf{R}^n)$, with $j = 1, \ldots n$.*

(a) *If $n < p$, $f$ has an ordinary derivative (in the sense (1)) for almost every $x^0$, when $f$ has been suitably modified on a set of measure zero.*

(b) *If $1 < p < n$, then $f$ has a derivative in the $L^q$ sense with $1/q = 1/p - 1/n$, for almost every $x^0$.*

**1.2 Proof of Theorem 1.** After multiplying by a smooth function with compact support we may assume that $f$ itself has compact support and that $f$ and the $\dfrac{\partial f}{\partial x_j}$ are in $L^p(\mathbf{R}^n)$, with the latter taken in the sense of distributions; this means that $f \in L_1^p(\mathbf{R}^n)$. We can also suppose that $p < \infty$, since the result in the case $p = \infty$ is then a consequence of the case $n < p < \infty$.

If $f$ were in $\mathcal{S}$ we would have the identity

$$(4) \qquad f = I_1\left(\sum_{j=1}^{n} R_j\left(\frac{\partial f}{\partial x_j}\right)\right).$$

(See §2.3 of Chapter V.)

Now our $f$ is in $L_1^p(\mathbf{R}^n)$, so it can be approached in the norm of that space by a sequence $\{f_m\}$ of such elements (see §2.1 of Chapter V);

since each $\dfrac{\partial f}{\partial x_j}$ also belongs to $L^{p'}(\mathbf{R}^n)$ for every $p' \leq p$, the proof of the approximation argument also shows that

$$\frac{\partial f_m}{\partial x_j} \to \frac{\partial f}{\partial x_j} \quad \text{in} \quad L^{p'}(\mathbf{R}^n).$$

Recall also that the $R_j$ are continuous from $L^{p'}(\mathbf{R}^n)$ to itself, if $1 < p' < \infty$ and $I_1$ is continuous from $L^{p'}(\mathbf{R}^n)$ to $L^{q'}(\mathbf{R}^n)$ if $1/q' = 1/p' - 1/n$, $1 < p' < n$, (see §1.2 of Chapter V). Thus altogether we have that the identity (4) holds for our $f$. For our purposes this can be restated as follows.

$$(5)\quad f(x) = \int_{\mathbf{R}^n} \frac{g(y)}{|x-y|^{n-1}}\, dy, \quad \text{with} \quad g \in L^{p'}(\mathbf{R}^n), \qquad 1 < p' \leq p.$$

Here we have set

$$g(x) = \frac{1}{\gamma(1)} \sum_{j=1}^{n} \left(R_j \frac{\partial f}{\partial x_j}\right)(x).$$

Of course the integral (5) converges absolutely for almost every $x$, (this can be seen by taking $p' < n$). Redefine $f$ to be the value of this integral wherever it converges absolutely, and for other points of $x$ (if there be such) we may define $f$ arbitrarily.

Next, since $g \in L^p(\mathbf{R}^n)$ the following two properties hold for almost every $x^0$:

$$(6)\qquad \frac{1}{r^n} \int_{|y| \leq r} |g(x^0 - y) - g(x^0)|^p \, dy \to 0, \quad \text{as} \quad r \to 0.$$

$$(7)\qquad \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} \frac{y_j}{|y|^{n+1}}\, g(x^0 - y)\, dy \quad \text{exists.}$$

The property (6) is merely a restatement of §5.7 of Chapter I; (7) is a special case of the theorem of the existence almost everywhere of singular integrals given in §4.5 of Chapter II.

It is at an $x^0$ where (6) and (7) hold simultaneously that we shall prove the existence of the appropriate derivative of $f$.

In order to simplify the notation we shall assume that $x^0 = 0$ and $g(x^0) = 0$. This can be achieved first by an appropriate translation and then by replacing $g(x)$ by $g(x) - g(x^0)\varphi(x)$, where $\varphi$ is a smooth function with compact support and with $\varphi(x^0) = 1$. It is clear that the result with $g$ replaced by $g - g(x^0)\varphi$ implies the desired result for $g$.

After these preliminaries are out of the way we see that our assumptions are reduced to

$$(6')\qquad \frac{1}{r^n} \int_{|y| \leq r} |g(y)|^p \, dy \to 0, \quad \text{as} \quad r \to 0$$

and

$$\alpha_j = \lim_{\varepsilon \to 0} (n-1) \int_{|y| \geq \varepsilon} g(y) \frac{y_j}{|y|^{n+1}}\, dy \quad \text{exists.} \tag{7'}$$

We set $\Lambda(x) = \sum_{j=1}^{n} \alpha_j x_j$, with $x = (x_1, \ldots, x_n)$. We remark that in view of the oft-mentioned estimation at infinity, namely that $g \in L^{p'}(\mathbf{R}^n)$, $1 < p' \leq p$, and the majorization (6′) near the origin, then the integral $\int_{\mathbf{R}^n} \frac{g(y)}{|y|^{n-1}}\, dy$ which represents $f(0)$ converges absolutely.

It will be our purpose to find the appropriate measure of smallness of

$$f(x) - f(0) - \sum_{j=1}^{n} \alpha_j x_j, \quad \text{as} \quad |x| \to 0.$$

By (5) we have

$$\begin{aligned} f(x) - f(0) &= \int_{|y| \leq 2|x|} g(y)\, |x-y|^{-n+1}\, dy - \int_{|y| \leq 2|x|} g(y)\, |y|^{-n+1}\, dy \\ &\quad + \int_{|y| > 2|x|} g(y)\{|x-y|^{-n+1} - |y|^{-n+1}\}\, dy \\ &= A_x - B_x + C_x. \end{aligned}$$

*1.2.1 Study of the integral $C_x$.* We deal with

$$C_x = \int_{|y| > 2|x|} g(y)\{|x-y|^{-n+1} - |y|^{-n+1}\}\, dy;$$

it will be the dominant term of the difference $f(x) - f(0)$. We shall see that as

$$|x| \to 0, \qquad C_x = \sum_{j=1}^{n} \alpha_j x_j + o(|x|).$$

Now the Taylor development of $|x - y|^{-n-1}$ in powers of $x_1, \ldots, x_n$ shows that

$$|x-y|^{-n+1} - |y|^{-n+1} = (-n+1) \sum_{j=1}^{n} x_j \frac{y_j}{|y|^{n+1}} + R(x, y),$$

where

$$|R(x, y)| \leq A \frac{|x|^2}{|y|^{n+1}}, \quad \text{for} \quad |y| \geq 2|x|.$$

Therefore

$$C_x = (-n+1) \sum_{j=1}^{n} x_j \int_{|y| > 2|x|} g(y) \frac{y_j}{|y|^{n+1}}\, dy + R_x,$$

where

$$|R_x| \leq A |x|^2 \int_{|y| > 2|x|} \frac{|g(y)|\, dy}{|y|^{n+1}}\ .$$

For any $\delta > 0$ we have

$$|x|\int_{|y|>2|x|}\frac{|g(y)|\,dy}{|y|^{n+1}} = |x|\int_{\delta>|y|>2|x|}\frac{|g(y)|\,dy}{|y|^{n+1}} + |x|\int_{|y|>\delta}\frac{|g(y)|\,dy}{|y|^{n+1}}.$$

The first integral is majorized by a constant multiple of $\delta^{-n}\int_{|y|\leq\delta}|g(y)|\,dy$, which tends to zero with $\delta$ by (6′). Once $\delta$ is fixed the second integral clearly tends to zero with $|x|$. Altogether we see therefore that

$$(8)\qquad C_x = \sum_{j=1}^{n}\alpha_j x_j + o(|x|), \quad \text{as} \quad |x| \to 0.$$

*1.2.2 Majorization of $A_x$ when $n < p < \infty$.* Let us consider $A_x = \int_{|y|<2|x|} g(y)\,|x-y|^{-n+1}\,dy$. We apply Hölder's inequality with the exponents $p$ and $r$, $1/p + 1/r = 1$; the hypothesis $n < p$ is equivalent with $(n-1)r < n$, and that $|y|^{-n+1}$ is locally in $L^r(\mathbf{R}^n)$. Thus since $|x-y| \leq 3\,|x|$ in this integral we have

$$\begin{aligned} A_x &\leq \left(\int_{|y|<2|x|}|g(y)|^p\,dy\right)^{1/p}\left(\int_{|y|\leq 3|x|} y^{(-n+1)r}\,dy\right)^{1/r} \\ &= o(|x|^{n/p}) \times \{C\,|x|^{(n+(-n+1)r)/r}\} \\ &= o(|x|). \end{aligned}$$

To summarize

$$(9)\qquad A_x = o(|x|) \quad \text{as} \quad |x| \to 0.$$

*1.2.3 Majorization of $A_x$ when $1 < p < n$.* For each $n > 0$, let $\chi_h(y)$ be the characteristic function of the ball $|y| \leq h$. Then assuming $|x| \leq h$, we have

$$|A_x| = \left|\int_{|y|\leq 2|x|}\frac{g(y)}{|x-y|^{n-1}}\,dy\right| \leq \int_{\mathbf{R}^n}\frac{|g(y)|\,\chi_{2h}(y)}{|x-y|^{n-1}}\,dy.$$

Hence by the $L^p$ theorem on fractional integration in §1.2 of Chapter V we have

$$\|\chi_h A_x\|_q \leq A\,\|g\chi_{2h}\|_p, \quad \text{where} \quad 1/q = 1/p - 1/n.$$

That is,

$$\int_{|x|\leq h}|A_x|^q\,dx \leq A^q\left(\int_{|y|\leq 2h}|g(y)|^p\,dy\right)^{q/p} = o(h^{nq/p}) = o(h^{n+q}),$$

by property (6′).

We obtain therefore

$$(10)\qquad \left(\frac{1}{h^n}\int_{|x|\leq h}|A_x|^q\,dx\right)^{1/q} = o(h), \quad \text{as} \quad h \to 0.$$

*1.2.4 Majorization of $B_x$.* Since

$$B_x = \int_{|y| \le 2x} \frac{g(y)}{|y|^{n-1}}\, dy,$$

we have that

$$|B_x| \le \sum_{j=-1}^{\infty} 2^{-j-1} \int_{|x| \le |y| \le 2^{-j}x} \le \sum_{j=-1}^{\infty} (2^{-j}|x|)^{-n+1} o((2^{-j}|x|)^n)$$

by property (6′). Thus

$$B_x = o(|x|) \sum_{j=-1}^{\infty} 2^{-j},$$

and therefore

$$B_x = o(|x|) \quad \text{as} \quad |x| \to 0. \tag{11}$$

The combination of (8), (9), (10), and (11) then shows that

$$f(x) - f(0) - \sum_{j=1}^{n} \alpha_j x_j = o(|x|)$$

if $n < p$, and

$$\left(h^{-n} \int_{|x| \le h} \left| f(x) - f(0) - \sum_{j=1}^{n} \alpha_j x_j \right|^q dx\right)^{1/q} = o(h) \quad \text{if} \quad 1 < p < n.$$

This proves our theorem.

*1.2.5* The theorem is also valid in the case $p = 1$ although the proof must be modified. This and other generalizations are stated in §6 below.

## 2. *The splitting of functions*

**2.1 Derivatives in the harmonic sense.** We come now to one of the main techniques used, a certain splitting of functions into their "good" and "bad" parts. Since we shall also be appealing to the theory of harmonic functions in an essential way it will be important to be able to achieve this splitting in the context of a notion of differentiability deriving from that theory. The idea we have in mind is in fact still more general than the definitions given in §1, and is described as follows.

Let $f$ be a locally integrable function defined in an open set $\Omega$. For a fixed $x^0 \in \Omega$ we modify $f$ outside a bounded open set containing $x^0$, by setting it equal to zero. For the resulting $f$ (now in $L^1(\mathbf{R}^n)$) we take its Poisson integral, $u(x, y) = P_y * f$. We shall say that $f$ has a *harmonic derivative* at $x^0$, if $u$ and $\dfrac{\partial u}{\partial x_1}, \dfrac{\partial u}{\partial x_2}, \ldots, \dfrac{\partial u}{\partial x_m}$ have non-tangential limits at

$x_0$. In this case, we define

$$f(x^0) = \lim_{(x,y)\to(x^0,0)} u(x, y), \quad \text{and} \quad \frac{\partial f}{\partial x_j}(x^0) = \lim_{(x,y)\to(x^0,0)} \frac{\partial u}{\partial x_j}(x, y)$$

where the variable point $(x, y)$ converges to $(x^0, 0)$ non-tangentially.

We need to make the following clarifying remarks about this definition.

(i) The notion of harmonic derivative is well defined in the sense that it is independent of the modification we have subjected $f$ to strictly away from $x^0$. In particular it is easy to verify that if $f$ is in $L^1(\mathbf{R}^n)$ and vanishes in a neighborhood of $x^0$, then $u$ and all its partial derivatives with respect to the $x_j$ have non-tangential limits zero at $x^0$.

(ii) If $f$ has derivative in the ordinary sense at $x^0$, or more generally in the $L^q$ sense (as defined by (3)) then $f$ has a derivative in the harmonic sense and the values of these derivatives coincide at $x^0$. To see this it suffices to assume that $f$ has a derivative in the $L^1$ sense. For appropriate constants $A_0, \alpha_1, \ldots, \alpha_n$ we have that if

$$f(x^0 + y) = A_0 + \sum_{j=1}^{n} \alpha_j y_j + \varepsilon(y),$$

then

$$\int_{|y|\le r} |\varepsilon(y)|\, dy = o(r^{n+1}), \quad \text{as} \quad r \to 0.$$

Observe that it is immediate from this that if we take $f(x^0)$ to be $A$, then $x^0$ is a Lebesgue point for $f$, and hence by Theorem 1 of Chapter VII (see page 197) it then follows that $u$ has $A$ as non-tangential limit at $x^0$. Next

$$\frac{\partial u}{\partial x_j}(x, y) = \int_{\mathbf{R}^n} P_y^j(t) f(x - t)\, dt,$$

with

$$P_y^j(t) = \frac{\partial}{\partial t_j} P_y(t), \qquad j = 1, \ldots, n.$$

Thus

$$\begin{aligned}
\frac{\partial u}{\partial x_j}(x^0 + h, y) &= \int P_y^j(t + h) f(x^0 - t)\, dt \\
&= \int P_y^j(t + h) A\, dt + \sum_{k=1}^{n} \int P_y^j(t + h)\alpha_k t_k\, dt \\
&\quad + \int P_y^j(t + h)\varepsilon(t)\, dt.
\end{aligned}$$

The first term vanishes and the second group of terms equals $\alpha_j$. This can be seen by a simple argument of passing the derivative $\dfrac{\partial}{\partial t_j}$ from $P_y(t)$ to $A$

and $\sum \alpha_k t_k$ respectively. That leaves only the last term; for this we take $|h| < \alpha y$ for some $\alpha$ (this is the non-tangential condition) and rewrite the integral as

$$(12) \quad \int_{|t| \le y} P_y^j(t+h)\varepsilon(t)\,dt + \int_{1>|t|>y} P_y^j(t+h)\varepsilon(t)\,dt + \int_{|t| \ge 1} P_y^j(t+h)\varepsilon(t)\,dt.$$

Since

$$P_y(t) = \frac{cy}{(|t|^2 + y^2)^{(n+1)/2}},$$

it is clear that $\dfrac{\partial P_y(t+h)}{\partial t_j}$ is bounded by the following two expressions, if $|h| < \alpha y$; first $cy^{-n-1}$, and secondly $cy/|t|^{n+2}$. We insert the first estimate for $P_j$ in the first integral and we get therefore that it is bounded by a constant multiple of $y^{-n-1}\int_{|t|\le y} |\varepsilon(t)|\,dt$, which tends to zero with $y$. Similarly the second integral is bounded by a constant multiple of

$$y \int_{1 \ge |t| \ge y} \frac{|\varepsilon(t)|}{|t|^{n+2}}\,dt \le \sum y \int_{2^{k+1}y > |t| > 2^k y} \frac{|\varepsilon(t)|\,dt}{|t|^{n+2}},$$

where the sum is taken over all non-negative integral $k$ so that $2^{k+1}y \le 2$. By assumption on $\varepsilon$ we get that each term of this sum is

$$y(2^k y)^{-n-2}\, o((2^{k+1}y)^{n+1}) = 2^{-k}o(1);$$

and therefore this integral tends to zero with $y$. The third integral trivially converges to zero with $y$, since $\varepsilon(t)$ is the sum of term which is bounded by $c\,|t|$ and a function in $L^1(\mathbf{R}^n)$. This proves our assertion.

(iii) Our last comment about the notion of harmonic derivative is this. Our definition does not require anything about the behavior of $\dfrac{\partial}{\partial y} u(x, y)$. In fact the existence of a non-tangential limit of $\dfrac{\partial u}{\partial y}$ at a given $x^0$ is *not* a consequence of ordinary differentiability at $x^0$. If however we had the differentiability (ordinary, in the $L^q$ sense, or harmonic) for a set of $x^0$ of positive measure, we could conclude that for almost all such $x^0$ the non-tangential limits of $\dfrac{\partial u}{\partial y}$ exist. We shall return to this deeper fact momentarily.

**2.2 The splitting.** Our theorem is as follows:

THEOREM 2. *Suppose $f$ is a given locally integrable function, and that for every point $x^0$ in a set of finite measure $E$, $f$ has a derivative in the* *harmonic sense. Then given any $\varepsilon > 0$ we can find a compact set $F$, so that $F \subset E$, $m(E - F) < \varepsilon$, and we can also write $f$ as $f = g + b$ where*

(1) $g \in L_1^\infty(\mathbf{R}^n)$

(2) *$b$ is zero on $F$.*

Observe that $g$ is the "good" function. In view of Theorem 1 it has an ordinary derivative almost everywhere. (By definition it also has essentially bounded first derivatives in $\mathbf{R}^n$ taken in the weak sense.) While $b$ is the "bad" function, it has the useful redeeming feature that it vanishes on the set $F$.

We shall now give the proof of Theorem 2. After reducing to a bounded subset of $E$ if necessary (for simplicity we call this subset also $E$), we may assume that $f$ vanishes outside a neighborhood of this set; thus $f \in L^1(\mathbf{R}^n)$. Let $u(x, y)$ be the Poisson integral of $f$, and then according to our assumptions $\dfrac{\partial u}{\partial x_1}, \ldots, \dfrac{\partial u}{\partial x_n}$, have non-tangential limits at every point of $E$. It then follows by Theorem 5 of Chapter VII, page 213, that $\dfrac{\partial u}{\partial y}$ has non-tangential limits for almost every $x^0$ in $E$. Also $u$ has non-tangential limits at almost every $x^0 \in E$ as we have seen (in particular at points of the Lebesgue set of $f$). Let us fix the parameters $\alpha$ and $h$ and consider the truncated cones $\Gamma_\alpha^h(x^0) = \{(x, y) : |x - x^0| < \alpha y, 0 < y < h\}$. Then in view of what has been said and because of the uniformization lemma in §1.3.1 of Chapter VII (page 201) we can conclude that we can find a compact set $F$ with the property that $F \subset E$, $m(E - F) < \varepsilon$ and $u$, $\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial u}{\partial x_1}, \ldots, \dfrac{\partial u}{\partial x_n}$ are all uniformly bounded in $\mathscr{R} = \bigcup_{x^0 \in F} \Gamma_\alpha^h(x^0)$. We can also assume that $u$ has the non-tangential limit $f(x^0)$ at every point $x^0 \in F$. Now in place of the truncated cones $\Gamma_\alpha^h(x^0)$ consider the infinite cones $\Gamma_\alpha(x^0) = \{(x, y) : |x - x^0| < \alpha y\}$. Since $u$ is the Poisson integral of an $L^1$ function, it follows also that $u, \dfrac{\partial u}{\partial y}, \dfrac{\partial u}{\partial x_1}, \ldots, \dfrac{\partial u}{\partial x_n}$ are all uniformly bounded in the half-space $y \geq h$, and thus in addition to $\mathscr{R}$ are also bounded in $\tilde{\mathscr{R}} = \bigcup_{x^0 \in F} \Gamma_\alpha(x^0)$.

Let us now look more carefully at $\tilde{\mathscr{R}}$. $\tilde{\mathscr{R}}$ is an open subset of $\mathbf{R}^{n+1}$ whose boundary is given by the hypersurface $y = \alpha^{-1}\delta(x)$ with $\delta(x) =$ dist $(x, F)$, being the distance from $x$ to $F$. Since $\delta(x)$ clearly satisfies the condition $|\delta(x) - \delta(x^1)| \leq |x - x^1|$ we see that $\tilde{\mathscr{R}}$ is a special Lipschitz domain in the terminology of §3.2 of Chapter VI, and therefore the extension theorem in §3.2 applies to $u$ restricted to $\tilde{\mathscr{R}}$. More explicitly let

$U(x, y)$ denote the restriction of $u$ to $\tilde{\mathscr{R}}$. Then of course $U$ is $C^\infty$ in $\tilde{\mathscr{R}}$ and $U$ and its first partial derivatives are bounded in $\tilde{\mathscr{R}}$; thus $U \in L_1^\infty(\tilde{\mathscr{R}})$. Let $\mathfrak{G}(U)$ be the extension of $U$ to all of $\mathbf{R}^{n+1}$. We have therefore that $\mathfrak{G}(U) \in L_1^\infty(\mathbf{R}^{n+1})$. Finally set $g$ to be the restriction of $\mathfrak{G}(U)$ to $\mathbf{R}^n$. Since the restriction of a function in $L_1^\infty(\mathbf{R}^{n+1})$ is in $L_1^\infty(\mathbf{R}^n)$ (this is an immediate consequence of §6.2(a) in Chapter V), we see that $g \in L_1^\infty(\mathbf{R}^n)$.

Because $u$ has a non-tangential limit equal to $f(x^0)$ at every point $(x^0, 0)$ in $\mathbf{R}^{n+1}$, where $x^0 \in F$, so does $U$; and since $\mathfrak{G}(U)$ is continuous it follows that $\mathfrak{G}(U)(x^0, 0) = f(x^0)$, for every $x^0 \in F$. Hence the same is true of $g$, its restriction to $\mathbf{R}^n = \{(x, 0)\}$; that is $g(x^0) = f(x^0)$ for each $x^0 \in F$. Writing then $b(x) = f(x) - g(x)$, we have achieved the desired conclusion.

## 3. *A characterization of differentiability*

**3.1 Boundedness of difference quotient.** Our first application of these techniques will be to prove a theorem which characterizes almost everywhere the notion of differentiability.

The reader should not overlook the striking analogy of the theorem stated below and the local version of Fatou's theorem (Theorem 3, p. 201 in Chapter VII). The application of this theorem on harmonic functions is a main idea in the proof that follows.

THEOREM 3. *Let $f$ be given in an open neighborhood of a given set $E$, then $f$ is differentiable (in the ordinary sense) at almost every point of $E$ if and only if*

$$f(x^0 + y) - f(x^0) = O(|y|) \quad as \quad |y| \to 0, \tag{13}$$

*for almost every $x^0 \in E$. It is of course not assumed that the constant appearing in the "$O$" above is uniform in $x^0$.*

The fact that differentiability at a given $x^0$ implies (13) is trivial, so we turn to the converse.

If we restrict ourselves to a subset $E_0$ of $E$ where $f$ is bounded and (13) applies, then it is apparent that outside an open neighborhood of this subset $f$ is still bounded, and we can modify $f$ so as to vanish outside this neighborhood. It will suffice to take $E_0$ to be bounded and the appropriate neighborhood of $E_0$ to be bounded also. Then it will be enough to show that the modified $f$ is differentiable almost everywhere in $E_0$. For ease of notation replace $E_0$ by $E$; and call the $f$ which is modified also $f$. Then this $f \in L^1(\mathbf{R}^n)$ and we let $u$ denote its Poisson integral. Just as in the argument of (ii) of §2.1 we see that at every point $x^0$ where (13) holds the $\dfrac{\partial u}{\partial x_j}$ are non-tangentially bounded at $x^0$. This is merely a repetition of the argument on page 248, with the appropriate $o$'s replaced by $O$'s. Therefore by the

Theorems 1 and 3 of Chapter VII the non-tangential limits of $u$ and $\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}$ exist at almost every point of $E$; thus $f$ has a harmonic derivative at almost every point of $E$. We can now invoke the splitting theorem of §2.2 above. Therefore for every $\varepsilon > 0$ there exists a compact set $F$, $F \subset E$, $m(E - F) < \varepsilon$, and a splitting $f = g + b$, with $g = f$ on $F$. Since $g \in L_1^\infty(\mathbf{R}^n)$, then Theorem 1 in §1.1 shows that $g$ has an ordinary derivative for every point in $\mathbf{R}^n$ and hence it suffices to show that $b$ is differentiable at almost every $x \in F$. However $b$ vanishes at all points of $F$ and thus we get in place of (13)

$$(13') \quad b(x^0 + y) = O(|y|), \quad \text{as} \quad |y| \to 0 \quad \text{for almost every } x^0 \in F.$$

We wish now to uniformize the relation (13′). For every integer $k$ therefore let $F_k$ be the set

$$F_k = \{x^0 : |b(x^0 + y)| \leq k\,|y|,\, |y| \leq 1/k\}.$$

Clearly $\bigcup_{k=1}^{\infty} F_k$ contains all points where (13′) holds and we are reduced to considering what happens for $x^0 \in F_k$.

*3.1.1* We must digress now because we have reached an unpleasant point in the argument; namely, the set $F_k$ since it is given by a continuum of inequalities is not necessarily Lebesgue measurable. One could have gotten around this difficulty by assuming $f$ to be everywhere continuous to begin with, but this assumption is an artificial one in the general context of our problems.

We proceed instead by the observation that the statement "almost every point of $E$ is a point of density of $E$" holds under suitable modification for non-measurable sets. In fact for any not-necessarily-measurable set $E$, denote by $m_e(E)$ its *exterior* (or *outer*) measure; by definition $m_e(E) = \inf m(F)$, $E \subset F$, where $F$ ranges over measurable sets. Clearly for any set $E$ there exists a measurable set $\tilde{E}$, so that $m_e(E) = m(\tilde{E})$ and more generally $m_e(E \cap F) = m(\tilde{E} \cap F)$ for every measurable set $F$.

Next we say that $x^0$ is a point of (*exterior*) *density* of $E$ if

$$\frac{m_e(E \cap B(x^0, r))}{m(B(x^0, r))} \to 1 \quad \text{as} \quad r \to 0$$

where $B(x^0, r)$ is the ball centered at $x^0$ of radius $r$. Obviously $x^0$ is a point of exterior density of $E$ if and only if it is a point of density of $\tilde{E}$. Thus except for a subset of $E$ of Lebesgue measure zero, the points of $E$ are points of exterior density of $E$.

*3.1.2* To conclude the proof of the theorem we need therefore only prove that at every point $x^0$ which is a point of exterior density of $F_k$ we have $b(x^0 + y) = o(|y|)$ as $|y| \to 0$. For then $b$ is differentiable at this point of $F_k$; and by taking the union over $k$, $b$ is differentiable except for a set of measure zero of $F$.

The argument now is merely a repetition of the ideas already anticipated in Chapter I in the proof of Proposition 2 (in §2.2, page 13). For a point $x^0$ of exterior density of $F_k$ and for a given $\varepsilon > 0$, we consider the "small" ball of center $x + y$, of radius $\varepsilon\,|y|$; also we consider the "large" ball of center $x$ and radius $|y| + \varepsilon\,|y|$. Now by that argument of point of density there exists a $z \in F_k$ belonging to the small ball, if $|y|$ is small enough. This means that within a distance of $\varepsilon\,|y|$ from the point $x^0 + y$ we can find a point of $F_k$. Thus $|b(x + y)| \leq k\varepsilon\,|y|$, by the definition of $F_k$, and since $\varepsilon$ is arbitrary this shows that $b(x + y) = o(|y|)$ as $y \to 0$.

**3.2 A refinement of the splitting.** Before we come to the next application we need to refine the conclusions of the splitting theorem in §2.2. Let $1 \leq q < \infty$, and suppose that $f$ has at every point of the set $E$ a derivative in the $L^q$ sense (as defined in §1.1), and thus $f$ has at every point of $E$ a derivative in the harmonic sense. Let $\varepsilon > 0$. According to the theorem in §2.2 we can find a set $F$, $F \subset E$, $m(E - F) < \varepsilon$ and $f = g + F$, so that $g \in L_1^\infty(\mathbf{R}^n)$ and $b(x^0) = 0$ for $x^0 \in F$. We can state more:

COROLLARY (OF THEOREM 2). *For almost every* $x^0 \in F$

$$\int_{|y| \leq 1} \frac{|b(x^0 + y)|}{|y|^{n-1}}\, dy < \infty. \tag{14}$$

*Proof.* In view of the fact that $g \in L_1^\infty(\mathbf{R}^n)$, we get that $g$ has at almost every point an ordinary (and thus also $L^q$) derivative. Hence $b$, which vanishes on $F$, has almost everywhere a derivative in the $L^q$ sense on $F$; that is in particular

$$\frac{1}{r^n}\int_{|y| \leq r} |b(x^0 + y)|^q\, dy = O(r^q), \quad \text{as} \quad r \to 0 \tag{15}$$

for almost every $x^0 \in F$.

We can also assume that after an easy modification (of the type made in the proof of Theorem 3) that $f \in L^q(\mathbf{R}^n)$. Thus after a further trivial modification of $g$ we can suppose that $b \in L^q(\mathbf{R}^n)$.

Now let $F_k$ be the *closed* set given by

$$F_k = \left\{x^0 : \frac{1}{r^n}\int_{|y| \leq r} |b(x^0 + y)|^q\, dy \leq kr^q,\ 0 < r \leq 1/k\right\}.$$

(Observe that in this case the sets $F_k$ are automatically measurable!)

We shall show that if $\delta(x)$ denotes the distance of $x$ from the closed set $F_k$ we have the inequality

$$\int_{|y|\leq 1} \frac{|b(x^0 + y)|\,dy}{|y|^{n+1}} \leq c \int_{|y|\leq 1} \frac{\delta(x^0 + y)\,dy}{|y|^{n+1}}, \qquad x^0 \in F \tag{16}$$

and therefore our corollary will follow from the theorem about the finiteness of the integral of Marcinkiewicz given in §2.3 of Chapter I, page 14.

To carry out the argument we write the complement of $F_k$ as a "disjoint" union of cubes $\{Q_j\}$ whose diameters are comparable to their distances from $F_k$, in accordance with Theorem 1 of Chapter VI.

We then have

$$\int_{|y|\leq 1} \frac{|b(x^0 + y)|\,dy}{|y|^{n+1}} = \int_{^cF_k \cap \{|x^0 - y| \leq 1\}} \frac{|b(y)|\,dy}{|x^0 - y|^{n+1}}$$
$$= \sum_j \int_{Q_j \cap \{|x^0 - y| \leq 1\}} \frac{|b(y)|\,dy}{|x^0 - y|^{n+1}}.$$

Now if $x^0 \in F_k$, then the quantity $\dfrac{1}{|x_0 - y|^{n+1}}$ is roughly constant in $y$, as $y$ ranges in $Q_j$; that is

$$\sup_{y\in Q_j} \frac{1}{|x^0 - y|^{n+1}} \leq c_1 \inf_{y\in Q_j} \frac{1}{|x^0 - y|^{n-1}} .$$

Also each $Q_j$ is contained in a ball $B_j$ centered at some point of $F_k$, with radius of $B_j$ comparable to the diameter of $Q_j$. Thus by the definition of $F_k$ it follows that

$$\int_{Q_j} |b(y)|^q\,dy \leq c_2 m(Q_j)^{1+q/n}$$

and then by Hölder's inequality

$$\int_{Q_j} |b(y)|\,dy \leq c_3 m(Q_j)^{1+1/n}.$$

Finally the diameter of $Q_j$ is comparable to the distance of any point in $Q_j$ from $F_k$. Thus $\int_{Q_j} |b(y)|\,dy \leq c_4 m(Q_j)\,\delta(y)$, $y \in Q_j$. Altogether then

$$\int_{Q_j \cap \{|x^0 - y| < 1\}} \frac{|b(y)|\,dy}{|x^0 - y|^{n+1}} \leq c \int_{Q_j \cap \{|x^0 - y| < 1\}} \frac{\delta(y)}{|x^0 - y|^{n-1}} .$$

Adding in $j$ we obtain (16) and hence our corollary.

REMARK. The argument above also proves that:

$$\int_{|y|\le 1} \frac{|b(x^0+y)|^q\,dy}{|y|^{n+q}} < \infty, \quad \text{for almost every } x^0 \in F. \tag{17}$$

The integral (17) is majorized by

$$\int_{|y|\le 1} \frac{(\delta(x^0+y))^q}{|y|^{n+q}}\,dy,$$

as the reasoning of the corollary shows.

At every point $x^0$ where (17) holds we evidently also have the conclusion

$$\frac{1}{r^n}\int_{|y|\le r} |b(x^0+y)|^q\,dy = o(r^q), \qquad r \to 0. \tag{18}$$

This has the interpretation that not only does $b$ vanish on $F$, but also at almost every point of $F$ the first derivatives of $b$ in the $L^q$ sense are zero.

**3.3 Preservation under the action of singular integrals.** We next consider a local analogue of the facts concerning the boundedness of singular integrals on operators $L^q(\mathbf{R}^n)$. We shall show that the pointwise notion of a derivative in the $L^q$ sense is stable almost everywhere under the action of appropriate singular integrals transformations.

We deal with the following class of operators; their kernels $K(x)$ satisfy:

(i) $K$ is of class $C^1$ away from the origin

(ii) $|K(x)| \le A/|x|^n$, $|\nabla K| \le A/|x|^{n-1}$, $x \ne 0$

(iii) if $T_\varepsilon(f) = \int_{|y|>\varepsilon} K(y)f(x-y)\,dy$, then for some fixed $q$,

$$\|T_\varepsilon(f)\|_q \le A_q\,\|f\|_q,$$

for $f \in L^q(\mathbf{R}^n)$, where $A_q$ is independent of $\varepsilon$. We also assume that $T_\varepsilon f$ converges in the $L^q$ norm to a limit $Tf$, as $\varepsilon \to 0$.

Among the examples of such transformation are those with $K(x) = \dfrac{\Omega(x)}{|x|^n}$ where $\Omega$ is homogeneous of degree 0, is of class $C^1$ on the unit sphere, and its mean-value on that sphere vanishes. This class of course includes the Riesz transforms, and the higher Riesz transforms of Chapter III.

THEOREM 4. *Let $1 < q < \infty$. Suppose $f \in L^q(R^n)$, and $f$ has a derivative in the $L^q$ sense at every point of a set $E$. If $T$ is a singular integral transformation of the above kind, then $Tf$ has a derivative in the $L^q$ sense at almost every point of $E$.*

It is natural to ask whether the notion of the ordinary derivative is also stable under the action of singular integral operators. However this is not so even for $n = 1$; see §6.8 below.

*3.3.1 Proof.* Let us remark first that if $f$ vanishes in a fixed neighborhood of a given point $x^0$, then it is easy to see that $Tf$, (after a suitable correction on a set of measure zero) has an ordinary derivative at every point of that neighborhood, and in particular at $x^0$. This remark shows that the question of whether $Tf$ is differentiable at $x_0$ depends only on the behavior of $f$ near $x_0$. We can therefore assume that the set $E$ is bounded. Using the splitting theorem we write $f = g + b$, where $g \in L_1^\infty(\mathbf{R}^n)$, $b = 0$ on $F$, $F \subset E$, with $m(E - F)$ small. It is easy to see that we can modify $g$ and $b$ in such a way so that $g$ has compact support without changing any other of the stated properties. Thus $g$ in addition belongs to $L_1^q(\mathbf{R}^n)$. Since $T$ is a bounded operator on $L^q$, we have therefore that $T(g)$ belongs to $L_1^q(\mathbf{R}^n)$. In fact the question whether a given $L^q$ function is in $L_1^q$ is a matter completely determined by the $L^q$ modulus of continuity of that function (see page 139); this is obviously preserved by $T$ because $T$ is bounded on $L^q$ and is translation-invariant.

Now that we know that $T(g) \in L_1^q(\mathbf{R}^n)$ we obtain from Theorem 1 of this chapter that $T(g)$ has at almost every point of $\mathbf{R}^n$ a derivative in the $L^q$ sense. It therefore suffices to consider $T(b)$ and show that it has an $L^q$ derivative at almost every point of the set $F$. We shall show that it indeed happens at every $x^0$ where the following two hold:

$$\text{(19)}\qquad \begin{aligned} &\frac{1}{r^n}\int_{|y|\le r} |b(x^0 + y)|^q \, dy = o(r^q), \quad \text{as} \quad r \to 0; \\ &\int_{|y|\le 1} \frac{|b(x^0 + y)|}{|y|^{n-1}} \, dy < \infty. \end{aligned}$$

We know by the discussion in §3.2 that (19) is valid for almost every $x^0$ in $F$. Observe that because $b \in L^q(\mathbf{R}^n)$ we get from the finiteness of the integral in (19) in addition that

$$\text{(20)}\qquad \int_{\mathbf{R}^n} \frac{|b(x^0 + y)|}{|y|^n} \, dy < \infty, \quad \text{and} \quad \int_{\mathbf{R}^n} \frac{|b(x^0 + y)|}{|y|^{n-1}} \, dy < \infty.$$

For simplicity of notation let us take $x^0 = 0$. Let us momentarily fix a positive number $r$. Then

$$(Tb)(x) = \int K(x - y)b(y)\, dy = \int_{|x-y|\ge 2r} K(x - y)b(y)\, dy$$
$$+ \lim_{\varepsilon\to 0} \int_{\varepsilon < |x-y| < 2r} K(x - y)b(y)\, dy = I_r^1(x) + I_r^2(x).$$

The integral giving $I_r^1(x)$ converges absolutely; $I_r^2$ converges in the $L^q$ norm by assumption, and $I_r^2 = (T - T_{2r})(b)$. Let us look at $I_r^2$ more carefully. If we make the restriction that $|x| \leq r$ then the integrand in $I_2^r$ involves only those $y$ for which $|y| \leq 3r$. Thus we can modify $b(y)$ by replacing it with $b(y)\chi_{3r}(y)$, where $\chi_{3r}$ is the characteristic function of the ball $|y| \leq 3r$. Hence

$$\int_{|x| \leq r} |I_r^2(x)|^q \, dx \leq \int_{\mathbf{R}^n} |(T - T_{2r})(b\chi_{3r})|^q \, dx \leq A_q^q \int_{\mathbf{R}^n} |b(y)\chi_{3r}(y)|^q \, dy$$

$$= A_q^q \int_{|y| \leq 3r} |b(y)|^q \, dy = o(r^{n+q}).$$

Here we have used the uniform boundedness of the $T_\varepsilon$ in $L^q(\mathbf{R}^n)$ norm, and the first property in (19) with $x^0 = 0$. We have therefore succeeded in showing that

$$\int_{|x| < r} |I_r^2(x)|^q \, dx = o(r^{n+q}), \quad \text{as} \quad r \to 0. \tag{21}$$

To study $I_r^1$ we use Taylor's expansion on $K$ in the form $K(x - y) = K(-y) + (x, \nabla K(-y)) + \varepsilon(r)\, |x|/|y|^{n+1}$. Here $|x| \leq r$, $|x - y| \geq r$, and $\varepsilon(r)$ tends to zero with $r$. We insert this in the integral defining $I_r^1$ and we have

$$I_r^1(x) = \int_{|x-y| \geq 2r} K(-y)b(y) \, dy + \int_{|x-y| \geq 2r} (x, \nabla K(-y))b(y) \, dy \tag{22}$$
$$+ \varepsilon'(r)\, |x| \int \frac{|b(y)|}{|y|^{n+1}} \, dy.$$

The first integral is equal to

$$\int_{\mathbf{R}^n} K(-y)b(y) \, dy + O\left(\int_{|y| < 3r} \frac{|b(y)|}{|y|^n} \, dy\right).$$

That $\int_{\mathbf{R}^n} K(-y)b(y)\, dy$ converges absolutely follows from the first inequality in (20) with $x^0 = 0$. We also get easily from (19) that

$$\int_{|y| \leq 3r} \frac{|b(y)|}{|y|^n} \, dy = o(r), \quad \text{as} \quad r \to 0.$$

The second integral in (22) is handled in the same way. It is equal to

$$\sum_{j=1}^n x_j \int_{\mathbf{R}^n} \frac{\partial K(-y)}{\partial y_j} b(y) \, dy + o(r),$$

as $r \to 0$ and these integrals converge by the second inequality in (20). For the same reason the third integral in (22) is also finite. Altogether then

$$I_r^1(x) = A_0 + \sum_{j=1}^n x_j A_j + o(|x|), \quad \text{as} \quad x \to 0 \tag{23}$$

with

$$A_0 = \int_{\mathbf{R}^n} K(-y)b(y)\,dy,\ A_j = \int_{\mathbf{R}^n} \frac{\partial K(-y)}{\partial y_j}\,b(y)\,dy, \quad j = 1, \ldots, n.$$

Combined with (21), we see that (23) implies our desired result.

## 4. *Desymmetrization principle*

We shall consider the idea of desymmetrization first in a rather general but abstruse form. Afterwards we will make various comments and give several illustrations to help clarify its meaning.

**4.1 A general theorem.** We shall be concerned with a function $U(x, y)$, $(x, y) \in \mathbf{R}^n \times \mathbf{R}^1_+ = \mathbf{R}^{n+1}_+$; $U$ is measurable in the $(n+1)$ dimensional upper-half-space, and for simplicity (since we are concerned with the behavior near $y = 0$) we shall assume that $U$ vanishes when $y \geq h$, for some fixed $h$, $h > 0$, and that $U$ is square integrable on every bounded subset of $\mathbf{R}^{n+1}_+$ which is at a positive distance from the boundary $\mathbf{R}^n$.

THEOREM 5. *Suppose that we are given a set $E \subset \mathbf{R}^n$, and that for every $x^0 \in E$ the following two conditions hold:*

$$\int_0^\infty y\,|U(x^0, y)|^2\,dy < \infty \tag{24}$$

$$\iint_{|t|<y} y^{1-n}\,|U(x^0 + t, y) + U(x^0 - t, y)|^2\,dt\,dy < \infty \tag{25}$$

*Then we can conclude that for almost every $x^0 \in E$ we have*

$$\iint_{|t|<y} y^{1-n}\,|U(x^0 + t, y)|^2\,dt\,dy < \infty. \tag{26}$$

By the usual arguments we may assume (upon reducing $E$ to a possibly smaller subset $F$) that $F$ itself is a compact set, and that the integrals (24) and (25) are uniformly bounded as $x^0$ ranges over $F$. It will then suffice to prove that (26) holds for almost every $x^0$ in $F$. That is we assume that

$$\int_0^\infty y\,|U(x^0, y)|^2\,dy \leq M, \quad \text{if} \quad x^0 \in F \tag{24'}$$

$$\iint_{|t|<y} y^{1-n}\,|U(x^0 + t, y) + U(x^0 - t, y)|^2\,dt\,dy \leq M, \qquad x^0 \in F. \tag{25'}$$

We integrate the inequality (25′) over $F$ and make the change of variables

$x^0 + t = u$ and $x^0 - t = v$. This gives

$$\iint_{(u+v)/2\in F} du\, dv \int_{|u-v|<2y} |U(u, y) + U(v, y)|^2\, y^{1-n}\, dy < \infty.$$

If we reduce the domains of integration by restricting the variable $v$ to lie in $F$ we certainly have

$$\iint_{(u+v)/2\in F,\, v\in F} du\, dv \int_{|u-v|<2y} |U(u, y) + U(v, y)|^2\, y^{1-n} dy < \infty. \tag{26}$$

Let us next integrate the inequality (24′) for $x^0$ over $F$, but making the change which is appropriate here, by setting $v$ in place of $x^0$. This gives

$$\int_{v\in F}\int_0^\infty |U(v, y)|^2\, y\, dy\, dv < \infty. \tag{27}$$

Now let

$$\begin{cases} I = \displaystyle\iint_{(u+v)/2\in F,\, v\in F} du\, dv \int_{|u-v|<2y} |U(v, y)|^2\, y^{1-n}\, dy \\ \mathscr{I} = \displaystyle\iint_{(u-v)/2\in F,\, v\in F} du\, dv \int_{|u-v|<2y} |U(u, y)|^2\, y^{1-n}\, dy. \end{cases} \tag{28}$$

We can majorize $I$ by dropping the restriction that $(u + v)/2 \in F$. Hence

$$\begin{aligned} I &\le \int_{\mathbf{R}^n} du \int_{v\in F} dv \int_{|u-v|<2y} |U(v, y)|^2\, y^{1-n}\, dy \\ &= \int_{v\in F}\int_0^\infty |U(v, y)|^2 \left\{\int_{|u-v|<2y} du\right\} dv\, y^{1-n}\, dy. \end{aligned}$$

Since the integral in brackets equals $cy^n$ we get that $I < \infty$ by (27). Therefore in view of (26) we have that $\mathscr{I} < \infty$.

Clearly $\mathscr{I}$ can be rewritten as a double integral,

$$\mathscr{I} = \int_{\mathbf{R}^n}\int_0^\infty |U(u, y)|^2\, \sigma(u, y)\, y^{1-n}\, du\, dy, \tag{29}$$

where

$$\sigma(u, y) = \int_{\substack{(u+v)/2\in F,\, v\in F \\ |u-v|<2y}} dv.$$

That is for fixed $y > 0$, and $u \in \mathbf{R}^n$, $\sigma(u, y)$ denotes the measure of the set of points $v$ in $\mathbf{R}^n$ lying in the ball of center $u$ and radius $2y$, and further restricted by the conditions that $v \in F$ and $(u + v)/2 \in F$.

The turning point of the proof will be to show that if $u^0$ is a point of density of $F$, then $\sigma(u, y) \sim m(B(u, 2y)) = c_1 y^n$, as the variable point $(u, y)$ tends to $(u^0, 0)$ non-tangentially.

Suppose for simplicity of notation that $u^0 = 0$ is a point of density of $F$. At this point $u^0$, the non-tangential approach will for us mean the restriction $|u| < y$. Let $\chi$ be the characteristic function of the set $F$ and $\tilde{\chi} = 1 - \chi$ that of the complementary set. Then

$$\sigma(u, y) = \int_{|u-v|<2y} \chi(v)\chi((u+v)/2)\,dv = \int_{|u-v|<2y} dv - \int_{|u-v|<2y} \tilde{\chi}(v)\,dv$$
$$- \int_{|u-v|<2y} \tilde{\chi}((u+v)/2)\,dv + \int_{|u-v|<2y} \tilde{\chi}(v)\tilde{\chi}((u+v)/2)\,du\,dv.$$

The second integral on the right-side of course equals $m(B(u, 2y)) = c_1 y^n$; it is therefore enough to show that the last three integrals are each $o(y^n)$. Recall that $|u| \le y$. Thus the second integral is dominated by

$$\int_{|v|\le 3y} \tilde{\chi}(v)\,dv = m({}^cF \cap B(0, 3y)).$$

which is $o(y^n)$ since 0 is a point of density of $F$. The third integral may be rewritten as $\int_{|u-x|<y} \tilde{\chi}(x)\,dx$, which is dominated by $\int_{|x|\le 2y} \tilde{\chi}(x)\,dx$ and is likewise $o(y^n)$. The fourth integral is of course dominated by the second integral and so is $o(y^n)$. Thus $\sigma(u, y) = c_1 y^n + o(y^n)$, as $y \to 0$ with $|u| < y$ and the assertion that $\sigma(u, y) \sim c_1 y^n$ is proved.

It follows that there is a closed subset $F_0$ of $F$ with $m(F - F_0)$ fixed but arbitrarily small, so that $\sigma(u, y) \ge c_2 y^n$, if $|u - u^0| < y$ and $0 < y < c_3$, for appropriate positive constants $c_2$ and $c_3$. Now let $\mathscr{R} = \bigcup_{u^0 \in F^0} \Gamma_1^{c_3}(u^0)$, where $\Gamma_1^{c_3}(u^0)$ is the truncated cone given by $\Gamma_1^{c_3}(u_3) = \{(u, y) : |u - u_0| < y, 0 < y < c_3\}$. The finiteness of (29) and what we have proved about $\sigma$ then implies the finiteness of

$$\iint_{\mathscr{R}} |U(u, y)|^2\, y\, du\, dy.$$

However, by a very simple reasoning we have already used in Chapter VI, (see page 208) the finiteness of the last integral implies the finiteness for almost every $u^0 \in F_0$ of

$$\iint_{\substack{|u-u^0|<y \\ y<c_3}} |U(u, y)|^2\, y^{1-n}\, du\, dy = \iint_{\substack{|t|<y \\ y<c_3}} |U(u^0 + t, y)|^2\, y^{1-n}\, dt\, dy.$$

The full integral, $\iint_{|t|<y} |U(u^0 + t, y)|^2 y^{1-n}\, dt\, dy$ then converges because

of the local square-integrability of $U$ that we assumed. Since the set $F_0$ differed from $F$ by a subset of arbitrarily small measure the proof of the theorem is therefore complete.

## 4.2 Remarks.

*4.2.1* The first set of comments are of a rather trifling nature. By the usual arguments (or by a closer examination of the proof) it can be seen that the assumption (25) can be replaced by the weaker assumption that

$$\iint_{|t|<\alpha y} y^{1-n}\, |U(x^0+t, y) + U(x^0-t, y)|^2\, dt\, dy < \infty$$

where the constant $\alpha$ which determines the opening of the cone may vary with the point $x^0$. One can also obtain the conclusion (which is only stronger in appearance) that

$$\iint_{|t|<\beta y} y^{1-n}\, |U(x^0+t, y)|^2\, dt\, dy < \infty$$

for every $\beta$, as $x^0$ ranges over almost all points of $E$.

There are also simple (and immediate) variants of the theorem where the sum $U(x^0+t, y) + U(x^0-t, y)$ is replaced by other combinations such as $U(x^0+t, y) - U(x^0-t, y)$; also the square which appears may be replaced in (24), (25), and (26) with $y|U(x^0, y)|^p$, $y^{1-n}\, |U(x^0+t, y) + U(x^0-t, y)|^p$ and $y^{1-n}\, |U(x^0+t, y)|^p$ respectively, with $p < \infty$. The variant for $p = \infty$ is stated separately below.

*4.2.2* Suppose that for every $x^0$ in a set $E$

$$\sup_{y>0} |U(x^0, y)| < \infty \quad \text{and} \quad \sup_{|t|<y} |U(x^0+t, y) + U(x^0-t, y)| < \infty.$$

Then for almost every $x^0$ in $E$, $\sup_{|t|<y} |U(x^0+t, y)|$ is also finite. The proof of this statement is similar to that of Theorem 5, and is conceptually even simpler. The reader should have no difficulty in carrying out the details of the argument. The one point here which may involve non-measurable sets is disposed of as in §3.1.1.

*4.2.3* There is a consequence of the statement in §4.2.2 which is worth treating separately. Suppose $f$ is defined near $x^0$. We consider the significant condition

$$f(x^0+t) + f(x^0-t) - 2f(x^0) = O(|t|) \quad \text{as} \quad t \to 0. \tag{30}$$

This condition is certainly satisfied if $f$ has an ordinary derivative at $x^0$, but the converse is not true. It can be shown, moreover, that there are $f$ which satisfy (30) uniformly for all $x^0$ without having an ordinary derivative at any $x^0$.

It turns out that while (30) by itself does not imply differentiability, it does play the role of a "Tauberian condition," allowing one to refine one form of differentiability to another form. We formulate one such result.

COROLLARY. *Suppose that $f$ has a derivative in the harmonic sense at every point $x^0$ of a given set $E$. (This would hold in particular if $f$ were differentiable in the $L^q$ sense for each $x^0 \in E$). Assume also that (30) holds for each $x^0 \in E$. Then $f$ has an ordinary derivative for almost every $x^0 \in E$.*

The proof is an easy application of what we have already done. By the splitting theorem in §2.2 we can write $f = g + b$, where $g$ is differentiable almost everywhere in the ordinary sense, and where $b$ vanishes on $F$, with $F \subset E$, and $m(E - F)$ small. It will suffice to show that $b$ has an ordinary derivative almost everywhere on $F$. Since $g$ is differentiable almost everywhere it satisfies condition (30) almost everywhere, and so $b$ satisfies condition (30) at almost every point of $F$. The crucial fact is that $b$ vanishes on $F$. Therefore the conditions becomes

$$b(x^0 + t) + b(x^0 - t) = O(|t|), \qquad t \to 0.$$

Now let $U(x, y) = \dfrac{b(x)}{y}$. Thus by the vanishing of $b$, we have

$$\sup_{y>0} |U(x^0, y)| = 0$$

for almost every $x^0 \in F$, and

$$\sup_{|t|<y} |U(x^0 + t, y) + U(x^0 - t, y)| < \infty,$$

for almost every $x^0 \in F$. By the statement in §4.2.2 we conclude that

$$\sup_{|t|<y} |U(x^0 + t, y)| < \infty,$$

which is

$$\sup_{t\neq 0} \frac{|b(x^0 + t)|}{|t|} < \infty,$$

i.e.

$$b(x^0 + t) = O(|t|), \quad \text{as} \quad t \to 0,$$

again for almost all $x^0 \in F$. Theorem 3 in §3.3 now allows us to conclude that $b$ has an ordinary derivative almost everywhere in $F$.

### 5. Another characterization of differentiability

**5.1** The theorem we intend to prove is as follows.

THEOREM 6. *Suppose $f \in L^2(\mathbf{R}^n)$. Then for almost every $x^0 \in \mathbf{R}^n$ the following two conditions are equivalent:*

(i) *$f$ has a derivative in the $L^2$ sense at $x^0$.*

(ii) $$\int_{\mathbf{R}^n} \frac{|f(x^0+t)+f(x^0-t)-2f(x^0)|^2\,dt}{|t|^{n+2}} < \infty.$$

A consequence of this theorem is the following corollary.

COROLLARY. *Suppose $f$ is given in an open neighborhood of a set $E$. Then $f$ has an ordinary derivative at almost every $x^0 \in E$, if and only if the following two conditions hold for almost every $x^0 \in E$.*

(a) $f(x^0+t)+f(x^0-t)-2f(x^0) = O(|t|)$, *as* $t \to 0$

(b) $$\int_{|t|\leq\delta} \frac{|f(x^0+t)+f(x^0-t)-2f(x^0)|^2\,dt}{|t|^{n+2}} < \infty$$

*where $\delta$ is a sufficiently small positive number (depending on $x^0$).*

**5.2** We shall need to establish a fact which may be viewed as the "global" analogue of our theorem (the relevant notions of global and local are discussed in the introductory remarks of this chapter).

PROPOSITION. *Suppose $f \in L_1^2(\mathbf{R}^n)$. Then*

$$(31)\qquad \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \frac{|f(x+t)+f(x-t)-2f(x)|^2}{|t|^{n+2}}\,dx\,dt = a_n\int_{\mathbf{R}^n}\sum_{j=1}^n \left|\frac{\partial f}{\partial x_j}\right|^2 dx.$$

The proof of this is best carried out by the Fourier transform. Thus let

$$\hat f(y) = \int_{\mathbf{R}^n} f(x)e^{2\pi i x\cdot y}\,dx, \qquad f(x) = \int_{\mathbf{R}^n}\hat f(y)e^{-2\pi i x\cdot y}\,dy,$$

taken in the $L^2$ sense. By Plancherel's theorem, and the properties of the Sobolov spaces $L_1^2(\mathbf{R}^n)$ we then have

$$\int_{\mathbf{R}^n}\left|\frac{\partial f}{\partial x_j}\right|^2 dx = 4\pi^2\int_{\mathbf{R}^n}|x_j|^2\,|\hat f(x)|^2\,dx.$$

Also

$$\int_{\mathbf{R}^n}|f(x+t)+f(x-t)-2f(x)|^2\,dx = 4\int_{\mathbf{R}^n}|\hat f(x)|^2\,|1-\cos 2\pi x\cdot t|^2 dx.$$

Hence

$$\int_{|t|>\varepsilon}\int_{\mathbf{R}^n} |f(x+t)+f(x-t)-2f(x)|^2\,dx\,\frac{dt}{|t|^{n+2}} = \int_{\mathbf{R}^n} |\hat{f}(x)|^2\,\mathscr{I}_\varepsilon(x)\,dx,$$

with

$$\mathscr{I}_\varepsilon(x) = 4\int_{|t|>\varepsilon} \frac{|1-\cos 2\pi x\cdot t|^2}{|t|^{n+2}}\,dt.$$

Now it is clear that as $\varepsilon \to 0$, $\mathscr{I}_\varepsilon(x)$ increases monotonically to

$$\mathscr{I}(x) = 4\int_{\mathbf{R}^n} \frac{|1-\cos 2\pi x\cdot t|^2}{|t|^{n+2}}\,dt,$$

which is obviously radial in $x$ and homogeneous of degree 2. Thus

$$\mathscr{I}(x) = b_n\,|x|^2, \quad \text{with} \quad b_n = 4\int_{\mathbf{R}^n} \frac{|1-\cos 2\pi t_1|^2}{|t|^{n+2}}\,dt.$$

Hence the left-side of (31) is $b_n \int_{\mathbf{R}^n} |x|^2\,|\hat{f}(x)|^2\,dx$, while the right side is $a_n 4\pi^2 \int_{\mathbf{R}^n} |x|^2\,|\hat{f}(x)|^2\,dx$. The proposition is therefore proved with $a_n = b_n/4\pi^2$. The reader should compare this proposition with Proposition 5 in §3.5 of Chapter V.

**5.3** A moment's reflection shows that it suffices to prove the theorem, and its corollary, under the assumption that $f$ vanishes outside a bounded set. We shall therefore make this assumption about $f$. Assume that $f$ has a derivative in the $L^2$ sense at each point of a set $E$. Then in view of the fact that this implies that $f$ has a harmonic derivative at each such point we get by Theorem 2 in §2.2 that we can split $f$ as $f = g + b$ with $g \in L_1^\infty(\mathbf{R}^n)$, and $b = 0$ on $F$, where $F \subset E$ and $m(E - F)$ is small. Nothing will be changed if we assume that $g$ (and thus $b$) also vanishes outside a bounded set. Since $g$ has bounded support then $g \in L_1^\infty(\mathbf{R}^n)$, and therefore by the above proposition

$$\int_{\mathbf{R}^n} \frac{|g(x^0+t)+g(x^0-t)-2g(x^0)|^2}{|t|^{n+2}}\,dt < \infty \tag{32}$$

for almost every $x^0 \in \mathbf{R}^n$.

Further, $g$ also has an ordinary derivative at almost every point in $\mathbf{R}^n$ (see Theorem 1), and therefore $b$ has a derivative in the $L^2$ sense for almost every point of $E$, and hence for almost every point of $F$. Since $b$ vanishes on $F$ we get by (17) in §3.2 that $\displaystyle\int_{|t|<1} \frac{|b(x^0+t)|^2}{|t|^{n+2}}\,dt < \infty$, for almost every $x^0$ in $F$. Since in any case $b \in L^2(\mathbf{R}^n)$ we see that for these $x^0$,

$$\int_{\mathbf{R}^n} \frac{|b(x^0 + t)|^2}{|t|^{n+2}} \, dt < \infty. \tag{33}$$

If we combine this with the fact that $b(x^0) = 0$ and (32) we obtain the conclusion (ii) of the theorem.

**5.4** We now come to the converse part of Theorem 6. Let $u(x, y) = P_y * f$ be the Poisson integral of $f$, and consider $\dfrac{\partial^2 u}{\partial y^2}$. For this purpose we need a favorable estimate of $\dfrac{\partial^2 P_y}{\partial y^2}$. Since

$$\frac{\partial^2 P_y}{\partial y^2} = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} (P_y(x)),$$

it suffices to observe that

$$\left| \frac{\partial^2}{\partial x_j^2} P_y(x) \right| \le y^{-n-2} \psi(x/y) \quad \text{where} \quad \psi(x) \le A(1 + |x|)^{-n-3}.$$

We remark also that

$$\int_{\mathbf{R}^n} \frac{\partial^2}{\partial y^2} P_y(x) \, dx = 0,$$

and what is crucial, that $\dfrac{\partial^2 P_y}{\partial y^2}(x)$ is radial, and hence even in $x$. In view of these facts

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{2} \int_{\mathbf{R}^n} \frac{\partial^2}{\partial y^2} P_y(t) [f(x + t) + f(x - t) - 2f(x)] \, dt \tag{34}$$

and hence

$$\left| \frac{\partial^2 u}{\partial y^2} \right| \le A' y^{-n-2} \int_{|t| \le y} |\Delta_t| \, dt + A' y \int_{|t| \ge y} \frac{|\Delta_t|}{|t|^{n+3}} \, dt = I_1(y) + I_2(y). \tag{35}$$

Here we have set $\Delta_t = f(x + t) + f(x - t) - 2f(x)$. We estimate $\displaystyle\int_0^\infty y \left| \frac{\partial^2 u}{\partial y^2} \right|^2 dy$ in terms of similar integrals for $I_1$ and $I_2$. By Schwarz's inequality,

$$|I_1(y)|^2 \le (A')^2 y^{-2n-4} \left( \int_{|t| \le y} \frac{|\Delta_t|^2 \, dt}{|t|^{n+1}} \right) \left( \int_{|t| \le y} |t|^{n-1} \, dt \right)$$

$$= B y^{-3} \int_{|t| \le y} \frac{|\Delta_t|^2}{|t|^{n+1}} \, dt.$$

Therefore

$$\int_0^\infty y\,|I_1(y)|^2\,dy \le B\int_0^\infty y^{-2}\int_{|t|\le y}\frac{|\Delta_t|^2}{|t|^{n+1}}\,dt\,dy = B\int_{\mathbf{R}^n}\frac{|\Delta_t|^2}{|t|^{n+2}}\,dt.$$

A similar argument works for $I_2$ and altogether this gives

$$(36)\qquad \int_0^\infty y\left|\frac{\partial^2 u}{\partial y^2}\right|^2 dy \le B'\int_{\mathbf{R}^n}\frac{|f(x+t)+f(x-t)-2f(x)|^2}{|t|^{n+2}}\,dt.$$

In view of the evenness of $\dfrac{\partial^2 P}{\partial y^2}$ we can also modify (34) and write it in a more extended form

$$(34')\qquad \frac{\partial^2 u}{\partial y^2}(x+\tau,y)+\frac{\partial^2 u}{\partial y^2}(x-\tau,y)$$
$$=\int_{\mathbf{R}^n}\frac{\partial^2}{\partial y^2}P_y(t+\tau)[f(x+t)+f(x-t)-2f(x)]\,dt,$$

where $\tau$ is arbitrary. Now if $|\tau|\le y$ we can make similar estimates on $\dfrac{\partial^2}{\partial y^2}P_y(t+\tau)$ and get that

$$(35')\qquad \left|\frac{\partial^2 u}{\partial y^2}(x+\tau,y)+\frac{\partial^2 u}{\partial y^2}(x-\tau,y)\right| \le A\{I_1(y)+I_2(y)\},\quad \text{if } |\tau|\le y.$$

Therefore by the same argument we obtain instead of (36) the conclusion that

$$(36')\qquad \iint_{|\tau|\le y}\left|\frac{\partial^2 u}{\partial y^2}(x+\tau,y)+\frac{\partial^2 u}{\partial y^2}(x-\tau,y)\right|^2 y^{1-n}\,d\tau\,dy$$
$$\le B\int_{\mathbf{R}^n}\frac{|f(x+t)+f(x-t)-2f(x)|^2}{|t|^{n+2}}\,dt.$$

We can now invoke the desymmetrization theorem in §4.1 at each point where the condition (ii) of the statement of Theorem 6 is valid. Here we have set $U(x,y)=\dfrac{\partial^2 u}{\partial y^2}(x,y)$ with $U=0$, if $y>1$. The conclusion is that at almost all such points $x^0$

$$(37)\qquad \iint_{|\tau|\le y,\,y\le 2}\left|\frac{\partial^2 u}{\partial y^2}(x^0+\tau)\right|^2 y^{1-n}\,d\tau\,dy<\infty$$

and therefore by the theory of the previous chapter (see §2.5, in particular p. 213), $\dfrac{\partial u}{\partial y}$ has a non-tangential limit for almost all such $x^0$; and finally because of §2.5 in Chapter VII, $u$ has a harmonic derivative at almost all points where condition (ii) holds.

We can now split $f$ as $g + b$, where $g$ is in $L_1^\infty(\mathbf{R}^n)$, and by the above we know that the integral condition holds almost everywhere for $g$ instead of $f$. Thus it holds for almost all relevant points for $b$; but $b$ also vanishes for these points. In summary we obtain that

$$(38) \quad \int_{\mathbf{R}^n} \frac{|b(x^0 + t) + b(x^0 - t)|^2}{|t|^{n+2}} \, dt < \infty, \qquad b(x^0) = 0, \quad x^0 \in F$$

where $F \subset E$, and $m(E - F)$ is small.

We now again involve the desymmetrization theorem, this time with $U(x, y) = y^{-2} b(x)$, if $y < 1$. Then we in fact have

$$\int_0^\infty y \, |U(x^0, y)|^2 \, dy = 0, \qquad x^0 \in F$$

and

$$\iint_{|t| < y} y^{1-n} \, |U(x^0 + t, y) + U(x^0 - t, y)|^2 \, dt \, dy < \infty,$$

as easy consequence of (38). The result is then that

$$\int_{|t| < 1} \frac{|b(x^0 + t)|^2}{|t|^{n+2}} \, dt < \infty,$$

for almost all $x^0$ in $F$, and at such points clearly

$$\int_{|t| < r} |b(x^0 + t)|^2 \, dt = o(r^{n+2}), \quad \text{as} \quad r \to 0$$

which means in particular that $b$ has a derivative in the $L^2$ sense. The converse is therefore also proved. The corollary is an immediate consequence of the theorem and the corollary in §4.2.3, page 261.

## 6. *Further results*

**6.1** Most of the results in this chapter have analogues for derivatives of higher order. The appropriate definitions are as follows. Let $k$ be integral, $k \geq 1$. We say that $f$ has an *ordinary derivative of order* $k$ at $x^0$ if there exists a polynomial of degree $\leq k$ in $y$, $P_{x_0}(y)$, so that

$$f(x^0 + y) - P_{x_0}(y) = o(|y|^k), \quad \text{as} \quad y \to 0.$$

Similarly $f$ will be said to have a *derivative of order* $k$ *in the* $L^q$ *sense* if

$$h^{-n} \int_{|y| < h} |f(x^0 + y) - P_{x_0}(y)|^q \, dy = o(h^{kq}),$$

as $h \to 0$. Finally if we assume $f$ to be integrable near $x^0$, and set equal to zero outside a neighborhood of $x^0$, we shall say that it has a *harmonic derivative of order* $k$ at $x^0$ if the $\left\{\frac{\partial^\alpha u}{\partial x^\alpha}(x, y)\right\}_{|\alpha| \leq k}$, all have non-tangential limits at $x^0$.

The generalization of Theorem 1 is the statement: If $1 < p \leq \infty$, and $f \in L_k^p(\mathbf{R}^n)$, then $f$ has an ordinary derivative of order $k$ for almost all points of $\mathbf{R}^n$, if $p > n/k$; however if $p < n/k$, then $f$ has a derivative of order $k$ in the $L^q$ sense for almost all points, where $1/q = 1/p - k/n$. The generalization of Theorem 2 is the splitting $f = g + b$, where $g \in L_k^\infty(\mathbf{R}^n)$, $b$ vanishes on $F$, where $F \subset E$, $m(E - F) < \epsilon$, and $f$ is assumed to have a derivative of order $k$ in the harmonic sense for all points of $E$. In generalizing Theorem 3 it suffices to assume that for each $x_0 \in E$, there exists a polynomial of degree $\leq k - 1$ in $y$, $\tilde{P}_{x_0}(y)$, so that

$$f(x + y) - \tilde{P}_{x_0}(y) = O(|y|^k), \quad \text{as} \quad y \to 0.$$

Then $f$ has an ordinary derivative of order $k$ for almost every $x^0 \in E$.

The condition (ii) for the kernel $K(x)$ which arises in Theorem 4 should be amplified to read

$$|\nabla^r K(x)| \leq A/|x|^{n+r}, \qquad 0 \leq r \leq k;$$

with this modification the singular integrals in question also preserve almost everywhere the differentiability of order $k$ in the $L^q$ sense.

Finally if $f \in L^2(\mathbf{R}^n)$ the condition characterizing the existence of derivatives of order $k$ in the $L^2$ sense is given by the finiteness for almost all points in question of

$$\int_{\mathbf{R}^n} \frac{|\Delta^k_{x_0}(t)|^2}{|t|^{n+2k}}\, dt.$$

Here $f(x^0 + t) - P_{x_0}(t) = R_{x_0}(t)$, and $\Delta^k_{x_0}(t) = R_{x_0}(t) + (-1)^{k-1} R_{x_0}(-t)$. For the above see Calderón and Zygmund [7], Stein and Zygmund [1], and Stein [8].

**6.2** The result of Theorem 1 holds also for $p = 1$, but requires a different argument. (Compare however the inequality in §2.5 of Chapter V, and the identity (18) in §2.3 of that chapter.) A further consequence of the above is the following theorem. If $f$ is of bounded variation on $\mathbf{R}^n$ in the sense of Tonelli, then for almost every point $f$ has a derivative in the $L^q$ sense, where

$$q = n/(n - 1).$$

See Calderón and Zygmund [6].

**6.3** The splitting theorem (of §2.2) can be given in somewhat sharper form if we assume differentiability in the sense of $L^q$. We formulate the result for derivatives of order 1. Suppose $f \in L^q(\mathbf{R}^n)$, $1 \leq q$, and for each point $x^0 \in F$, with $F$ compact,

$$h^{-n} \int_{|y| \leq h} |f(x^0 + y) - f(x^0)|^q \, dy \leq A h^q, \qquad 0 < h < \infty$$

with $A$ independent of $x^0$. Then $f = g + b$, where $g$ is continuously differentiable and it and its first partial derivatives are bounded; $b = 0$ on $F$. For the general formulation for higher derivatives and further details see Calderón and Zygmund [8].

**6.4** In view of the measurability difficulties discussed in §3.1.1 it may be of interest to state the following theorem. Suppose $f$ is Lebesgue measurable on $\mathbf{R}^n$. Then the set of points where $f$ has an ordinary derivative is Lebesgue measurable (see Haslam-Jones [1]).

The fact that the set $E = \left\{x\colon \limsup_{h\to 0}\left|\frac{f(x+h)-f(x)}{h}\right| < \infty\right\}$ is Lebesgue measurable may be seen as follows: For each integer $k$ let

$$E_k = \{x : |f(x+h) - f(x)| \le k\,|h|, \quad \text{for all} \quad |h| < 1/k\}.$$

Then $E = \bigcup E_k$; also because we have taken the open balls $|h| < 1/k$, it follows that $f|_{E_k}$ is continuous and $E_k$ is closed.*

**6.5** Suppose $k$ is odd. We shall say that $f$ has a *symmetric derivative of order $k$, in the $L^q$ sense*, at $x^0$, if there exists a polynomial $P_{x_0}(y)$, in $y$ of degree $\le k$, such that

$$h^{-n}\int_{|y|\le h} |f(x_0+y) - f(x_0-y) - P_{x_0}(y)|^q\,dy = o(h^{kq})$$

as $y \to 0$. If $k$ is even we make the same definition except that $f(x_0 + y) - f(x_0 - y)$ must be replaced by $f(x_0 + y) + f(x_0 - y)$.

THEOREM: *Suppose that for each $x^0 \in E$, $f$ has a symmetric derivative of order $k$ in the $L^q$ sense. Then for almost every $x^0$ in $E$, $f$ has a derivative of order $k$ in the $L^q$ sense.*

For $n = 1$, $k = 1$, and for the ordinary notion of differentiability, this theorem goes back to Khintchine [1]. For the one-dimensional form see M. Weiss [1]. The general formulation can be deduced from the methods of the present chapter. Assume for example that $k = 1$. Let $u$ be the Poisson integral of $f$ and consider $\frac{\partial u}{\partial x_j} = \frac{\partial P_y}{\partial x_j} * f$. Since the kernel $\frac{\partial P}{\partial x_j}$ is odd we can write

$$\frac{\partial u}{\partial x_j}(x+\tau, y) + \frac{\partial u}{\partial x_j}(x-\tau, y) = \int_{\mathbf{R}^n} \frac{\partial}{\partial x_j} P_y(t+\tau)[f(x+t) - f(x-t)]\,dt$$

and our assumption shows that $\frac{\partial u}{\partial x_j}(x^0+\tau, y) + \frac{\partial u}{\partial x_j}(x^0-\tau, y)$ tends to a limit as $|\tau| \le y$ and $y \to 0$, for $x^0 \in E$. Thus by the desymmetrization theorem in §4.2.2 we get that $\frac{\partial u}{\partial x_j}$ is non-tangentially bounded at $x^0$, for almost all $x^0 \in E$. Therefore $f$ has a harmonic derivative of order 1 almost everywhere in $E$ and hence by the splitting lemma one can reduce matters to the special case where $f$ vanishes on $E$.

* I am indebted to H. Federer for this argument.

More generally for $k$ even we consider $\dfrac{\partial^k u}{\partial y^k}(x+\tau, y) + \dfrac{\partial^k}{\partial y^k} u(x-\tau, y)$; while for odd $k$ one takes

$$\frac{\partial^k}{\partial y^{k-1}\,\partial x_j} u(x+\tau, y) + \frac{\partial^k}{\partial y^{k-1}\,\partial x_j} u(x-\tau, y), \qquad j = 1, \ldots, n;$$

then one applies the results of Chapter VII to show that $f$ has almost everywhere on $E$ a harmonic derivative of order $k$, thus reducing matters to the special case where $f$ vanishes on $E$. For this special case, which is much easier, the kind of arguments used in §4 apply; see for instance Stein and Zygmund [1, Lemma 14].

**6.6** Suppose $f$ has a derivative in the $L^q$ sense at each $x^0 \in E$. Then the first partial derivatives in the $L^q$ sense of $f$ exist for almost every $x^0 \in E$. More particularly if

$$\frac{1}{h^n}\int_{|y|\le h} |f(x^0+y) - f(x^0) - \sum \alpha^j_{x^0} y_j|^q\, dy = o(h^q), \qquad h \to 0$$

for $x^0 \in E$, then if $e_j$ is the unit vector $(0, \ldots, 1, 0, \ldots, 0)$

$$\frac{1}{h}\int_{|y_j|\le h} |f(x^0+e_j y_j) - f(x^0) - \alpha^j_{x^0} y_j|^q\, dy_j = o(h^q)$$

for almost all $x^0 \in E$. For this and related results see M. Weiss [2] and [3].

**6.7** Suppose $2 \le q < \infty$. Let $f \in L^q(\mathbf{R}^n)$. Then $f$ has a derivative of order one in the $L^q$ sense for almost all $x^0 \in E$ if and only if the following two conditions are satisfied for almost all $x^0 \in E$:

$$(1)\quad \int_{\mathbf{R}^n} \frac{|f(x^0+t) + f(x^0-t) - 2f(x^0)|^q}{|t|^{n+q}}\, dt < \infty$$

$$(2)\quad \int_{\mathbf{R}^n} \frac{|f(x^0+t) + f(x^0-t) - 2f(x^0)|^2}{|t|^{n+2}}\, dt < \infty.$$

See Stein and Zygmund [1]; also Wheeden [1], Neugebauer [1].

**6.8** The following example shows that Theorem 4 in §3.3 does not extend to the notion of the ordinary derivative (i.e. the case $q = \infty$). We consider the case of $\mathbf{R}^1$, with $T$ the Hilbert transform.

Consider first a function $F_0(x)$ defined on $\mathbf{R}^1$ with the properties: (1) $F_0(x) = (\log 1/|x|)^{1-\varepsilon}$ for $|x| \le 1/2$, $(0 < \varepsilon < 1)$; $F_0(x)$ vanishes outside a compact set and is smooth away from the origin; also $F_0(x) \ge 0$, $x \in \mathbf{R}^1$. Let $\tilde{F}_0(x)$ denote the Hilbert transform of $F_0$. It is not difficult to see that $\tilde{F}_0$ (after a suitable modification on a set of measure zero) is absolutely continuous on $\mathbf{R}^1$ and $\dfrac{d\tilde{F}_0}{dx} \in L^1(\mathbf{R})$. Write $F(x) = \sum 2^{-k} F_0(x + r_k)$, where $r_1, r_2, r_k, \ldots$ is an enumeration of the rationals. Then $\tilde{F} = \sum 2^{-k} \tilde{F}_0(x + r_k)$ is the Hilbert transform of $F$

and is absolutely continuous, with $\dfrac{d\tilde{F}}{dx} \in L^1(\mathbf{R}^1)$. Thus $\tilde{F}$ has an ordinary derivative for almost every $x$, but since $F$ is unbounded near every point, it has an ordinary derivative nowhere.

For the periodic analogue of $F_0$ we may take $f_0$ with

$$f_0(x) \sim \sum_{n>1} \frac{\cos nx}{n(\log n)^\varepsilon}, \qquad \tilde{f}_0(x) \sim \sum_{n>1} \frac{\sin nx}{n(\log n)^\varepsilon}.$$

For a treatment of these series see Zygmund [8, Chapter V].

## *Notes*

*Section 1.* The notion of a function differentiable at a given point in the $L^p$ sense was first studied systematically by Calderón and Zygmund [7]. Part (b) of Theorem 1 is due to them, but part (a) dealing with the ordinary derivative is older; see for example Cesari [1]. The reader is also referred to the work of Federer [1] for a variety of topics related to the material in this chapter.

*Section 2.* The idea of splitting of functions, in the context of ordinary differentiability of one variable appears first in Marcinkiewicz [1]. This basic technique was extended to $n$-dimensions in Calderón and Zygmund [7]; the presentation given here, which is based on the theory of harmonic functions, has several essential advantages over the previous methods. It is due to Zygmund and the author, and is sketched in the author's survey, Stein [8].

*Section 3.* Theorem 3 is a famous theorem of Denjoy, Rademacher and Stepanov; see Saks [2, Chapter IX].

The proof given here is, of course, not the standard one, relying as it does on the notion of the harmonic derivative. For §3.2, as well as variants of Theorem 4, see Calderón and Zygmund [7].

*Section 5.* The original proof of Theorem 6 and its corollary is in Stein and Zygmund [1]; see also Wheeden [1]. The argument given here is also due to Zygmund and the author, and is sketched in Stein [8].

# Appendices

## A. Some inequalities

We shall summarize here some well-known inequalities that are used systematically above. Further details may be found in Zygmund [8], Chapter I, and Hardy, Littlewood, and Polya [1].

**A.1** Minkowski's inequality for integrals states in effect that the norm of an integral is not greater than the integral of the corresponding norms. In explicit form, for the case of $L^p$ spaces, this can be restated as follows. Let $1 \leq p < \infty$, then

$$\left(\int_{\mathscr{Y}} \left(\int_{\mathscr{X}} |F(x, y)| \, dx\right)^p dy\right)^{1/p} \leq \int_{\mathscr{X}} \left(\int_{\mathscr{Y}} |F(x, y)|^p \, dy\right)^{1/p} dx.$$

Here $F(x, y)$ is a measurable function on the $\sigma$-finite product measure space $\mathscr{X} \times \mathscr{Y}$; $dx$ and $dy$ are the measures on $\mathscr{X}$ and $\mathscr{Y}$ respectively.

**A.2** Young's inequality for convolutions is as follows: Let $h = f * g$, then

$$\|h\|_q \leq \|f\|_p \|g\|_r,$$

where $1 \leq p, q, r \leq \infty$ and $1/q = 1/p + 1/r - 1$.

Two noteworthy special cases arise; first $r = 1$, then $p = q$; also when $r$ is the conjugate index to $p$ (namely $1/p + 1/r = 1$), then $q = \infty$. In this case it can also be shown that $h$ is continuous.

**A.3** The following is a general integral inequality of wide application.

Let $(Tf)(x) = \int_0^\infty K(x, y) f(y) \, dy$. Here $K$ is assumed to be homogeneous of degree $-1$; that is $K(\lambda x, \lambda y) = \lambda^{-1} K(x, y)$, for $\lambda > 0$. In addition we assume that $\int_0^\infty |K(1, y)| \, y^{-1/p} \, dy = A_K < \infty$. Then

$$\|Tf\|_p \leq A_K \|f\|_p.$$

Here the norms $\|\cdot\|_p$ are those of $L^p(0, \infty; dx)$, and $1 \leq p \leq \infty$.

To prove this we write $(Tf)(x) = \int_0^\infty K(1, y) f(yx) \, dy$ and use Minkowski's inequality for integrals.

An interesting special case, the *Hilbert integral*, arises if $K(x, y) = \dfrac{1}{x + y}$.

**A.4** Another useful instance of §A.3 is the pair of inequalities due to Hardy

$$\left(\int_0^\infty \left(\int_0^x f(y)\,dy\right)^p x^{-r-1}\,dx\right)^{1/p} \le p/r\left(\int_0^\infty (yf(y))^p y^{-r-1}\,dy\right)^{1/p},$$

$$\left(\int_0^\infty \left(\int_x^\infty f(y)\,dy\right)^p x^{r-1}\,dx\right)^{1/p} \le p/r\left(\int_0^\infty (yf(y))^p y^{r-1}\,dy\right)^{1/p}.$$

Here $f \geq 0$, $p \geq 1$, and $r > 0$.

## *B. The Marcinkiewicz interpolation theorem*

**B.1** We extend here the theorem given in §4 of Chapter I. We assume that $p_0, p_1, q_0, q_1$ are given exponents, with $1 \leq p_i \leq q_i \leq \infty$, $p_0 < p_1$, and $q_0 \neq q_1$. $T$ is a sub-additive transformation which is defined on $L^{p_0}(\mathbf{R}^n) + L^{p_1}(\mathbf{R}^n)$. We recall the definition that $T$ is of *weak type* $(p_i, q_i)$. This means that there is a constant $A_i$, so that for every $f \in L^{p_i}(\mathbf{R}^n)$

$$m\{x: |Tf(x)| > \alpha\} \leq \left(\frac{A_i \|f\|_{p_i}}{\alpha}\right)^{q_i}, \quad \text{all} \quad \alpha > 0.$$

If $q_i = \infty$, it means that $\|Tf\|_{q_i} \leq A_i \|f\|_{p_i}$.

THEOREM. *Suppose $T$ is simultaneously of weak-types $(p_0, q_0)$ and $(p_1, q_1)$. If $0 < \theta < 1$, and $1/p = \dfrac{(1-\theta)}{p_0} + \theta/p_1$, $1/q = \dfrac{(1-\theta)}{q_0} + \theta/q_1$, then $T$ is of type $(p, q)$, namely*

$$\|Tf\|_q \leq A \|f\|_p, \qquad f \in L^p(\mathbf{R}^n).$$

*Here $A = A(A_i, p_i, q_i, \theta)$, but it does not otherwise depend on $T$ or $f$.*

From the proof given below it is easy to see that the theorem can be extended to the following situations: First the underlying measure space $\mathbf{R}^n$ of the $L^{p_i}(\mathbf{R}^n)$, can be replaced by a general measure space (and the measure space occurring in the domain of $T$ need not be the same as the one entering in the range of $T$). Secondly the sub-additivity condition can be replaced by the more general condition $|T(f_1 + f_2)(x)| \leq K\{|Tf_1(x)| + |Tf_2(x)|\}$. A less superficial generalization of the theorem can be given in terms of the notion of Lorentz spaces, which unify and generalize the usual $L^p$ spaces and the weak-type spaces. For a discussion of this more general form of the Marcinkiewicz interpolation theorem see *Fourier Analysis*, Chapter V.

**B.2** Suppose $h$ is a given measurable function on $\mathbf{R}^n$. We have already used the notion of its distribution function $\lambda(\alpha)$, defined by $\lambda(\alpha) = m\{x: |h(x)| > \alpha\}$, with $m$ Lebesgue measure on $\mathbf{R}^n$. For the proof of the above theorem and in other situations it is useful to consider the concept of the non-increasing re-arrangement of $h$. This is a function $h^*$, defined on $(0, \infty)$ which has the same distribution function as $h$, but which is non-increasing on $(0, \infty)$. $h^*$ is defined as $h^*(t) = \inf\{\alpha, \lambda(\alpha) \leq t\}$. Both $h^*$ and $\lambda$ are non-negative non-increasing functions, and are continuous on the right. $h^*$ and $h$ have the same distribution function, since the set where $h^*(t) > \alpha$ is the interval $0 \leq t < \lambda(\alpha)$, which of course has measure $\lambda(\alpha)$. Thus

$$\|h^*(t)\|_p = \left(\int_0^\infty |h^*(t)|^p\, dt\right)^{1/p} = \left(\int_{\mathbf{R}^n} |h(x)|^p\, dx\right)^{1/p} = \|h\|_p.$$

**B.3** We shall also make use of some integral inequalities for functions on $(0, \infty)$.

The first is the set of Hardy inequalities given in Appendix A.4. In addition we also need the observation that

$$\left(\int_0^\infty [t^{1/p}h(t)]^{q_2}\frac{dt}{t}\right)^{1/q_2} \leq A\left(\int_0^\infty [t^{1/p}h(t)]^{q_1}\frac{dt}{t}\right)^{1/q_1}$$

wherever $h$ is a non-negative non-increasing function on $(0, \infty)$, $0 < p \leq \infty$, and $q_1 \leq q_2 \leq \infty$. $A = A(p, q_1, q_2)$ does not depend on $h$. To prove this inequality assume that the integral on the right side is normalized to be one. Integrating only between $t/2$ and $t$ in this integral (and using the fact that $h$ is at least $h(t)$ in that interval) we get $\sup_{0<t} t^{1/p}h(t) \leq A_1$. This is the desired result for $q_2 = \infty$. The general result, where $q_1 \leq q_2 < \infty$, then follows by writing

$$\int_0^\infty [t^{1/p}h(t)]^{q_2}\frac{dt}{t} \leq \sup_t\, [t^{1/p}h(t)]^{q_2-q_1}\int_0^\infty [t^{1/p}h(t)]^{q_1}\frac{dt}{t}.$$

**B.4** We come now to the proof of the Marcinkiewicz theorem. We let $\sigma$ denote the slope of the line segment in $\mathbf{R}^n$ joining the points $(1/p_0, 1/q_0)$ with $(1/p_1, 1/q_1)$. Since $(1/p, 1/q)$ lies on this segment, we have

$$\sigma = \frac{1/q_0 - 1/q}{1/p_0 - 1/p} = \frac{1/q - 1/q_1}{1/p - 1/p_1}.$$

For any $t > 0$, we split an arbitrary function $f \in L^p(\mathbf{R}^n)$ as follows:

$$f = f^t + f_t$$

where $f^t(x) = f(x)$, if $|f(x)| > f^*(t^\sigma)$, and $f^t(x) = 0$ otherwise; $f_t = f - f^t$. It follows easily that

$$(f^t)^*(y) \leq f^*(y), \quad \text{if} \quad 0 \leq y \leq t^\sigma \quad \text{and}$$
$$(f^t)^*(y) = 0, \quad \text{if} \quad y > t^\sigma.$$

Also

$$(f_t)^*(y) \leq f^*(t^\sigma), \quad \text{if} \quad y \leq t^\sigma \quad \text{and}$$
$$(f_t)^*(y) \leq f^*(y), \quad \text{if} \quad y \geq t^\sigma.$$

Now if $f = f_1 + f_2$, then as is easily verified

$$(T(f))^*(t) \leq (Tf_1)^*(t/2) + (Tf_2)^*(t/2).$$

The weak-type $(p_0, q_0)$ assumption on $T$ implies that whenever $f \in L^{p_0}(\mathbf{R}^n)$, then $(Tf)^*(t) \le A_0 t^{-1/q_0} \|f\|_{p_0}$. Similarly if $f \in L^{p_1}(\mathbf{R}^n)$, $(Tf)^*(t) \le A_1 t^{-1/q_1} \|f\|_{p_1}$. We now use the decomposition $f = f^t + f_t$, $f \in L^p(\mathbf{R}^n)$, and since $p_0 < p < p_1$, $f^t \in L^{p_0}(\mathbf{R}^n)$, and $f_t \in L^{p_1}(\mathbf{R}^n)$. Inserting this in the above gives

$$T(f)^*(t) \le A_0 (2/t)^{1/q_0} \|f^t\|_{p_0} + A_1 (2/t)^{1/q_1} \|f_t\|_{p_1}.$$

However, $\|T(f)\|_q = \|Tf^*\|_q$, and by the inequality in §B.3 the latter is majorized by a constant multiple of

$$\left( \int_0^\infty (t^{1/q}(Tf)^*(t))^p \frac{dt}{t} \right)^{1/p},$$

since $p \le q$, because $p_i \le q_i$.

Applying the previous estimate for $(Tf)^*(t)$, reduces the majorization of $\|Tf\|_q$ to a constant multiple of

$$(1) \qquad \left\{ \int_0^\infty [t^{1/q - 1/q_0} \|f^t\|_{p_0}]^p \frac{dt}{t} \right\}^{1/p} + \left\{ \int_0^\infty [t^{1/q - 1/q_1} \|f_t\|_{p_1}]^p \frac{dt}{t} \right\}^{1/p}.$$

In view of the estimate $(f^t)^*(y) \le f^*(y)$, if $y \le t^\sigma$ and $(f^t)^*(y) = 0$, if $y > t^\sigma$ made earlier, we have that

$$\|f^t\|_{p_0} = \left( \int_0^\infty (y^{1/p_0}(f^t)^*(y))^{p_0} \frac{dy}{y} \right)^{1/p_0} \le \text{constant} \times \int_0^\infty y^{1/p_0}(f^t)^*(y) \frac{dy}{y}$$

$$\le \text{constant} \times \int_0^{t^\sigma} y^{1/p_0} f^*(y) \frac{dy}{y}.$$

We insert this estimate for $\|f^t\|_{p_0}$ in the first bracket in (1) above. After the indicated change of variables ($t^\sigma \to t$), and the application of the first of the Hardy inequalities (in Appendix A.4), we see that the first term in (1) is majorized by a constant $\times \|f\|_p$. A similar argument for the second term concludes the proof of the theorem.

## C. Some elementary properties of harmonic functions*

**C.1** A useful form of the maximum principle can be stated as follows. *Suppose $u$ is a real-valued function of class $C^2$ in a bounded region $\mathscr{R}$, and let $u$ be continuous in $\bar{\mathscr{R}}$. We assume that $\Delta(u) \ge 0$ in $\mathscr{R}$. If $u \le 0$ on the boundary of $\mathscr{R}$, then $u \le 0$, throughout $\mathscr{R}$.*

In proving this assertion it is convenient to make the stronger assumption that $\Delta u > 0$ in $\mathscr{R}$. We can reduce to this special case by considering instead of $u$, $u + \varepsilon |x|^2 - \delta$, where $\varepsilon > 0$, $\delta > 0$ and both $\varepsilon$ and $\delta$ are small. Let us assume then that $\Delta u > 0$ in $\mathscr{R}$. If it were not true that $u \le 0$ in $\mathscr{R}$, then $u$ would have to attain a positive maximum at some point $x^0 \in \mathscr{R}$. Since $(\Delta u)(x^0) > 0$, it must be

* For a discussion of the elementary properties of harmonic functions see also *Fourier Analysis*, Chapter II.

true that $\frac{\partial^2 u}{\partial x_j^2}(x^0) > 0$ for at least one $j$. By the maximum property $\frac{\partial u}{\partial x_j}(x^0) = 0$ and so by Taylor's theorem

$$u(x^0 + \xi e_j) - u(x^0) = \frac{\xi^2}{2}\frac{\partial^2 u}{\partial x_j^2}(x^0) + o(\xi^2),$$

where $e_j$ is the unit vector along the $x_j$ direction and $\xi$ is small. This of course contradicts the fact that $u(x^0)$ is the maximum value of $u$ in $\mathscr{R}$.

**C.2** Suppose that $u$ is harmonic in $\mathscr{R}$. Then

$$u(x^0) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} u(x^0 + y'r)\, d\sigma(y').$$

Here $d\sigma(y')$ is the element of volume on the unit sphere $S^{n-1}$ in $\mathbf{R}^n$; $\omega_{n-1}$ is the total volume of that sphere; and $r$ is sufficiently small so that the ball of radius $r$ with center $x^0$ lies entirely in $\mathscr{R}$. This is the *mean-value property* of harmonic functions. This fact follows, as is well-known, from Green's identity

$$\int_{\mathscr{S}} (u\, \Delta v - v\, \Delta u)\, dx = \int_{\partial\mathscr{S}} \left(\frac{u\,\partial v}{\partial n} - \frac{v\,\partial u}{\partial n}\right) d\tau$$

where $\mathscr{S}$ is the region contained between the spheres of radius $r$ and radius $\varepsilon$ centered at the point $x^0$ ($\varepsilon$ is small); $d\tau$ is the surface element on $\partial\mathscr{S}$, which is the union of these two spheres, and $\frac{\partial}{\partial n}$ is the differentiation in the outward normal direction. We take $v(x) = |x - x^0|^{-n+2} = r^{-n+2}$. So $\Delta v = 0$ in $\mathscr{S}$ and $v = 0$ on larger of the two boundary spheres. Letting $\varepsilon \to 0$ and collecting terms gives the desired result.

**C.3** The mean-value property and the device of regularization lead immediately to a variety of useful estimates for harmonic functions. Let $\varphi$ be a fixed $C^\infty$ function in $\mathbf{R}^n$, which is radial, is supported in the unit ball, and is normalized, namely $\int_{\mathbf{R}^n} \varphi(x)\, dx = 1$. Then by integration we get as an immediate consequence of the mean-value property,

$$u(x^0) = \int_{\mathbf{R}^n} u(x^0 - y)\varphi_r(y)\, dy,$$

where $\varphi_r(y) = r^{-n}\varphi(y/r)$.

Here we assume that as before the distance of $x^0$ from the boundary of $\mathscr{R}$ exceeds $r$. The above can be rewritten as $u(x^0) = \int_{\mathbf{R}^n} u(y)\varphi_r(x^0 - y)\, dy$ and thus $\left(\frac{\partial}{\partial x}\right)^\alpha u(x^0) = \int_{\mathbf{R}^n} u(y) \left(\frac{\partial}{\partial x^0}\right)^\alpha \varphi_r(x^0 - y)\, dy$. From this and Schwarz's inequality we get the following inequality for harmonic functions in $n$ variables,

$$\left| \left(\frac{\partial}{\partial x}\right)^\alpha u(x^0) \right| \leq A_\alpha r^{-n/2-|\alpha|} \left(\int_{B_r} |u(y)|^2\, dy\right)^{1/2}.$$

$B_r$ denotes the ball of radius $r$ centered at $x^0$.

**C.4** We prove here the assertion made in §3.1.5 of Chapter III, namely that if $f$ has the spherical harmonic development

$$f(x) \sim \sum Y_k(x)$$

and $\int_{S^{n-1}} |Y_k(x)|^2 \, d\sigma(x) = O(k^{-N})$, for every $N$ as $k \to \infty$, then $f$ can be corrected on a set of measure zero so as to be a $C^\infty$ function on $S^{n-1}$. To prove this it suffices to verify the inequality

$$(*) \qquad \sup_{|x|=1} \left| \frac{\partial^\alpha Y_k(x)}{\partial x^\alpha} \right| \leq A'_\alpha k^{(n/2+|\alpha|)} \left( \int_{S^{n-1}} |Y_k(x)|^2 \, d\sigma(x) \right)^{1/2}.$$

Let us normalize $Y_k$ by assuming that $\int_{S^{n-1}} |Y_k(x)|^2 \, d\sigma(x) = 1$. $Y_k$ is of course homogeneous of degree 0. If we set $P_k(x) = |x|^k Y_k(x)$, then $P_k$ is a solid harmonic of degree $k$. Now

$$\int_{|x|\leq 1+\varepsilon} |P_k(x)|^2 \, dx = \left( \int_{S^{n-1}} |Y_k(x)|^2 \, d\sigma(x) \right) \int_0^{1+\varepsilon} r^{2k+n-1} \, dr \leq (1+\varepsilon)^{2k+n}.$$

We invoke here the final inequality of §C.3, and take $u = P_k$ with $x^0$ any point on $S^{n-1}$ and $B_r$ a ball of radius $\varepsilon$ centered at $x^0$. The result is

$$\left| \left( \frac{\partial}{\partial x} \right)^\alpha P_k(x^0) \right| \leq A_\alpha \varepsilon^{-n/2-|\alpha|} (1+\varepsilon)^{k+n/2}.$$

$\varepsilon$ is at our disposal and we choose it to be equal to $1/k$. Then $(1+\varepsilon)^{k+n/2} \leq$ constant, and we have

$$(**) \qquad \sup_{|x|=1} \left| \left( \frac{\partial}{\partial x} \right)^\alpha P_k(x) \right| \leq A''_\alpha k^{n/2+|\alpha|}.$$

Finally since $Y_k(x) = |x|^{-k} P_k(x)$, the Leibnitz identity shows that (**) implies (*).

## D. *Inequalities for Rademacher functions*

It is our purpose here to provide a proof for the inequalities for Rademacher functions asserted in §5.2 of Chapter IV. See also Zygmund [8], chapter V.

**D.1** Let $\mu, a_0, a_1, \ldots, a_N$, be real numbers. Then because the Rademacher functions are mutually independent variables we have

$$\int_0^1 \exp \mu \sum_{m=0}^{N} a_m r_m(t) \, dt = \prod_{m=0}^{N} \int_0^1 \exp \mu a_m r_m(t) \, dt.$$

However in view of their definition, $\int_0^1 e^{\mu a_m r_m(t)} \, dt = \cosh \mu a_m$. If we now make use of the simple inequality $\cosh x \leq e^{x^2}$, we obtain

$$\int_0^1 e^{\mu F(t)} \, dt \leq \exp \mu \sum a_m^2,$$

with $F(t) = \sum a_m r_m(t)$.

**D.2** Let us make the normalizing assumption that $\sum_{n=0}^{N} a_m^2 = 1$. Then since $e^{\mu|F|} \le e^{\mu F} + e^{-\mu F}$, we have

$$\int_0^1 e^{\mu|F(t)|}\, dt \le 2e^{\mu^2}.$$

Let $\lambda(\alpha) = m\{t : |F(t)| > \alpha\}$, be the distribution function of $|F|$. If we take $\mu = \alpha/2$ in the above inequality we have $\lambda(\alpha) \le 2e^{(\alpha/2)^2 - (\alpha/2)\alpha}$ and so $\lambda(\alpha) \le 2e^{-\alpha^2/4}$. From this it follows immediately that

$$\left(\int_0^1 |F(t)|^p\, dt\right)^{1/p} = \|F\|_p \le A_p \le Ap^{1/2}, \quad \text{for} \quad p < \infty$$

and so in general

$$\|F(t)\|_p \le A_p\left(\sum_{m=0}^{N} |a_m|^2\right)^{1/2}, \qquad p < \infty.$$

**D.3** We shall now extend the last inequality to several variables. The case of two variables is entirely typical of the inductive procedure used in the proof of the general case.

We can also limit ourselves to the situation when $p > 2$, since for the case $p \le 2$ the desired inequality is a simple consequence of Hölder's inequality and the orthogonality of the Rademacher functions.

We have

$$F(t_1, t_2) = \sum_0^N \sum_0^N a_{m_1 m_2} r_{m_1}(t_1) r_{m_2}(t_2) = \sum_0^N F_{m_1}(t_2) r_{m_1}(t_1).$$

Now by what has just been proved

$$\int_0^1 |F(t_1, t_2)|^p\, dt_1 \le A_p^p\left(\sum_{m_1} |F_{m_1}(t_2)|^2\right)^{p/2}.$$

Integrate this with respect to $t_2$, and use the fact that for any sequence of functions

$$\int_0^1 \left(\sum |F_{m_1}(t_2)|^2\right)^{p/2} dt_2 \le \left(\sum_{m_1}\left(\int_0^1 |F_{m_1}(t_2)|^p\, dt_2\right)^{2/p}\right)^{p/2}.$$

This assertion is merely a restatement of a special case of Minkowski's inequality (in Appendix A.1) for the exponent $p/2$, and the function $|F_{m_1}(t_2)|^2$.

However $F_{m_1}(t_2) = \sum_{m_2} a_{m_1 m_2} r_{m_2}(t_2)$, and therefore the case already proved shows that

$$\left(\int_0^1 |F_{m_1}(t_2)|^p\, dt_2\right)^{2/p} \le A_p^2 \sum_{m_2} a_{m_1 m_2}^2.$$

Inserting this in the above gives

$$\int_0^1 \int_0^1 |F(t_1, t_2)|^p\, dt_1\, dt_2 \le A_p^{2p} \sum_{m_2} \sum_{m_1} a_{m_1 m_2}^2$$

which leads to the desired inequality

$$\|F\|_p \leq A'_p \|F\|_2, \qquad p < \infty.$$

**D.4** The converse inequality

$$\|F\|_2 \leq B_p \|F\|_p, \qquad p > 0$$

is a simple consequence of the direct inequality.

In fact for any $p > 0$, (here we may assume $p < 2$) define the exponent $r > 2$ by the rule that the exponent 2 should be midway (in the appropriate sense), between $p$ and $r$. That is, take $1/2 = (1/2)(1/p + 1/r)$. By Hölder's inequality

$$\|F\|_2 \leq \|F\|_p^{1/2} \|F\|_r^{1/2}.$$

We already know that $\|F\|_r \leq A'_r \|F\|_2$, $r > 2$. We therefore get

$$\|F\|_2 \leq (A'_r)^2 \|F\|_p,$$

which is the required converse inequality.

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# INDEX

# 目　录

# 第一章　实变理论的若干基本概念

实变理论的基础是同集合与函数的概念以及应用于它们的积分与微分过程相联系着的。这些思想的主要方面早在本世纪初已经发扬光大，而它们的若干进一步的应用则是晚近才发展的。我们要考虑的是后者中使我们感兴趣的那部分理论。为此，我们特别考虑以下几个主要内容：

(a) 关于积分的微分的Lebesgue定理。在这方面有关性质的研究，最有效的是归到由此引出的“极大函数”的研究，它的基本特征反映在一个“弱型”不等式，它是所考虑情形的特征。

(b) 某些覆盖引理。一般说来，其意义在于用不相重叠①的方块或球体的并集来覆盖任意给定的一个（开）集合，至于这些方块或球体的选择则取决于所考虑的问题的需要。一个合适的例子就是Whitney引理（定理3）。有时往往只需要覆盖住集合的一部分，正象普通的简单的覆盖引理那样，它是用来证明上面提到的弱型不等式的。

(c) 任意点集的“一般”点邻近的性质。这里最简单的概念是全密点的概念。进一步改进的性质最好是用某种形式的积分来刻划，这方面首先系统地为Marcinkiewicz所研究。

(d) 函数分成大小两部分的分解。经常发生的是，这种性质较诸以分解本身为目的更具有技巧性。象在本章的第一个定理中那样，它在证明$L^p$不等式时特别有用。第一个定理证明中的这一部分，在本章§4讨论的Marcinkiewicz内插定理中将得到系统化，同样情形见附录B。

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① 我们称两集合不相重叠，是指它们的内部不相交。——译者注

## §1 极大函数

**1.1** 根据Lebesgue基本定理，对于几乎所有的$x$，关系式

$$\lim_{r\to0}\frac{1}{m(B(x,r))}\int_{B(x,r)}f(y)dy=f(x) \tag{1}$$

成立，只要函数$f(x)$在$\boldsymbol{R}^n$上确定并且局部可积。这里用的记号$B(x,r)$表示以$x$为中心$r$为半径的球体，而$m(B(x,r))$表示它的测度。为了研究极限(1)，我们考虑它的量的类似，即用"$\sup\limits_{r>0}$"来代替"$\lim\limits_{r\to0}$"，这就是**极大函数**$M(f)$。由于极大函数的性质是用相对大小来表达的，并不包括任何正、负值的相消，因此我们还应当用$|f|$来代替$f$。这样我们定义

$$M(f)(x)=\sup_{r>0}\frac{1}{m(B(x,r))}\int_{B(x,r)}|f(y)|dy. \tag{2}$$

应当注意，对于任意给定的$x$，并不排斥$M(f)(x)$为无穷的可能性。

从一个取极限的表达式转为相应的极大函数是一种经常重复发生的情形。这里我们的第一个例子，(2)，将显示出它是最基本的一个。

**1.2** 对于函数的相对大小，我们现在将感兴趣于给出具体的表达式。现设$g(x)$在$\boldsymbol{R}^n$上有定义(可测)，对于每一非负数$\alpha$，考虑满足$|g(x)|$大于$\alpha$的全体点$x$组成的集合：

$$\{x:|g(x)|>\alpha\}.$$

这个集合的测度定义为函数$\lambda(\alpha)$，它就是我们所要的表达式，称之为$|g|$的**分布函数**。

特别是，当$\alpha$增长时，$\lambda(\alpha)$的减小程度描述了函数的相对大；这对局部的研究是很主要的。当$\alpha$趋向于零时，$\lambda(\alpha)$的增长情形反映了函数"在无穷远"的相对小；这对整体的研究是很重

要的，举例来说，当函数支在有界集合上时，它就没有意思了。

任何仅仅涉及 $g$ 的大小的量都可以用分布函数$\lambda(\alpha)$来表示。例如，设$g\in L^p$，则

$$\int_{R^n}|g(y)|^p dy=-\int_0^\infty \alpha^p d\lambda(\alpha),$$

而若$g\in L^\infty$，则

$$\|g\|_\infty=\inf\{\alpha;\ \lambda(\alpha)=0\}.$$

一个与分布函数有关的事实将立即有用，这就是：若 $g$ 可积，则

$$\lambda(\alpha)\leqslant A/\alpha,\qquad 其中 A=\int_{R^n}|g(y)|dy.$$

事实上，

$$\int_{R^n}|g(y)|dy\geqslant\int_{|g|>\alpha}|g(y)|dy\geqslant\alpha\lambda(\alpha),$$

这就是所要证明的。

1.3 有了以上的定义我们可以来叙述我们的第一个定理了。它给出了关于极大函数的主要结果，并作为推论得到由(1)表达的关于积分的几乎处处可微性。

**定理1** 设 $f$ 在$R^n$上有定义。

(a) 若$f\in L^p(R^n)$，$1\leqslant p\leqslant\infty$，则函数 $M(f)$ 几乎处处有限。

(b) 若$f\in L^1(R^n)$，则对于每一$\alpha>0$，

$$m\{x;\ M(f)(x)>\alpha\}\leqslant\frac{A}{\alpha}\int_{R^n}|f|dx,$$

其中 $A$ 是仅与维数 $n$ 有关的常数($A=5^n$就行)。

(c) 若 $f\in L^p(R^n)$，$1<p\leqslant\infty$，则 $M(f)\in L^p(R^n)$，且

$$\|M(f)\|_p\leqslant A_p\|f\|_p,$$

其中$A_p$只与 $p$ 及维数 $n$ 有关。

**推论1** 若$f\in L^p(R^n)$，$1\leqslant p\leqslant\infty$，或更一般，若 $f$ 局部可积，则对于几乎所有的 $x$，下述等式

$$\lim_{r\to 0}\frac{1}{m(B(x,r))}\int_{B(x,r)}f(y)dy=f(x)$$

成立。

1.4　在证明定理之前我们作以下几点澄清。

(a)　与$p>1$的情形相反，变换$f\to M(f)$当$p=1$时在$L^1(\boldsymbol{R}^n)$并不有界。假如$f$不恒为零（等价于零），$Mf$不可能在整个$\boldsymbol{R}^n$上可积，这只要对$M(f)(x)\geqslant c|x|^{-n}(|x|\geqslant 1)$这一事实作简单观察就可以知道。甚至即使我们限于考虑$\boldsymbol{R}^n$的任一有界子集时，也只有当$f$满足比可积性更强的条件时才能使$M(f)$可积（见以下§5.2）。

(b)　我们所得的结果(b)比$f\to M(f)$在$L^1(\boldsymbol{R}^n)$有界要弱，这一点可从§1.2中的提示看出；由于这个原因(b)才被称为**弱型**估计。这一估计对于$M(f)$的分布函数，当$f$是$L^1(\boldsymbol{R}^n)$的任一函数时，是最佳可能的。这可以从下述情形看到：在定义(2)中用测度$d\mu$来代替$|f(y)|dy$，这里$d\mu$是“Dirac测度”，它的总测度1全部集中于原点。于是有$M(d\mu)(x)=c|x|^{-n}$，其中$c^{-1}$等于单位球体的测度。这时的分布函数$\lambda(\alpha)$正好是$1/\alpha$。然而我们总可以找到一串正的可积函数序列$\{f_m(x)\}$，其$L^1$模都是1，使之弱收敛于测度$d\mu$。所以我们不能期望有实质上强于(b)的估计，否则在极限过程中将会导致对$M(d\mu)(x)$也有相似的强的估计。

**1.5　定理1及其推论的证明**　这里我们将预先承认在§1.6中叙述并在§1.7中证明的“Vitali型”覆盖引理，并用来证明定理1及它的推论。我们记

$$E_\alpha=\{x:\ M(f)(x)>\alpha\},$$

根据$M(f)$的定义，对于每个$x\in E_\alpha$，存在以$x$为中心的球体$B_x$，使得

$$\int_{B_x}|f(y)|dy>\alpha m(B_x)\ . \tag{3}$$

但是一方面由式(3)可得$m(B_x)<(1/\alpha)\|f\|_1$对所有这样的$x$成立；另一方面当$x$取遍$E_\alpha$中的点时，相应的$B_x$构成的并集覆盖了$E_\alpha$。于是利用§1.6中的覆盖引理，从这族球体中可选出一串球体的序列$\{B_k\}$，它们彼此不相重叠，且具有性质

$$\sum_{k=0}^{\infty} m(B_k) \geqslant C m(E_\alpha) \tag{4}$$

(例如取$C=5^{-n}$即行)。应用式(3)与(4)于每一个不相重叠的球体，我们得到

$$\int_{\cup B_k} |f(y)|\,dy > \alpha \sum_k m(B_k) \geqslant \alpha C m(E_\alpha) .$$

但上述不等式左端第一项被$\|f\|_1$所控制，取$A=1/C$，即得定理1的(b)，从而(a)的关于$p=1$的情形也就成立。现在我们要同时证明结论(a)($M(f)(x)$的几乎处处有限)与结论(c)($L^p$不等式)中$1<p\leqslant\infty$的情形。$p=\infty$的情形是平凡的，这时界$A_\infty=1$。因而我们可假定$1<p<\infty$。我们将运用分解函数为大、小两部分的技巧，这可以作为在本章开始时曾提到的一个简单例子。我们定义$f_1(x)$如下：$f_1(x)=f(x)$，当$|f(x)|\geqslant\alpha/2$；$f_1(x)=0$，其它地方。于是，我们依次可得$|f(x)|\leqslant|f_1(x)|+\alpha/2$，$M(f)(x)\leqslant M(f_1)(x)+\alpha/2$，从而有

$$\{x: M(f)(x)>\alpha\}\subset\{x: M(f_1)(x)>\alpha/2\},$$

最后有

$$m(E_\alpha)=m\{x: M(f)(x)>\alpha\}\leqslant\frac{2A}{\alpha}\|f_1\|_1,$$

亦即

$$m(E_\alpha)=m\{x: M(f)(x)>\alpha\}\leqslant\frac{2A}{\alpha}\int_{|f|>\alpha/2}|f|\,dx. \tag{5}$$

由于当$f\in L^p$时，$f_1\in L^1$，上述最后一步是利用了定理的结论(b)。我们现在令$g=M(f)$，并设$g$的分布函数为$\lambda$，利用§1.2中所

述，并进行分部积分①，我们有

$$\int_{\mathbf{R}^n}(M(f))^p dx=-\int_0^\infty \alpha^p d\lambda(\alpha)=p\int_0^\infty \alpha^{p-1}\lambda(\alpha)d\alpha.$$

利用(5)可得

$$\|M(f)\|_p^p=p\int_0^\infty \alpha^{p-1}m(E_\alpha)d\alpha$$

$$\leqslant p\int_0^\infty \alpha^{p-1}\left(\frac{2A}{\alpha}\int_{|f|>\alpha/2}|f(x)|dx\right)d\alpha.$$

上式最后的重积分，可用交换积分顺序来计算，先对$\alpha$积分，由于$p>1$，则内层积分为

$$\int_0^{2|f(x)|}\alpha^{p-2}d\alpha=\frac{1}{p-1}|2f(x)|^{p-1}\qquad(p>1).$$

于是重积分的值为

$$\frac{2Ap}{p-1}\int_{\mathbf{R}^n}|f(x)||2f(x)|^{p-1}dx=(A_p)^p\int_{\mathbf{R}^n}|f(x)|^p dx,$$

这里

$$A_p=2\left(\frac{5^n p}{p-1}\right)^{1/p}\quad(1<p<\infty).$$

定理(1)的结论(c)即证得，从而(a)也就成立（$1<p<\infty$）。$A_p$的表达式可用来得到

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① 利用$E_\alpha=\{x:|g(x)|>\alpha\}$的特征函数$\chi_{E_\alpha}(x)$，可直接证明如下：

$$p\int_0^\infty \alpha^{p-1}\lambda(\alpha)d\alpha=p\int_0^\infty \alpha^{p-1}\left\{\int_{\mathbf{R}^n}\chi_{E_\alpha}(x)dx\right\}d\alpha$$

$$=\int_{\mathbf{R}^n}\left\{\int_0^\infty p\alpha^{p-1}\chi_{E_\alpha}(x)d\alpha\right\}dx$$

$$=\int_{\mathbf{R}^n}\left\{\int_0^{|g(x)|}p\alpha^{p-1}d\alpha\right\}dx$$

$$=\int_{\mathbf{R}^n}|g(x)|^p dx,$$

——译者注。

$$A_p = O\left(\frac{1}{p-1}\right) \quad (p \to 1),$$

这在某些应用上是有用的。

现在我们来证明推论。很容易把我们的考虑归到 $p=1$ 的情形，如果我们把原来的函数乘上球体的特征函数，并用可数个球体的并集包含整个$\boldsymbol{R}^n$的话。现记

$$(6) \qquad f_r(x) = \frac{1}{m(B(x,r))}\int_{B(x,r)} f(y)dy, \qquad r>0.$$

我们知道①，当$r\to 0$时，$\|f_r - f\|_1 \to 0$，如果 $f\in L^1(\boldsymbol{R}^n)$。于是 $f_{r_k}\to f$ 几乎处处成立，这里$\{r_k\}$是适当选取的趋向于零的序列。现在剩下来要考虑的是证明 $\lim\limits_{r\to 0} f_r(x)$ 几乎处处存在就行了。为此，对于每个$g\in L^1$ 与 $x\in\boldsymbol{R}^n$，记

$$(7) \qquad \Omega g(x) = |\limsup_{r\to 0} g_r(x) - \liminf_{r\to 0} g_r(x)|,$$

这里$g_r$的定义如同$f_r$一样。$\Omega g$表示函数$\{g_r\}$当$r\to 0$时的振动；

若 $g$ 连续且具有紧支集，则$g_r\to g$一致成立，从而$\Omega g$这时就恒等于零。

其次若$g\in L^1(\boldsymbol{R}^n)$，则由定理(1)的结论(b)可知

$$m\{x: 2M(g)>\varepsilon\} \leqslant \frac{2A}{\varepsilon}\|g\|_1.$$

而显然

$$\Omega g(x) \leqslant 2Mg(x),$$

于是

$$(8) \qquad m\{x = \Omega g(x)>\varepsilon\} \leqslant \frac{2A}{\varepsilon}\|g\|_1 \quad (g\in L^1(\boldsymbol{R}^n)).$$

最后对于任意的$f\in L^1(\boldsymbol{R}^n)$，可把 $f$ 写成 $f=g+h$，其中 $h$

① 这是恒等逼近的一个特殊性质，详细的讨论见第三章§2.2。它的有关部分可以运用，并无逻辑循环的现象发生。

是连续函数且具有紧支集，而$\|g\|_1$可以任意小。注意到$\Omega f \leqslant \Omega g + \Omega h$，而$\Omega h \equiv 0$。因此由(8)式可得

$$m\{x: \ \Omega f(x) > \varepsilon\} \leqslant \frac{2A}{\varepsilon} \|g\|_1.$$

由于这时$\|g\|_1$可以预先取得任意小，我们即得 $\Omega f = 0$几乎处处成立，亦即极限 $\lim\limits_{r \to 0} f_r(x)$ 几乎处处存在。

值得提出的是以下关于推论的证明的小结性意见：以上证明所用的方法带有很大的普遍性并且经常碰到。也就是说，关于几乎处处收敛性的证明是由两部分组成的，一部分比较深刻而且已经包含了结果的实质，这就是象定理的(b)（或(c)）那样的不等式所表达的部分。另一部分一般来说比较简单，但也同样具有实质性，即先在所考虑的函数空间的一个稠密子集上来证明几乎处处收敛性，在我们的情形就是具有紧支集的连续函数全体。

**1.6 一个覆盖引理** 除了决定性的一步即覆盖引理的证明推延至今外，定理 1 与它的推论的证明已经完成。不仅由于引理的叙述简单或其应用广泛，而且由于数学文献中有大量的它的变式，都证实了这个引理的基础性。读者可能会注意到它的叙述与证明紧密联系于一个更精密但可能更有名的Vitali① 引理。

**引理** 设$E$是$\boldsymbol{R}^n$的一个可测子集，为一族具有有界直径的球体$\{B_j\}$所覆盖。则从这族球体中可挑选出不相重叠的子序列$B_1, B_2, \cdots, B_k, \cdots$（有限或可数无限），使得

$$\sum_k m(B_k) \geqslant C m(E)$$

其中$C$是正常数，仅与维数$n$有关；$C = 5^{-n}$就行。

**1.7** 我们从描述怎样选取$B_1, B_2, \cdots, B_k, \cdots$开始来证明这个引理。我们选择$B_1$使得它实质上尽可能大，即使$B_1$的直径大于

① Vitali的这个引理可以在§5.4中找到。

或等于$B_j$直径所构成的数集的上确界之半。当然这样选取的$B_1$以及其后选取的$B_k$都不是唯一的，这对我们无关紧要。假如我们已经选取了$B_1, \cdots, B_k$，我们要来挑选$B_{k+1}$使与$B_1, B_2, \cdots, B_k$不相重叠。我们仍旧选择实质上尽可能大的球体，即取$B_{k+1}$与$B_1, B_2, \cdots, B_k$不相重叠，并使$B_{k+1}$的直径大于或等于球族$B_j$中与$B_1, B_2, \cdots, B_k$不相重叠的球体的直径所组成的数集的上确界之半。

这样，我们得到了一串球体的序列$B_1, B_2, \cdots, B_k, \cdots$。原则上这个序列可以有限，例如到$B_k$为止；这样的情形是可能的，如果在球族$\{B_j\}$中已不存在与$B_1, B_2, \cdots, B_k$都不相重叠的球体了。

现在有两种可能情形，一种是$\sum\limits_k m(B_k) = \infty$，另一种是$\sum\limits_k m(B_k) < \infty$。在前一种情形不论$E$的测度是有限或是无限，引理结论都已达到。因此我们只要考虑$\sum\limits_k m(B_k) < \infty$的情形。

现设$B_k^*$是与$B_k$同心而直径5倍于$B_k$的直径的球体。我们宣称

$$\bigcup_k B_k^* \supset E. \tag{9}$$

为要证明式(9)，我们只要指出，对于覆盖$E$的球族$\{B_j\}$中任一固定的$B_j$，都有$\bigcup\limits_k B_k^* \supset B_j$。事实上，我们可以假定这个固定的$B_j$不在序列$B_1, B_2, \cdots, B_k, \cdots$中，否则就不必证明了。由于$\sum\limits_k m(B_k) < \infty$，则$B_k$的直径当$k \to \infty$时趋向于零。从而我们取第一个$k$满足：

$$\operatorname{diam}(B_{k+1}) < \frac{1}{2}(\operatorname{diam}(B_j)) .$$

这样$B_j$必定与$B_1, B_2, \cdots, B_k$这$k$个球中至少一个球有重叠部分；要不然这与$B_{k+1}$的选择原则不符。现设$B_j$与$B_{j_0}$有重叠$(1\leqslant j_0 \leqslant k)$，又根据$k$的取法，可知

$$\operatorname{diam}(B_{j_0}) \geqslant \frac{1}{2}(\operatorname{diam}(B_j)) .$$

于是，从几何上明确地有$B^*_{j_0} \supset B_j$。这样我们已证得(9)，从而有

$$m(E) \leqslant \sum_k m(B^*_k) = 5^n \sum_k m(B_k) ,$$

这就证明了引理。

**1.8 Lebesgue 集合** 前面证明的微分定理是关于在球体上平均的极限。但这定理有一推广，其中平均可以在更广泛的集合族上取。这推广实质上是该定理的简单的引伸。

设$\mathscr{F}$是一族$R^n$的可测子集。我们称这族是**正则的**，假如存在常数$c>0$，使得当$S\in\mathscr{F}$时，有$S\subset B$且$m(S)\geqslant cm(B)$，其中$B$是一个适当的以原点为中心的开球。这种正则族的例子有：(a) 所有集合$\delta U$，$0<\delta<\infty$，构成的集合族，其中$U$是一个有界集，$m(U)>0$，而$\delta U$是集合$U$的展缩；(b) 所有这样的方体构成的集合族，其中每一方体到原点的距离小于等于一个常数乘该方体的直径；(c) 上述集合族的任何子族。类似于以原点为中心的所有球体构成的集合族，对于集合族$\mathscr{F}$我们定义相应的极大函数

$$M_{\mathscr{F}}(f)(x) = \sup_{S\in\mathscr{F}} \frac{1}{m(S)}\int_S |f(x-y)|\,dy.$$

显然有$M_{\mathscr{F}}(f)(x)\leqslant c^{-1}M(f)(x)$，因此$M_{\mathscr{F}}$满足象$M$在定理1中所具有的所有结论。于是重复推论中的论证就可得到，当$f$局部可积时，下式

$$(10) \qquad \lim_{\substack{S\in\mathscr{F}\\ m(S)\to 0}} \frac{1}{m(S)}\int_S f(x-y)\,dy = f(x)$$

对几乎所有的 $x$ 都成立。

所有这些都很简单，但根据以下的理由，这些结论还不完全令人满意。给定一个固定的局部可积函数 $f$，我们虽已证明关系式(10)几乎处处成立，但这个测度为零的例外集依赖于所给的正则集族$\mathscr{F}$。假如我们能够找到统一的一个测度为零的例外集，它只与所给函数 $f$ 有关，使得除去这个例外集中的点以外，关系式(10)对于所有的正则族$\mathscr{F}$都成立，这样就更好了。而这个角色是由 $f$ 的Lebesgue集合的余集来担任的，所谓Lebesgue集合是由满足下述关系式的点 $x$ 所构成：

$$(11)\qquad \lim_{r\to 0}\frac{1}{m(B(x,r))}\int_{B(x,r)}|f(y)-f(x)|dy=0,$$

其中$B(x,r)$仍是以 $x$ 为中心 $r$ 为半径的球体。

为要证明(11)对几乎所有的 $x$ 成立，我们考虑关系式

$$(11')\qquad \lim_{r\to 0}\frac{1}{m(B(x,r))}\int_{B(x,r)}|f(y)-c|dy=|f(x)-c|,$$

对于每一固定的常数 $c$，它对几乎所有的 $x$ 成立。亦即存在例外集$E_c$，$m(E_c)=0$，当$x\overline{\in}E_c$时，关系式(11′)成立。令$c_1,c_2,\cdots,c_n,\cdots$是全部有理数的一个排列，若 $x\overline{\in}E=\bigcup\limits_n E_{c_n}$，则式(11′)对任何有理数 $c$ 都成立，由连续性即对任何实数 $c$ 都成立。特别，式(11)也就成立，这表示当 $x$ 属于 $E$ 的余集时，$x$ 就在 $f$ 的Lebesgue集合中。

但注意到当$S$属于正则集族$\mathscr{F}$时，我们有

$$\left|\frac{1}{m(S)}\int_S f(x-y)dy-f(x)\right|=\left|\frac{1}{m(S)}\int_S[f(x-y)-f(x)]dy\right|$$

$$\leqslant\frac{1}{m(S)}\int_S|f(x-y)-f(x)|dy$$

$$\leqslant c^{-1}\frac{1}{m(B(x,r))}\int_{B(x,r)}|f(y)-f(x)|dy.$$

于是关于任一正则集族，积分的可微性在 $f$ 的Lebesgue集合上统一地建立了。

关于非正则族的一个讨论参看后面的§5.3.

## §2 可测集的一般点邻近的性质

**2.1** 在这一节里我们要处理正测度集合的各种性质，这些性质证实这样的观察：这种集合的“一般”点几乎完全为该集合的其它点所包围。最简单的具体例子蕴含在**全密点**的定义中。

设 $E$ 是一给定的可测集，$x\in \boldsymbol{R}^n$，我们称 $x$ 为 $E$ 的全密点，如果

$$\lim_{r\to 0}\frac{m\{E\cap B(x,r)\}}{m\{B(x,r)\}}=1. \tag{12}$$

自然对于任一 $x$，上述极限不一定为1，甚至极限根本不存在；当式(12)中的极限为0时，则根据我们的定义可知，$x$ 是 $E$ 的余集的全密点。现在让我们运用微分定理(即§1.3中定理1的推论)到 $E$ 的特征函数 $\chi_E$ 上去，便立即得到下述命题：

**命题1** 对于几乎所有的点 $x\in E$，极限(12)成立，即几乎所有的点 $x\in E$ 是它的全密点，而 $E$ 的余集的几乎所有的点都不是 $E$ 的全密点。

注意，如果限制我们的注意力于 $\chi_E$ 的 Lebesgue 集的点时，那末我们应得到类似于命题1但较强的结论。在式(12)中出现的球体同样可以换成正则族中的集合，其意义如§1.8所示。

**2.2** 现在我们只限于考虑闭集合，当然这些闭集仍然是任意的。至于这样限制的理由是显然的，因为在以后的研究中，许多结果要通过到一个集合 $E$ 的距离来刻划，如果 $E$ 不是闭集，到 $E$ 的距离实质上是到它的闭包 $\bar{E}$ 的距离，而 $E$ 和 $\bar{E}$ 从测度论观点看

是可以完全不同的。同时，限于闭集时，并没有在应用中造成严重的障碍。闭集已经是够一般的，特别是任何可测集可以用它所包含的闭集来近似，使得相应的差集的测度任意小。

为反映出我们新加的限制，我们用$F$来表示$R^n$的一般的闭子集。令$\delta(x)=\delta(x,F)$表示点$x$到集合$F$的距离。显然$\delta(x)=0$当且仅当$x\in F$。又当$x\in F$时，我们有$\delta(x+y)\leqslant|y|$。可是一般来说，这个结果对于$F$中几乎所有的$x$，还可以改进为$\delta(x+y)=o(|y|)$，其中"小$o$"的意义是：对于任意给定的$\varepsilon>0$，存在$\eta=\eta_\varepsilon>0$，使当$|y|\leqslant\eta$时，$\delta(x+y)\leqslant\varepsilon|y|$。

**命题2** 设$F$是一闭集，则对于几乎所有的$x\in F$，$\delta(x+y)=o(|y|)$。这一结论特别当$x$是$F$的全密点时成立。

我们讲述这个命题，主要是因为它是全密点概念的一个简单解释。另外这命题还有一个应用，但这要很后才能讲到。

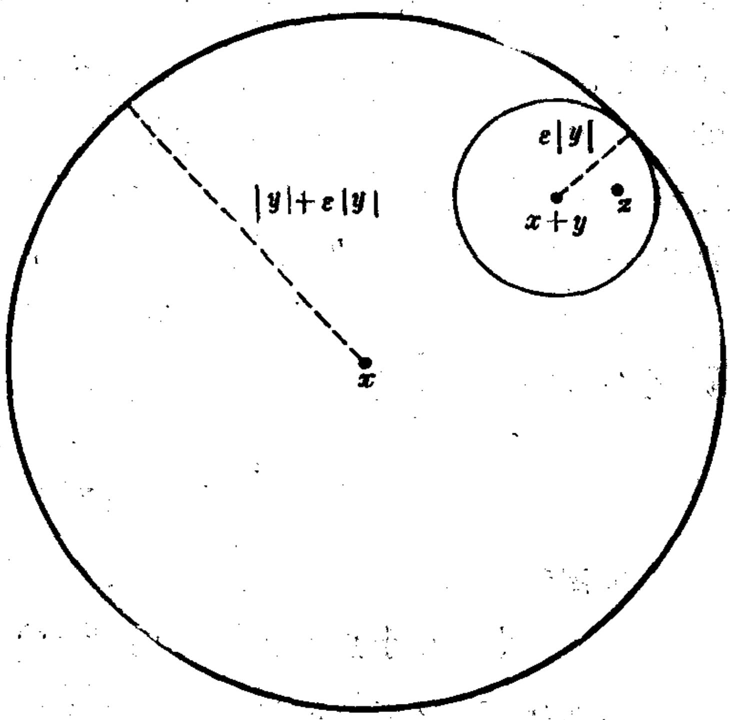

**图1 全密点论证**
大圆表示$B(x,|y|+\varepsilon|y|)$，小圆表示$B(x+y,\varepsilon|y|)$

现在来证明这个命题，设$x$是$F$的全密点，$\varepsilon>0$任意给定。考虑以$x+y$为中心的小球，其半径为$\varepsilon|y|$；以及以$x$为中心$|y|+\varepsilon|y|$为半径的大球(见图1)，显然有$B(x+y,\varepsilon|y|)\subset B(x,|y|+\varepsilon|y|)$。我们要证明当$|y|$充分小时，存在$F$的点$z$，使得$z\in B(x+y,\varepsilon|y|)$。要不然，$F\cap B(x+y,\varepsilon|y|)=\varnothing$，从而有

$$\frac{m(F\cap B(x,|y|+\varepsilon|y|))}{m(B(x,|y|+\varepsilon|y|))}$$

$$\leqslant\frac{m(B(x,|y|+\varepsilon|y|))-m(B(x+y,\varepsilon|y|))}{m(B(x,|y|+\varepsilon|y|))}$$

$$\leqslant 1-\left(\frac{\varepsilon}{1+\varepsilon}\right)^n,$$

当 $y$ 充分小时，这将与式(12)发生矛盾。因此存在 $z\in F$ 同时又属于 $B(x+y,\varepsilon|y|)$，这表示 $\delta(x+y)\leqslant\varepsilon|y|$，当 $|y|$ 充分小，即 $\delta(x+y)=o(|y|)$。

**2.3 Marcinkiewicz 积分** 我们将给出关于可测集的一般点几乎为它的另外的点全部包围这一直观的另一表达形式。这一形式将与微分定理无关，但在许多问题中有重要意义，从而有同样的重要性。事实上，下面考虑的积分，首先为 Marcinkiewicz 系统地处理过，并在奇异积分理论(这将在下一章讨论)中起着决定性作用，同时在本书处理的其它问题中起着同样的作用。

我们仍用 $F$ 表示闭集，$\delta(x)$ 表示 $x$ 到 $F$ 的距离。现在我们要研究如下形式的积分

$$I(x)=\int_{|y|<1}\frac{\delta(x+y)}{|y|^{n+1}}dy. \tag{13}$$

**定理2** (a) 当 $x$ 属于 $F$ 的余集时，$I(x)=\infty$；

(b) 对于几乎所有的 $x\in F$，$I(x)<\infty$。

上述结论(a)是显然的，因为 $F$ 的余集是一个开集，当 $x$ 属于这个余集时，在 $y=0$ 的一个邻域内有 $\delta(x+y)\geqslant c>0$。上述的结论(b)是该定理的兴趣所在，它实质上说明命题 2 中的估计 $\delta(x+y)=o(|y|)$ 能从平均上改进使得积分(13)收敛。

上述定理将成为下述引理的简单推论，它是同一事实的进一步的定量表示：

**引理** 设 $F$ 是一闭集，其余集具有有限测度。令

$$I_*(x)=\int_{R^n}\frac{\delta(x+y)}{|y|^{n+1}}dy, \tag{14}$$

则对于几乎所有的 $x\in F$，$I_*(x)<\infty$，并且有

$$\int_F I_*(x)dx\leqslant c\cdot m({}^cF) \quad ({}^cF\text{是}F\text{的余集})。 \tag{15}$$

2.4 在引理的证明中，我们只需证明式(15)就够了，因为被积函数是非负的缘故。同样理由，式(15)中的积分可以交换积分顺序，这样就可以完成我们的证明。现把细节写出来：

$$\begin{aligned}\int_F I_*(x)dx&=\int_F\int_{R^n}\frac{\delta(x+y)}{|y|^{n+1}}dydx\\&=\int_F\int_{R^n}\frac{\delta(y)}{|x-y|^{n+1}}dydx\\&=\int_F\int_{{}^cF}\frac{\delta(y)}{|x-y|^{n+1}}dydx\\&=\int_{{}^cF}\left(\int_F\frac{dx}{|x-y|^{n+1}}\right)\delta(y)dy。\end{aligned}$$

现在考虑

$$\int_F\frac{dx}{|x-y|^{n+1}} \quad (y\in{}^cF)。$$

当 $x$ 在 $F$ 上变动，$|x-y|$ 的最小值应是 $\delta(y)$，即 $y$ 到 $F$ 的距离。于是有

$$\begin{aligned}\int_F\frac{dx}{|x-y|^{n+1}}&\leqslant\int_{|x|>\delta(y)}\frac{dx}{|x|^{n+1}}\\&\leqslant c(\delta(y))^{-1}。\end{aligned}$$

这就表示

$$\int_F I_*(x)dx \leqslant \int_{^cF} c(\delta(y))^{-1}\cdot\delta(y)dy = cm(^cF),$$

引理证毕。

定理 2 可如下由引理得出。设 $B_m$ 表示以原点为中心、$m$ 为半径的球体，记 $F_m = F \cup {}^cB_m$，于是 $F_m$ 是闭的，它的余集具有有限测度(因为余集包含在 $B_m$ 中)。这样，我们可应用引理于 $F_m$。令 $\delta_m(x)$表示 $x$ 到 $F_m$ 的距离，而 $\delta(x)$仍表示 $x$ 到 $F$ 的距离。当 $|y| \leqslant 1$，$x \in B_{m-2} \cap F$ 时，$\delta(x+y) = \delta_m(x+y)$。从而由引理可得 $I(x) < \infty$ 对于几乎所有的 $x \in F \cap B_{m-2}$ 成立。令 $m \to \infty$ 即得所要的结果。

在定理与引理的其它形式中，我们在这里介绍以下的一种(另一形式将在本章末 §5中讨论)。现在我们用下式代替上述 $I(x)$：

$$I^{(\lambda)}(x) = \int_{|y|\leqslant 1} \frac{\delta^\lambda(x+y)}{|y|^{n+\lambda}} dy,$$

其中 $\lambda > 0$。同样，上述 $I_*(x)$也可以用

$$I_*^{(\lambda)}(x) = \int_{R^n} \frac{\delta^\lambda(x+y)}{|y|^{n+\lambda}} dy, \qquad \lambda > 0$$

代替，这时关于 $I^\lambda(x)$与 $I_*^\lambda(x)$用以上的方法可以得到类似的结果。

## §3 $R^n$中的开集分解为立方体

**3.1** 本章所述理论中的一个基本工具是把给定的集合分解成为不相重叠的立方体(或球体)的并集。在 §1.6的覆盖引理中我们已经用到这一类型的工具，只是十分粗略的。

3.1.1 我们现在要考虑如下有关的一般性问题，它不涉及测度理论，而是关于 $R^n$ 中一般闭子集 $F$ 的几何结构问题：能否把 $F$ 的余集理解为不相重叠的立方体的并集(按正规的形式)？当

$n=1$ 时答案当然是对的，因为每一开集唯一地等同于不相交的开区间的并集。当 $n\geqslant 2$ 时，情况就不这么简单了，任一开集可以用无限多的方法理解为不相重叠的方体的并集(这时方体理解为闭的方体，而不相重叠指的是内部不相交)。但存在着极为有用的分解，虽非正规但令人满意。我们印象中这首先是由 Whitney 引进的。现在叙述如下：

**定理3** 设 $F$ 是 $\boldsymbol{R}^n$ 中的非空闭子集。则 $F$ 的余集 $\Omega$ 是一串不相重叠的方体 $Q_k$ 的并集，每一方体的边平行于坐标轴，其直径与它到 $F$ 的距离大体上成比例。更明确地说：

(a) $\Omega={}^cF=\bigcup\limits_{k=1}^{\infty}Q_k$；

(b) $Q_j^0\cap Q_k^0=\varnothing\ (j\neq k)$ ($Q_j^0$ 表示方体 $Q_j$ 的内部)；

(c) 存在两个常数 $c_1, c_2>0$ (我们可取 $c_1=1$，$c_2=4$)，使得

$$c_1(\text{diameter } Q_k)\leqslant \text{distance}(Q_k, F)\leqslant c_2(\text{diameter } Q_k),$$

其中 diameter $A$ 表示集合 $A$ 的直径，distance$(A, B)$ 表示集合 $A$ 与集合 $B$ 的距离(见图 2)。

3.1.2 我们在这里叙述这个定理，主要是为了教学法上的安排。严格地说我们要到后面第六章才需要用到它。由于引理的证明稍微复杂，我们把它推迟到后面去。

在这里给出这个定理的另一个理由，是因为它能帮助我们搞清楚下面定理的意义。即我们要考虑 Calderón 与 Zygmund 的基本引理，它能给出 § 2 中的极大函数理论的另一论证，而它的主要作用，对我们来说，还在于它对下一章奇异积分的应用。

3.2 **定理4** 设 $f$ 是 $\boldsymbol{R}^n$ 上的非负可积函数，又设 $\alpha$ 是一正的常数。于是存在 $\boldsymbol{R}^n$ 的一个分解，使得

(a) $\boldsymbol{R}^n=F\cup\Omega$，$F\cap\Omega=\varnothing$；

(b) $f(x)\leqslant\alpha$ 在 $F$ 上几乎处处成立；

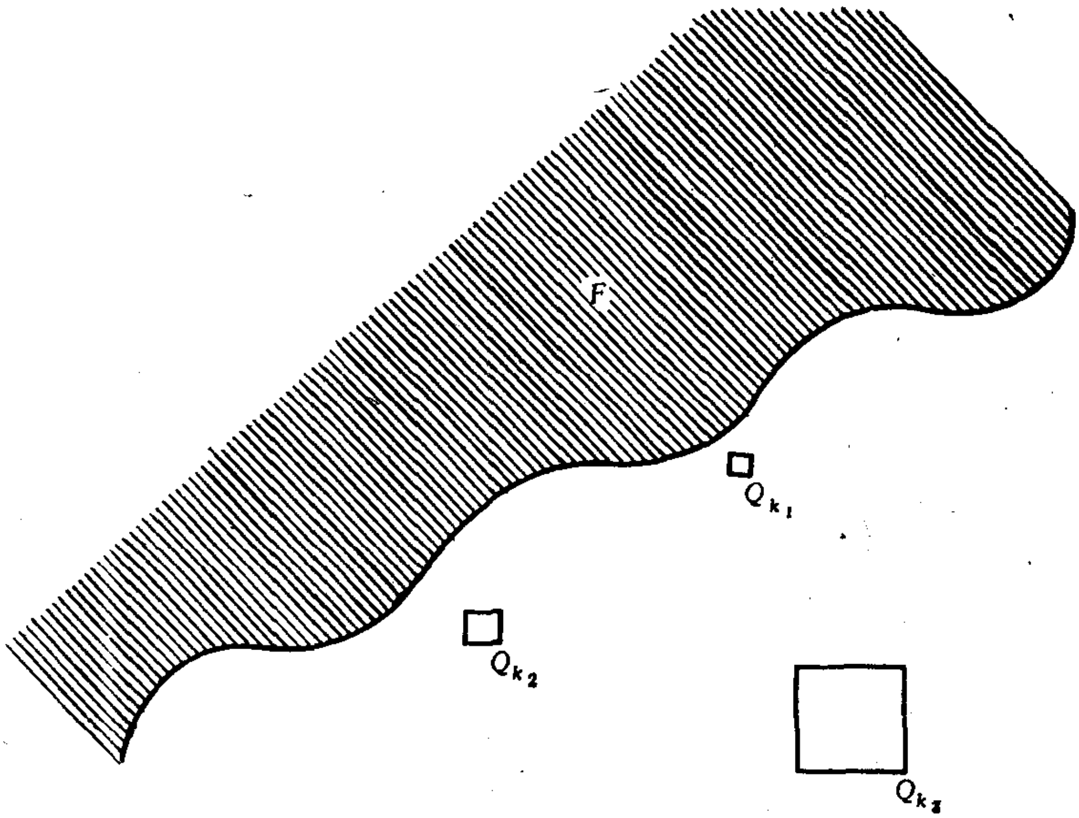


图 2 分解F的余集为立方体，其直径近似于它们到F的距离

(c) $\Omega$ 是不相重叠的立方体 $Q_k$ 的并集：$\Omega=\bigcup_k Q_k$，而对每个 $Q_k$ 有

$$a<\frac{1}{m(Q_k)}\int_{Q_k} f(x)dx\leqslant 2^n a. \tag{16}$$

3.3 我们分解 $\boldsymbol{R}^n$ 成为相等方体构成的网，方体间彼此不重叠，它们的公共直径足够大，使得

$$\frac{1}{m(Q')}\int_{Q'} fdx\leqslant a$$

对方体网中的每个方体 $Q'$ 都成立。

令 $Q'$ 是方体网中的固定方体，我们把它等分为 $2^n$ 个小方体，即把 $Q'$ 的每条边等分为二，连接相对边的分点而成。令 $Q''$ 是这

种新的小方体中的一个，于是有两种可能的情形：

第一种情形

$$\frac{1}{m(Q'')}\int_{Q''} f dx \leqslant a;$$

第二种情形

$$\frac{1}{m(Q'')}\int_{Q''} f dx > a.$$

在第二种情形，我们对 $Q''$ 不再进行细分，并把 $Q''$ 就选作某个 $Q_k$，即定理叙述中所需的。这时不等式(16)自然成立，因为

$$a < \frac{1}{m(Q'')}\int_{Q''} f dx \leqslant \frac{1}{2^{-n} m(Q')}\int_{Q'} f dx \leqslant 2^n a.$$

在第一种情形，我们对 $Q''$ 继续进行 $2^n$ 等分，并重复以上的步骤，直到不再出现第一种情形，否则继续细分下去。我们对开始的每个 $Q'$ 都进行这样的步骤，然后对每个满足第一种情形的 $Q''$ 都继续进行这样的步骤，而对所有出现第二种情形的 $Q_k$，取其并集并记为 $\Omega = \bigcup_k Q_k$，再令 $F = {}^c\Omega$。我们要证对于几乎所有的 $x \in F$，$f(x) \leqslant a$ 成立。事实上根据微分定理(参看§1.8中的形式)，我们有

$$f(x) = \lim_{Q} \frac{1}{m(Q)}\int_{Q} f(y) dy,$$

其中极限是考虑所有含有点 $x$ 的方体 $Q$，当其直径趋向于零时的情形。但由 $F$ 的定义，所有含有 $x \in F$ 的方体都属于第一种情形，从而定理证毕。

3.4 我们现在叙述一个直接推论，它在下一章有用。

**推论** 设 $f, a, F, \Omega$ 以及 $Q_k$ 都如定理 4 中所述，则存在仅与维数 $n$ 有关的两个常数 $A$ 与 $B$，使得

(a) $m(\Omega) \leqslant \frac{A}{a}\|f\|_1$;

(b) $\frac{1}{m(Q_k)}\int_{Q_k} f dx \leqslant B\alpha$;

成立，同时定理 4 中的(a)与(b)仍成立。

事实上，由式(16) 我 们 可 取 $B = 2^n$，即得(b)。同样，由(16)可得

$$m(\Omega) = \sum_k m(Q_k) < \frac{1}{\alpha}\int_\Omega f dx \leqslant \frac{1}{\alpha}\|f\|_1,$$

取 $A = 1$，即得(a)。

**3.5** 上述定理 4 的推论可以不从定理 4 导出，而利用极大函数定理 1 以及把开集分解成为不相重叠的方体的并集的定理给以另外的证明。这个比较不直接的证法有利于弄清楚把 $\boldsymbol{R}^n$ 分为 $F$ 与 $\Omega$ 的作用。我们知道在 $F$ 上有 $f(x) \leqslant \alpha$ 几乎处处成立，但这个事实并不能决定 $F$。事实上，$F$ 是为这样的事实决定的：极大函数在其上满足 $M(f)(x) \leqslant \alpha$。因此我们取 $F = \{x: M(f)(x) \leqslant \alpha\}$，从而

$$\Omega = E_\alpha = \{x; M(f)(x) > \alpha\}.$$

于是由定理 1 的结果(b)可知 $m(\Omega) \leqslant (A/\alpha)\|f\|_1$，而 $A$ 取$5^n$就行。注意到由 $F$ 的定义可知 $F$ 是闭的，于是我们可以依照定理 3 选取方体 $Q_k$，并有 $\Omega = \bigcup_k Q_k$，而其直径近似地与它到 $F$ 的距离成比例。现取定这样一个 $Q_k$，并设 $p_k \in F$ 满足

$$\text{distance}(F, Q_k) = \text{distance}(p_k, Q_k).$$

令 $B_k$ 是以 $p_k$ 为中心并包含 $Q_k$ 的内部的最小球体。令

$$\gamma_k = \frac{m(B_k)}{m(Q_k)}.$$

由于 $p_k \in F = \{x: M(f)(x) \leqslant \alpha\}$，我们有

$$\alpha \geqslant (M(f))(p_k) \geqslant \frac{1}{m(B_k)}\int_{B_k} f dx$$

$$\geqslant\frac{1}{\gamma_k m(Q_k)}\int_{Q_k} f dx.$$

利用定理 3 中(c)的不等式，由初等几何可以知道，

$$\gamma_k=\frac{m(B_k)}{m(Q_k)}$$

对一切 $k$ 都不超过一个固定的常数。这样，我们就给出了推论的另一证明。

值得注意的是上述第二个证法还给了我们一个没有预计到的利益，即方体 $Q_k$ 到 $F$ 的距离与它们本身的直径是大致相当的。

**3.6** 最后一个注记涉及本定理与前述定理 1 的关系。我们将会看到前者也蕴含着后者，而且用不着 § 1.6中的覆盖引理。对于这一点可参看本章末的 § 5.1。

## § 4 $L^p$空间的一个内插定理

**4.1** 我们在这里要把定理 1 证明中的一部分论证加以定型化。我们所要考虑的是证明中的这样一部分，它从不等式(b)推导出 $L^p$不等式。这个思想将引导我们到 Marcinkiewicz 的内插定理，或者更准确地说，是内插定理的一个基本的特殊情形。内插定理的更普遍的形式将在后面的附录 B 中给出。

我们需要以下几个定义。设 $T$ 是从 $L^p(\boldsymbol{R}^n)$到 $L^q(\boldsymbol{R}^n)$的算子，$1\leqslant p\leqslant\infty$，$1\leqslant q\leqslant\infty$。我们称 $T$ 是$(p,q)$型的，如果

$$\|T(f)\|_q\leqslant A\|f\|_p,\qquad f\in L^p(\boldsymbol{R}^n),\tag{17}$$

其中 $A$ 与 $f$ 无关。类似地称 $T$ 是弱$(p,q)$型的，如果

$$m\{x:\ |T(f)(x)|>\alpha\}\leqslant\left(\frac{A\|f\|_p}{\alpha}\right)^q\qquad (q<\infty),\tag{18}$$

其中 $A$ 与函数 $f$，正数 $\alpha$ 都无关。

当 $q=\infty$时，称 $T$ 是弱$(p,q)$型的，假如 $T$ 是$(p,q)$型的话。

注意到式(17)蕴含(18)，所以$(p,q)$型概念强于弱$(p,q)$型概念。事实上，当$q<\infty$时，

$$\alpha^q m\{x: |T(f)(x)|>\alpha\}\leqslant\int_{R^n}|T(f)|^q dx$$
$$=\|T(f)\|_q^q\leqslant(A\|f\|_p)^q.$$

有时还必须处理定义在几个不同的$L^p$空间上的算子$T$。这时我们用$L^{p_1}(\boldsymbol{R}^n)+L^{p_2}(\boldsymbol{R}^n)$来代表这样的函数$f$组成的空间，它可以写成$f=f_1+f_2$，其中$f_1\in L^{p_1}(\boldsymbol{R}^n)$而$f_2\in L^{p_2}(\boldsymbol{R}^n)$。现设$p_1<p_2$，我们可以看出

$$L^p(\boldsymbol{R}^n)\subset L^{p_1}(\boldsymbol{R}^n)+L^{p_2}(\boldsymbol{R}^n)$$

对一切满足$p_1\leqslant p\leqslant p_2$的$p$都成立。事实上，当$f\in L^p(\boldsymbol{R}^n)$时，令$\gamma$为一固定的正数。我们记

$$f_1(x)=\begin{cases}f(x), & |f(x)|>\gamma,\\ 0, & |f(x)|\leqslant\gamma;\end{cases}$$

$$f_2(x)=\begin{cases}f(x), & |f(x)|\leqslant\gamma,\\ 0, & |f(x)|>\gamma.\end{cases}$$

于是

$$\int|f_1(x)|^{p_1}dx=\int|f_1(x)|^p|f_1(x)|^{p_1-p}dx$$
$$\leqslant\gamma^{p_1-p}\int|f(x)|^p dx,$$

这是因为$p_1-p\leqslant0$。类似地可得

$$\int|f_2(x)|^{p_2}dx=\int|f_2(x)|^p|f_2(x)|^{p_2-p}dx$$
$$\leqslant\gamma^{p_2-p}\int|f(x)|^p dx.$$

我们方才运用的把函数分成大小两部分的思想就是下述定理证明的主要思想。

4.2 **定理5** 设$1<r\leqslant\infty$。若$T$是一个从$L^1(\boldsymbol{R}^n)+L^r(\boldsymbol{R}^n)$到$\boldsymbol{R}^n$的可测函数空间的次可加算子，它既是弱(1,1)型又是弱

$(r,r)$型的，则它也一定是$(p,p)$型的，其中 $p$ 满足$1<p<r$。更明确地说，假如对于一切 $f,g\in L^1(\boldsymbol{R}^n)+L^r(\boldsymbol{R}^n)$，有

(a) $|T(f+g)(x)|\leqslant|T(f)(x)|+|T(g)(x)|$;

(b) $m\{x:\ |T(f)(x)|>\alpha\}\leqslant\dfrac{A_1}{\alpha}\|f\|_1,\quad f\in L^1(\boldsymbol{R}^n)$;

(c) $m\{x:\ |T(f)(x)|>\alpha\}\leqslant\left(\dfrac{A_r}{\alpha}\|f\|_r\right)^r,\quad f\in L^r(\boldsymbol{R}^n)$,

$r<\infty$（当 $r=\infty$时，我们假定（17）形式的不等式成立）。

则我们有

$$\|T(f)\|_p\leqslant A_p\|f\|_p,\qquad f\in L^p(\boldsymbol{R}^n)$$

对一切$1<p<r$成立，其中 $A_p$ 只与 $A_1,A_r$, $p$ 及 $r$ 有关。

4.3 我们将限于 $r<\infty$的情形来证明这个定理。在 $r=\infty$ 的情形，下述证明需稍加变更，我们将留给有兴趣的读者作为练习来考虑。这一情形实际上已经隐含在§1.5所给的证明中了。

现在设 $f\in L^p(\boldsymbol{R}^n)$。我们要估计分布函数 $\lambda(\alpha)=m\{x:\ |T(f)(x)|>\alpha\}$。暂时固定 $\alpha$。如同上面所述，我们可以分解 $f=f_1+f_2$，使得 $f_1\in L^1(\boldsymbol{R}^n)$，$f_2\in L^r(\boldsymbol{R}^n)$，这个分解实际上是以 $\gamma>0$为高度对$|f|$进行分割的。那时 $\gamma$ 是任意取定的，我们现在让 $\gamma$ 等于 $\alpha$。由于$|T(f)(x)|\leqslant|T(f_1)(x)|+|T(f_2)(x)|$，我们有

$$\{x:|T(f)(x)|>\alpha\}$$
$$\subset\left\{x:\ |T(f_1)(x)|>\frac{\alpha}{2}\right\}\cup\left\{x:\ |T(f_2)(x)|>\frac{\alpha}{2}\right\},$$

从而有

$$\begin{aligned}\lambda(\alpha)&=m\{x:\ |T(f)(x)|>\alpha\}\\&\leqslant m\{x:\ |T(f_1)(x)|>\alpha/2\}\\&\quad+m\{x:\ |T(f_2)(x)|>\alpha/2\}.\end{aligned}$$

于是根据假定(b)与(c)，

$$\lambda(\alpha)\leqslant\frac{A_1}{\alpha/2}\int|f_1(x)|dx+\frac{A_r^r}{(\alpha/2)^r}\int|f_2(x)|^r dx.$$

由 $f_1$ 与 $f_2$ 的定义我们可得

$$(19)\quad \lambda(\alpha)\leqslant\frac{2A_1}{\alpha}\int_{|f|>\alpha}|f(x)|dx+\left(\frac{2A_r}{\alpha}\right)^r\int_{|f|\leqslant\alpha}|f|^r dx.$$

现在我们已经知道

$$\int_{R^n}|T(f)|^p dx=-\int_0^\infty \alpha^p d\lambda(\alpha)$$

$$=p\int_0^\infty \alpha^{p-1}\lambda(\alpha)\,d\alpha,$$

在式(19)两端各乘 $p\alpha^{p-1}$，然后关于 $\alpha$ 在 $(0,\infty)$ 积分。注意到

$$\int_0^\infty \alpha^{p-1}\alpha^{-1}\left\{\int_{|f|>\alpha}|f|dx\right\}d\alpha=\int_{R^n}|f|\left\{\int_0^{|f|}\alpha^{p-2}d\alpha\right\}dx$$

$$=\frac{1}{p-1}\int_{R^n}|f|^p dx\quad(p>1),$$

以及

$$\int_0^\infty \alpha^{p-1}\alpha^{-r}\left\{\int_{|f|\leqslant\alpha}|f|^r dx\right\}d\alpha=\int_{R^n}|f|^r\left\{\int_{|f|}^\infty \alpha^{p-1-r}d\alpha\right\}dx$$

$$=\frac{1}{r-p}\int_{R^n}|f|^r|f|^{p-r}dx\quad(p<r),$$

我们即得

$$\|T(f)\|_p\leqslant A_p\|f\|_p,\quad (A_p)^p=\left(\frac{2A_1}{p-1}+\frac{(2A_r)^r}{r-p}\right)p.$$

应当指出在极大函数的情形，界 $A_p$ 满足不等式

$$A_p\leqslant A/(p-1)\quad(p\to1).$$

上述定理的一个例子当然是定理 1 的(c)。而定理 1 的(b)表示算子 $f\to M(f)$ 是弱(1,1)型，至于 $f\to M(f)$ 为 $(\infty,\infty)$ 型是显然的。另一重要的关于 Marcinkiewicz 内插定理的应用将在奇异

积分中碰到，这是下一章的主题。

## §5 进一步的结果

**5.1** 定理4（在§3中）可以用来给出定理1的关于极大函数基本不等式(b)的另一个证明。事实上，对于$f\geqslant 0$，$f\in L^1(\boldsymbol{R}^n)$以及$\alpha>0$，令$\Omega=\bigcup_k Q_k$，如定理4所示，则

$$m(\Omega)\leqslant\frac{1}{\alpha}\int f dx.$$

令$Q_k^*$表示与$Q_k$同一中心但直径为其两倍的方体。显然有

$$m\left(\bigcup_k Q_k^*\right)\leqslant\frac{2^n}{\alpha}\int f dx,$$

且当$x\in\bigcup_k Q_k^*$时，存在适当的常数$c$，使$M(f)(x)\leqslant c\alpha$；也就是

$$m\{x: M(f)(x)>c\alpha\}\leqslant\frac{2^n}{\alpha}\int f(x)dx.$$

说详细的情形可参看 Calderón 与 Zygmund[1]，114—115。

**5.2** (a) 设$f$支于有限球体$B\subset\boldsymbol{R}^n$。若$|f|\log(2+|f|)$在$B$可积，则$M(f)\in L^1(B)$。事实上

$$\int_B M(f)dx\leqslant m(B)+\int_{M(f)>1}M(f)dx,$$

而

$$\int_{M(f)>1}M(f)dx=\int_1^\infty\lambda(\alpha)d\alpha+\lambda(1),$$

其中

$$\lambda(\alpha)=m\{x: M(f)>\alpha\}\leqslant\frac{2A}{\alpha}\int_{|f|>\alpha/2}|f|dx$$

（由§1的式(5)），参看Wiener[1]。

(b) 上述关于$\lambda(\alpha)$的不等式本质上是可逆的。事实上：对于适当的常数$c>0$，

$$m\{x: M(f)(x)>c\alpha\} \geqslant \frac{2^{-n}}{\alpha}\int_{|f|>\alpha}|f|dx.$$

为了证明这一点，我们应用定理 4 到$|f|$与$\alpha$。这就导致

$$2^n\alpha \geqslant \frac{1}{m(Q_k)}\int_{Q_k}|f|dx>\alpha.$$

若$x\in Q_k$，则$M(f)(x)>c\alpha$，从而

$$m\{x: M(f)(x)>c\alpha\} \geqslant \sum_k m(Q_k) \geqslant \frac{2^{-n}}{\alpha}\int_{\cup Q_k}|f|dx.$$

但是当 $x\notin \cup Q_k$时，$|f|\leqslant\alpha$；从而

$$\int_{\cup Q_k}|f|\ dx \geqslant \int_{|f|>\alpha}|f|dx.$$

这样就得到了所要的不等式。

(c) 上述(a)有逆定理。若$f$支在球体$B$上，则$M(f)\in L^1$蕴含了$|f|\log(2+|f|)$在$B$上可积。为证明这结果，把上述(b)中关于$m\{x: M(f)(x)>c\alpha\}$的不等式两端进行积分，像证明(a)部分那样即得所要结果。对于(b)与(c)参看Stein[12]。

(d) 更一般情形，若$|f|$支于$B$，则

$$M(f)(\log(2+M(f)))^k\in L^1(B)$$
$$\Longleftrightarrow |f|(\log(2+|f|))^{k+1}\in L^1(B)\quad (k\geqslant 0).$$

5.3 我们考虑下述问题，即当$S\in\mathscr{F}$时，下面的关系式是否几乎处处成立：

$$(*)\qquad \lim_{\operatorname{diam}(S)\to 0}\frac{1}{m(S)}\int_S f(x-y)dy=f(x),$$

其中$\mathscr{F}$是包含原点的矩形构成的适当的集合族。

(a) 当$\mathscr{F}$是所有的矩形构成的集合族，则$f$即使是有界时，

关系式(*)仍可以不成立。可参看 O. Nikodym[1], Busemann-Feller[1]。

(b) 当$\mathscr{F}$是所有其边平行于坐标轴的矩形构成的集合族时，则对于某些可积函数 $f$，关系式(*)不成立。参看 Saks[1]。

(c) 然而对于所有其边平行于坐标轴构成的集合族，当 $f\in L^p(\boldsymbol{R}^n)(p>1)$时，关系式(*)成立。事实上，假如我们定义

$$\tilde{M}(f)(x)=\sup_{S\in\mathscr{F}}\frac{1}{m(S)}\int_S|f(x-y)|dy.$$

当 $S$ 属于所述的集合族时，$\|\tilde{M}(f)\|_p\leqslant A_p\|f\|_p$，$1<p\leqslant\infty$。这不等式可以通过定理 1 关于一维的$L^p$不等式的 $n$ 次应用得到。

(d) 设$\mathscr{F}$是一个参数的单调的矩形族，每个矩形的边都平行于坐标轴，即 $\mathscr{F}=\{S_t\}_{0<t<\infty}$，且$S_{t_1}\subset S_{t_2}$，当 $t_1\leqslant t_2$。这时对 $f\in L^p(\boldsymbol{R}^n)$，关系式(*)成立。这是由于对这样的单调矩形族，§1.6中的覆盖引理的一个类似结果是成立的。对于(c),(d)以及进一步的有关结果，可参看Zygmund[8]的第 X Ⅶ 章。

**5.4 Vitali覆盖引理** 设可测集 $E$ 为一族球体$\{B_\alpha\}$按下述意义所覆盖，即对于每一$x\in E$与每一正数$\varepsilon$，存在 $B_{\alpha_0}\in\{B_\alpha\}$，使得 $x\in B_{\alpha_0}$，且 $m(B_{\alpha_0})<\varepsilon$。于是存在互不相交的球体的子序列 $B_1,B_2,\cdots,B_k$，$\cdots$，使得

$$m(E-\bigcup_k B_k)=0.$$

关于这一结果以及有关的推广可参看Saks[2]第 4 章。

**5.5** (F. Riesz)设$F(x)$ 是实的连续的单变量有界函数，令$\Omega$表示满足

$$\sup_{h>0}\frac{F(x+h)-F(x)}{h}>a$$

的点 $x$ 所构成的集合。则

$$\Omega=\bigcup_{k=1}^{\infty} I_k,$$

其中$I_k=(a_k,b_k)$，且$\{F(b_k)-F(a_k)\}/(b_k-a_k)=\alpha$。这个引理可用以给出定理 4 的另一个证明，如果我们令

$$F(x)=\int_0^x f(t)\,dt \qquad \text{(当然指一维的情形)。}$$

这时不等式(16)当即为等式

$$\alpha=\frac{1}{m(I_k)}\int_{I_k} f dx$$

所替代。参看 Riesz 与 Nagy[1] 第 1 章。

**5.6** (a) 不等式(15)的一个加强形式如下：设$\psi\geqslant 0$，则

$$\int_F I_*(x)\psi(x)dx\leqslant\int_{^cF} M(\psi)(x)dx,$$

其中$M(\psi)$是函数$\psi$的极大函数。这表示$I_*(x)\in L^p(F)$对于一切$1\leqslant p<\infty$都成立。假如我们规定$\psi\log(2+\psi)$可积(§5.2)，则可得

$$\int_F \exp(aI_*(x))dx<\infty$$

($a>0$适当取定)。

(b) $I_*(x)$的一个变式是

$$\mathscr{I}_*(x)=\int_{\boldsymbol{R}^n}\frac{\delta(x+y)dy}{[|y|+\delta(x+y)]^{n+1}}.$$

这时我们有：(i) $\mathscr{I}_*(x)\geqslant cI_*(x)$，$x\in F$；(ii) $\mathscr{I}_*(x)<\infty$对几乎所有$x\in\boldsymbol{R}^n$成立；(iii) 更进一步，存在一个正常数$a$，使对每一个有限球体$B$，

$$\int_B \exp(a\mathscr{I}_*(x))dx<\infty.$$

关于这方面可参看Carleson[3]。

**5.7** 设$f\in L^p(\boldsymbol{R}^n)$，$1\leqslant p<\infty$。这时由§1.8的论证略经改

变即得

$$\lim_{r\to 0}\frac{1}{m(B(x,r))}\int_{B(x,r)}|f(y)-f(x)|^p dy=0$$

对几乎所有的 $x$ 都成立。

5.8 设$f_1,f_2,\cdots,f_n,\cdots$是$L^p(\boldsymbol{R}^n)$中的一个函数序列，满足

$$\left(\sum_j|f_j(x)|^2\right)^{1/2}\in L^p(\boldsymbol{R}^n).$$

令$M(f_j)$表示$f_j$的极大函数。于是

$$\left\|\left(\sum_j|M(f_j)|^2\right)^{1/2}\right\|_p\leqslant A_p\left\|\left(\sum_j|f_j|^2\right)^{1/2}\right\|_p,\quad 1<p<\infty.$$

关于$A_p$的估计有 $A_p=O((p-1)^{-1})(p\to 1)$ 以及 $A_p=O(p^{1/2})$ $(p\to\infty)$。这些估计是最佳可能的。当 $f_j$ 取作不相交的方体的特征函数时，上述结果实质上包含了§5.6的结果，参看Fefferman与Stein[1]。

## 注　释

**节1** 关于积分与微分的基础事项，参看Saks[2]。当$n=1$时，原始的极大函数定理属于Hardy-Littlewood[1]，其 $n$ 维论述见Wiener[1]。关于§1.6中的覆盖引理见Marcinkiewicz-Zygmund[3]与 Wiener[1]。读者如果把这个引理与《富里叶分析》第Ⅱ章§3中的进一步改进的论述作一比较，将会发觉是有教益的。这改进的论述基于 Besicovitch 的思想。关于其它的推广可参看Hewitt[1]与Stein[11]。

**节2** Marcinkiewicz 积分首先出现在Marcinkiewicz[1]，[2]与[3]；亦可参看Zygmund[8]，第Ⅳ章。它的 $n$ 维形式的系统运用见Calderón-Zygmund[7]。

**节3** §3.2 中的分解定理见 Calderón-Zygmund[1]。它与

Whitney分解的紧密联系似乎首先在Stein[10]中指出。

**节5** 内插定理5属于Marcinkiewicz[5]，其更广泛的论述如附录B所示则属于Zygmund，但那里给出的证明属于Hunt[1]，亦可参看《富里叶分析》第V章所给出的更广泛的处理

# 第二章 奇异积分

奇异积分理论来源的一个基本例子是 Hilbert 变换。$f$ 的 Hilbert 变换定义为

$$(*)\qquad \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{f(x-y)}{y}dy,$$

其中非绝对收敛的积分应了解为一个合适的极限过程。在这里我们挑出Hilbert变换理论的若干特征，使得我们能够根据它们描绘一下在这章将要讨论的 $n$ 维奇异积分。

(a) $L^2$ **理论**。无论是在这里还是在高维的一般情形，我们讨论的是同平移可交换的算子。因此，卷积、Fourier 变换与Plancherel 定理等（简言之，$\boldsymbol{R}^n$ 上调和分析中一些基本的）工具是不可少的；我们将从扼要叙述这些内容开始这一章。

(b) $L^p$ **理论**。算子$(*)$以及我们要讨论的 $n$ 维推广的一个基本性质是 $L^p$ 有界性，$1<p<\infty$。在 Hilbert变换的情形，这个经典定理是由 M. Riesz 用复函数论证明的。这个方法不适用于一般的情形，在后者$L^p$理论将从$L^1$理论作为一个推论得到。

(c) $L^1$**理论**。Hilbert 变换并不是$L^1$的有界算子。然而，却有一个替代它的结果，那就是弱(1,1)型。在一般情形有类似的情况。证明弱(1,1)型的实变技巧，在 Hilbert 变换的情形是由 Besicovitch 和 Titchmarsh 首创的，而在 $n$ 维理论的研究中由 Calderón 和 Zygmund 作了进一步的发展。这些方法的叙述可以说构成了这一章的核心。自然，我们一定会用到第一章的一般的实变理论。

(d) Hilbert**变换的特殊性质**。它们是：

1° 算子$(*)$不仅同平移可交换，而且同展缩 $x\to\delta x(\delta>0)$

可交换。因此自然地，描述其 $n$ 维推广的定理本质上对展缩也是不变的。进一步，类似(*)这样一些对展缩保持不变的算子，构成了一个重要的子类，其理论更加清楚且具有深远意义。这是§4的内容。

2° 与解析函数的联系。在变换(*)(或它的某种 $n$ 维推广)与解析函数（或其 $n$ 维推广）之间存在一个特殊的联系。这个联系的含义及其涉及旋转不变的有关性质，将在下一章中叙述。

## §1 $R^n$ 上调和分析某些内容的回顾

在这里，我们不加证明地叙述 $\boldsymbol{R}^n$ 上调和分析理论中的某些初等事实，在局部紧的Abel群的情形，很容易得到它们的自然推广。

1.1 同已经用过的空间$L^p(\boldsymbol{R}^n)(1\leqslant p\leqslant\infty)$一起，我们考虑由在无穷远趋向于0的连续函数组成的空间$C_0(\boldsymbol{R}^n)$（带着通常的上确界模）；以及它的对偶空间$\mathscr{B}(\boldsymbol{R}^n)$，大家知道它等同于由所有有限测度 $d\mu$ 组成的 Banach 空间，其模为

$$\|d\mu\|=\int_{R^n}|d\mu|.$$

空间 $L^1$ 通过等距变换 $f(x)\to f(x)dx$($dx$ 表示 Lebesgue 测度) 可以看成$\mathscr{B}(\boldsymbol{R}^n)$的子空间。

$\mathscr{B}$上的一个基本运算是卷积。如果$\mu_1,\mu_2\in\mathscr{B}$，那末$\mu=\mu_1*\mu_2$由

$$\mu(f)=\int_{R^n}\int_{R^n}f(x+y)d\mu_1(x)d\mu_2(y)$$

定义。

我们有$\mu_1*\mu_2=\mu_2*\mu_1$，并且 $\|\mu\|\leqslant\|\mu_1\|\|\mu_2\|$。当一个因子限制在$L^1(\boldsymbol{R}^n)$时，卷积运算的值域仍在$L^1(\boldsymbol{R}^n)$。因此，当 $f\in L^1(\boldsymbol{R}^n)$时，

$$g=f*\mu=\int_{R^n}f(x-y)d\mu(y)$$

对几乎所有的 $x$ 绝对收敛，并且$g\in L^1(\boldsymbol{R}^n)$，还有

$$\|g\|_1\leqslant\|f\|_1\|\mu\|.$$

类似地，如果$f\in L^p(\boldsymbol{R}^n)$，那末

$$\int_{\boldsymbol{R}^n}f(x-y)d\mu(y)$$

也属于$L^p$，并且$\|g\|_p\leqslant\|f\|_p\|\mu\|$。这就是说，当$\mu\in\mathscr{B}$时，我们刚刚讨论的变换

$$f\to\int f(x-y)d\mu(y)$$

是$L^p(\boldsymbol{R}^n)$有界的，并且同平移$x\to x+h$可交换。这类变换用下面的定理来刻划：

1.2 **命题1** 设$T$是 $L^1(\boldsymbol{R}^n)$ 到自身的有界线性变换。则$T$同平移可交换的必要充分条件是在$\mathscr{B}(\boldsymbol{R}^n)$中存在一个测度$\mu$，使得$T(f)=f*\mu$对所有 $f\in L^1(\boldsymbol{R}^n)$成立，并且还有$\|T\|=\|\mu\|$。

1.3 对每个测度$\mu\in\mathscr{B}(\boldsymbol{R}^n)$，我们可以定义它的 Fourier 变换$\hat{\mu}(y)$为

$$\hat{\mu}(y)=\int_{\boldsymbol{R}^n}e^{2\pi i x\cdot y}d\mu(x)①.$$

特别地，Fourier变换对所有$f\in L^1(\boldsymbol{R}^n)$有定义，并且

$$\hat{f}(y)=\int_{\boldsymbol{R}^n}e^{2\pi i x\cdot y}f(y)dy\in C_0(\boldsymbol{R}^n).$$

Fourier 变换的基本性质是，如果$\mu=\mu_1*\mu_2$，那末

$$\hat{\mu}(y)=\hat{\mu}_1(y)\hat{\mu}_2(y).$$

如果 $f\in L^1(\boldsymbol{R}^n)\cap L^2(\boldsymbol{R}^n)$，那末 $\hat{f}\in L^2(\boldsymbol{R}^n)$，并且$\|\hat{f}\|_2=\|f\|_2$。Fourier 变换可以通过连续性推广到整个 $L^2(\boldsymbol{R}^n)$，使它们成为$L^2(\boldsymbol{R}^n)$的酉变换。用连续性还可以得到，如果 $g=f*\mu$，其中 $f\in L^2(\boldsymbol{R}^n)$，$\mu\in\mathscr{B}(\boldsymbol{R}^n)$，那末$\hat{g}(y)=\hat{f}(y)\hat{\mu}(y)$。

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① 当把以后出现的公式与别的教程中的公式比较时，请注意，我们按习惯在指数中加上了因子$+2\pi$。

1.4 命题1的$L^2$类似是下面的定理。

**命题2** 设$T$是$L^2(\boldsymbol{R}^n)$到自身的有界线性变换。则$T$同平移可交换的必要充分条件是存在有界可测函数$m(y)$（“乘子”），使得$(T(f))^\wedge(y)=m(y)\hat{f}(y)$对所有$f\in L^2(\boldsymbol{R}^n)$成立，并且还有$\|T\|=\|m\|_\infty$。

注意，对$T$还在$L^1(\boldsymbol{R}^n)$有界的特殊情形，有

$$m(y)=\hat{\mu}(y),$$

其中$T(f)=f*\mu$。

## §2 奇异积分：核心部分

上述两个命题表明，在$L^1$或$L^2$有界的平移不变算子的构造是既简单又好理解的。在某些$L^p$($p\neq2$但又不是所有的$p$)有界的平移不变算子的研究是既费劲又未完成的。然而，对一类很重要的算子，很多研究却已完成。这类算子包含了带奇异核的卷积算子，其核仅在一个有限点(原点)与无穷远点具有奇性。那些核的奇性位置在不同地方或者比孤立点更一般的类似研究，是一个似乎只能留待进一步理论研究的重要问题。

下面的定理显示了主要结果的本质。然而，同§3中的完全发展比起来，它的叙述缺乏某种普遍性。现在我们来做这个不大自然的工作，只是由于它有助于了解理论的主要思想。

2.2 **定理1** 令$K\in L^2(\boldsymbol{R}^n)$。假设

(a) $K$的Fourier变换是本质有界的

$$|\hat{K}(x)|\leqslant B. \tag{1}$$

(b) $K$在原点之外属于$C^1$，并且

$$|\nabla K(x)|\leqslant B/|x|^{n+1}. \tag{2}$$

对$f\in L^1\cap L^p$，令

$$T(f)(x)=\int_{\boldsymbol{R}^n}K(x-y)f(y)dy. \tag{3}$$

则存在常数$A_p$，使得

(4) $$\|T(f)\|_p\leqslant A_p\|f\|_p,\quad 1<p<\infty.$$

因此，人们可以用连续性把$T$开拓到整个$L^p$。常数$A_p$仅仅依赖于$p,B$与维数$n$。特别地，它不依赖于$K$的$L^2$模。

以下的说明是重要的。$K\in L^2$的假设是为了保证$Tf$在$L^p$的一个稠密子集（在这里是$L^1\cap L^p$）有直接的定义，它可以用其它的假设（例如$K\in L^1+L^2$）代替。

在应用中条件$K\in L^2$是无所谓的，这是由于它可以用一个合适的极限过程代替；而且还由于定理1中最后的界并不依赖于$K$的$L^2$模。参看下面§3.2的定理2。

2.3 **证明** 第一步。$T$是弱(2,2)型的。

用Fourier变换我们知道，当$f\in L^1\cap L^2$时，

$$(T(f))^\wedge(y)=\hat{K}(y)\hat{f}(y),$$

因此由假设(a)与Plancherel定理

(5) $$\|T(f)\|_2\leqslant B\|f\|_2.$$

根据(5)，$T$有唯一的到整个$L^2$的扩张，使得(5)仍然保持成立。于是用第一章§4.1的说明，我们得到

(6) $$m\{x:\ |T(f)(x)|>\alpha\}\leqslant\left(\frac{B^2}{\alpha^2}\right)\int_{R^n}|f|^2dx,$$

$$f\in L^2(R^n).$$

2.4 第二步。$T$是弱(1,1)型的。

对$f\in L^1(R^n)$，我们通过分解$f$为$f=g+b$来研究$T(f)$，其中$g$是"好的"部分（good part），它在$f$本质上小的集合上等于$f$；$b$是"坏的"部分（bad part），定义在$f$本质上是大的集合上。好的部分$g$归结到$L^2(R^n)$，而$L^2$的上述结果(6)就给出$T(g)$的一个合适的估计。在研究Hilbert变换时，人们已能意会到处理大的$b$的想法。事实上，在积分

(7) $$\int_{-L}^{+L}\frac{b(y)\,dy}{x-y}$$

中，一个初等的（但便利的）估计的主要障碍是积分 $1/x$ 时出现对数，也就是

$$\int_h^L \frac{dx}{x} \sim \log\frac{1}{h}, \qquad h \to 0.$$

想法就是用

$$\int_{-L}^{+L}\left[\frac{1}{x-y}-\frac{1}{x}\right]b(y)\,dy \tag{8}$$

代替式(7)，这样做是可以的，如果

$$\int_{-L}^{+L} b(y)\,dy = 0.$$

注意

$$\left|\frac{1}{x-y}-\frac{1}{x}\right| \sim \frac{1}{x^2},$$

只要 $x$ 显然与区间$[-L,L]$分离开来（譬如说，$|x|\geqslant 2L$）；而另一方面

$$L\int_{|x|\geqslant 2L}\frac{dx}{x^2}\leqslant 1.$$

在这个方法中人们可以避免对数的困难，只要 $b$ 在一个合适的区间（在 $\boldsymbol{R}^n$ 的情形是方体）中的积分为零。这就是 $b$ 在下面(11)中所表示的性质。

一旦用式(8)代替(7)，我们就必须对出现的方体把式(8)的类似项加起来。这个和归结到用关于距离函数的Marcinkiewicz积分（第一章的式(14)）来控制，而借助于那一章§2.3的引理，就会对完成我们的估计提供必要的信息。

2.4.1 我们转到细节。我们需要找出常数 $C$，使得①

① 为了书写不等式方便，直到本章末，我们将采取以下的约定，$C$ 表示一个一般的常数（在不同的地方出现时不必是相同的），但它只依赖于定理假设中的常数 $B$ 与维数 $n$。

$$(9)\qquad m\{x:\ |T(f)(x)|>\alpha\}\leqslant\frac{C}{\alpha}\int_{R^n}|f(x)|dx,\qquad f\in L^1(\boldsymbol{R}^n).$$

为此，固定$\alpha$，对这个$\alpha$与$|f(x)|$，应用第一章§3.4中定理4的推论。这样，我们有$\boldsymbol{R}^n=F\cup\Omega$，$F\cap\Omega=\varnothing$；$|f(x)|\leqslant\alpha$，$x\in F$；$\Omega=\bigcup_{j=1}^{\infty}Q_j$，其中$Q_j$互不重叠；

$$m(\Omega)\leqslant\frac{C}{\alpha}\int_{R^n}|f|dx\ \text{ 并且 }\ \frac{1}{m(Q_j)}\int_{Q_j}|f|dx\leqslant C\alpha.$$

于是我们令

$$(10)\qquad g(x)=\begin{cases}f(x), & x\in F,\\ \dfrac{1}{m(Q_j)}\displaystyle\int_{Q_j}f(x)dx, & x\in Q_j,\end{cases}$$

它几乎处处定义了$g(x)$。由此以及$f(x)=g(x)+b(x)$，可知

$$(11)\qquad \begin{aligned}&b(x)=0, && x\in F,\\ &\int_{Q_j}b(x)dx=0, && \text{对每个方体}Q_j.\end{aligned}$$

现在由于$T(f)=T(g)+T(b)$，便有

$$\begin{aligned}&m\{x:\ |T(f)(x)|>\alpha\}\\ &\quad\leqslant m\left\{x:\ |T(g)(x)|>\frac{\alpha}{2}\right\}+m\left\{x:\ |T(b)(x)|>\frac{\alpha}{2}\right\},\end{aligned}$$

从而只要对不等式右边的两项建立类似于式(9)所要求的不等式。

2.4.2 $T(g)$的估计。我们有$g\in L^2(\boldsymbol{R}^n)$，因为由式(10)

$$\begin{aligned}\|g\|_2^2&=\int_{R^n}|g(x)|^2dx=\int_F|g(x)|^2dx+\int_\Omega|g(x)|^2dx\\ &\leqslant\int_F\alpha|f(x)|dx+C^2\alpha^2m(\Omega)\\ &\leqslant(C^2A+1)\alpha\|f\|_1.\end{aligned}$$

现在如果把$L^2$理论的不等式(6)用到$g$，我们就得到

(12) $$m\{x: |T(g)(x)|>\alpha/2\}\leqslant\frac{C}{\alpha}\|f\|_1.$$

2.4.3 $T(b)$ 的估计。记

$$b_j(x)=\begin{cases}b(x), & x\in Q_j\\ 0, & x\notin Q_j.\end{cases}$$

这样，$b(x)=\sum_j b_j(x)$，而$T(b)(x)=\sum_j T(b_j)(x)$，其中

(13) $$T(b_j)(x)=\int_{Q_j}K(x-y)b_j(y)dy.$$

对 $x\in F\left(=\bigcup_j Q_j\text{的余集}\right)$我们有可能得到式(13)的一个方便的估计。首先，由于

$$\int_{Q_j}b_j(y)dy=0,$$

我们有

$$T(b_j)(x)=\int_{Q_j}[K(x-y)-K(x-y^j)]b_j(y)dy,$$

其中$y^j$是方体$Q_j$的中心。又由于$|\nabla K|\leqslant B|x|^{-n-1}$，故

$$|K(x-y)-K(x-y^j)|\leqslant C\frac{\text{diameter}(Q_j)}{|x-\bar{y}^j|^{n+1}},$$

其中$\bar{y}^j$是联结$y^j$到$y(\in Q)$的线段中的一(变化)点，diameter$(Q_j)$表示$Q_j$的直径。

其次，我们用第一章§3.4中关于 $Q_j$ 的直径与它到$F$的距离是可比较的说明。这意味着，当$x$是$F$的固定点，而$y$在$Q_j$变化时，距离$\{|x-y|\}$都是相互可比较的。因此

$$|T(b_j)(x)|\leqslant C\cdot\text{diameter}(Q_j)\int_{Q_j}\frac{b(y)dy}{|x-y|^{n+1}}.$$

然而

$$\int_{Q_j}|b(y)|dy\leqslant\int_{Q_j}|f(y)|dy+Ca\int_{Q_j}dy,$$

故
$$\int_{Q_j}|b(y)|dy\leqslant(1+C)am(Q_j).$$

这就有下面的推论。如果用$\delta(y)$表示$y$到$F$的距离，由于

$$\text{diameter}(Q_j)\cdot m(Q_j)\leqslant C\int_{Q_j}\delta(y)dy,$$

那末

$$|T(b_j)(x)|\leqslant Ca\int_{Q_j}\frac{\delta(y)}{|x-y|^{n+1}}dy,\quad x\in F.$$

最后

$$(14)\qquad |T(b)(x)|\leqslant Ca\int_{R^n}\frac{\delta(y)}{|x-y|^{n+1}}dy,\quad x\in F.$$

这个通过Marcinkiewicz积分对$T$的控制，正是我们前面指出过的。剩下来的就比较容易了。

用§2.3的引理就有

$$(15)\qquad \int_F|T(b)(x)|dx\leqslant Cam(\Omega)\leqslant C\|f\|_1.$$

从这个不等式直接推出

$$(16)\qquad m\{x\in F:\ |T(b)(x)|>a/2\}\leqslant\frac{2C}{a}\|f\|_1.$$

然而，由于

$$m({}^cF)=m(\Omega)\leqslant\frac{C}{a}\|f\|_1,$$

我们就得到$T(b)$的估计，这就是

$$m\{x:\ |T(b)(x)|>a/2\}\leqslant\frac{C}{a}\|f\|_1.$$

把它与对$T(g)$的类似结果(12)结合起来，我们就得到了式(9)，也就是说$T$是弱(1,1)型的。

**2.5 最后一步．$L^p$ 不等式．**

(a) 对 $p=2$，看§2.3。

(b) 对$1<p<2$，充分地只要对 $r=2$验证内插定理（第一章§4.2)的假设。$T$显然对$L^1(\boldsymbol{R}^n)+L^2(\boldsymbol{R}^n)$有定义并且是线性的。由§2.4它是弱(1,1)型的，而由§2.3它是弱(2,2)型的，并且界都仅仅依赖于$B$与维数$n$（$B$是定理假设中出现的常数)。因此，由内插定理推出

$$\|T(f)\|_p\leqslant A_p\|f\|_p,\quad 1<p<2,\ f\in L^p,$$

其中$A_p$只依赖于$B$，$p$与$n$。

(c) 对$2<p<\infty$，我们利用$L^p$与$L^q$的对偶性，

$$\frac{1}{p}+\frac{1}{q}=1,$$

以及定理对$L^q$是已经证明了的这一事实．注意：如果$\psi$局部可积，又

$$\sup\left|\int\psi\varphi dx\right|=A<\infty,$$

其中上确界取自一切有紧支集的连续函数且满足$\|\varphi\|_q=1$，那末$\psi\in L^p$ 并且$\|\psi\|_p=A$。

于是，取$f\in L^1\cap L^p(2<p<\infty)$，$\varphi$如上所述．由$K\in L^2$以及我们对$f$与$\varphi$的选择，二重积分

$$\int_{\boldsymbol{R}^n}\int_{\boldsymbol{R}^n}K(x-y)f(y)\varphi(x)dxdy$$

绝对收敛；因此它的值等于

$$I=\int_{\boldsymbol{R}^n}f(y)\left(\int_{\boldsymbol{R}^n}K(x-y)\varphi(x)dx\right)dy.$$

但定理对$1<q<2$是成立的（这时用核$K(-x)$代替$K(x)$，而且带有相同的常数$A_q$)。故

$$\int_{\mathbf{R}^n} K(x-y)\varphi(x)dx$$

属于$L^q$，并且它的$L^q$模被$A_q\|\varphi\|_q = A_q$控制。Hölder 不等式表明

$$\left|\int_{\mathbf{R}^n} T(f)\varphi dx\right| = |I| \leqslant A_q\|f\|_p,$$

对上述一切 $\varphi$ 取上确界就给出结果

$$\|T(f)\|_p \leqslant A_q\|f\|_p, \qquad 2<p<\infty。$$

这样，定理证毕。

## §3 奇异积分：前面结果的某些推广与变形

3.1 定理 1 的假设有两个不同的部分。一个用来处理 $L^2$ 理论，这就是假设(a)；它已经写成最一般的但并非最有用的形式。第二个假设(b)，是用来处理弱(1,1)型估计的，却可以做某些改进。叙述这种改进的意义是要表明，使定理 1 一类推理成立的本质上最弱的条件是什么？这条件是下面的

$$(2') \qquad \int_{|x|\geqslant 2|y|} |K(x-y)-K(x)|dx \leqslant B', \quad |y|>0。$$

条件(2)蕴含了(2′)是读者不难验证的。我们有定理 1 证明方法的下述推论。

**推论** 用上面的(2′)代替定理 1 的条件(b)(等式(2))，定理 1 的结论保持成立，其中界 $B$ 用 $B'$ 代替。

论证与定理 1 的相同，只有弱(1,1)型不等式的证明作下述的某些改变。

在这个证明的变化中，我们将不用方体 $Q_j$ 的直径同它到 $F = {}^c\left(\bigcup_j Q_j\right)$的距离是可比较的这一事实(只是由于我们想表明，这个事实在此实际上是不必要的)。我们用一个简单的办法来避免这一点，那就是对每个方体$Q_j$考虑 $Q_j^*$，它与 $Q_j$ 有相同的中心 $y^j$，但延伸了$2n^{1/2}$倍。我们有

(a) $Q_j \subset Q_j^*$；设 $\Omega^* = \bigcup_j Q_j^*$，则 $\Omega \subset \Omega^*$，并且 $m(\Omega^*) \leqslant (2n^{1/2})^n m(\Omega)$；若 $F^* = {}^c\Omega^*$，则 $F^* \subset F$。

(b) 若 $x \overline{\in} Q_j^*$，则 $|x-y^j| \geqslant 2|y-y^j|$，对所有 $y \in Q_j$，这从几何上看是显然的。

另外一个区别是不用距离积分控制 $|T(b)(x)|$，而是直接估计它；代替 $F$ 考虑集合 $F^*$，作为推论可以得到一个有利的估计。

就像在定理中那样

$$T(b_j)(x) = \int_{Q_j} [K(x-y) - K(x-y^j)] b_j(y) dy,$$

我们有

$$\int_{F^*} |T(b)(x)| dx$$

$$\leqslant \sum_j \int_{x \overline{\in} Q_j^*} \int_{y \in Q_j} |K(x-y) - K(x-y^j)| |b(y)| dy dx.$$

然而由(b)，对 $y \in Q_j$

$$\int_{x \overline{\in} Q_j^*} |K(x-y) - K(x-y^j)| dx$$

$$\leqslant \int_{|x'| \geqslant 2|y'|} |K(x'-y') - K(x')| dx' \leqslant B',$$

这里用到了假设。因此

$$(17) \qquad \int_{F^*} |T(b)(x)| dx \leqslant B' \sum_j \int_{Q_j} |b(y)| dy \leqslant C \|f\|_1.$$

这就回到了定理 1 证明中的式(15)，剩下的同以前的一样。

3.2 在我们的叙述中，还有一些可以认为是不能令人满意的，这是由于下列有关的几点：$T$ 的 $L^2$ 有界性是假设而不是作为 $K$ 的某种条件的推论；一个像 $K \in L^2$ 这样的外加条件包含在假设中；为此我们的结果不能直接处理那些“主值的”奇异积分，它

们的存在是由于正负值的抵消。然而，从我们所做的，现在有一个比较简单的方法得到下面的定理，它包含了人们感兴趣的情形。

**定理 2** 假设核$K(x)$满足条件

$$
(18)\qquad \begin{aligned} &|K(x)|\leqslant B|x|^{-n}, && 0<|x|,\\ &\int_{|x|>2|y|}|K(x-y)-K(x)|dx\leqslant B, && 0<|y| \end{aligned}
$$

并且

$$
(19)\qquad \int_{R_1<|x|<R_2}K(x)dx=0,\qquad 0<R_1<R_2<\infty.
$$

对$f\in L^p(\boldsymbol{R}^n)$，$1<p<\infty$，令

$$
(20)\qquad T_\varepsilon(f)(x)=\int_{|y|>\varepsilon}f(x-y)K(y)dy,\qquad \varepsilon>0.
$$

则

$$
(21)\qquad \|T_\varepsilon(f)\|_p\leqslant A_p\|f\|_p,
$$

其中$A_p$与$f$及$\varepsilon$无关。还有，对每个$f\in L^p(\boldsymbol{R}^n)$，极限$\lim\limits_{\varepsilon\to0}T_\varepsilon(f)=T(f)$按$L^p$模存在。算子$T$因此有定义并满足不等式(21)。

前面提到的抵消，包含在条件(19)中。这个假设，连同式(18)，使我们有可能证明$L^2$有界性并由此证明截断积分(20)的$L^p$收敛性。

**3.3** 关于$L^2$有界性我们有下述引理。

**引理** 假设$K$满足上述定理的条件，其中的上界为$B$。令

$$
K_\varepsilon(x)=\begin{cases} K(x), & |x|\geqslant\varepsilon,\\ 0, & |x|<\varepsilon. \end{cases}
$$

则显然$K_\varepsilon\in L^2(\boldsymbol{R}^n)$；对Fourier变换来说，我们有估计

$$
(22)\qquad \sup_y|\hat{K}_\varepsilon(y)|\leqslant CB,\qquad \varepsilon>0,
$$

其中$C$只依赖于维数$n$。

我们首先对特殊情形$\varepsilon=1$证明不等式(22)。

注意，这里需要一个差不多是显然的估计，就是$K_1(x)$满足与$K(x)$同样的条件(18)及(19)，只是界$B$需要用$CB$来代替，其中$C$只依赖于维数$n$。

其次

$$\begin{aligned}\hat{K}_1(y)&=\lim_{R\to\infty}\int_{|x|<R}e^{2\pi i x\cdot y}K_1(x)dx\\&=\int_{|x|<1/|y|}e^{2\pi i x\cdot y}K_1(x)dx\\&\quad+\lim_{R\to\infty}\int_{1/|y|<|x|<R}e^{2\pi i x\cdot y}K_1(x)dx\\&=I_1+I_2.\end{aligned}$$

然而

$$\begin{aligned}&\int_{|x|<1/|y|}e^{2\pi i x\cdot y}K_1(x)dx\\&\quad=\int_{|x|<1/|y|}[e^{2\pi i x\cdot y}-1]K_1(x)dx,\end{aligned}$$

这是由于$K_1$满足条件(19)。因此

$$|I_1|\leqslant C|y|\int_{|x|<1/|y|}|x||K_1(x)|dx\leqslant C'B,$$

这里用了(18)。

为估计$I_2$，选择$z=z(y)$使得$e^{2\pi i y\cdot z}=-1$。这是办得到的，只要取$z=\frac{1}{2}\frac{y}{|y|^2}$，它满足$|z|=\frac{1}{2|y|}$。但

$$\int_{R^n}K_1(x)e^{2\pi i y\cdot x}dx=\frac{1}{2}\int_{R^n}[K_1(x)-K_1(x-z)]e^{2\pi i x\cdot y}dx,$$

因此

$$\lim_{R\to\infty}\int_{1/|y|<|x|<R}K_1(x)e^{2\pi i x\cdot y}dy$$

$$= \frac{1}{2}\lim_{R\to\infty}\int_{1/|y|\leqslant|x|\leqslant R}[K_1(x)-K_1(x-z)]e^{2\pi i x\cdot y}dx$$

$$-\frac{1}{2}\int_{\left\{\substack{1/|y|\leqslant|x+z|\\|x|\leqslant 1/|y|}\right\}} K_1(x)e^{2\pi i x\cdot y}dx.$$

后一个积分区域包含在环体$1/2|y|\leqslant|x|\leqslant 1/|y|$内，由$|K_1(x)|\leqslant B|x|^{-n}$知它有界。前一个积分被

$$\frac{1}{2}\int_{|x|\geqslant 1/|y|}|K_1(x-z)-K_1(x)|dx$$

控制。由于$|z|=(2|y|)^{-1}$，把类似于(18)的条件用到$K_1$，这个积分也有界$CB$。把$I_1$与$I_2$加起来就得到引理对$K_1$的证明。为了过渡到一般的$K_\varepsilon$，我们用一个简单的事实，它在本章叙述的整个理论中都有意义。

设$\tau_\varepsilon$是$\varepsilon$倍的展缩，$\varepsilon>0$，即$\tau_\varepsilon(f)(x)=f(\varepsilon x)$。这样，如果$T$是卷积算子

$$T(f)=\varphi*f=\int_{\mathbf{R}^n}\varphi(x-y)f(y)dy,$$

那末$\tau_{\varepsilon^{-1}}T\tau_\varepsilon$是带有核$\varphi_\varepsilon$的卷积算子，其中$\varphi_\varepsilon(x)=\varepsilon^{-n}\varphi(\varepsilon^{-1}x)$。在我们这里，$T$对应于核$K(x)$，$\tau_{\varepsilon^{-1}}T\tau_\varepsilon$就对应于核$\varepsilon^{-n}K(\varepsilon^{-1}x)$。注意，如果$K$满足我们定理的假设，那末$\varepsilon^{-n}K(\varepsilon^{-1}x)$也满足这些假设，并且具有**相同的界**（对本章所有定理的假设，可以作类似的说明）。现在对于给定的$K$，令$K'=\varepsilon^nK(\varepsilon x)$，它满足我们引理的条件且带有相同的界$B$。取

$$K_1'=\begin{cases}K'(x), & |x|>1,\\ 0, & |x|<1,\end{cases}$$

我们知道$|\widehat{K_1'(y)}|\leqslant CB$。$\varepsilon^{-n}K_1'(\varepsilon^{-1}x)$的Fourier变换是$\widehat{K_1'(\varepsilon y)}$，它还是有界$CB$的；然而$\varepsilon^{-n}K_1'(\varepsilon^{-1}x)=K_\varepsilon(x)$，引理因此证毕。

3.4　我们现在可以来证明定理 2 了。由$K$满足条件(18)与

(19)，知 $K_\varepsilon(x)$ 满足同样的条件，其界不大于 $CB$。我们在引理的证明中对 $K_1$ 指出过这点，而用上述的展缩方法，就可以从 $K_1$ 转到$K_\varepsilon$。然而，显然每个 $K_\varepsilon\in L^2(\boldsymbol{R}^n)$，$\varepsilon>0$。因此用§3.1的推论就证明了(21)；这就是说算子的 $L^p$ 模是一致有界的。另外，假设$f_1$是有连续微商与紧支集的连续函数，则

$$
\begin{aligned}
T_\varepsilon(f_1)(x) &= \int_{|y|>\varepsilon} K(y)f_1(x-y)dy \\
&= \int_{|y|>1} K(y)f_1(x-y)dy \\
&\quad + \int_{1>|y|>\varepsilon} K(y)[f_1(x-y)-f_1(x)]dy,
\end{aligned}
$$

这是由于抵消条件(19)。第一个积分表示一个 $L^p$ 函数，因为它是$L^1$函数$f_1$与$L^p$函数$K(y)$的卷积，这里用到了$|K(y)|\leqslant B|y|^{-n}$，当$|y|>1$。第二个积分的支集是 $x$ 的一个紧集，并且当 $\varepsilon\to 0$ 时对 $x$ 一致收敛，这是由于从 $f_1$ 的可微性知$|f_1(x-y)-f_1(x)|\leqslant A|y|$。综合这些就得到$T_\varepsilon(f_1)$当$\varepsilon\to 0$时按$L^p$模收敛。

最后，对任意$f\in L^p$，我们可以写出$f=f_1+f_2$，其中 $f_1$如上述，而$\|f_2\|_p$ 很小。把基本不等式(21)用到 $f_2$，我们就看到$\lim\limits_{\varepsilon\to 0}T_\varepsilon(f)$对 $L^p$ 模存在；极限算子也满足不等式(21)是显然的，这就完成了定理的证明。

我们指出，核 $K(x)=1/\pi x$，$x\in\boldsymbol{R}^1$，显然满足定理 2 的假设。因此证明了Hilbert变换的下述意义的存在性：若 $f\in L^p(\boldsymbol{R}^1)$，$1<p<\infty$，则

$$
\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|y|>\varepsilon}\frac{f(x-y)}{y}dy
$$

按$L^p$ 模存在，并且最后所得到的算子是 $L^p$ 有界的。这个例子的比较完整的研究以及它的某些重要的 $n$ 维类似，将在下一节叙述。

## §4 同展缩可交换的奇异积分算子

**4.1** 在Abel 群中一个基本的算子类是由同（群的）平移可交换的算子构成的集合。然而，$R^n$不仅是 Abel 群，而且除平移过程外还有某些熟知的作用在其上并与其Euclid 构造有关的变换群。我们这里想谈的变换是展缩 $\tau_\varepsilon$: $x\to\varepsilon x$，$\varepsilon>0$，它们作用在函数上相应于$\tau_\varepsilon(f)(x)=f(\varepsilon x)$，这是前面已讨论过的。

我们将对那些不仅同平移可交换而且同展缩可交换的算子感兴趣。其中我们将研究定理 2 影响所及的奇异积分算子。

设$T$对应于核$K(x)$，则正如我们指出过的，$\tau_{\varepsilon^{-1}}T\tau_\varepsilon$对应于核 $\varepsilon^{-n}K(\varepsilon^{-1}x)$。因此，如果 $\tau_{\varepsilon^{-1}}T\tau_\varepsilon=T$，那末这等于要求 $K(\varepsilon x)=\varepsilon^{-n}K(x)$，$\varepsilon>0$，即$K$是$-n$次齐次的。用另一种方式写出来

(23) $$K(x)=\frac{\Omega(x)}{|x|^n},$$

其中$\Omega$是 0 次齐次的，即$\Omega(\varepsilon x)=\Omega(x)$，$\varepsilon>0$。$\Omega$ 的这个条件等价于它沿着从原点出发的射线取常数值；特别地，$\Omega$ 由它在单位球面$S^{n-1}$的限制完全决定。让我们试着通过$\Omega$来解释一下定理 2 的条件。首先，根据(18)$\Omega$必须是有界的，因此它在$S^{n-1}$可积。抵消条件(19)相当于

(24) $$\int_{S^{n-1}}\Omega(x)d\sigma=0,$$

其中$d\sigma$是$S^{n-1}$的Euclid测度。精确的条件(18)不易通过$\Omega$转述；然而，显然它要求的是$\Omega$的某种连续性。这里我们将满足于研究$\Omega$适合下述来自(18)的“Dini 型”条件：

(25) $$\int_0^1\frac{\omega(\delta)d\delta}{\delta}<\infty,$$

其中 $$\sup_{\substack{|x-x'|\leqslant\delta\\|x|=|x'|=1}}|\Omega(x)-\Omega(x')|=\omega(\delta).$$

自然，任意属于$C^1$类或甚至仅仅是Lipschitz连续的$\Omega$都满足条件(25)①。

4.2 **定理 3** 设$\Omega$是 0 次齐次并满足抵消条件(24)以及上面的光滑性条件(25)。对$1<p<\infty$与$f\in L^p(\boldsymbol{R}^n)$，令

$$T_\varepsilon(f)(x)=\int_{|y|>\varepsilon}\frac{\Omega(y)}{|y|^n}f(x-y)dy.$$

(a) 则存在界$A_p$(与 $f$ 以及$\varepsilon$无关)，使得

$$\|T_\varepsilon(f)\|_p\leqslant A_p\|f\|_p.$$

(b) $\lim\limits_{\varepsilon\to 0}T_\varepsilon(f)=T(f)$按$L^p$模存在，并且

$$\|T(f)\|_p\leqslant A_p\|f\|_p.$$

(c) 如果$f\in L^2(\boldsymbol{R}^n)$，那末 $f$ 与$T(f)$的 Fourier 变换满足关系$(T(f))^\wedge(x)=m(x)\hat{f}(x)$，其中$m$是 0 次齐次函数。更明确些

$$m(x)=\int_{S^{n-1}}\left[\frac{\pi i}{2}\operatorname{sign}(x\cdot y)+\log\left(\frac{1}{|x\cdot y|}\right)\right]\Omega(y)d\sigma(y),\quad |x|=1. \tag{26}$$

定理的结论(a)与(b)是定理 2 的直接推论，只要我们证明形如 $\Omega(x)/|x|^n$ 的$K(x)$满足

$$\int_{|x|\geqslant 2|y|}|K(x-y)-K(x)|dx\leqslant B,$$

其中$\Omega$满足条件(25)。事实上

$$K(x-y)-K(x)=\left(\frac{\Omega(x-y)-\Omega(x)}{|x-y|^n}\right)+\Omega(x)\left(\frac{1}{|x-y|^n}-\frac{1}{|x|^n}\right).$$

上式右边第二项满足所要求的估计是由于

$$\int_{|x|\geqslant 2|y|}\left|\frac{1}{|x-y|^n}-\frac{1}{|x|^n}\right|dx\leqslant C$$

① 关于这条件的推广参看本章末的§6.10.

以及$\Omega$有界。为估计第一项，我们注意，$x-y$与$x$到单位球的投影的距离

$$\left|\frac{x-y}{|x-y|}-\frac{x}{|x|}\right|$$

是被$C|y/x|$控制的，只要$|x|\geqslant 2|y|$。因此对应于第一项的积分就被

$$C'\int_{|x|\geqslant 2|y|}\omega\left(C\frac{|y|}{|x|}\right)\frac{dx}{|x|^n}=C''\int_0^{C/2}\frac{\omega(\delta)}{\delta}d\delta<\infty$$

控制。

**4.3** 由于$T$是$L^2$上同平移可交换的有界算子，由§1.4的命题知道，$T$可以通过一个乘子$m$来实现，使得$(T(f))^\wedge$由$\hat{f}$乘以$m$得到。对这样的算子，它们同展缩可交换的事实等价于乘子具有0次的齐次性。对我们的特殊算子来说，不仅有$m$的存在性，而且还有乘子通过核表达的显式。这公式的推导如下。令

$$0<\varepsilon<\eta<\infty,\quad K_{\varepsilon,\eta}(x)=\begin{cases}\dfrac{\Omega(x)}{|x|^n}, & \varepsilon\leqslant|x|\leqslant\eta,\\ 0, & \text{其它。}\end{cases}$$

显然，$K_{\varepsilon,\eta}\in L^1(\boldsymbol{R}^n)$。若$f\in L^2(\boldsymbol{R}^n)$，则

$$(K_{\varepsilon,\eta}*f)^\wedge=K^\wedge_{\varepsilon,\eta}(y)\hat{f}(y)。$$

我们将证明两个关于$K^\wedge_{\varepsilon,\eta}(x)$的事实。

(i) $\sup|K^\wedge_{\varepsilon,\eta}(x)|\leqslant A$，其中$A$与$\varepsilon,\eta$无关。

(ii) 当$x\neq 0$时，$\lim\limits_{\substack{\varepsilon\to 0\\ \eta\to\infty}}K^\wedge_{\varepsilon,\eta}(x)=m(x)$（见式(26)）。

为此，引入极坐标是方便的。令$x=Rx'$，$R=|x|$，$x'=x/|x|\in S^{n-1}$，而$y=ry'$，$r=|y|$，$y'=y/|y|\in S^{n-1}$。我们还需要一个辅助积分

$$I_{\varepsilon,\eta}(x,y')=\int_\varepsilon^\eta[\exp(2\pi iRrx'\cdot y')-\cos(2\pi Rr)]\frac{dr}{r},\qquad R>0,$$

其虚部为

$$\int_{\varepsilon}^{\eta} \frac{\sin 2\pi Rr(x' \cdot y')}{r} dr,$$

通过分部积分可以证明它是一致有界的，并且趋向于

$$\left(\int_0^{\infty} \frac{\sin t}{t} dt\right) \operatorname{sign}(x' \cdot y') = \frac{\pi}{2} \operatorname{sign}(x' \cdot y').$$

其实部也是有界的，且绝对值的界为 $C\log(1/|x' \cdot y'|) + C$，这也可以用分部积分证明。还有

$$\lim_{\substack{\varepsilon \to 0 \\ \eta \to \infty}} \operatorname{Re}(I_{\varepsilon,\eta}(x, y')) = \log \frac{1}{|x' \cdot y'|},$$

这是由于

$$\lim_{\substack{\varepsilon \to 0 \\ \eta \to \infty}} \int_{\varepsilon}^{\eta} \frac{h(\lambda r) - h(\mu r)}{r} dr = h(0) \log \frac{\mu}{\lambda},$$

其中 $\mu, \lambda > 0$，$h$ 是合适的函数。在这里，$h(r) = \cos 2\pi r$，$\lambda = R|x' \cdot y'|$，而 $\mu = R$。

现在

$$K^{\wedge}_{\varepsilon,\eta}(x) = \int_{S^{n-1}} \left(\int_{\varepsilon}^{\eta} e^{2\pi i R r x' \cdot y'} \Omega(y') \frac{dr}{r}\right) d\sigma(y').$$

由于

$$\int_{S^{n-1}} \Omega(y') d\sigma(y') = 0,$$

我们可以在给出 $K^{\wedge}_{\varepsilon,\eta}(x)$的积分中引进因子 $\cos 2\pi rR$（它与$y'$无关）。这就给出

$$K^{\wedge}_{\varepsilon,\eta}(x) = \int_{S^{n-1}} I_{\varepsilon,\eta}(x, y') \Omega(y') d\sigma(y').$$

由$I_{\varepsilon,\eta}$刚证明的性质知

$$|K^{\wedge}_{\varepsilon,\eta}(x)| \leqslant A \int_{S^{n-1}} \left[1 + \log \frac{1}{|x' \cdot y'|}\right] |\Omega(y')| d\sigma(y'),$$

由它推出(i)（$K^{\wedge}_{\varepsilon,\eta}(x)$一致有界），因为$\Omega$本身有界。根据刚刚说明的$I_{\varepsilon,\eta}(x)$当$\varepsilon \to 0, \eta \to \infty$时的极限与控制收敛定理，当 $x \neq 0$

时，我们有

$$\lim_{\substack{\varepsilon\to 0\\ \eta\to\infty}} \hat{K}_{\varepsilon,\eta}(x)=m(x),$$

这就是(ii)。

由Plancherel定理知，当 $f\in L^2(\boldsymbol{R}^n)$时，$K_{\varepsilon,\eta}*f$ 当 $\eta\to\infty$，$\varepsilon\to 0$时按$L^2$模收敛，其极限的Fourier 变换是 $m(x)\hat{f}(x)$。然而，若固定 $\varepsilon$ 而令 $\eta\to\infty$，则显然

$$\int K_{\varepsilon,\eta}(y)f(x-y)dy$$

处处收敛到

$$\int_{|y|<\varepsilon} K(y)f(x-y)dy,$$

它就是 $T_\varepsilon(f)$。

现在令$\varepsilon\to 0$，我们便得到结论(c)，从而定理证毕。

4.4 注意，(c) 部分的证明对很一般的 $\Omega$ 是成立的。记 $\Omega=\Omega_e+\Omega_o$，其中$\Omega_e$是$\Omega$的偶部，$\Omega_e(x)=\Omega_e(-x)$，而 $\Omega_o$是奇部，$\Omega_o(-x)=-\Omega_o(x)$。这时，由于正弦积分的一致有界性，我们只要求

$$\int_{S^{n-1}}|\Omega_o(y')|d\sigma(y')<\infty,$$

即奇部的可积性。对偶部，证明要求

$$\int_{S^{n-1}}|\Omega_e(y')|\log\frac{1}{|x'\cdot y'|}d\sigma(y')$$

的一致有界性。

这个说明对定理 2 的某些推广是有启发性的（见 §6.5）。

在此，我们不多说关于定理 3 中的变换在$L^1(\boldsymbol{R}^n)$与$L^\infty(\boldsymbol{R}^n)$都是无界的。在 Hilbert 变换的情形，可以从区间$(a,b)$的特征函数的变换这一明显例子看出，这时变换等于

$$\frac{1}{\pi}\log\left|\frac{x-a}{x-b}\right|.$$

其它例子在 §6.1中叙述。

**4.5** 定理3保证了奇异积分变换

$$\lim_{\varepsilon\to 0}\int_{|y|>\varepsilon}\frac{\Omega(y)}{|y|^n}f(x-y)dy \tag{27}$$

按$L^p$模收敛的存在性。这个结果的自然补充是几乎处处收敛。在对应于Hilbert变换的经典情形，几乎处处收敛的结果却先于$L^p$的结果。前者是作为保证有界调和函数边值几乎处处存在的Fatou定理的推论而得到的。现在，几乎处处的结果实际上是已证明的极限(27)按$L^p$模存在的一个推论。这时，如同在包含几乎处处收敛的其它问题中那样，最好还是考虑相应的极大函数。

**定理4** 假设$\Omega$满足前面定理的条件。对$f\in L^p(\boldsymbol{R}^n)$，$1\leqslant p<\infty$，考虑

$$T_\varepsilon(f)(x)=\int_{|y|>\varepsilon}\frac{\Omega(y)}{|y|^n}f(x-y)dy,\quad \varepsilon<0.$$

(积分对每个$x$是绝对收敛的。)

(a) $\lim\limits_{\varepsilon\to 0}T_\varepsilon(f)(x)$对几乎所有的$x$是存在的。

(b) 令$T^*(f)(x)=\sup\limits_{\varepsilon>0}|T_\varepsilon(f)(x)|$。若$f\in L^1(\boldsymbol{R}^n)$，则映射$f\to T^*(f)$是弱(1,1)型的。

(c) 若$1<p<\infty$，则$\|T^*(f)\|_p\leqslant A_p\|f\|_p$。

**4.6** 定理4的证明分成三步。首先，不等式(c)有一个证明，就是作为已证明的$\lim\limits_{\varepsilon\to 0}T_\varepsilon$按$L^p$模存在以及"恒等逼近"的某些一般性质的一个比较容易的推论而得到。因此，我们把(c)的证明推到下一章，在那里我们较系统地研究这些课题。

4.6.1 让我们进行第二步，导致结论(b)的最困难一步。在这里证明基本上像在证明奇异积分弱(1,1)型结果，特别是在§3.1给出的变形那样。我们简单地回顾一下，为的是避免那些曾经验证过的细节的重复。对给定的$\alpha>0$，我们像§2.4那样分解$f=g+b$。对每个方体$Q_j$，我们还考虑具有同中心但伸展了$2n^{1/2}$倍的方体$Q_j^*$。下面补充的关于这些方体的几点几何说明几乎是显

然的。

(iii) 假设 $x\in {}^cQ_j^*$ (特别地如果 $x\in F^*$)，并设对某个 $y\in Q_j$，$|x-y|=\varepsilon$。这时以 $x$ 为中心，半径为$\gamma_n\varepsilon$的闭球包含了$Q_j$，即 $\overline{B(x,r)}\supset Q_j$，其中$r=\gamma_n\varepsilon$。

(iv) 在(iii)的同样假设下，我们有$|x-y|\geqslant\gamma'_n\varepsilon$，对每个$y\in Q_j$。

上述$\gamma_n$与$\gamma'_n$只依赖于维数 $n$，而与每个特殊的方体无关。

4.6.2 有了这些说明与§2.4后面的论证，我们来证明，对 $x\in F^*$，有

$$
(28)\quad \sup_{\varepsilon>0}|T_\varepsilon(b)(x)|\leqslant\sum_j\int_{Q_j}|K(x-y)-K(x-y^j)||b(y)|dy
$$

$$
+C\sup_{r>0}\frac{1}{m(B(x,r))}\int_{B(x,r)}|b(y)|dy,
$$

其中

$$
K(x)=\frac{\Omega(x)}{|x|^n}.
$$

这时，式(28)右边出现了极大函数，这是证明中新的主要因素。为了证明式(28)，固定 $x\in F^*$与$\varepsilon>0$。现在方体$Q_j$被分成三类：

(a) 对所有$y\in Q_j$，$|x-y|<\varepsilon$；

(b) 对所有$y\in Q_j$，$|x-y|>\varepsilon$；

(c) 存在$y\in Q_j$，使得$|x-y|=\varepsilon$。

我们现在来估计$T_\varepsilon(b)(x)$。

$$
(29)\qquad T_\varepsilon(b)(x)=\sum_j\int_{Q_j}K_\varepsilon(x-y)b(y)dy.
$$

情形(a)。$K_\varepsilon(x-y)=0$，当$|x-y|<\varepsilon$。因此式(29)中 $Q_j$上的积分为 0。

情形(b)。$K_\varepsilon(x-y)=K(x-y)$，当$|x-y|>\varepsilon$。因此在$Q_j$上的这个积分等于

$$\int_{Q_j} K(x-y)b(y)dy = \int_{Q_j} [K(x-y)-K(x-y^j)]b(y)dy.$$

这项的绝对值被下式控制

$$\int_{Q_j} |K(x-y)-K(x-y^j)||b(y)|dy,$$

它出现在式(28)的右边。

情形(c)。由(iii)，我们有

$$\left|\int_{Q_j} K_\varepsilon(x-y)b(y)dy\right| \leqslant \int_{Q_j} |K_\varepsilon(x-y)||b(y)|dy$$

$$= \int_{B(x,r)\cap Q_j} |K_\varepsilon(x-y)||b(y)|dy,$$

其中$r=\gamma_n\varepsilon$。然而

$$|K_\varepsilon(x-y)| \leqslant \left|\frac{\Omega(x-y)}{(x-y)^n}\right| \leqslant \frac{B}{(\gamma'_n)^n\varepsilon^n},$$

这是由于(iv)与$\Omega$有界。因此在这种情形

$$\left|\int_{Q_j} K_\varepsilon(x-y)b(y)dy\right| \leqslant \frac{C}{m[B(x,r)]}\int_{B(x,r)\cap Q_j} |b(y)|dy.$$

对所有方体求和，我们最后得到

$$|T_\varepsilon(b)(x)| \leqslant \sum_j \int_{Q_j} |K(x-y)-K(x-y^j)||b(y)|dy$$

$$+C\frac{1}{m(B(x,r))}\int_{B(x,r)} |b(y)|dy, \quad r=\gamma_n\varepsilon.$$

对$\varepsilon$取上确界就给出式(28)。

这个不等式可以写成

$$|T^*(b)(x)| \leqslant \sum + CM(b)(x), \qquad x\in F^*.$$

因此

$$m\{x\in F^*: |T^*(b)(x)|>\alpha/2\}$$
$$\leqslant m\{x\in F^*: \sum>\alpha/4\} + m\{x\in F^*: CM(b)(x)>\alpha/4\}.$$

上述不等式右边的两个测度都小于等于$\frac{c}{a}\|b\|_1$。第一个是由于类似于§3.1中(17)的不等式对$\sum$成立；第二个是由于极大函数$M$的弱(1,1)型（第一章§1.3的定理）。$T^*$的弱(1,1)型可以像§3.1中$T$的同样性质的证明一样推出来（或者比较详细地像§2.4等式(15)以后所叙述的那样）。

4.6.3 定理证明的最后一步，从$T^*$的不等式过渡到几乎处处极限的存在，用第一章§1.5所叙述的熟知程式推得。更准确些，对任意$f\in L^p(R^n)$，$1\leqslant p<\infty$，令

$$\Lambda(f)(x)=|\limsup_{\varepsilon\to 0} T_\varepsilon(f)(x)-\liminf_{\varepsilon\to 0} T_\varepsilon(f)(x)|.$$

显然，$\Lambda(f)(x)\leqslant 2T^*(f)(x)$。现在记$f=f_1+f_2$，其中$f_1$有紧支集并且属于$C^1$类，而$\|f_2\|_p\leqslant\delta$。我们在§3.4中曾指出过，$T_\varepsilon f_1$当$\varepsilon\to 0$时一致收敛。因此$\Lambda(f_1)(x)\equiv 0$。但$\Lambda(f)(x)\leqslant\Lambda(f_1)(x)+\Lambda(f_2)(x)$，而

$$\|\Lambda(f_2)\|_p\leqslant 2A_p\|f_2\|_p\leqslant 2A_p\delta,\qquad 1<p<\infty.$$

这表明$\Lambda(f_2)=0$几乎处处，因此$\Lambda(f)=0$几乎处处，从而当$1<p<\infty$时，$\lim\limits_{\varepsilon\to 0} T_\varepsilon(f)$几乎处处存在。在$p=1$时，类似地我们有

$$m\{x:\Lambda(f)(x)>a\}\leqslant\frac{A}{a}\|f_2\|_1\leqslant\frac{A\delta}{a},$$

因此也有$\Lambda f(x)=0$几乎处处，它蕴含了$\lim\limits_{\varepsilon\to 0} T_\varepsilon(f)(x)$几乎处处存在。

## §5 向量值的类似

5.1 有趣的是，这章中对函数取实值或复值的结果，可以推广到函数在Hilbert空间中取值的情形。我们叙述这种推广，因为它在许多问题中很有用。在第四章Littlewood-Paley理论中将给出一个应用。

我们从简单复习这方面的积分理论的某些内容开始。

令$\mathscr{H}$是可分的Hilbert空间。这时从$R^n$到$\mathscr{H}$的函数$f(x)$是可测的，如果数值函数$(f(x),\varphi)$是可测的，其中$(\ ,\ )$表示$\mathscr{H}$的内积，而$\varphi$表示$\mathscr{H}$的任意向量。如果$f(x)$是这样的可测函数，那末$|f(x)|$也可测（作为一个非负值函数），其中$|\cdot|$表示$\mathscr{H}$的模。这样，$L^p(R^n,\mathscr{H})$定义为$R^n$到$\mathscr{H}$的可测函数的等价类，使得模

$$\|f\|_p=\left(\int_{R^n}|f(x)|^p dx\right)^{1/p}$$

有限，当$p<\infty$；当$p=\infty$有类似的定义，并且

$$\|f\|_\infty=\operatorname{ess\,sup}|f(x)|.$$

另外，设$\mathscr{H}_1$与$\mathscr{H}_2$是两个可分的Hilbert空间，而$B(\mathscr{H}_1,\mathscr{H}_2)$表示由$\mathscr{H}_1$到$\mathscr{H}_2$的有界线性算子组成的Banach空间，带有通常的算子模。我们说从$R^n$到$B(\mathscr{H}_1,\mathscr{H}_2)$的函数$f(x)$是可测的，如果$f(x)\varphi$对每个$\varphi\in\mathscr{H}_1$是$\mathscr{H}_2$可测的。这时，$|f(x)|$也是可测的，并且我们可以定义空间$L^p(R^n,B(\mathscr{H}_1,\mathscr{H}_2))$如前（这里$|\cdot|$仍表示模，但是在$B(\mathscr{H}_1,\mathscr{H}_2)$内）。关于内积的通常事实这时也都成立。例如，假设$K(x)\in L^q(R^n,B(\mathscr{H}_1,\mathscr{H}_2))$，而$f\in L^p(R^n,\mathscr{H}_1)$，则

$$g(x)=\int_{R^n}K(x-y)f(y)dy$$

对几乎所有的$x$按$\mathscr{H}_2$模收敛，并且

$$|g(x)|\leqslant\int_{R^n}|K(x-y)f(y)|dy$$

$$\leqslant\int_{R^n}|K(x-y)|\,|f(y)|dy.$$

还有$\|g\|_r\leqslant\|K\|_q\|f\|_p$，当$\frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1$，其中$1\leqslant r\leqslant\infty$。

5.2 假设$f(x)\in L^1(R^n,\mathscr{H})$，我们可以定义它的Fourier变换

$$\hat{f}(y)=\int_{R^n}e^{2\pi i x\cdot y}f(x)dx,$$

它是$L^\infty(R^n,\mathscr{H})$的元素。如果$f\in L^1(R^n,\mathscr{H})\cap L^2(R^n,\mathscr{H})$，那末$\hat{f}(y)\in L^2(R^n,\mathscr{H})$，并且$\|\hat{f}\|_2=\|f\|_2$。Fourier 变换可以连续开拓为Hilbert空间$\mathscr{H}$到自身的酉变换。

这些事实可以容易地通过在$\mathscr{H}$引入任意的正交基从数值的情形得到。

5.3 现在假设$\mathscr{H}_1$与$\mathscr{H}_2$是两个给定的 Hilbert 空间。假设$f(x)$取值在$\mathscr{H}_1$，而$K(x)$取值在$B(\mathscr{H}_1,\mathscr{H}_2)$。这时

$$T(f)(x)=\int_{R^n}K(y)f(x-y)dy$$

有定义，取值在$\mathscr{H}_2$。

**定理5** 本章的结果，特别是定理 1，它的推论，以及定理 2 到定理 4 在下述一般情形都成立，其中$f$取值在$\mathscr{H}_1$，$K$取值在$B(\mathscr{H}_1,\mathscr{H}_2)$而$T(f)$与$T_\varepsilon(f)$取值在$\mathscr{H}_2$，只是所有的绝对值分别用$\mathscr{H}_1$, $B(\mathscr{H}_1,\mathscr{H}_2)$或$\mathscr{H}_2$的模代替。

这个定理不能用显然的办法看成是上面已研究过的数值情形的推论。然而，它的证明沒有什么新东西，只是重复数值情形的推理，只要我们注意到上面几段所做的说明。事实上，这个似乎有点冒失的论断是可以通过对证明的耐心复习而得到验证的；但是，如果读者沒那么大的兴趣，那他可以从本章末所列的文献中找到必要的细节。

通过验证可以得到几点澄清：

(a) 最后得到的界与 Hilbert 空间$\mathscr{H}_1$或$\mathscr{H}_2$无关，而只依赖于$B$, $p$与$n$，就像数值的情形那样。

(b) 大多数推理适用于最一般的适当定义了的 Banach 空间值函数。Hilbert 空间的构造只是在$L^2$理论，也就是应用§5.2叙述的 Plancherel 定理的变形时用到。

Hilbert空间的构造还用在下面的推论。

**推论** 在定理 5 相同的假设下，如果加上

$$\|T(f)\|_2=c\|f\|_2,\quad c>0,\quad f\in L^2(R^n,\mathscr{H}_1),$$

那末$\|f\|_p \leqslant A'_p \|T(f)\|_p$，当$f \in L^p(\boldsymbol{R}^n, \mathscr{H}_1)$，$1<p<\infty$。

**证明** 我们注意$L^2(\boldsymbol{R}^n, \mathscr{H}_j)$是Hilbert空间。事实上，用$(\ ,\ )_j$表示$\mathscr{H}_j$的内积，$j=1,2$。用$\langle\ ,\ \rangle_j$表示相应的$L^2(\boldsymbol{R}^n, \mathscr{H}_j)$的内积，也就是说

$$\langle f, g\rangle_j = \int_{\boldsymbol{R}^n} (f(x), g(x))_j dx.$$

现在$T$是从Hilbert空间$L^2(\boldsymbol{R}^n, \mathscr{H}_1)$到Hilbert空间$L^2(\boldsymbol{R}^n, \mathscr{H}_2)$的有界线性变换，因此，根据内积的一般理论，存在唯一的从$L^2(\boldsymbol{R}^n, \mathscr{H}_2)$到$L^2(\boldsymbol{R}^n, \mathscr{H}_1)$的共轭变换$\tilde{T}$，它满足特征性质

$$\langle T(f_1), f_2\rangle_2 = \langle f_1, \tilde{T}(f_2)\rangle_1, \qquad f_j \in L^2(\boldsymbol{R}^n, \mathscr{H}_j).$$

但我们的假设等价于恒等式

$$\langle T(f), T(g)\rangle_2 = c^2\langle f, g\rangle_1, \qquad \text{对所有} f, g \in L^2(\boldsymbol{R}^n, \mathscr{H}_1).$$

因此，应用共轭的定义$\langle \tilde{T}T(f), g\rangle_1 = c^2\langle f, g\rangle_1$，从而假设可改写成

$$\tilde{T}T(f) = c^2 f, \qquad f \in L^2(\boldsymbol{R}^n, \mathscr{H}_1). \tag{30}$$

$\tilde{T}$又是与$T$同一类的算子，但它把在$\mathscr{H}_2$取值的函数变到在$\mathscr{H}_1$取值的函数，并且它的核$\tilde{K}(x)$就是$\tilde{K}(x) = K^*(-x)$，其中“$*$”表示对$B(\mathscr{H}_1, \mathscr{H}_2)$中的元素取共轭。

形式地看这是显然的

$$\begin{aligned}\langle T(f_1), f_2\rangle_2 &= \int_{\boldsymbol{R}^n}\int_{\boldsymbol{R}^n} (K(x-y)f_1(y), f_2(x))_2 dy dx \\ &= \int_{\boldsymbol{R}^n}\int_{\boldsymbol{R}^n} (f_1(y), K^*(-(y-x))f_2(x))_1 dx dy \\ &= \langle f_1, \tilde{T}(f_2)\rangle_1.\end{aligned}$$

这等式可以通过一个简单的极限过程验证。我们不打算让这些常规的细节烦扰读者。

这就是说，我们只需加上一个说明，$K^*(-x)$满足与$K(x)$相同的条件，因而对它有类似于$K$的结论（带有相同的界）。故由式(30)

$$c^2\|f\|_p = \|\tilde{T}T(f)\|_p \leqslant A_p\|T(f)\|_p,$$

这就证明了推理，其中 $A'_p = A_p/c^2$。

特别地，把这个推论应用到 § 4 的奇异积分；这时所要求的条件就是，乘子$m(x)$的绝对值是常数。例如，$T$ 是Hilbert变换，$K(x)=1/\pi x$，这时 $m(x)=i\,\mathrm{sign}\,x$ 就是这种情形。这个注释的一种推广见下面的 § 6.6。

## § 6　进一步的结果

**6.1**　设$K(x)=\Omega(x)/|x|^n$如定理 3 的那样，且 $\Omega\neq 0$。

(a) 若$f\in L^1(\boldsymbol{R}^n)$，$f\geqslant 0$，则 $T(f)\overline{\in} L^1(\boldsymbol{R}^n)$，当 $f\not\equiv 0$。**提示**：$m(x)\hat{f}(x)$不可能在 0 连续，由于 $m(x)$是 0 次齐次且不等于常数，又 $\hat{f}(0)>0$。

(b) 存在在单位球$B$外为 0 的连续函数，使得$T(f)$在 $B$ 的每一点附近是无界的。

**6.2**　(a) 若$A_p$是定理1,2或3中$T$的$L^p$界，则$A_p\leqslant A/(p-1)$，当$1<p\leqslant 2$；而$A_p\leqslant Ap$，当$2\leqslant p<\infty$(见第一章 § 4末的注)。

(b) 若 $f$ 以球 $B$ 为支集，且$|f|\log(2+|f|)$ 在 $B$ 可积，则 $T(f)$在 $B$ 可积。

(c) 若 $f$ 有界且支集为$B$，则对适当的$a>0$，$e^{a|T(f)|}$ 在 $B$ 可积。**提示**：写出

$$\int e^{a|T(f)|}dx = \sum \frac{a^n}{n!}\|T(f)\|_n^n,$$

并用本题的(a)。

(d) 同样的结果对定理 4 的极大算子 $T^*$ 成立。关于这些结果见Calderón 与Zygmund[1]，Zygmund[8]的第Ⅻ章。

**6.3**　设

$$T(f)(x) = \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} K(x,y)f(y)dy,$$

其中$|K(x,y)|\leqslant A/|x-y|^n$。假设$T$在$L^p(\boldsymbol{R}^n)$有界，则$T$在测度$|x|^a dx$（代替$dx$）的$L^p$空间是有界的，其中$-n<a<n(p-1)$。见Stein[2]；易见这对$(1+|x|)^a dx$蕴含了相同的结果。

**6.4** 下面的方法把极大函数，第一章的微分定理以及本章的许多奇异积分统一起来。

设$L(x)$在$\boldsymbol{R}^n$可积，并假设$L(x)=0$，当$|x|\geqslant 2$，

$$\int_{\boldsymbol{R}^n} L(x)dx=0$$

且

$$\int_{\boldsymbol{R}^n}|L(x-y)-L(x)|dx\leqslant B|y|.$$

对任意整数对$i$，$j$，用

$$L_{i,j}(x)=\sum_{k=-i}^{j}2^{nk}L(2^k x)$$

定义$L_{i,j}$。记$T_{i,j}(f)=L_{i,j}*f$。如果$T_*(f)=\sup\limits_{i,j}|T_{i,j}(f)(x)|$，那末

(a) $f\to T_*(f)$是弱$(1,1)$型的；

(b) $f\to T_*(f)$是$L^p$有界的，$1<p<\infty$；

(c) 对$f\in L^p, 1\leqslant p<\infty$，$\lim\limits_{\substack{i\to\infty\\ j\to\infty}}T_{i,j}(f)$几乎处处以及按$L^p$模$(1<p)$存在。

两个有趣的例子是：

(i) $L(x)=1-2^n$，当$|x|\leqslant 1$；而$L(x)=1$，当$1\leqslant|x|\leqslant 2$，其它地方为0。这时

$$T_{i,0}(f)(x)=2^{-ni}\int_{|y|\leqslant 2^{i+1}}f(x-y)dy$$
$$-2^n\int_{|y|\leqslant 1}f(x-y)dy.$$

(ii) $L(x)=\dfrac{\Omega(x)}{|x|^n}$，当$1\leqslant|x|\leqslant 2$，其它地方为0。这时

$$T_{i,j}(f)(x)=\int_{2^{-i}<|y|<2^{j+1}}\frac{\Omega(y)}{|y|^n}f(x-y)dy$$

(详见Cotlar[2])。

**6.5** (a) 设$y'$是$R^n$的单位向量。$y'$方向的Hilbert变换可以定义为$\lim_{\varepsilon\to 0} H_{y'}^{(\varepsilon)}(f)(x)$，其中

$$H_{y'}^{(\varepsilon)}(f)(x)=\int_{|t|>\varepsilon}\frac{f(x-y't)}{t}dt$$

$$=\int_{\varepsilon}^{\infty}\frac{[f(x-y't)-f(x+y't)]dt}{t}.$$

这时$\|H_{y'}^{(\varepsilon)}(f)\|_p\leqslant A_p\|f\|_p$，当$f\in L^p(R^n)$，$1<p<\infty$，其中$A_p$与$y'$及$\varepsilon$无关。

(b) 假设$\Omega(y')$是0次齐次，在单位球面$S^{n-1}$可积的奇函数，即$\Omega(y')=-\Omega(-y')$。令

$$T_\varepsilon(f)(x)=\int_{|y|>\varepsilon}\frac{\Omega(y)}{|y|^n}f(x-y)dy.$$

则

$$T_\varepsilon=\frac{1}{2}\int_{S^{n-1}}\Omega(y')H_{y'}^{(\varepsilon)}d\sigma(y'),$$

并且

$$\|T_\varepsilon(f)\|_p\leqslant\left(\frac{1}{2}A_p\int_{S^{n-1}}|\Omega(y')|d\sigma(y')\right)\|f\|_p.$$

(c) 一个类似的但比较困难的结果对$\Omega$是偶的情形成立。这时要求$|\Omega(y')|\log(2+|\Omega(y')|)$在$S^{n-1}$可积。

(d) 当$f\in L^1(R^n)$，对这里考虑的一般的$\Omega$，$T_\varepsilon(f)$的性质仍是未解决的问题。

上述(a),(b)与(c)的细节见Calderón与Zygmund[3]。理论的这部分在《富理叶分析》第Ⅵ章也讲到，只是缺乏某些一般性。

**6.6** 假设$m(x)$是0次齐次且在$S^{n-1}$连续。对$f\in L^2(R^n)$，用$(T(f))^\wedge(x)=m(x)\hat{f}(x)$定义$T(f)$。假设$\|T(f)\|_p\leqslant A_p\|f\|_p$，当$f\in L^2\cap L^p$对某个$p(1<p<\infty)$成立。若$|m(x)|\geqslant c>0$，则也有$\|f\|_p\leqslant B_p\|T(f)\|_p$（见Calderón与Zygmund[4]，Hörmander[1]， Benedek, Calderón与 Panzone[1]).

6.7 设$T$是Hilbert 变换(*), $\chi_E$表示$R^1$上的有限测度子集$E$的特征函数。则$T\chi_E$的分布函数只依赖于$E$的测度；更准确些，若$\lambda(\alpha)$是这个分布函数，则

$$\lambda(\alpha) = \frac{2m(E)}{\sinh \pi\alpha}.$$

见Stein与Weiss[1]。

6.8 正如曾经指出过的，展缩$x \to \varepsilon x = (\varepsilon x_1, \varepsilon x_2, \cdots, \varepsilon x_n)$在本章中起着重要的作用。存在本章结果的许多变形，其中这种齐次型被一种各向异性所代替，即$x \to \varepsilon \circ x = (\varepsilon^{a_1} x_1, \varepsilon^{a_2} x_2, \cdots, \varepsilon^{a_n} x_n)$，其中$a_1, a_2, \cdots, a_n$是固定的正指数，而$\varepsilon > 0$。这时核$K(x) \to \varepsilon^n K(\varepsilon x)$的作用由$K(x) \to \varepsilon^a K(\varepsilon \circ x)$代替，其中$a = a_1 + a_2 + \cdots + a_n$。细节见Jones[1], Fabes与Riviere[1], Kreé[1]以及Besov, Il'in与Lizorkin[1]。

6.9 设$T$与$K(x) = \Omega(x)/|x|^n$如定理3所述。假设$0 < \alpha < 1$，$f$是紧支集的连续函数，满足

$$|f(x+t) - f(x)| \leqslant A|t|^\alpha.$$

这时，设$g(x) = T(f)$，则也有$|g(x+t) - g(x)| \leqslant B|t|^\alpha$。（提示：当$\Omega$充分光滑时证明是“初等的”；见Privalov[1], Calderón与Zygmund[2]，一般情况见Taibleson[1]。）

6.10 设$\Omega$是0次齐次，在单位球上可积且

$$\int_{S^{n-1}} \Omega d\sigma = 0.$$

假设

$$\sup_{|r| < \delta} \int_{S^{n-1}} |\Omega(r(x')) - \Omega(x')| d\sigma \leqslant \omega(\delta),$$

其中

$$\int_0^\varepsilon \frac{\omega(\delta) d\delta}{\delta} < \infty.$$

这里$r$表示绕原点的旋转，而$|r|$表示$r$到恒同旋转的距离，用

的是旋转群的任意光滑的Riemann度量。则

(a) $\int_{S^{n-1}}|\Omega|\log^+|\Omega|d\sigma<\infty$，从而§6.5的 $L^p$理论可以应用；

(b) 若 $K(x)=\Omega(x)/|x|^n$，则

$$\int_{|x|>2|y|}|K(x-y)-K(x)|dx\leqslant B,$$

因而§3.1的$L^1$理论也可以应用。见 Calderón，M.Weiss 与 Zygmund[1]。

6.11 §3.3推理的微小的改变可证明下面事实。假设 $K(x)$ 是给定的函数，满足假定：

(a) $\int_{|x|<R}|x||K(x)|dx\leqslant BR,\qquad 0<R<\infty$；

(b) $\int_{|x|>2|y|}|K(x-y)-K(x)|dx\leqslant B$；

(c) $\left|\int_{R_1<|x|<R_2}K(x)\,dx\right|\leqslant B,\qquad 0<R_1<R_2<\infty$。

令

$$K_{\varepsilon,\eta}(x)=\begin{cases}K(x), & \varepsilon<|x|<\eta,\\ 0, & \text{其它。}\end{cases}$$

则$|\hat{K}_{\varepsilon,\eta}(x)|\leqslant CB$，其中$C$与$\varepsilon,\eta$无关。见Benedek，Calderón与Panzone[1]。

6.12 假设$f$有界并具有有界支集，令

$$I^s(f)(x)=\int_{R^n}f(x-y)|y|^{-n+\sigma+it}dy,$$

$\sigma>0$，$s=\sigma+it$。则对每个 $\varepsilon>0$

$$I^s(f)(x)=\int_{|y|<\varepsilon}\{f(x-y)-f(x)\}|y|^{-n+s}dy$$
$$+\int_{|y|>\varepsilon}f(x-y)|y|^{-n+s}dy+\omega_{n-1}\left(\frac{\varepsilon^s}{s}\right)f(x).$$

这表明

$$I^{it}f(x)=\lim_{\sigma\to 0}I^{\sigma+it}(f)(x)$$

存在，只要$t\neq 0$，并且$f$属于$C^1$类。最后，对$K(x)=|x|^{-n+it}$应用上面§6.11以及本章的定理1，可知算子

$$f\to I_\varepsilon^{it}(f)=\int_{|y|>\varepsilon}f(x-y)|y|^{-n+it}dy$$

是$L^p(R^n)$有界的，$1<p<\infty$，且对$\varepsilon$一致。通过选择$\varepsilon$趋向于0的一个合适序列，我们看到$I^{it}$可以开拓成为$L^p(R^n)$的有界算子，$1<p<\infty$。用第三章的§3.3还可以得到

$$(I^{it}(f))^\wedge(x)=\gamma_{0,it}|x|^{-it}\hat{f}(x),$$

其中 $$\gamma_{0,it}=\pi^{n/2-it}\frac{\Gamma(it/2)}{\Gamma(n/2-it/2)}.$$

有关结果见Muckenhoupt[1]。

6.13　令$K_1(x)=\dfrac{\Omega_1(x)}{|x|^n}$与$K_2(x)=\dfrac{\Omega_2(x)}{|x|^m}$是属于在§4考虑的类型的核，分别定义在$R^n$与$R^m$上。在$L^p(R^{n+m})$，通过$T_{\varepsilon,\delta}=T_\varepsilon^1\otimes T_\delta^2$来定义变换$f\to T_{\varepsilon,\delta}(f)$，其中

$$T_\varepsilon^1(f)(x^1)=\int_{|y^1|>\varepsilon}\frac{\Omega_1(y^1)}{|y^1|^n}f(x^1-y^1)dy^1,$$

与 $$T_\delta^2(f)(x^2)=\int_{|y^2|>\delta}\frac{\Omega_2(y^2)}{|y^2|^m}f(x^2-y^2)dy^2.$$

（$T_{\varepsilon,\delta}$可以认为是对$R^{n+m}$上的函数取$T_\varepsilon^1$与$T_\delta^2$的复合；其中$T_\varepsilon^1$作用在前$n$个变量上，$T_\delta^2$作用在后$m$个变量上.）

(a) 若$f\in L^p(R^{n+m})$，则

$$\lim_{\substack{\varepsilon\to 0\\ \delta\to 0}}T_{\varepsilon,\delta}(f)(x)=T(f)(x)$$

几乎处处与按$L^p$模存在，其中$1<p<\infty$。（见Sokol-Sokolowski[1]与Cotlar[1]。然而后一篇文章在$L\log L$的情形包含了一个错误的推理。）

(b) 当$f\in L\log L$类时，还有一类似的但比较精密的结果。对$T^1$与$T^2$是一维的Hilbert变换，见Zygmund[3]；然而，他的方

法用了复函数论。一般的情形见 Fefferman[2]。

6.14 设$K$是一个紧支集的广义函数；它在原点以外等于一局部可积函数，并假定它的 Fourier 变换$\hat{K}$是函数。假设对固定的$\theta$，$0\leqslant\theta<1$，有

(a) $|\hat{K}(x)|\leqslant A(1+|x|)^{-n\theta/2}$；

(b) $\int_{|x|\geqslant 2|y|^{1-\theta}}|K(x-y)-K(x)|\,dx\leqslant A$。

则算子$f\to K*f$对具紧支集的$C^\infty$函数有定义，可以开拓为一弱(1,1)型与$L^p$有界的变换，其中$1<p<\infty$。Fefferman[1]。

一个例子是算子

$$\lim_{\varepsilon\to 0}\int_{\varepsilon<|y|<1}K(y)f(x-y)dy,$$

其中$K(x)=|x|^{-n}\exp\{i|x|^{-\gamma}\}$，$\gamma>0$。这同第四章§7.4所叙述的乘子变换密切相关。

## 注　释

**节1**　本节中所复习的材料的一个详细说明可以在《富里叶分析》第Ⅰ章中找到。从抽象的局部紧 Abel 群的观点来研究，见 Rudin[1]，Hewitt 与 Ross[1]。

**节2，节3，节4与节5**　最早的方法(一维)是用复函数论，详见 Zygmund[7，第Ⅶ章]与[8，第Ⅶ章]，其中可以找到更多的历史上的文献目录。Hilbert 变换的实变方法要回溯到 Besicovitch[1]和[2]，Titchmarsh[1]与 Marcinkiewicz[1]。现在的$n$维理论源于 Calderón–Zygmund[1]。它在 Cotlar[2]，Stein[3]，Hörmander[1]，Schwartz[1]，Benedek，Calderón 与 Panzone[1]及其它文献中都得到了发挥。读者还可以查阅综合性文章 Zygmund[5]与 Calderón[7]。

# 第三章 Riesz 变换，Poisson积分与球调和函数系

至此，跟随我们的读者迫于与理论中的某些较为技术性的内容打交道，而且不得不一步一步地朝着一个似乎是枯燥且无所得益的方向攀登。如果在若干地方他感到不愿意再继续进行，这是可以理解的。

本章的目的，一方面是要通过对已经熟悉的领域的审视以使读者具有信心，同时也想利用这一机会给读者介绍一些以后需要的工具。

于是在这里，我们的论述在方式上自然与前两章不同。实际上，主要力量将放在理论的重要形式方面以及详尽地研究某些重要的例子。这些形式方面以及特别的例子的背景有两点，这是我们要给予扼要说明的。从本质上说，$\boldsymbol{R}^n$上的旋转群在调和分析理论中起着决定性的作用，就像平移与展缩变换群所起的那样。如果从这一观点出发，考虑最简单的，非平凡的而且是旋转“不变的”算子，就导致了Riesz 变换。与此相关的是经典调和分析($\boldsymbol{R}^1$上)与复变函数论之间的内在联系，企图通过调和函数把它尽可能地推广到$\boldsymbol{R}^n$，也会使我们回到Riesz变换上来。

## §1 Riesz 变换

**1.1** 让我们从Hilbert 变换

$$H(f)(x)=\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|y|>\varepsilon}\frac{f(x-y)}{y}dy$$

开始。这是$\boldsymbol{R}^1$中的情形，其中

$$K(x)=\frac{1}{\pi x},\qquad \Omega(x)=\frac{1}{\pi}\operatorname{sign}x=\frac{1}{\pi}\frac{x}{|x|}.$$

根据上一章§4.2 的式 (26)，我们由 Fourier 变换直接得到 $(H(f))^\wedge(x)=m(x)\hat{f}(x)$，其中乘子 $m(x)$ 是 $i\operatorname{sign}x$。由此易知，$H$ 是 $L^2(\boldsymbol{R}^1)$ 上的酉变换，且 $H^2=-I$。

现在，联系到展缩运算 $\tau_\delta(f)(x)=f(\delta x)$，在一维的情形注意到它对一切非零的 $\delta$(正的或负的) 都有定义是方便的。因此，当 $\delta>0$ 时，显然有

$$\begin{aligned}H\tau_\delta(f)(x)&=\lim_{\varepsilon\to0}\frac{1}{\pi}\int_{|y|>\varepsilon}\frac{f(\delta x-\delta y)}{y}dy\\&=\lim_{\varepsilon'\to0}\frac{1}{\pi}\int_{|y|>\varepsilon'}\frac{f(\delta x-y)}{y}dy\\&=\tau_\delta H(f)(x),\end{aligned}$$

即 $H\tau_\delta=\tau_\delta H$；同样明显的是，当 $\delta<0$ 时，有 $\tau_\delta H=-H\tau_\delta$。

事实上，这些简单的展缩"不变"性与显见的平移不变性就刻划了Hilbert 变换的特征。

**命题 1** 假设 $T$ 是 $L^2(\boldsymbol{R}^1)$ 上的一个有界算子，满足下列性质：

(a) $T$ 与平移是可交换的；

(b) $T$ 与正展缩(即 $\delta>0$)是可交换的；

(c) $T$ 与反射 $f(x)\to f(-x)$ 是反交换的。

则 $T$ 是Hilbert 变换的常数倍。

证明是不困难的。事实上，因为 $T$ 与平移可交换，所以，根据上一章§1.4 的命题存在有界函数 $m(x)$，使得 $(T(f))^\wedge(x)=m(x)\hat{f}(x)$。再用 $\mathscr{F}$ 表示 Fourier 变换，$\mathscr{F}(f)=\hat{f}$。我们有

$$\begin{aligned}(\mathscr{F}\tau_\delta)(f)(y)&=\int_{-\infty}^{+\infty}e^{2\pi ixy}f(\delta x)dx\\&=|\delta|^{-1}\int_{-\infty}^{+\infty}e^{2\pi ixy/\delta}f(x)dx\end{aligned}$$

$$= |\delta|^{-1}(\tau_{\delta^{-1}}\mathscr{F})(f)(y).$$

即$\mathscr{F}\tau_\delta = |\delta|^{-1}\tau_{\delta^{-1}}\mathscr{F}$。

乘子的定义可以形式地写为$\mathscr{F}T = m\mathscr{F}$(这里的$m$表示用$m$作乘法运算!)，而假设(b)与(c)又可以导出$T\tau_\delta = \mathrm{sign}(\delta)\tau_\delta T$，结合这几个等式就得到

$$\tau_\delta m = \tau_\delta \mathscr{F}T\mathscr{F}^{-1} = |\delta|^{-1}\mathscr{F}\tau_{\delta^{-1}}T\mathscr{F}^{-1} = \delta^{-1}\mathscr{F}T\tau_{\delta^{-1}}\mathscr{F}^{-1}$$
$$= \mathrm{sign}(\delta)\mathscr{F}T\mathscr{F}^{-1}\tau_\delta = \mathrm{sign}(\delta)m\tau_\delta.$$

从而 $\tau_\delta m = \mathrm{sign}(\delta)m\tau_\delta$，即若 $\delta\neq 0$，有 $m(\delta x) = \mathrm{sign}(\delta)m(x)$。

这就是说，$m(x) =$ 常数 $\times\,\mathrm{sign}(x)$，命题证毕。

命题的证明附带说明：只有那些与所有上述算子（平移、正的与负的展缩）可交换的 $L^2(\boldsymbol{R}^1)$ 上的有界线性变换才等于恒等算子的常数倍。这一说明与命题一起形象地表明了 Hilbert 变换在$\boldsymbol{R}^1$ 上的调和分析中所起的特殊作用。现在，我们来寻求在 $\boldsymbol{R}^n$ 中有着类似结构特征的算子。

1.2 我们先对展缩、旋转变换与 $n$ 维Fourier 变换之间的相互关系作一些考察。设$\mathscr{F}(f) = \hat{f}$以及$\delta>0$，我们有

$$(\mathscr{F}\tau_\delta)(f)(x) = \int_{\boldsymbol{R}^n} e^{2\pi i x\cdot y}f(\delta y)dy$$

$$= \delta^{-n}\int_{\boldsymbol{R}^n} e^{2\pi i x\cdot y/\delta}f(y)dy = \delta^{-n}(\tau_{\delta^{-1}}\mathscr{F})(f)(x).$$

从而形式地可以写成

$$\mathscr{F}\tau_\delta = \delta^{-n}\tau_{\delta^{-1}}\mathscr{F}. \tag{1}$$

现在用$\rho$表示 $\boldsymbol{R}^n$ 中绕原点的任一（真正的或广义的）旋转变换，也用$\rho$表示其对函数的作用，$\rho(f)(x) = f(\rho^{-1}x)$，则得

$$(\mathscr{F}\rho)(f)(x) = \int_{\boldsymbol{R}^n} e^{2\pi i x\cdot y}f(\rho^{-1}y)dy = \int_{\boldsymbol{R}^n} e^{2\pi i x\cdot \rho y}f(y)dy$$

$$= \int_{\boldsymbol{R}^n} e^{2\pi i \rho^{-1}x\cdot y}f(y)dy = (\rho\mathscr{F})(f)(x),$$

即

(2) $$\mathscr{F}\rho=\rho\mathscr{F}.$$

我们还需要下述初等事实。设 $m(x)=(m_1(x),m_2(x),\cdots,m_n(x))$ 是 $\boldsymbol{R}^n$ 上的 $n$ 函数组，对任一旋转 $\rho$，记其矩阵表示为 $\rho=(\rho_{jk})$。假定变换 $m$ 类似向量的作用，那末形式上我们可以记为

$$m(\rho^{-1}x)=\rho(m(x)),$$

更明显些就是

(3) $$m_j(\rho^{-1}x)=\sum_k \rho_{jk}m_k(x),\quad \text{对每一个旋转}\rho.$$

**引理** 假设 $m$ 是零次齐次函数，即对 $\delta>0$，$m(\delta x)=m(x)$。若变换 $m$ 满足式(3)，则存在常数 $c$，使得 $m(x)=cx/|x|$；即

(4) $$m_j(x)=c\frac{x_j}{|x|}.$$

为了证明这一结论，我们注意到只需考虑 $x$ 在单位球上的情形就可以了。现在设 $e_1,\cdots,e_n$ 为坐标轴的单位向量，并记 $c=m_1(e_1)$。易知若 $j\neq 1$，则 $m_j(e_1)=0$。事实上，对于任一保持 $e_1$ 不变的旋转 $\rho$，式(3)给出

$$m_j(e_1)=\sum_{k=2}^{n}\rho_{jk}m_k(e_1),\qquad j=2,\cdots,n.$$

这就是说，$n-1$ 维向量 $(m_2(e_1),m_3(e_1),\cdots,m_n(e_1))$ 对所有 $n-1$ 维向量空间上的旋转都是不变的，因而有 $m_2(e_1)=m_3(e_1)=\cdots=m_n(e_1)=0$。把这代入式(3)中，就得到 $m_j(\rho^{-1}e_1)=\rho_{j1}m_1(e_1)=c\rho_{j1}$。但若 $\rho^{-1}e_1=x$，则 $\rho_{j1}=x_j$，故 $m_j(x)=cx_j$，$(|x|=1)$，这就证明了引理。

令人惊奇的事实是，只是在 $n=1$ 与 $n=2$ 时才需要整个旋转群，而当 $n\geqslant 3$ 时真正的旋转就足够了，而这时真正的旋转子群在少一维的空间中仍在其单位球面 $S^{n-2}$ 上可传递的。

现在我们可以定义 $n$ 维 Riesz 变换了。对 $f\in L^p(\boldsymbol{R}^n)$，$1\leqslant p<\infty$，令

(5) $$R_j(f)(x)=\lim_{\varepsilon\to 0}c_n\int_{|y|>\varepsilon}\frac{y_j}{|y|^{n+1}}f(x-y)dy,\quad j=1,\cdots,n,$$

其中
$$c_n=\frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}}.$$
这样，$R_j$ 是由核 $K_j(x)=\Omega_j(x)/|x|^n$ 定义的，而 $\Omega_j(x)=c_n x_j/|x|$。

下面，我们来导出相应于 Riesz 变换的乘子，它们确实是满足定义的。让我们回忆上一章 § 4.2的式(26)，就是

$$(6)\qquad m(x)=\int_{S^{n-1}}\Gamma(x\cdot y)\Omega(y)d\sigma(y),\qquad |x|=1,$$

其中
$$\Gamma(t)=\frac{\pi i}{2}\operatorname{sign}t+\log\left|\frac{1}{t}\right|.$$
注意，从$\Omega$到$m$的映射(6)是与旋转可交换的，这是因为核$\Gamma(x\cdot y)$仅仅依赖于 $x$ 与 $y$ 的内积。另外，易知核

$$(K_1(x),\cdots,K_n(x))=c_n\left(\frac{x_1}{|x|^{n+1}},\frac{x_2}{|x|^{n+1}},\cdots,\frac{x_n}{|x|^{n+1}}\right)$$

满足变换规律(3)（用$K_j$代替$m_j$）。因此，根据刚才提到的变换$K_j\to m_j$是同旋转可交换的，知其乘子也满足式(3)。又因每个$m_j$是零次齐次的，所以由上引理知

$$m_j(x)=c\frac{x_j}{|x|}.$$

而在我们这一特殊情形（加上我们所选的常数）中，$c=i$。这是由于在式(6)中对某固定点计算$m_j$，上述结论等价于

$$(7)\qquad \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}=\int_{S^{n-1}}|\cos\theta|d\sigma(y),$$

其中 $\theta$ 是变动的单位向量 $y$ 与某固定方向的夹角。我们可以直接计算这一积分，或者干脆引用后文的一般结果即定理 5。总之，我们得到

$$(8)\qquad (R_j(f))^\wedge(x)=i\frac{x_j}{|x|}\hat{f}(x),\qquad j=1,\cdots,n,$$

我们可以把作用到Riesz变换的变换规律(3)用更反映本质的方式表达出来。具体地说，即

$$(9)\qquad \rho R_j\rho^{-1}(f)=\sum_k \rho_{jk}R_k(f).$$

这一陈述说明，在$R^n$中的旋转变换下，Riesz 算子类似于向量分量以同样的方式进行变换。式(9)的证明是直接的。可以直接应用Riesz变换的定义(5)，或者根据式(8)借助于它们的 Fourier变换。这样，如果我们用符号记 $\hat{R}_j=m_j$，那末式 (9) 成为

$$\rho(m_j\rho^{-1}(f))=\sum_k \rho_{jk}m_k(f),$$

即

$$m_j(\rho^{-1}x)=\sum_k \rho_{jk}m_k(x).$$

这些考察使我们得到一个逆命题。

**命题2** 设 $T=(T_1,T_2,\cdots,T_n)$是 $n$ 个 $L^2(R^n)$上的有界变换组。若有

(a) 每个$T_j$同$R^n$的平移可交换；

(b) 每个$T_j$同$R^n$的展缩可交换；

(c) 对于$R^n$的每个旋转 $\rho=(\rho_{jk})$，有

$$\rho T_j\rho^{-1}(f)=\sum_k \rho_{jk}T_k(f).$$

则$T_j$是Riesz变换的常数倍，即存在一个常数 $c$，使得

$$T_j=cR_j,\qquad j=1,2,\cdots,n.$$

命题证明的所有部分，实际上已经讨论过了。现在我们把它综合如下：(i) 因为 $T_j$ 在 $L^2(R^n)$ 上有界，而且同平移可交换，所以它们都可以用有界的乘子来表示。记作 $\hat{T_j}=m_j$。(ii) 由$T_j$同展缩可交换，又根据展缩与Fourier 变换之间的关系式(1)，我们有 $m_j(\delta x)=m_j(x)$，$\delta>0$；即每个$m_j$ 是零次齐次的。(iii) 最后，关系式(3)是假设(c)的推论，从而由引理可得命题的结论。

**1.3 一个应用** Riesz变换的重要应用之一是，它们可用于联系一个函数的偏微商的各种组合之间的媒介。Riesz 变 换的这一作用特别地将在第五章中叙述。这里我们只作两个较简单的例解，这些例子不仅本身是有趣的，而且反映了在椭圆微分算子理论中所作的一类典型估计的特征。

**命题3** 假设 $f$ 属于$c^2$类且具有紧支集，令

$$\Delta f = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2},$$

则我们有先验估计

$$\left\| \frac{\partial^2 f}{\partial x_j \partial x_k} \right\|_p \leqslant A_p \|\Delta f\|_p, \qquad 1 < p < \infty. \tag{10}$$

这一命题是Riesz变换的$L^p$有界性以及恒等式

$$\frac{\partial^2 f}{\partial x_j \partial x_k} = -R_j R_k \Delta f \tag{11}$$

的一个直接推论。

为了证明式（11），我们利用Fourier 变换。设 $f$ 的 Fourier 变换是$\hat{f}(x)$，

$$\hat{f}(x) = \int_{R^n} e^{2\pi i x \cdot y} f(y) dy,$$

则$\dfrac{\partial f}{\partial x_j}$的Fourier变换是$-2\pi i x_j \hat{f}(x)$，从而有

$$\begin{aligned}\left(\frac{\partial^2 f}{\partial x_j x_k}\right)^\wedge (x) &= -4\pi^2 x_j x_k \hat{f}(x) \\ &= -\left(\frac{i x_j}{|x|}\right)\left(\frac{i x_k}{|x|}\right)(-4\pi |x|^2) \hat{f}(x) \\ &= -(R_j R_k \Delta f)^\wedge,\end{aligned}$$

这就是式(11)。

现在，再看下述在二维位势理论的另一个有趣的应用。

**命题4** 设 $f$ 属于$\boldsymbol{R}^2$上的 $C^1$类，且具有紧支集，则我们有先

验估计

$$\left\|\frac{\partial f}{\partial x_1}\right\|_p+\left\|\frac{\partial f}{\partial x_2}\right\|_p\leqslant A_p\left\|\frac{\partial f}{\partial x_1}+i\frac{\partial f}{\partial x_2}\right\|_p,\qquad 1<p<\infty.$$

不用说，这一命题仅当 $f$ 是复值时才有意义。

命题 4 的证明与命题 3 极其相似，只是这里所用的恒等式是

$$\frac{\partial f}{\partial x_j}=-R_j(R_1-iR_2)\left(\frac{\partial f}{\partial x_1}+i\frac{\partial f}{\partial x_2}\right),\qquad j=1,2.$$

有关这些特殊事实的一个比较系统的表达将在下面的 §3.5 中给出。

## §2 Poisson 积分与恒等逼近

**2.1** 现在我们要来介绍一个概念，它在今后的大部分工作中是必不可少的。我们已经知道了调和函数的理论。这一理论的应用程式可叙述如下。设想$\boldsymbol{R}^n$ 是$(n+1)$维上半空间$\boldsymbol{R}^{n+1}_+$ 的边界超平面，用坐标的记法

$$\boldsymbol{R}^{n+1}_+=\{(x,y):\ x\in\boldsymbol{R}^n,\ y>0\}.$$

考虑$\boldsymbol{R}^n$上函数 $f$ 的Poisson积分，这个Poisson积分实际上是下述$\boldsymbol{R}^{n+1}_+$上Dirichlet问题的解：寻求定义在 $\boldsymbol{R}^{n+1}_+$ 上的一个调和函数$u(x,y)$，它在$\boldsymbol{R}^n$上的边值（在适当的意义下）是$f(x)$。

这一问题的形式解几乎可以在$L^2$理论的推演中给出。

事实上，设 $f\in L^2(\boldsymbol{R}^n)$，且$\hat{f}$是它的Fourier 变换。考虑

$$u(x,y)=\int_{t\in\boldsymbol{R}^n}\hat{f}(t)e^{-2\pi i t\cdot x}e^{-2\pi|t|y}\,dt,\qquad y>0. \tag{12}$$

由于$\hat{f}\in L^2(\boldsymbol{R}^n)$以及$e^{-2\pi|t|y}$ 对 $y>0$ 随$|t|$增大而速降，上述积分是绝对收敛的。同理，它可对 $x$ 与 $y$ 微分任意次，并且这些微分运算可以在积分号下实现。因此

$$\Delta u=\frac{\partial^2 u}{\partial y^2}+\sum_{k=1}^{n}\frac{\partial^2 u}{\partial x_k^2}=0,$$

这是因为对每个固定的 $t$，因子$e^{-2\pi i t\cdot x}e^{-2\pi|t|y}$ 是满足这个等式的。

再由Plancherel定理可知，当$y\to 0$时，$u(x,y)$按$L^2(\boldsymbol{R}^n)$模收敛到$f(x)$。

Dirichlet问题的这一个解也可以通过不明显地用Fourier变换的方法来给出。为此，我们定义Poisson核$P_y(x)$：

$$(13)\qquad P_y(x)=\int_{\boldsymbol{R}^n}e^{-2\pi i t\cdot x}e^{-2\pi|t|y}\,dt,\qquad y>0.$$

这时前面所得的$u(x,y)$就可以写成卷积

$$(14)\qquad u(x,y)=\int_{\boldsymbol{R}^n}P_y(t)f(x-t)\,dt.$$

我们称此 $u$ 为 $f$ 的 Poisson 积分。

Poisson 核还有一个明显的表示式。

**命题 5**

$$(15)\qquad P_y(x)=\frac{c_n y}{(|x|^2+y^2)^{(n+1)/2}},\qquad c_n=\frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}},$$

这里的$c_n$与 Riesz 变换定义（前面的式(5)）中出现的常数相同。熟知的公式（15）可证明如下。我们引用两个恒等式：

$$(\alpha)\qquad \int_{\boldsymbol{R}^n}e^{-\pi\delta|t|^2}e^{-2\pi i t\cdot x}\,dt=\delta^{-n/2}e^{-\pi|x|^2/\delta},\qquad \delta>0;$$

$$(\beta)\qquad e^{-\gamma}=\frac{1}{\sqrt{\pi}}\int_0^\infty\frac{e^{-u}}{\sqrt{u}}e^{-\gamma^2/4u}\,du,\qquad \gamma>0.$$

首先，由变量替换，公式$(\alpha)$可直接化为 $\delta=1$ 的众所周知的特殊情形。其次，公式$(\beta)$表示指数$e^{-\gamma}$是指数族$e^{-\gamma^2/4u}$$(0<u<\infty)$的加权平均，而这正是**从属性原理**① 的一个重要特例。为了证明$(\beta)$，写出

$$e^{-\gamma}=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{e^{i\gamma x}}{1+x^2}\,dx,$$

① 参见Bochner[2]，第四章。

再把因子$1/(1+x^2)$表成$\int_0^\infty e^{-(1+x^2)u}du$。这就导致二重积分

$$e^{-\gamma}=\frac{1}{\pi}\int_0^\infty e^{-u}\left(\int_{-\infty}^{+\infty}e^{i\gamma x}e^{-ux^2}dx\right),$$

算出内层积分就可得到式($\beta$)。现在让我们再回到$P_y(x)$。根据式($\beta$)以及式(13)，我们有

$$P_y(x)=\frac{1}{\sqrt{\pi}}\int_{R^n}\left(\int_0^\infty\frac{e^{-u}}{\sqrt{u}}e^{-\pi^2|t|^2y^2/u}du\right)e^{-2\pi it\cdot x}dt.$$

对$\delta=\pi y^2/u$应用式($\alpha$)，我们得到

$$P_y(x)=\frac{1}{\pi^{(n+1)/2}}\int_0^\infty e^{-u}e^{-|x|^2u/y^2}y^{-n}u^{(n-1)/2}du$$

$$=\frac{y}{(\pi(|x|^2+y^2))^{(n+1)/2}}\int_0^\infty e^{-u}u^{(n-1)/2}du,$$

这就是所要求的Poisson核表达式。

下面我们列出 Poisson 核的一些性质，这些性质现在看起来已经较为明显了。

(a) $P_y(x)>0$。

(b) $\int_{R^n}P_y(x)dx=1$，$y>0$；更一般地，

$$P_y^\wedge(x)=e^{-2\pi|x|y},$$

这只要应用 Fourier 变换的反演公式到式(13)即可。

(c) $P_\varepsilon(x)$是$-n$次齐次的：

$$P_\varepsilon(x)=P_1(x/\varepsilon)\varepsilon^{-n},\qquad \varepsilon>0.$$

(d) $P_y(x)$是$|x|$的递降函数，而且对$1\leqslant p\leqslant\infty$有$P_y(x)\in L^p(R^n)$。

(e) 设$f\in L^p(R^n)$，$1\leqslant p\leqslant\infty$，则由式(14)给出的$f$的Poisson积分$u$是$R_+^{n+1}$上的调和函数。这是下述事实的简单推论：$P_y(x)$是$R_+^{n+1}$上的调和函数；而后者可从式(13)直接得到。

(f) Poisson 核具有"半群性质"：

$$P_{y_1} * P_{y_2} = P_{y_1+y_2}, \quad y_1>0, \ y_2>0.$$

这可从(b)中的Fourier 变换公式直接导出。

下述定理已充分反映出 Poisson 积分边界性质的重要内容。

**定理1** 假设$f \in L^p(\boldsymbol{R}^n)$，$1 \leqslant p \leqslant \infty$，且令$u(x,y)$是它的 Poisson 积分。则

(a) $\sup\limits_{y>0}|u(x,y)| \leqslant M(f)(x)$，其中$M(f)$是第一章 §1 中所述的极大函数。

(b) 对几乎所有的$x$，$\lim\limits_{y \to 0} u(x,y) = f(x)$。

(c) 若$p<\infty$，则当$y \to 0$时$u(x,y)$按$L^p(\boldsymbol{R}^n)$模收敛到$f(x)$。

现在，这一定理的证明将用更加一般的命题来代替，后者对于一大类恒等逼近都是成立的。

2.2 设$\varphi$是$\boldsymbol{R}^n$上的一个可积函数，且令$\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$，$\varepsilon>0$。

**定理2** 假设$\varphi$的最小下降径向控制是可积的，即令$\psi(x) = \sup\limits_{|y| \geqslant |x|} |\varphi(y)|$，假定

$$\int_{\boldsymbol{R}^n} \psi(x)dx = A < \infty.$$

则对同一常数$A$，我们有

(a) $\sup\limits_{\varepsilon>0}|(f*\varphi_\varepsilon)(x)| \leqslant AM(f)(x)$，$f \in L^p(\boldsymbol{R}^n)$，$1 \leqslant p \leqslant \infty$。

(b) 若再假定

$$\int_{\boldsymbol{R}^n} \varphi(x)dx = 1,$$

则几乎处处有$\lim\limits_{\varepsilon \to 0}(f*\varphi_\varepsilon)(x) = f(x)$。

(c) 若$p<\infty$，且$\|\varphi\|_p = 1$，则当$\varepsilon \to 0$时，$\|f*\varphi_\varepsilon - f\|_p \to 0$。

在第一章中，我们已经考虑过一种特殊情形，那时，$\varphi(x)$是单位球$B$的特征函数除以$B$的测度，即

$$\varphi(x)=\frac{1}{m(B)}\chi_B.$$

本定理的要点就是要将问题约化为这一基本的特殊情形。

我们先给出(c)的证明。注意，这一证明实际上在$\varphi$仅为可积这一较弱条件下就是成立的。(当然，规一化

$$\int_{R^n}\varphi dx=1$$

仍是需要的。) 首先我们指出，若$f\in L^p(\mathbf{R}^n)$，$p<\infty$，并且记$\|f(x-y)-f(x)\|_p=\Delta(y)$，则当$y\to 0$时$\Delta(y)\to 0$ ①。若$f_1$是具有紧支集的连续函数，则对$f_1$，此断言是$f_1(x-y)$一致收敛(当$y\to 0$时)到$f_1(x)$的推论。在一般情形，记$f=f_1+f_2$，其中$f_1$如上所述，而$\|f_2\|_p\leqslant\delta$；这是可能的，因为这样的$f_1$在$L^p(p<\infty)$中稠密。因此我们有$\Delta(y)\leqslant\Delta_1(y)+\Delta_2(y)$，其中$\Delta_1(y)\to 0$(当$y\to 0$时)，而$\Delta_2(y)\leqslant 2\delta$。这就说明对一般的$f\in L^p(\mathbf{R}^n)$，$p<\infty$，有$\Delta(y)\to 0$。现在因为

$$\int_{R^n}\varphi_\varepsilon(x)dx=\int_{R^n}\varphi(x)dx=1,$$

所以有

$$f*\varphi_\varepsilon-f=\int_{R^n}[f(x-y)-f(x)]\varphi_\varepsilon(y)dy.$$

故

$$\|f*\varphi_\varepsilon-f\|_p\leqslant\int_{R^n}\Delta(y)|\varphi_\varepsilon(y)|dy$$

$$=\int_{R^n}\Delta(\varepsilon y)|\varphi(y)|dy\to 0;$$

这后一事实的根据是 Lebesgue 控制收敛定理以及当$\varepsilon\to 0$时有

① 这一命题说的是从$\mathbf{R}^n$到$L^p(\mathbf{R}^n)$的映射$y\to f(x-y)$是连续的.

$\Delta(\varepsilon y)\to 0$。定理的结论(c)证毕。现在来证明结论(a)。让我们稍微改变一下记号，令 $\psi(r)=\psi(x)$，当 $|x|=r$；因为 $\psi(x)$是径向函数，所以这样的记法不会产生混乱。注意

$$\int_{r/2<|x|<r}\psi(x)\,dx\geqslant\psi(r)\int_{r/2<|x|<r}dx=\psi(r)cr^{n}.$$

因此由假设$\psi\in L^1$(以及$\psi(r)$是下降函数这一事实)推得

$$r^n\psi(r)\to 0,\qquad r\to 0 \text{ 或 } r\to\infty.$$

为了证明(a)，我们需要指出

(16) $$(f*\psi_\varepsilon)(x)\leqslant AM(f)(x),$$

其中$f\geqslant 0$，$f\in L^p(\mathbf{R}^n)$，$1\leqslant p\leqslant\infty$，$\varepsilon>0$ 以及

$$A=\int_{\mathbf{R}^n}\psi(x)\,dx.$$

由于结论(16)显然是平移不变的(关于$f$)，而且也是展缩不变的(关于$\psi$)，因此只要证明

(17) $$(f*\psi)(0)\leqslant AM(f)(0)$$

即可。

为证式(17)，不妨假定$M(f)(0)<\infty$。记

$$\lambda(r)=\int_{x\in S^{n-1}}f(rx)\,d\sigma(x),\qquad \Lambda(r)=\int_{|x|<r}f(x)\,dx,$$

因此

$$\Lambda(r)=\int_0^r\lambda(t)t^{n-1}dt.$$

我们有

$$(f*\psi)(0)=\int_{\mathbf{R}^n}f(x)\psi(x)\,dx=\int_0^\infty\lambda(r)\psi(r)r^{n-1}dr$$

$$=\lim_{\substack{\varepsilon\to 0\\ N\to\infty}}\int_\varepsilon^N\lambda(r)\psi(r)r^{n-1}dr$$

$$=-\lim_{\substack{\varepsilon\to 0\\ N\to\infty}}\int_\varepsilon^N\Lambda(r)\,d\psi(r),$$

其中的最后一个等式用到了分部积分。本来还有误差

$$\Lambda(N)\psi(N)-\Lambda(\varepsilon)\psi(\varepsilon),$$

但这一项当$\varepsilon\to 0$以及$N\to\infty$时是趋向于零的。这里用到了前面提及的$\psi$的性质以及

$$\Lambda(r)=\int_{|x|<r}f(x)dx\leqslant Vr^nM(f)(0),$$

其中$V$是单位球的体积。于是

$$f*\psi(0)=\int_0^\infty \Lambda(r)d(-\psi(r))\leqslant VM(f)(0)\int_0^\infty r^n d(-\psi(r)),$$

这就得到式(17)，从而式(16)得证。

几乎处处收敛性质(6)可用大家熟知的方法证明如下：首先我们知道，若$f_1$是具有紧支集的连续函数，则当$\varepsilon\to 0$时$(f_1*\varphi_\varepsilon)(x)$是一致收敛于$f_1(x)$的。其次，对于$f\in L^p(\boldsymbol{R}^n)$，$1\leqslant p<\infty$，我们有$f=f_1+f_2$，其中$f_1$如上所述，$f_2$的$L^p$范数很小。剩下的推理则与第一章§1.5中式(6)以后的叙述雷同。这样，我们就得到了$\lim\limits_{\varepsilon\to 0}f_\varepsilon(x)$几乎处处存在且等于$f(x)$。最后，对于$f$是有界的情形，我们取定任意一个球$B$，并把问题归结为证明，对几乎所有的$x\in B$，$\lim\limits_{\varepsilon\to 0}(f*\varphi_\varepsilon)(x)=f(x)$。令$B_1$是任意一个严格包含$B$的球，并记$\delta$为从$B$到$B_1$的余集的距离。令

$$f_1(x)=\begin{cases}f(x), & x\in B_1,\\ 0, & x\overline{\in}B_1,\end{cases}$$

并写$f(x)=f_1(x)+f_2(x)$。则$f_1(x)\in L^1(\boldsymbol{R}^n)$，因此，对于$f_1$上述的逼近结论成立。而对$x\in B$，我们有

$$|(f_2*\varphi_\varepsilon)(x)|=\left|\int f_2(x-y)\varphi_\varepsilon(y)dy\right|$$

$$\leqslant\int_{|y|>\delta>0}|f_2(x-y)\varphi_\varepsilon(y)|dy$$

$$\leqslant \|f\|_\infty \int_{|y| \geq \delta/\varepsilon} |\varphi(y)|\, dy \to 0, \qquad \varepsilon \to 0.$$

定理 2 证毕。

定理 1 可从定理 2 直接得出，因为这时$\varphi(x) = \psi(x) = P_1(x)$，再加上 Poisson 核的性质(a)—(d)。

还有一些关于定理 2 结果的变形。当然，这对于 Poisson 积分也同样有效。第一个变形叙述如下，但不给证明，这是因为它容易从前面的论证得到。

**推论** 假设$f(x)$在$\boldsymbol{R}^n$上连续且有界，则在$\boldsymbol{R}^n$的紧集上，$(f * \varphi_\varepsilon)(x)$一致收敛到$f(x)$。

特别地，这个推论表明，如果在$\boldsymbol{R}^n$上给定一个有界且连续的函数$f$，那末就可以找到一个在$\boldsymbol{R}^{n+1}_+$的闭包上连续且在其内部调和的函数$u(x, y)$，当它限制在边界上时其值就是给定的$f$。因此，在这种情形下，Dirichlet 问题是可解的。

第二个变形稍微困难。它是用有限 Borel 测度代替可积函数的一个类似结果，我们将在§4.1中作一概述。

**2.3 共轭调和函数系** 现在我们要把Riesz 变换与调和函数理论，特别是与 Poisson 积分联系起来。由于在这里我们的主要兴趣是形式的完整，因此仅限于$L^2$的情形。（$L^p$的情形以及有关结果将在§4.3与§4.4中叙述。）

**定理3** 设$f$以及$f_1, f_2, \cdots, f_n$都属于$L^2(\boldsymbol{R}^n)$，并且令其 Poisson 积分分别为$u_0(x, y) = P_y * f$，$u_1(x, y) = P_y * f_1$，…，$u_n(x, y) = P_y * f_n$，则

$$f_j = R_j(f), \quad j = 1, \cdots, n \tag{18}$$

成立的必要充分条件是下述广义 Cauchy-Riemann 方程成立：

$$\begin{cases} \displaystyle\sum_{j=0}^{n} \frac{\partial u_j}{\partial x_j} = 0, \\ \displaystyle\frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j}, \qquad j \neq k, \quad x_0 = y. \end{cases} \tag{19}$$

注意，至少在局部，方程组等价于存在调和函数$H$（有$n+1$个变量），使得

$$u_j=\frac{\partial H}{\partial x_j},\quad j=0,1,2,\cdots,n.$$

定理3是这样一类定理中的一个，其证明近乎是明显的，但其阐述却有某些令人感兴趣的地方。

假设$f_j=R_j(f)$，则

$$\hat{f}_j(t)=\frac{it_j}{|t|}\hat{f}(t).$$

因此，由式(12)知

$$u_j(x,y)=\int_{\mathbf{R}^n}\hat{f}(t)\frac{it_j}{|t|}e^{-2\pi it\cdot x}e^{-2\pi|t|y}dt,$$

$$j=1,\cdots,n.$$

由于积分的快速收敛，故方程(19)可根据积分号下求微分而直接验证。

反之，令

$$u_j(x,y)=\int_{\mathbf{R}^n}\hat{f}_j(t)e^{-2\pi it\cdot x}e^{-2\pi|t|y}dt,$$

$$j=0,1,\cdots,n.$$

则由$\dfrac{\partial u_0}{\partial x_j}=\dfrac{\partial u_j}{\partial x_0}=\dfrac{\partial u_j}{\partial y}$，$j=1,\cdots,n$，可知

$$-2\pi it_j\hat{f}_0(t)e^{-2\pi|t|y}=-2\pi|t|\hat{f}_j(t)e^{-2\pi|t|y},$$

因此，$\hat{f}_j(t)=\dfrac{it_j}{|t|}\hat{f}_0(t)$，从而

$$f_j=R_j(f_0)=R_j(f),\qquad j=1,\cdots,n.$$

上述定理表明，与复变函数理论相似，我们应该有兴趣于研究满足方程组(19)的调和函数系。我们将在第五、七与八章中再回到这一观点上来。

**2.4 一个插篇** 现在我们暂时中断主要论题，转回到第二章§4.6关于奇异积分论述中的一个未解决的问题。那里的情况是这样的：考虑核$K(x)=\Omega(x)/|x|^n$，其中$\Omega$是零次齐次函数，且它在单位球上的限制满足抵消性质(24)以及光滑性质(25)(见第47页)。涉及到的问题是，极限

$$\lim_{\varepsilon\to 0}T_\varepsilon(f)=\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\frac{\Omega(y)}{|y|^n}f(x-y)dy$$

是几乎处处存在的。

当时，我们用到了下述未予证明的引理。

**引理** 若$T^*(f)(x)=\sup_{\varepsilon>0}|T_\varepsilon(f)(x)|$，则

$$\|T^*(f)\|_p\leqslant A_p\|f\|_p,\qquad 1<p<\infty.$$

令 $T(f)(x)=\lim_{\varepsilon\to 0}T_\varepsilon(t)(x)$，这里的极限是依$L^p$模取的。第二章§4.2中定理3保证了这一极限的存在性。

现在，我们将通过证明

$$T^*(f)(x)\leqslant MT(f)(x)+CM(f)(x)$$

来证明引理。

设$\varphi$是$\boldsymbol{R}^n$上非负的光滑函数，其支集在单位球内且

$$\int_{\boldsymbol{R}^n}\varphi dx=1.$$

此外，$\varphi$还是径向函数，且对$|x|$来说是下降的。考虑

$$K_\varepsilon(x)=\begin{cases}\dfrac{\Omega(x)}{|x|^n}, & |x|\geqslant\varepsilon,\\ 0, & |x|<\varepsilon.\end{cases}$$

这又导致考察另一函数$\Phi$：

(20) $$\Phi=\varphi*K-K_1,$$

其中

$$\varphi*K=\lim_{\varepsilon\to 0}\varphi*K_\varepsilon=\lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon}K(x-y)\varphi(y)dy.$$

我们需要证明 $\Phi$ 的最小下降径向控制是可积的(从而可应用定理 2)。事实上，在$|x|<1$时，有 $\Phi=\varphi*K$，它等于

$$\int_{R^n}K(y)\varphi(x-y)dy \quad 或 \quad \int_{R^n}K(y)[\varphi(x-y)-\varphi(x)]dy,$$

由 $\varphi$ 的光滑性可知它还是有界的。当$1\leqslant|x|\leqslant 2$时，$\Phi(x)=K*\varphi-K(x)$，同理它也是有界的。最后，当$|x|\geqslant 2$时，我们有

$$\Phi(x)=\int_{R^n}K(x-y)\varphi(y)dy-K(x)$$

$$=\int_{|y|<1}[K(x-y)-K(x)]\varphi(y)dy,$$

根据第二章§4.2中的估计有

$$\Phi(x)\leqslant C'\frac{\omega(c/|x|)}{|x|^n}.$$

由于$\omega(\delta)$是递增函数且

$$\int_0^1\frac{\omega(\delta)d\delta}{\delta}<\infty,$$

我们就证明了关于$\Phi$的结论。因为奇异积分算子 $\varphi\to\varphi*K$ 是与展缩可交换的，所以从式(20)可知

(21) $\quad \varphi_\varepsilon*K-K_\varepsilon=\Phi_\varepsilon$， 其中$\Phi_\varepsilon(x)=\varepsilon^{-n}\Phi(x/\varepsilon)$。

现在，我们断言对任意的$f\in L^p(R^n)$，$1<p<\infty$，有

(22) $$(\varphi_\varepsilon*K)*f(x)=T(f)*\varphi_\varepsilon(x),$$

这里的等式是对每个 $x$ 都成立的。事实上，我们首先注意到

(23) $\quad (\varphi_\varepsilon*K_\delta)*f(x)=T_\delta(f)*\varphi_\varepsilon(x)$，对任意的$\delta>0$，

这是因为对每个 $x$ 来说，式(23)的两端都等于绝对收敛的二重积分

$$\int_{z\in R^n}\int_{|y|>\delta}K(y)f(z-y)\varphi_\varepsilon(x-z)dzdy.$$

其次，$\varphi_\varepsilon\in L^q(R^n)$，$1<q<\infty$，且 $1/p+1/q=1$，因此当 $\delta\to 0$

时，$\varphi_\varepsilon * K_\delta$依$L^p$模收敛到$\varphi_\varepsilon * K$，$T_\delta(f)$依$L^p$模收敛到$T(f)$。这就证明了式(22)。再由式(21)得

$$T_\varepsilon(f) = T(f) * \varphi_\varepsilon - f * \Phi_\varepsilon.$$

通过对$\varepsilon$取上确界并应用定理2的(a)，我们就可得到关于$T^*(f)$所要的控制。再对$f \to T(f)$以及极大函数$M$作$L^p$估计，就完成了引理的证明。

## §3 高阶Riesz变换与球调和函数系

3.1 现在我们回到原来的主题上来，考虑形如

$$T(f)(x) = \lim_{\varepsilon \to 0} \int_{|y| > \varepsilon} \frac{\Omega(y)}{|y|^n} f(x-y)\,dy \tag{24}$$

的特殊变换，其中$\Omega$是0次齐次函数且其在$S^{n-1}$上的积分为零。我们已经研究过例子

$$\Omega_j(y) = c\frac{y_j}{|y|}, \quad j = 1, \cdots, n.$$

对于$n=1$，$\Omega(y) = c\,\mathrm{sign}\,y$，而且这是仅有的可能性。对于$n>1$的情形，则有表示式(见式(6)，第70页)

$$m(x) = \int_{S^{n-1}} \Gamma(x \cdot y)\Omega(y)\,d\sigma(y), \quad |x| = 1,$$

其中$m$是由变换(24)产生的乘子。

我们曾经说过，映射$\Omega \to m$是与旋转可交换的。因此，我们将从在旋转作用下分解的观点来考察$S^{n-1}$上的函数(特别是空间$L^2(S^{n-1})$)。众所周知，这种分解要借助于球调和函数，因此我们从简述它们的性质开始。

一如既往，我们总是在$\boldsymbol{R}^n$上进行讨论，并且考虑$\boldsymbol{R}^n$上的多项式，它们还是调和的。我们定义$\mathscr{H}_k$是一切$k$次齐次调和多项式组成的线性空间，称为**$k$次球体调和系**。为了方便起见，我们将把这些多项式限制在单位球$S^{n-1}$的球面上，且在其上定义标

准的内积：

$$(P,\ Q)=\int_{S^{n-1}}P(x)\overline{Q(x)}\,d\sigma(x).$$

我们有下列结论。

3.1.1 有限维空间系$\{\mathscr{H}_k\}_{k=0}^{\infty}$是相互正交的。事实上，若$P\in\mathscr{H}_k$，且$Q\in\mathscr{H}_j$，则由Green定理知

$$(k-j)\int_{S^{n-1}}P\overline{Q}\,d\sigma(x)=\int_{S^{n-1}}\left(\overline{Q}\frac{\partial P}{\partial\nu}-P\frac{\partial\overline{Q}}{\partial\nu}\right)d\sigma(x)$$

$$=\int_{|x|<1}[\overline{Q}\Delta P-P\Delta\overline{Q}]dx=0.$$

其中$\frac{\partial}{\partial\nu}$表示外法线方向的微分，而$\Delta=\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}$是Laplace算子。

3.1.2 若$P$是任意一个$k$次齐次多项式(不必一定调和)，则$P=P_1+|x|^2P_2$，其中$P_1$是$k$次齐次调和多项式，$P_2$是$k-2$次齐次多项式。为了证明这一点，推理如下。设$\mathscr{P}_k$为一切$k$次齐次多项式组成的线性空间，并记

$$P(x)=\sum_{\alpha}a_\alpha x^\alpha,$$

其中$\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$，$\alpha_1+\alpha_2+\cdots+\alpha_n=k$，且

$$x^\alpha=x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}.$$

对每个这样的多项式，都有一个微分算子

$$P\left(\frac{\partial}{\partial x}\right)=\sum_{\alpha}a_\alpha\left(\frac{\partial}{\partial x}\right)^\alpha$$

与之对应，其中

$$\left(\frac{\partial}{\partial x}\right)^\alpha=\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

在$\mathscr{P}_k$上，定义一个正内积

$$\langle P,Q\rangle=P\left(\frac{\partial}{\partial x}\right)\overline{Q}.$$

注意，两个不同的单项式$x^\alpha$与$x^{\alpha'}$关于它总是正交的，并且

$$\langle P, P\rangle = \sum_\alpha |a_\alpha|^2 \alpha!,$$

其中$\alpha! = (\alpha_1!)(\alpha_2!)\cdots(\alpha_n!)$。

令$|x|^2\mathscr{P}_{k-2}$是$\mathscr{P}_k$的子空间，由一切形如$|x|^2P_2$的多项式组成，其中$P_2\in\mathscr{P}_{k-2}$。则其正交补(对于$\langle\cdot,\cdot\rangle$而言)恰是$\mathscr{H}_k$。事实上，$P_1$属于上述正交补当且仅当对一切$P_2$，有$\langle|x|^2P_2, P_1\rangle = 0$。但是

$$\langle|x|^2P_2, P_1\rangle = \left(P_2\left(\frac{\partial}{\partial x}\right)\Delta\right)\overline{P}_1 = \langle P_2, \Delta P_1\rangle,$$

因此$\Delta P_1$是零，从而得

$$\mathscr{P}_k = \mathscr{H}_k \oplus |x|^2\mathscr{P}_{k-2},$$

这就证明了3.1.2的结论。

3.1.3 令$H_k$为$\mathscr{H}_k$在单位球面上的限制而组成的线性空间。$H_k$中的元是$k$次**球面调和函数**。于是

$$L^2(S^{n-1}) = \sum_{k=0}^\infty H_k.$$

这里$L^2$空间用的是通常的测度，无穷直和是依Hilbert空间理论中的意义取的。因为我们已经证明了子空间系$H_k$的相互正交性，所以只需指出：每个$f\in L^2(S^{n-1})$均可用$H_k$中元素的有限线性组合按$L^2$模逼近即可。这从下面的论述就能知道。运用3.1.2中的推理并再一次用到$P_2$上，重复这一过程就得到：若$P$是一个多项式，则有

$$P(x) = P_1(x) + |x|^2P_2(x) + |x|^4P_3(x) + \cdots,$$

其中每个$P_j$都是调和多项式。当我们令$|x| = 1$时，就看到任一多项式在单位球面上的限制是球面调和函数的有限线性组合。

由于多项式在单位球面上的限制在$L^2(S^{n-1})$中按模是稠密的，故结论成立。这一事实也可重述如下。若$f\in L^2(S^{n-1})$，则$f$有展开式

$$(25)\qquad f(x)=\sum_{k=0}^{\infty}Y_k(x),\quad Y_k\in H_k,$$

其中收敛是按$L^2(S^{n-1})$模说的，而且还有

$$\int_{S^{n-1}}|f(x)|^2d\sigma(x)=\sum_k\int_{S^{n-1}}|Y_k(x)|^2d\sigma(x).$$

3.1.4 令$\Delta_S$为球面Laplace算子。若$Y_k(x)\in H_k$，则

$$\Delta_S Y_k(x)=-k(k+n-2)Y_k(x).$$

事实上，如果$Y(x)$是任意一定义在球面上的函数，那末$\Delta_S Y(x)$等于通常的Laplace算子作用于$Y(x)$的限制，不过，这里的$Y(x)$是作为零次齐次函数定义在球面的邻域上。因此，我们必须计算$\Delta(|x|^{-k}P_k(x))$，其中$P_k\in\mathscr{H}_k$。而这就是

$$|x|^{-k}\Delta P_k+P_k\Delta(|x|^{-k})+2\sum_{j=1}^{n}\frac{\partial}{\partial x_j}|x|^{-k}\frac{\partial}{\partial x_j}P_k,$$

只要我们在求微分的过程中利用

$$\sum_{j=1}^{n}x_j\frac{\partial}{\partial x_j}P_k=kP_k$$

（这是关于齐次函数的Euler定理），就得到了我们的结论。

3.1.5 设$f$有展开式（25），则$f$（必要时，在校正一个零测集上的值以后）在$S^{n-1}$上是无穷次可微的充分必要条件是

$$(26)\qquad \int_{S^{n-1}}|Y_k(x)|^2d\sigma(x)=O(k^{-N}),$$

$$k\to\infty,\ \text{对每个固定的}N.$$

为了证明这一结论，我们记

$$f(x)=\sum_{k=0}^{\infty}a_kY_k^0(x),$$

其中$Y_k^0(x)$是规一化了的$Y_k$，即$Y_k^0$的$L^2$模为1；这时条件等价于当$k\to\infty$时有$a_k=O(k^{-N/2})$。设$f$属于$C^2$类，应用Green定理可得

$$\int_{S^{n-1}} \Delta_S f\, Y_k^0 d\sigma = \int_{S^{n-1}} f \Delta_S Y_k^0 d\sigma \ .$$

因此当 $f$ 是无穷次可微时，由3.1.4便有

$$\int_{S^{n-1}} (\Delta_S^r) f \cdot Y_k^0 d\sigma = \int_{S^{n-1}} f(\Delta_S^r Y_k^0) d\sigma$$
$$= a_k[-k(k+n-2)]^r,$$

即对每一个 $r$，$a_k = O(k^{-2r})$。故式（26）成立。为证其逆，我们注意，式（26）不仅蕴含了 $f \in L^2(S^{n-1})$，而且对每个正整数 $r$，$(\Delta_S)^r f$（在适当定义后）也属于 $L^2(S^{n-1})$，这就能够推出，可以校正 $f$ 在一零测集上的值，使其成为无穷次可微的。但这里不加以证明；我们将在附录C中给出这种纯粹技术性的论证。

**3.2** 在作了上述关于球调和函数系一些基本事实的简单回顾以后，让我们回到特殊奇异积分的研究上来。首先我们讨论球调和函数系与 Fourier 变换之间的关系，实际上，这就是研究在旋转与 Fourier 变换的同时作用下空间 $L^2(\boldsymbol{R}^2)$ 的分解。它来自 Hecke 的精彩的恒等式。

**定理 4** 假设 $P_k(x)$ 是 $k$ 次齐次调和多项式，则

(27) $$\mathscr{F}(P_k(x)\, e^{-\pi|x|^2}) = i^k P_k(x) e^{-\pi|x|^2}.$$

这个需要证明的恒等式可以改写为

(28) $$\int_{\boldsymbol{R}^n} P_k(x) \exp[-\pi|x|^2 + 2\pi i x \cdot y] dx$$
$$= i^k P_k(x) e^{-\pi|y|^2}.$$

显然式（28）的左边等于 $Q(y)e^{-\pi|y|^2}$，其中 $Q$ 是一个多项式，这一点只要将微分算子 $P_k\left(\dfrac{\partial}{\partial y}\right)$ 作用到恒等式

$$\int_{\boldsymbol{R}^n} \exp[-\pi|x|^2 + 2\pi i x \cdot y] dx = \exp(-\pi|y|^2)$$

的两边便可得到。因此问题在于证明 $Q(y) = P_k(iy)$。注意

$$Q(y) = \int_{\boldsymbol{R}^n} P_k(x) \exp[-\pi\{(x_1 - iy_1)^2$$

$$+(x_2-iy_2)^2+\cdots+(x_n-iy_n)^2\}]dx.$$

而这个积分等于

$$\int_{R^n} P_k(x+iy)\, e^{-\pi|x|^2}dx,$$

这是因为 $P_k(x)\exp\left[-\pi\sum_{j=1}^{n}x_j^2\right]$是解析且是速降的，所以在 $C^n$ 中可作积分围道的移动。基于同样的理由，我们还可以用 $y$ 代替 $iy$，得到

$$Q\left(\frac{y}{i}\right)=\int_{R^n} P_k(x+y)\ e^{-\pi|x|^2}dx.$$

现在，由 $P_k$ 是调和函数可知，它在以 $y$ 为中心任一球面上的平均值等于 $P_k(y)$；而因子 $e^{-\pi|x|^2}$在这样的球面上是常数，且其在 $\boldsymbol{R}^n$ 上的积分又等于 1。于是得到 $Q(y/i)=P_k(y)$，定理证毕。

这定理还蕴含了如下所述的一个自身的推广，其意义在于把 $L^2(\boldsymbol{R}^n)$分解中的各个分量（对不同的 $n$）联系起来。

若 $f$ 是一个径向函数，则我们记为$f=f(r)$，其中 $|x|=r$。

**推论** 设 $P_k(x)$是 $\boldsymbol{R}^n$ 上一个 $k$ 次齐次调和多项式。假定$f$是径向函数且 $P_k(x)f(r)\in L^2(\boldsymbol{R}^n)$，则 $P_k(x)f(r)$的 Fourier 变换也是形如 $P_k(x)g(r)$的函数，其中 $g$ 是一个径向函数。而且由此导出的变换 $f\to g$，$\mathscr{F}_{n,k}(f)=g$，本质上仅依赖于 $n+2k$。更确切地说，我们有 Bochner 关系

(29) $$\mathscr{F}_{n,k}=i^k\mathscr{F}_{n+2k,0}.$$

**证明** 考虑径向函数的 Hilbert 空间

$$\mathscr{R}=\left\{f(r):\ \|f\|^2=\int_0^\infty|f(r)|^2r^{2k+n-1}dr<\infty\right\}$$

赋以括号内所指的模。现在取定 $P_k(x)$，并假定 $P_k$ 是规一化的，即

$$\int_{S^{n-1}}|P_k(x)|^2d\sigma(x)=1.$$

这时，显然在$\mathscr{R}$中的元素$f$与$L^2(\boldsymbol{R}^n)$中的元素$f(|x|)P_k(x)$之间存在一个酉对应，并且$f(|x|)$属于$L^2(\boldsymbol{R}^{n+2k})$。我们必须证明的是：

(30) $\mathscr{F}_{n,k}(f)(r)=i^k\mathscr{F}_{n+2k,0}(f)(r)$，对任意$f\in\mathscr{R}$。

首先，若$f(r)=e^{-\pi r^2}$，则式(30)是上述定理(见式(27))的一个直接推论。其次，考虑$e^{-\pi\delta r^2}$，其中$\delta>0$固定。根据$P_k$的齐次性以及展缩与Fourier变换的相互关系(见§1.2的式(1))，我们相继得到

$$\begin{aligned}\mathscr{F}(P_k(x)\,e^{-\pi\delta|x|^2})&=\delta^{-k/2}\mathscr{F}\,(P_k(\delta^{1/2}x)e^{-\pi\delta|x|^2})\\&=i^k\delta^{-k/2-n/2}P_k(x/\delta^{1/2})e^{-\pi|x|^2/\delta}\\&=i^k\delta^{-k-n/2}P_k(x)e^{-\pi|x|^2/\delta}.\end{aligned}$$

这就说明$\mathscr{F}_{n,k}(e^{-\pi\delta r^2})=i^k\delta^{-k-n/2}\,e^{-\pi r^2/\delta}$，从而对于$f(r)=e^{-\pi\delta r^2}$，$\delta>0$，证明了式(30)。

最后，为了完成整个推论的证明，只需注意，$\{e^{-\pi\delta r^2}\}_{0<\delta<\infty}$的线性组合在$\mathscr{R}$中是稠密的。事实上，假如这结论不真，那末存在一个非零的$g\in\mathscr{R}$，使得

$$\int_0^\infty e^{-\pi\delta r^2}g(r)r^{2k+n-1}dr=0,\ \text{对一切}\ \delta>0.$$

作变量替换$r^2\to r$将其化为Fourier-Laplace变换，并根据一个众所周知的推理就可得到$g\equiv0$。推论证毕。

3.3 现在来实现我们讨论调和函数系的主要目标。

**定理 5** 设$P_k(x)$是$k$次齐次调和多项式，$k\geqslant1$。则带有核$P_k(x)/|x|^{k+n}$的变换(24)的乘子是

$$\gamma_k\frac{P_k(x)}{|x|^k},\ \text{其中}\ \gamma_k=i^k\pi^{n/2}\frac{\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{k+n}{2}\right)}.$$

注意，当$k\geqslant1$时，$P_k(x)$在球面上是与常数正交的(见§3的3.1.1)，因此它在球面上的平均值为零。

定理的结论可以复述为

$$\left(\frac{P_k(x)}{|x|^{k+n}}\right)^\wedge=\gamma_k\frac{P_k(x)}{|x|^k}. \tag{31}$$

这样，它可由下述密切相关的事实导出

$$\left(\frac{P_k(x)}{|x|^{k+n-\alpha}}\right)^\wedge=\gamma_{k,\alpha}\frac{P_k(x)}{|x|^{k+\alpha}}, \tag{32}$$

其中
$$\gamma_{k,\alpha}=i^k\pi^{\frac{n}{2}-\alpha}\frac{\Gamma(k/2+\alpha/2)}{\Gamma(k/2+n/2-\alpha/2)}.$$

**引理** 恒等式（32）在下述意义下成立：对每个$\varphi$有

$$\int_{\mathbf{R}^n}\frac{P_k(x)}{|x|^{k+n-\alpha}}\Phi(x)dx \tag{33}$$

$$=\gamma_{k,\alpha}\int_{\mathbf{R}^n}\frac{P_k(x)}{|x|^{k+\alpha}}\varphi(x)dx,$$

其中$\varphi$及其 Fourier 变换$\Phi$均在$\infty$处速降，而且它对一切整数$k$以及$0<\alpha<n$均有效。

注意，式（32）的左、右两端在$0<\alpha<n$时都是局部可积的，因此式（33）中两端的积分都绝对收敛。

应该指出，上述定理及其所依据的引理实际上都是一般结论（29）的特殊情形；这里只是以"广义函数"代替$L^2$的说法罢了。结论（29）还有其它的推广，但我们不再继续讨论了。

**3.4** 现在回到引理的证明上来。在§3.2中我们已经看到

$$\mathscr{F}(P_k(x)e^{-\pi\delta|x|^2})=i^k\delta^{-k-n/2}P_k(x)e^{-\pi|x|^2/\delta},$$

因此，当$\delta>0$时有

$$\int_{\mathbf{R}^n}P_k(x)e^{-\pi\delta|x|^2}\Phi(x)dx$$

$$=i^k\delta^{-k-n/2}\int_{\mathbf{R}^n}P_k(x)e^{-\pi|x|^2/\delta}\varphi(x)dx.$$

在等式两端各乘以$\delta$的适当幂次（具体地说是$\delta^{\beta-1}$，$\beta=(k$

$+n-\alpha)/2)$，再对 $\delta$ 取积分。

对$\beta>0$利用等式

$$\int_0^\infty e^{-\pi\delta|x|^2}\delta^{\beta-1}d\delta=(\pi|x|^2)^{-\beta}\Gamma(\beta),$$

便知上式左端的积分等于

$$\Gamma\left(\frac{k+n-\alpha}{2}\right)\pi^{-(k+n-\alpha)/2}\int_{R^n}\frac{P_k(x)}{|x|^{k+n-\alpha}}\varphi(x)dx.$$

右端相应的积分是

$$i^k\Gamma\left(\frac{k+\alpha}{2}\right)\pi^{-(k+\alpha)/2}\int_{R^n}\frac{P_k(x)}{|x|^{k+\alpha}}\phi(x)dx.$$

这就推导出了式（33）。注意，由于 $0<\alpha<n$ 以及 $\varphi$，$\Phi$ 的速降（用估计$|\varphi(x)|\leqslant A(1+|x|)^{-n}$以及$|\Phi(x)|\leqslant A(1+|x|)^{-n}$ 就足够了），上述推理中出现的二重积分是绝对收敛的。故上述形式上的推理确实证明了引理。

为了证明定理，我们假设$k\geqslant 1$，并且进一步限制 $\varphi$ 使得$\Phi$ 也是光滑的（只要有$\Phi$ 在原点附近的可微性就足够了）。这时有

$$\lim_{\substack{\alpha\to 0\\ \alpha>0}}\int_{R^n}\frac{P_k(x)}{|x|^{k+n-\alpha}}\Phi(x)dx \tag{34}$$

$$=\lim_{\varepsilon\to 0}\int_{|x|>\varepsilon}\frac{P_k(x)}{|x|^{k+n}}\Phi(x)dx.$$

事实上，根据$P_k$在以原点为中心的任一球面上的积分为零就得到

$$\int_{R^n}\frac{P_k(x)}{|x|^{k+n-\alpha}}\Phi(x)dx$$

$$=\int_{|x|\leqslant 1}\frac{P_k(x)}{|x|^{k+n-\alpha}}[\Phi(x)-\Phi(0)]dx$$

$$+\int_{|x|>1}\frac{P_k(x)}{|x|^{k+n-\alpha}}\Phi(x)dx.$$

令$\alpha\to0$取极限，对于上式右边的第一个积分，我们得到

$$\int_{|x|<1}\frac{P_k(x)}{|x|^{k+n}}[\varphi(x)-\varphi(0)]dx$$

$$=\lim_{\varepsilon\to0}\int_{\varepsilon\leq|x|<1}\frac{P_k(x)}{|x|^{k+n}}\varphi(x)dx,$$

这就证明了式(34)。最后，设$f$是任一具有紧支集的充分光滑的函数，且对固定的$x$记$f(x-y)=\varphi(y)$。由于$(\hat{\varphi})^\wedge(y)=\varphi(-y)$，我们有$\hat{\varphi}(y)=\hat{f}(y)e^{-2\pi ix\cdot y}$。这时，我们的结果变成

(35) $$\lim_{\varepsilon\to0}\int_{|y|\geq\varepsilon}\frac{P_k(y)}{|y|^{k+n}}f(x-y)dy$$

$$=\gamma_k\int_{R^n}\frac{P_k(y)}{|y|^k}\hat{f}(y)e^{-2\pi ix\cdot y}dy.$$

现在，根据乘子$m$的定义，我们又有

$$\lim_{\varepsilon\to0}\int_{|y|\geq\varepsilon}\frac{P_k(y)}{|y|^{k+n}}f(x-y)dy$$

$$=\int_{R^n}m(y)\hat{f}(y)e^{-2\pi ix\cdot y}dy,$$

其中两边的积分是依$L^2$意义收敛的。由于上述$f$在$L^2$中稠密，故得

$$m(y)=\gamma_k\frac{P_k(y)}{|y|^k}.$$

定理证毕。

对于固定的$k$，$k\geqslant1$，令

$$\Omega(y)=\frac{P_k(y)}{|y|^k}$$

并让$P_k$取遍$k$次齐次调和多项式，这时算子(24)的(有限维)线性空间形成Riesz变换的一个自然推广；Riesz变换相当于$k=1$的特殊情形，对于$k>1$的情形，我们称它们为高价Riesz

变换①；它们同样也可用其不变性质来表征（见§4.8）。

3.5 现在我们来考虑两类定义在$L^2(\boldsymbol{R}^n)$上（稍后也可定义于$L^p(\boldsymbol{R}^n)$，$1<p<\infty$）的变换。第一类由所有形如

$$(36)\qquad T(f)=c\cdot f+\lim_{\varepsilon\to 0}\int_{|y|>\varepsilon}\frac{\Omega(y)}{|y|^n}f(x-y)dy$$

的变换所组成，其中$c$是常数；$\Omega$是零次齐次函数，它在$S^{n-1}$上是无穷次可微的，且其在$S^{n-1}$上的平均值为零。第二类是由

$$(37)\qquad (T(f))^{\wedge}(y)=m(y)\hat{f}(y)$$

确定的变换$T$，其中乘子$m$是零次齐次的且在球面上是无穷次可微的。

**定理6** 由式(36)与(37)定义的两类变换是相同的。

首先，假定$T$具有式(36)的形式。根据第二章§4.2中的定理（也可参见本章公式(6)），$T$是形如式(37)的变换，其中$m$是零次齐次函数，并且

$$(38)\qquad m(x)=c+\int_{S^{n-1}}\Gamma(x\cdot y)\Omega(y)d\sigma(y),\qquad |x|=1.$$

现在写出球调和函数展开式

$$(39)\qquad \begin{aligned}\Omega(y)&=\sum_{k=1}^{\infty}Y_k(y), & m(x)&=\sum_{k=0}^{\infty}\tilde{Y}_k(x),\\ \Omega_N(y)&=\sum_{k=1}^{N}Y_k(y), & m_N(x)&=\sum_{k=0}^{N}\tilde{Y}_k(x).\end{aligned}$$

由刚才所证的定理，当$\Omega=\Omega_N$时，有$m(x)=m_N(x)$，其中

$$\tilde{Y}_k(x)=\gamma_kY_k(x),\qquad k\geqslant 1.$$

但是

$$m_M(x)-m_N(x)=\int_{S^{n-1}}\Gamma(x\cdot y)[\Omega_M(y)-\Omega_N(y)]d\sigma(y),$$

从而进一步有，当$M,N\to\infty$时，

① 我们称$k$为高阶Riesz变换的阶数。

$$\sup_{x\in S^{n-1}} |m_M(x)-m_N(x)|$$

$$\leqslant\left(\sup_x\int_{S^{n-1}}|\Gamma(x\cdot y)|^2 d\sigma(y)\right)^{\frac{1}{2}}$$

$$\times\left(\int_{S^{n-1}}|\Omega_M-\Omega_N|^2 d\sigma(y)\right)^{\frac{1}{2}}\to 0,$$

这是因为从

$$\Gamma(t)=\frac{\pi i}{2}\operatorname{sign} t+\log\frac{1}{|t|},$$

可知

$$\sup_x\int_{S^{n-1}}|\Gamma(x\cdot y)|^2 d\sigma(y)=\int_{S^{n-1}}|\Gamma(x\cdot y)|^2 d\sigma(y)$$

$$=c_1+c_2\int_0^\pi |\log|\cos\theta||^2(\sin\theta)^{n-2}d\theta$$

$$<\infty.$$

这就说明

$$m(x)=c+\sum_{k=1}^{\infty}\gamma_k Y_k(x).$$

现在，根据$\Omega$的无穷次可微性，对每一个固定的$N$，有

$$\int_{S^{n-1}}|Y_k(x)|^2 d\sigma(x)=O(k^{-N}),\quad k\to\infty.$$

而由$\gamma_k$的显式我们看到$\gamma_k\approx k^{-n/2}$，故$m(x)$在单位球面上也是无穷次可微的。

反之，设$m(x)$在单位球面上是无穷次可微的，且其球调和展开如式（39）。令

$$c=\tilde{Y}_0\quad 以及\quad Y_k(x)=\frac{1}{\gamma_k}\tilde{Y}_k(x),$$

则由式（39）给出的$\Omega(y)$在球面上的平均是零，且在其上仍是

无穷次可微的。但如前所述，相应于这一变换的乘子是 $m$。定理证毕。

作为这个定理的一个应用以及关于奇异积分变换的最后的一点说明，在这里我们将对 § 1.3 中所述的偏微商估计给出一个推广。

设 $P(x)$ 是 $\boldsymbol{R}^n$ 上的一个 $k$ 次齐次多项式。若 $P(x)$ 只在原点取零值，则我们称 $P$ 是椭圆的。对于任意多项式 $P$，我们还考虑它的相应微分多项式：设

$$P(x)=\sum_{\alpha} a_{\alpha} x^{\alpha},$$

记

$$P\left(\frac{\partial}{\partial x}\right)=\sum_{\alpha} a_{\alpha}\left(\frac{\partial}{\partial x}\right)^{\alpha},$$

其中

$$\left(\frac{\partial}{\partial x}\right)^{\alpha}=\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n},$$

对应于单项式 $x^{\alpha}=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$（它是 $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n$ 次的）。

**推论** 设 $P$ 是椭圆的 $k$ 次齐次多项式。令 $\left(\frac{\partial}{\partial x}\right)^{\alpha}$ 是任一 $k$ 次微分单项式。假定 $f$ 是具有紧支集的 $k$ 次连续可微的函数，则我们有先验估计

$$\left\|\left(\frac{\partial}{\partial x}\right)^{\alpha} f\right\|_p \leqslant A_p\left\|P\left(\frac{\partial}{\partial x}\right) f\right\|_p, \quad 1<p<\infty. \tag{40}$$

为了证明这一结论，我们注意，就像在 § 1.3 那样，$\left(\frac{\partial}{\partial x}\right)^{\alpha} f$ 的 Fourier 变换与 $P\left(\frac{\partial}{\partial x}\right) f$ 的 Fourier 变换之间有下述关系

$$P(y)\left[\left(\frac{\partial}{\partial x}\right)^{\alpha} f\right]^{\wedge}(y)=y^{\alpha}\left(P\left(\frac{\partial}{\partial x}\right) f\right)^{\wedge}(y),$$

因为$P(y)$除在原点外是非零的，所以$y^a/P(y)$是零次齐次的且在单位球面上是无穷次可微的。于是

$$\left(\frac{\partial}{\partial x}\right)^a f = T\left(P\left(\frac{\partial}{\partial x}\right)f\right),$$

其中$T$是形如式（37）的一个变换。由定理6，$T$也是式（36）型的变换，根据第二章的结论，我们得到了估计（40）。这个结果的一个推广将在下一章的§7.9中阐述。

## §4 进一步的结果

**4.1** 我们的目的是要证明，对$L^1(\boldsymbol{R}^n)$的某些结果可以推广到$\boldsymbol{R}^n$上的有限Borel测度，即$\mathscr{B}(\boldsymbol{R}^n)$：

(a) 设$d\mu\in\mathscr{B}(\boldsymbol{R}^n)$以及

$$M(d\mu)(x)=\sup\frac{1}{m(B(x,r))}\int_{B(x,r)}|d\mu|,$$

则

$$m\{x:\ M(d\mu)(x)>\alpha\}\leqslant\frac{A}{\alpha}\int_{\boldsymbol{R}^n}|d\mu|.$$

其论证与可积函数的情形相同。

(b) 若$d\mu$是纯奇异的，则对几乎所有的$x$，有

$$\lim_{r\to 0}\frac{1}{m(B(x,r))}\int_{B(x,r)}d\mu=0.$$

**提示**：记$d\mu=d\mu_1+d\mu_2$，其中$d\mu_1$的支集是闭零测集$F$，而$\|d\mu_2\|\leqslant\delta$。从而对每个$x\overline{\in}F$有

$$\lim_{r\to 0}\frac{1}{m(B(x,r))}\int_{B(x,r)}d\mu_1=0.$$

这种类型的一个较为一般的结果，对于在定理2中出现的任一恒等逼近，都是成立的，特别是对Poisson积分。

(c) 设

$$T_\varepsilon(d\mu)(x) = \int_{|x-y|>\varepsilon} \frac{\Omega(x-y)}{|x-y|^n} d\mu(y),$$

其中 $\Omega$满足第二章定理 3 与定理 4 的条件。则 $\lim\limits_{\varepsilon\to 0} T_\varepsilon(d\mu)(x)$ 几乎处处存在。参阅例如 Zygmund[8]；关于 (c)，见 Calderón 与 Zygmund[1]。

4.2 设 $u(x,y)$在 $R_+^{n+1}$ 上调和。

(a) 若 $1<p\leqslant\infty$，则 $u(x,y)$是一个 $L^p(R^n)$函数的 Poisson 积分的充分必要条件是

$$\sup_{y>0} \|u(x,y)\|_p < \infty.$$

(b) $u(x,y)$是$\mathscr{B}(R^n)$中一个测度的 Poisson 积分的充分必要条件是

$$\sup_{y>0} \|u(x,y)\|_1 < \infty.$$

参阅例如 Stein 与 Weiss[2]，也可参阅第七章§1的1.2.1。

4.3 设 $f\in L^2(R^n), f_j = R_j(f)$，且 $u_j(x,y)$是 $f_j$ 的Poisson 积分，则

$$u_j(x,y) = \int_{R^n} Q_y^{(j)}(t) f(x-t)\,dt,$$

其中

$$Q_y^{(j)}(x) = c_n \frac{x_j}{(|x|^2+y^2)^{(n+1)/2}}.$$

4.4 这个结果以及§2.3定理 3 都可推广到 $L^p(R^n)$，$1<p<\infty$。详见 Horváth[1]；也可参阅第七章 §3.2。$n=1$ 的情形见 Titchmarch[1]，第五章。

4.5 注意到下列易证的事实是有意义的。

(a) 设 $\mathscr{A}$ 是 $L^2(R^n)$上的算子代数，它是由 Riesz 变换 $R_1, R_2, \cdots, R_n$（代数地）生成的，则每个高阶 Riesz 变换皆属于 $\mathscr{A}$。

(b) $\mathscr{A}$ 的闭包（在强算子拓扑意义下）等同于 $L^2(R^n)$上与平移以及展缩可交换的有界变换的代数。

**4.6** 从§4.6到§4.8我们皆假定$n\geqslant3$。($n=1,2$的情形须作少许修改。)令$SO(n)$是$\boldsymbol{R}^n$中真正的旋转群，$SO(n-1)$是使$x_1$轴保持不变的旋转子群。则对每一个$k$，多项式$P_k(x)\in\mathscr{H}_k$的子空间（它由$SO(n-1)$固定，即$P_k(\rho^{-1}x)=P_k(x)$，$\rho\in SO(n-1)$）恰是一维的。见《富里叶分析》的第Ⅳ章。

**4.7** 设$V$是一个有限维Hilbert空间，$\rho\to R_\rho$是一个从$SO(n)$到$V$上酉变换群的连续同态。偶对$(R_\rho,V)$称为$SO(n)$的表示。若没有$V$的非平凡子空间在$R_\rho(\rho\in SO(n))$下不变，则称它是不可约的。对于两种表示$(R_\rho^{(1)},V_1)$与$(R_\rho^{(2)},V_2)$，若存在一个酉对应$U$，$U$: $V_1\longleftrightarrow V_2$，使得$U^{-1}R_\rho^{(2)}U=R_\rho^{(1)}$。则称此两种表示是等价的。

(a) 令$V=\mathscr{H}_k$（$k$次齐次调和多项式的线性空间）。定义

$$(R_\rho P(x))=P(\rho^{-1}x),\quad \rho\in SO(n),\quad P\in\mathscr{H}_k.$$

这一表示是不可约的。

(b) $SO(n)$的一个不可约表示（$R_\rho,V$）与我们上述从球调和函数系所得的表示等价，当且仅当存在一个非零$v\in V$，使得

$$R_\rho(v)=v,\quad \text{一切}\ \rho\in SO(n-1).$$

关于旋转群表示的一般理论见Weyl[1]，Boerner[1]。应用Frobenius关于紧群的互反定理(见Weil[1])，§4.7可由§4.6导出。

**4.8** 令$(R_\rho,V)$是$SO(n)$的不可约表示，如§4.7。设$f\to T(f)$是一个从$L^2(\boldsymbol{R}^n)$到$L^2(\boldsymbol{R}^n,V)$的有界线性变换；$T$将复值函数变换到取值于$V$的函数。

(a) 若$T$与平移、展缩，以及变换（按$(R_\rho,V)$），依下述意义可交换：

$$\rho T\rho^{-1}(f)=R_\rho T(f).$$

则有$T\equiv0$，除非（$R_\rho,V$）等价于从球调和函数系所得出的表示（如§4.7(a)）。

(b) 若$(R_\rho,V)$是由$k$次球调和函数系产生的，则$T$除一

常数因子外是确定的。特别地，若$\beta_1, \beta_2, \cdots, \beta_N$是$V$上线性泛函的一个基，则在$k \geqslant 1$时每个$\beta_j T(f)$是$k$阶 Riesz 变换（当$k=0$时，$T$是恒等变换的常数倍）。

4.9 下述事实在后文中是有用的：假设$u(x, y)$是$f \in L^p(\boldsymbol{R}^n)$的 Poisson 积分，则

$$\sup_{y>0}\left|y \frac{\partial u(x, y)}{\partial x_j}\right| \leqslant A M(f)(x).$$

**提示**：应用定理 2 于

$$\varphi(x)=\frac{\partial}{\partial x_j}\left[P_1(x)\right].$$

这类结果的更一般情形是

$$\sup_{y>0}\left|y^{|\alpha|} D u(x, y)\right| \leqslant A_\alpha M(f)(x),$$

其中$D$是任意一个总次数为$|\alpha|$的$x$与$y$的微分单项式。

## 注释

**节1，节2** 奇异积分（“Riesz 变换型”的）与类似于§1.3中的估计之间的联系已有很长的历史了，例如，对$p=2$参见 Friedrichs[1]；对一般$p$，见 Calderón 与 Zygmund[1]。认识到 Riesz 变换与共轭调和函数系（§2.3）的联系要回溯到 Horváth[1]。关于$n$维 Poisson 积分的 Fourier 分析请看 Bochner[1]，[2]，以及 Bochner 与 Chandrasekharan[1]。它与极大函数的关系，推广了 Hardy 与 Littlewood 的古典结果，见 K. T. Smith[1]。§2.4中的论证来自 Calderón与Zygmund[1]。

**节 3** 主要结果是定理 4 及其推论。前者隐含于 Hecke[1] 的工作中，并由 Bochner 将它明显地表达出来，他还导出了推论，见 Bochner[2]；还有 Calderón 与 Zygmund[5]，以及 Calderón[3]。这些课题的某些精细的研究，特别是球调和函数系以及与 Bessel 函数系的联系，可在《富里叶分析》的第 Ⅳ 章

中找到详尽的叙述。我们必须记住，这里所定义的 Fourier 变换相应于《富里叶分析》一书的 Fourier 逆变换。

奇异积分借助于它们的符号进行运算的原始思想是在 § 3.5 中给出的，虽然后一概念并未在那里明显定义。进一步的详情参见 Mihlin [1]，Calderón 与 Zygmund [5]。以后包括偏微分算子的发展见于 Calderón [3]，[5]，Seeley [1]，Kohn 与 Nirenberg [1]，Unterberger 与 Bokobza [1]，以及 Hörmander[2]。这一课题的历史概况可在 Seeley[2]中找到。希望读者能查阅一些早期作者的工作，特别是 Girand 在这方面的贡献。

# 第四章 Littlewood-Paley理论与乘子

一维Fourier级数的Littlewood-Paley理论及其应用是该学科意义最重大的进展之一。这个理论最初沿着三个主要方向进行，每个方向本身都是很有意义的：

(a) 辅助的 $g$ 函数。除去应用以外，它还说明了这样一个原则，就是刻划各种分析的结果（例如$L^p$模的有限性，极限几乎处处存在，等等），通常最有效的方法是用适当的二次式。

(b) 通过Fourier分析进行的函数的"二进"分解。

(c) Marcinkiewicz乘子定理，它给出关于$L^p$乘子的非常有用的充分条件。

这个理论主要是在1930年至1939年间由Littlewood，Paley，Zygmund和Marcinkiewicz等人发展起来的，但它依赖于复变函数论，因此它的完全展开被局限在一维的情形。$n$ 维理论是最新的并且部分地受到在第一章与第二章中介绍的实变技巧的刺激。

然而，我们并不想让读者停留在上述极其简单的图象上。我们早就知道一些重要的 $n$ 维结果可以从一维理论推出。不过 $n$ 维理论与一维情形比较起来只是部分地取得了成功，总的说来还有许多工作尚待去做。（最后这一点在§6.2中会再次谈到。）

直到现在已有好几条可能的途径来获得我们在这里要叙述的主要结果，但我们故意不选取最短和最直接的途径；我们希望将要遵循的这条较长的路线会更有启发性。这样读者将有更好的机会来考察下面详述的复杂技巧的全部作用。

## §1 Littlewood-Paley的 $g$ 函数

1.1 $g$ 函数是一个（非线性）算子；它使人们能够根据$R^n$

上一个函数的Poisson积分的性质来给出它的$L^p$模的有用的特征。这个特征不仅在这一章用到，而且在讨论函数空间的下一章中也用到。$g$函数的定义如下。设$f \in L^p(R^n)$，并用$u(x,y)$记它的Poisson积分，这就像第三章§2所定义的，

$$u(x,y) = \int_{R^n} P_y(t) f(x-t)\, dt.$$

用$\Delta$表示$R_+^{n+1}$中的Laplace算子，即

$$\Delta = \frac{\partial^2}{\partial y^2} + \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2},$$

$\nabla$是相应的梯度，

$$|\nabla u(x,y)|^2 = \left|\frac{\partial u}{\partial y}\right|^2 + |\nabla_x u(x,y)|^2,$$

这里
$$|\nabla_x u(x,y)|^2 = \sum_{j=1}^{n} \left|\frac{\partial u}{\partial x_j}\right|^2.$$

用这些记号我们定义$g(f)(x)$为

(1) $$g(f)(x) = \left(\int_0^\infty |\nabla u(x,y)|^2 y\, dy\right)^{1/2}.$$

关于$g$的基本结果是下面的定理。

**定理1** 设$f \in L^p(R^n)$，$1<p<\infty$，则$g(f)(x) \in L^p(R^n)$，且

(2) $$A'_p \|f\|_p \leqslant \|g(f)\|_p \leqslant A_p \|f\|_p.$$

**1.2** 最好是从简单的情形$p=2$开始。对于$f \in L^2(R^n)$，我们有

$$\|g(f)\|_2^2 = \int_{y=0}^{\infty} \int_{R^n} y\, |\nabla u(x,y)|^2\, dx\, dy.$$

当我们先对$x$积分时，这个两重积分可以用Green定理（如以后在§2.1将看到的那样）或用Plancherel公式来处理。事实上，由于恒等式

$$u(x,y)=\int_{\mathbf{R}^n}\hat{f}(t)e^{-2\pi i t\cdot x}e^{-2\pi|t|y}dt,$$

我们有

$$\frac{\partial u}{\partial y}=\int_{\mathbf{R}^n}-2\pi|t|\hat{f}(t)e^{-2\pi i t\cdot x}e^{-2\pi|t|y}dt,$$

以及

$$\frac{\partial u}{\partial x_j}=\int_{\mathbf{R}^n}-2\pi i t_j\hat{f}(t)e^{-2\pi i t\cdot x}e^{-2\pi|t|y}dt.$$

这样

$$\int_{\mathbf{R}^n}|\nabla u(x,y)|^2dx=\int_{\mathbf{R}^n}8\pi^2|t|^2|\hat{f}(t)|^2e^{-4\pi|t|y}dt,\quad y>0,$$

因此

$$\|g(f)\|_2^2=\int_{\mathbf{R}^n}|\hat{f}(t)|^2\left\{8\pi^2|t|^2\int_0^\infty e^{-4\pi|t|y}ydy\right\}dt$$

$$=\frac{1}{2}\|\hat{f}\|_2^2.$$

故

(3) $$\|g(f)\|_2=2^{-1/2}\|f\|_2.$$

在这里引进以下两个“部分” $g$ 函数可能是合适的，其中一个与对 $y$ 的微商有关而另一个与对 $x$ 的微商有关，

(4)
$$g_1(f)(x)=\left(\int_0^\infty\left|\frac{\partial u(x,y)}{\partial y}\right|^2ydy\right)^{1/2},$$
$$g_x(f)(x)=\left(\int_0^\infty|\nabla_x u(x,y)|^2ydy\right)^{1/2}.$$

注意$g^2=g_1^2+g_x^2$，以及更有意思的是，式(3)的证明也表明了

$$\|g_1(f)\|_2=\|g_x(f)\|_2=\frac{1}{2}\|f\|_2.$$

用$g_1$或$g_x$代替 $g$，也可以建立整个理论，同时这三个函数还由Riesz变换紧密地联系起来(见 §7.1)。

1.3 当$p\neq 2$时，$L^p$不等式将作为第二章§5中取值于Hilbert空间的函数的奇异积分理论的推论而得到。我们确定现在要考虑的Hilbert空间$\mathscr{H}_1$与$\mathscr{H}_2$。$\mathscr{H}_1$是由复数组成的一维Hilbert空间。为了定义$\mathscr{H}_2$，先定义$\mathscr{H}_2^0$为$(0,\infty)$上带有测度$ydy$的$L^2$空间，即

$$\mathscr{H}_2^0=\left\{f: |f|^2=\int_0^\infty |f(y)|^2 y dy<\infty\right\}.$$

令$\mathscr{H}_2$是$n+1$个$\mathscr{H}_2^0$的直和；这样$\mathscr{H}_2$的元可以表示成有$n+1$个分量的向量，其分量属于$\mathscr{H}_2^0$。因为$\mathscr{H}_1$与复数域相同，所以$B(\mathscr{H}_1,\mathscr{H}_2)$自然可以认为与$\mathscr{H}_2$恒同。现在令$\varepsilon>0$，并暂时让它固定。

定义

$$K_\varepsilon(x)=\left(\frac{\partial P_{y+\varepsilon}(x)}{\partial y},\frac{\partial P_{y+\varepsilon}(x)}{\partial x_1},\cdots,\frac{\partial P_{y+\varepsilon}(x)}{\partial x_n}\right).$$

注意到对每个固定的$x$，$K_\varepsilon(x)\in\mathscr{H}_2$。这等价于

$$\int_0^\infty\left|\frac{\partial P_{y+\varepsilon}(x)}{\partial y}\right|^2 y dy<\infty$$

及

$$\left|\int_0^\infty \frac{\partial P_{y+\varepsilon}(x)}{\partial x_j}\right|^2 y dy<\infty,\quad j=1,\cdots,n.$$

然而由Poisson核的表达式（见第74页）容易看到$\dfrac{\partial P_y}{\partial y}$与$\dfrac{\partial P_y}{\partial x}$都囿于

$$\frac{A}{(|x|^2+y^2)^{(n+1)/2}}.$$

因此①

$$|K_\varepsilon(x)|^2\leqslant A^2(n+1)\int_0^\infty\frac{ydy}{(|x|^2+(y+\varepsilon)^2)^{n+1}}\leqslant A_\varepsilon,$$

并且$|K_\varepsilon(x)|^2\leqslant A|x|^{-2n}$。于是

---

① 注意，在这里对于每个$x$，符号$|K_\varepsilon(x)|$表示$K_\varepsilon(x)$在$\mathscr{H}_2$中的模。

(5) $$|K_\varepsilon(x)|\in L^2(R^n).$$

类似地

$$\left|\frac{\partial K_\varepsilon(x)}{\partial x_j}\right|^2\leqslant A\int_0^\infty\frac{ydy}{(|x|^2+(y+\varepsilon)^2)^{n+2}}$$

$$\leqslant A\int_0^\infty\frac{ydy}{(|x|^2+y^2)^{n+2}}=A'|x|^{-2n-2}.$$

因此

(6) $$\left|\frac{\partial K_\varepsilon(x)}{\partial x_j}\right|\leqslant\frac{A}{|x|^{n+1}},$$

其中 $A$ 与$\varepsilon$无关。

现在考虑由

$$T_\varepsilon(f)(x)=\int_{R^n}K_\varepsilon(t)f(x-t)dt$$

定义的算子$T_\varepsilon$。函数 $f$ 是复值的(取值于$\mathscr{H}_1$中)，而$T_\varepsilon(f)(x)$取值于$\mathscr{H}_2$中。注意到

(7) $$|T_\varepsilon(f)(x)|=\left(\int_0^\infty|\nabla u(x,y+\varepsilon)|^2ydy\right)^{1/2}\leqslant g(f)(x).$$

因此，若 $f\in L^2(R^n)$，由式(3)得 $\|T_\varepsilon(f)(x)\|_2\leqslant 2^{-1/2}\|f\|_2$；故

(8) $$|\hat{K}_\varepsilon(x)|\leqslant 2^{-1/2}$$

(此式也可以直接验证)。

由于式(5)，(6)及(8)，并由第二章的定理 5；我们可以应用第二章定理 1 的Hilbert空间形式。其结论是

$$\|T_\varepsilon(f)\|_p\leqslant A_p\|f\|_p,\quad 1<p<\infty,$$

其中$A_p$与$\varepsilon$无关。由式(7)，对每个 $x$，当 $\varepsilon\to0$ 时$|T_\varepsilon(f)(x)|$增大到$g(f)(x)$，于是最后我们得到

(9) $$\|g(f)\|_p\leqslant A_p\|f\|_p,\quad 1<p<\infty.$$

1.4 我们很想直接从式(9)以及第二章 § 5.3 中 的推论导出

反向不等式

$$(10)\qquad A'_p\|f\|_p\leqslant\|g(f)\|_p,\quad 1<p<\infty.$$

但这需要一个新的论证，因为对应于 $g$ 的算子是作为极限得到的，而这个极限并非由截断核产生的主值(见式(7))；事实上，这里所用的极限过程更为自然，因为在这里主值实际上是不相关的。尽管如此，问题还是能够立刻解决而没有任何困难。取$g_1$代替 $g$。这样对于$f\in L^2(R^n)$，等式

$$\|g_1(f)\|_2=\frac{1}{2}\|f\|_2$$

通过极化便导出恒等式

$$4\int_{R^n}\int_0^\infty y\frac{\partial u_1(x,y)}{\partial y}\overline{\frac{\partial u_2(x,y)}{\partial y}}dydx$$

$$=\int_{R^n}f_1(x)\overline{f_2(x)}dx,$$

这里$f_1,f_2\in L^2(R^n)$，而$u_j$是 $f_j$ 的Poisson积分，$j=1,2$。这个恒等式又导出不等式

$$\frac{1}{4}\left|\int_{R^n}f_1(x)\overline{f_2(x)}\,dx\right|\leqslant\int_{R^n}g_1(f_1)(x)g_1(f_2)(x)dx.$$

现在再假定$f_1\in L^p(R^n)$及$f_2\in L^q(R^n)$，其中$\|f_2\|_q\leqslant 1$，且

$$\frac{1}{p}+\frac{1}{q}=1.$$

则由Hölder不等式以及结果(9)，得到

$$(11)\qquad \left|\int_{R^n}f_1(x)\overline{f_2(x)}\,dx\right|\leqslant 4\|g_1(f_1)\|_p\|g_1(f_2)\|_q$$

$$\leqslant 4A_q\|g_1(f_1)\|_p.$$

现在对属于 $L^2\cap L^q$ 并满足$\|f_2\|_q\leqslant 1$ 的 $f_2$，在(11)中取上确界，就得到所要的结果(10)，其中

$$A'_p=\frac{1}{4A_q},$$

但这里 $f$ 限制在 $L^2\cap L^p$ 中。通过一个简单的极限过程可以过渡到一般情形。令 $f_m$ 是 $L^2\cap L^p$ 中的一列函数，它们按 $L^p$ 模收敛到 $f$ ($L^p$ 的一个一般元素)。注意到

$$|g(f_m)(x)-g(f_n)(x)|\leqslant g(f_n-f_m)(x).$$

故 $g(f_n)$ 按 $L^p$ 模收敛到 $g(f)$，并且我们得到关于 $f$ 的不等式(10)，这是从 $f_n$ 的相应不等式推出的。顺便还证明了以下结果，我们把它叙述成一个推论。

**推论** 假设 $f\in L^2(\boldsymbol{R}^n)$，及 $g_1(f)(x)\in L^p(\boldsymbol{R}^n)$，$1<p<\infty$。则 $f\in L^p(\boldsymbol{R}^n)$，且 $A'_p\|f\|_p\leqslant\|g_1(f)\|_p$。

1.5 应当指出上述结果有若干十分简单的变形：

(a) 用 $g_x(f)$ 代替 $g(f)$ 后该结果仍成立。正向不等式 $\|g_x(f)\|_p\leqslant A_p\|f\|_p$ 自然是关于 $g$ 的不等式的推论。反向不等式可用关于 $g_1$ 的同样的方法证得。

(b) 对于任意整数 $k$，$k>1$，定义

$$g_k(f)(x)=\left(\int_0^\infty\left|\frac{\partial^k u(x,y)}{\partial y^k}\right|^2 y^{2k-1}dy\right)^{1/2}.$$

则 $L^p$ 不等式对于 $g_k$ 也成立。(a)与(b)在下面的§7.2中将作更系统的叙述。

(c) 为了今后的目的，注意以下结论是有益的：对于每个 $x$，有 $g_k(f)(x)\geqslant A_k g_1(f)(x)$，这里界 $A_k$ 只依赖于 $k$。

事实上，由Poisson积分公式容易验证，若 $f\in L^p(\boldsymbol{R}^n)$，$1\leqslant p\leqslant\infty$，则对每个 $x$，

$$\frac{\partial^k u(x,y)}{\partial y^k}\to 0,\quad y\to\infty.$$

因此

$$\frac{\partial^k u(x,y)}{\partial y^k}=-\int_y^\infty\frac{\partial^{k+1}u(x,s)}{\partial s^{k+1}}s^k\frac{ds}{s^k}.$$

用Schwarz不等式

$$\left|\frac{\partial^k u(x,y)}{\partial y^k}\right|^2 \leqslant \left(\int_y^\infty \left|\frac{\partial^{k+1} u(x,s)}{\partial s^{k+1}}\right|^2 s^{2k} ds\right)\left(\int_y^\infty s^{-2k} ds\right)$$

便得到

$$(g_k(f)(x))^2 \leqslant \frac{1}{2k-1}(g_{k+1}(f)(x))^2,$$

对 $k$ 用归纳法便证得所要求的结论。

## §2 函 数 $g_1^*$

2.1 $g$ 函数的 $L^p$ 不等式的上述证明，实质上并不依赖于调和函数论，尽管该函数是用 Poisson 积分定义的。实际上真正用到的全部事实就是Poisson 核是合适的恒等逼近。(见上节末尾的注，也见 §7.2.)

然而还有另外一种方法，它不要用奇异积分理论，但却紧密依赖于调和函数的特征性质。因为它的思想可以用于第二章的方法失效的其它地方，所以我们在这里介绍它（更确切地说，我们只介绍不等式(9)在$1<p\leqslant 2$时的那部分）。整个考虑基于以下的三条引理。

**引理1** 假设 $u$ 调和并且是严格正的。则

(12) $$\Delta(u)^p = p(p-1)u^{p-2}|\nabla u|^2.$$

**引理2** 假设 $F(x,y)$ 在$\overline{R}_+^{n+1}$中连续，在$R_+^{n+1}$ 中属于 $C^2$，并且在无穷远处适当小。则

(13) $$\int_{R_+^{n+1}} y\Delta F(x,y)\,dxdy = \int_{R^n} F(x,0)\,dx.$$

**引理3** 若$u(x,y)$是 $f$ 的Poisson积分，则

(14) $$\sup_{y>0}|u(x,y)| \leqslant M(f)(x).$$

引理 1 的证明是关于微分法的简单练习。它的主要意义是式(12)的右边不包含 $u$ 的任何二阶微商。

为了证明引理 2，我们用Green定理

$$\int_D (u\Delta v - v\Delta u)dxdy = \int_{\partial D}\left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right)d\sigma,$$

这里$D = B_r \cap R_+^{n+1}$，其中 $B_r$ 是 $R^{n+1}$ 上中心在原点，半径为 $r$ 的球。我们取$v = F$，而$u = y$。假如有

$$\int_D y\Delta F(x,y)dxdy \to \int_{R_+^{n+1}} y\Delta F(x,y)dxdy$$

及

$$\int_{\partial D_0}\left(y\frac{\partial F}{\partial \nu} - \frac{\partial y}{\partial \nu}F\right)d\sigma \to 0, \quad r \to \infty,$$

我们便得到结果(13)。这里$\partial D_0$是$D$的边界中球面部分。例如，若$\Delta F \geqslant 0$，且对某个$\varepsilon > 0$，$|F| \leqslant O((|x| + y)^{-n-\varepsilon})$以及$|\nabla F| = O((|x| + y)^{-n-1-\varepsilon})$，当$|x| + y \to \infty$，这当然满足上述要求。

第三个引理是读者已经熟悉了的控制式(见第三章 § 2.1中的定理的(a)的部分)。

在建立这些事实以后，不等式$\|g(f)\|_p \leqslant A_p\|f\|_p (1 < p \leqslant 2)$的证明很快就可以完成了。

首先假定$f \geqslant 0$，无限次可微且具有紧支集。对 Poisson 核的研究说明 $f$ 的 Poisson 积分 $u$ 在 $R_+^{n+1}$ 中是严格正的，并且控制式$u(x,y) = O((|x| + y)^{-n})$ 及$|\nabla u| = O((|x| + y)^{-n-1})$，当$|x| + y \to \infty$时成立。先用引理 1，然后用引理 3 与假设 $1 < p \leqslant 2$，我们有

$$g(f)(x)^2 = \int_0^\infty y|\nabla u(x,y)|^2 dy$$

$$= \frac{1}{p(p-1)}\int_0^\infty yu^{2-p}\Delta(u)^p dy$$

$$\leqslant \frac{1}{p(p-1)}[M(f)(x)]^{2-p}\int_0^\infty y\Delta(u)^p dy.$$

我们可以把此式写成

(15) $$g(f)(x) \leqslant C_p(M(f)(x))^{(2-p)/2}(I(x))^{1/2},$$

其中 $$I(x)=\int_0^\infty y\Delta(u)^p dy.$$

然而，由引理 2，

$$\int_{R^n} I(x)dx=\int_{R_+^{n+1}} y\Delta(u)^p dxdy \tag{16}$$

$$=\int_{R^n} u^p(x,0)dx=\|f\|_p^p.$$

对于 $p=2$，这就立刻给出所要的结果。现在假设 $1<p<2$。由式(15)

$$\int_{R^n}(g(f)(x))^p dx\leqslant C_p^p\int_{R^n}(M(f)(x))^{p(2-p)/2}(I(x))^{p/2}dx$$

$$\leqslant C_p^p\left(\int_{R^n}((M(f)(x))^p dx\right)^{1/r'}$$

$$\times\left(\int_{R^n} I(x)dx\right)^{1/r},$$

这里对指数 $r$ 与$r'$ 用了Hölder不等式，

$$\frac{1}{r}+\frac{1}{r'}=1\quad(1<r<2),$$

这是可以的，因为取$r=2/p$，就有

$$\left(\frac{2-p}{2}\right)pr'=p\quad 及\quad \frac{rp}{2}=1.$$

由式(16)，上述不等式最后的因子是$\|f\|_p^{p/r}$；根据第一章的极大定理，倒数第二个因子被 $C'_p\|f\|_p^{p/r'}$ 所控制。把这两个估计代进去，对于任何无限次可微且具有紧支集的正函数 $f$，便得到

$$\|g(f)\|_p\leqslant A_p\|f\|_p,\quad 1<p\leqslant 2.$$

对于一般的$f\in L^p(R^n)$(为了简单起见假设 $f$ 是实值的)，把它分解成正部和负部，$f=f^+-f^-$；这时只需要对 $f^+$与$f^-$ 中的每一个用一列无限次可微且具有紧支集的正函数按模来逼近。我们把证明的细节省略了。

2.2　不幸的是刚才所给的简洁的证明对于 $p>2$ 不成立。然

而，存在着这同一思想的更复杂的变形，它对于$p>2$的情形有效，但在这里我们就不转述它了[①]。

不过，我们将用上述想法来得到$g$函数不等式的一个重要推广。我们来研究正函数$g_\lambda^*$的不等式，其中$g_\lambda^*$定义如下

(17) $$(g_\lambda^*(f)(x))^2=\int_0^\infty\int_{t\in\boldsymbol{R}^n}\left(\frac{y}{|t|+y}\right)^{\lambda n}\times|\nabla u(x-t,y)|^2y^{1-n}dtdy.$$

2.3 在进一步进行之前，我们作一些说明，这将有助于理解复杂的表达式(17)的含意。

首先，将证明$g_\lambda^*(f)(x)$逐点控制了$g(f)(x)$。为了更好地了解这点，还要引进另一个量，粗略地说它在$g$与$g_\lambda^*$的中间。它的定义如下。令$\Gamma$是$\boldsymbol{R}_+^{n+1}$中顶点在原点并包含$(0,1)$在它内部的固定锥体。$\Gamma$的确切形式无关紧要，但为了确定起见，让我们选定$\Gamma$为正圆锥：

$$\Gamma=\{(t,y)\in\boldsymbol{R}_+^{n+1}:\ |t|<y,\ y>0\}.$$

对于任何$x\in\boldsymbol{R}^n$，令$\Gamma(x)$是$\Gamma$经平移后使得它的顶点在$x$处的锥体。现在定义正函数$S(f)(x)$为

(18) $$[S(f)(x)]^2=\int_{\Gamma(x)}|\nabla u(t,y)|^2y^{1-n}dydt=\int_\Gamma|\nabla u(x-t,y)|^2y^{1-n}dydt.$$

我们即将证明

(19) $$g(f)(x)\leqslant CS(f)(x)\leqslant C_\lambda g_\lambda^*(f)(x).$$

对联系这三个量的不等式我们能作什么样的解释呢？考虑调和函数的三种相应的向边界的趋向会得到启发。

(a) 当$u(x,y)$是$f(x)$的Poisson积分时，通过令$y\to0$($x$固定)，得到向边界点$x\in\boldsymbol{R}^n$的最简单的趋向。这是垂直趋向。我

① 见本章末所引的文献，也见第七章§3的3.3.2中的论证。

们知道，对于这种趋向，适当的极限几乎处处存在。

(b) 较广的可能是允许可变点$(t, y)$通过任意锥体$\Gamma(x)$（其顶点为$x$）趋向$(x, 0)$。这是非切向趋向，它在以后对我们是十分重要的（在第七章与第八章）。读者可能已认识到，$S$函数与$g$函数的关系在某种意义上类似于非切向趋向与垂直趋向之间的关系；应当说明，$S$函数本身有着明确的意义，但我们现在不多说了①。

(c) 最后，最广的可能是允许可变点$(t, y)$以任意的方式趋向$(x, 0)$，即没有限制的趋向。函数$g_\lambda^*$有着类似的作用：它考虑Poisson积分的没有限制的趋向。

注意到$g_\lambda^*(x)$依赖于$\lambda$。对于每个$x$，$\lambda$越小$g_\lambda^*(x)$越大，而且这个性质使得$g_\lambda^*$的$L^p$有界性精密地依赖于$p$与$\lambda$的正确关系。上述最后这一点也许就是对$g_\lambda^*$的主要关心之处，并且是造成对它的研究比对$g$（或$S$）更为困难的原因。

在做了这些有启发性而不精确的说明之后，让我们回到实地。现在对我们来说唯一的事情是证明论断(19)。因为积分(17)不小于只在$\Gamma$上取的积分，而在$\Gamma$上

$$\left(\frac{y}{|t|+y}\right)^{\lambda n} \geqslant \left(\frac{1}{2}\right)^{\lambda n},$$

所以不等式$CS(f)(x) \leqslant C_\lambda g_\lambda^*(f)(x)$是显然的。论断的不平凡部分是：

$$g(f)(x) \leqslant CS(f)(x).$$

只要对$x=0$证明这个不等式就够了。让我们用$B_y$表示在$R_+^{n+1}$上中心为$(0, y)$并且与锥体$\Gamma$的边界相切的球，于是$B_y$的半径与$y$成比例。另外偏微商$\frac{\partial u}{\partial y}$与$\frac{\partial u}{\partial x_k}$像$u$一样是调和函数。这样由平

① 见第七章。

均值定理

$$\frac{\partial u(0,y)}{\partial y}=\frac{1}{m(B_y)}\int_{B_y}\frac{\partial u(x,s)}{\partial y}dxds$$

（这里$m(B_y)$是$B_y$的$n+1$维测度，即$m(B_y)=cy^{n+1}$，其中$c$是适当的常数）。由Schwarz不等式

$$\left|\frac{\partial u(0,y)}{\partial y}\right|^2\leqslant\frac{1}{m(B_y)}\int_{B_y}\left|\frac{\partial u(x,s)}{\partial y}\right|^2dxds.$$

积分这个不等式便得到

$$\int_0^\infty y\left|\frac{\partial u(0,y)}{\partial y}\right|^2dy\leqslant\int_0^\infty c^{-2}y^{-n}\left(\int_{B_y}\left|\frac{\partial u(x,s)}{\partial y}\right|^2dxds\right)dy.$$

然而$(x,s)\in B_y$显然意味着$c_1s\leqslant y\leqslant c_2s$，其中$c_1$与$c_2$是两个正常数。于是除去一个倍数外，上述积分被

$$\int_\Gamma\left(\int_{c_1s}^{c_2s}y^{-n}dy\right)\left|\frac{\partial u(x,s)}{\partial y}\right|^2dxds$$

所控制。换句话说，这就是

$$\int_0^\infty y\left|\frac{\partial u(0,y)}{\partial y}\right|^2dy\leqslant c'\int_\Gamma\left|\frac{\partial u(x,y)}{\partial y}\right|^2y^{1-n}dxdy.$$

对于微商

$$\frac{\partial u}{\partial x_j},\qquad j=1,\cdots,n,$$

同样的结论也成立，把相应的估计式加起来就证明了我们的论断。

2.4 现在我们可以叙述关于$g_\lambda^*$的结果了。

**定理2.** 设$\lambda$是大于1的参数。假设$f\in L^p(\boldsymbol{R}^n)$。则

(a) 对每个$x\in\boldsymbol{R}^n$，$g(f)(x)\leqslant C_\lambda g_\lambda^*(f)(x)$；

(b) 若$1<p<\infty$，及$p>2/\lambda$，则

(20) $$\|g_\lambda^*(f)\|_p\leqslant A_{p,\lambda}\|f\|_p.$$

定理的(a)部分已经证明过了。

当 $p\geqslant 2$ 时不等式是 $g$ 函数的相应不等式的相当容易的推论。现在我们来证明这一点。对于$p\geqslant 2$的情形，仅假定 $\lambda>1$ 是适当的。

设$\psi$是$\boldsymbol{R}^n$上的一个正函数；我们断言

$$(21)\qquad \int_{\boldsymbol{R}^n}(g_\lambda^*(f)(x))^2\psi(x)dx \leqslant A_\lambda\int_{\boldsymbol{R}^n}(g(f)(x))^2M(\psi)(x)dx.$$

式（21）的左边等于

$$\int_{y=0}^{\infty}\int_{t\in\boldsymbol{R}^n}y|\nabla u(t,y)|^2\left[\int_{x\in\boldsymbol{R}^n}\psi(x)[|t-x|+y]^{-\lambda n}\times y^{\lambda n}y^{-n}dx\right]dtdy,$$

因此为了得到式(21)，必须证明

$$(22)\qquad \sup_{y>0}\int_{\boldsymbol{R}^n}\psi(x)[|t-x|+y]^{-\lambda n}y^{\lambda n}y^{-n}dx \leqslant A_\lambda M(\psi)(t).$$

然而由第三章§2.2中的定理2，我们知道，对于适当的$\varphi$，有

$$\sup_{\varepsilon>0}(\psi*\varphi_\varepsilon)(t)\leqslant AM(\psi)(t),$$

其中 $\varphi_\varepsilon(x)=\varepsilon^{-n}\varphi(x/\varepsilon)$。在这里我们实际上有 $\varphi(x)=(1+|x|)^{-\lambda n}$，$\varepsilon=y$，当 $\lambda>1$ 时定理的条件是满足的。这就得到了式（22），从而证明了式（21）。

当$p=2$时，只要把函数$\psi=1$代入不等式（21）中，并利用关于 $g$ 的$L^2$结果便立即推得定理2的结论。现在假定$2<p$；我们设

$$\frac{1}{q}+\frac{2}{p}=1,$$

并在式（21)的左边对于所有满足$\psi\in L^q(\boldsymbol{R}^n)$且$\|\psi\|_q\leqslant 1$的$\psi\geqslant 0$取上确界。于是式（21）的左边给出$\|g_\lambda^*(f)\|_p^2$；而Hölder不等式给出该式右边的一个估计：

$$A_\lambda\|g(f)\|_p^2\|M(\psi)\|_q.$$

由 $g$ 函数的不等式$\|g(f)\|_p\leqslant A_p'\|f\|_p$；注意当 $p<\infty$时，有

$q>1$，由极大函数的定理有

$$\|M(\psi)\|_q \leqslant A_q'' \|\psi\|_q \leqslant A_q''.$$

把这些代入上面的估计中，便得到结果：

$$\|g_\lambda^*(f)\|_p \leqslant A_{p,\lambda} \|f\|_p, \quad 2 \leqslant p < \infty, \ \lambda > 1.$$

2.5 采用§2.1中对于 $g$ 的推理就可以证得 $p<2$ 时的不等式。在本节中引理 1 与引理 2 可以同样地应用，但是为了控制 Poisson 积分向边界的沒有限制的趋向，我们需要引理3的一个更一般的形式。

下面将首次出现一些紧密依赖于 $L^p$ 类的结果。问题与极大函数的一个变形有关，其定义如下。设 $\mu \geqslant 1$，并记 $M_\mu(f)(x)$ 为

$$(23) \quad M_\mu(f)(x) = \left(\sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|^\mu dy\right)^{1/\mu}.$$

则 $M_1(f)(x) = M(f)(x)$，以及 $M_\mu(f)(x) = ((M|f|^\mu)(x))^{1/\mu}$。于是由极大函数的定理立即推得

$$(23') \quad \|M_\mu(f)\|_p \leqslant A_{p,\mu} \|f\|_p, \qquad \text{对于 } p>\mu.$$

如同在 $\mu=1$的特殊情形那样，这个不等式对于 $p \leqslant \mu$ 不成立。

2.5.1 代替引理3的引理如下。

**引理 4** 设 $f \in L^p(R^n)$，$p \geqslant \mu$，$\mu \geqslant 1$；如果 $u(x,y)$ 是 $f$ 的 Poisson 积分，那末

$$(24) \quad |u(x-t,y)| \leqslant A\left(1+\frac{|t|}{y}\right)^n M(f)(x),$$

更一般地

$$(24') \quad |u(x-t,y)| \leqslant A_\mu \left(1+\frac{|t|}{y}\right)^{n/\mu} M_\mu(f)(x).$$

我们从推导式（24）开始。

注意到式（24）是对展缩$(x,t,y) \to (x\delta, t\delta, y\delta)$不变的；因此只要在 $y=1$时证明式（24）就够了。

在 Poisson 核中令 $y=1$，有

$$P_1(x) = \frac{c_n}{(1+|x|^2)^{(n+1)/2}},$$

而对每个 $t$，$u(x-t,1)=f(x)*P_1(x-t)$。第三章的定理 2（在§2.2中）表明$|u(x-t,1)|\leqslant A_t M(f)(x)$，其中

$$A_t=\int Q_t(x)\,dx,$$

而$Q_t(x)$是$P_1(x-t)$的最小下降径向控制函数，即

$$Q_t(x)=c_n\cdot\sup_{|x'|\geqslant|x|}\left\{\frac{1}{(1+|x'-t|^2)^{(n+1)/2}}\right\}.$$

对 $Q_t(x)$我们有简单的估计，$Q_t(x)\leqslant c_n$，当$|x|\leqslant 2|t|$，而$Q_t(x)\leqslant A'(1+|x|^2)^{-(n+1)/2}$，当$|x|\geqslant 2|t|$。由此显然 $A_t\leqslant A(1+|t|)^n$，这就证明了式（24）。

由于

$$u(x-t,y)=\int_{s\in\mathbf{R}^n}P_y(s)f(x-t-s)\,ds$$

与

$$\int_{\mathbf{R}^n}P_y(s)\,ds=1,$$

我们有

$$|u(x-t,y)|^\mu\leqslant\int_{s\in\mathbf{R}^n}P_y(s)|f(x-t-s)|^\mu ds=U(x-t,y),$$

其中 $U$ 是$|f|^\mu$的 Poisson 积分。对 $U$ 应用式（24），就得到

$$|u(x-t,y)|\leqslant A^{1/\mu}\left(1+\frac{|t|}{y}\right)^{n/\mu}(M(|f|^\mu)(x))^{1/\mu}$$

$$=A_\mu\left(1+\frac{|t|}{y}\right)^{n/\mu}M_\mu(f)(x).$$

引理证毕。

2.5.2 现在我们对$1<p<2$ 以及限制 $p>2/\lambda$ 的情形来证明不等式（20）。

首先注意，我们总能找到一个 $\mu$，$1\leqslant\mu<p$，使得若令

$$\lambda' = \lambda - \frac{2-p}{\mu},$$

则仍有 $\lambda' > 1$。事实上若 $\mu = p$，则

$$\lambda - \frac{2-p}{\mu} > 1,$$

这是因为 $\lambda > 2/p$；因此经过 $\mu$ 的微小变化后这个不等式仍能保持。选定这样的 $\mu$ 以后，由引理 4 我们有

$$(25) \qquad |u(x-t,y)|\left(\frac{y}{y+|t|}\right)^{n/\mu} \leqslant AM_\mu(f)(x).$$

下面就像在 § 2.1中我们论述函数 $g$ 时那样进行。

$$(26) \qquad (g_\lambda^*(f)(x))^2 = \frac{1}{p(p-1)}\int_{\mathbf{R}_+^{n+1}} y^{1-n}\left(\frac{y}{y+|t|}\right)^{\lambda n} u^{2-p}$$
$$\times |\Delta u^p| dt dy \leqslant A^{2-p}(M_\mu(f)(x))^{2-p} I^*(x),$$

其中

$$I^*(x) = \int_{\mathbf{R}_+^{n+1}} y^{1-n}\left(\frac{y}{y+|t|}\right)^{\lambda' n} \Delta u^p(x-t,y) dt dy,$$

显然

$$\int_{\mathbf{R}^n} I^*(x)dx = \int_{\mathbf{R}_+^{n+1}}\int_{x\in\mathbf{R}^n} y^{1-n}\left(\frac{y}{y+|t-x|}\right)^{\lambda' n}$$
$$\times \Delta u^p(t,y) dx dt dy$$
$$= C_{\lambda'}\int_{\mathbf{R}_+^{n+1}} y\Delta u^p(t,y) dt dy.$$

在最后一步用了下面的事实：当 $\lambda' > 1$，

$$y^{-n}\int_{\mathbf{R}^n}\left(\frac{y}{y+|x|}\right)^{\lambda' n} dx = \int_{\mathbf{R}^n}\frac{dx}{(1+|x|)^{\lambda' n}} = C_{\lambda'} < \infty.$$

由引理 2

$$\int_{R^n} I^*(x)dx = C_{\lambda'}\|f\|_p^p. \tag{27}$$

这样式（26）起着式（15）的作用，而式（27）起着式（16）的作用。

如果我们用式（23′）中 $M_\mu(f)$的 $L^p$界代替 $M(f)$的 $L^p$界，那末就像在§2.1中那样结束我们的证明。

## §3 乘子（第一型）

3.1 $g$ 函数与 $g_\lambda^*$函数理论的第一个应用是在乘子的研究方面。下面介绍的定理（定理 3）是乘子定理的“初步的”形式。在§6中将介绍一个“最后的”形式，在那里还将对这两种形式作出比较。

设 $m$ 是 $\boldsymbol{R}^n$上的有界可测函数。则下述 Fourier 变换之间的关系

$$(T_m(f))^\wedge(x) = m(x)\hat{f}(x), \qquad f\in L^2\cap L^p \tag{28}$$

可以定义一个线性变换 $T_m$，它的定义域是 $L^2(\boldsymbol{R}^n)\cap L^p(\boldsymbol{R}^n)$。

如果对任意的 $f\in L^2\cap L^p$，$T_m(f)$也属于 $L^p$（注意，它自然是属于 $L^2$的），并且 $T_m$有界，即

$$\|T_m(f)\|_p\leqslant A\|f\|_p, \qquad f\in L^2\cap L^p, \tag{29}$$

（其中 $A$ 与 $f$ 无关），那末我们就称 $m$ 是 $L^p$乘子（$1\leqslant p\leqslant\infty$）。

使得式（29）成立的最小的 $A$ 称为乘子的模。注意，若式（29）对 $p<\infty$成立，则 $T_m$可以唯一地有界开拓到 $L^p$，并且仍满足相同的不等式。我们把这个开拓也记作 $T_m$。

我们用$\mathscr{M}_p$表示带有上述模的乘子类。按照逐点的乘法，它显然是一个 Banach 代数。

我们从某些例子开始，由 $T_m$同平移可交换，以及第二章

§1.2与§1.4的命题，可直接得出下面的事实。

**例1** $\mathscr{M}_2$是由全体有界可测函数组成的类，并且其乘子模与$L^\infty(\boldsymbol{R}^n)$模相等。

**例2** $\mathscr{M}_1$是由$\mathscr{B}(\boldsymbol{R}^n)$（有限Borel测度）中元素的Fourier变换组成的函数类，并且其$\mathscr{M}_1$模与$\mathscr{B}(\boldsymbol{R}^n)$模相等。

第二章与第三章的奇异积分理论允许我们断言以下事实。

**例3** 假设$m$是0次齐次的。如果或者$m$在球面上无限次可微，或者更一般地，$m$可表成第二章中等式（26）的形式（可差一个常数），那末$m\in\mathscr{M}_p$，$1<p<\infty$。

现在回到某些一般性的研究。

乘子的一个基本的对偶性包含在下述的命题中，这种对偶性（在第二章§2.5中我们在稍微不同的条件下运用过）反映了$L^p$空间的对偶性。

**命题** 假设

$$\frac{1}{p}+\frac{1}{p'}=1,\qquad 1\leqslant p\leqslant\infty,$$

则$\mathscr{M}_p=\mathscr{M}_{p'}$带有相同的模。

**证明** 用$\sigma$表示对合变换$\sigma(f)(x)=\bar{f}(-x)$。可以直接验证$\sigma^{-1}T_m\sigma=T_{\bar{m}}$；因此当$m$属于$\mathscr{M}$时，$\bar{m}$也属于$\mathscr{M}$；而且$\bar{m}$与$m$有相同的模。现在用Plancherel公式，

$$\int_{\boldsymbol{R}^n}T_m(f)\cdot\bar{g}\,dx=\int_{\boldsymbol{R}^n}m(x)\hat{f}(x)\overline{\hat{g}(-x)}\,dx$$

$$=\int_{\boldsymbol{R}^n}\hat{f}(x)\overline{\bar{m}\hat{g}(-x)}\,dx$$

$$=\int_{\boldsymbol{R}^n}f\,\overline{T_{\bar{m}}(g)}\,dx,\qquad \text{对任意}\ f,g\in L^2(\boldsymbol{R}^n),$$

再假设$f\in L^{p'}(\boldsymbol{R}^n)$，$g\in L^p(\boldsymbol{R}^n)$且$\|g\|_p\leqslant1$，则

$$\left|\iint_{R^n} T_{\bar{m}}(f)\cdot\bar{g}\,dx\right|\leqslant\|f\|_{p'}\|T_m(g)\|_p\leqslant A\|f\|_{p'},$$

这里 $A$ 是乘子 $m$（或 $\bar{m}$）在$\mathscr{M}$中的模。对上述所有的 $g$ 取上确界，就得到

$$\|T_m(f)\|_{p'}\leqslant A\|f\|_{p'}.$$

因此 $m$ 属于 $\mathscr{M}_{p'}$，并且它的 $\mathscr{M}_{p'}$ 模不大于它的 $\mathscr{M}_p$模；由于情况对 $p$ 与 $p'$是对称的，故这两个模相等。

我们曾经指出过，若 $m$ 是乘子（在 $\mathscr{M}_p$中），则在 $L^p(\boldsymbol{R}^n)$ 中有界的变换 $T_m$是同平移可交换的。逆命题也成立：设 $T$ 是 $L^p(\boldsymbol{R}^n)(p<\infty)$上的有界线性变换，它同平移可交换；则存在一个 $m\in\mathscr{M}_p$使得 $T_m=T$。这个事实的证明将在后面§7.3中略述。

在关于乘子的这些说明之后，我们要告诉读者，乘子类 $\mathscr{M}_p$ 的更深入的结构（除去对应于 $p=1,2$,或 $\infty$ 的“平凡的”情形以外），在很大程度上仍然是不知道的。甚至连在 $\boldsymbol{R}^1$中也是这样。然而，下面我们将要得到的是一个重要的充分条件，它在很大程度上附带包含了例 3 中所引的结果。

**3.2 定理 3** 假设$m(x)$在 $\boldsymbol{R}^n$ 中原点的余集上属于 $C^k$，这里 $k$ 是整数，$k>n/2$。还假定对于每个微分单项式

$$\left(\frac{\partial}{\partial x}\right)^\alpha,\qquad \alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n),$$

其中 $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n$，有

$$(30)\qquad \left|\left(\frac{\partial}{\partial x}\right)^\alpha m(x)\right|\leqslant B|x|^{-|\alpha|},\qquad \text{对任意}|\alpha|\leqslant k.$$

则 $m\in\mathscr{M}_p$，$1<p<\infty$；即$\|T_m(f)\|_p\leqslant A_p\|f\|_p$。

证明将表明，界 $A_p$只与 $B,p$ 及 $n$ 有关。

定理的证明引导出它的一个推广。我们把这叙述成一个推论。

**推论** 上述条件（30）可以用较弱的假定

$$
(31)\quad \begin{aligned}
&|m(x)|\leqslant B',\\
&\sup_{0<R<\infty} R^{2|\alpha|-n}\int_{R<|x|\leqslant 2R}\left|\left(\frac{\partial}{\partial x}\right)^{\alpha} m(x)\right|^{2}dx\leqslant B',\ |\alpha|\leqslant k
\end{aligned}
$$

来代替。

现在我们叙述与这个定理有关的两个实例：

**例1** $m(x)=|x|^{it}$，这里 $t$ 是实数。这个例子与第五章§1的 Riesz 位势有关。也见第二章§6.12。

**例2** $m(x)$是0次齐次的，并且在单位球面上属于 $C^k$。（也见第三章§3.5。）

定理（与推论）可以归结为下面的引理。它的陈述既说明了这里所考虑的乘子变换的本质，也指出了 $g$ 函数及其变形所起的作用。

**引理** 在定理3（或它的推论）的假定下，对每个 $f\in L^2(\boldsymbol{R}^n)$，我们设

$$F(x)=T_m(f)(x)。$$

则

$$(32)\qquad g_1(F,x)\leqslant B_\lambda g_\lambda^*(f,x),\qquad 其中\ \lambda=2k/n。$$

由引理看来，$g$ 函数及其变形是可以同时处理所有所考虑的乘子的特征表达式。另一方面，关系式（32）是点态的这一事实说明，在很大程度上映射 $T_m$是“半局部的”。

定理可由引理如下推得。关于 $k$ 的假定使得 $\lambda>1$。于是定理2告诉我们

$$\|g_\lambda^*(f,x)\|_p\leqslant A_{\lambda,p}\|f\|_p,\qquad 2\leqslant p<\infty,\ f\in L^2\cap L^p。$$

然而由定理 1（见§1.4中的推论），

$$A_p'\|F\|_p\leqslant\|g_1(F,x)\|_p;$$

因此

$$\|F\|_p=\|T_m(f)\|_p\leqslant A_p\|f\|_p,\qquad 2\leqslant p<\infty 及\ f\in L^2\cap L^p,$$

即 $m\in\mathscr{M}_p,2\leqslant p<\infty$。由对偶性，即§3.1中的命题，我们也有 $m\in\mathscr{M}_p,1<p\leqslant 2$，这就给出了定理的结论。

3.3 现在我们来证明引理。

用 $u(x,y)$ 表示 $f$ 的 Poisson 积分，$U(x,y)$ 表示 $F$ 的Poisson 积分。然后用"∧"表示关于变量 $x$ 的 Fourier 变换，我们有

$$\hat{u}(x,y)=e^{-2\pi|x|y}\hat{f}(x),$$

以及

$$\hat{U}(x,y)=e^{-2\pi|x|y}\hat{F}(x)=e^{-2\pi|x|y}m(x)\hat{f}(x).$$

定义

$$M(x,y)=\int_{R^n}e^{-2\pi i x\cdot t}e^{-2\pi|t|y}m(t)dt.$$

这样显然

$$\hat{M}(x,y)=e^{-2\pi|x|y}m(x),$$

因此

$$\hat{U}(x,y_1+y_2)=\hat{M}(x,y_1)\hat{u}(x,y_2),\qquad y=y_1+y_2,y_1>0.$$

这可以写成

$$U(x,y_1+y_2)=\int_{R^n}M(t,y_1)u(x-t,y_2)dt.$$

在这个等式中对 $y_1$微商 $k$ 次，对 $y_2$微商一次，然后令 $y_1=y_2=y/2$。这就给出恒等式

$$U^{(k+1)}(x,y)=\int_{R^n}M^{(k)}(t,y/2)u^{(1)}(x-t,y/2)dt. \tag{33}$$

（上标表示对 $y$ 的微商。）

3.3.1 借助这个恒等式将不难证明引理.关于 $m$ 的假定(30)（或 (31)）需要转移到 $M(x,y)$去。这就是：

$$|M^{(k)}(t,y)|\leqslant B'y^{-n-k}; \tag{34}$$

$$\int_{R^n}|t|^{2k}|M^{(k)}(t,y)|^2dt\leqslant B'y^{-n}. \tag{34'}$$

事实上，由 $M$ 的定义推得

$$|M^{(k)}(x,y)|\leqslant B(2\pi)^k\int_{R^n}|t|^k e^{-2\pi|t|y}dt$$

$$=B'\int_0^\infty r^k e^{-2\pi r y}r^{n-1}dr=B''y^{-n-k}.$$

这就是式（34）。

为了证明式（34′），让我们仔细验证

$$\int_{R^n}|t^\alpha M^{(k)}(t,y)|^2dt\leqslant B'y^{-n},$$

其中 $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ 满足 $\alpha_1+\alpha_2+\cdots+\alpha_n=k$，而

$$t^\alpha=t_1^{\alpha_1}t_2^{\alpha_2}\cdots t_n^{\alpha_n}.$$

由 Plancherel 定理

$$\|t^\alpha M^{(k)}(t,y)\|_2=\left\|\left(\frac{\partial}{\partial x}\right)^\alpha(|x|^k m(x)e^{-2\pi|x|y})\right\|_2.$$

而

$$y^{2r}\int_{R^n}|x|^{2r}e^{-4\pi|x|y}dx\leqslant Cy^{-n},\qquad 0\leqslant r,$$

又由假定（30）与 Leibniz 法则，得

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha(|x|^k m(x))\right|\leqslant B'|x|^{k-|\alpha|},$$

其中 $|\alpha|\leqslant k$。于是再对

$$\left(\frac{\partial}{\partial x}\right)^\alpha(|x|^k m(x)e^{-2\pi|x|y})$$

用 Leibniz 法则便有

$$|t^\alpha M^{(k)}(t,y)|_2^2\leqslant B'y^{-n},\qquad |\alpha|=k,$$

这就证明了式（34′）。

3.3.2 回到恒等式（33），对每个 $y$ 把积分区域分成两部分：$|t|\leqslant y/2$ 及 $|t|\geqslant y/2$。在第一个区域用关于 $M^{(k)}$ 的估计式

(34)，在第二个区域用估计式(34′)。再由 Schwarz 不等式就得到

$$|U^{(k+1)}(x,y)|^2\leqslant Ay^{-n-2k}\int_{|t|<y/2}|u^{(1)}(x-t,y/2)|^2dt$$

$$+Ay^{-n}\int_{|t|>y/2}\frac{|u^{(1)}(x-t,y/2)|^2}{|t|^{2k}}dt$$

$$=I_1(y)+I_2(y).$$

现在

$$(g_{k+1}(F,x))^2=\int_0^\infty|U^{(k+1)}(x,y)|^2y^{2k+1}dy$$

$$\leqslant\sum_{j=1}^2\int_0^\infty I_j(y)y^{2k+1}dy.$$

然而

$$\int_0^\infty I_1(y)y^{2k+1}dy\leqslant B\int_0^\infty\int_{|t|<y/2}|u^{(1)}(x-t,y/2)|^2y^{-n+1}dtdy$$

$$\leqslant B'\int_\Gamma|\nabla u(x-t,y)|^2y^{-n+1}dtdy$$

$$=B'(S(f,x))^2\leqslant B_\lambda(g_\lambda^*(f,x))^2.$$

类似地

$$\int_0^\infty I_2(y)y^{2k+1}dy$$

$$\leqslant B'\int_0^\infty\int_{|t|>y}y^{-n+2k+1}|t|^{-2k}|\nabla u(x-t,y)|^2dtdy$$

$$\leqslant B''(g_\lambda^*(f,x))^2,\quad 其中 n\lambda=2k.$$

这说明 $g_{k+1}(F,x)\leqslant B_\lambda g_\lambda^*(f,x)$。然而由 § 1.5 的 (c)，我们知道 $g_1(F,x)\leqslant A_kg_{k+1}(F,x)$ 。于是证明了引理，由此定理3证毕。

注意推论的证明与定理的证明相同，只要做一点微小的变动：在引理中，估计式

$$y^{2r}\int_{R^n}|x|^{2r}e^{-4\pi|x|y}dx\leqslant Cy^{-n},$$

必须用

$$y^{2r}\int_{R^n}|x|^{2r}|m_0(x)|^2e^{-4\pi|x|y}dx\leqslant C'y^{-n}$$

来代替，其中 $m_0$满足不等式

$$\sup_{0<R<\infty}R^{-n}\int_{R<|x|<2R}|m_0(x)|^2dx\leqslant 1.$$

## §4 部分和算子的应用

4.1 现在叙述Littlewood-Paley 理论的第二个主要工具（第一个是函数 $g$ 与 $g^*$的用法）. 在这里 $n$ 维理论已经比一维情形受到更多的限制，但是我们把关于这点的进一步讨论推迟到§4.3.

设 $\rho$ 表示 $\boldsymbol{R}^n$ 中任意的矩形. 矩形的含意（在本章的其余部分）将是指其边平行于坐标轴但可以是无限的矩形，即 $n$ 个区间的Cartesian 乘积. 对每个矩形 $\rho$，用 $S_\rho$ 表示"部分和算子"，也就是对应于 $m=\chi_\rho=$ 矩形 $\rho$ 的特征函数的乘子算子，即

(35) $$S_\rho(f)^\wedge=\chi_\rho\cdot\hat{f},\qquad f\in L^2(\boldsymbol{R}^n)\cap L^p(\boldsymbol{R}^n).$$

对于这个算子我们有以下定理.

**定理4** 若$1<p<\infty$，则

$$\|S_\rho(f)\|_p\leqslant A_p\|f\|_p,\qquad f\in L^2\cap L^p.$$

常数 $A_p$与矩形 $\rho$ 无关.

不过我们将需要这个定理的更广的形式，当我们用取值于Hilbert 空间的函数代替复值函数时就出现这种形式.

设$\mathscr{H}$是序列 Hilbert 空间，

$$\mathscr{H}=\left\{(c_j)_{j=1}^{\infty}:\ \left(\sum_j|c_j|^2\right)^{1/2}=|c|<\infty\right\}.$$

则我们可以把一个函数 $f\in L^p(\boldsymbol{R}^n,\mathscr{H})$表示成为一个序列

$$f(x)=(f_1(x),\cdots,f_n(x),\cdots),$$

这里每个 $f_j$是复值的，并且

$$|f(x)|=\left(\sum_{j=1}^{\infty}|f_j(x)|^2\right)^{1/2}.$$

设 $\mathfrak{R}$ 是矩形列，$\mathfrak{R}=\{\rho_j\}_{j=1}^{\infty}$。则我们可以用等式

$$S_{\mathfrak{R}}(f)=\left(S_{\rho_1}(f_1),\cdots,S_{\rho_j}(f_j),\cdots\right) \tag{36}$$

来定义算子 $S_{\mathfrak{R}}$，其中 $f=(f_1,f_2,\cdots,f_j,\cdots)$。这个算子把$L^2(\boldsymbol{R}^n,\mathscr{H})$映入到自身中。

定理4的推广如下。

**定理4′** 设 $f\in L^2(\boldsymbol{R}^n,\mathscr{H})\cap L^p(\boldsymbol{R}^n,\mathscr{H})$。则

$$\|S_{\mathfrak{R}}(f)\|_p\leqslant A_p\|f\|_p,\quad 1<p<\infty, \tag{37}$$

其中 $A_p$与矩形族 $\mathfrak{R}$ 无关。

**4.2** 这个定理将通过好几个步骤来证明，其中前两步已经包含了问题的本质。

4.2.1 **第一步**。这时 $n=1$，并且矩形 $\rho_1,\rho_2,\cdots,\rho_j,\cdots$是半无限区间 $(-\infty,0)$。

我们回想 Hilbert 变换 $f\to H(f)$，它对应于乘子 $i\operatorname{sign}x$（见第三章）。显然

$$S_{(-\infty,0)}=\frac{I+iH}{2}, \tag{38}$$

这里 $I$ 是恒等算子，而 $S_{(-\infty,0)}$ 是对应于区间 $(-\infty,0)$ 的部分和算子。整个证明依赖于下述引理。

**引理** 设

$$f(x)=(f_1(x),\cdots,f_j(x),\cdots)\in L^2(\boldsymbol{R}^n,\mathscr{H})\cap L^p(\boldsymbol{R}^n,\mathscr{H}).$$

记$\tilde{H}(f)(x)=(H(f_1)(x),\cdots,H(f_j)(x),\cdots)$。则

$$\|\tilde{H}(f)\|_p\leqslant A_p\|f\|_p,\quad 1<p<\infty, \tag{39}$$

其中 $A_p$是标量情形时，即当$\mathscr{H}$是一维空间时的常数。

我们用 Hilbert 变换的向量值形式，就像在第二章§5中很一般地叙述的那样，设 Hilbert 空间 $\mathscr{H}_1$与 $\mathscr{H}_2$都恒等于$\mathscr{H}$。在 $\boldsymbol{R}^1$ 中取

$$K(x) = I \cdot \frac{1}{\pi x},$$

其中 $I$ 是 $\mathscr{H}$ 上的恒等映射，则核 $K(x)$ 满足第二章定理3与定理5的一切条件。进一步

$$\lim_{\varepsilon \to 0} \int_{|y| \geqslant \varepsilon} K(y) f(x-y) dy = \widetilde{H}(f)(x),$$

于是引理证毕。（另一证明将在后面的 §7.12给出。）

现在假如所有矩形是区间（$-\infty, 0$），则由于式（38），

$$S_{\Re} = \frac{I + i\widetilde{H}}{2},$$

根据引理，在这种情况下定理得证。

4.2.2 **第二步**。这里 $n=1$，并且矩形是区间$(-\infty, a_1)$，$(-\infty, a_2), \cdots, (-\infty, a_j), \cdots$。

注意$(f(x)e^{-2\pi i x \cdot a})^{\wedge} = \hat{f}(x+a)$，所以

$$H(e^{-2\pi i x \cdot a} f)^{\wedge} = i \operatorname{sign} x \hat{f}(x+a),$$

因此

$$[e^{2\pi i x \cdot a} H(e^{-2\pi i x \cdot a} f)]^{\wedge} = i \operatorname{sign}(x-a) \hat{f}(x).$$

于是我们看到

$$(40) \quad S_{(-\infty, a_j)}(f_j)(x) = \frac{f_j + ie^{2\pi i x \cdot a_j} H(e^{-2\pi i x \cdot a_j} f_j)}{2}.$$

如果现在用记号 $e^{-2\pi i x \cdot a} f$ 来记

$$(e^{-2\pi i x \cdot a_1} f_1, \cdots, e^{-2\pi i x \cdot a_j} f_j, \cdots),$$

其中 $f = (f_1, \cdots, f_j, \cdots)$，那末式（40）可以改写成

$$(41) \qquad S_{\Re}(f) = \frac{f + ie^{2\pi i x \cdot a} \widetilde{H}(e^{-2\pi i x \cdot a} f)}{2},$$

再次运用引理，在这种情形便推得所要的结果。

4.2.3 **第三步**。一般的 $n$，但矩形 $\rho_j$是半空间 $x_1 < a_j$，即 $\rho_j = \{x:\ x_1 < a_j\}$，

用$S^{(1)}_{(-\infty,a_j)}$表示在$L^2(R^n)$上的算子，它只作用于变量$x_1$，其作用由$S_{(-\infty,a_j)}$给定。我们断定

$$S_{p_j}=S^{(1)}_{(-\infty,a_j)}. \tag{42}$$

事实上，对于乘积形式

$$f'(x_1)f''(x_2,\cdots,x_n)$$

的$L^2$函数，这个等式是显然的；而上述函数生成的线性子集在$L^2$中稠密，故等式（42）成立。

现在对每个固定的$x_2,x_3,\cdots,x_n$，用第二步得到的$L^p$不等式。在这个不等式两边取$p$次幂，然后再对$x_2,\cdots,x_n$积分。这就给出了现在这种情形所要的结果。注意到如果用半空间$\left\{x:\ x_1>a_j\right\}_{j=1}^{\infty}$代替半空间$\left\{x:\ x_1<a_j\right\}_{j=1}^{\infty}$；或者如果用$x_2$轴代替$x_1$轴，等等，那末所要结果仍成立。

4.2.4 最后一步。注意任意所考虑的有限矩形是$2n$个半空间的交，其中每个半空间的边界是与$R^n$的一个坐标轴垂直的超平面。于是应用第三步的结果$2n$次便证得定理，其中矩形族$\mathfrak{R}$由有限矩形组成。由于所得到的界不依赖于族$\mathfrak{R}$，通过一个显然的极限过程，我们可以过渡到一般的情形，其中$\mathfrak{R}$可能包含无限的矩形。

**4.3 一些问题** 我们想对定理4与定理4′的局限性作一些评注。当$n=1$时，定理论述相应于区间的部分和算子。因为区间是$R^1$中唯一的"正则"集：它们是唯一的凸集，唯一的连通集，等等，所以在一维的情形不能期望更多的东西。

然而当$n>1$，情况就根本改变了。其边平行于坐标轴的矩形只是非常特殊的集合，而我们只考虑这样的矩形便使得定理的一般性受到很大的局限。因此已证明的结果只是一维结果的$n$重叠加，而不是真正的$n$维结果。

为了说明这一点，我们叙述两个本质上涉及 $n$ 维理论的特殊的未完全解决的问题。它们本身是很有意义的，而其中每一个问题的解决都一定会带来许多进一步的结果。

**问题 A** 设 $B$ 是 $\boldsymbol{R}^n$ 中的单位球。我们能否在定理4中用球 $B$ 代替矩形 $\rho$？

现在仅知道只有在范围

$$\frac{2n}{n+1}<p<\frac{2n}{n-1}$$

内问题的回答可能是肯定的，但是除去 $p=2$ 以外，还没有确定的回答。见后面 §7.7与 §7.8。

**问题 B** 定理4′的矩形能否用经过任意旋转所得的矩形来代替？

可以证明，问题A的肯定的解决将会包含问题B对于同一个 $p$ 的解决。也可以证明，对于区间

$$\frac{2n}{n+1}\leqslant p\leqslant\frac{2n}{n-1}$$

以外的 $p$，问题B的回答是否定的①。

**4.4** 现在我们叙述定理4′的一个连续性方面的类似。设 $(\Gamma,d\gamma)$ 是一个 $\sigma$ 有限测度空间，并考虑在 $\Gamma$ 上平方可积的函数组成的 Hilbert 空间 $\mathscr{H}$，即 $\mathscr{H}=L^2(\Gamma,d\gamma)$。元素

$$f\in L^p(\boldsymbol{R}^n,\mathscr{H})$$

是 $\boldsymbol{R}^n\times\Gamma$ 上的复值函数 $f(x,\gamma)=f_\gamma(x)$，这些函数是关于 $x$ 与 $\gamma$ 可测的，并且对于它们有

$$\left(\int_{\boldsymbol{R}^n}\left(\int_\Gamma|f(x,\gamma)|^2d\gamma\right)^{p/2}dx\right)^{1/p}=\|f\|_p<\infty\quad（若p<\infty）。$$

与 §4.1类似，令 $\mathfrak{R}=\{\rho_\gamma\}_{\gamma\in\Gamma}$，并设映射 $\gamma\to\rho_\gamma$ 是从 $\Gamma$ 到矩形的可测函数，即对于每个 $\gamma$ 给出 $\rho_\gamma$ 顶点的分量的数值函数都是可测的。

① Y.Meyer，私人通信。

设 $f\in L^2(\boldsymbol{R}^n, \mathscr{H})$。则我们按照规则

$$F(x,\gamma)=S_{\rho_\gamma}(f_\gamma)(x)\quad (f_\gamma(x)=f(x,\gamma))$$

来定义 $F=S_{\Re}(f)$。

**定理4″** 对于 $f\in L^2(\boldsymbol{R}^n,\mathscr{H})\cap L^p(\boldsymbol{R}^n,\mathscr{H})$，有

$$\|S_{\Re}(f)\|_p\leqslant A_p\|f\|_p,\quad 1<p<\infty,$$

其中界 $A_p$ 不依赖于测度空间 $(\Gamma,\gamma)$ 或函数 $\gamma\to\rho_\gamma$。

这个定理的证明完全是定理4′证明的重复。读者如果愿意，也可以从定理4′经过一个极限过程得到它。

## §5 二进分解

**5.1** 我们现在考虑把 $\boldsymbol{R}^n$ 分解成矩形的一种典型的分解。首先，在 $\boldsymbol{R}^1$ 的情形，我们把 $\boldsymbol{R}^1$ 分解成为“不相重叠”的区间（即它们的内部不相交）$[2^k,2^{k+1}]$ $(-\infty<k<\infty)$ 以及 $[-2^{k+1},-2^k]$ $(-\infty<k<\infty)$ 的并集。这两组区间构成 $\boldsymbol{R}^1$ 的二进分解，其中一组合并成为正半直线，另一组合并成为负半直线。（严格地说，原点被漏掉了；但为了用语简单起见，我们仍称它为 $\boldsymbol{R}^1$ 的分解。）得到了 $\boldsymbol{R}^1$ 的这个分解后，我们对 $\boldsymbol{R}^n$ 取对应的乘积分解。于是 $\boldsymbol{R}^n$ 写成“不相重叠”的矩形的并集，这些矩形是在每个坐标轴的二进分解中出现的区间的乘积。这就是 $\boldsymbol{R}^n$ 的二进分解。所得的矩形族用 $\Delta$ 表示。回忆对每个矩形按式(35)定义的部分和算子 $S_\rho$，这时按照一种明显的意义（例如，$L^2$ 收敛），有

$$\sum_{\rho\in\Delta}S_\rho=\text{恒等算子}.$$

还是在 $L^2$ 情形，不同的块 $(S_\rho(f),\ \rho\in\Delta)$ 显得是独立的；它们当然互相正交。确切地说：$f$ 的 $L^2$ 模可以精确地通过 $S_\rho(f)$ 的 $L^2$ 模给出，即

$$\sum_{\rho\in\Delta}\|S_\rho(f)\|_2^2=\|f\|_2^2 \tag{43}$$

(对于 $R^n$ 的任何分解这都是正确的)。对于一般的 $L^p$ 情形不能期望这么多，但是仍然可以建立下述重要定理。

**定理5** 假设 $f\in L^p(R^n)$，$1<p<\infty$。则

$$\left(\sum_{\rho\in\Delta}|S_\rho f(x)|^2\right)^{\frac{1}{2}}\in L^p(R^n)$$

并且比式 $\left\|\left(\sum_{\rho\in\Delta}|S_\rho(f)(x)|^2\right)^{1/2}\right\|_p\Big/\|f\|_p$ 包含在两个(与 $f$ 无关的)界之间。

5.2 在依据二次式研究 $L^p$ 模时，Redemacher 函数提供了一种非常有用的方法。这些函数 $r_0(t),r_1(t),\cdots,r_m(t),\cdots$ 在区间 $(0,1)$ 上定义如下：对于 $0\leqslant t\leqslant 1/2$，$r_0(t)=1$，而对于 $1/2<t\leqslant 1$，$r_0(t)=-1$；$r_0$ 在单位区间外面按周期性延拓，即 $r_0(t+1)=r_0(t)$。一般地，$r_m(t)=r_0(2^m t)$。Rademacher 函数系在 $[0,1]$ 上是正交的(并且实际上相互独立)。它们在应用上的重要性来自以下事实。设 $\sum_{m=0}^{\infty}|a_m|^2<\infty$，又设 $F(t)=\sum_{m=0}^{\infty}a_m r_m(t)$。则对于每个 $p<\infty$，有 $F(t)\in L^p[0,1]$，并且对于 $p<\infty$，

$$A_p\|F\|_p\leqslant\|F\|_2=\left(\sum_{m=0}^{\infty}|a_m|^2\right)^{\frac{1}{2}}\leqslant B_p\|F\|_p \tag{44}$$

对两个正常数 $A_p$ 与 $B_p$ 成立。

因此对于能够按照 Rademacher 函数系展开的函数，它的所有 $L^p$ 模是可以比较的，其中 $p<\infty$。

我们也将需要式(44)的 $n$ 维形式。考虑 $R^n$ 中的单位立方体 $Q$，$Q=\{t=(t_1,t_2,\cdots,t_n):\ 0\leqslant t_j\leqslant 1\}$。设 $m$ 是非负整数的 $n$ 元数组 $m=(m_1,m_2,\cdots,m_n)$。定义

$$r_m(t)=r_{m_1}(t_1)r_{m_2}(t_2)\cdots r_{m_n}(t_n).$$

记 $F(t)=\sum_m a_m r_m(t)$。令

$$\|F\|_p=\left(\int_Q |F(t)|^p dt\right)^{1/p},$$

则对任何$\sum\limits_m |a_m|^2<\infty$，我们也有式(44)。这些事实的证明并不太长，但最好不在这点上离开本题。因此我们把证明推迟到以后在一个附录中介绍①。

**5.3** 现在我们来给出定理本身的证明。我们把它分成几步。

5.3.1 在这里我们说明，只要对于$f\in L^2(\mathbf{R}^n)\cap L^p(\mathbf{R}^n)$证明不等式

$$\left\|\left(\sum_{\rho\in\Delta}|S_\rho(f)(x)|^2\right)^{1/2}\right\|_p\leqslant A_p\|f\|_p \quad (1<p<\infty) \tag{45}$$

就足够了。为此，令$g\in L^2(\mathbf{R}^n)\cap L^q(\mathbf{R}^n)$，$\frac{1}{p}+\frac{1}{q}=1$，并考虑恒等式：

$$\sum_{\rho\in\Delta}\int_{\mathbf{R}^n}S_\rho(f)\overline{S_\rho(g)}\,dx=\int_{\mathbf{R}^n}f\bar{g}\,dx,$$

它可由式(43)通过极化推得。用Schwarz不等式，然后用Hölder不等式

$$\left|\int_{\mathbf{R}^n}f\bar{g}\,dx\right|\leqslant\int_{\mathbf{R}^n}\left(\sum_\rho|S_\rho(f)|^2\right)^{\frac{1}{2}}\left(\sum_\rho|S_\rho(g)|^2\right)^{\frac{1}{2}}dx$$

$$\leqslant\left\|\left(\sum_\rho|S_\rho(f)|^2\right)^{\frac{1}{2}}\right\|_p\left\|\left(\sum_\rho|S_\rho(g)|^2\right)^{\frac{1}{2}}\right\|_q.$$

对所有满足$\|g\|_q\leqslant 1$的上述$g$取上确界，不等式的左边给出$\|f\|_p$。由于对一切$p$（特别地对$q$）假定式(45)成立，因此上式右边被$\left\|\left(\sum\limits_\rho|S_\rho(f)|^2\right)^{\frac{1}{2}}\right\|_p A_q$所控制。于是

① 附录D。

(46) $$B_p\|f\|_p\leqslant\left\|\left(\sum_\rho|S_\rho(f)|^2\right)^{\frac{1}{2}}\right\|_p.$$

为了解除 $f\in L^2$ 这个附加的假定，对于 $f\in L^p$ 取 $f_j\in L^2\cap L^p$ 使得 $\|f_j-f\|_p\to 0$；对于 $f_j$ 以及 $f_j-f_{j'}$ 用不等式(45)与(46)；通过一个简单的极限论证，对 $f$ 也就得到式(45)与(46)。

5.3.2 在这里我们对于 $n=1$ 来证明不等式(45)。先介绍一些记号。设 $\Delta_1$ 如 § 5.1中所说的是 $\boldsymbol{R}^1$ 中的二进区间族；我们把它们列出来：$I_0, I_1, \cdots, I_m, \cdots$（在这里次序是不重要的）。对于每个 $I\in\Delta_1$，我们考虑部分和算子 $S_I$，以及立刻要定义的它的一个修改。设 $\varphi$ 是 $C^1$ 类中一个确定的函数①，具有以下性质：

$$\begin{cases}\varphi(x)=1, & 1\leqslant x\leqslant 2,\\ \varphi(x)=0, & x\leqslant 1/2 \text{ 或 } x\leqslant 4.\end{cases}$$

假设 $I$ 是任意二进区间，并设它的形式为 $[2^k, 2^{k+1}]$。按照

(47) $$(\tilde{S}_I(f))^\wedge(x)=\varphi(2^{-k}x)\hat{f}(x)=\varphi_I(x)\hat{f}(x)$$

来定义 $\tilde{S}_I$。也就是，$\tilde{S}_I$，像 $S_I$ 那样是乘子变换，它的乘子在区间 $I$ 上等于1；但又与 $S_I$ 不一样，$\tilde{S}_I$ 的乘子是光滑的。

当 $I=[-2^{k+1}, -2^k]$ 时，对 $\tilde{S}_I$ 可作类似的定义。由于 $S_I$ 的乘子是 $I$ 的特征函数，我们得到

(48) $$S_I\tilde{S}_I=S_I.$$

现在对每个 $t\in[0,1]$，考虑乘子变换

$$\tilde{T}_t=\sum_{m=0}^{\infty}r_m(t)\tilde{S}_{I_m}.$$

即对每个 $t$，$\tilde{T}_t$ 是其乘子为 $m_t(x)$ 的乘子变换，其中

(49) $$m_t(x)=\sum_m r_m(t)\varphi_{I_m}(x).$$

---

① 在这个证明的其余部分，它保持固定。

由$\varphi_{I_m}$的定义，显然对于任意 $x$ 在和式(49)中至多只能有三项不为零。并且容易看出

(50) $$|m_t(x)|\leqslant B, \qquad \left|\frac{dm_t(x)}{dx}\right|\leqslant\frac{B}{|x|},$$

其中 $B$ 与 $t$ 无关。因此由乘子定理(§3中定理3)，

(51) $$\|\tilde{T}_t)f)\|_p\leqslant A_p\|f\|_p, \qquad f\in L^2\cap L^p,$$

其中 $A_p$ 与 $t$ 无关。由此显然得出

$$\left(\int_0^1\|\tilde{T}_t(f)\|_p^p dt\right)^{\frac{1}{p}}\leqslant A_p\|f\|_p.$$

然而，由 Rademacher 函数的性质(44)，

$$\int_0^1\|\tilde{T}_t(f)\|_p^p dt=\int_0^1\int_{R^1}\left|\sum_m r_m(t)\tilde{S}_{I_m}(f)(x)\right|^p dxdt$$

$$\geqslant A'_p\int_{R^1}\left(\sum_m|\tilde{S}_{I_m}(f)(x)|^2\right)^{p/2}dx.$$

于是我们有

(52) $$\left\|\left(\sum_m|\tilde{S}_{I_m}(f)\ |^2\right)^{\frac{1}{2}}\right\|_p\leqslant B_p\ \|f\|_p.$$

现在对 $\mathfrak{R}=\Delta_1$ 应用关于部分和的一般定理，即定理 4′；然后用式(48)，得到

(53) $$\left\|\left(\sum_m|\tilde{S}_{I_m}(f)\ |^2\right)^{1/2}\right\|_p\leqslant C_p\ \|f\|_p,$$

它是不等式(45)的一维情形，也就是我们这一步所要证明的。

5.3.3 仍是在一维情形，记 $T_t$ 为算子

$$T_t=\sum_m r_m(t)S_{I_m}.$$

我们断言

(54) $$\|T_t(f)\|_p\leqslant A_p\|f\|_p, \qquad 1<p<\infty,$$

其中 $A_p$ 不依赖于 $t$，而$f\in L^2\cap L^p$。

记$T_t^N=\sum_{m=0}^{N} r_m(t) S_{I_m}$，只要用$T_t^N$代替$T_t$证明式(54)成立(并且其中$A_p$不依赖于$N$与$t$)就够了。由于每个$S_{I_m}$是$L^2$与$L^p$上的有界算子，我们有$T_t^N(f)\in L^2\cap L^p$，因此对它可应用已经证明了的式(46)(在$n=1$的情形)。于是，再用式(53)，得到

$$B_p\|T_t^N(f)\|_p\leqslant\left\|\left(\sum_{m=0}^{N}|S_{I_m}(f)|^2\right)^{1/2}\right\|_p\leqslant C_p\|f\|_p.$$

令$N\to\infty$，便证得式(54)。

5.3.4 现在我们转向$n$维情形，并定义$T_{t_1}^{(1)}$为只作用于变量$x_1$上的算子$T_{t_1}$。若$f\in L^2(\boldsymbol{R}^n)\cap L^p(\boldsymbol{R}^n)$，则对于几乎所有固定的$x_2,\cdots,x_n$，有

$$x_1\to f(x_1,x_2,\cdots,x_n)\in L^2(\boldsymbol{R}^1)\cap L^p(\boldsymbol{R}^1).$$

于是由不等式(54)得到

$$\int_{\boldsymbol{R}^1}|T_{t_1}^{(1)}(f)(x_1,x_2,\cdots,x_n)|^p dx_1 \leqslant A_p^p\int_{\boldsymbol{R}^1}|f(x_1,\cdots,x_n)|^p dx_1, \tag{55}$$

将上式两边对$x_2,\cdots,x_n$积分，便得到

$$\|T_{t_1}^{(1)}(f)\|_p\leqslant A_p\|f\|_p,\qquad f\in L^2\cap L^p, \tag{56}$$

其中$A_p$与$t_1$无关。用$x_2$代替$x_1$，或$x_3$代替$x_1$，等等，上述不等式当然也成立。

5.3.5 现在我们进行证明的最后一步。先叙述一些新的记号。用$\Delta$表示$\boldsymbol{R}^n$中全体二进矩形的集合。把任意$\rho\in\Delta$写成$\rho=I_{m_1}\times I_{m_2}\times\cdots\times I_{m_n}$，其中$I_0,I_1,\cdots,I_m,\cdots$表示前面(任意地)列出过的二进区间。记$m=(m_1,m_2,\cdots,m_n)$(其中每个$m_j\geqslant0$)，这时$\rho_m=I_{m_1}\times I_{m_2}\times\cdots\times I_{m_n}$。

现在对变量$x_1$，用算子$T_{t_1}^{(1)}$，再逐次地对$x_2,x_3$等等用类似的算子。其结果是

$$\|T_t(f)\|_p\leqslant A_p^n\|f\|_p. \tag{57}$$

这里

$$T_t = \sum_{p_m \in \Delta} r_m(t) S_{p_m},$$

其中 $r_m(t) = r_{m_1}(t_1) \cdots r_{m_n}(t_n)$同§5.2中所叙述的一样。上述不等式对于单位立方体$Q$中的$(t_1, t_2, \cdots, t_n)$一致地成立。

把不等式自乘$p$次幂，并对$t$积分，用式(44)中所援引的Rademacher函数系的性质。类似于式(52)的证明，便得到

$$\left\| \left( \sum_{p_m \in \Delta} |S_{p_m}(f)|^2 \right)^{1/2} \right\|_p \leqslant A_p \|f\|_p,$$

当$f \in L^2(\boldsymbol{R}^n) \cap L^p(\boldsymbol{R}^n)$。再加上§5的5.3.1的说明，定理5证毕。

## §6 Marcinkiewicz乘子定理

**6.1** 现在介绍第二型的乘子定理。这个形式在很大程度上综合了§4与§5中叙述的思想，因此它是整个理论中最重要的结果之一。为了说明白起见，我们先叙述一维情形。

**定理6** 设$m$是$\boldsymbol{R}^1$上的有界函数，它在每个不含原点的有限区间上是有界变差的。假设

(a) $|m(x)| \leqslant B$, $-\infty < x < \infty$;

(b) $\int_I |dm(x)| \leqslant B$，对每个二进区间$I$。

则$m \in \mathscr{M}_p$, $1 < p < \infty$；更确切地说，当$f \in L^2 \cap L^p$时，有

$$\|T_m(f)\|_p \leqslant A_p \|f\|_p,$$

其中$A_p$只依赖于$B$和$p$。

为了介绍一般定理，我们把$\boldsymbol{R}^1$分成它的两条半直线，$\boldsymbol{R}^2$分成它的四个象限，一般地，$\boldsymbol{R}^n$分成它的$2^n$个"卦限"。这样，$\boldsymbol{R}^n$的第一卦限是由所有坐标为严格正的点$x$组成的开"矩形"。我们假设$m(x)$在每个这样的卦限上有定义，并且它连同它的直到$n$阶的偏微商在这些卦限上连续。这时$m$可以在由其中一个或多个

变量的坐标为零的点所组成的集合上无定义。

对每个$k\leqslant n$，按以下明显的方式我们把$\boldsymbol{R}^k$嵌入$\boldsymbol{R}^n$中：$\boldsymbol{R}^k$是所有形如$(x_1, x_2, \cdots, x_k, 0, \cdots, 0)$的点组成的子空间。

**定理6′** 设$m$是$\boldsymbol{R}^n$上如上所述的有界函数。还设

(a) $|m(x)|\leqslant B$;

(b) 对每个$0<k\leqslant n$，当$\rho$在$\boldsymbol{R}^k$的二进矩形的范围内变化时，

$$\sup_{x_{k+1},\cdots,x_n}\int_\rho\left|\frac{\partial^k m}{\partial x_1\partial x_2\cdots\partial x_k}\right|dx_1\cdots dx_k\leqslant B;$$

(当$k=n$，"sup"记号消失。)

(c) 对于变量$x_1, x_2, \cdots, x_n$的$n!$个排列当中的每一个，类似于(b)的条件成立。

则$m\in\mathscr{M}_p$，$1<p<\infty$；更确切地说，当$f\in L^2\cap L^p$时，有

$$\|T_m(f)\|_p\leqslant A_p\|f\|_p,$$

其中$A_p$只与$B$, $p$及$n$有关。

**6.2 评注** 在证明定理之前我们说明某些技术性的问题；还要说明这个定理与§3的第一个乘子定理之间的关系。

6.2.1 当$n=1$时，定理6似乎比定理6′强，因为定理6′的假设要求$m$在原点以外连续可微，而定理6只要求$m$在与原点分离的区间上是有界变差的。然而，实际上两个结果是一样强的，因为对任何满足定理6假设的$m$，可以找到一个序列$\{m_j(x)\}$(其中$m_j(x)$事实上在离开原点处无穷次可微)，对于它们定理6′的界关于$j$一致成立，并且使得几乎处处有$m_j(x)\to m(x)$。这就推得$T_{m_j}\to T_m$，而根据这种方法就可以建立上述论断。我们把详细处理留给有兴趣的读者。

6.2.2 另外一个学究气的评注如下。读者可能已经推测到，在定理5，定理6与定理6′中二进矩形的定义里，关于2的幂的作用并没有必不可少的东西。事实上，可以用其它矩形代替二进矩形；例如，对顶点坐标而言，用$\left\{-\lambda_k\right\}_{k=-\infty}^{k=\infty}$与$\left\{\lambda_k\right\}_{k=-\infty}^{k=\infty}$来

代替$\left\{-2^k\right\}_{k=-\infty}^{k=\infty}$与$\left\{2^k\right\}_{k=-\infty}^{k=\infty}$，其中$\lambda_{k+1}/\lambda_k\geqslant r>1$，对一切$k$。不过这时得到的结论并不比在二进情形所得到的强。(也见下面§7.10。)

6.2.3 更有意义的是比较定理6′与第一个乘子定理，即定理3及其推论。显然当$n=1$时，定理6′较为强些。然而当$n\geqslant2$时，它们互相重叠而互不包含。对于$n\geqslant2$的这种差别还可以通过考虑简单的不变性来说明。定理3处理的乘子类对展缩$m(x)\to m(\varepsilon x)(\varepsilon>0)$不变，也对旋转$m(x)\to m(\rho^{-1}x)$不变。定理6′的乘子类对旋转并不是不变的，但对更大一群的展缩$m(x)\to m(\varepsilon\circ x)$不变，其中$(\varepsilon\circ x)=(\varepsilon_1x_1,\varepsilon_2x_2,\cdots,\varepsilon_nx_n)$，$x=(x_1,\cdots,x_n)$，而$\varepsilon_j$是独立的非零的数。

6.2.4 尽管如此，在各种应用中，定理6′似乎是两者中更为有用的一个。例如对于出现在椭圆型微分方程中的那些典型的乘子(见第三章§1.3，§3.5，也见本章的§7.9)，两个定理同样都可应用。

然而对出现在抛物型方程中的乘子

$$\frac{x_1}{x_1+i(x_2^2+x_3^2+\cdots+x_n^2)}$$

只有定理6′可以应用。乘子

$$\frac{|x_1|^{a_1}|x_2|^{a_2}\cdots|x_n|^{a_n}}{(x_1^2+x_2^2+\cdots+x_n^2)^{a/2}}$$

$$(a=a_1+a_2+\cdots+a_n,\ 其中a_j>0)$$

在分数次位势空间的研究中是典型的，对它也只能用定理6′。

6.2.5 最后，两个定理都有非常明显的缺陷，这从§4.3中提出的问题可以看到。似乎需要的是一种意义更为重大的理论，它判定乘子属于某个$\mathscr{M}_p$，$p\neq2$，但并非属于一切$\mathscr{M}_p$，$1<p<\infty$。对于这个困难的任务来说工具显得太少。唯一容易想到的是围绕函数$g_\lambda^*$做尽可能的发展。

6.2.6 定理3与定理6′的局限性可以进一步说明如下。考虑$R^n$中任意多面体的特征函数.用同定理4一样的方法,可以证明它是$L^p$乘子,$1<p<\infty$。但这个简单的例子并不在定理3或定理6′的范围内。

**6.3 证明** 我们只在$n=2$的情形证明定理6′。对一般情形来说这已经是十分典型的了。我们这样做只是为了避免记号复杂。

设$f\in L^2(R^2)\cap L^p(R^2)$,并记$F=T_m(f)$,即

$$F(x)^\wedge=m(x)\hat{f}(x).$$

用$\Delta$表示二进矩形,并对每个$\rho\in\Delta$,记

$$f_\rho=S_\rho(f),\qquad F_\rho=S_\rho F,$$

于是$F_\rho=T_m(f_\rho)$。

根据二进分解定理(定理5),只要证明

$$\left\|\left(\sum_{\rho\in\Delta}|F_\rho|^2\right)^{1/2}\right\|_p\leqslant C_p\left\|\left(\sum_{\rho\in\Delta}|f_\rho|^2\right)^{1/2}\right\|_p \tag{58}$$

就够了。

$\Delta$中的矩形分别来自第一、第二、第三与第四象限。在式(58)的左边的估计中分别考虑属于每个象限的矩形,而从现在起我们假定矩形属于第一象限。

我们将用一个包含$f_\rho$以及部分和算子的积分来表示$F_\rho$。这种可能性就是本证明的基本思想。

固定$\rho$并假设

$$\rho=\{(x_1,x_2):\ 2^k\leqslant x_1\leqslant 2^{k+1},\ 2^l\leqslant x_2\leqslant 2^{l+1}\}.$$

则对于$(x_1,x_2)\in\rho$,我们有恒等式

$$m(x_1,x_2)=\int_{2^k}^{x_1}\int_{2^l}^{x_2}\frac{\partial^2 m(t_1,t_2)}{\partial t_1\partial t_2}dt_1dt_2+\int_{2^k}^{x_1}\frac{\partial}{\partial t_1}m(t_1,2^l)dt_1$$
$$+\int_{2^l}^{x_2}\frac{\partial}{\partial t_2}m(2^k,t_2)dt_2+m(2^k,2^l).$$

现在用 $S_t$ 表示对应于矩形 $2^k<x_1<t_1$，$2^l<x_2<t_2$ 的乘子变换。类似地令 $S^{(1)}_{t_1}$ 表示对应于区间 $2^k<x_1<t_1$ 的乘子，$S^{(2)}_{t_2}$ 也类似。事实上 $S_t=S^{(1)}_{t_1}\cdot S^{(2)}_{t_2}$。这时上述等式显然化为

$$(59)\qquad S_\rho T_m=\int_{2^l}^{2^{l+1}}\int_{2^k}^{2^{k+1}}S_t\frac{\partial^2 m}{\partial t_1\partial t_2}dt_1dt_2$$

$$+\int_{2^k}^{2^{k+1}}S^{(1)}_{t_1}\frac{\partial}{\partial t_1}m(t_1,2^l)dt_1$$

$$+\int_{2^l}^{2^{l+1}}\cdots+m(2^k,2^l)S_\rho.$$

注意 $S_\rho T_m(f)=F_\rho$，以及 $S^{(1)}_{t_1}S_\rho=S^{(1)}_{t_1}$，$S^{(2)}_{t_2}S_\rho=S^{(2)}_{t_2}$，$S_tS_\rho=S_t$，并用 Schwarz 不等式与定理的假设，就给出

$$(60)\qquad |F_\rho|^2\leqslant B'\left\{\iint_\rho|S_t(f_\rho)|^2\left|\frac{\partial^2 m}{\partial t_1\partial t_2}\right|dt_1dt_2\right.$$

$$+\int_{I_1}|S^{(1)}_{t_1}(f_\rho)|^2\left|\frac{\partial}{\partial t_1}m(t_1,2^l)\right|dt_1$$

$$\left.+\int_{I_2}|S^{(2)}_{t_2}(f_\rho)|^2\left|\frac{\partial}{\partial t_2}m(2^k,t_2)\right|dt_2+|f_\rho|^2\right\}$$

$$=\Im^1_\rho+\Im^2_\rho+\Im^3_\rho+\Im^4_\rho,\qquad \text{其中 } \rho=I_1\times I_2.$$

为了估计 $\left\|\left(\sum_\rho|F_\rho|^2\right)^{1/2}\right\|_p$，我们用 §4.4 的定理 4″ 分别估计在式(60)右边的四项中每一项的贡献。为了在 $\Im^1_\rho$ 的情形应用该定理，我们取第一象限为 $\Gamma$，以及

$$d\gamma=\left|\frac{\partial^2 m(t_1,t_2)}{\partial t_1\partial t_2}\right|dt_1dt_2;$$

在二进矩形上函数 $\gamma\to\rho_\gamma$ 是常量。由于对每个矩形

$$\int_\rho d\gamma=\int_\rho\left|\frac{\partial^2 m(t_1,t_2)}{\partial t_1\partial t_2}\right|dt_1dt_2\leqslant B,$$

故

$$\left\|\left(\sum_{\rho}|\mathfrak{I}_{\rho}^{1}|\right)^{1/2}\right\|_{p}\leqslant C_{p}\left\|\left(\sum_{\rho}|f_{\rho}|^{2}\right)^{1/2}\right\|_{p}.$$

类似地对 $\mathfrak{I}^2$, $\mathfrak{I}^3$ 与 $\mathfrak{I}^4$ 进行估计，这就完成了定理的证明。

## §7 进一步的结果

7.1 设 $R_1, R_2, \cdots, R_n$ 是 Riesz 变换。则

(a) $(g(f,x))^2=(g_1(f)(x))^2+\sum_{j=1}^{n}(g_1(R_jf)(x))^2$ ;

(b) $g_1^2(f)(x)\leqslant\sum_{j=1}^{n}(g_x(R_jf)(x))^2$ .

7.2 (a) 设 $\varphi$ 在 $\boldsymbol{R}^n$ 中连续可微，并且

1° $|\varphi(x)|\leqslant A(1+|x|)^{-n-\delta}$;

2° $\left|\dfrac{\partial\varphi}{\partial x_j}\right|\leqslant A(1+|x|)^{-n-\delta}$，对每个 $j=1,\cdots,n$;

3° $\displaystyle\int_{R^n}\left|\frac{\partial\varphi(x+t)}{\partial x_j}-\frac{\partial\varphi(x)}{\partial x_j}\right|dx\leqslant A|t|^{\delta}$，对某个 $\delta>0$。

定义 $f_\varepsilon(x)=f*\varphi_\varepsilon$，其中 $\varphi_\varepsilon(x)=\varepsilon^{-n}\varphi(x/\varepsilon)$。则

$$\left\|\left(\int_0^\infty\varepsilon\left|\frac{\partial f_\varepsilon}{\partial\varepsilon}\right|^2d\varepsilon\right)^{\frac{1}{2}}\right\|_p\leqslant A_p\|f\|_p,\quad f\in L^p,\ 1<p<\infty.$$

如果再加上

$$\left\|\left(\int_0^\infty\varepsilon\left|\frac{\partial f_\varepsilon}{\partial\varepsilon}\right|^2d\varepsilon\right)^{\frac{1}{2}}\right\|_2=C\|f\|_2,\quad C>0,$$

那末上式的反向不等式也成立。对于 $\left(\int_0^\infty\varepsilon\left|\dfrac{\partial f_\varepsilon}{\partial x_k}\right|^2d\varepsilon\right)^{\frac{1}{2}}$ 也有类似的结果。（关于有密切关系的结 果，见 Benedek, Calderón, 以及 Panzone[1]。）

(b) $\boldsymbol{R}^1$中的一个例子由

$$\left(\int_0^\infty \frac{|F(x+t)+F(x-t)-2F(x)|^2}{t^3}dt\right)^{\frac{1}{2}}$$

给出，其中

$$F(x)=\int_0^x f(t)dt.$$

见 Marcinkiewicz[2]，Zygmund[1]；关于其推广也见 Stein[3]，Hörmander[1]。

**7.3** 设$T$是$L^p(\boldsymbol{R}^n)$到自身的一个有界线性变换，$1\leqslant p\leqslant\infty$，它同平移可交换。则存在一个有界函数$m$，使得$(T(f))^\wedge(x)=m(x)\hat{f}(x)$，对任何$f\in L^2\cap L^p$成立。

**证明的概要** (a) 因为$T$与平移可交换，对适当的$f$与$g$，$(T(f))*g=T(f*g)$。于是$T(f)*g=f*T(g)$。

(b) 设$\frac{1}{p}+\frac{1}{q}=1$，并设$f$与$g$都属于$L^p(\boldsymbol{R}^n)\cap L^q(\boldsymbol{R}^n)$。

则卷积$T(f)*g$与$f*T(g)$表示连续函数，因此它们在每个点，特别在原点处相等。于是

$$\int_{\boldsymbol{R}^n}T(f)(x)g(-x)dx=\int_{\boldsymbol{R}^n}T(g)(x)f(-x)dx.$$

通常的对偶论证表明$T$在$L^q$上有界，然后根据内插定理（第一章定理 4），$T$也是在$L^2$上有界的。最后应用第二章 §1.4 中的命题。也见《富里叶分析》第 I 章。

**7.4** §7.4—§7.6 给出若干不能用本章的方法来处理的乘子的有意义的说明。

设$m(x)$是$\boldsymbol{R}^n$中一个函数，它的形式为

$$m(x)=\varphi_0(x)\frac{e^{i|x|^\alpha}}{(1+|x|^2)^\beta},\quad 1\geqslant\alpha>0,\ \beta>0,$$

其中$\varphi_0$是一个光滑函数，它在原点附近为零，并且对于足够大

的 $x$，它等于 1．假设 $n=1$ 与 $a=1$ 不能同时成立。

(a) 若 $|1/2-1/p|<\theta$，则 $m(x)\in\mathscr{M}_p$。其中 $\theta=2\beta/an$；

(b) 若 $|1/2-1/p|>\theta$，则 $m(x)\overline{\in}\mathscr{M}_p$。

在例外的情形（$n=1$，$a=1$），

(a′) 当 $\beta=0$ 时，$m\in\mathscr{M}_p\Longleftrightarrow 1<p<\infty$；

(b′) 当 $\beta>0$ 时，$m\in\mathscr{M}_p$，对一切 $p$。

见 Hirschmann[2]，Wainger[1]，Hörmander[3]，Stein[8]，以及 Fefferman[1]。

**7.5** 假设 $m(x)\in\mathscr{M}_p(\boldsymbol{R}^n)$，并且在 $\boldsymbol{R}^k$ 的每个点都是连续的，$k<n$（$\boldsymbol{R}^k$ 看作 $\boldsymbol{R}^n$ 的子空间）。则限制于 $\boldsymbol{R}^k$ 上的 $m(x)$ 属于 $\mathscr{M}_p(\boldsymbol{R}^k)$。见 deLeeuw[1]。

**7.6** 设 $m(x)=(m_1*m_2)(x)$，这里 $m_1\in L^r(\boldsymbol{R}^n)$，而 $m_2\in L^{r'}(\boldsymbol{R}^n)$，其中 $1/r+1/r'=1$。则当 $2\leqslant r\leqslant\infty$ 时，$m\in\mathscr{M}_p$，$|1/2-1/p|\leqslant 1/r$。见 Hahn[1]。

**7.7** (a) 令 $\chi_B$ 是 $\boldsymbol{R}^n$ 中单位球的特征函数。则若

$$p\leqslant\frac{2n}{n+1}\quad\text{或}\quad p\geqslant\frac{2n}{n-1}$$

时，$\chi_B\overline{\in}\mathscr{M}_p$。见 Herz[1]。

(b) 更一般地：设 $m(x)$ 是径向函数，并假设 $m\in\mathscr{M}_p(\boldsymbol{R}^n)$。则当

$$p<\frac{2n}{n+1}\quad\text{或}\quad p>\frac{2n}{n-1}$$

时，$m$ 处处连续，原点可能除外。

**提示**：设 $f\in L^p(\boldsymbol{R}^n)$，$p<2n/(n+1)$，及 $f$ 是径向函数。则 $\hat{f}$ 连续，可能原点除外。为了证明这个论断，用 $\hat{f}$ 的 Bessel 积分表示式，如同在《富里叶分析》第Ⅳ章中那样。

**7.8** 考虑函数

$$m_\delta(x)=\begin{cases}(1-|x|^2)^\delta, & |x|\leqslant 1,\\ 0, & |x|>1\end{cases}$$

是否$L^p(\boldsymbol{R}^n)$ 乘子的问题，即是否$m_\delta\in\mathscr{M}_p$，当$\delta=0$时，这就是§4.3所讨论的问题，也是§7.7中讨论的问题。下述正面的结果是已知的。

(a) 若$\delta>\dfrac{n-1}{2}\left|1-\dfrac{2}{p}\right|$，则$m_\delta\in\mathscr{M}_p$。见Stein[1]。

(b) 近来在Fefferman[1]中这个结果有了重要的改进。他的一个特殊结果是，当$n=2$时，只要

$$\delta>2(n-1)\left|\frac{1}{p}-\frac{3}{4}\right|,$$

以及$1\leqslant p<6/5$。这个结果本质上对于$p$在$1\leqslant p<6/5$范围中是最好的可能。对于$n$维情形，$n\geqslant 3$，有着类似的结果，当$1\leqslant p<4n/(3n+1)$时它本质上是最好的可能。

7.9 设$P(x)$是$\boldsymbol{R}^n$中的$k$次多项式。如果$P(x)$依下述意义是椭圆型的，即它的$k$次齐次的部分除了在原点外都不为零，又$f$是$\boldsymbol{R}^n$中具有紧支集的任意$k$次连续可微函数。则我们有不等式

$$\left\|\left(\frac{\partial}{\partial x}\right)^\alpha f\right\|_p\leqslant A_p\left[\left\|P\left(\frac{\partial}{\partial x}\right)f\right\|_p+\|f\|_p\right],\qquad 1<p<\infty,$$

只要$|\alpha|\leqslant k$。

**提示**：令$\varphi(x)$是光滑函数，它在$P$的零点集的一个邻域内为零，并且在一个足够大的球外为1。则$\varphi(x)x^\alpha/P(x)$满足乘子定理3或定理6′的条件；$x^\alpha(1-\varphi(x))$是一个$L^1$函数的Fourier变换，而最后，

$$x^\alpha\hat{f}(x)=x^\alpha(1-\varphi(x))\hat{f}(x)+\varphi(x)\frac{x^\alpha}{P(x)}\cdot P(x)\hat{f}(x).$$

也见第三章的§3.5以及Agmon，Douglis与Nirenberg[1]。

7.10 (a) 定理6的条件(a)与(b)同下述条件等价

$$|m(x)|\leqslant B',$$

$$\sup_{0<R}\frac{1}{R}\int_{|x|<R}|x||dm(x)|\leqslant B'.$$

(b) 定理6′的条件的类似说法是什么？

7.11 §2.4中定理2的结果可以加强如下．设$1<p<2$及$p=2/\lambda$，则映射$f\to g_\lambda^*(f)$是弱$(p,p)$型的。见Fefferman[1]。一个更早的结果（对于一个类似的极大函数）将在第七章§4.5中叙述。

7.12 设$T$是$L^p(R^n)$到自身的一个有界算子。$\mathscr{H}$是任意Hilbert空间。则$T$具有一个唯一的“开拓”$T\otimes I$，结果是$L^p(R^n,\mathscr{H})$到自身的算子，它具有性质$(T\otimes I)(\varphi f(x))=\varphi\cdot T(f)(x)$，对任意$\varphi\in\mathscr{H}$，以及$f\in L^p(R^n)$．并且$T\otimes I$在$L^p(R^n,\mathscr{H})$上的模与$T$在$L^p(R^n)$上的模相同。Marcinkiewicz 与 Zygmund[1]，Zygmund[8]第XV章。

## 注　释

节1 经典理论（使用复方法）在Zygmund[8]的第XIV章和第XV章中叙述。在那里可以找到进一步的历史上的文献索引。关于$n$维$g$函数的定理见Stein[3]；进一步的推广由Hörmander[1]，Schwartz[1]，及Benedeck，Calderón，与Panzone[1]给出。

节2 函数$g^*$由Zygmund在[1]中系统地研究过，而$n$维理论由Stein在[6]与[10]中作了系统的研究．§2.1中叙述的特殊方法取自Stein[10]；一个有关的思想由Gasper[1]独立地发展。如同在Stein[13]中那样，这一方法是理论的各种推广的出发点。

节3 最初的Marcinkiewicz乘子定理出现在Marcinkiewicz[4]中．在该文章中定理是在周期的情形下给出的，这个定理的非周期变形由Mihlin[2]，Hörmander[1]，及Kree[1]给出。定理3

的叙述与 Hörmander 的叙述相同；然而本书的证明，如运用 $g$ 函数与 $g^*$ 函数的比较，却是不同的，而且本书的证明方法可以在各种其它情况下采用，这在后面第七章中将会看到。

关于乘子的一般讨论，也见Edward[1]。

**节4，节5，及节6** 定理4′的一维形式在Zygmund[8]第XV章中给出。这里给出的较一般形式事实上是这种特殊情形的一个简单的推论。

定理6′的证明是在 Marcinkiewicz[4]中最初对周期情形所作论证的一个简单的改进。也见 Lizorkin[1]以及 Kree[1]。

# 第五章　通过函数空间描述的可微性

在本章中我们将研究在Banach空间中可以得到最好描述的函数的可微性与光滑性。

这种研究的动力之一是基于它作为分析中各种不同问题的一个有用工具而有着广泛的应用，虽然我们要做的许多工作在思想和已经发展的方法方面实际上是预示了的。事实上，这样一些技巧，如Marcinkiewicz内插定理、调和函数的应用以及Littlewood-Paley的$g$函数，都是下面将要详述的理论的重要内容。

我们研究的函数空间是下列的：

(a) Sobolev空间$L_k^p(\mathbf{R}^n)$。它在许多问题中很有用。它由$\mathbf{R}^n$中全体其直到$k$阶的微商均属于$L^p(\mathbf{R}^n)$的函数组成；自然，$k$是非负整数。

将要考虑的两种类型的函数空间是试图"推广"Sobolev空间到$k$是非整数次的情形。

(b) 位势空间$\mathscr{L}_\alpha^p(\mathbf{R}^n)$，由全体$L^p$函数的$\alpha$阶"位势"组成。当$\alpha$是整数且$1<p<\infty$时，这些空间等价于Sobolev空间。

(c) 空间$\Lambda_\alpha^{p,q}$。这是通过$L^p$连续模定义的函数空间。因此它们是空间$L_k^p(\mathbf{R}^n)$的比较容易定义的"推广"，从而是很有用的。然而，它们并不是Sobolev空间的真正推广，故要把Sobolev空间同$\Lambda_\alpha^{p,q}(\mathbf{R}^n)$以及$\mathscr{L}_\alpha^p(\mathbf{R}^n)$进行比较。这个比较可以认为是本章研究的中心问题之一。正是在这里要用到第四章的Littlewood-Paley理论。

我们将从研究分数次的Laplace算子$(-\Delta)^{\alpha/2}$开始。它以及它的变形$(I-\Delta)^{\alpha/2}$体现了我们将用到的重要的典型方法。

## §1 Riesz 位势

1.1 一个充分光滑且在无穷远很小的函数 $f$ 的 Fourier 变换，同 Laplace 算子对它的作用

$$\Delta f = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2},$$

通过关系式

$$(1) \qquad (-\Delta f)^\wedge(x) = 4\pi^2 |x|^2 \hat{f}(x)$$

联系起来。

由此，只要在 $|x|^2$ 中用一般的指数 $\beta$ 代替指数 2（至少是形式地），分数次 Laplace 算子就被

$$(2) \qquad ((-\Delta)^{\beta/2} f)^\wedge(x) = (2\pi|x|)^\beta \hat{f}(x)$$

定义了。

在这里负幂 $\beta$，$-n<\beta<0$，将有着特殊的意义。这时形式定义的算子可以通过一个积分算子实现。也就是说，符号上稍作改变，我们有

$$(3) \qquad I_\alpha(f) = (-\Delta)^{-\alpha/2}(f), \qquad 0<\alpha<n,$$

其中我们通过

$$(4) \qquad I_\alpha(f)(x) = \frac{1}{\gamma(\alpha)} \int_{R^n} |x-y|^{-n+\alpha} f(y) dy$$

定义 Riesz 位势，这里

$$\gamma(\alpha) = \pi^{n/2} 2^\alpha \Gamma(\alpha/2) \Big/ \Gamma\left(\frac{n}{2} - \frac{\alpha}{2}\right).$$

这些形式上的运算要有一个准确的含义。

为此，方便的是函数类 $\varphi \in \mathscr{S}$，它在 $R^n$ 上无限次可微，它的所有阶微商乘上多项式后仍保持有界。

**引理1** 设 $0<\alpha<n$。

(a) 函数 $|x|^{-n+\alpha}$ 的 Fourier 变换是函数 $\gamma(\alpha)(2\pi)^{-\alpha}|x|^{-\alpha}$，其意义是指

(5) $$\int_{\mathbf{R}^n} |x|^{-n+\alpha}\overline{\varphi(x)}\,dx = \int_{\mathbf{R}^n} \gamma(\alpha)(2\pi)^{-\alpha}|x|^{-\alpha}\,\overline{\varphi^\wedge(x)}\,dx,$$

当$\varphi\in\mathscr{S}$。

(b) 等式$(I_\alpha f)^\wedge = (2\pi|x|)^{-\alpha}\hat{f}(x)$成立，其意义是指

$$\int_{\mathbf{R}^n} I_\alpha(f)(x)\overline{g(x)}\,dx = \int_{\mathbf{R}^n} \hat{f}(x)(2\pi|x|)^{-\alpha}\overline{g^\wedge(x)}\,dx,$$

当$f, g\in\mathscr{S}$。

引理的第一部分只是第三章§3.3结果的重述，因为

$$\gamma(\alpha) = \gamma_{0,\alpha}(2\pi)^\alpha.$$

部分(b)可从部分(a)直接推出，只要写出

$$\frac{(2\pi)^\alpha}{\gamma(\alpha)}\int_{\mathbf{R}^n} f(x-y)|y|^{-n+\alpha}dy$$

$$= \int_{\mathbf{R}^n} \hat{f}(-y)|y|^{-\alpha}e^{+2\pi i x\cdot y}dy,$$

(它是式(5)的改写) 并在等式两边乘上$\overline{g(x)}$后积分。

我们现在叙述两个可以从引理1得到的恒等式，它们反映了位势$I_\alpha$的本质。

(6) $$I_\alpha(I_\beta f) = I_{\alpha+\beta}(f),$$

其中$f\in\mathscr{S}$，$\alpha>0$，$\beta>0$，$\alpha+\beta<n$。

(7) $$\Delta(I_\alpha f) = I_\alpha(\Delta f) = -I_{\alpha-2}(f),$$

其中$f\in\mathscr{S}$，$n\geqslant 3$，$n\geqslant\alpha\geqslant 2$。

式(6)与式(7)的验证没有实质的困难，最好是留给有兴趣的读者自己去做。

式(6)的一个简单推论是Beta积分的$n$维变形

(8) $$\int_{\mathbf{R}^n} |1-y|^{-n+\alpha}|y|^{-n+\beta}dy = \frac{\gamma(\alpha)\gamma(\beta)}{\gamma(\alpha+\beta)},$$

其中$0<\alpha$，$0<\beta$，$\alpha+\beta<n$。

**1.2 位势的$L^p$不等式** 直到现在我们只是从形式的观点考

虑了 Riesz 位势。特别地，我们只是对在无穷远充分小的十分光滑的函数进行运算。但由于 Riesz 位势是积分算子，自然要求它们作用在 $L^p(\boldsymbol{R}^n)$空间。

为此，我们提出下面的问题。给定 $\alpha$，$0<\alpha<n$，对什么样的数对 $p$ 与 $q$，算子 $f\to I_\alpha(f)$是 $L^p(\boldsymbol{R}^n)$ 到 $L^q(\boldsymbol{R}^n)$ 有界的？也就是说，什么时候不等式

$$\|I_\alpha(f)\|_q\leqslant A\|f\|_p \tag{9}$$

成立？

存在一个必要条件，它只是核 $(\gamma(\alpha))^{-1}|y|^{-n+\alpha}$ 齐次性的反映。事实上，考虑展缩算子

$$\tau_\delta(f)(x)=f(\delta x),\qquad \delta>0.$$

显然

$$\tau_\delta^{-1}I_\alpha\tau_\delta=\delta^{-\alpha}I_\alpha,\qquad \delta>0. \tag{10}$$

又有

$$\begin{aligned}\|\tau_\delta(f)\|_p&=\delta^{-n/p}\|f\|_p,\\ \|\tau_\delta^{-1}I_\alpha(f)\|_q&=\delta^{n/q}\|I_\alpha(f)\|_q.\end{aligned} \tag{11}$$

因此式（9）可能成立只有当

$$\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}. \tag{12}$$

下面我们将看到，这条件也是充分的，除了两个例外情形。

这两个情形就是 $p=1$（这时 $q=n/(n-\alpha)$）与$q=\infty$（这时$p=n/\alpha$）。让我们考虑 $p=1$。不难看出，预想的不等式

$$\|I_\alpha(f)\|_{n/(n-\alpha)}\leqslant A\|f\|_1 \tag{13}$$

不能成立。假使（13）成立，我们就可以用正的函数列，它们的积分等于1而支集收敛到原点（“恒等逼近”）来代替 $f$。一个简单的极限论证就表明，它蕴含了

$$\left\|\frac{1}{\gamma(\alpha)}|x|^{-n+\alpha}\right\|_{n/(n-\alpha)}\leqslant A<\infty,$$

这意味着

$$\int_{R^n} |x|^{-n} dx < \infty,$$

这是一个矛盾。

第二个典型情形出现在$q=\infty$。一个直接的理由是，这是刚刚讨论的$p=1$情形的对偶。$q=\infty$的否定也可以如下直接看出：令

$$f(x) = |x|^{-\alpha}\left(\log\frac{1}{|x|}\right)^{-(\alpha/n)(1+\varepsilon)}, \qquad |x|<\frac{1}{2};$$

而$f(x)=0$，$|x|>1/2$，其中$\varepsilon$是正的但很小。这时$f\in L^p(R^n)$，$p=n/\alpha$，因为

$$\int_{|x|<1/2} |x|^{-n}\left(\log\frac{1}{|x|}\right)^{-1-\varepsilon} dx < \infty.$$

然而$I_\alpha(f)$在原点附近不是本质有界的，因为

$$I_\alpha(f)(0) = \frac{1}{\gamma(\alpha)}\int_{|x|<1/2} |x|^{-n}\left(\log\frac{1}{|x|}\right)^{-(\alpha/n)(1+\varepsilon)} dx = \infty,$$

只要$\frac{\alpha}{n}(1+\varepsilon)\leqslant 1$。

作了上述考虑之后，我们可以写出正面的定理：**分数次积分的 Hardy-Littlewood-Sobolev 定理**。

**定理1** 设$0<\alpha<n$，$1\leqslant p<q<\infty$，$\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$。

(a) 若$f\in L^p(R^n)$，则定义$I_\alpha(f)$的积分（4）对几乎所有的$x$绝对收敛。

(b) 若加上$1<p$，则

$$\|I_\alpha(f)\|_q \leqslant A_{p,q}\|f\|_p.$$

(c) 若$f\in L^1(R^n)$，则

$$m\{x:\ |I_\alpha(f)|>\lambda\} \leqslant \left(\frac{A\|f\|_1}{\lambda}\right)^q$$

对所有 $\lambda>0$ 成立。也就是说，变换 $f\to I_\alpha(f)$ 是弱$(1,q)$型的$(1/q=1-\alpha/n)$。

**1.3 定理1的证明** 记 $K(x)=|x|^{-n+\alpha}$。代替 $f\to I_\alpha(f)$，我们考虑 $f\to K*f$，它们间的差别只是一个常数倍。让我们分解 $K$ 为 $K_1+K_\infty$，其中

$$K_1(x)=K(x),\ 当\ |x|\leqslant\mu;\quad K_1(x)=0,\ 当\ |x|>\mu.$$
$$K_\infty(x)=K(x),\ 当\ |x|>\mu;\quad K_\infty(x)=0,\ 当\ |x|\leqslant\mu.$$

在这里 $\mu$ 是一个待定的正常数。我们有

$$K*f=K_1*f+K_\infty*f.$$

表示 $K_1*f$ 的积分**几乎处处**绝对收敛，因为它是 $L^1$ 函数$(K_1)$与 $L^p$ 函数的卷积。类似地，表示 $K_\infty*f$ 的积分**处处**收敛，因为它是 $L^p$ 函数$(f)$与共轭空间 $L^{p'}$ 的函数 $(K_\infty)$ 的卷积，事实上，如果

$$\frac{1}{p}+\frac{1}{p'}=1,$$

那末

$$\|K_\infty\|_{p'}^{p'}=\int_{|x|\geqslant\mu}|x|^{(-n+\alpha)p'}dx<\infty,$$

这是因为 $(-n+\alpha)p'<-n$ 等价于 $q<\infty$。因此定理的(a)得证。

下面我们用一个类似的但较详尽的理由来证明，如果

$$1\leqslant p<q<\infty,\qquad \frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n},$$

那末变换 $f\to K*f$ 是弱$(p,q)$型的，其意义是指

$$(14)\qquad m\{x:\ |K*f|>\lambda\}\leqslant\left(A_{p,q}\frac{\|f\|_p}{\lambda}\right)^q,$$

$f\in L^p(\boldsymbol{R}^n)$，对所有 $\lambda>0$。

我们注意，不妨在不等式的左边用 $2\lambda$ 代替 $\lambda$，并对 $\|f\|_p=1$，证明不等式(14)就可以了。现在

$$m\{x:|K*f|>2\lambda\}$$
$$\leqslant m\{x:|K_1*f|>\lambda\}+m\{x:|K_\infty*f|>\lambda\},$$

这是由于$K*f=K_1*f+K_\infty*f$。然而

$$m\{x: |K_1*f|>\lambda\}\leqslant\frac{\|K_1*f\|_p^p}{\lambda^p}\leqslant\frac{\|K_1\|_1^p\|f\|_p^p}{\lambda^p}=\frac{\|K_1\|_1^p}{\lambda^p}.$$

但是

$$\|K_1\|_1=\int_{|x|<\mu}|x|^{-n+\alpha}dx=c_1\mu^\alpha.$$

另外，

$$\|K_\infty*f\|_\infty\leqslant\|K_\infty\|_{p'}\|f\|_p\leqslant\|K_\infty\|_{p'}.$$

然而

$$\|K_\infty\|_{p'}=\left(\int_{|x|>\mu}(|x|^{-n+\alpha})^{p'}dx\right)^{1/p'}=c_2\mu^{-n/q}.$$

因此，$\|K_\infty\|_{p'}=\lambda$，只要$c_2\mu^{-n/q}=\lambda$，即$\mu=c_3\lambda^{-q/n}$。故只要这样选择$\mu$，就有$\|K_\infty*f\|_\infty\leqslant\lambda$，即$m\{x: |K_\infty*f|>\lambda\}=0$。从而

$$m\{x: |K*f|>2\lambda\}\leqslant\left(c_1\frac{\mu^\alpha}{\lambda}\right)^p=c_4\mu^{-q}=c_4\left(\frac{\|f\|_p}{\lambda}\right)^q$$

（由于$\|f\|_p=1$）。这就是式(14)，故变换$f\to K*f$是弱$(p,q)$型的。特别地取$p=1$，就给出定理 的(c)，而(b)则由Marcinkiewicz 内插定理推出（见附录B）。

1.4 评注 回顾定理1的证明，作下面的评注是合适的。在定理1的证明中关于算子$f\to K*f$要确定的东西并不是核$K$的特殊构造。实际上涉及到的只是$K$的分布函数（用第一章的术语）。一个比较仔细的检验将表明，我们只需要这样的事实

$$m\{x: |K(x)|>\lambda\}\leqslant A\lambda^{-n/(n-\alpha)},$$

也就是说，核是“弱$n/(n-\alpha)$型”的。

如果我们有更强一些的假设$K\in L^{n/(n-\alpha)}$，我们就得到结果

$$\|K*f\|_q\leqslant A\|f\|_p,\quad \frac{1}{q}=\frac{1}{p}+\frac{1}{r}-1,\ r=\frac{n}{n-\alpha}.$$

这本质上就是熟知的Young不等式，它在$p=1$或$q=\infty$时也成立（见附录A）。

## §2 Sobolev 空间，$L_k^p(R^n)$

现在我们来研究函数同它的偏微商的关系。我们将要用到的偏微商的概念是由广义函数论给出的一般概念，并且合适的定义是通过 $R^n$ 上无限次可微且具有紧支集的函数空间$\mathscr{D}$给出的。

令

$$\frac{\partial^\alpha}{\partial x^\alpha}=\frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}$$

是一微分单项式，它的阶是$|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n$。假设给定两个 $R^n$ 上局部可积的函数 $f$ 与 $g$。如果

$$(15)\qquad \int_{R^n} f(x)\frac{\partial^\alpha\varphi(x)}{\partial x^\alpha}dx=(-1)^{|\alpha|}\int_{R^n} g(x)\varphi(x)dx,$$

对所有 $\varphi\in\mathscr{D}$，那末就说 $\dfrac{\partial^\alpha f}{\partial x^\alpha}=g$（如果需要避免混淆，就加上说明“在弱意义下”）。

分部积分表明，若 $f$ 有直到$|\alpha|$阶的连续偏微商，并且按通常意义$\dfrac{\partial^\alpha f}{\partial x^\alpha}=g$，则我们期望的这个关系式(15)确实是成立的。

自然，并非每一个局部可积函数在这种意义下皆有微商；例如，考虑 $f(x)=e^{i/|x|^n}$。然而，当偏微商存在时，它们由(15)几乎处处完全决定。

对任意非负整数 $k$，Sobolev 空间 $L_k^p(R^n)=L_k^p$ 定义为函数$f$的空间，其中 $f\in L^p(R^n)$，并且所有在上述意义下的$\dfrac{\partial^\alpha f}{\partial x^\alpha}$存在且$\dfrac{\partial^\alpha f}{\partial x^\alpha}\in L^p(R^n)$，只要$|\alpha|\leqslant k$。这个函数空间可以通过表达式

(16) $$\|f\|_{L_k^p}=\sum_{|\alpha|\leq k}\left\|\frac{\partial^\alpha f}{\partial x^\alpha}\right\|_p \qquad \left(\frac{\partial^0 f}{\partial x^0}=f\right)$$

成为赋范空间。

这样得到的赋范空间是完备的。证明如下。若$\{f_m\}$是$L_k^p$的Cauchy序列，则对每个$\alpha$，$|\alpha|\leqslant k$，$\left\{\frac{\partial^\alpha f_m}{\partial x^\alpha}\right\}$是Cauchy序列。如果

$$f^{(\alpha)}=\lim_m\frac{\partial^\alpha f_m}{\partial x^\alpha}$$

（按$L^p$模取极限），那末显然对任意的$\varphi\in\mathscr{D}$有

$$\int_{R^n}f\frac{\partial^\alpha\varphi}{\partial x^\alpha}dx=-(-1)^{|\alpha|}\int_{R^n}f^{(\alpha)}\varphi dx,$$

这就是所要证明的。

$L_k^p$函数的一个等价特征用起来常常是很方便的，这个特征并不明显包含在(15)给出的弱微商概念中。

**命题 1** 设$1\leqslant p<\infty$。则$f\in L_k^p$当且仅当存在序列$\{f_m\}$，使得

(a) 每个$f_m\in\mathscr{D}$；

(b) $\|f-f_m\|_p\to 0$；

(c) 对每个$\alpha$，$|\alpha|\leqslant k$，$\left\{\frac{\partial^\alpha f_m}{\partial x^\alpha}\right\}$按$L^p$模收敛。

条件(a)，(b)与(c)的充分性是显然的。事实上，令

$$f^{(\alpha)}=\lim_{m\to\infty}\frac{\partial^\alpha f_m}{\partial x^\alpha},\qquad f^{(0)}=f;$$

由于

$$\int f_m\frac{\partial^\alpha\varphi}{\partial x^\alpha}dx=(-1)^{|\alpha|}\int\frac{\partial^\alpha f_m}{\partial x^\alpha}\varphi dx,$$

我们得到

$$\int f\frac{\partial^\alpha\varphi}{\partial x^\alpha}dx=(-1)^{|\alpha|}\int f^{(\alpha)}\varphi dx.$$

这就证明了 $f\in L^p_k$。

反过来是更有意思的。证明所要求的推理在包含正则化方法的大量问题中是典型的。

为此设 $\psi$ 是 $\mathscr{D}$ 中的一个固定函数，满足

$$\int_{R^n}\psi(x)dx=1.$$

对每个 $\varepsilon>0$，考虑 $\psi_\varepsilon(x)$，由 $\psi_\varepsilon(x)=\varepsilon^{-n}\psi(x/\varepsilon)$定义，并对每个 $f\in L^p$，令 $f_\varepsilon=f*\psi_\varepsilon$。函数族$\{f_\varepsilon\}$是 $f$ 的正则化，这是指：

(a) $\|f_\varepsilon-f\|_p\to 0$，当$\varepsilon\to 0$；

(b) 每个 $f_\varepsilon$ 都是无限次可微的；

(c) 若 $f$ 有偏微商$\dfrac{\partial^\alpha f}{\partial x^\alpha}$（在弱意义下），则

$$\frac{\partial^\alpha f_\varepsilon}{\partial x^\alpha}=\left(\frac{\partial^\alpha f}{\partial x^\alpha}\right)_\varepsilon=\frac{\partial^\alpha f}{\partial x^\alpha}*\psi_\varepsilon.$$

(a) 在更一般的条件下即 $\psi$ 可积时成立，这在第三章 § 2.2 已经看到。

(b) 由于

$$f_\varepsilon(x)=\int_{R^n}f(y)\psi_\varepsilon(x-y)dy,$$

用积分号下求微商，显然$f_\varepsilon$是无限次可微的。

(c) 让我们导出它的微分

$$\frac{\partial^\alpha}{\partial x^\alpha}f_\varepsilon(x)=\int_{R^n}f(y)\frac{\partial^\alpha}{\partial x^\alpha}(\psi_\varepsilon(x-y))dy$$

$$=(-1)^{|\alpha|}\int_{R^n}f(y)\frac{\partial^\alpha}{\partial y^\alpha}(\psi_\varepsilon(x-y))dy.$$

对每个 $x$，函数$y\to\psi_\varepsilon(x-y)$属于$\mathscr{D}$，因此用定义(15)就得到

$$\frac{\partial^\alpha}{\partial x^\alpha}f_\varepsilon(x)=\int\left(\frac{\partial^\alpha f}{\partial y^\alpha}\right)\psi_\varepsilon(x-y)dy=\frac{\partial^\alpha f}{\partial x^\alpha}*\psi_\varepsilon,$$

我们现在可以应用性质(a)到$\frac{\partial^\alpha f}{\partial x^\alpha}*\psi_\varepsilon$。我们看出$\frac{\partial^\alpha f_\varepsilon}{\partial x^\alpha}$当$\varepsilon\to 0$时按$L^p$模收敛。因此函数$\{f_\varepsilon\}$给出了所要求的逼近，除了它们中每个并非具有紧支集以外，故需要做最后的修改。设$\eta$是一固定的无穷次可微且具有紧支集的函数，满足$\eta(0)=1$。考虑双参数族$\{\eta(\delta x)f_\varepsilon(x)\}$，$\varepsilon>0$，$\delta>0$。首先，选择$\varepsilon$使得$\frac{\partial^\alpha f_\varepsilon}{\partial x^\alpha}$充分接近它的极限。其次，对固定后的$\varepsilon$，选择$\delta$充分小使得$\frac{\partial^\alpha}{\partial x^\alpha}(\eta(\delta x)f_\varepsilon(x))$充分接近于$\frac{\partial^\alpha f_\varepsilon}{\partial x^\alpha}$。由于每个$\eta(\delta x)f_\varepsilon(x)$是无限次可微且具有紧支集，命题由此得证。

对于$p=\infty$的情形，存在一个平行的命题，但它需要作常规的修改，因为光滑函数在$L^\infty(\boldsymbol{R}^n)$空间不是稠密的。

对$k=1$而$1\leqslant p\leqslant\infty$，或对一般的$k$而$p=\infty$，可以在下面的§6.1与§6.2找到其它的特征。

2.1.1 就命题1证明中谈到的，$\psi$具有紧支集的要求并不是必要的。我们可以，例如取

$$\psi(x)=\frac{c_n}{(1+|x|^2)^{(n+1)/2}}$$

来实现证明(在优美方面作一点小牺牲)。这时$f_\varepsilon=f*\psi_\varepsilon$等于$u(x,\varepsilon)$，其中$u(x,y)$是$f$的Poisson积分(见第三章§2)。

在其它一些问题中(例如第四章§3的3.2.4)用紧支集的$\psi$作正则化起着最本质的作用。这种正则化具有这样的性质，$f_\varepsilon(x)$仅仅依赖于$f$在$x$的一个小邻域上的值。

**2.2 Sobolev 定理** 刚才考虑的Sobolev空间的重要性在于，通过它们我们可以用一个相对简单的方法描写出，偏微商"大小"的限制如何决定了问题中函数的相应的性质。一个一般性的定理可以叙述如下。

**定理2** 设$k$是正整数，而

$$\frac{1}{q}=\frac{1}{p}-\frac{k}{n}.$$

(a) 若 $q<\infty$(即 $p<n/k$)，则$L_k^p(\boldsymbol{R}^n)\subset L^q(\boldsymbol{R}^n)$，并且自然的包含变换是连续的；

(b) 若$q=\infty$(即 $p=n/k$)，则$f\in L_k^p(\boldsymbol{R}^n)$在$\boldsymbol{R}^n$紧子集上的限制属于$L^r(\boldsymbol{R}^n)$，对任意$r<\infty$；

(c) 若$p>n/k$，则对每个$f\in L_k^p(\boldsymbol{R}^n)$，可以修改它在一零测集上的值，使所得到的函数是连续的。

**2.3** 为了证明这定理，要求我们找到一个合适的方法，通过函数的偏微商来表示函数。为此，我们提供一个纯粹形式的方法。运算永远是在类$\mathscr{S}$(或$\mathscr{D}$)的函数上进行。

对检验函数 $f$，我们来考虑它的 Fourier 变换。这时$\dfrac{\partial f}{\partial x_j}$的 Fourier 变换是 $-2\pi ix_j\hat{f}(x)$。现在回忆第三章§1.2的 Riesz 变换。$R_j$的效果在 Fourier 变换那里来看就是用$\dfrac{ix_j}{|x|}$作乘法（见公式(8)）。因此

$$\left(R_j\left(\frac{\partial}{\partial x_j}f\right)\right)^{\wedge}(x)=2\pi\frac{x_j^2}{|x|}\hat{f}(x).$$

根据公式(3)我们就得到

$$f=I_1\left(\sum_{j=1}^{n}R_j\left(\frac{\partial}{\partial x_j}f\right)\right). \tag{17}$$

这个通过一阶偏微商表示函数的公式包含了两个因素：Riesz 变换与一阶位势。前者是在$L^p(\boldsymbol{R}^n)$封闭的算子。后者变$L^p(\boldsymbol{R}^n)$为$L^q(\boldsymbol{R}^n)$，对合适的 $p$ 与 $q$（定理1)。这一点揭示了隐藏在这定理背后的本质技巧。

然而，存在一个与等式(17)密切相关的比较简单的途径，但它避免用到 Riesz 变换的太深的理论，这是基于初等等式

(18) $$f(x)=\frac{1}{\omega_{n-1}}\sum_{j=1}^{n}\int_{R^n}\frac{\partial f(x-y)}{\partial x_j}\frac{y_j}{|y|^n}dy,$$

其中$\omega_{n-1}$是球$S^{n-1}$的“面积”。

公式(18)证明如下。我们从一维公式

$$f(x)=\int_0^\infty f'(x-t)dt$$

出发，当$f$是检验函数时它显然成立。它的$n$维类似是它的一个直接推论，这就是

(19) $$f(x)=\int_0^\infty(\nabla f(x-\xi t),\xi)dt,$$

其中$\xi$是任意的单位向量，$\nabla f$是分量为

$$\left(\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\cdots,\frac{\partial f}{\partial x_n}\right)$$

的向量。在(19)中对$\xi$在单位球上进行积分，得到

$$f(x)=\frac{1}{\omega_{n-1}}\int_{\xi\in S^{n-1}}\int_0^\infty\nabla(f(x-\xi t),\xi)dtd\xi.$$

从极坐标变换到直角坐标就知公式(18)成立。公式(18)将在下面应用。然而，值得费点时间利用这个机会指出与(17)同类的一些其它公式，当然也只是一样形式地给出来。

首先，假如我们想通过二阶偏微商来表示$f$，那末我们可以通过一个特殊的偏微商组合

$$\frac{\partial^2 f}{\partial x_1^2}+\frac{\partial^2 f}{\partial x_2^2}+\cdots+\frac{\partial^2 f}{\partial x_n^2}=\Delta f$$

来做到这一点。这等式就是

$$f=-I_2(\Delta f),$$

它刚巧是公式(7)的一个特殊情形。用这办法我们得到了位势理论的经典公式①。

① 至少在$n\geqslant 3$的情形，$n=2$的情形可以通过一个极限过程单独研究，并通过对数位势给出表达式。

另外一个有用的公式是

(20) $$\text{若}F=I_1(f)\text{，则}\frac{\partial F}{\partial x_j}=-R_j(f).$$

事实上根据引理1，取$g=\frac{\partial\varphi}{\partial x_j}$，我们有

$$\int I_1(f)\overline{\frac{\partial\varphi}{\partial x_j}}dx=\int\hat{f}(x)(2\pi|x|)^{-1}2\pi i x_j\overline{\hat{\varphi}(x)}dx$$

$$=\int\hat{f}(x)i\frac{x_j}{|x|}\overline{\hat{\varphi}}(x)dx=\int R_j(f)\varphi dx.$$

因此

$$\int I_1(f)\frac{\partial\varphi}{\partial x_j}dx=\int R_j(f)\varphi dx,$$

从而式(20)得证，至少当$f\in\mathscr{S}$。

在需要的时候(见下面§6.3)，还有可能推广(20)到较广的函数类。

**2.4** 我们首先对$k=1$, $1<p,q<\infty$证明定理。假设$f\in\mathscr{D}$。这时等式(18)表明

$$|f(x)|\leqslant A\sum_{j=1}^{n}\int_{R^n}\left|\frac{\partial f(x-y)}{\partial x_j}\right||y|^{-n+1}dy.$$

因此用定理1($\alpha=1$的情形)我们有

(21) $$\|f\|_q\leqslant A'\sum_{j=1}^{n}\left\|\frac{\partial f}{\partial x_j}\right\|_p,\qquad\frac{1}{q}=\frac{1}{p}-\frac{1}{n}.$$

现在，设$f$是$L_1^p(R^n)$中任意函数。根据命题1，存在$\mathscr{D}$的元素序列$\{f_m\}$，使得按$L^p$模$f_m\to f$，并且$\frac{\partial f_m}{\partial x_j}$按$L^p$模收敛，极限$\lim\limits_{m\to\infty}\frac{\partial f_m}{\partial x_j}$必须等于$\frac{\partial f}{\partial x_j}$，因为

$$\lim_m \int \frac{\partial f_m}{\partial x_j} \varphi dx = -\lim_m \int f_m \frac{\partial \varphi}{\partial x_j} dx = -\int f \frac{\partial \varphi}{\partial x_j} dx$$

$$= \int \frac{\partial f}{\partial x_j} \varphi dx, \quad 对每个\varphi \in \mathscr{D}.$$

代入式(21)我们得到

$$\|f_m - f_{m'}\|_q \leqslant A' \sum_{j=1}^{n} \left\| \frac{\partial f_m}{\partial x_j} - \frac{\partial f_{m'}}{\partial x_j} \right\|_p,$$

因此序列$f_m$也按$L^q$收敛，而且极限必定也等于$f$。故$f \in L^q(\boldsymbol{R}^n)$，还有

$$\|f\|_q \leqslant A' \sum_{j=1} \left\| \frac{\partial f}{\partial x_j} \right\|_p \leqslant A' \|f\|_{L_1^p(\boldsymbol{R}^n)}, \quad f \in L_1^p(\boldsymbol{R}^n).$$

这就证明了$f \in L^q(\boldsymbol{R}^n)$，并且从$L_1^p(\boldsymbol{R}^n)$到$L^q(\boldsymbol{R}^n)$的包含变换是连续的。

其次我们考虑$k=1$但$q=\infty$($p=n/k=n$)或$p>n/k=n$的情形。在这两种情形中定理2有关的结论(即(b)与(c))都有局部的特征，因而我们可以简单地把问题归结为$f$(从而它的偏微商)具有紧支集的情形。这样，给定任意紧集$K$，令$\eta$是在$K$上等于1的$\mathscr{D}$内的函数。对于$f \in L_1^p(\boldsymbol{R}^n)$，考虑$\eta \cdot f$。

充分地我们只要对显然还属于$L_1^p(\boldsymbol{R}^n)$的$\eta f$证明结论(b)与(c)。为了证明$\eta f \in L_1^p(\boldsymbol{R}^n)$，只要验证微商$\frac{\partial}{\partial x_j}(\eta f)$在弱意义下等于

$$\frac{\partial \eta}{\partial x_j} f + \eta \frac{\partial f}{\partial x_j}.$$

然而

$$\int \frac{\partial}{\partial x_j}(\eta f) \varphi dx = -\int \eta f \frac{\partial \varphi}{\partial x_j} dx$$

$$= -\int f\frac{\partial(\varphi\eta)}{\partial x_j}dx + \int\varphi f\frac{\partial\eta}{\partial x_j}dx$$

$$= \int\left(\frac{\partial\eta}{\partial x_j}f + \eta\frac{\partial f}{\partial x_j}\right)\varphi dx,$$

这就是所要证的。

我们因此从$f\in L_1^p(R^n)$与它的由命题1给定的逼近序列$\{f_m\}$出发。显然，$\eta f_m$是$\eta f$的逼近序列。现在选择$R$充分大，使得当$K_1$是$\eta$的紧支集时，$K_1-K$包含在以原点为中心，以$R$为半径的球内。

从等式(18)又推出

$$(22)\quad |\eta(x)f_m(x)|\leqslant A\sum_{j=1}^{n}\int_{|y|\leqslant R}\left|\frac{\partial(\eta f_m)}{\partial x_j}(x-y)\right||y|^{-n+1}dy,$$

$$x\in K.$$

我们现在用Young不等式，它断言，若

$$\mathscr{C}(x)=\int_{R^n}\mathscr{A}(x-y)\mathscr{B}(y)dy,$$

则$\|\mathscr{C}\|_r\leqslant\|\mathscr{A}\|_p\|\mathscr{B}\|_s$，其中$1/r=1/p+1/s-1$。我们令

$$\mathscr{A}=A\sum_{j=1}^{n}\left|\frac{\partial(\eta f_m)}{\partial x_j}\right|;$$

$\mathscr{B}(y)=|y|^{-n+1}$，当$|y|\leqslant R$，而$\mathscr{B}(y)=0$在其它地方；再令$s$是任意指数$<n/(n-1)$。注意$\|\mathscr{B}\|_s<\infty$，这是由于

$$\|\mathscr{B}\|_s^s=\int_{|y|\leqslant R}|y|^{(-n+1)s}dy<\infty,$$

因为$(-n+1)s>-n$。

因此Young不等式表明

$$\int_K|f_m|^r dx\leqslant\int|\eta f_m|^r dx\leqslant\|\mathscr{C}\|_r^r$$

$$\leqslant A'\left\|\sum_{j=1}^{n}\left|\frac{\partial(\eta f_m)}{\partial x_j}\right|\right\|_p^r.$$

类似地

$$\int_K|f_m-f_{m'}|^r dx\leqslant A'\left\|\sum_{j=1}^{n}\left|\frac{\partial(\eta(f_m-f_{m'}))}{\partial x_j}\right|\right\|_p^r.$$

这样，我们看到，按$L^p$模收敛的序列$\{f_m\}$也按$L^r$模收敛，只要限制在集合$K$上。因此限制在$K$上时$f$属于$L^r$。

在$p=n$时，条件$s<n/(n-1)$与$r<\infty$相同，因为

$$\frac{1}{r}=\frac{1}{p}+\frac{1}{s}-1.$$

所以定理的断言(b)得证(自然，假设$k=1$)。

(c)的证明与刚才进行的十分类似，所不同的是在这里我们要用估计式

$$\sup_x|\mathscr{C}(x)|\leqslant\|\mathscr{A}\|_p\|\mathscr{B}\|_{p'},\qquad \frac{1}{p}+\frac{1}{p'}=1,$$

它显然能从Hölder不等式推出。注意，如果$p>n$，那末

$$\|\mathscr{B}\|_{p'}^{p'}=\int_{|y|\leqslant R}|y|^{(-n+1)p'}\,dy<\infty.$$

因此，类似于上述，我们得到

$$\sup_{x\in K}|f_m(x)-f_{m'}(x)|\leqslant A'\left\|\sum_{j=1}^{n}\left|\frac{\partial\eta(f_m-f_{m'})}{\partial x_j}\right|\right\|_p.$$

这表明，连续函数$\{f_m(x)\}$在任意紧集上一致收敛，因而$f$可以看成连续的。

**2.5 情形 $p=1$** 在$k=1$的假设下，定理的结论除了例外情形$p=1$外已全部证明了。所用过的推理在这时自然是不能用的，因为定理1的结论(b)对$p=1$是不成立的。这时需要不同的想法，而它来自下面精细的不等式

$$\text{(23)}\qquad \|f\|_q\leqslant\left(\prod_{j=1}^{n}\left\|\frac{\partial f}{\partial x_j}\right\|_1\right)^{1/n},\qquad \frac{1}{q}=1-\frac{1}{n},\ f\in\mathscr{D}.$$

我们对 $n$ 用归纳法证明不等式(23)。$n=1$时是显然的，因为它意味着$\|f\|_\infty\leqslant\|f'\|_1$，而这可以从

$$f(x)=\int_{-\infty}^{x}f'(t)dt$$

直接推出。

我们假设不等式(23)对$n-1$成立。为了用归纳法，我们把$x\in\boldsymbol{R}^n$写成$x=(x_1,x')$，$x'\in\boldsymbol{R}^{n-1}$，$x_1\in\boldsymbol{R}^1$。我们令

$$I_j(x_1)=\int_{\boldsymbol{R}^{n-1}}\left|\frac{\partial f(x_1,x')}{\partial x_j}\right|dx',\qquad j=2,\cdots,n$$

与

$$I_1(x')=\int_{\boldsymbol{R}^1}\left|\frac{\partial f(x_1,x')}{\partial x_1}\right|dx_1.$$

现在假设 $q$ 是对应于 $n$ 的指标($q=n/(n-1)$)，而$q'$是对应于$n-1$的指标($q'=(n-1)/(n-2)$)。用$n-1$的假设，我们有

$$\text{(24)}\qquad\left(\int_{\boldsymbol{R}^{n-1}}|f(x_1,x')|^{q'}dx'\right)^{1/q'}\leqslant\left(\prod_{j=2}^{n}I_j(x_1)\right)^{1/(n-1)}.$$

显然，$|f(x)|\leqslant I_1(x')$(这又一次用一维的结果)。因此

$$|f|^q\leqslant(I_1(x'))^{1/(n-1)}|f|,$$

这是由于

$$q=\frac{1}{n-1}+1.$$

对共轭指标$n-1$与$q'$用Hölder不等式，就有

$$\int_{\boldsymbol{R}^{n-1}}|f|^q dx\leqslant\int_{\boldsymbol{R}^{n-1}}(I_1(x'))^{1/(n-1)}|f|dx'$$

$$\leqslant\left(\int_{\boldsymbol{R}^{n-1}}I_1(x')dx'\right)^{\frac{1}{n-1}}\left(\int_{\boldsymbol{R}^{n-1}}|f|^{q'}dx'\right)^{1/q'}.$$

把(24)代入上式，得到

$$\int_{R^{n-1}} |f|^q dx' \leqslant \left(\int_{R^{n-1}} I_1(x')dx'\right)^{\frac{1}{n-1}} \left(\prod_{j=2}^{n} J_j(x_1)\right)^{\frac{1}{n-1}}.$$

对$x_1$积分并再次用Hölder不等式，就有

$$\int_{R^1}\left(\prod_{j=2}^{n} I_j(x_1)\right)^{1/(n-1)} dx_1 \leqslant \prod_{j=2}^{n}\left(\int_{R^1} I_j(x_1)dx_1\right)^{1/(n-1)}$$

$$\leqslant \prod_{j=2}^{n}\left\|\frac{\partial f}{\partial x_j}\right\|_1^{1/(n-1)}.$$

最后的结果是

$$\int_{R^n} |f|^q dx \leqslant \left(\prod_{j=1}^{n}\left\|\frac{\partial f}{\partial x_j}\right\|_1\right)^{1/(n-1)},$$

它就是所要求的不等式(23)，因为$q=n/(n-1)$。

如果用这样的事实，

$$\left(\prod_{j=1}^{n} a_j\right)^{1/n} \leqslant \frac{1}{n}\sum_{j=1}^{n} a_j, \qquad a_j \geqslant 0,$$

那末作为不等式(23)的一个推论就有

$$\|f\|_q \leqslant \frac{1}{n}\sum_{j=1}^{n}\left\|\frac{\partial f}{\partial x_j}\right\|_1, \qquad f\in\mathscr{D}, \quad \frac{1}{q}=1-\frac{1}{n}.$$

这个结果与§2.4用到的理由表明，$L_1^1(\boldsymbol{R}^n)\subset L^q(\boldsymbol{R}^n)$，并且包含变换是连续的。

**2.6** 为了得到定理的结论，我们可以用归纳法，证明$k\geqslant 2$的情形可以归结为$k=1$。例如，让我们看定理的断言(a)。显然$f\in L_k^p(\boldsymbol{R}^n)$的假设蕴含了$f\in L_{k-1}^p(\boldsymbol{R}^n)$与$\frac{\partial f}{\partial x_j}\in L_{k-1}^p(\boldsymbol{R}^n)$。因此，定理的$k-1$情形蕴含了$f\in L^{q'}(\boldsymbol{R}^n)$与$\frac{\partial f}{\partial x_j}\in L^{q'}(\boldsymbol{R}^n)$，其中

$$\frac{1}{q'}=\frac{1}{p}-\frac{k-1}{n},$$

即$f\in L_1^{q'}(\mathbf{R}^n)$。由$k=1$的情形又推出$f\in L^q(\mathbf{R}^n)$，其中

$$\frac{1}{q}=\frac{1}{q'}-\frac{1}{n}=\frac{1}{p}-\frac{k-1}{n}-\frac{1}{n}=\frac{1}{p}-\frac{k}{n}.$$

相应的包含变换也都是连续的。(b)与(c)可以类似地证明。

最后，还需要作一点说明。定理 2 对$p=1$成立，这和与之密切相关的定理 1 不同。然而当$q=\infty$时(情形(b))，一般说来结论$f\in L^\infty$是不正确的（这时情况又类似于定理 1）。详见本章末的§6.3。

## §3 Bessel 位势

正如我们已看到的那样，Riesz 位势$I_\alpha$导出了十分精细且有用的公式。然而，公式也有缺陷，这可以解释如下。上述位势$I_\alpha$的重要性在于他起着“光滑算子”的作用。为此目的，核$|x|^{-n+\alpha}/\gamma(\alpha)$的**局部性质**($|x|\to 0$)是很合适的，而**整体性质**($|x|\to\infty$)就不大合适，并且$\alpha$愈大就愈麻烦。

躲开这个窘境的办法是修改 Riesz 位势，既保留它重要的局部性质，又消除它在无穷远处的不合适的地方。存在好几种大致等价的方法来做这件事，而最简单又最自然的方法就是用“严格正”的算子$I-\Delta$（$I$是恒同算子）代替“非负”算子$-\Delta$，并且类似于

$$I_\alpha=(-\Delta)^{-\alpha/2},$$

定义 Bessel 位势$\mathscr{J}_\alpha$为

$$\mathscr{J}_\alpha=(I-\Delta)^{-\alpha/2}.$$

为使讨论按逻辑顺序进行，我们必须从推导Bessel位势的核开始，也就是说，找出函数$G_\alpha(x)$，满足

$$(G_\alpha(x))^\wedge=(1+4\pi^2|x|^2)^{-\alpha/2}.$$

推导的出发点是这样的想法(在第三章§3.2已经用过)，$|x|$的一个“一般的”函数可以通过$\{e^{-\pi\delta|x|^2}\}_{\delta>0}$表示。在这里我们的简单的恒等式正是

(25) $$(4\pi)^{-\alpha/2}(1+4\pi^2|x|^2)^{-\alpha/2}$$

$$=\frac{1}{\Gamma(\alpha/2)}\int_0^\infty e^{-\frac{\delta}{4\pi}(1+4\pi^2|x|^2)}\delta^{\alpha/2}\frac{d\delta}{\delta},\quad \alpha>0,$$

它无非是下述事实

$$t^{-a}=\frac{1}{\Gamma(a)}\int_0^\infty e^{-t\delta}\delta^a\frac{d\delta}{\delta}$$

的一个表现而已，其中取$a=\alpha/2>0$。

这样，我们可以列出下表，其中右边所列的是左边对应者的Fourier 变换，其中$a=\alpha/2$。

(i) $e^{-\pi|x|^2}$ $\qquad\qquad$ $e^{-\pi|x|^2}$

(ii) $e^{-\pi\delta|x|^2}$ $\qquad\qquad$ $e^{-\pi|x|^2/\delta}\delta^{-n/2}$

(iii) $\int_0^\infty e^{-\pi\delta|x|^2}\delta^a\frac{d\delta}{\delta}$ $\qquad\qquad$ $\int_0^\infty e^{-\pi|x|^2/\delta}\delta^{-n/2}\delta^a\frac{d\delta}{\delta}$

(iv) $\Gamma(a)(\pi|x|^2)^{-a}$ $\qquad\qquad$ $\Gamma\left(\frac{n}{2}-a\right)\left(\pi|x|^2\right)^{a-\frac{n}{2}}$

由(25)与(ii)，

(v) $(1+4\pi^2|x|^2)^{-\alpha/2}$ $\qquad\qquad$ $\frac{1}{(4\pi)^{\alpha/2}}\frac{1}{\Gamma(\alpha/2)}\int_0^\infty e^{-\pi|x|^2/\delta}\times e^{-\delta/4\pi}\delta^{(-n+\alpha)/2}\frac{d\delta}{\delta}$

因此我们定义$G_\alpha(x)$为

(26) $$G_\alpha(x)=\frac{1}{(4\pi)^{\alpha/2}}\frac{1}{\Gamma(\alpha/2)}\times\int_0^\infty e^{-\pi|x|^2/\delta}e^{-\delta/4\pi}\delta^{(-n+\alpha)/2}\frac{d\delta}{\delta}.$$

**命题 2**

(a) 对每个$\alpha>0$，$G_\alpha(x)\in L^1(\boldsymbol{R}^n)$；

(b) $G_\alpha^\wedge(x)=(1+4\pi^2|x|^2)^{-\alpha/2}$.

**证明** 由于

$$\int_{R^n} e^{-\pi|x|^2/\delta}dx=\delta^{n/2},$$

应用Fubini定理到等式(26)就有

$$\int_{R^n} G_\alpha(x)dx=\frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)}\int_0^\infty e^{-\pi/4\delta}\delta^{\alpha/2}\frac{d\delta}{\delta}=1,\ \alpha>0,$$

因而第一个结论得证。为了证明第二个结论，我们用上述(i)—(v)的论证。事实上，如果我们让$f$代表上表中左边所列的，而$\hat{f}$表示右边的对应者，那末对$\varphi\in\mathscr{S}$，我们有

$$\int_{R^n} f(x)\ \phi(x)dx=\int_{R^n}\hat{f}(x)\varphi(x)dx. \tag{27}$$

首先，我们取 $f(x)=e^{-\pi\delta|x|^2}$，$\hat{f}(x)=e^{-\pi|x|^2/\delta}\delta^{-n/2}$；然后 $f(x)=e^{-\delta/4\pi}e^{-\pi\delta|x|^2}$，$\hat{f}(x)=e^{-\delta/4\pi}e^{-\pi|x|^2/\delta}\delta^{-n/2}$。对于这样选择的$f$与$\hat{f}$，在等式两边对$\delta^{\alpha/2}\frac{d\delta}{\delta}$积分(见式(25))。交换积分次序(用Fubini定理)，就有

$$\int_{R^n}(1+4\pi^2|x|^2)^{-\alpha/2}\varphi(x)dx=\int_{R^n} G_\alpha(x)\ \phi(x)dx.$$

由于$G_\alpha\in L^1(\boldsymbol{R}^n)$，这就证明了$\hat{G}_\alpha(x)=(1+4\pi^2|x|^2)^{-\alpha/2}$。

一个也可以从上表推出的类似于等式(26)的结果是

$$\frac{|x|^{-n+\alpha}}{\gamma(\alpha)}=\frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)}\int_0^\infty e^{-\pi|x|^2/\delta}\delta^{(-n+\alpha)/2}\frac{d\delta}{\delta}. \tag{28}$$

如果我们用$e^{-\delta/4\pi}=1+o(e^{-\delta/4\pi})$，$\delta\to0$，再比较等式(26)与(28)，我们得到

$$(29)\qquad G_\alpha(x)=\frac{|x|^{-n+\alpha}}{\gamma(\alpha)}+o(|x|^{-n+\alpha}),\qquad |x|\to 0,$$

只要$0<\alpha<n$。

对定义$G_\alpha(x)$的积分的一个直接检验还可证明

$$(30)\qquad G_\alpha(x)=O(e^{-c|x|}),\qquad |x|\to\infty,$$

对某个$c>0$，因此核$G_\alpha$当$|x|\to\infty$是速降的。

事实上$e^{-\pi|x|^2/\delta}e^{-\delta/4\pi}$有极大值$e^{-|x|}$（它在$\delta=2\pi|x|$达到）。又当$|x|\geqslant 1$时，显然

$$e^{-\pi|x|^2/\delta}e^{-\delta/4\pi}\leqslant e^{-\pi/\delta}e^{-\delta/4\pi}.$$

综合这两点，当$|x|\geqslant 1$时，有

$$e^{-\pi|x|^2/\delta}e^{-\delta/4\pi}\leqslant e^{-|x|/2}e^{-\pi/2\delta}e^{-\delta/8\pi}.$$

现在把这式代入公式(26)，就有

$$(4\pi)^{\alpha/2}\Gamma(\alpha/2)G_\alpha(x)\leqslant e^{-|x|/2}\int_0^\infty e^{-\pi/2\delta}e^{-\delta/8\pi}\delta^{(-n+\alpha)/2}\frac{d\delta}{\delta}.$$

这就是公式(30)，其中$c=1/2$。

3.1.1　核$G_\alpha$还可以有另外的积分表示。它表明，这本质上是"第三类"Bessel函数（见下面§6.5）；这是它的来源。然而，我们并不需要Bessel函数的任何性质，因此，术语"Bessel位势"对我们来说只有历史痕迹的意义。

**3.2　Riesz位势与Bessel位势的关系**　可以从定义本身、也可以从渐近关系(29)推测，在Bessel位势与Riesz位势之间存在密切的关系。这关系可用下面的引理准确地给出。

**引理2**　设$\alpha>0$

(a) 存在$R^n$上的有限测度$\mu_\alpha$，使得它的Fourier变换等于

$$\hat{\mu}_\alpha(x)=\frac{(2\pi|x|)^\alpha}{(1+4\pi^2|x|^2)^{\alpha/2}}.$$

(b) 存在$R^n$上的一对有限测度$\nu_\alpha$与$\lambda_\alpha$，使得

$$(1+4\pi^2|x|^2)^{\alpha/2}=\nu_\alpha^\wedge(x)+(2\pi|x|)^\alpha\lambda_\alpha^\wedge(x).$$

事实上，引理的第一部分是说下面的形式商算子

$$\frac{(-\Delta)^{\alpha/2}}{(I-\Delta)^{\alpha/2}},\qquad \alpha>0 \tag{31}$$

在每个$L^p(\boldsymbol{R}^n)$是有界的，其中$1\leqslant p\leqslant\infty$。第二部分是说，在某种意义下同样的结果对(31)的逆算子也是对的。

为了证明(a)，我们用展式

$$(1-t)^{\alpha/2}=1+\sum_{m=1}^{\infty}A_{m,\alpha}t^m, \tag{32}$$

它当$|t|<1$时成立。当$m$充分大时，$A_{m,\alpha}$是同号的，因此

$$\sum_m|A_{m,\alpha}|<\infty,$$

这是由于$(1-t)^{\alpha/2}$当$t\to1$时$(\alpha\geqslant0)$仍保持有界。令

$$t=\frac{1}{1+4\pi^2|x^2|},$$

这时

$$\left(\frac{4\pi^2|x|^2}{1+4\pi^2|x|^2}\right)^{\alpha/2}=1+\sum_{m=1}^{\infty}A_{m,\alpha}(1+4\pi^2|x|^2)^{-m}. \tag{33}$$

然而$G_{2m}(x)\geqslant0$并且

$$\int_{R^n}G_{2m}(x)e^{2\pi i x\cdot y}dx=(1+4\pi^2|y|^2)^{-m}.$$

我们曾指出过

$$\int_{R^n}G_{2m}(x)dx=1,$$

因而$\|G_{2m}\|_1=1$。

这样，从$\sum\limits_m|A_{m,\alpha}|$收敛推出，如果$\mu_\alpha$由

(34) $$\mu_\alpha = \delta_0 + \left(\sum_{m=1}^{\infty} A_{m,\alpha} G_{2m}(x)\right) dx$$

定义，其中 $\delta_0$ 是在原点的 Dirac 测度，那末 $\mu_\alpha$ 表示一有限测度。进一步，根据(33)

(35) $$\mu_\alpha^\wedge(x) = \frac{(2\pi|x|)^\alpha}{(1+4\pi^2|x|^2)^{\alpha/2}}.$$

现在我们借助 Wiener 定理的 $n$ 维形式，这就是：如果 $\Phi_1 \in L^1(\boldsymbol{R}^n)$，而 $\Phi_1^\wedge(x)+1$ 无处为0，那末存在 $\Phi_2 \in L^1(\boldsymbol{R}^n)$，使得

$$(\Phi_1^\wedge(x)+1)^{-1} = \Phi_2^\wedge(x)+1.$$

为了我们的目的，我们记

$$\Phi_1(x) = \sum_{m=1}^{\infty} A_{m,\alpha} G_{2m}(x) + G_\alpha(x).$$

由(35)我们看到

$$\Phi_1^\wedge(x) + 1 = \frac{(2\pi|x|)^\alpha + 1}{(1+4\pi^2|x|^2)^{\alpha/2}},$$

它无处为 0 。因此对一个合适的 $\Phi_2 \in L^1$，

$$(1+4\pi^2|x|^2)^{\alpha/2} = (1+(2\pi|x|)^\alpha)[\Phi_2^\wedge(x)+1],$$

从而得到所要求的结论，只要取 $\nu_\alpha = \lambda_\alpha = \delta_0 + \Phi_2(x)dx$。

**3.3 空间 $\mathscr{L}_\alpha^p$** 现在我们要比较系统地研究单参数算子族 $\{\mathscr{J}_\alpha\}_\alpha$。

对任意 $\alpha \geqslant 0$ 与 $f \in L^p(\boldsymbol{R}^n)$，$1 \leqslant p \leqslant \infty$，我们定义 $\mathscr{J}_\alpha(f)$ 为 $\mathscr{J}_\alpha(f) = G_\alpha * f$，当 $\alpha > 0$，而 $\mathscr{J}_0(f) = f$。注意到

$$\|G_\alpha\|_1 = 1\left(= \int_{\boldsymbol{R}^n} G_\alpha(x)dx\right),$$

我们看到这个卷积实际上是有定义的，并且

(36) $$\|\mathscr{J}_\alpha(f)\|_p \leqslant \|f\|_p, \quad 1 \leqslant p \leqslant \infty.$$

显然还有

(37) $$\mathscr{J}_\alpha\cdot\mathscr{J}_\beta=\mathscr{J}_{\alpha+\beta},\qquad \alpha\geqslant 0,\ \beta\geqslant 0.$$
这是由于 $G_\alpha*G_\beta=G_{\alpha+\beta}$，正如命题 2 所证明的。

我们在这里主要是要定义**位势空间** $\mathscr{L}^p_\alpha$。用符号写出来就是

(38) $$\mathscr{L}^p_\alpha(\boldsymbol{R}^n)=\mathscr{J}_\alpha(L^p(\boldsymbol{R}^n)),\qquad 1\leqslant p\leqslant\infty,\ \alpha\geqslant 0.$$

换句话说，$\mathscr{L}^p_\alpha(\boldsymbol{R}^n)$是$L^p(\boldsymbol{R}^n)$的子空间，包含所有可以写成 $f=\mathscr{J}_\alpha(g)$的 $f$，其中 $g\in L^p(\boldsymbol{R}^n)$，$f$ 的 $\mathscr{L}^p_\alpha$ 模记为 $\|f\|_{p,\alpha}$，它用 $g$ 的 $L^p$ 模定义，即

(39) $$\|f\|_{p,\alpha}=\|g\|_p,\qquad f=\mathscr{J}_\alpha(g).$$
为了说明这样给出的$\|f\|_{p,\alpha}$的定义是相容的， 我们只要注意，如果 $\mathscr{J}_\alpha(g_1)=\mathscr{J}_\alpha(g_2)$，那末 $g_1=g_2$。事实上，当 $\varphi\in\mathscr{S}$，用 Fubini 定理，有
$$\int\mathscr{J}_\alpha(g_1)\varphi(x)dx=\iint G_\alpha(x-y)g_1(y)\varphi(x)dxdy$$
$$=\int g_1\mathscr{J}_\alpha(\varphi)dx.$$
因此 $\mathscr{J}_\alpha(g_1)=\mathscr{J}_\alpha(g_2)$蕴含了
$$\int_{\boldsymbol{R}^n}(g_1-g_2)\mathscr{J}_\alpha(\varphi)dx=0,\qquad \text{对所有 }\varphi\in\mathscr{S}.$$
实际上$\mathscr{J}_\alpha$ 从$\mathscr{S}$到自身是映上的。因为假设给定 $\psi\in\mathscr{S}$，取$\Phi(x)=\hat{\psi}(x)(1+4\pi^2|x|^2)^{-\alpha/2}$，由 $\hat{\psi}\in\mathscr{S}$知$\Phi$ 也一样，因此 $\varphi\in\mathscr{S}$。但 $\hat{\psi}(x)=\Phi(x)(1+4\pi^2|x|^2)^{\alpha/2}$，故 $\psi=\mathscr{J}_\alpha(\varphi)$。这就证明了
$$\int(g_1-g_2)\psi dx=0,\qquad \text{对所有 }\psi\in\mathscr{S},$$
从而 $g_1=g_2$。

这个定义以及式 (36) 的一个直接**推论**就是：

(40) $$\mathscr{L}^p_\beta\subset\mathscr{L}^p_\alpha,\quad \text{并且 }\|f\|_{p,\alpha}\leqslant\|f\|_{p,\beta},\qquad \text{当}\beta>\alpha.$$
还有

(41) $\mathscr{J}_\beta$ 是 $\mathscr{L}^p_\alpha$ 到 $\mathscr{L}^p_{\alpha+\beta}$ 的一同构变换，当 $\alpha\geqslant 0$，$\beta\geqslant 0$。

在复述这些关于 $\mathscr{L}^p_\alpha$ 的例行性质之后，我们来讨论比较重要的推论。我们回到在 § 2 中提到过的原始想法，就是位势同偏微

商的联系。在我们这里反映出来就是位势空间的大小同 Ŝobolev 空间的大小之间存在着密切的关系。

**定理 3** 假设 $k$ 是正整数，$1<p<\infty$，则

$$\mathscr{L}_k^p(\boldsymbol{R}^n)=L_k^p(\boldsymbol{R}^n)$$

在下述意义下成立：$f\in\mathscr{L}_k^p(\boldsymbol{R}^n)$ 当且仅当 $f\in L_k^p(\boldsymbol{R}^n)$，并且由(39)与(16)分别给出的两个模是等价的。

空间 $\mathscr{L}_k^p$ 与 $L_k^p$ 的这种一致性，当 $p=1$ 或 $p=\infty$时是不成立的。见下面 § 6.6的讨论。

**3.4 定理 3 的证明** 定理 3 的证明基于下述引理。

**引理3** 假设$1<p<\infty$，$\alpha\geqslant1$。则 $f\in\mathscr{L}_\alpha^p(\boldsymbol{R}^n)$当且仅当 $f\in\mathscr{L}_{\alpha-1}^p(\boldsymbol{R}^n)$，并且对每个 $j$，

$$\frac{\partial f}{\partial x_j}\in\mathscr{L}_{\alpha-1}^p(\boldsymbol{R}^n).$$

进一步，模$\|f\|_{p,\alpha}$ 与

$$\|f\|_{p,\alpha-1}+\sum_{j=1}^m\left\|\frac{\partial f}{\partial x_j}\right\|_{\alpha,p-1}$$

是等价的。

首先设 $f\in\mathscr{L}_\alpha^p(\boldsymbol{R}^n)$，这时 $f=\mathscr{J}_\alpha(g)$，$g\in L^p$。

我们要证

$$\frac{\partial f}{\partial x_j}=\mathscr{J}_{\alpha-1}(g^{(j)}),\tag{42}$$

其中 $g^{(j)}=-R_j(\mu_1*g)$，只要 $f=\mathscr{J}_\alpha(g)$。

当 $g$ (从而 $f$ )属于$\mathscr{S}$时，这是可以直接验证的。事实上，这时

$$\begin{aligned}\left(\frac{\partial f}{\partial x_j}\right)^\wedge(x)&=-2\pi i x_j\hat{f}(x)\\&=-2\pi i x_j(1+4\pi^2|x|^2)^{-\alpha/2}\,\hat{g}(x)\\&=(1+4\pi^2|x|^2)^{-(\alpha-1)/2}\,\hat{g}^j(x),\end{aligned}$$

其中

$$\hat{g}^j(x)=-\frac{ix_j}{|x|}\,\frac{(2\pi|x|)}{(1+4\pi^2|x|^2)^{1/2}}\hat{g}(x).$$

当 $g\in\mathscr{S}$ 时，这就证明了（42）。在一般情形，当 $g\in L^p(\boldsymbol{R}^n)$，存在序列 $g_m\in\mathscr{S}$，使得按 $L^p$ 模 $g_m\to g$。变换 $g\to\mu_1*g$，从而 $g\to R_j(\mu_1*g)$ 按 $L^p$ 模是有界的。第一个变换的有界性是由于 $\mu_1$ 是有限测度然后再根据引理 2；第二个的有界性是由于当 $1<p<\infty$ 时 Riesz 变换 $R_j$ 是有界的（见第二章与第三章 § 1)。这就证明了序列 $\left\{\dfrac{\partial f_m}{\partial x_j}\right\}$ 按 $\mathscr{L}^p_{\alpha-1}$ 模收敛，并且式(42)成立。因此，$\dfrac{\partial f}{\partial x_j}\in\mathscr{L}^p_{\alpha-1}$，并且

$$\sum_{j=1}^{n}\left\|\frac{\partial f}{\partial x_j}\right\|_{p,\alpha-1}=\sum_{j=1}^{n}\|g^{(j)}\|_p$$
$$\leqslant A_p\|g\|_p=A_p\|f\|_{p,\alpha}.$$

把这与显然的估计 $\|f\|_{p,\alpha-1}\leqslant\|f\|_{p,\alpha}$（见(40)）合起来，就得到

(43) $$\|f\|_{p,\alpha-1}+\sum_{j=1}^{n}\left\|\frac{\partial f}{\partial x_j}\right\|_{p,\alpha-1}\leqslant A_p^1\|f\|_{p,\alpha}.$$

为了证明其逆，我们首先注意，若 $f$ 与 $\dfrac{\partial f}{\partial x_j}$ 全都属于 $\mathscr{L}^p_{\alpha-1}$，则

(44) $$f=\mathscr{J}_{\alpha-1}(g),\quad 并且\frac{\partial f}{\partial x_j}=\mathscr{J}_{\alpha-1}\left(\frac{\partial g}{\partial x_j}\right),$$

其中 $\dfrac{\partial g}{\partial x_j}$ 在弱意义下存在，并且 $g$ 与 $\dfrac{\partial g}{\partial x_j}$ 属于 $L^p$。

事实上，假设

$$\frac{\partial f}{\partial x_j}=\mathscr{J}_{\alpha-1}(g^{(j)}).$$

设 $\varphi$ 与 $\varphi'\in\mathscr{S}$。这时

$$\int_{R^n}f\varphi'dx=\int_{R^n}\mathscr{J}_{\alpha-1}(g)\varphi'dx=\int_{R^n}g\mathscr{J}_{\alpha-1}(\varphi')dx.$$

类似地

$$\int_{R^n} \frac{\partial f}{\partial x_j} \varphi dx = \int_{R^n} g^{(j)} \mathscr{J}_{\alpha-1}(\varphi) dx.$$

但

$$\int_{R^n} f \frac{\partial \varphi}{\partial x_j} dx = -\int_{R^n} \frac{\partial f}{\partial x_j} \varphi dx,$$

它对 $\varphi \in \mathscr{D}$ 成立；通过一个简单的极限过程可把它推广到 $\varphi$ 属于 $\mathscr{S}$ 的情形。因此，对

$$\varphi' = \frac{\partial \varphi}{\partial x_j}$$

有

$$\int_{R^n} g \frac{\partial}{\partial x_j} (\mathscr{J}_{\alpha-1} \varphi) dx = -\int_{R^n} g^{(j)} \mathscr{J}_{\alpha-1}(\varphi) dx,$$

这是由于对每个 $\varphi \in \mathscr{S}$，

$$\frac{\partial}{\partial x_j} \mathscr{J}_{\alpha-1}(\varphi) = \mathscr{J}_{\alpha-1}\left(\frac{\partial \varphi}{\partial x_j}\right),$$

这可以通过 Fourier 变换验证。然而，正如我们指出过的，变换 $\varphi \to \mathscr{J}_{\alpha-1}(\varphi)$ 在 $\mathscr{S}$ 是映上的，因此对任意 $\psi \in \mathscr{S}$（特别地当 $\psi \in \mathscr{D}$），我们有

$$\int_{R^n} g \frac{\partial \psi}{\partial x_j} dx = -\int_{R^n} g^{(j)} \psi dx,$$

这就证明了（44）。

由于 $g \in L_1^p$，根据§2.1的命题我们可以逼近它；这就给出 $\mathscr{D}$ 中（从而 $\mathscr{S}$ 中）的序列 $\{g_m\}$，使得

$$g_m \to g \quad 与 \quad \frac{\partial g_m}{\partial x_j} \to \frac{\partial g}{\partial x_j}$$

按 $L^p$ 模成立。我们可以令

$$g_m = \mathscr{J}_1(h_m), \qquad h_m \in \mathscr{S}.$$

根据引理 2 的(b)，对 $\alpha = 1$，有

$$h_m = \nu_1 * g_m + \lambda_1 * \left( \sum_{j=1}^{n} R_j \left( \frac{\partial}{\partial x_j} g_m \right) \right),$$

因此

$$\|h_m\|_p \leqslant A_p \left[ \|g_m\|_p + \sum_{j=1}^{n} \left\| \frac{\partial g_m}{\partial x_j} \right\|_p \right], \quad 1 < p < \infty,$$

这里用了 $R_j$ 的有界性。

然而 $f_m = \mathscr{J}_\alpha(h_m)$，这是因为 $f_m = \mathscr{J}_{\alpha-1}(g_m)$，因此

$$\|f_m\|_{p,\alpha} = \|h_m\|_p.$$

从而

$$\|f_m\|_{p,\alpha} \leqslant A_p \left[ \|g_m\|_p + \sum_{j=1}^{n} \left\| \frac{\partial g_m}{\partial x_j} \right\|_p \right].$$

当用 $f_m - f_{m'}$ 代替 $f_m$，用 $g_m - g_{m'}$ 代替 $g_m$ 时，同样的不等式也成立。这表明 $f_m$ 按 $\mathscr{L}_\alpha^p$ 模也收敛；取极限，我们得到 $f \in \mathscr{L}_\alpha^p$，并且

$$\|f\|_{p,\alpha} \leqslant A_p \left[ \|g\|_p + \sum_{j=1}^{n} \left\| \frac{\partial g}{\partial x_j} \right\|_p \right]$$

$$\leqslant A_p \left[ \|f\|_{p,\alpha-1} + \sum_{j=1}^{n} \left\| \frac{\partial f}{\partial x_j} \right\|_{p,\alpha-1} \right].$$

结合(43)，这就完成了引理 3 的证明。

定理 3 的证明现在是直接的了。当 $\alpha = k = 0$时，$L_k^p$ 与 $\mathscr{L}_\alpha^p$ 的相同是十分显然的。然而，显然当 $k \geqslant 1$时 $f \in L_k^p(\boldsymbol{R}^n)$，当且仅当 $f$ 与 $\frac{\partial f}{\partial x_j} \in L_{k-1}^p(\boldsymbol{R}^n)$，$j = 1, 2, \cdots, n$。模

$$\|f\|_{L_k^p(\boldsymbol{R}^n)} \text{ 与 } \|f\|_{L_{k-1}^p(\boldsymbol{R}^n)} + \sum_{j=1}^{n} \left\| \frac{\partial f}{\partial x_j} \right\|_{L_{k-1}^p(\boldsymbol{R}^n)}$$

显然也是等价的。因此，$L_k^p$ 等同于 $\mathscr{L}_k^p$ 可从 $k = 0$ 推到 $k = 1$， 2, ….

**3.5 连续模** 假设 $f\in L^p(\boldsymbol{R}^n)$。我们引入 $L^p$ 的连续模：

$$\omega_p(t)=\|f(x+t)-f(x)\|_p,$$

其中 $L^p$ 模是对变量 $x$ 取的。我们知道当 $|t|\to 0$时，$\omega_p(t)\to 0$，只要$1\leqslant p<\infty$（见第三章§2.2）。

我们向自己提一个自然的问题：$f\in\mathscr{L}^p_\alpha(\boldsymbol{R}^n)$这一性质能否通过 $\omega_p(t)$的无穷小阶（当$|t|\to 0$）来刻划？如果可能，这将给出 $\mathscr{L}^p_\alpha$ 元素用其光滑性来表示的简单特征。

不幸，除了某些特殊情况外，这个希望是不能实现的。空间 $\mathscr{L}^p_\alpha$ 可以用连续模刻划的特殊情形是简单的并且值得写出来。这些情形只出现在 $\alpha$ 是整数（这时 $p$ 可以任意）或 $p=2$（这时 $\alpha$ 可以任意）。我们只对 $\alpha$ 小的情形写出细节，这些情形已是很典型的了。

**命题3** 假设$1<p<\infty$，则$f\in\mathscr{L}^p_1(\boldsymbol{R}^n)$当且仅当 $f\in L^p(\boldsymbol{R}^n)$，并且 $\omega_p(t)=O(|t|)$，当$|t|\to 0$。

设 $f\in\mathscr{D}$。令 $t=|t|t'$，$|t'|=1$。这时

$$f(x+t)-f(x)=\int_0^{|t|}(\nabla f,t')(x+st')ds.$$

因此用 Minkowski 不等式有

$$\|f(x+t)-f(x)\|_p\leqslant|t|\sum_{j=1}^n\left\|\frac{\partial f}{\partial x_j}\right\|_p. \tag{45}$$

§2.1 的逼近方法（命题1）允许我们把（45）推广到任意 $f\in L^p_1(\boldsymbol{R}^n)$。由定理3知，这对$f\in\mathscr{L}^p_1(\boldsymbol{R}^n)$ 也成立，其中 $1<p<\infty$，从而 $f\in\mathscr{L}^p_1(\boldsymbol{R}^n)$蕴含了 $\omega_p(t)=O(|t|)$。

反过来，若 $\omega_p(t)=O(|t|)$，当$|t|\to 0$，则当 $e_j$ 是沿 $x_j$ 轴的单位向量时，序列

$$\left\{\frac{f(x+e_j/m)-f(x)}{1/m}\right\}$$

按 $L^p(\boldsymbol{R}^n)$ 模一致有界。由 $L^p(1<p)$ 单位球的弱紧致性，我们可以找出子序列 $\{m_k\}$ 与函数 $f^{(j)}\in L^p(\boldsymbol{R}^n)$，使得

$$\frac{f(x+e_j/m_k)-f(x)}{1/m_k}\to f^{(j)}$$

弱收敛。特别地

$$\int_{\boldsymbol{R}^n}\left[\frac{f(x+e_j/m_k)-f(x)}{1/m_k}\right]\varphi(x)dx$$

$$=\int_{\boldsymbol{R}^n}f(x)\left[\frac{\varphi(x-e_j/m_k)-\varphi(x)}{1/m_k}\right]dx$$

$$\to\int_{\boldsymbol{R}^n}f^{(j)}\varphi dx=-\int_{\boldsymbol{R}^n}f\frac{\partial\varphi}{\partial x_j}dx.$$

这就证明了 $\dfrac{\partial f}{\partial x_j}\in L^p(\boldsymbol{R}^n)$，从而 $f\in L_1^p(\boldsymbol{R}^n)=\mathscr{L}_1^p(\boldsymbol{R}^n)$。

**命题4** 假设 $0<\alpha<1$，则 $f\in\mathscr{L}_\alpha^2(\boldsymbol{R}^n)$ 当且仅当 $f\in L^2(\boldsymbol{R}^n)$ 且

$$\int_{\boldsymbol{R}^n}\frac{(\omega_2(t))^2}{|t|^{n+2\alpha}}dt<\infty.$$

注意，由于 $\omega_p(t)\leqslant 2\|f\|_p$，因此积分

$$\int_{\boldsymbol{R}^n}\frac{(\omega_2(t))^2}{|t|^{n+2\alpha}}dt$$

只在原点附近是要研究的。

根据 Plancherel 定理，$f=\mathscr{J}_\alpha(g)$，$g\in L^2(\boldsymbol{R}^n)$ 等价于

$$\int_{\boldsymbol{R}^n}|\hat{f}(x)|^2(1+4\pi^2|x|^2)^\alpha dx<\infty. \tag{46}$$

再用 Plancherel 定理

$$(\omega_2(t))^2=\|f(x+t)-f(x)\|_2^2$$

$$=\int_{R^n}|\hat{f}(x)|^2|e^{-2\pi i x\cdot t}-1|^2dx.$$

因此

$$\int_{R^n}\frac{(\omega_2(t))^2}{|t|^{n+2\alpha}}dt=\int_{R^n}|\hat{f}(x)|^2\mathscr{I}(x)dx,$$

其中

$$\mathscr{I}(x)=\int_{R^n}\frac{|e^{-2\pi i x\cdot t}-1|^2}{|t|^{n+2\alpha}}dt. \tag{47}$$

积分 $\mathscr{I}(x)$的估计是容易的。首先，$\mathscr{I}(x)=\mathscr{I}(\rho x)$，其中 $\rho$ 是绕原点的任意旋转。因此 $\mathscr{I}(x)=\mathscr{I}_0(|x|)$。其次，由齐次性有 $\mathscr{I}(x)=|x|^{2\alpha}\mathscr{I}(\eta)$，其中 $\eta$ 是任意固定的单位向量。显然，常数 $\mathscr{I}(\eta)$满足性质$0<\mathscr{I}(\eta)<\infty$；$\mathscr{I}(\eta)$的有限性从$|e^{-2\pi i\eta\cdot t}-1|\leqslant 2$以及$|e^{-2\pi i\eta\cdot t}-1|\leqslant c|t|$推出，因此积分 $\mathscr{I}(\eta)$ 收敛。故条件 $f\in L^2(\boldsymbol{R}^n)$与

$$\int_{R^n}\frac{(\omega_2(t))^2}{|t|^{n+2\alpha}}dt<\infty$$

等价于

$$\int_{R^n}|\hat{f}(x)|^2dx<\infty \quad 与 \quad \int_{R^n}|x|^{2\alpha}|\hat{f}(x)|^2dx<\infty.$$

这两个条件显然等价于式（46），从而命题得证。

3.5.1 有趣的是，命题 3 对 $p=2$，$\alpha=1$ 并不是（至少用现在叙述的方式）命题 4 当 $\alpha\to 1$ 的极限情形。然而存在 $\alpha=1$ 的类似于命题 4 的一种说法，它的形式预示了某些我们以后感兴趣的东西。

我们考虑一种微稍不同的连续模 $\tilde{\omega}_p(t)$，由

$$\tilde{\omega}_p(t)=\|f(x+t)+f(x-t)-2f(x)\|_p$$

定义。好处在于 $\tilde{\omega}_p(t)$永远不大于 $\omega_p(t)$，因为 $\tilde{\omega}_p(t)\leqslant\omega_p(t)+$

$\omega_p(-t)$，而另一方面它有时却可能真正小于 $\omega_p(t)$①。

**命题5** 假设$0<\alpha<2$。则 $f\in \mathscr{L}^2_\alpha(R^n)$当且仅当 $f\in L^2(R^n)$ 与

$$\int_{R^n}\frac{(\bar{\omega}_2(t))^2}{|t|^{n+2\alpha}}dt<\infty.$$

这个命题的证明几乎完全和上面的一样。唯一的区别是积分(47)（它仅当$0<\alpha<1$时收敛）代之以

$$\int_{R^n}\frac{|e^{-2\pi i x\cdot t}+e^{2\pi i x\cdot t}-2|^2}{|t|^{n+2\alpha}}dt,$$

它当$0<\alpha<2$时收敛（还见第八章§5.2的命题）。

3.5.2 一般说来，整个 $\mathscr{L}^p_\alpha(R^n)$空间不能通过连续模刻划，然而存在某些有趣的联系，这将在下面§5中看到。假设 $0<\alpha<1$。这时，当 $f\in \mathscr{L}^p_\alpha$，就有

$$\int_{R^n}\frac{(\omega_p(t))^p dt}{|t|^{n+\alpha p}}<\infty,\qquad 当 p\geqslant 2$$

与

$$\int_{R^n}\frac{(\omega_p(t))^2 dt}{|t|^{n+2\alpha}}<\infty,\qquad 当 p\leqslant 2.$$

反过来，假设 $f\in L^p$，则 $f\in \mathscr{L}^p_\alpha$，如果

$$\int_{R^n}\frac{(\omega_p(t))^p dt}{|t|^{n+\alpha p}}<\infty,\qquad p\leqslant 2,$$

或者如果

$$\int_{R^n}\frac{(\omega_p(t))^2 dt}{|t|^{n+2\alpha}}<\infty,\qquad p\geqslant 2.$$

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① 例如，设 $f\in\mathscr{S}$。这时 $\omega_p(t)=O(|t|)$，但若

$$\frac{\omega_p(t)}{|t|}\to 0,\quad |t|\to 0,$$

则 $f\equiv 0$。然而却永远有 $\bar{\omega}_p(t)=O(|t|^2)$，$|t|\to 0$．需要更深入了解可见下面§4的4.3.1.

这表明，对用连续模定义的函数空间进行研究，将是很有意思的。我们将从考虑这种空间的最简单而又最熟悉的例子开始。这就是 Lipschitz（或 Hölder）连续函数空间①。

## §4 Lipschitz 连续函数空间 $\Lambda_\alpha$

**4.1** 我们从 $0<\alpha<1$ 的情形开始。我们定义 $\Lambda_\alpha$ 如下：

$$\Lambda_\alpha=\{f:\ f\in L^\infty(\boldsymbol{R}^n),\ \text{并且}\ \omega_\infty(t)=\|f(x+t)-f(x)\|_\infty\leqslant A|t|^\alpha\}.$$

$\Lambda_\alpha$ 模就由

$$\|f\|_{\Lambda_\alpha}=\|f\|_\infty+\sup_{|t|>0}\frac{\|f(x+t)-f(x)\|_\infty}{|t|^\alpha} \tag{48}$$

给出。

要注意的第一件事是，$\Lambda_\alpha$ 的函数可以认为是连续的，因此 $|f(x+t)-f(x)|\leqslant A|t|^\alpha$ 对每个 $x$ 成立。更准确些有

**命题6** 对每个 $f\in\Lambda_\alpha$，总可以在一零测度集合上改值，使它变为连续的。

证明可以用 §2.1的正则化方法。可以用任何一种正则化，在这里我们将用 Poisson 积分（第三章 §2）。因此考虑

$$u(x,y)=\int_{\boldsymbol{R}^n}P_y(t)f(x-t)dt,$$

$$P_y(t)=\frac{c_ny}{(|t|^2+y^2)^{(n+1)/2}}.$$

这时

$$u(x,y)-f(x)=\int_{\boldsymbol{R}^n}P_y(t)[f(x-t)-f(x)]dt,$$

因此

① 函数满足 Hölder 条件或 Lipschitz 条件，这两个术语是一样地通用的。在我们这里引用的是后者。

$$\|u(x,y)-f(x)\|_\infty \leqslant \int_{R^n} P_y(t)\omega_\infty(-t)dt$$

$$\leqslant Ac_n y \int_{R^n} \frac{|t|^\alpha}{(|t|^2+y^2)^{(n+1)/2}} dt$$

$$=A'y^\alpha \quad (\text{当 } \alpha<1)。$$

特别地，$\|u(x,y_1)-u(x,y_2)\|_\infty \to 0$，当 $y_1, y_2 \to 0$，而由于 $u(x,y)$ 对 $x$ 连续，因而 $u(x,y)$ 当 $y\to 0$ 时一致收敛。故 $f(x)$ 可以认为是连续的。

**4.2 特征** 同刚才考虑的情况相反，Poisson 积分的特殊性质由于下面的理由将起着更决定性的作用。我们从用 Poisson 积分 $u(x,y)$ 给出 $f\in\Lambda_\alpha$ 的特征开始。

**命题7** 假设 $f\in L^\infty(\boldsymbol{R}^n)$，且 $0<\alpha<1$，则 $f\in\Lambda_\alpha(\boldsymbol{R}^n)$ 当且仅当

$$\left\|\frac{\partial u(x,y)}{\partial y}\right\|_\infty \leqslant Ay^{-1+\alpha}. \tag{49}$$

**补充**：如果 $A_1$ 是使 (49) 成立的最小数，那末 $\|f\|_\infty + A_1$ 与 $\|f\|_{\Lambda_\alpha}$ 给出等价模。

我们用下列关于 Poisson 核的容易验证的事实

$$\begin{aligned}&\int_{R^n}\left|\frac{\partial P_y(x)}{\partial y}\right| dx \leqslant \frac{c}{y};\\ &\int_{R^n}\frac{\partial P_y(x)}{\partial y}dx=0, \qquad y>0.\end{aligned} \tag{50}$$

第一个是由于对 $\frac{\partial P_y}{\partial y}$，有明显的估计

$$\left|\frac{\partial P_y}{\partial y}\right| \leqslant c' y^{-n-1}, \qquad \left|\frac{\partial P_y}{\partial y}\right| \leqslant c'|x|^{-n-1}.$$

第二个是由于 $\int_{R^n} P_y(x)dx=1$。因此

$$\frac{\partial u(x,y)}{\partial y}=\int_{R^n}\frac{\partial P_y(t)}{\partial y}f(x-t)dt$$

$$=\int_{R^n}\frac{\partial P_y(t)}{\partial y}[f(x-t)-f(x)]dt.$$

故

$$\left\|\frac{\partial u}{\partial y}\right\|_\infty\leqslant\|f\|_{\Lambda_\alpha}\int_{R^n}\left|\frac{\partial P_y}{\partial y}\right||t|^\alpha dt$$

$$=c'\|f\|_{\Lambda_\alpha}y^{-1+\alpha}.$$

反过来，虽然没多大困难，但却有启发性，它揭示了所讨论的空间的本质特征。这可从下面的引理与接着的说明中看出来。

**引理4** 假设 $f\in L^\infty(\boldsymbol{R}^n)$，$0<\alpha<1$，则单独的条件（49）等价于 $n$ 个条件

$$(51)\qquad \left\|\frac{\partial u(x,y)}{\partial x_j}\right\|_\infty\leqslant A'y^{-1+\alpha},\qquad j=1,\cdots,n.$$

**补充**：(49)中的最小 $A$ 同(51)中的最小 $A'$ 可比较。

我们知道，在

$$\frac{\partial u}{\partial y}\text{ 与 }\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2},\cdots,\frac{\partial u}{\partial x_n}$$

之间的联系是通过 Riesz 变换现实的，见第三章 §2.3。然而，Riesz 变换在有界函数类中并不封闭（见第二章§6.1(b)），因此当 $\alpha=1$时，条件（49）与（51）是不一样的。不过，这个引理的意思是说，Riesz 变换在 $\Lambda_\alpha$ 却是有界的（这点的初等证明同 $L^p$（$1<p<\infty$）有界性时的困难证明形成鲜明的对比，见第二章§6.9）。

为了证明引理，我们要用下列估计

$$(52)\qquad \left\|\frac{\partial P_y}{\partial y}\right\|_1\leqslant cy^{-1},\qquad \left\|\frac{\partial P_y}{\partial x_j}\right\|_1\leqslant cy^{-1}\qquad (y>0),$$

第一个已在公式（50）中出现过；第二个可同样证明。

由 $P_y=P_{y_1}*P_{y_2}$，$y=y_1+y_2$，$y_j>0$，我们有

$$u(x,y)=P_{y_1}*u(x,y_2),$$

因此取 $y_1=y_2=y/2$，

$$\frac{\partial^2 u}{\partial y\partial x_j}=\left(\frac{\partial P_{y/2}}{\partial x_j}\right)*\left(\frac{\partial u}{\partial y}\right)_{y/2}.$$

根据（52），从假设 $\left\|\frac{\partial u}{\partial y}\right\|_\infty\leqslant Ay^{-1+\alpha}$ 就推出

（53）
$$\left\|\frac{\partial^2 u}{\partial y\partial x_j}\right\|_\infty\leqslant A_1y^{-2+\alpha}.$$

然而由（52），

$$\left\|\frac{\partial}{\partial x_j}u(x,y)\right\|_\infty=\left\|\frac{\partial P_y}{\partial x_j}*f\right\|_\infty\leqslant\left\|\frac{\partial P_y}{\partial x_j}\right\|_1\|f\|$$
$$\leqslant cy^{-1}\|f\|_\infty.$$

因此

$$\frac{\partial}{\partial x_j}u(x,y)\to 0,\qquad 当 y\to\infty,$$

从而
$$\frac{\partial}{\partial x_j}u(x,y)=-\int_y^\infty\frac{\partial^2 u(x,y')}{\partial y\partial x_j}dy'.$$

由（53）就得到

$$\left\|\frac{\partial u}{\partial x_j}\right\|_\infty\leqslant A_2y^{-1+\alpha},\qquad \alpha<1.$$

反过来，假设(51)满足。同上理由有

$$\left\|\frac{\partial^2 u}{\partial x_j^2}\right\|_\infty\leqslant A_3y^{-2+\alpha},\qquad j=1,2,\cdots,n.$$

然而，由于 $u$ 是调和的，即

$$\frac{\partial^2 u}{\partial y^2}=-\sum_{j=1}^{n}\frac{\partial^2 u}{\partial x_j^2},$$

就有

$$\left\|\frac{\partial^2 u}{\partial y^2}\right\|_\infty\leqslant A_4 y^{-2+\alpha},$$

由一个类似的积分论证就推出

$$\left\|\frac{\partial u}{\partial y}\right\|_\infty\leqslant A_5 y^{-1+\alpha}.$$

我们现在可以证明命题 7 的反过来那部分了。假设

$$\left\|\frac{\partial}{\partial y}u(x,y)\right\|_\infty\leqslant A y^{-1+\alpha}.$$

引理又证明了

$$\left\|\frac{\partial}{\partial x_j}u(x,y)\right\|_\infty\leqslant A' y^{-1+\alpha}.$$

我们写出

$$\begin{aligned}f(x+t)-f(x)=&\{u(x+t,y)-u(x,y)\}\\&+\{f(x+t)-u(x+t,y)\}\\&-\{f(x)-u(x,y)\}.\end{aligned}$$

在这里 $y$ 不必依赖于 $t$，但最好选择 $y=|t|$。这样

$$|u(x+t,y)-u(x,y)|\leqslant\int_L|\nabla u(x+s,y)|ds,$$

其中 $L$ 是连接 $x$ 与 $x+t$ 的线段（长为 $|t|$）。因此

$$\begin{aligned}|u(x+t,y)-u(x,y)|&\leqslant|t|\sum_{j=1}^{n}\|u_{x_j}(x,y)\|_\infty\\&\leqslant A_5|t|\,|t|^{-1+\alpha}=A_6|t|^\alpha.\end{aligned}$$

还有

$$f(x+t)-u(x+t,y)=-\int_0^y\frac{\partial}{\partial y'}u(x+t,y')dy',$$

因而

$$|f(x+t)-u(x+t,y)|\leqslant y\int_0^y\left\|\frac{\partial u}{\partial y'}\right\|_\infty dy'$$
$$\leqslant A_6 y^\alpha = A_6|t|^\alpha.$$

对 $f(x)-u(x,y)$ 有类似的估计，从而结束命题的证明。

4.2.1 读者可以毫无困难地用证明引理 4 的方法证明下面的引理。

**引理 5** 假设 $f\in L^\infty(\boldsymbol{R}^n)$ 而 $0<\alpha$。令 $k$ 与 $l$ 是两个都超过 $\alpha$ 的整数。则条件

$$\left\|\frac{\partial^k u(x,y)}{\partial y^k}\right\|_\infty\leqslant A_k y^{-k+\alpha}\quad 与\quad \left\|\frac{\partial^l u(x,y)}{\partial y^l}\right\|_\infty\leqslant A_l y^{-l+\alpha}$$

是等价的。更进一步，使上面不等式成立的最小的 $A_k$ 与 $A_l$ 是可比较的。

这引理的好处很快就会看到。

**4.3** $\Lambda_\alpha$，**对所有** $\alpha>0$ 现在可以对任意 $\alpha>0$ 定义 $\Lambda_\alpha(\boldsymbol{R}^n)$ 了。假设 $k$ 是超过 $\alpha$ 的最小整数。令

$$(54)\qquad \Lambda_\alpha=\left\{f\in L^\infty(\boldsymbol{R}^n):\ \left\|\frac{\partial^k}{\partial y^k}u(x,y)\right\|_\infty\leqslant Ay^{-k+\alpha}\right\}.$$

设 $A_k$ 表示在（54）的不等式中出现的最小的 $A$，则我们用

$$(55)\qquad \|f\|_{\Lambda_\alpha}=\|f\|_\infty+A_k$$

来定义 $\Lambda_\alpha$ 的模。

根据命题 7，当 $0<\alpha<1$ 时，这个定义与前面的等价，并且两种模也是等价的。引理 5 还表明，我们可以用 $\dfrac{\partial^l u(x,y)}{\partial y^l}$ 的相应估计代替 $\dfrac{\partial^k u(x,y)}{\partial y^k}$，其中 $l$ 是任意大于 $\alpha$ 的整数。

关于条件(54)，需要作一点说明。估计

$$\left\|\frac{\partial^k}{\partial y^k}u(x,y)\right\|_\infty\leqslant Ay^{-k+\alpha}$$

只在 $y$ 接近于 0 是才有意义，这是由于

$$\left\|\frac{\partial^k}{\partial y^k}u(x,y)\right\|_\infty \leqslant Ay^{-k}$$

（在远离零的地方这是比较强的）可从 $f\in L^\infty$ 推出（如同引理 4 的论证）。这就允许我们断言 $\Lambda_\alpha\subset\Lambda_{\alpha'}$，当 $\alpha>\alpha'$。

对 $\Lambda_\alpha(\alpha>0)$的定义（54）与(55)，当把它们与在 $0<\alpha<1$ 给出的定义比较时，样子显得比较做作。读者不必担心这一点，因为刚刚给出的定义只是暂时的，在下面两个命题中将叙述自然的特征。

我们首先考虑 $0<\alpha<2$ 的情形。我们需要研究二阶差分，就像在 § 3的3.5.1的命题 5 中那样。

**命题 8** 假设 $0<\alpha<2$。则 $f\in\Lambda_\alpha$ 当且仅当 $f\in L^\infty(\boldsymbol{R}^n)$ 并且

$$\|f(x+t)+f(x-t)-2f(x)\|_\infty\leqslant A|t|^\alpha.$$

表达式

$$\|f\|_\infty+\sup_{|t|>0}\frac{\|f(x+t)+f(x-t)-2f(x)\|_\infty}{|t|^\alpha}$$

与 $\Lambda_\alpha$ 模是等价的。

我们需要下列关于 Poisson 核微分的性质：

(a) $$\int_{R^n}\frac{\partial^2 P_y(t)}{\partial y^2}dt=0;$$

(b) $$\frac{\partial^2 P_y(t)}{\partial y^2}=\frac{\partial^2 P_y(-t)}{\partial y^2};$$

(c) $$\left|\frac{\partial^2 P_y(t)}{\partial y^2}\right|\leqslant cy^{-n-2};$$

(d) $$\left|\frac{\partial^2 P_y(t)}{\partial y^2}\right|\leqslant c|t|^{-n-2}.$$

验证(c)与(d)的详细计算，最好是留给读者去做，但指出

$\dfrac{\partial^2 P_y(t)}{\partial y^2}$也是$-n-2$次齐次的，也许会有所帮助。

从上述性质我们看到

$$\frac{\partial^2 u(x,y)}{\partial y^2}=\frac{1}{2}\int_{R^n}\frac{\partial^2}{\partial y^2}P_y(t)[f(x+t)+f(x-t)-2f(x)]dt,$$

因此

$$\left\|\frac{\partial^2 u(x,y)}{\partial y^2}\right\|_\infty\leqslant\frac{Ac}{2}\left\{y^{-n-2}\int_{|t|<y}|t|^\alpha dt\right.$$

$$\left.+\int_{|t|\geqslant y}|t|^{-n-2+\alpha}dt\right\},$$

故

$$\left\|\frac{\partial^2 u}{\partial y^2}\right\|_\infty\leqslant A' y^{-2+\alpha},\qquad \alpha<2.$$

为证明其逆，记

$$(\Delta_t^2 F)(x)=F(x+t)+F(x-t)-2F(x),$$

并注意，若 $F$ 有二阶连续偏微商，则

$$\Delta_t^2 F(x)=\int_0^{|t|}\left\{\int_{-s}^{+s}\frac{d^2}{d\tau^2}(F(x+t'\tau))d\tau\right\}ds,$$

其中 $t'=t/|t|$。由它直接推出

(56) $$\|\Delta_t^2 F\|_\infty\leqslant|t|^2\left\{\sum_{i,j}\left\|\frac{\partial^2 F}{\partial x_i\partial x_j}\right\|_\infty\right\}.$$

由定义(54)，显然，若 $f\in\Lambda_\alpha$，则 $f\in\Lambda_{\alpha'}$，其中 $\alpha'<\alpha$。如果选择 $\alpha'<1$，那末由命题(2)与(7)的结果，我们有

(57) $\|u(x,y)-f(x)\|_\infty\to 0$，$y\|u_y(x,y)\|_\infty\to 0$，当 $y\to 0$。

现在可以得到恒等式

(58) $$f(x)=u(x,0)=\int_0^y y'\frac{\partial^2}{\partial y'^2}u(x,y')\,dy'$$

$$-y\frac{\partial}{\partial y}u(x,y)+u(x,y),$$

这只要注意上式右边对 $y$ 的微商恒为 0，再用(57)的端点条件。然而引理 4 与 5 的论证表明，不等式

$$\left\|\frac{\partial^2 u(x,y)}{\partial y^2}\right\|_\infty \leqslant Ay^{-2+\alpha}$$

蕴含着

$$\left\|\frac{\partial^2 u}{\partial x_i \partial x_j}\right\|_\infty \leqslant A' y^{-2+\alpha}, \quad \left\|\frac{\partial^3 u}{\partial y \partial x_i \partial x_j}\right\|_\infty \leqslant A' y^{-3+\alpha}.$$

这样，由 (56)与(58)

$$\|\Delta_t^2 f\|_\infty \leqslant A''\left\{\int_0^y y'(y')^{-2+\alpha}dy' + y^{-2+\alpha}|t|^2\right\}.$$

取 $y=|t|$，就得到所要求的结果

$$\|\Delta_t^2 f\|_\infty \leqslant A''|t|^\alpha, \quad 0<\alpha.$$

**命题 9** 假设 $\alpha>1$。则 $f\in\Lambda_\alpha$ 当且仅当 $f\in L^\infty$ 且

$$\frac{\partial f}{\partial x_j}\in\Lambda_{\alpha-1}, \quad j=1,\cdots,n.$$

模 $\|f\|_{\Lambda_\alpha}$ 与

$$\|f\|_\infty+\sum_{j=1}^n\left\|\frac{\partial f}{\partial x_j}\right\|_{\Lambda_{\alpha-1}}$$

是等价的。

为简单起见，让我们假设 $1<\alpha\leqslant 2$；其它的情形类似推得。

首先注意 $\frac{\partial f}{\partial x_j}\in L^\infty$。我们有

$$\left\|\frac{\partial^3 u}{\partial y^3}\right\|_\infty \leqslant Ay^{-3+\alpha},$$

正如大家已知的，它蕴含着

$$\left\|\frac{\partial^3 u}{\partial y^2 \partial x_j}\right\|_\infty \leqslant A y^{-3+\alpha}.$$

我们限制 $0<y\leqslant 1$，这时

$$\left\|\frac{\partial^3 u}{\partial y^2 \partial x_j}\right\|_\infty \leqslant A y^{-1-\beta},$$

其中 $\beta<1$。对 $y$ 积分就给出

$$\left\|\frac{\partial^2 u}{\partial y \partial x_j}\right\|_\infty \leqslant A y^{-\beta} + A \left\|\left[\frac{\partial^2 u}{\partial y \partial x_j}\right]_{y=1}\right\|_\infty.$$

再一次积分就给出，$\left\{\frac{\partial}{\partial x_j}u(x,y)\right\}$ 按 $L^\infty$模 是 Cauchy 序 列

（当 $y\to 0$），因此它的极限可以看作$\frac{\partial f}{\partial x_j}$。这推理还给出界

$$\left\|\frac{\partial f}{\partial x_j}\right\|_\infty \leqslant C\|f\|_{\Lambda_\alpha}.$$

由于 $f$ 的（弱）微商是$\frac{\partial f}{\partial x_j}$，后者的 Poisson 积分是 $\frac{\partial u}{\partial x_j}$。但

$$\left\|\frac{\partial^3 u}{\partial y^2 \partial x_j}\right\|_\infty \leqslant A y^{-3+\alpha}.$$

因此$\frac{\partial f}{\partial x_j}\in\Lambda_{\alpha-1}$。反过来的蕴含关系可用同样方法证明。

最后这个命题把空间 $\Lambda_\alpha$ 的研究归结到 $0<\alpha\leqslant 1$ 的情形。

4.3.1 一个例子。谈到 $\Lambda_\alpha$，$0<\alpha\leqslant 1$，下面的注释是必要的。首先，当 $0<\alpha<1$，命题 8 证明了，当 $f\in L^\infty$ 时，条件

$$\|f(x+t)-f(x)\|_\infty \leqslant A|t|^\alpha$$

与

$$\|f(x+t)+f(x-t)-2f(x)\|_\infty \leqslant A'|t|^\alpha$$

是等价的，但当 $\alpha=1$ 时，情形就不同了。

**例** 存在 $f\in L^\infty$ 使得

$$\|f(x+t)+f(x-t)-2f(x)\|_\infty\leqslant A|t|,\qquad |t|>0,$$

但 $\|f(x+t)-f(x)\|_\infty\leqslant A'|t|$ 对任意 $A'$ 不成立。

人们可以用大缺项级数构造这样的例子，甚至更特别些用 Weierstrass-Hardy 不可微函数。为此，考虑单变量函数

$$f(x)=\sum_{k=1}^{\infty}a^{-k}e^{2\pi i a^k x},$$

其中 $a>1$；如简单起见，取 $a$ 为整数，这时 $f$ 是周期的①。现在

$$f(x+t)+f(x-t)-2f(x)$$
$$=2\sum_k a^{-k}[\cos 2\pi a^k t-1]\,e^{2\pi i a^k x},$$

因此

$$\|f(x+t)+f(x-t)-2f(x)\|_\infty$$
$$\leqslant 2\sum_{a^k|t|<1}a^{-k}A(a^k t)^2+4\sum_{a^k|t|\geqslant 1}a^{-k}\leqslant A'|t|,$$

这里我们只用了事实：$|\cos 2\pi a^k t-1|\leqslant A(a^k t)^2$ 以及 $\leqslant 2$。

然而，如果有 $\|f(x+t)-f(x)\|_\infty\leqslant A'|t|$，那末对 $L^2$ 的周期函数用 Bessel 不等式就有

$$(A'|t|)^2\geqslant\int_0^1|f(x+t)-f(x)|^2dx$$
$$=\sum a^{-2k}\,|e^{2\pi i a^k t}-1|^2$$
$$\geqslant\sum_{a^k|t|<1}|e^{2\pi i a^k t}-1|^2.$$

但在 $a^k|t|\leqslant 1$ 的范围内，

$$|e^{2\pi i a^k t}-1|\geqslant c(a^k t)^2,$$

因而产生矛盾

---

① 结果对 $a$ 是非整数时也成立。

$$(A'|t|)^2 \geqslant c|t|^2 \sum_{a^k |t| < 1} 1.$$

**4.4** $\mathscr{J}_\beta$: $\Lambda_\alpha \to \Lambda_{\alpha+\beta}$ 我们现在将把 Bessel 位势同 Lipschitz 空间联系起来。

**定理 4** 假设 $\alpha > 0$，$\beta \geqslant 0$，则 $\mathscr{J}_\beta$ 同构地映 $\Lambda_\alpha$ 到 $\Lambda_{\alpha+\beta}$。

这里的所谓"同构"，并不是说 $\|f\|_{\Lambda_\alpha}$ 与 $\|\mathscr{J}_\beta f\|_{\Lambda_{\alpha+\beta}}$ 是相同的，而仅仅是说他们是等价的。

在§3.3中我们已经注意到变换 $\mathscr{J}_\beta$ 是1-1的。为证明 $\Lambda_\alpha$ 在 $\mathscr{J}_\beta$ 作用下的像在 $\Lambda_{\alpha+\beta}$ 内，并且变换是连续的，我们推理如下。设 $u$ 是 $f$ 的 Poisson 积分，而 $U$ 是 $\mathscr{J}_\beta(f) = G_\beta * f$ 的 Poisson 积分。这时 $u = P_y * f$，$U = P_y * G_\beta * f$。因此，$U(x,y) = G_\beta(x,y) * f(x)$，其中 $G_\beta(x,y)$ 是 $G_\beta(x)$ 的 Poisson 积分。$G_\beta(x,y)$ 的下列性质将在下面§5.4中证明。

设 $l$ 是整数且 $l > \beta$，则

$$\left\| \frac{\partial^l G_\beta(x,y)}{\partial y^l} \right\|_1 \leqslant A y^{-l+\beta}, \qquad y > 0. \tag{59}$$

然而，我们知道 $P_{y_1+y_2} = P_{y_1} * P_{y_2}$，$y_1 > 0$，$y_2 > 0$；因此

$$\begin{aligned} U(x, y_1+y_2) &= P_{y_1+y_2} * G_\beta * f = P_{y_1} * G_\beta * P_{y_2} * f \\ &= G_\beta(x, y_1) * u(x, y_2). \end{aligned}$$

设 $k$ 是大于 $\alpha$ 的最小整数，在上式中对 $y_1$ 微分 $l$ 次，对 $y_2$ 微分 $k$ 次，结果就是

$$\frac{\partial^{k+l} U(x,y)}{\partial y^{k+l}} = \frac{\partial^l}{\partial y_1^l} G_\beta(x, y_1) * \frac{\partial^k}{\partial y_2^k} u(x, y_2), \qquad y = y_1 + y_2.$$

注意到(59)，取 $y_1 = y_2 = y/2$，我们得到

$$\left\| \frac{\partial^{k+l} U(x,y)}{\partial y^{k+l}} \right\|_1 \leqslant A\left(\frac{y}{2}\right)^{-l+\beta} \cdot A'\left(\frac{y}{2}\right)^{-k+\alpha}.$$

这里用到 $f \in \Lambda_\alpha$ 蕴含了

$$\left\| \frac{\partial^k}{\partial y^k} u(x,y) \right\|_\infty \leqslant A y^{-k+\alpha} \qquad \text{(见定义(54))}.$$

另外，显然$\mathscr{J}_\beta(f)\in L^\infty$，当$f\in L^\infty$。因此$\mathscr{J}_\beta f\in\Lambda_{\alpha+\beta}$。并且证明过程还表明$\|\mathscr{J}_\beta f\|_{\Lambda_{\alpha+\beta}}\leqslant C\|f\|_{\Lambda_\alpha}$。

下面我们要证明$\Lambda_\alpha$在$\mathscr{J}_2$作用下的像是整个$\Lambda_{\alpha+2}$。为此设$f\in\Lambda_{\alpha+2}$。这时$f\in\Lambda_\alpha$，且$\Delta f\in\Lambda_\alpha$，后者是§4.3命题9的推论。因此$(I-\Delta)f\in\Lambda_\alpha$。然而$\mathscr{J}_2[(I-\Delta)f]=f$。为证明这等式，只要验证

$$\int_{\mathbf{R}^n}[\mathscr{J}_2(I-\Delta)f]\varphi dx=\int_{\mathbf{R}^n}f\varphi dx,$$

当$\varphi\in\mathscr{S}$。但

$$\int_{\mathbf{R}^n}[\mathscr{J}_2(I-\Delta)f]\varphi dx=\int_{\mathbf{R}^n}(I-\Delta)f\mathscr{J}_2(\varphi)dx$$

$$=\int_{\mathbf{R}^n}f(I-\Delta)\mathscr{J}_2(\varphi)dx=\int_{\mathbf{R}^n}f\varphi dx,$$

这是由于$(I-\Delta)\mathscr{J}_2(\varphi)=\varphi$，它显然可用Fourier变换证明。因为$\mathscr{J}_2$是映上的，$\mathscr{J}_{2-\beta}$是1-1的，而$\mathscr{J}_2=\mathscr{J}_{2-\beta}\mathscr{J}_\beta$，当$0<\beta<2$，所以$\mathscr{J}_\beta$在$\beta$的这个范围是映上的。作重复用这样的$\mathscr{J}_\beta$，知对任意正$\beta$，$\mathscr{J}_\beta$是映上的。再用闭图象定理便证得定理①。

## §5 空间 $\Lambda_\alpha^{p,q}$

5.1 类似于我们对$\Lambda_\alpha$的定义，以及由§3.5中命题4的推动，我们来定义空间$\Lambda_\alpha^{p,q}$，其中$1\leqslant p,q\leqslant\infty$。我们从$0<\alpha<1$的情形开始。这时$\Lambda_\alpha^{p,q}$由$L^p$中的函数组成，它们的模

$$\|f\|_p+\left(\int_{\mathbf{R}^n}\frac{(\|f(x+t)-f(x)\|_p)^q dt}{|t|^{n+\alpha q}}\right)^{1/q}\tag{60}$$

是有限的。当$q=\infty$时，表达式(60)用通常的极限过程来解释，也就是

① 我们用闭图象定理只是为了很快证明$\mathscr{J}_\beta$的逆变换的连续性。正如读者可以想到的，只要稍作努力，问题也可以直接处理。

(60′) $$\|f\|_p+\sup_{|t|>0}\frac{\|f(x+t)-f(x)\|_p}{|t|^\alpha}.$$

由此可见，$\Lambda_\alpha^{\infty,\infty}=\Lambda_\alpha$。而命题 4 实际上说的是 $\Lambda_\alpha^{2,2}=\mathscr{L}_\alpha^2$，$0<\alpha<1$（以后将看到，$\Lambda_\alpha^{2,2}=\mathscr{L}_\alpha^2$ 对一切 $\alpha$ 成立）.

由命题 7，8 和 9 与引理 4，5，以及定理 4 给出的空间 $\Lambda_\alpha$ 的基本性质，作显然的修改后，对空间 $\Lambda_\alpha^{p,q}$ 也成立。一般地我们把它写出来，仅仅对命题 7 的类似的直接部分给出详细证明。

**命题 7′** 假设 $f\in L^p(\boldsymbol{R}^n)$，又 $0<\alpha<1$。则 $f\in\Lambda_\alpha^{p,q}$ 当且仅当

(61) $$\left(\int_0^\infty\left(y^{1-\alpha}\left\|\frac{\partial}{\partial y}u(x,y)\right\|_p\right)^q\frac{dy}{y}\right)^{1/q}<\infty.$$

$\Lambda_\alpha^{p,q}$ 模等价于模

$$\|f\|_p+\left(\int_0^\infty\left(y^{1-\alpha}\left\|\frac{\partial u}{\partial y}\right\|_p\right)^q\frac{dy}{y}\right)^{1/q}.$$

我们有

$$\frac{\partial}{\partial y}u(x,y)=\int_{\mathbf{R}^n}\frac{\partial P_y(t)}{\partial y}[f(x-t)-f(x)]dt,$$

据根曾用过的初等估计

$$\left|\frac{\partial P_y(t)}{\partial y}\right|\leqslant c'y^{-n-1},\qquad\left|\frac{\partial P_y(t)}{\partial y}\right|\leqslant c'|t|^{-n-1},$$

我们看出

$$\left\|\frac{\partial}{\partial y}u(x,y)\right\|_p\leqslant c'y^{-n-1}\int_{|t|<y}\|f(x+t)-f(x)\|_p dt$$

$$+c'\int_{|t|>y}\|f(x+t)-f(x)\|_p\frac{dt}{|t|^{n+1}}.$$

令 $t=r\xi\in\boldsymbol{R}^n$，其中 $r=|t|$，而 $|\xi|=1$。由

$$\|f(x+t)-f(x)\|_p=\omega_p(t)=\omega_p(r\xi),$$

我们取

$$\Omega(r)=\int_{S^{n-1}}\omega_p(r\xi)d\xi,$$

上面的不等式变成

$$\left\|\frac{\partial u}{\partial y}\right\|_p\leqslant c'\,y^{-n-1}\int_0^y\Omega(r)r^{n-1}dr+c'\int_y^{\infty}\Omega(r)r^{-2}dr$$

（因为 $dt=d\xi r^{n-1}dr$）。

因此用 Hardy 不等式（见附录 A）

$$\left(\int_0^{\infty}\left(y^{1-\alpha}\left\|\frac{\partial u}{\partial y}\right\|_p\right)^q\frac{dy}{y}\right)^{1/q}\leqslant c\left(\int_0^{\infty}[\Omega(r)r^{-\alpha}]^q\,\frac{dr}{r}\right)^{1/q}.$$

然而根据 Hölder 不等式

$$\Omega(r)^q\leqslant c\int_{S^{n-1}}(\omega(r\xi))^q d\xi\text{①},$$

把这代入上式就得到界

$$c''\left(\int_{S^{n-1}}\int_0^{\infty}(\omega(r\xi))^q r^{-\alpha q}\frac{dr}{r}d\xi\right)^{1/q}$$

$$=c''\left(\int_{R^n}\frac{\|f(x+t)-f(x)\|_p^q dt}{|t|^{n+\alpha q}}\right)^{1/q}.$$

用同样的方法可以证明

**引理 4′** 假设 $f\in L^p(R^n)$, $0<\alpha<1$，则单个条件（61）等价于 $n$ 个条件

$$\left(\int_0^{\infty}\left(y^{1-\alpha}\left\|\frac{\partial u}{\partial x_j}\right\|_p\right)^q\frac{dy}{y}\right)^{1/q}<\infty,\qquad j=1,2,\cdots,n. \tag{62}$$

如果式（61）中的量用（62）中 $n$ 个量的和来代替，那末所得到的模与 $\Lambda_\alpha^{p,q}$ 原来的模等价。

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① 当 $q=\infty$ 时用 $\Omega(r)\leqslant\sup\limits_{\xi}\omega(r\xi)$。

在进一步讨论以前，我们叙述一个比较一般的类似于引理4′的结果，这就是不等式

$$(62') \qquad \left(\int_0^\infty (y^{k-\alpha}\|D^k u\|_p)^q \frac{dy}{y}\right)^{1/q}$$

$$\leqslant A\left(\int_0^\infty \left(y^{k-\alpha}\left\|\frac{\partial^k u}{\partial y^k}\right\|_p\right)^q \frac{dy}{y}\right)^{1/q},$$

其中 $k$ 是正整数，$0<\alpha<k$，$D^k$ 是 $x_1, x_2, \cdots, x_n, y$ 的 $k$ 阶微分单项式。证明可用相同的方法得到。

像以前的程序一样，我们接着对任意的 $\alpha>0$，定义空间 $\Lambda_\alpha^{p,q}(\boldsymbol{R}^n)$。设 $k$ 是大于 $\alpha$ 的最小整数。我们令

$$(63) \qquad \Lambda_\alpha^{p,q}(\boldsymbol{R}^n) = \Big\{f \in L^p(\boldsymbol{R}^n):$$

$$\left(\int_0^\infty \left(y^{k-\alpha}\left\|\frac{\partial^k u}{\partial y^k}\right\|_p\right)^q \frac{dy}{y}\right)^{1/q} < \infty\Big\}.$$

$\Lambda_\alpha^{p,q}$ 模定义为

$$\|f\|_{\Lambda_\alpha^{p,q}} = \|f\|_q + \left(\int_0^\infty \left(y^{k-\alpha}\left\|\frac{\partial^k u}{\partial y^k}\right\|_p\right)^q \frac{dy}{y}\right)^{1/q}.$$

就像对特殊情形 $\Lambda_\alpha$ 所做的那样，我们指出，当用任意整数 $l$ $(l>\alpha)$ 代替整数 $k$ 时，可以得到一个等价的定义与等价模（这已蕴含在（62′）中）。类似于命题 8，9 与定理 4，我们叙述下列的命题及定理。

**命题 8′** 假设$0<\alpha<2$，则 $f\in\Lambda_\alpha^{p,q}$ 当且仅当 $f\in L^p(\boldsymbol{R}^n)$，且

$$\left(\int_{\boldsymbol{R}^n} \frac{(\|f(x+t)+f(x-t)-2f(x)\|_p)^q}{|t|^{n+\alpha q}} dt\right)^{1/q} < \infty.$$

表达式

$$\|f\|_p + \left(\int_{\boldsymbol{R}^n} \frac{(\|f(x+t)+f(x-t)-2f(x)\|_p)^q}{|t|^{n+\alpha q}} dt\right)^{1/q}$$

与 $\Lambda_\alpha^{p,q}$ 模等价①。

**命题 9′** 假设 $\alpha>1$。则 $f\in\Lambda_\alpha^{p,q}$ 当且仅当 $f\in L^p(\boldsymbol{R}^n)$ 且 $\dfrac{\partial f}{\partial x_j}\in\Lambda_{\alpha-1}^{p,q}$，模 $\|f\|_{\Lambda_\alpha^{p,q}}$ 与 $\|f\|_p+\sum_{j=1}^n\left\|\dfrac{\partial f}{\partial x_j}\right\|_{\Lambda_{\alpha-1}^{p,q}}$ 等价。

**定理 4′** 假设 $\alpha>0,\beta\geqslant0$。则 $\mathscr{J}_\beta$ 同构地把 $\Lambda_\beta^{p,q}$ 映到 $\Lambda_{\alpha+\beta}^{p,q}$。

**5.2 $\Lambda_\alpha^{p,q}$ 的进一步认识** 在这些机械的初步工作之后，我们对 $\Lambda_\alpha^{p,q}$ 作某些更有兴趣的研究。第一个问题是，指标 $\alpha$, $p$ 与 $q$ 起什么作用？粗略地回答如下。首先，指标 $p$ 表示用到的基本模；其次，$\alpha$ 给出所具有的光滑性的阶，而指标 $q$ 表示这个光滑性的阶的二阶（更精细的）修正。准确的结果如下。

**命题10** 结论 $\Lambda_{\alpha_1}^{p,q_1}(\boldsymbol{R}^n)\subset\Lambda_{\alpha_2}^{p,q_2}(\boldsymbol{R}^n)$ 成立，如果：(a) $\alpha_1>\alpha_2$（这时 $q_1$ 与 $q_2$ 没有限制），或者 (b) $\alpha_1=\alpha_2$ 而 $q_1\leqslant q_2$。

证明基于下述引理，它无非是调和函数通常的极值原理的一种变形而已。

**引理 6** 假设 $f\in L^p(\boldsymbol{R}^n)$，则对于任意 $k$，函数

$$\left\|\frac{\partial^k}{\partial y^k}u(x,y)\right\|_p$$

是 $y$ 的非增函数，$0<y<\infty$。

首先考虑 $k=0$ 的情形。由于 $P_{y_1}*P_{y_2}=P_{y_1+y_2}$，我们有

$$u(x,y_1+y_2)=P_{y_1}*u(x,y_2),\tag{64}$$

因此 $\|u(x,y_1+y_2)\|_p\leqslant\|P_{y_1}\|_1\|u(x,y_2)\|_p$。根据 $\|P_{y_1}\|_1=1$，我们得到 $\|u(x,y_1+y_2)\|_p\leqslant\|u(x,y_2)\|_p$，这就是所要证的。为了证明引理的一般情形，在等式（64）中对 $y_2$ 微分 $k$ 次，再进行类似

① 对 $q=\infty$，我们把

$$\left(\int_{\boldsymbol{R}^n}\frac{(\|f(x+t)+f(x-t)-2f(x)\|_p)^q}{|t|^{n+\alpha q}}dt\right)^{1/q}$$

理解为

$$\sup_{|t|>0}\frac{\|f(x+t)+f(x-t)-2f(x)\|_p}{|t|^\alpha},$$

的推理。

让我们来证明命题的(b)((a)可类似地进行，无论如何，结论总是粗糙一些)．假设 $q_1<\infty$，并且

(65) $$\left(\int_0^\infty\left(y^{k-\alpha}\left\|\frac{\partial^k u}{\partial y^k}\right\|_p\right)^{q_1}\frac{dy}{y}\right)^{1/q_1}=A.$$

这时

$$\int_{y_0/2}^{y_0} y^{(k-\alpha)q_1}\left\|\frac{\partial^k u}{\partial y^k}\right\|_p^{q_1}\frac{dy}{y}\leqslant A^{q_1}.$$

然而，根据引理 $\left\|\frac{\partial^k u}{\partial y^k}\right\|_p$ 在上述积分的端点 $(y=y_0)$ 取它的极小值。因此我们有

$$\left\|\frac{\partial^k}{\partial y^k}u(x,y_0)\right\|_p^{q_1}\int_{y_0/2}^{y_0} y^{(\alpha-k)q_1}\frac{dy}{y}\leqslant A^{q_1},$$

即

(66) $$\left\|\frac{\partial^k u}{\partial y^k}\right\|_p\leqslant cAy^{-k+\alpha}.$$

换句话说，就是 $f\in\Lambda_\alpha^{p,q_1}$ 蕴含了 $f\in\Lambda_\alpha^{p,\infty}$。把式(66)与(65)合起来，就容易证明

$$\left(\int_0^\infty\left(y^{k-\alpha}\left\|\frac{\partial^k u}{\partial y^k}\right\|_p\right)^{q_2}\frac{dy}{y}\right)^{1/q_2}<\infty,\qquad q_2\geqslant q_1,$$

从而 $f\in\Lambda_\alpha^{p,q_2}$。

这种类型的其它包含关系，见后面的§6.7。

**5.3 $\mathscr{L}_\alpha^p$ 与 $\Lambda_\alpha^{p,q}$ 的比较** 我们现在转到我们的主要目标之一，它说明在§4与§5所做的许多准备工作是必要的。位势空间 $\mathscr{L}_\alpha^p$ 与 Lipschitz 空间 $\Lambda_\alpha^{p,q}$ 之间的比较达到了本章最深刻的认识，而且顺便说，这是唯一用到第四章 Littlewood-Paley 理论的地方。结果如下：

**定理 5** 假设$1<p<\infty$且 $\alpha>0$，则

(a) $\mathscr{L}_\alpha^p\subset\Lambda_\alpha^{p,p}$，当 $p\geqslant 2$；

(b) $\mathscr{L}_\alpha^p\subset\Lambda_\alpha^{p,2}$，当 $p\leqslant 2$；

(c) $\Lambda_\alpha^{p,p}\subset\mathscr{L}_\alpha^p$，当 $p\leqslant 2$；

(d) $\Lambda_\alpha^{p,2}\subset\mathscr{L}_\alpha^p$，当 $p\geqslant 2$。

这种类型的更好的包含关系是不可能的，这一事实包含在后面的§6.8与§6.9中。

由于算子$\mathscr{J}_\beta$给出空间的同构性(见定理 4 与§3.3的(41))，只要对$\alpha$的任何特殊值证明包含关系就可以了。对我们来说，取 $\alpha=1$ 是方便的。由于§3.3的定理3，空间$\mathscr{L}_1^p$等价于$\dot{L}$(当$1<p<\infty$)，因此我们只要对$\alpha=1$并用$L_1^p$代替$\mathscr{L}_1^p$来证明包含关系。

$\Lambda_1^{p,q}$的模可以通过$f$的 Poisson 积分$u$的二阶微商表示，因此我们考虑 Littlewood-Paley 函数的下列变形：

$$
(67)\qquad \begin{cases}\mathscr{G}_p(x)=\left(\displaystyle\int(y\,|\,\nabla^2u(x,y)\,|)^p\frac{dy}{y}\right)^{1/p}, & p<\infty,\\ \mathscr{G}_\infty(x)=\displaystyle\sup_{y>0}y\,|\,\nabla^2u(x,y)\,|,\end{cases}
$$

其中

$$
|\nabla^2u(x,y)|^2=\sum_{k=0}^n\sum_{j=0}^n\left|\frac{\partial^2}{\partial x_j\partial x_k}u(x,y)\right|^2,\qquad x_0=y.
$$

假设

$$
\frac{\partial f}{\partial x_j}\in L^p(\boldsymbol{R}^n),\quad j=1,\cdots,n.
$$

由于$u$是$f$的 Poisson 积分，正如我们曾经在§4.3中指出过的，$\dfrac{\partial f}{\partial x_j}$的 Poisson 积分是$\dfrac{\partial u}{\partial x_j}$。回忆$g$函数的定义(见第四章§1.1)，我们看出

$$
\left[g\left(\frac{\partial f}{\partial x_j}\right)(x)\right]^2=\sum_{k=0}^n\int_0^\infty y\left|\frac{\partial^2}{\partial x_j\partial x_k}u(x,y)\right|^2dy,\qquad x_0=y.
$$

然而，

$$\frac{\partial^2 u}{\partial y^2} = -\sum_{j=1}^{n}\frac{\partial^2 u}{\partial x_j^2},$$

因此

$$\mathscr{G}_2(x) \leqslant c\sum_{j=1}^{n} g\left(\frac{\partial f}{\partial x_j}\right)(x).$$

由第三章§4.9，

$$\sup_{y>0}\left|y\frac{\partial^2}{\partial y^2}u(x,y)\right| \leqslant A\sum_{j=1}^{n} M\left(\frac{\partial f}{\partial x_j}\right)(x).$$

因此

$$\|\mathscr{G}_2(x)\|_p \leqslant A_p\sum_{j=1}^{n}\left\|\frac{\partial f}{\partial x_j}\right\|_p \tag{68}$$

以及

$$\|\mathscr{G}_\infty(x)\|_p \leqslant A_p\sum_{j=1}^{n}\left\|\frac{\partial f}{\partial x_j}\right\|_p. \tag{69}$$

关于$\mathscr{G}_2$的不等式成立是因为第四章的定理1，而关于$\mathscr{G}_\infty$的则是第一章极大定理的一个推论。现在，显然有

$$\mathscr{G}_p^p(x) \leqslant \mathscr{G}_2^2(x)\mathscr{G}_\infty^{p-2}(x), \quad p\geqslant 2.$$

由此

$$\mathscr{G}_p(x) \leqslant \mathscr{G}_2^{2/p}(x)\mathscr{G}_\infty^{(p-2)/p}(x) = \mathscr{G}_2^{\theta}(x)\mathscr{G}_\infty^{1-\theta}(x)$$

（其中 $\theta = 2/p$）。

用Hölder不等式以及（68）与（69），就有

$$\|\mathscr{G}_p(x)\|_p \leqslant \|\mathscr{G}_2(x)\|_p^{\theta}\|\mathscr{G}_\infty(x)\|_p^{1-\theta} \leqslant A_p\sum_{j=1}^{n}\left\|\frac{\partial f}{\partial x_j}\right\|_p.$$

特别地，我们就有

$$\int_0^\infty\left(y\left\|\frac{\partial^2 u}{\partial y^2}\right\|_p\right)^p\frac{dy}{y} < \infty, \quad 2\leqslant p<\infty.$$

这就表明了，当$2\leqslant p<\infty$时，$L_1^p(\boldsymbol{R}^n)\subset\Lambda_1^{p,p}(\boldsymbol{R}^n)$，从而结

论(a)得证。

为了证明(b)，我们用下面的积分形式的 Minkowski 不等式：当 $F(x,y)\geqslant 0$，$r\geqslant 1$时

$$(70)\qquad \left(\int_0^\infty\left\{\int_{R^n}F(x,y)dx\right\}^r ydy\right)^{\frac{1}{r}}$$

$$\leqslant\int_{R^n}\left(\int_0^\infty F^r(x,y)ydy\right)^{1/r}dx,$$

这就是说，积分的模不大于模的积分，取 $r=2/p$（这里 $p\leqslant 2$），以及 $F(x,y)=|\nabla u(x,y)|^p$。这时(70)可以改写为

$$\int_0^\infty y\||\nabla^2 u|\|_p^2dy\leqslant\left(\int_{R^n}(\mathscr{G}_2(x))^pdx\right)^{2/p},$$

而由(68)知，当 $f\in L_1^p$时，上式右边是有限的。故 $f\in\mathscr{L}_1^p$ 蕴含了 $f\in\Lambda_1^{p,2}$，当 $p\leqslant 2$，从而(b)得证。

上面的推理还证明了 $\|f\|_{\Lambda_1^{p,q}}\leqslant A_p\|f\|_{L_1^p}$，其中取 $q=p$，当 $2\leqslant p<\infty$；取 $q=2$，当$1<p\leqslant 2$。

我们现在来证明反过来的结论(c)与(d)，方法是在 $f$ 属于 $L_1^p$ 的前提下，建立**先验**不等式

$$(71)\qquad \|f\|_{L_1^p}\leqslant A_p\|f\|_{\Lambda_1^{p,q}},$$

其中取 $q=p$，当$1<p\leqslant 2$；取 $q=2$，当$2\leqslant p<\infty$。

这只要把刚刚进行的推理反过来就行。事实上，当 $r\leqslant 1$ 时，积分的 Minkowski 不等式表明，把(70)中的不等号反过来也成立。这样，我们就有

$$\left(\int_{R^n}(\mathscr{G}_p(x))^pdx\right)^{2/p}\leqslant\int_0^\infty y\||\nabla^2u|\|_p^2dy,\qquad p\geqslant 2.$$

但是由于

$$A'_p\left\|\frac{\partial f}{\partial x_j}\right\|_p\leqslant\|\mathscr{G}_2(x)\|_p,$$

这是根据第四章定理1的逆，我们对$2\leqslant p<\infty$得到了式(71)。

类似地，当$1<p\leqslant 2$时，$\mathscr{G}_2(x)\leqslant\mathscr{G}_p^\theta(x)\mathscr{G}_\infty^{1-\theta}(x)$，其中 $\theta=$

$p/2$，用 Hölder 不等式就有

$$\|\mathscr{G}_2(x)\|_p \leqslant \|\mathscr{G}_p(x)\|_p^\theta \|\mathscr{G}_\infty(x)\|_p^{1-\theta}.$$

再一次用 Littlewood-Paley 定理，$\|\mathscr{G}_2\|_p$ 超过 $A'_p\left\|\frac{\partial f}{\partial x_j}\right\|_p$，由（69）就得到

$$\left(A'' \sum \left\|\frac{\partial f}{\partial x_j}\right\|_p\right)^\theta \leqslant \|\mathscr{G}_p(x)\|_p^\theta.$$

然而

$$\begin{aligned}\|\mathscr{G}_p(x)\|_p &= \left(\int_0^\infty y^p \|\nabla^2 u\|_p^p \frac{dy}{y}\right)^{1/p}\\ &\leqslant c\left(\int_0^\infty y^p \left\|\frac{\partial^2 u}{\partial y^2}\right\|_p^p \frac{dy}{y}\right)^{1/p}\\ &\leqslant c\|f\|_{\Lambda_1^{p,p}},\end{aligned}$$

这是根据（62′）与 $\Lambda_1^{p,p}$ 模的定义。因此，（71）对 $1<p\leqslant 2$ 也就证明了。

最后，为了去掉 $f\in L_1^p$ 的限制，代替 $f$ 我们考虑 $u(x,\varepsilon)$，$\varepsilon>0$。显然，当 $f\in\Lambda_1^{p,q}$ 时，$u(x,\varepsilon)\in L_1^p(\boldsymbol{R}^n)$（因为这时 $f\in L^p(\boldsymbol{R}^n)$）。因此，由（71）

$$\|u(x,\varepsilon)\|_{L_1^p} \leqslant A_p \|u(x,\varepsilon)\|_{\Lambda_1^{p,q}} \leqslant A_p\|f\|_{\Lambda_1^{p,q}}.$$

故 $u(x,\varepsilon)$按 $L^p$ 模收敛，并且它的 $L_1^p$ 模一致有界。由此，对每个 $j$，有

$$\int_{\boldsymbol{R}^n} \frac{\partial}{\partial x_j} u(x,\varepsilon)\varphi dx \to -\int_{\boldsymbol{R}^n} f(x)\frac{\partial \varphi}{\partial x_j}dx,$$

只要 $\varphi\in\mathscr{D}$，而线性泛函

$$\varphi \to \int_{\boldsymbol{R}^n} f\frac{\partial\varphi}{\partial x_j}dx$$

按 $L^{p}(\boldsymbol{R}^n)$的共轭模是有界的。故用 Riesz 表示定理，存在 $g_j\in L^p$使得

$$\int_{\boldsymbol{R}^n} f\frac{\partial\varphi}{\partial x_j}dx = -\int_{\boldsymbol{R}^n} g_j\varphi dx.$$

这就证明了 $f\in L_1^p$，从而定理证完。

**5.4 一个留下来的问题** 我们回到留至现在的式(59)的证明。回顾在§5.1给出的 $\Lambda_\beta^{p,q}$ 的定义，我们只需证明下面的事实

$$G_\beta(x)\in\Lambda_\beta^{1,\infty},\qquad \beta>0。\tag{72}$$

让我们首先考虑$0<\beta<1$的情形。由于 $G_\beta\in L^1(\boldsymbol{R}^n)$，根据式(60′)，我们要证明

$$\int_{\boldsymbol{R}^n}|G_\beta(x+t)-G_\beta(x)|dx\leqslant A|t|^\beta。$$

写出

$$\int_{\boldsymbol{R}^n}|G_\beta(x+t)-G_\beta(x)|dx$$

$$=\int_{|x|<2|t|}|\cdot|dx+\int_{|x|>2|t|}|\cdot|dx。$$

上式右端第一个积分被

$$\int_{|x|<2|t|}[|G_\beta(x+t)|+|G_\beta(x)|]dx$$

$$\leqslant 2\int_{|x|<3|t|}|G_\beta(x)|dx$$

控制。由于 $G_\beta(x)\leqslant c|x|^{-n+\beta}$(见§3.1的式(29)与(30))，我们得到

$$2\int_{|x|<3t}|G_\beta(x)|dx\leqslant A|t|^\beta。$$

另外，通过对§3.1的公式(26)进行微分，很快就得到

$$\left|\frac{\partial G_\beta}{\partial x_j}\right|=c|x_j|\int_0^\infty e^{-\pi|x|^2/\delta}e^{-\delta/4\pi}\delta^{(\beta-n-2)/2}\frac{d\delta}{\delta}$$

$$\leqslant c|x_j|\int_0^\infty e^{-\pi|x|^2/\delta}\delta^{(\beta-n-2)/2}\frac{d\delta}{\delta}$$

$$=c'|x_j||x|^{-n+\beta-2}\leqslant c'|x|^{-n+\beta-1}。$$

因此，$|G_\beta(x+t)-G_\beta(x)|\leqslant c''|t||x|^{-n+\beta-1}$，只要$|x|>2|t|$。从而

$$\int_{|x|\geqslant 2|t|}|G_\beta(x+t)-G_\beta(x)|dx\leqslant A|t|^\beta,$$

只要$0<\beta<1$。故式（59）因而（72）当$0<\beta<1$时得证。

为转到一般情形，我们注意，当 $r$ 是正整数时，$G_{\beta r}=G_\beta*G_\beta*\cdots*G_\beta$（$r$ 个），并且 $P_y=P_{y_1}*P_{y_2}*\cdots*P_{y_r}$，其中 $y=y_1+y_2+\cdots+y_r$，$y_k>0$。因此

$$G_{\beta r}(\cdot,y)=G_\beta(\cdot,y_1)*G_\beta(\cdot,y_2)*\cdots*G_\beta(\cdot,y_r).$$

现在分别对 $y_1,y_2,\cdots,y_r$ 微分这个关系式一次，并令 $y_1=y_2=\cdots=y_r=y/r$。结果是

$$\left\|\frac{\partial^r G_{\beta r}(x,y)}{\partial y^r}\right\|_1\leqslant Ay^{-\beta r}.$$

由于$0<\beta<1$是任意的，我们便得到式（59）所要求的估计（用 $\beta r$ 代替 $\beta$），而这蕴含了式（72）。

## §6 进一步的结果

**6.1** $f$ 属于 $L_1^p(R^n)$，当且仅当 $f\in L^p(R^n)$，并且：(a) 可以在一零测集修改 $f$ 的值，使它成为按 Tonelli 意义是连续的；(b) $\dfrac{\partial f}{\partial x_j}\in L^p(R^n),j=1,\cdots,n$（微商几乎处处存在）。

**6.2** $f$ 属于 $L_k^\infty(R^n),k\geqslant 1$，当且仅当 $f$ 在一零测集上经过修改，使得它满足下列两条件中的任一个：

(a) $f$ 有阶数$\leqslant k-1$的连续偏微商，并且若

$$g=\frac{\partial^\alpha f}{\partial x^\alpha},\qquad |\alpha|\leqslant k-1,$$

则

$$\sup_x|g(x)|<\infty,\quad 而且\ \sup_{x,x'}\frac{|g(x)-g(x')|}{|x-x'|}<\infty.$$

(b) 存在序列$\{\varphi_n\}$，$\varphi_n\in\mathscr{D}$，使得在每个紧集上 $\varphi_n\to f$ 一致成立，并且

$$\sup_{|\alpha|\leqslant k}\ \sup_{n}\left\|\frac{\partial^{\alpha}\varphi_n}{\partial x^{\alpha}}\right\|_{\infty}<\infty.$$

（提示：分别参看§3.5中命题3与§2.1中命题1的证明。）

**6.3** 假设$1<p<\infty$而$1/p=k/n$，则存在$f\in L_k^p(\boldsymbol{R}^n)$，它在每一点的邻域都不是本质有界的。

**提示**：例如，考虑$n=2$，$k=1$(从而$p=2$)的情形。令

$$\varphi(x)=|x|^{-1}\left(\log\frac{1}{|x|}\right)^{-1},\qquad |x|\leqslant 1/2;$$

在其它地方$\varphi=0$。记$f_0=I_1(\varphi)$。这时

$$\frac{\partial f_0}{\partial x_j}=R_j(\varphi)\in L^2.$$

然而，$f_0$在原点附近无界。人们还可以比较直接地构造一个类似的$f_0$，这就是当$x$很小时，

$$f_0(x)=\log\log\frac{1}{|x|},$$

并且$f_0$是正的、光滑的、具有紧支集且其支集与原点分离。最后令

$$f(x)=\sum_{k=1}^{\infty}2^{-k}f_0(x-r_k),$$

其中$\{r_k\}$是$\boldsymbol{R}^n$的稠密集。

**6.4** 下面是§2.5中不等式（23）的推广。假设$1<k<n$。这时对$1/q=1-k/n$，$f\in\mathscr{D}$

$$\|f\|_q\leqslant\left(\prod_{i_1,\cdots,i_k}\left\|\frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}\right\|_1\right)^{1/\binom{n}{k}}$$

其中乘积对$i_1,i_2,\cdots,i_k$取遍$1,2,\cdots,n$的

$$\binom{n}{k}=\frac{n!}{k!(n-k)!}$$

个不同的组合。

**提示**：例如，考虑 $k=n-1$。记

$$I_j(x_j)=\int_{R^k}\left|\frac{\partial^{n-1}f}{\partial\hat{x}_j^{n-1}}\right|d\hat{x}_j,$$

其中符号 $\hat{x}_j$ 表示 $x_j$ 以外的变量。显然，$|f(x)|\leqslant I_j(x_j)$，因此

$$|f(x)|^n\leqslant\prod_{j=1}^{n}I_j(x_j).$$

再积分。（如果不是从

$$f(x)=\int_{-\infty}^{x}f'(t)dt$$

而是从
$$f(x)=\frac{1}{2}\int_{-\infty}^{+\infty}\operatorname{sign}(x-t)f'(t)dt$$

出发，那末上述等式可以改进到多一个因子$2^{-k}$。）

6.5 关于 Bessel 核的公式（26）的另一个形式是

$$G_\alpha(x)=c_\alpha e^{-|x|}\int_0^\infty e^{-|x|t}\left(t+\frac{t^2}{2}\right)^{(n-\alpha-1)/2}dt,$$

当 $0<\alpha<n+1$，其中

$$c_\alpha^{-1}=(2\pi)^{(n-1)/2}2^{\alpha/2}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{n-\alpha+1}{2}\right).$$

见 Aronszajn 与 Smith[1]。

6.6 下面描述 $L_k^p(R^n)$与 $\mathscr{L}_k^p(R^n)$之间在端点情形 $p=1$ 与 $p=\infty$可能的包含关系。

(a) 当 $n=1$时，$L_k^p(R^1)=\mathscr{L}_k^p(R^1)$，如果 $k$ 是偶数，且 $p=1$ 或$\infty$。

(b) 当 $n>1$时，$L_k^p(R^n)\subset\mathscr{L}_k^p(R^n)$，如果 $k$ 是偶数且 $p=1$ 或$\infty$；反过来的包含关系对 $p=1$ 与 $p=\infty$均不成立。

(c) 对所有的 $n$，当 $k$ 是奇数时，$L_k^p(R^n)\subset\mathscr{L}_k^p(R^n)$ 与 $\mathscr{L}_k^p(R^n)\subset L_k^p(R^n)$都不成立。

**提示**：对(a)用这样的事实，$f\in L^p(\boldsymbol{R}^1)$ 与 $\frac{d^2f}{dx^2}\in L^p(\boldsymbol{R}^1)$ 蕴含了$\frac{df}{dx}\in L^p(\boldsymbol{R}^1)$。为了证明$\mathscr{L}_k^p(\boldsymbol{R}^n)\not\subset L_k^p(\boldsymbol{R}^n)$，用高阶 Riesz 变换在 $L^1$ 与 $L^\infty$的无界性（见第二章§6.1）。例如，为了看 $L_1^\infty(\boldsymbol{R}^n)\not\subset\mathscr{L}_1^\infty(\boldsymbol{R}^n)$，用函数 $G_{n+1}(x)$。从公式（26）容易推出，$G_{n+1}$与$\frac{\partial G_{n+1}}{\partial x_j}\in L^\infty$，因此 $G_{n+1}\in L_1^\infty$。然而 $G_{n+1}\overline{\in}\mathscr{L}_1^\infty(\boldsymbol{R}^n)$，这是因为

$$G_n(x)\approx\log\frac{1}{|x|},\qquad 当|x|\to 0,$$

因此 $G_n\overline{\in}L^\infty$。当$|x|\to 0$时

$$G_n\approx\log\frac{1}{|x|}$$

的事实也可以从公式（26）推得，就像用同样的方法证明(29)一样。对这种联系有用的特殊函数，由 Wainger[1]研究过。

**6.7**

(a) $\Lambda_{\alpha_1}^{p_1,q}(\boldsymbol{R}^n)\subset\Lambda_{\alpha_2}^{p_2,q}(\boldsymbol{R}^n)$，当 $\alpha_1\geqslant\alpha_2$ 并且

$$\alpha_1-\frac{n}{p_1}=\alpha_2-\frac{n}{p_2}.$$

(b) 若 $f\in\Lambda_{\alpha_j}^{p_j,q_j}(\boldsymbol{R}^n)$，其中 $j=0,1$，则 $f\in\Lambda_\alpha^{p,q}(\boldsymbol{R}^n)$，其中

$$\alpha=\alpha_0(1-\theta)+\alpha_1\theta,$$

$$\frac{1}{p}=\frac{1}{p_0}(1-\theta)+\frac{\theta}{p_1},\qquad \frac{1}{q}=\frac{1}{q_0}(1-\theta)+\frac{\theta}{q_1},$$

对每个 $0<\theta<1$。见 Hardy-Littlewood[2]，Taibleson[2]。

**6.8** 假设

$$f_{a,\sigma}(x)=e^{-\pi x^2}\sum_{k=1}^{\infty}a^{-ka}k^{-\sigma}e^{2\pi i a^k x},\qquad x\in\boldsymbol{R}^1,$$

其中 $a>1$ 是整数。

(a) $f_{a,\sigma}\in\mathscr{L}_a^p(R^1)\Longleftrightarrow\sigma>1/2$，对$1\leqslant p<\infty$；

(b) $f_{a,\sigma}\in\Lambda_a^{p,q}(R^1)\Longleftrightarrow\sigma>1/q$，对$1\leqslant p\leqslant\infty$。

因此 $\mathscr{L}_a^p(R^1)\not\subset\Lambda_a^{p,q}(R^1)$，当 $q<2$；而 $\Lambda_a^{p,q}(R^1)\not\subset\mathscr{L}_a^p(R^1)$，当 $q>2$。

6.9 令 $g_{\alpha,\delta,p}(x)=|x|^{\alpha-n/p}(\log(1/|x|))^{-\delta}$，当$|x|<1/2$，并假设 $g_{\alpha,\delta,p}$ 在原点之外是光滑的且有紧支集，又设 $\alpha<n/p$。

(a) $g_{\alpha,\delta,p}\in\mathscr{L}_a^p(R^n)\Longleftrightarrow\delta p>1$；

(b) $g_{\alpha,\delta,p}\in\Lambda_a^{p,q}(R^n)\Longleftrightarrow\delta q>1$。

因此，$\mathscr{L}_a^p(R^n)\not\subset\Lambda_a^{p,q}(R^n)$，当 $q<p$；而 $\Lambda_a^{p,q}(R^n)\not\subset\mathscr{L}_a^p(R^n)$，当 $q>p$。同§6.8与§6.9密切有关的例子，见 Taibleson[2]。

6.10 假设$0<\alpha<2$。则 $f\in\mathscr{L}_a^p(R^n)\Longleftrightarrow f\in L^p$，并且

(a) $\lim\limits_{\varepsilon\to0}I_\varepsilon\equiv\lim\limits_{\varepsilon\to0}\int_{|t|\geqslant\varepsilon}\frac{[f(x+t)-f(x)]}{|t|^{n+\alpha}}dt$ 按 $L^p$ 模收敛，

当 $1\leqslant p<\infty$。

(b) $I_\varepsilon$ 按 $L^\infty$ 模保持有界，当 $p=\infty$。见 Stein[7]；还有 Wheeden[2]。

**提示**：当 $f\in\mathscr{D}$时，验证

$$\lim_{\varepsilon\to0}\int_{|t|\geqslant\varepsilon}\frac{f(x+t)-f(x)}{|t|^{n+\alpha}}dt=c_\alpha(-\Delta)^{\alpha/2}f.$$

反过来，假设 $f=\mathscr{J}_\alpha(g)$，$g\in L^p$，则 $f=I_\alpha(\gamma)$，$\gamma\in L^p$（见§3.2）。还有

$$\int_{|t|\geqslant\varepsilon}\frac{f(x+t)-f(x)}{|t|^{n+\alpha}}dt=\int K_\varepsilon(t)\gamma(x+t)dt,$$

其中 $K_\varepsilon(t)=\varepsilon^{-n}K(t/\varepsilon)$。还可以证明$|K(x)|\leqslant A|x|^{-n+\alpha}$与$|K(x)|\leqslant A|x|^{-n+\alpha-2}$，从而 $K\in L^1(R^n)$。

6.11 (a) 空间 $\mathscr{L}_a^p(R^n)$对逐点乘法组成一个代数，当且仅当$\mathscr{L}_a^p(R^n)$的每个元素是连续的，而这成立当且仅当 $p>n/\alpha$。

(b) 设 $\chi_K$ 是任一凸集$K\subset R^n$ 的特征函数，则变换 $f\to\chi_K f$

是在 $\mathscr{L}_\alpha^p(\boldsymbol{R}^n)$ 连续的，只要$0\leqslant\alpha<1/p$。对这些以及有关的结果见 Strichartz[1]。对 $p=2$，也可见 Hirschmann[2]。

6.12 假设 $F=I_\alpha(f), 0<\alpha<1$，

$$\mathscr{D}_\alpha(F)(x)=\left(\int_{\boldsymbol{R}^n}\frac{|F(x+t)-F(x)|^2}{|t|^{n+2\alpha}}dt\right)^{1/2},$$

则

$$A_\alpha g_1(f)(x)\leqslant\mathscr{D}_\alpha(F)(x)\leqslant B_\alpha g_\lambda^*(f)(x),$$

其中 $\lambda<1+2\alpha/n$，函数 $g_1$与 $g_\lambda^*$ 在第四章已经定义(分别见 § 1.2 与 § 2.2)，$A_\alpha$ 与 $B_\alpha$ 是合适的常数。

**提示**：分别用 $U(x,y)$与 $u(x,y)$记 $F$ 与 $f$ 的 Poisson 积分。由于

$$\frac{\partial^2 U}{\partial y^2}=\int\frac{\partial^2 P_y(t)}{\partial y^2}[F(x+t)-F(x)]dt,$$

简单的估计给出

$$\int_0^\infty y^{3+2\alpha}\left|\frac{\partial^2 U}{\partial y^2}\right|^2 dy\leqslant c_1(\mathscr{D}_\alpha(F))^2.$$

另外

$$\frac{\partial^2 u(x,y)}{\partial y^2}=\frac{1}{\Gamma(1-\alpha)}\int_0^\infty\frac{\partial^2 U(x,y+s)}{\partial y^2}s^\alpha ds$$

$$=\frac{1}{\Gamma(1-\alpha)}\int_y^\infty\frac{\partial^2 U(x,s)}{\partial y^2}(s-y)^\alpha ds.$$

因此

$$\int_0^\infty y\left|\frac{\partial u}{\partial y}\right|^2 dy\leqslant c_2\int_0^\infty y^{3+2\alpha}\left|\frac{\partial^2 U}{\partial y^2}\right|^2 dy,$$

从而 $A_\alpha g_1(f)(x)\leqslant\mathscr{D}_\alpha(F)(x)$。反过来，我们有

$$|F(x+t)-F(x)|\leqslant\int_{L_1}|\nabla U|ds+\int_{L_2}|\nabla U|ds+\int_{L_3}|\nabla U|ds,$$

其中 $L_1, L_2$与 $L_3$分别是联结$(x,0)$到$(x,y)$，$(x+t,0)$到$(x+t,y)$，

$(x+t,y)$到$(x,y)$的线段。然而

$$U(x,y)=\frac{1}{\Gamma(\alpha)}\int_0^\infty u(x,y+s)s^{-1+\alpha}ds,$$

因此

$$|\nabla U(x,y)|\leqslant\frac{1}{\Gamma(\alpha)}\int_0^\infty|\nabla u(x,y+s)|s^{-1+\alpha}ds.$$

如果把这个估计代入上式，令$y=|t|$，并进行积分，化简以后便得结果$\mathscr{D}_\alpha(F)(x)\leqslant B_\alpha g_\lambda^*(f)(x)$，对$\lambda<1+2\alpha/n$。也可见下面§6.13引的文献。

**6.13** (a) 假设$0<\alpha<1$并且$2n/(n+2\alpha)<p<\infty$（特别地，当$2\leqslant p<\infty$时后者成立），则$f\in\mathscr{L}_\alpha^p(\boldsymbol{R}^n)$当且仅当$f\in L^p(\boldsymbol{R}^n)$且$\mathscr{D}_\alpha(f)\in L^p(\boldsymbol{R}^n)$。还有$\|f\|_{p,\alpha}$与$\|f\|_p+\|\mathscr{D}_\alpha(f)\|_p$可比较。

(b) 类似的结果在较大的范围$0<\alpha<2$成立，如果$\mathscr{D}_\alpha(f)$用

$$\left(\int_{\boldsymbol{R}^n}\frac{|f(x+t)+f(x-t)-2f(x)|^2}{|t|^{n+2\alpha}}dt\right)^{\frac{1}{2}}$$

代替。§6.12以及§6.13(a)与(b)的结果在Stein[7]中叙述。更早的（一维的）理论见Marcinkiewicz[2]，Zygmund[1]与Hirschmann[1]。对临界情形$p=2n/(n+2\alpha)$的晚近的更强的结果见Fefferman[1]。

(c) 上述(a)的一个变形对所有$1<p<\infty$中的$p$成立，如果$\mathscr{D}_\alpha(f)$用

$$\left(\int_0^\infty\left\{\int_B|f(x+rt)-f(x)|dt\right\}^2\frac{dr}{r^{1+2\alpha}}\right)^{1/2}$$

代替，其中$B$是单位球。见Strichartz[1]。

**6.14** 令$\beta>\alpha$.则$T$是同平移可交换的从$\Lambda_\alpha(\boldsymbol{R}^n)$到$\Lambda_\beta(\boldsymbol{R}^n)$的有界线性变换，当且仅当$T$具有形式$T(f)=K*f$，其中$K\in\Lambda_{\beta-\alpha}^{1,\infty}(\boldsymbol{R}^n)$。对$n=1$情形，见Zygmund[6]，而一般情形见Taibleson[2]。

6.15 (a) 假设$T$是上面§6.14所讨论的那种变换，则$T$把$L^p(\boldsymbol{R}^n)$有界地变到$L^q(\boldsymbol{R}^n)$，如果$1<p$，$q<\infty$，并且

$$\frac{1}{q}=\frac{1}{p}-\frac{\beta-\alpha}{n}.$$

(b) 然而，存在一个同平移可交换并且把$\Lambda_\alpha(\boldsymbol{R}^n)$有界地变到$\Lambda_\alpha(\boldsymbol{R}^n)$的$T$，它不是$L^p(\boldsymbol{R}^n)$有界的，当$p\neq 2$。见Stein与Zygmund[2]；这个方向更早的结果是在Hardy与Littlewood[3]中。

6.16 最后，我们研究**有界平均振动**函数空间；它的意义在于，当许多结果在通常的极限情形即在$L^\infty$不成立时，这函数类经常可以作为$L^\infty$的替代物。

(a) 假设$f$在$\boldsymbol{R}^n$定义。我们说它是（在$\boldsymbol{R}^n$）有界平均振动的（简称为BMO），如果存在常数$M$，使得

$$\frac{1}{m(Q)}\int_Q|f(x)-a_Q|dx\leqslant M$$

对每个$\boldsymbol{R}^n$中的立方体$Q$成立，其中$a_Q$是$f$在$Q$的平均值。注意，每个有界函数是BMO的；然而$\log|x|$属于BMO，可见反过来不成立。这例子在某种程度上是典型的，这可以从下面的估计式看出来：

$$m\{x\in Q: |f(x)-a_Q|>\alpha\}\leqslant ce^{-c\alpha/M}m(Q),$$

对任意$\alpha>0$。特别地，若$f$是BMO，则

$$\int_Q e^{a|f|}dx<\infty,$$

对每个$Q$以及合适的正数$a$。见John-Nirenberg[1]。

(b) 记$T$是在第二章定理1、它的推论、或定理2、定理3中所研究的任一种奇异积分变换。假设$f$有界，则$T(f)$是BMO的。见Stein[8]。

6.17 (a) 假设$f$是BMO的，则$\mathscr{J}_\alpha(f)\in\Lambda_\alpha(\boldsymbol{R}^n)$，$\alpha>0$。

(b) 假设 $f$ 是弱 $p$ 型的，即 $m\{x:|f(x)|>\lambda\}\leqslant A\lambda^{-p}$，$0<\lambda<\infty$，对 $1<p<\infty$，则 $\mathscr{I}_\alpha(f)\in\mathrm{BMO}$，如果 $\alpha=n/p$（同本章的定理 1 比较）。

见 Stein 与 Zygmund[2]。

**6.18** 假设 $f\in\Lambda_1^{\infty,2}$，即假设 $f\in L^\infty(\boldsymbol{R}^n)$，并且

$$\int_{\boldsymbol{R}^n}\frac{\|f(x+t)+f(x-t)-2f(x)\|_\infty^2}{|t|^{n+2}}dt<\infty,$$

则 $\dfrac{\partial f}{\partial x_j}\in\mathrm{BMO}$，$j=1,\cdots,n$。

**提示**：用 § 4 与 § 5 的论证可以证明，这里的假设蕴含了存在函数 $\delta(s)$，$0<s<\infty$，使得

(a) $\delta(s)$ 在 $0<s<\infty$ 非降；

(b) $\displaystyle\int_0^1\frac{\delta^2(s)}{s}ds<\infty$；

(c) $\|f(x+t)+f(x-t)-2f(x)\|_\infty\leqslant t\delta(|t|)$。

完成这点以后，人们就可以改写在 John 与 Nirenberg[1] 中对

$$\delta(s)=\left(\log\frac{1}{s}\right)^{-\frac{1}{2}-\varepsilon},\ \varepsilon>0\quad\left(s<\frac{1}{2}\right)$$

所给出的论证。一个早期的有关结果见 M.Weiss 与 Zygmund [1]。

## 注 释

**节 1** 关于 Riesz 位势见 M.Riesz[2]，一个较早的一维说法见 Weyl[1]。定理 1 的 $L^p$ 不等式，$n=1$ 时属于 Hardy-Littlewood[2]，对一般的 $n$ 属于 Sobolev[1]。分数次积分变换是弱 $(1,n/(n-\alpha))$ 型的事实，最早出现在 Zygmund[4]。关于 Lorentz 空间的一般研究见 O'Neil[1]. 本书中给出的简单证明取自 Muckenhoupt 与 Stein[1]。

**节 2** 定理 2 见 Sobobv[1]。然而 $p=1$ 的情形，直到后来 Gagliardo[2]与 Nirenberg[1]才得到解决。

**节 3** Bessel 位势与对应的空间$\mathscr{L}_\alpha^p$是由 Aronszajn 与Smith [1]以及 Calderón[4]引入的。联系 Bessel 与 Riesz 位势的引理2在 Stein[7]中叙述。$\mathscr{L}_k^p$ 与 $L_k^p(1<p<\infty)$的等同，在 Calderón[4]中证明，而命题 4 中给出的 $\mathscr{L}_\alpha^2$ 的特征取自 Aronszajn 与 Smith [1]。

**节 4 与节 5** 在 $n=1$的情形，这里给出的大部分结果，是由 Hardy-Littlewood[2]和[3]，Zygmund[6]，Hirschmann[1]用这种或那种方式（有时只是含蓄地）叙述并证明的。第一次对空间$\Lambda_\alpha^{p,q}$($n$ 维) 的明白叙述见 Besov[1]；然而，这曾经在 Gagliardo 一篇有意义的文章[1]中谈过。在这两节中给出的叙述，主要依赖于 Taibleson[2]的系统研究；特别地，定理 4′与 5 是属于他的。读者还可以参阅 Nikolskĭi 稍早的综合性文章[1]。

# 第六章 开拓与限制

当我们想把 $\boldsymbol{R}^n$ 上调和分析的结果应用到各种其它问题上去时，往往遇到下面的情况。设 $S$ 是 $\boldsymbol{R}^n$ 的一个子集（$S$ 的性质将在后面规定），考虑某个我们已经研究过的 $\boldsymbol{R}^n$ 上函数的Banach空间。那末出现两个问题。**限制**问题：把这个Banach空间中的函数限制在 $S$ 上，所产生的函数空间是什么？还有与之密切相关的**开拓**问题：给定一个定义在 $S$ 上的特殊的函数空间，如何能把这些函数开拓到 $\boldsymbol{R}^n$ 上去？

处理这些问题的方法与结果因集合 $S$ 的性质不同而不同，虽然有一些重叠。我们将选出三种真正有意义的情形：

(a) 集合 $S$ 是一个任意的闭集 $F$。合适的函数空间是由下述函数所组成的，它们具有直到某阶的连续偏微商，带着它们的连续模的界。这时归结到Whitney 型的开拓，除了细节以外我们沿用他的结构。

(b) 集合 $S$ 是一个区域($\boldsymbol{R}^n$ 中的开子集)，它的边界满足某种最小光滑性条件。假如区域有一个光滑的(例如 $C^\infty$)边界，开拓会极其容易，一个比较简单的构造就能完成这个任务。这里所给的开拓的主要之点在于只需假定边界总共只有一阶可微性，而得到所有阶可微性的开拓。我们的介绍（在 § 3）是基于与Calderón在这方面最初引进的方法不相同的一种想法，他的方法的要点在后面 § 4.8中简略叙述。

(c) 集合 $S$ 是 $\boldsymbol{R}^n$ 的一个线性子流形 $\boldsymbol{R}^m$，从限制问题的观点来看，把 $\boldsymbol{R}^n$ 上的函数看作 $\boldsymbol{R}^m$ 上的函数，一般来说其光滑性会有损失。由于 $\boldsymbol{R}^m$ 在 $\boldsymbol{R}^n$ 中的Lebesgue测度为零，这又产生一个问题，对 $\boldsymbol{R}^n$ 上的函数如何给出它们在 $\boldsymbol{R}^m$ 上的自然定义，使得其限制是

完全确定的。这种困难在情形(a)不会出现，因为那里只处理连续函数。而这里所考虑的函数可能处处不连续，不过它们具有某种平均连续性。令人惊异的是出现这样的事实，对 $\boldsymbol{R}^n$ 合适的函数空间(作为限制)在性质上可以大大不同于对 $\boldsymbol{R}^n$ 合适的函数空间。

本章将从详述把$\boldsymbol{R}^n$中任意开集分解成适当的“不相重叠的”方体的并集开始。这种分解的用处在第一章中已指出过。在这里我们将再次应用它。单位分解以它为基础，并作为类型(a)的开拓的主要工具。如果只从隐含的意义来说，它再次出现在类型(b)的开拓中。

## §1 开集分解成立方体

在下文中，$F$ 表示$\boldsymbol{R}^n$中任意的非空闭集，$\Omega$是它的余集。**立方体**是指 $\boldsymbol{R}^n$ 中其边平行于坐标轴的闭立方体，如果两个这样的立方体内部不相交就称之为**不相重叠的**。对于这样的立方体 $Q$，piam$(Q)$表示它的直径，dist$(Q,F)$表示它到$F$的距离。

1.1 **定理 1** 设给定 $F$。则存在立方体的集合$\mathscr{F}$，$\mathscr{F}=\{Q_1,Q_2,\cdots,Q_k,\cdots\}$，使得

(1) $$\bigcup_k Q_k=\Omega=({}^cF);$$

(2) $Q_k$互不重叠；

(3) $$c_1\operatorname{diam}(Q_k)\leqslant\operatorname{dist}(Q_k,F)\leqslant c_2\operatorname{diam}(Qk).$$

常数$c_1$和$c_2$与$F$无关。实际上可以取$c_1=1$，$c_2=4$。

1.2 考虑 $\boldsymbol{R}^n$ 中坐标为整数的格点。由格点确定一个**网格** $\mathscr{M}_0$，它是立方体的集合：即所有其顶点是上述的格点，边长为单位长的立方体。网格$\mathscr{M}_0$导出这种网格的双向无穷链 $\{\mathscr{M}_k\}_{-\infty}^{\infty}$，其中$\mathscr{M}_k=2^{-k}\mathscr{M}_0$。

网格 $\mathscr{M}_k$ 中的每一个立方体通过把边长平分，产生出网格

$\mathscr{M}_{k+1}$中的 $2^n$ 个立方体。$\mathscr{M}_k$ 中每个立方体的边长为 $2^{-k}$，从而直径为$\sqrt{n}\,2^{-k}$。

除网格$\mathscr{M}_k$外，我们还考虑层$\Omega_k$，其定义为

$$\Omega_k=\{x:\ c2^{-k}<\operatorname{dist}(x,F)\leqslant c2^{-k+1}\};$$

$c$ 是即将确定的正常数。显然 $\Omega=\bigcup\limits_{k=-\infty}^{\infty}\Omega_k$。

现在我们对立方体作初次的选择，并用$\mathscr{F}_0$表示所得的集合。选择的办法如下。我们考虑$\mathscr{M}_k$的立方体(每一个这样的立方体其大小约为 $2^{-k}$)，如果这个网格中某个立方体与$\Omega_k$相交，它就包含在$\mathscr{F}_0$中($\Omega_k$的点与$F$的距离约为$2^{-k}$)，也就是取

$$\mathscr{F}_0=\bigcup_k\{Q\in\mathscr{M}_k:\ Q\cap\Omega_k\neq 0\}。$$

于是便有

$$\bigcup_{Q\in\mathscr{F}_0}Q=\Omega。$$

对于适当选取的$c$，有

(3) $\qquad \operatorname{diam}(Q)\leqslant\operatorname{dist}(Q,F)\leqslant 4\operatorname{diam}(Q),\quad Q\in\mathscr{F}_0$。

我们先来证明公式(3)。假设$Q\in\mathscr{M}_k$；则$Q$的直径$=\sqrt{n}\,2^{-k}$。由$Q\in\mathscr{F}_0$，知存在$x\in Q\cap\Omega_k$。因此

$$\operatorname{dist}(Q,F)\leqslant\operatorname{dist}(x,F)\leqslant c2^{-k+1},$$

并且

$$\operatorname{dist}(Q,F)\geqslant\operatorname{dist}(x,F)-\operatorname{diam}(Q)>c\,2^{-k}-\sqrt{n}\,2^{-k}。$$

选取$c=2\sqrt{n}$，便得到式(3)。

然后，由式(3)知立方体$Q\in\mathscr{F}_0$与$F$不相交，并且显然覆盖了$\Omega$。这就证明了公式(1)。注意到集合$\mathscr{F}_0$除了它的立方体不一定互不重叠以外，具有我们所要求的全部性质。为了完成定理的证明，我们需要改进得到$\mathscr{F}_0$的选择办法，去掉那些实际上不必要的立方体。

需要作以下简单的观察。设$Q_1$与$Q_2$是两个立方体(分别取自

网格$\mathscr{M}_{k_1}$与$\mathscr{M}_{k_2}$)。如果$Q_1$与$Q_2$是重叠的，那末这两者中必定有一个包含在另一个里面（特别，当$k_1 \geq k_2$，有$Q_1 \subset Q_2$)。

现在从任一立方体$Q \in \mathscr{F}_0$开始，考虑$\mathscr{F}_0$中包含它的最大立方体。由不等式(3)看出，对于$\mathscr{F}_0$中包含$Q$的任意立方体$Q'$，有$\operatorname{diam}(Q') \leqslant 4\operatorname{diam}(Q)$。而且任意两个包含$Q$的立方体$Q'$与$Q''$显然具有非空的交集。于是由上述观察得知，对每个立方体$Q \in \mathscr{F}_0$，在$\mathscr{F}_0$中存在一个唯一的包含它的最大立方体。按这种取法，这些最大立方体也是互不重叠的。用$\mathscr{F}$表示$\mathscr{F}_0$中最大立方体的集合。显然

(1) $$\bigcup_{Q \in \mathscr{F}} Q = \Omega;$$

(2) $\mathscr{F}$中的立方体是互不重叠的；

(3) $$\operatorname{diam}(Q) \leqslant \operatorname{dist}(Q, F) \leqslant 4\operatorname{diam}(Q), \quad Q \in \mathscr{F}.$$

定理1证毕。

**1.3 单位分解** 现在我们观察一下立方体族$\mathscr{F}$，它的存在性是定理1保证了的。如果$\mathscr{F}$中两个不同的立方体$Q_1$与$Q_2$的边界具有共同点，就称它们为**相连的**（提醒读者注意，$\mathscr{F}$中两个不同的立方体总是不相重叠的）.

**命题1** 假设$Q_1$与$Q_2$相连，则

$$\frac{1}{4}\operatorname{diam}(Q_2) \leqslant \operatorname{diam}(Q_1) \leqslant 4\operatorname{diam}(Q_2).$$

我们已知$\operatorname{dist}(Q_1, F) \leqslant 4\operatorname{diam}(Q_1)$。由$Q_1$与$Q_2$相连便知

$$\operatorname{dist}(Q_2, F) \leqslant 4\operatorname{diam}(Q_1) + \operatorname{diam}(Q_1) = 5\operatorname{diam}(Q_1).$$

但是$\operatorname{diam}(Q_2) \leqslant \operatorname{dist}(Q_2, F)$，因此$\operatorname{diam}(Q_1) \leqslant 5\operatorname{diam}(Q_2)$。然而对某个整数$k$，有$\operatorname{diam}(Q_2) = 2^k \operatorname{diam}(Q_1)$，故$\operatorname{diam}(Q_1) \leqslant 4\operatorname{diam}(Q_2)$，由对称性，命题便得到了证明。

现在令$N = (12)^n$。在下文中所需要的$N$的确切大小是不重要的；事实上只要求它仅与维数$n$有关，而特别是与闭集$F$无关。

**命题 2** 假设$Q\in\mathscr{F}$。则在$\mathscr{F}$中至多有$N$个立方体是与$Q$相连的。

如果立方体$Q$属于网格$\mathscr{M}_k$，那末容易看到有$3^n$个立方体（包含$Q$在内）属于$\mathscr{M}_k$并与$Q$相连。另外，网格$\mathscr{M}_k$中的每个立方体至多能够包含$\mathscr{F}$中$4^n$个直径$\geqslant(1/4)\mathrm{diam}(Q)$的立方体。把它与命题1综合起来，便得到命题2的证明。

现在用$Q_k$表示$\mathscr{F}$中任意的立方体。设$x^k$是它的中心，$l_k$是它的边长。当然$\mathrm{diam}(Q_k)=\sqrt{n}\,l_k$。对于任意的但在下文中保持固定的$\varepsilon$，$0<\varepsilon<1/4$，用$Q_k^*$表示与$Q_k$有相同中心，但扩大$(1+\varepsilon)$倍的立方体；即，

$$Q_k^*=(1+\varepsilon)[Q_k-x^k]+x^k.$$

显然$Q_k\subset Q_k^*$，并且立方体$Q_k^*$不再是互不重叠的。然而下述命题成立。

**命题 3** $\Omega$的每个点至多包含在$N$个立方体$Q_k^*$中。

设$Q$与$Q_k$是$\mathscr{F}$中两个立方体。我们断言只有当$Q_k$与$Q$相连时$Q_k^*$与$Q$相交。事实上考虑$Q_k$以及$\mathscr{F}$中所有与$Q_k$相连的立方体的并集；由于所有这些立方体的直径$\geqslant(1/4)\mathrm{diam}(Q_k)$，显然这个并集包含$Q_k^*$。因此只有当$Q$与$Q_k$相连时，$Q$才与$Q_k^*$相交。然而任意点$x\in\Omega$必属于某个立方体$Q$，故由命题2，至多有$N$个立方体$Q_k^*$包含$x$。

证明还表明，$\Omega$的每一点包含在一个至多与$N$个立方体$Q_k^*$相交的小邻域内。

现在用$Q_0$表示中心在原点的单位立方体。取定一个具有下述性质的属于$C^\infty$的函数$\varphi$：$0\leqslant\varphi\leqslant1$；$\varphi(x)=1$，当$x\in Q_0$；$\varphi(x)=0$，当$x\overline{\in}(1+\varepsilon)Q_0$。

令$\varphi_k$表示调整到立方体$Q_k$上的函数$\varphi$；即

$$\varphi_k(x)=\varphi\left(\frac{x-x^k}{l_k}\right).$$

注意$x^k$是$Q_k$的中心，$l_k$是$Q_k$的边长。因此$\varphi_k(x)=1$，当

$\in Q_k$; $\varphi_k(x)=0$，当 $x \overline{\in} Q_k^*$。并且还有

$$\left|\frac{\partial^\alpha}{\partial x^\alpha}\varphi_k(x)\right| \leqslant A_\alpha(\operatorname{diam}(Q_k))^{-|\alpha|}. \tag{4}$$

现在对$x\in {}^cF$定义$\varphi_k^*(x)$为

$$\varphi_k^*(x)=\frac{\varphi_k(x)}{\Phi(x)},\quad \text{其中 } \Phi(x)=\sum_k \varphi_k(x).$$

于是显然的恒等式

$$\sum_k \varphi_k^*(x)\equiv 1,\quad x\in {}^cF \tag{5}$$

定义了我们所要的单位分解。

## §2 Whitney 型的开拓定理

**2.1 正则化距离** Whitney 开拓定理的思想隐含在刚才叙述的单位分解（5)中，并且由即将叙述的正则化距离函数的构造进一步暗示出来。

设$F$是$\boldsymbol{R}^n$中任意闭集。沿用第一章的记号，$\delta(x)$表示$x$到$F$的距离。虽然这个函数在$F$上光滑（它在此处为零），但一般来说在${}^cF$上它不会比 Lipschitz 不等式$|\delta(x)-\delta(y)|\leqslant|x-y|$所表示的有更多的可微性。

为了若干应用，可以考虑用正则化距离来代替$\delta(x)$，它对于$x\in {}^cF$，即$x$离开$F$时，是光滑的。此外它实质上具有与$\delta(x)$相同的轮廓。

下述定理保证了它的存在性。

**定理 2** 存在一个定义在${}^cF$中的函数$\Delta(x)=\Delta(x,F)$，使得

(a) $c_1\delta(x)\leqslant\Delta(x)\leqslant c_2\delta(x)$， $x\in {}^cF$;

(b) 在${}^cF$中$\Delta(x)$属于$C^\infty$，且

$$\left|\frac{\partial^{\alpha}}{\partial x^{\alpha}}\Delta(x)\right|\leqslant B_{\alpha}(\delta(x))^{1-|\alpha|}.$$

其中$B_{\alpha}$，$c_1$和$c_2$与$F$无关。

$\Delta(x)$的构造一下就可以给出。事实上令

$$\Delta(x)=\sum_k \operatorname{diam}(Q_k)\varphi_k(x). \tag{6}$$

我们注意，当$x\in Q_k$时，由不等式(3)有

$$\begin{aligned}\delta(x)=\operatorname{dist}(x,F)&\leqslant\operatorname{dist}(Q_k,F)+\operatorname{diam}(Q_k)\\&\leqslant 5\operatorname{diam}(Q_k).\end{aligned}$$

当$x\in Q_k^*$时，由式(3)又有

$$\begin{aligned}\delta(x)&\geqslant\operatorname{dist}(Q_k,F)-\frac{1}{4}\operatorname{diam}(Q_k)\\&\geqslant\frac{3}{4}\operatorname{diam}(Q_k).\end{aligned}$$

总结起来：

$$\begin{aligned}&x\in Q_k, \quad \delta(x)\leqslant 5\operatorname{diam}(Q_k);\\&x\in Q_k^*, \quad \delta(x)\geqslant\frac{3}{4}\operatorname{diam}(Q_k).\end{aligned} \tag{7}$$

然而，当$x\in Q_k$时，$\varphi_k(x)=1$，因此

$$\Delta(x)\geqslant\operatorname{diam}(Q_k)\geqslant\frac{\delta(x)}{5}.$$

另一方面，每个$x$至多位于$N$个$Q_k^*$中，于是

$$\Delta(x)\leqslant\sum_{x\in Q_k^*}\operatorname{diam}(Q_k)\leqslant\frac{4}{3}N\delta(x).$$

这就证明了结论(a)，其中$c_1=1/5$，$c_2=4N/3$。

为了证明结论(b)，可类似地讨论，但要引用不等式(4)，并且注意(与式(7)类似)，当$x\in Q_k^*$时，$\delta(x)\leqslant 6\operatorname{diam}(Q_k)$。这就

给出了所要的结果，其中$B_\alpha = A_\alpha N 6^{|\alpha|-1}$。

现在我们不再继续讨论这个构造，并把它的应用推到§3。在这里我们想指出，由(b)给出的$\Delta(x)$微商的界，虽然当$x$趋向于$F$时它们会失效，但一般来说已是最好的可能了。在$R^1$的情形，取$F$为开集$\bigcup_{j=-\infty}^{\infty}(2^{-j}, 2^{-j+1})$的余集，就可以看到这一点。

在上述每个区间上，正则化距离函数经过一段长为$2^{-j-1}$的距离时，必定从零上升到至少$c_1 2^{-j-1}$，所以其一阶微商必定达到一个不小于$c_1$的值；按同样取法它也必定达到比$-c_1$小的值，所以二阶微商必定在该区间某处有至少为$c_1 2^{j+1}$的值，等等。

**2.2 第一个开拓算子$\mathscr{E}_0$** 设$F$是$R^n$中一个闭集。我们的目的是找出算子$\mathscr{E}_0$，它把定义在$F$上的函数开拓成为定义在$R^n$上的函数。它的主要特性可通过包含一阶或更低的可微性的函数空间的术语表示出来。因此，$\mathscr{E}_0$是要求更高阶可微性的开拓算子系统中最简单的一个。

$\mathscr{E}_0$的定义如下。考虑集合$F$以及定理1中所给出的立方体族$\{Q_k\}$。对于立方体$Q_k$，确定$F$中一个点$p_k$，它具有性质

$$\operatorname{dist}(Q_k, F) = \operatorname{dist}(Q_k, p_k).$$

$F$是闭集，这样的点$p_k$当然存在，虽然它可能不唯一。任意选定一个达到最小距离的点就可以了。事实上任意选取$p_k \in F$，使得$p_k$到$Q_k$的距离与$Q_k$到$F$的距离是可以比较的，这样作也可以，不过前面那样取更简单些。

现在设$f$给定在$F$上。考虑函数$\mathscr{E}_0(f)$，其定义为$\mathscr{E}_0(f)(x) = f(x)$，$x \in F$，以及

$$\mathscr{E}_0(f)(x) = \sum_k f(p_k)\varphi_k^*(x), \quad x \in {}^cF, \tag{8}$$

这里$\{\varphi_k^*(x)\}$是在§1.3末尾所叙述的单位分解。

容易看出，当$x \in {}^cF$时，它至多属于$N$个立方体$Q_k^*$，又由于$\varphi_k^*$的支集在$Q_k^*$中，因此式(8)中的和实际上是有限和，从而

$\mathscr{E}_0(f)(x)$是完全确定的。现在给出$\mathscr{E}_0$的第一个特性。

**命题** 假设$f$是在$F$上给定的函数。则$\mathscr{E}_0(f)$是$f$到$\mathbf{R}^n$上的开拓。另外，假定$f$在$F$上还是连续的，则$\mathscr{E}_0(f)$在$\mathbf{R}^n$上连续，并在${}^cF$中实际上属于$C^\infty$。

按定义$\mathscr{E}_0(f)$是$f$的一个开拓。为了证明$\mathscr{E}_0(f)$的连续性，以及为了得到后面的估计，用下述约定的记号是方便的：设$A$与$B$是两个正量；$A\approx B$就表示$A$与$B$是**可比较的**。在本章中这意味着存在两个正常数$c_1$与$c_2$，使得$c_1A\leqslant B\leqslant c_2A$；常数$c_1$与$c_2$可以与维数$n$有关，但与集合$F$，立方体$Q_k$，函数$f$等无关。

按上述记号，我们首先看到

(9) 若$x\in Q_k^*$，则$|x-p_k|\approx\operatorname{diam}(Q_k)$。

还有

(10) $\operatorname{dist}(Q_k^*,F)\approx\operatorname{diam}(Q_k)$ （见式(7)）。

若$y\in F$，$x\in Q_k^*$，则$|y-p_k|\leqslant|y-x|+|p_k-x|$。但显然$|y-x|\geqslant\operatorname{dist}(Q_k^*,F)$，故由式(9)与(10)知：

(11) 若$y\in F$及$x\in Q_k^*$，则$|y-p_k|\leqslant c|y-x|$。

现在我们可以证明$\mathscr{E}_0(f)$的连续性了。已知每个点$x\in{}^cF$属于一个至多与$N$个立方体$Q_k^*$相交的邻域。由于每个函数$\varphi_k^*$在${}^cF$中属于$C^\infty$，因此$\mathscr{E}_0(f)(x)$在${}^cF$中属于$C^\infty$，当然在${}^cF$中是连续的。

现在设$y$是$F$中一个固定点。我们要证明$\mathscr{E}_0(f)(x)$在$x=y$处连续。考虑$\mathscr{E}_0(f)(y)-\mathscr{E}_0(f)(x)=f(y)-\mathscr{E}_0(f)(x)$，其中$x\to y$。当$x$属于$F$时，这个差就是$f(y)-f(x)$，问题就化为$f$在$F$上已知的连续性。因此可以假定$x\in{}^cF$，$x\to y$。注意当$x\in{}^cF$时，$\sum\limits_k\varphi_k^*(x)\equiv1$，故

$$f(y)-\mathscr{E}_0(f)(x)=f(y)-\sum_k f(p_k)\varphi_k^*(x)$$

$$= \sum_{k} (f(y) - f(p_k))\varphi_k^*(x).$$

根据(11)并注意$\varphi_k^*(x)$的支集在$Q_k^*$中，便得到

$$|f(y) - \mathscr{E}_0(f)(x)| \leqslant \sup_{y' \in F} |f(y) - f(y')| \to 0,$$

当$x \to y$，其中$|y - y'| \leqslant c|y - x|$。

2.2.1 定理3。我们想进一步用Banach空间的术语来表示线性算子$f \to \mathscr{E}_0(f)$的连续性。最合适的函数空间是通过连续模给出的函数空间，特别是Lipschitz空间。为此，设$0 < \gamma \leqslant 1$，定义

$$\mathrm{Lip}(\gamma, \boldsymbol{R}^n) = \{f:\ |f(x)| \leqslant M,\ |f(x) - f(y)| \leqslant M|x - y|^\gamma,$$

$$x, y \in \boldsymbol{R}^n\}.$$

若取以上定义中最小的$M$为范数，则$\mathrm{Lip}(\gamma, \boldsymbol{R}^n)$就成为Banach空间①。

注意到当$0 < \gamma < 1$时，$\mathrm{Lip}(\gamma, \boldsymbol{R}^n)$与第五章§4中研究的空间$\Lambda_\gamma = \Lambda_\gamma^{\infty, \infty}$等价。然而重要的是要注意当$\gamma \to 1$时情况并不一样。即$\mathrm{Lip}(1, \boldsymbol{R}^n)$同构于$L_1^\infty(\boldsymbol{R}^n)$，它是由在$\boldsymbol{R}^n$上有界且其一阶微商也有界的函数组成的空间，而不同构于$\Lambda_1 (= \Lambda_1^{\infty, \infty})$；见第五章§4的4.3.1及§6.2。

当$F$是任意闭集时，类似地定义$\mathrm{Lip}(\gamma, F)$，它由定义在$F$上并满足

(12) $|f(x)| \leqslant M$，且$|f(x) - f(y)| \leqslant M|x - y|^\gamma$, $x, y \in F$

的函数$f$所组成。用以上最小的$M$作为范数，$\mathrm{Lip}(\gamma, F)$也是Banach空间。

**定理3** 当$0 < \gamma \leqslant 1$时，线性开拓算子$\mathscr{E}_0$把$\mathrm{Lip}(\gamma, F)$连续地映入$\mathrm{Lip}(\gamma, \boldsymbol{R}^n)$中。这个映射的范数有界，其界与闭集$F$无关。

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① 当$\gamma > 1$时，上面定义的空间只由常数组成。

2.2.2 为了证明定理，我们从不等式

(13) $$\left|\frac{\partial^{\alpha}}{\partial x^{\alpha}}\varphi_k^*(x)\right|\leqslant A'_{\alpha}(\operatorname{diam}(Q_k))^{-|\alpha|}$$

开始。它可以作为§1.3中$\varphi_k$的类似不等式(4)的容易推论而得到；我们把这简单的细节留给读者。

现在假定$f$满足不等式(12)，其中$M=1$。注意对任意的$f$，$\mathscr{E}_0(f)$在${}^cF$中属于$C^{\infty}$。下面我们需要估计式

(14) $$\left|\frac{\partial}{\partial x_i}\mathscr{E}_0(f)(x)\right|\leqslant c(\delta(x))^{\gamma-1},$$

$$i=1,\cdots,n,\quad x\in{}^cF,$$

其中$\delta(x)$表示$x$到$F$的距离。

事实上

$$\frac{\partial}{\partial x_i}\mathscr{E}_0(f)(x)=\sum_k f(p_k)\frac{\partial\varphi_k^*(x)}{\partial x_i}$$

$$=\sum_k(f(p_k)-f(y))\frac{\partial\varphi_k^*(x)}{\partial x_i},$$

这是因为由式(5)知

$$\sum_k\frac{\partial\varphi_k^*}{\partial x_i}(x)\equiv 0.$$

对任意$x\in{}^cF$，选取$y$是$F$中最靠近$x$的点，即$|x-y|=\delta(x)$。

其次考虑使$x\in Q_k^*$的那些立方体$Q_k^*$。至多有$N$个这样的立方体，如在(11)中曾指出过的(见§2.2)，对于这些立方体总有

$$|y-p_k|\leqslant c|x-y|=c\delta(x).$$

因此

$$\left|\frac{\partial}{\partial x_i}\mathscr{E}_0(f)(x)\right|\leqslant A_1\sum_{x\in Q_k^*}|f(p_k)-f(y)|(\operatorname{diam}(Q_k))^{-1}.$$

但显然，若$x\in Q_k^*$，则$\delta(x)\approx\operatorname{diam}(Q_k)$。于是

$$\left|\frac{\partial}{\partial x_i}\mathscr{E}_0(f)(x)\right|\leqslant A'\Big(\sum_{x\in Q_k^*}|p_k-y|^\gamma\Big)\delta^{-1}(x)$$

$$\leqslant c'\delta(x)^{\gamma-1},$$

这就证明了式(14)。

对于远离$F$的点，估计式(14)是合用的。对于靠近$F$的点，我们注意到，若$y\in F$，$x\in {}^cF$，则

$$\begin{aligned}\mathscr{E}_0(f)(y)-\mathscr{E}_0(f)(x)&=f(y)-\mathscr{E}_0(f)(x)\\&=\sum_k(f(y)-f(p_k))\varphi_k^*(x);\end{aligned}$$

故由(11)

$$\begin{aligned}(15)\qquad&|\mathscr{E}_0(f)(y)-\mathscr{E}_0(f)(x)|\\&\leqslant\sup_{|y-p_k|\leqslant c|y-x|}|f(y)-f(p_k)|\\&\leqslant c|y-x|^\gamma,\qquad y\in F,\ x\in {}^cF.\end{aligned}$$

现在设$y$和$x$都在${}^cF$中。令$L$为连接它们的线段，我们考虑两种情形：(a) $L$到$F$的距离超过$L$的长度($=|x-y|$)；(b) $L$到$F$的距离不大于$L$的长度。在情形(a)，由式(14)简单地有

$$\begin{aligned}|\mathscr{E}_0(f)(y)-\mathscr{E}_0(f)(x)|&\leqslant|y-x|\sup_{x'\in L}|\nabla\mathscr{E}_0(f)(x')|\\&\leqslant c|y-x|\sup_{x'\in L}(\delta(x'))^{\gamma-1}.\end{aligned}$$

因为在这种情形$\delta(x')>|y-x|$，$x'\in L$。于是

$$|\mathscr{E}_0(f)(y)-\mathscr{E}_0(f)(x)|\leqslant c|y-x|^\gamma.$$

在情形(b)，我们可找到点$x'\in L$，与点$y'\in F$，使得$|x'-y'|\leqslant|y-x|$。因此$|y'-x|\leqslant2|y-x|$，且$|y'-y|\leqslant2|y-x|$。如果把(15)应用于$\mathscr{E}_0(f)(y')-\mathscr{E}_0(f)(x)$以及$\mathscr{E}_0(f)(y')-\mathscr{E}_0(f)(y)$，那末再次得到

$$|\mathscr{E}_0(f)(y)-\mathscr{E}_0(f)(x)|\leqslant c'|y-x|^\gamma.$$

最后当 $x$ 与 $y\in F$，显然有 $|\mathscr{E}_0(f)(y)-\mathscr{E}_0(f)(x)|\leqslant|y-x|^{\gamma}$。

还可看出，如果 $f$ 的绝对值以 1 为上界，那末 $\mathscr{E}_0(f)$的绝对值同样以 1 为上界。注意到上面得出的所有的界都与闭集 $F$ 无关，这就完全证明了定理 3。

2.2.3 **一个推论。** 定理 3 的证明引导出它本身的一个简单推广。设$\omega(\delta)$ $(0<\delta<\infty)$是**连续模**，即它是$\delta$的一个正的增函数；假定它是**正则的**，其意义是指：(a) 当$\delta\to 0$时$\omega(\delta)/\delta$是增加的；(b) $\omega(2\delta)\leqslant c\omega(\delta)$。(特别地，条件(a)排除了$\omega(\delta)=\delta^{\gamma}$ $(\gamma>1)$的可能。条件(b)使结果的叙述较为简练。)定义

$$\mathrm{Lip}(\omega,F)=\{f:\ |f(x)|\leqslant M,\\ |f(x)-f(y)|\leqslant M\omega(|x-y|),\ x,y\in F\}$$

以最小的$M$作为范数。

**推论** $\mathscr{E}_0$是$\mathrm{Lip}(\omega,F)$到$\mathrm{Lip}(\omega,\boldsymbol{R}^n)$上的连续映射。

证明只不过是定理3证明的重复。注意，条件 $\omega(2\delta)\leqslant c\omega(\delta)$，以及它的非减性质蕴含着对每个正数$c_1$，都存在一个正数 $c_2$，使得

$$\omega(c_1\delta)\leqslant c_2\omega(\delta),\qquad 0<\delta<\infty.$$

**2.3 开拓算子 $\mathscr{E}_k$** 为了把 §2.2 的结果推广到高阶可微的情形，首先当 $\gamma>1$ 时需要相应地定义 $\mathrm{Lip}(\gamma,F)$。为此设$k$是非负整数并设$k<\gamma\leqslant k+1$。

我们说定义在$F$上的函数 $f$ 是属于 $\mathrm{Lip}(\gamma,F)$的，如果存在定义在$F$上的函数$f^{(j)}$，$0\leqslant|j|\leqslant k$，其中$f^{(0)}=f$，使得当记

$$(16)\qquad f^{(j)}(x)=\sum_{|j+l|\leqslant k}\frac{f^{(j+l)}(y)}{l\,!}(x-y)^l+R_j(x,y)$$

时，有

$$(17)\qquad |f^{(j)}(x)|\leqslant M,\qquad |R_j(x,y)|\leqslant M|x-y|^{\gamma-|j|},\\ \text{对一切}x,y\in F,\ |j|\leqslant k.$$

关于这个定义有几点需要解释的。$j$ 与 $l$ 表示多重指标 $j=$

$(j_1, j_2, \cdots, j_n)$，$l = (l_1, l_2, \cdots, l_n)$，而$j! = j_1! \cdot j_2! \cdot \cdots \cdot j_n!$；$|j| = j_1 + j_2 + \cdots + j_n$；$x^l = x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}$。

注意到函数$f = f^{(0)}$不能唯一地确定$f^{(j)}$ $(0<|j|\leqslant k)$（例如考虑$f$定义在有限集$F$上的情形）；因此，为了避免模糊，当我们讲$\mathrm{Lip}(\gamma, F)$的一个元时，实际上是指集合$\{f^{(j)}(x)\}_{|j|\leqslant k}$。$\mathrm{Lip}(\gamma, F)$中一个元的范数是取使不等式（17）都成立的最小$M$。但当$F = \boldsymbol{R}^n$时我们不用这个说法。$\mathrm{Lip}(\gamma, \boldsymbol{R}^n)$将只是指由$f = f^{(0)}$所组成的线性空间；当然对于$f$要求存在$f^{(j)}$满足式(16)和(17)。范数还是取满足（17）的最小$M$。这样做纯粹是为了记号简单，它与$\mathrm{Lip}(\gamma, F)$的一般定义一致，因为容易看到，当$F = \boldsymbol{R}^n$时，$f^{(j)}$由$f$唯一确定，$1\leqslant|j|$。

更特别地，按照刚才所说的，若$f \in \mathrm{Lip}(\gamma, \boldsymbol{R}^n)$，则$f$连续有界，并且具有阶数不超过$k$的连续有界偏微商；此外

$$\frac{\partial^j f}{\partial x^j} = f^{(j)} \quad (|j|\leqslant k),$$

并且对$|j| = k$，函数$f^{(j)}$属于§ 2的2.2.1中考虑的空间$\mathrm{Lip}(\gamma\text{-}k, \boldsymbol{R}^n)$。其逆命题也是正确的而且容易证明。因此，当$\gamma$不是整数时，$\mathrm{Lip}(\gamma, \boldsymbol{R}^n)$与$\Lambda_\gamma$等价；见第五章§4，特别是第190页的命题9。当$\gamma$是整数，即$\gamma = k+1$时，$\mathrm{Lip}(k+1, \boldsymbol{R}^n)$与$L^\infty_{k+1}(\boldsymbol{R}^n)$等价；见第五章§6.2。

现在设$\{f^{(j)}\}_{|j|\leqslant k}$是定义在$F$上的函数集合。对任意这样的集合，线性映射$\mathscr{E}_k$将给出一个定义在$\boldsymbol{R}^n$上的函数$\mathscr{E}_k(f^{(j)})$，它显然是$f^{(0)} = f$到$\boldsymbol{R}^n$上的一个开拓。为了记号简单起见，我们还用$f$表示这个开拓。$\mathscr{E}_k$定义如下：

$$(18) \qquad \begin{cases} \mathscr{E}_k(f^{(j)}) = f^{(0)}(x), & x \in F, \\ \mathscr{E}_k(f^{(j)}) = \sum_i{}' P(x, p_i)\varphi_i^*(x), & x \in {}^cF. \end{cases}$$

$P(x, y)$表示$f$在$y\in F$的Taylor展开式，它是$x$的多项式，即

$$P(x,y)=\sum_{|l|\leqslant k}\frac{f^{(l)}(y)(x-y)^l}{l!},\quad x\in \boldsymbol{R}^n,\ y\in F.$$

$p_i$如§2.2中的一样，是$F$中达到$F$到立方体$Q_i$的最小距离的点。最后，记号$\sum'$表示只对靠近$F$的那些立方体$Q_i$求和；更确切地说，是对那些与$F$的距离不超过1的立方体求和。

**定理4** 假设$k$是一个非负整数，$k<\gamma\leqslant k+1$，$F$是$\boldsymbol{R}^n$中的闭集。则映射$\mathscr{E}_k$是$\mathrm{Lip}(\gamma,F)$到$\mathrm{Lip}(\gamma,\boldsymbol{R}^n)$的连续映射，它给出$f^{(0)}$到整个$\boldsymbol{R}^n$的开拓。这个映射的范数具有不依赖于$F$的界。

2.3.1 除了刚才引进的记号$P(x,y)$外，用$P_j(x,y)$记

$$\sum_{|j+l|\leqslant k}\frac{f^{(j+l)}(y)}{l!}(x-y)^l$$

是方便的，其中$|j|\leqslant k$。显然

$$f^{(j)}(x)=P_j(x,y)+R_j(x,y),\qquad x,y\in F.$$

我们有$P(x,y)=P_0(x,y)$，与此一致，记$R(x,y)=R_0(x,y)$。

**引理** 设$b,a\in F$，$x\in \boldsymbol{R}^n$，则

$$P(x,b)-P(x,a)=\sum_{|l|\leqslant k}R_l(b,\ a)\frac{(x-b)^l}{l!},$$

更一般地，

$$P_j(x,b)-P_j(x,a)=\sum_{|j+l|\leqslant k}R_{j+l}(b,a)\frac{(x-b)^l}{l!}.$$

为了证明引理，我们把$x$的多项式$P(x,b)-P(x,a)$写成它在$b$点的Taylor展开式。于是$(x-b)^l/l!$的系数是

$$\frac{\partial^l}{\partial x^l}(P(x,b)-P(x,a))|_{x=b}.$$

然而

$$\frac{\partial^l}{\partial x^l}(P(x,y))=P_l(x,y),$$

且$P_l(b,b)=f^{(l)}(b)$，这就对$j=0$的情形证明了引理。$j\neq0$的情形当然可以看作已经证明的情形的特例。

现在把注意力转向和数$\sum'$并看它与$\sum$有何区别. 注意式(7),(10)及性质(3), 过去的阐述表明

(19) 若 $x\in Q_k^*$, 则 $\delta(x)=\operatorname{dist}(x,F)\approx\operatorname{dist}(Q_k,F)$
$$\approx\operatorname{dist}(Q_k^*,F).$$

因此对某个足够小的正数$c_1$, 只要 $\delta(x)\leqslant c_1$, 则和数$\sum'$取遍所有立方体, 而对另一个足够大的正常数$c_2$, 只要 $\delta(x)\geqslant c_2$, 则和数$\sum'$中沒有一项出现, 从而当$\delta(x)\geqslant c_2$时 $f$ 为零. 最后, 当$c_1\leqslant\delta(x)\leqslant c_2$ 时, 出现的项数是有界的, 根据式(13)所给出的$\varphi_i^*(x)$微商的界以及式 (17) 所给出的$f^{(j)}$的界, 我们看到

$$\left|\frac{\partial^\alpha}{\partial x^\alpha}f(x)\right|\leqslant A'_\alpha M,\qquad \text{一切 } \alpha.$$

故可以把考虑的范围限制于靠近$F$的$x$, 即$\delta(x)\leqslant c_1$. 我们还假定规一化$M=1$.

2.3.2 现在来证明下列各式成立:

(a) $|f(x)-P(x,a)|\leqslant A|x-a|^\gamma,\qquad x\in\boldsymbol{R}^n,\ a\in F;$

(a′) $|f^{(j)}(x)-P_j(x,a)|$
$$\leqslant A|x-a|^{\gamma-|j|},\qquad x\in\boldsymbol{R}^n, a\in F, |j|\leqslant k;$$

(b) $|f^{(j)}(x)|\leqslant A,\qquad |j|\leqslant k;$

(b′) $|f^{(j)}(x)|\leqslant A(\delta(x))^{\gamma-k-1},\qquad x\in{}^cF,\ |j|=k+1.$

为了证明(a), 首先注意到当$x\in F$时它对于$A=1$成立, 这是假定的一个结论. 设$x\in{}^cF$及 $\delta(x)\leqslant c_1$ (由前面所作过的注, 其它情形是显然的). 这时

$$f(x)-P(x,a)=\sum_i\{P(x,p_i)-P(x,a)\}\varphi_i^*(x),$$

运用引理, 并注意定理的假定, 就有

$$|f(x)-P(x,a)|\leqslant\sum_{|l|\leqslant k}\sum|p_i-a|^{\gamma-|l|}|x-a|^{|l|},$$

其中内层的 (无指标的) 和号是对使$x\in Q_i^*$的那些立方体$Q_i$ (至多有 $N$ 个) 取的, 由式(11), $|p_i-a|\leqslant c_1|x-a|$, 从而我们证

明了 (a)。

在(a′)的证明中，我们仍可只考虑点$x\in {}^cF$，且有$\delta(x)\leqslant c_1$。这时

$$f^{(i)}(x)=\frac{\partial^i f(x)}{\partial x^i}.$$

于是

$$f^{(i)}(x)=\sum_i \frac{\partial^i P(x,p_i)}{\partial x^i}\varphi_i^*(x)+\text{其它项}.$$

如果我们不理“其它项”，并注意到

$$\frac{\partial^j}{\partial x^j}P(x,p_i)=P_j(x,p_i),$$

那末可以像(a)那样得到(a′)。其它项是由像

(20) $$\sum_i P_{j-l}(x,p_i)\frac{\partial^l}{\partial x^l}\varphi_i^*(x)$$

这样的表达式组成的和，其中$0<|l|$，且$l_i\leqslant j_i$，$i=1,\cdots,n$。因为

$$\sum_i \frac{\partial^l}{\partial x^l}\varphi_i^*(x)=0,$$

所以这些项又等于

(21) $$\sum_i\{P_{j-l}(x,p_i)-P_{j-l}(x,a)\}\frac{\partial^l}{\partial x^l}\varphi_i^*(x).$$

与前面同样的论证，并注意$\frac{\partial^l}{\partial x^l}\varphi_i^*(x)$的估计式 (13) 以及不等式 (19)，便证明了(a′)。

如果取$F$中到$x$的距离为有界的一个点作为点$a$，则不等式 (b) (还是对$\delta(x)\leqslant c_1$) 是(a′)的直接的结论。(顺便指出，在不等式 (b) 的证明过程中，定义$f$用的是$\sum_i{}' P(x,p_i)\varphi_i^*(x)$而不是在所有立方体上求和，这是要紧的。)

最后来证明(b′)。在$\delta(x)\leqslant c_1$处取微商，便得到$f^{(i)}(x)$等于

形如（20）的表达式之和。注意$|j|=k+1$，必定有$|l|>0$，要不然$P_j(x,p_i)\equiv 0$。于是每个和可以重新写成形如（21）的和式，式中我们选择$a$是$F$中达到离$x$的最小距离的点。根据引理以及不等式（11），（13）与（19），我们推得每一个和式（21）被形如

$$A|p_i-a|^{\gamma-|j|+|l|}(\delta(x))^{l-1}\leqslant A'\delta(x)^{\gamma-k-1}$$

的项的有限和所控制。(b′)也就证毕。

2.3.3 有了不等式(a)，(a′)，(b)与(b′)以后，现在可以来完成定理4的证明了．$k=0$的情形当然就是定理3（见§2的2.2.1）。我们将详细研究$k=1$的情形，它已经是很典型的了；这时有$1<\gamma\leqslant 2$。

不等式(a)表明函数$f$在$F$的每点处有一阶偏微商，它们就是$f^{(j)}(x)$，其中$|j|=1$，然而$f$在${}^cF$中属于$C^\infty$，因此

$$\frac{\partial^j f}{\partial x^j}=f^{(j)}$$

对${}^cF$中每点存在。不等式(a′)表明所得的$f^{(j)}$在$\boldsymbol{R}^n$上连续，$|j|=1$。现在令$g$表示这些一阶偏微商之一，则(a′)与(b′)分别意味着

$$|g(x)-g(a)|\leqslant A|x-a|^{\gamma-1},\quad x\in\boldsymbol{R}^n,\ a\in F$$

及

$$\left|\frac{\partial}{\partial x_i}g(x)\right|\leqslant A(\delta(x))^{\gamma-2},\quad i=1,\cdots,n,\ x\in{}^cF.$$

这两个不等式与$k=0$时定理证明中的式(15)及（14）形式相同（但以$\gamma-1$代替$\gamma$）。仿效那里的论证推知每个$g(x)\in\mathrm{Lip}(\gamma-1,\boldsymbol{R}^n)$，这就是当$k=1$时想要的结果。对于$k\geqslant 2$，证明可以用归纳法进行。归纳的步骤与刚才给出的$k=1$的情形很类似。

类似于§2的2.2.3中推论的本定理的变形见后面§4.6；也可见§4.7，那里给出另一种变形。

## §3 对于具有最小光滑边界的区域的开拓定理

设$D$是$\boldsymbol{R}^n$中的开集。我们的目的是描述一个算子$\mathfrak{E}$，它把定义在$D$上的函数开拓到$\boldsymbol{R}^n$上。在同时开拓所有阶的可微性这个意义上来说，这个即将给出的算子是有普遍性的。这是与算子系$\mathscr{E}_k$对比而言。随着$k$的增加$\mathscr{E}_k$变得愈加复杂，因为我们需要对任意闭集来进行开拓。如果$D$的边界满足某种最小光滑性，这大约等价于说它属于Lip1类，那末$\mathfrak{E}$的构造将是可能的。我们将会看到，这个条件实际上不能放宽。因此令人惊异的是，边界的一阶可微性粗略地说正是允许所有阶可微性开拓的必要条件。

**3.1 定理的叙述** 在这里适用的函数空间是Sobolev空间$L_k^p(D)$，与特殊情形$D=\boldsymbol{R}^n$相类似，它的定义如下。令$C_0^\infty(D)$表示$C^\infty$函数类，其支集是紧的，包含在$D$中。对定义在$D$上的局部可积函数$f$，如果有局部可积函数$g$，使得

$$\int_D f\frac{\partial^\alpha \varphi}{\partial x^\alpha}dx=(-1)^{|\alpha|}\int_D g\varphi dx,\qquad 对一切\varphi\in C_0^\infty(D),$$

那末称$f$具有局部可积的（弱）微商

$$\frac{\partial^\alpha f}{\partial x^\alpha}=g\text{.}$$

若$g\in L^p(D)$，则称$\frac{\partial^\alpha f}{\partial x^\alpha}\in L^p(D)$。若$k$是一个整数，则

$$L_k^p(D)=\left\{f\in L^p(D):\ \frac{\partial^\alpha f}{\partial x^\alpha}\in L^p(D),\ 一切|\alpha|\leqslant k\right\}.$$

对其中的等价类，范数由

$$\|f\|_{L_k^p(D)}=\sum_{|\alpha|\leqslant k}\left\|\frac{\partial^\alpha f}{\partial x^\alpha}\right\|_{L^p(D)}$$

给出。

我们的目的是证明下述定理。

**定理5** 设$D$是一个区域,它的边界满足§3.3中(a),(b)与(c)所给出的最小光滑性条件。则存在一个线性算子$\mathfrak{E}$,它把$D$上的函数映成$\boldsymbol{R}^n$上的函数,具有性质

(a) $\mathfrak{E}(f)|_D=f$,即$\mathfrak{E}$是一个开拓算子;

(b) 对一切$p$,$1\leqslant p\leqslant\infty$,及一切非负整数$k$,$\mathfrak{E}$把$L_k^p(D)$连续地映入$L_k^p(\boldsymbol{R}^n)$。

注意,对于这些区域,定理还解决了$L_k^p(\boldsymbol{R}^n)$的限制问题。事实上,设$D$是$\boldsymbol{R}^n$中的任意区域,显然$L_k^p(\boldsymbol{R}^n)$的任意元素限制在$D$上必属于$L_k^p(D)$。

**3.2 一种基本的特殊情形** 这个定理的证明的要点包含在一种基本的特殊情形中,我们单独阐述与讨论它。

为此目的,为了记号简便起见,把考虑的空间$\boldsymbol{R}^n$改变为$\boldsymbol{R}^{n+1}$。$\boldsymbol{R}^{n+1}$中的点记为$(x,y)$,其中$x\in\boldsymbol{R}^n$及$y\in\boldsymbol{R}^1$。我们研究的区域$D$($\boldsymbol{R}^{n+1}$中的开集)是**特殊的**Lipschitz区域,其定义如下。

令$\varphi$:$\boldsymbol{R}^n\to\boldsymbol{R}^1$是一个满足Lipschitz条件的函数:

(22) $$|\varphi(x)-\varphi(x')|\leqslant M|x-x'|,\quad 一切x,x'\in\boldsymbol{R}^n.$$

按照这个函数,我们可以定义特殊的Lipschitz区域,它由位于$\boldsymbol{R}^{n+1}$中的超曲面$y=\varphi(x)$之上的点集组成,即

(23) $$D=\{(x,y)\in\boldsymbol{R}^{n+1}:\ y>\varphi(x)\}.$$

使式(22)成立的最小的$M$称为这个特殊的Lipschitz区域的界。

我们要考虑的特殊情形可以阐述如下。

**定理5′** 设$D$是$\boldsymbol{R}^{n+1}$中一个特殊的Lipschitz区域。则存在一个把$D$上适当的函数开拓成$\boldsymbol{R}^{n+1}$上的函数的线性开拓算子$\mathfrak{E}$,它把$L_k^p(D)$连续地映入$L_k^p(\boldsymbol{R}^{n+1})$,$1\leqslant p\leqslant\infty$,$k$为整数。而且这些映射的范数有界,其界只依赖于数$n$,可微的阶数$k$,以及特殊Lipschitz区域本身的界。

首先指出,对区域边界来说,Lipschitz条件(22)实质上是

最好的可能。假定在$\boldsymbol{R}^2$中考虑由$\varphi(x)=|x|^\gamma(\gamma<1)$决定的 区域，即 $D=\{(x,y):\ y>|x|^\gamma\}$。这里 $\varphi$ 满足 $\gamma$ 阶的 Lipschitz 条 件，只是在原点附近违反条件（22）.让我们在$D$中原点附近设$f(x,y)=y^\beta$，远离原点处$f\in C^\infty$，且具有有界的支集。注意只要

$$\frac{1}{\gamma}+2(\beta-1)>-1,$$

便有 $f\in L_1^{2+\varepsilon}(D)$ 对某个 $\varepsilon$ 成立，不论 $\gamma$ 怎样接近 1，总有非负的 $\beta$ 使上述不等式成立。然而如果开拓定理对这 种 类 型 的$D$成立，那末 $f$ 的延拓必定属于 $L_1^{2+\varepsilon}(\boldsymbol{R}^2)$，而由 Sobolev 定理（第五章§2.2中的定理2）它必须连续，这是一个矛盾。

3.2.1 **纲要**。让我们考虑区域$D$，以及位于它的闭包之外的点$(x,y)$。问题是要定义 $\mathfrak{E}(f)(x,y)$（在该处 $y<\varphi(x)$），其中$f$是在$D$内给定的，对于确定的 $x$，我们将用 $f$ 在 $\varphi(x)<y$ 的线段上的适当的平均值来定义$\varphi(x)>y$ 处的 $\mathfrak{E}(f)(x,y)$。为实现这一想法需要两件事情。首先，要有一个适当的权函数，使能用它来定义平均。其次，要有一种方法来避免差 $\varphi(x)-y$ 至多容许有一阶可微性（关于 $x$）这个困难。下述两个引理照顾到这些事情。

**引理1** 存在一个定义在$[1,\infty)$上的连续函数$\psi$，它在$\infty$处是速降的，即$\psi(\lambda)=O(\lambda^{-N})$，当 $\lambda\to\infty$，对每个$N$，同时还满足性质

$$\int_1^\infty\psi(\lambda)d\lambda=1,\qquad \int_1^\infty\lambda^k\psi(\lambda)d\lambda=0,\qquad k=1,2,\cdots.$$

**引理2** 令$F=\overline{D}$。设 $\Delta(x,y)$是§2.1中定理2给出的到$F$的正则化距离。则存在一个常数 $c$（它只依赖于$D$的Lipschitz界），使得若 $(x,y)\in{}^cF$，则$c\Delta(x,y)\geq\varphi(x)-y$。

为了记号简单起见，令$\delta^*=2c\Delta$，这样

$$\delta^*(x,y)\geq2(\varphi(x)-y).$$

现在写下将被证明是关于$D$的开拓算子的表示式。设$(x,y)\in{}^c\overline{D}$，令

(24) $$\mathfrak{E}(f)(x,y)=\int_1^\infty f(x,y+\lambda\delta^*(x,y))\psi(\lambda)d\lambda,$$

其中的积分将按照适当的极限意义来定义。

我们的计划如下。首先在§3的3.2.2中给出两个引理的证明。其次说明由式(24)定义的算子$\mathfrak{E}$达到了定理5′的目的，即当$D$是特殊的Lipschitz区域时它是开拓算子。最后，对应于所考虑的更一般区域，$\mathfrak{E}$将通过对应于特殊区域的算子来构造。

3.2.2 引理1与引理2的证明。可以给出一个满足引理1结论的初等函数，即

$$\psi(\lambda)=\frac{e}{\pi\lambda}\operatorname{Im}(e^{-\omega(\lambda-1)^{1/4}}),$$

这里$\omega=e^{-i\pi/4}$。

事实上，我们考虑在沿着实轴从1到$+\infty$切开的复平面上的单值解析函数$e^{-\omega(z-1)^{1/4}}$。我们取围线$\gamma$，它在切缝上面从$+\infty$到1，围绕1作一个无限小的迴线，然后在切缝下面转向$+\infty$。根据$e^{-\omega(z-1)^{1/4}}$当$z\to\infty$时速降，于是由Cauchy定理就得到

$$-\frac{1}{2\pi i}\int_\gamma\frac{1}{z}e^{-\omega(z-1)^{1/4}}dz=e^{-\omega(z-1)^{1/4}}\Big|_{z=0}=e^{-1},$$

而

$$\frac{1}{2\pi i}\int_\gamma\frac{z^k}{z}e^{-\omega(z-1)^{1/4}}dz=0,\qquad k=1,2,\cdots.$$

为了证明引理2，我们要用到关于$D$的边界的Lipschitz条件的一个简单的几何解释。令$\Gamma_-$是顶点在原点的（下部的）锥体，它由下式给出，

$$\Gamma_-=\{(x,y):\ M|x|<|y|,\ y<0\}.$$

对于任意点$p\in\boldsymbol{R}^{n+1}$，用$\Gamma_-(p)$表示$\Gamma_-$经过平移使其顶点为$p$的锥体(见图3)。

现在直接看出，Lipschitz条件(22)蕴含了若$p$是$D$的边界上的任一点，即$p=(x_1,y_1)$，其中$y_1=\varphi(x_1)$，则$\Gamma_-(p)\subset{}^c\overline{D}={}^cF$。

另外用$(x,y)$表示${}^c\bar{D}$中的任意点，而用$p=(x,\varphi(x))$表示位于它之上的$D$的边界上的点。自然$(x,y)\in\Gamma_-(p)$，而且$\bar{D}$中沒有点比$\Gamma_-(p)$的边界更接近$(x,y)$。显然$(x,y)$位于圆锥$\Gamma_-(p)$的中心轴上，一个简单的几何论证表明，这个最小距离必定至少为

$$\frac{\varphi(x)-y}{\sqrt{1+M^{-2}}}.$$

因此　　$$\delta(x,y)\geqslant(1+M^{-2})^{-1/2}(\varphi(x)-y),$$

又由定理 2 我们有$c_1\delta\leqslant\Delta$，且定理的证明表明可以取$c_1=1/5$(见§2.1)、故$c\Delta(x,y)\geqslant\varphi(x)-y$，其中$c=5(1+M^{-2})^{1/2}$.

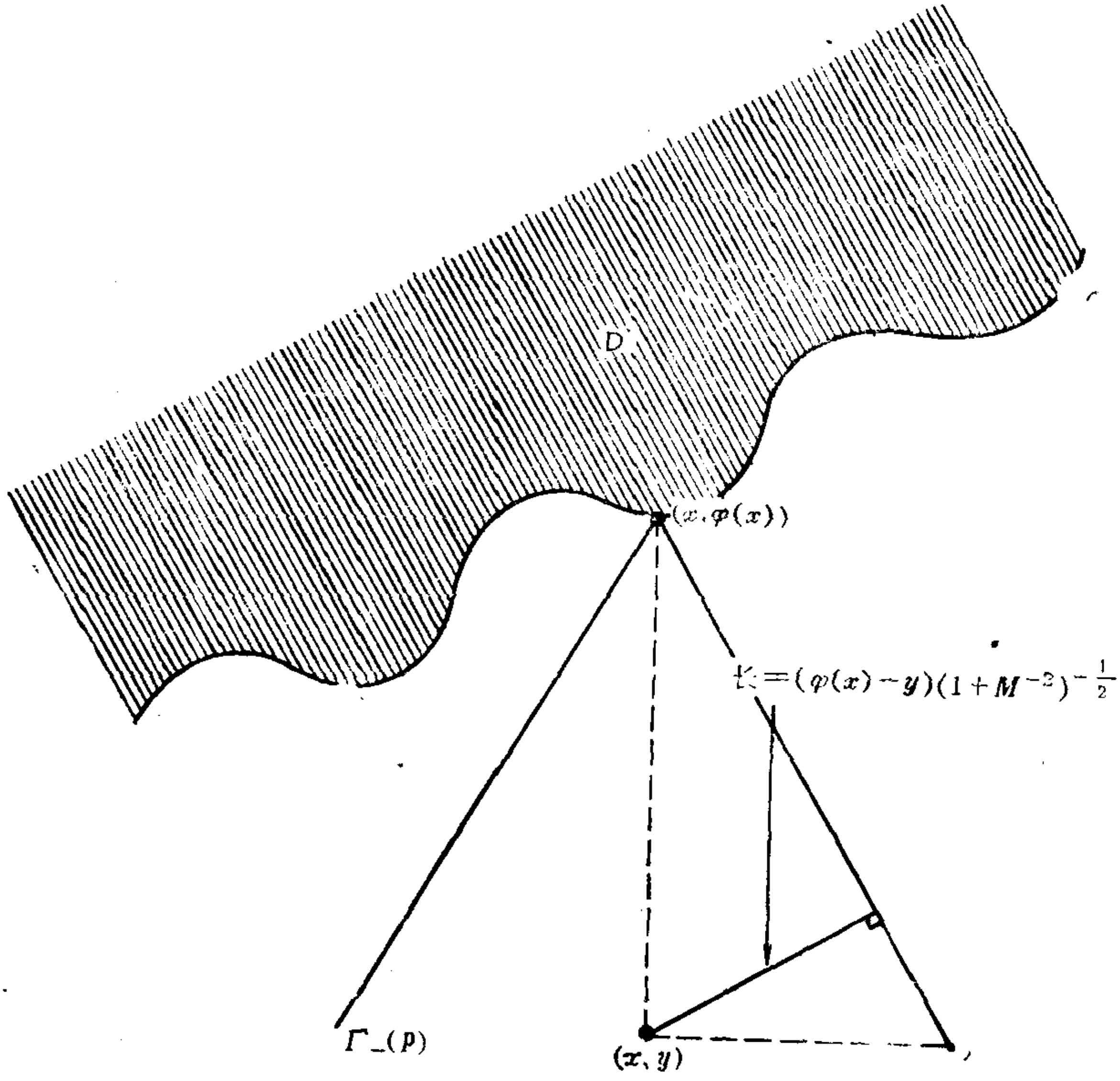


图3　Lipschitz区域$D$及其外部的锥体$\Gamma_-(p)$

3.2.3　有了引理 1，引理 2 以及定义$\mathfrak{E}$的式（24），我们现

在来证明定理5′. 假设$f\in L_k^p(D)$；还设$f$在$D$中属于$C^\infty$，$f$以及它的所有偏微商在$\bar{D}$中连续且有界. 当然不是所有$f\in L_k^p(D)$都满足上述这组条件，但我们的意图在于指出对于这样的$f$有不等式

$$(25)\qquad \|\mathfrak{E}(f)\|_{L_k^p(R^{n+1})}\leqslant A_{k,n}(M)\|f\|_{L_k^p(D)}.$$

有了这个**先验**不等式，就能够通过取极限来处理$L_k^p(D)$中一般的$f$.

现在定义　$\mathfrak{E}(f)(x,y)=f(x,y)$，当$(x,y)\in\bar{D}$，以及

$$\mathfrak{E}(f)(x,y)=\int_1^\infty\psi(\lambda)f(x,y+\lambda\delta^*(x,y))d\lambda,\quad (x,y)\in{}^c\bar{D}.$$

注意到由于$\delta^*(x,y)\geqslant 2(\varphi(x)-y)$这个事实，得到

$$\begin{aligned}y+\lambda\delta^*(x,y)&\geqslant y+2(\varphi(x)-y)\\&=\varphi(x)+\varphi(x)-y>\varphi(x),\qquad \lambda\geqslant 1.\end{aligned}$$

此式及所假定的$f$的有界性说明了定义$\mathfrak{E}$的积分是完全确定的.

现在，令$D$同前面一样，而$D_-=\{(x,y):\ \varphi(x)>y\}$，也就是严格地位于所述 Lipschitz 超平面下方的那些点. 这时当然有$\bar{D}\cup\bar{D}_-=\boldsymbol{R}^n$，但$\bar{D}$与$\bar{D}_-$相交. $f$的那些暂时认为不成问题的性质保证了$\mathfrak{E}(f)$以及它的所有偏微商在$\bar{D}$中连续. 另外，可以得知，对于这样的$f$，$\mathfrak{E}(f)$在$D_-$中属于$C^\infty$，并且它的一切偏微商连续（且在$\bar{D}_-$中有界）. 我们来看$\dfrac{\partial^2\mathfrak{E}(f)}{\partial x_j^2}$，其论证完全具有代表性. 进行微商就给出

$$(26)\qquad \begin{aligned}\frac{\partial^2\mathfrak{E}(f)}{\partial x_j^2}=&\int_1^\infty f_{jj}(\cdot)\psi(\lambda)d\lambda+\int_1^\infty f_{jy}(\cdot)\lambda\delta_j^*\psi(\lambda)d\lambda\\&+\int_1^\infty f_{yy}(\cdot)(\lambda\delta_j^*)^2\psi(\lambda)d\lambda\\&+\int_1^\infty f_y(\cdot)\lambda\delta_{j\,j}^*\psi(\lambda)d\lambda.\end{aligned}$$

这里我们使用了简写：记号$f_{jj}$是指$\dfrac{\partial^2 f}{\partial x_j^2}$；类似地$\dfrac{\partial f}{\partial y}$用$f_y$来表示；

$(\cdot)$代替了$(x, y+\lambda\delta^*)$。

由于$f$在$D$中所假定的可微性，上述等式表明，对于$(x,y)\in D_-$，$\dfrac{\partial^2}{\partial x_i^2}\mathfrak{E}(f)(x,y)$是完全确定的.（事实上，显然$\mathfrak{E}(f)(x,y)$在$D_-$中属于$C^\infty$.）其次，令$(x,y)$趋向于$D_-$的边界点，即$(x^0,y^0)\in\bar{D}_-\cap\bar{D}$，则$\delta^*(x,y)\to 0$，并由$\delta_i^*$保持有界（见定理2的结论(b)），而

$$\int_1^\infty \psi(\lambda)d\lambda=1,\quad \int_1^\infty \lambda\psi(\lambda)d\lambda=0,\quad \int_1^\infty \lambda^2\psi(\lambda)d\lambda=0,$$

故等式（26）右边的前三项收敛到一个总数，等于

$$\lim_{\substack{(x,y)\in D\\(x,y)\to(x^0,y^0)}}\frac{\partial^2 f(x,y)}{\partial x_i^2}.$$

等式（26）的最后一项的困难在于它包含了正则化距离$\delta^*$的超过一阶的微商，于是不再是有界的了，然而我们可以写出

$$f_y(x,y+\lambda\delta^*)=f_y(x,y+\delta^*)+(\lambda-1)\delta^* f_{yy}(x,y+\delta^*)$$
$$+O([(\lambda-1)\delta^*]^2).$$

将它代入式（26）的最后一项中，给出两个恒为零的积分以及一个余项，它就是

$$O\left((\delta^*)^2\delta^*_{x_i x_i}\int_1^\infty(\lambda-1)^2\lambda\psi(\lambda)d\lambda\right).$$

$\psi$是速降的，所以由定理2得知，当$(x,y)\to(x^0,y^0)$时这个总量是$O(\delta)\to 0$。

概括起来，因为$f$以及它的所有偏微商在$\bar{D}$中连续（且有界），所以关于$\mathfrak{E}(f)$同样的事实在$\bar{D}_-$中成立，并且$f$与$\mathfrak{E}(f)$在$\bar{D}\cap\bar{D}_-$上互相一致，所有它们的偏微商也是这样。这也表明了$\mathfrak{E}(f)\in C^\infty(\boldsymbol{R}^{n+1})$，这实际上意味着$\mathfrak{E}(f)$的两部分，一部分来自$D$，另一部分来自$D_-$，事实上已经正确地连接起来了。现在让我们来证明这个结论，先说明$\mathfrak{E}(f)$属于$C^1$类。（$\mathfrak{E}(f)$的连续性已经证明过了。）

必须证明，对于任意点$v\in \bar{R}^{n+1}$，当$u\to v$时

$$\mathfrak{E}(f)(u)-\mathfrak{E}(f)(v)=(u-v)\cdot(\nabla\mathfrak{E}(f))(u)+o(|u-v|).$$

如果 $v$ 是在$D$内或 $D_-$ 内，就没有什么要进一步证明的了。所以假定 $v$ 在这两个区域的（共同）边界上。设$u\in \bar{D}_-$；若$u\in \bar{D}$其论证完全类似。我们断言 $v$ 与 $u$ 可以用一条折线段来连接，这条折线段除 $v$ 与 $u$ 外完全在$D_-$内，并且它的总长不大于 $c|v-u|$。实际上存在一个点 $w\in D_-$，使得连接 $v$ 与 $w$，以及 $w$ 与 $u$ 的线段具有所要求的性质。为了寻找这样的 $w$，注意到或者 $u\in \bar{\Gamma}_-(v)$（则我们可以取 $w=u$），要不然锥体$\bar{\Gamma}_-(v)$与 $\bar{\Gamma}_-(u)$必定相交。其交集中到 $u$，$v$ 距离之和最小的点可以作为 $w$。现在

$$\mathfrak{E}(f)(u)-\mathfrak{E}(f)(w)=(u-w)(\nabla\mathfrak{E}(f))(u)+o(|u-w|),$$

还有

$$\mathfrak{E}(f)(w)-\mathfrak{E}(f)(v)=(w-v)(\nabla\mathfrak{E}(f))(w)+o(|w-v|).$$

把两式相加，并利用以下事实，在$\bar{D}_-$中当$|u-v|\to 0$时

$$(\nabla\mathfrak{E}(f))(w)-(\nabla\mathfrak{E}(f))(u)=o(1),$$

便给出所要的结果。类似地可知对于每个 $k$，$\mathfrak{E}(f)\in C^k(R^{n+1})$。

下一步是证明不等式(25)，办法是通过对每个确定的 $x$ 证明相应的不等式，然后对 $x$ 积分所得的结果。

首先考虑$k=0$的情形。让我们固定$x^0$，并假定（为了记号方便起见）$\varphi(x^0)=0$。这时

$$|\mathfrak{E}(f)(x^0,y)|\leqslant A\int_1^\infty|f(x^0,y+\lambda\delta^*)|\frac{d\lambda}{\lambda^2},\qquad y<0.$$

当然这里用了 $|\psi(\lambda)|\leqslant A/\lambda^2$。因为 $\delta^*\geqslant 2(\varphi(x)-y)$，现在是 $\delta^*\geqslant 2|y|$。当然一般来说还有$\varphi(x)-y\geqslant(x,y)$到$\bar{D}$的距离，所以在我们所考虑的情形有$\delta^*\leqslant a|y|$。令$s=y+\lambda\delta^*$，其中 $y$ 是固定的，则$ds=\delta^*d\lambda$，前面的不等式变成

$$|\mathfrak{E}(f)(x^0,\ y)|\leqslant A\delta^*\int_{|y|}^\infty|f(x^0,s)|(s-y)^{-2}ds,\qquad y<0.$$

因此

(27) $$|\mathfrak{E}(f)(x^0,y)|\leqslant Aa|y|\int_{|y|}^{\infty}|f(x^0,s)|\frac{ds}{s^2},\qquad y<0.$$

于是由 Hardy 不等式（见附录 A，第344页）便证得

$$\left(\int_{-\infty}^{0}|\mathfrak{E}(f)(x^0,y)|^p dy\right)^{1/p}\leqslant A'\left(\int_0^{\infty}|f(x^0,y)|^p dy\right)^{1/p}.$$

为了去掉条件 $\varphi(x^0)=0$，对 $y$ 作适当的平移变换便得到

$$\left(\int_{-\infty}^{\infty}|\mathfrak{E}(f)(x^0,y)|^p dy\right)^{1/p}\leqslant A\left(\int_{\varphi(x^0)}^{\infty}|f(x^0,y)|^p dy\right)^{1/p}.$$

将上式两边取 $p$ 次幂并对 $x^0\in \boldsymbol{R}^n$ 积分，就给出 $k=0$ 的不等式（25）。$k>0$ 时的证明类似。例如考虑 $k=2$ 的情形。这里对 $\frac{\partial^2\mathfrak{E}}{\partial x_j^2}$ 的研究仍然是典型的。现在除了要用 $|\psi(\lambda)|\leqslant A/\lambda^4(\lambda\geqslant 1)$ 以外，式（26）右边前三项可用同样的方法来处理。只有最后一项需要单独处理。我们写出

(28) $$f_y=f_y(x^0,y+\delta^*)+\int_{y+\delta^*}^{y+\lambda\delta^*}f_{yy}(x^0,t)dt$$

并把它代入式（26）中。根据关于 $\psi$ 的正交性条件，其中含有 $f_y(x^0,y+\delta^*)$ 的那个积分的对应项为零。因此问题化为估计

$$|y|^{-1}\int_1^{\infty}\left\{\int_{y+\delta^*}^{y+\lambda\delta^*}|f_{yy}(x^0,t)|dt\right\}\lambda^{-3}d\lambda.$$

交换积分次序把它化为前面的情形（与式(27)的右边类似），就解决了 $k=2$ 的情形。

对于一般的 $k$，我们在式（24）中在积分号下取微商，就得到 $f$ 的各阶微商。出现在 $f$ 上的微商的总阶数总是小于 $k$（它最低可以到 1），我们写出这个微商在点 $(x^0,y+\delta^*)$ 带有 $k$ 阶积分余项的Taylor展开式，然后像前面那样继续进行。

事实上设出现在 $f$ 上的微商阶数为 $k_0$，其中 $k_0<k$。令 $g$ 是 $f$ 的 $k_0$ 阶偏微商。这时

$$g(x^0,y+\lambda\delta^*)=\sum_{j=0}^{l-1}\frac{((\lambda-1)\delta^*)^j}{j!}\left.\frac{\partial^j g}{\partial y^j}\right|_{(x^0,y+\delta^*)}$$

$$+\frac{1}{l!}\int_{\delta^*}^{\lambda\delta^*}(\lambda\delta^*-t)^{l-1}\frac{\partial^l}{\partial t^l}g(x^0,y+t)dt,$$

其中$k_0+l=k$。上式右端只有带积分的项不为零，但它被

$$A(\lambda\delta^*)^{l-1}\int_{\delta^*}^{\lambda\delta^*}\left|\frac{\partial^l}{\partial t^l}g(x^0,y+t)\right|dt$$

所控制，其余的论证与前面的一样。

我们应该注意，在证明式（25）的全部推理中，区域$D$对式（25）中出现的界的唯一影响来自它在式（22）中的 Lipschitz 界$M$。

3.2.4　定理5′证明的最后一步是要去掉$f$在$D$上属于$C^\infty$以及它与它的所有偏微商在$\bar{D}$连续有界这一限制。

为此假设$\eta\in C^\infty(\boldsymbol{R}^{n+1})$是非负的，它的积分值为1，并且$\eta$的支集严格地包含在锥体$\Gamma_-$的内部。对于任意$\varepsilon>0$，记$\eta_\varepsilon(u)=\varepsilon^{-n-1}\eta(u/\varepsilon)(u\in\boldsymbol{R}^{n+1})$，及

$$f_\varepsilon(u)=\int f(u-v)\eta_\varepsilon(v)dv,$$

其中$f$属于$L_k^p(D)$。注意当$u\in\bar{D}$时，该积分只包含$u-v\in D$，因此是完全确定的。此外，由于$\eta_\varepsilon$的支集严格地包含在$\Gamma_-$的内部，我们看到这个积分在$\bar{D}$的一个邻域定义了$f_\varepsilon(u)$，且在该处$f_\varepsilon(u)$属于$C^\infty$。显然还有

$$\left\|\frac{\partial^\alpha f_\varepsilon}{\partial x^\alpha}\right\|_{L^p(D)}\leqslant\left\|\frac{\partial^\alpha f}{\partial x^\alpha}\right\|_{L^p(D)},\qquad|\alpha|\leqslant k,$$

故

(29)
$$\|f_\varepsilon\|_{L_k^p(D)}\leqslant\|f\|_{L_k^p(D)}.$$

现在若$p<\infty$，我们知道按$L_k^p(D)$范数$f_\varepsilon\to f$。对于$p=\infty$，下述较弱的陈述就足够了，即若$k\geqslant1$，则按$L_{k-1}^\infty$范数$f_\varepsilon\to f$。（见第五章§2.1与§6.2中与此有密切联系的事实。）

现在令 $\mathfrak{E}(f)=\mathfrak{E}(f_\varepsilon)$，则由式（25）与（29）我们有

（30） $$\|\mathfrak{E}_\varepsilon(f)\|_{L_k^p(R^{n+1})}\leqslant A_{k,n}(M)\|f\|_{L_k^p(D)}.$$

显然，$\|f_\varepsilon-f_{\varepsilon'}\|_{L_{k-1}^p(D)}\to 0$，当 $\varepsilon$，$\varepsilon'\to 0$。所以，$\mathfrak{E}_\varepsilon(f)$是$L_{k-1}^p(\boldsymbol{R}^{n+1})$中的 Cauchy 列，它的极限也满足式(30)，这意味着我们证明了算子 $\mathfrak{E}=\lim\mathfrak{E}_\varepsilon$ 的存在性，它推广了原来只作用在$\bar{D}$上的$C^\infty$ 函数的算子 $\mathfrak{E}$。这显然就是所要的开拓算子，它满足式(25)。于是定理5′完全证毕。

**3.3 一般情形** 解决了特殊Lipschitz 区域的情形以后，我们从 $\boldsymbol{R}^{n+1}$ 转回到 $\boldsymbol{R}^n$，把术语稍作修改将是方便的，即把特殊Lipschitz 区域的旋转也称之为特殊 Lipschitz 区域。这种区域的Lipschitz界的概念按明显的方法定义，它当然是旋转不变的。

现在令$D$是$\boldsymbol{R}^n$中的开集，并令$\partial D$是它的边界。如果存在$\varepsilon>0$，整数$N$，$M>0$，以及开集列$U_1,U_2,\cdots,U_n,\cdots$，使得

(a) 若 $x\in\partial D$，则对某个 $i$ 有 $B(x,\varepsilon)\subset U_i$；$B(x,\varepsilon)$ 是中心为 $x$，半径为 $\varepsilon$ 的球；

(b) $\boldsymbol{R}^n$ 中沒有一点包含在$N$个以上的$U_i$中；

(c) 对于每个 $i$ 存在一个特殊 Lipschitz 区域 $D_i$，它的界不超过$M$，使得

$$U_i\cap D=U_i\cap D_i.$$

那末就称 $\partial D$ 是**最小光滑的**。

上述区域的一些例子为：

**例 1** 设$D$是$\boldsymbol{R}^n$中的有界区域，它的边界属于嵌入 $\boldsymbol{R}^n$ 中的$C^1$ 类。这时只需要有限多个$U_i$。

**例 2** $D$是任意的有界开凸集。还是只需要有限多个$U_i$。

**例 3** $D\subset\boldsymbol{R}^1$，且 $D=\bigcup\limits_j I_j$，这里$I_j$是互不相交的开区间。如果存在一个$\delta>0$，使得$I_j$的长$\geqslant\delta$，以及若 $j\neq k$，$\mathrm{dist}(I_j,I_k)\geqslant\delta$，那末必满足条件 (a)—(c)。在这个例子中如果有无限多个$I_j$，就要求有无限多个$U_i$。上述$I_j$的长$\geqslant\delta$，$\mathrm{dist}(I_j,I_k)\geqslant\delta$等条件也都是必须的。读者容易验证，如果要对$L_1^1$有有界开拓，就要

求有条件 $I_j$ 的长$\geqslant\delta$，而为了对$L_1^\infty$有有界开拓，就需要条件

$$\operatorname{dist}(I_j, I_k)\geqslant\delta.$$

3.3.1 我们证明定理 5 的办法是把它归结为特殊 Lipschitz 区域的定理5′. 然而证明是有点巧妙的。

对于任意集合$U\subset\boldsymbol{R}^n$及任意$\varepsilon>0$，记$U^\varepsilon=\{x:B(x,\varepsilon)\subset U\}$. 注意到$U^\varepsilon\subset U$，并且条件 (a) 可以重新叙述为 $\bigcup\limits_i U_i^\varepsilon\supset\partial D$. 令 $\eta(x)$表示$\boldsymbol{R}^n$中一个确定的 $C^\infty$ 函数，它的积分值为 1，支集包含在单位球中。记 $\eta_\varepsilon(x)=\varepsilon^{-n}\eta(x/\varepsilon)$. 假定 $\chi_i$ 是 $U_i^{3\varepsilon/4}$ 的特征函数，并令

$$\lambda_i(x)=(\chi_i*\eta_{\varepsilon/4})(x).$$

$\lambda_i$的下述性质是明显的：

(a) 每个 $\lambda_i$ 的支集在 $U_i$ 中；

(b) 若 $x\in U_i^{\varepsilon/2}$，特别地若 $x\in U_i^\varepsilon$，则$\lambda_i(x)=1$；

(c) 每个 $\lambda_i\in C^\infty$ 具有所有阶的有界微商，并且微商的界可以不依赖于 $i$. (这些界只依赖于$\eta_{\varepsilon/4}$的相应的微商的$L^1$范数.)

此外还考虑三个别的开集，它们分别覆盖$D$的某个邻域，$D$的边界以及$D$内远离边界的部分，即

$$U_0=\left\{x:\operatorname{dist}(x,D)<\frac{\varepsilon}{4}\right\},$$

$$U_+=\left\{x:\operatorname{dist}(x,\partial D)<\frac{3}{4}\varepsilon\right\},$$

$$U_-=\left\{x\in D:\operatorname{dist}(x,\partial D)>\frac{\varepsilon}{4}\right\}.$$

记这些集合的特征函数为 $\chi_0,\chi_+$ 和$\chi_-$，并像前面那样把这些函数正则化，即 $\lambda_0=\chi_0*\eta_{\varepsilon/4}$，$\lambda_+=\chi_+*\eta_{\varepsilon/4}$，$\lambda_-=\chi_-*\eta_{\varepsilon/4}$. 显然 $\lambda_0(x)=1$，当$x\in\bar{D}$；$\lambda_+(x)=1$，当 $\operatorname{dist}(x,\partial D)\leqslant\varepsilon/2$；以及 $\lambda_-(x)=1$，当$x\in D$以及$\operatorname{dist}(x,\partial D)\geqslant\varepsilon/2$. 而且$\lambda_0$，$\lambda_+$及$\lambda_-$的支集分别位于 $D$ 的 $\varepsilon/2$ 邻域内、$\partial D$ 的 $\varepsilon$ 邻域内，以及 $D$ 内。这些函数以

及它们所有的偏微商也都在 $R^n$ 中有界。现在令

$$\Lambda_+=\lambda_0\left(\frac{\lambda_+}{\lambda_++\lambda_-}\right) \quad 及 \quad \Lambda_-=\lambda_0\left(\frac{\lambda_-}{\lambda_++\lambda_-}\right).$$

我们看到 $\lambda_0$ 的支集包含在使 $\lambda_++\lambda_-\geqslant 1$的集合中，因此 $\Lambda_+$与 $\Lambda_-$的所有阶偏微商也在$R^n$有界. 而且当 $x\in\bar{D}$时，$\Lambda_++\Lambda_-=1$，而在$D$的 $\varepsilon/2$ 邻域之外，$\Lambda_++\Lambda_-=0$。

注意覆盖了$D$的边界的开集 $U_1,U_2,\cdots,U_i,\cdots$ 都有与它们相联系的特殊 Lipschitz 区域 $D_1,D_2,\cdots,D_i,\cdots$。我们记 $\mathfrak{E}^i$ 是对于 $L_k^p(D_i)$的开拓算子，它的性质由定理5′给出。

在作好这一切准备之后，我们最后可以写出所要的对于$D$的开拓算子 $\mathfrak{E}$了. 实际上对于$f\in L^p(D)$，定义

$$(31)\qquad \mathfrak{E}(f)(x)=\Lambda_+(x)\left\{\frac{\sum_{i=1}^{\infty}\lambda_i(x)\mathfrak{E}^i(\lambda_i f)}{\sum_{i=1}^{\infty}\lambda_i^2(x)}\right\}+\Lambda_-(x)f(x).$$

注意以下事实:

(d) 对 $\Lambda_+$ 支集中的 $x$ (更一般地，若 $\operatorname{dist}(x,\partial D)\leqslant\varepsilon/2$)，则至少有一个 $i$ ，使得 $x\in U_i^{\varepsilon/2}$，因此对这样的 $x$ 有 $\sum_i\lambda_i^2(x)\geqslant 1$ (见上面的(b))；

(e) 对于每个 $x$，和式 (31) 至多包含$N+1$个非零项 (这是由于覆盖 $\{U_i\}$ 的条件(b))；

(f) 由于 $\Lambda_-$ 的支集包含在$D$内，项 $\Lambda_-(x)f(x)$是完全确定的；

(g) 由于 $\lambda_i f$ 给定在特殊 Lipschitz 区域 $D_i$ 内，项 $\mathfrak{E}^i(\lambda_i f)$ 是完全确定的；

(h) 显然对 $x\in D$，$\mathfrak{E}(f)(x)=f(x)$。

为了证明基本的不等式

$$(32)\qquad \|\mathfrak{E}(f)\|_{L_k^p(R^n)}\leqslant A_{kn}(D)\|f\|_{L_k^p(D)},\qquad f\in L_k^p(D),$$

我们需要以下的观察。

**命题** 假设 $A(x)=\sum_{i=1}^{\infty}a_i(x)$，并且对于每个 $x$ 至多有 $N$ 项 $\{a_i(x)\}$ 非零。则

$$\|A(x)\|_p\leqslant N^{1-1/p}\left(\sum_i\|a_i(x)\|_p^p\right)^{1/p},\quad p<\infty,$$

以及

$$\|A(x)\|_\infty\leqslant N\sup_i\|a_i(x)\|_\infty,\qquad p=\infty.$$

符号 $\|\cdot\|_p$ 表示标准的 $L^p$ 范数。这里 $L^\infty$ 情形和 $L^1$ 情形都是显然的。一般的情形也没有多大的困难，它可以从

$$|A(x)|^p\leqslant N^{p-1}\sum_{i=0}^{\infty}|a_i(x)|^p$$

推出，而后者也是 Hölder 不等式的明显推论。

我们首先对 $k=0$ 证明式 (32)。用 $\lambda_i$ 的性质 (a)—(c)，然后用 (d)—(h)，同时用上述命题以及定理 5′ ($k=0$)，得到，当 $p<\infty$ 时，由于 $\left(\sum_i\lambda_i\right)^{1/p}\leqslant N^{1/p}$，

$$\begin{aligned}\|\mathfrak{E}(f)\|_p&\leqslant N^{1-1/p}\left(\sum_i\int_{U_i}|\mathfrak{E}^i(\lambda_i f)|^p\,dx\right)^{1/p}\\&\quad+\left(\int_D|f(x)|^p dx\right)^{1/p}\\&\leqslant AN^{1-1/p}\left(\sum_i\int_D|\lambda_i f|^p dx\right)^{1/p}\\&\quad+\left(\int_D|f|^p dx\right)^{1/p}\\&\leqslant AN\left(\int_D|f|^p dx\right)^{1/p}+\left(\int_D|f|^p dx\right)^{1/p}.\end{aligned}$$

对 $p=\infty$，类似的推理成立。这就对 $k=0$ 证明了式(32)。由于 $\lambda_i(i=1,2,\cdots)$，$\Lambda_+$ 及 $\Lambda_-$ 的每一个确定的偏微商都是一致有界的，故对所有 $k$ 可进行完全类似的推理。从而定理 5 证毕。

## §4 进一步的结果

下面§4.1至§4.5，讨论$\mathscr{L}_\alpha^p(\boldsymbol{R}^n)$中的函数在线性子流形上的限制。

4.1 设$f$是$\boldsymbol{R}^n$上的局部可积函数。假如

$$\lim_{\varepsilon\to 0}\frac{1}{m(B_\varepsilon)}\int_{B_\varepsilon} f(x^0-t)\,dt$$

存在，其中$B_\varepsilon$表示中心在原点，半径为$\varepsilon$的球，我们就说$f$**在$x^0$点能够严格定义**。这时我们重新定义（如果必要）$f$在$x^0$处为这个值。假如在每个有可能这样作的点处都这样作了，我们就称$f$是**严格定义了的**。根据第一章的基本定理，每个局部可积的函数都能严格定义，并且与原来的函数几乎处处相等。为了研究$\mathscr{L}_\alpha^p$的限制，下面的引理是重要的。

**引理** 设$f\in\mathscr{L}_\alpha^p(\boldsymbol{R}^n)$，$f=G_\alpha*\varphi$，其中$\alpha>0$，$1\leqslant p\leqslant\infty$。假定$x^0$是这样一个点，使得表示$f$的积分

$$\int_{\mathbf{R}^n} G_\alpha(x-t)\varphi(t)\,dt=f(x)$$

在该点绝对收敛。则

$$\frac{1}{m(B_\varepsilon)}\int_{B_\varepsilon}|f(x^0-t)-f(x^0)|\,dt\to 0,\quad \varepsilon\to 0.$$

因此$f$在点$x^0$处能够严格定义。

4.2 我们考虑$\boldsymbol{R}^n$中的$m$维线性子流形。不失一般性，假定它是$\boldsymbol{R}^n$中这样的点组成的子空间$\boldsymbol{R}^m$，其前$m$个坐标是任意的，而后$n-m$个坐标为零。现在假定$\alpha>(n-m)/p$，$1\leqslant p\leqslant\infty$，且$f\in\mathscr{L}_\alpha^p(\boldsymbol{R}^n)$。则除去在$\boldsymbol{R}^m$中一个$m$维Lebesgue测度为零的子集外，$f$能在$\boldsymbol{R}^m$中所有点严格定义。如果我们用$\mathscr{R}(f)$表示这个限制，那末$\mathscr{R}(f)\in L^p(\boldsymbol{R}^m)$，并且$\mathscr{L}_\alpha^p(\boldsymbol{R}^n)$到$L^p(\boldsymbol{R}^m)$的映射$f\to\mathscr{R}(f)$是连续的，即

$$\|\mathscr{R}(f)\|_{L^p(\mathbf{R}^m)} \leqslant A\|f\|_{\mathscr{L}^p_\alpha(\mathbf{R}^n)}.$$

**4.3** 上面叙述的$\mathscr{R}(f)$不仅属于$L^p(\mathbf{R}^n)$，而且还属于适当的$\Lambda$空间。更确切地说，令$\beta=\alpha-(n-m)/p>0$，$1<p<\infty$。若$f\in\mathscr{L}^p_\alpha(\mathbf{R}^n)$，则$\mathscr{R}(f)\in\Lambda^{p,p}_\beta(\mathbf{R}^m)$，并且映射$f\to\mathscr{R}(f)$是连续的，即

$$\|\mathscr{R}(f)\|_{\Lambda^{p,p}_\beta(\mathbf{R}^m)} \leqslant A\|f\|_{\mathscr{L}^p_\alpha(\mathbf{R}^n)}.$$

**4.4** §4.3的逆也成立。因此$\mathscr{L}^p_\alpha(\mathbf{R}^n)$的元素在$\mathbf{R}^m$的限制恰好由$\Lambda^{p,p}_\beta(\mathbf{R}^m)$组成，其中$\beta=\alpha-(n-m)/p$。对于$m=n-1$的情形，我们给出这个逆的一个明确的叙述。设$\varphi$是$\mathbf{R}^{n-1}$上具有紧支集的一个确定的$C^\infty$函数，使得

$$\int_{\mathbf{R}^{n-1}}\varphi(x)dx=1,$$

而$\eta$是$\mathbf{R}^1$上具有紧支集的一个确定的$C^\infty$函数，满足$\eta(0)=1$。对于$\mathbf{R}^{n-1}$上的每个局部可积函数$f$，考虑它到$\mathbf{R}^n$的开拓，这个开拓由

$$E(f)(x,y)=\eta(y)\int_{\mathbf{R}^{n-1}}f(x-yt)\varphi(t)dt,$$

$$(x,y)\in\mathbf{R}^{n-1}\times\mathbf{R}^1=\mathbf{R}^n$$

给定。

可以证明，$f\in\Lambda^{p,p}_\beta(\mathbf{R}^{n-1})$意味着$E(f)\in\mathscr{L}^p_\alpha(\mathbf{R}^n)$，并且

$$\|E(f)\|_{\mathscr{L}^p_\alpha(\mathbf{R}^n)} \leqslant A\|f\|_{\Lambda^{p,p}_\beta(\mathbf{R}^{n-1})}.$$

这里假定$\beta=\alpha-1/p>0$。

这个开拓的一个更确切的形式如下。令$k$是不超过$\beta$的最大整数。设$\eta_0(y),\cdots,\eta_k(y)$是$\mathbf{R}^1$上具有紧支集的$C^\infty$函数，具有性质

$$\frac{d^l}{dy^l}\eta_j(y)\bigg|_{y=0}=\delta_{jl},\qquad 0\leqslant j,l\leqslant k.$$

假定$f_j\in\Lambda^{p,p}_{\beta-j}(\mathbf{R}^{n-1})$，$0\leqslant j\leqslant k$。设

$$F(x,y)=\sum_{j=0}^{k}\eta_j(y)\int_{R^{n-1}}f_j(x-yt)\varphi(t)dt,$$

则$F\in\mathscr{L}_a^p(R^n)$，

$$\|F\|_{\mathscr{L}_a^p(R^n)}\leqslant A\left\{\sum_{j=0}^{k}\|f_j\|_{\Lambda_{\beta-j}^{p,p}(R^{n-1})}\right\},$$

且

$$\mathscr{R}\left(\frac{\partial^j F(x,y)}{\partial y^j}\right)=f_j(x),\quad j=0,\cdots,k,$$

其中$\mathscr{R}$表示在超平面$y=0$上的限制。前面的结果见 Stein[7]，后面的论述在 Aronszajn, Mulla, 及 Szeptycki[1] 及 Lizorkin[2] 中。对于背景看 Gagliardo[1] 及 Aronszajn 及 Smith[1]。

4.5 对于空间$\Lambda_a^{p,q}(R^n)$，类似的结果成立。若

$$a>(n-m)/p,$$

则空间$\Lambda_\beta^{p,q}(R^n)$中的元素在$R^m$上的限制属于$\Lambda_a^{p,q}(R^m)$，其中$\beta=a-(n-m)/p$. 反之，$\Lambda_\beta^{p,q}(R^m)$中每个函数可以延拓到$R^n$上，使它成为$\Lambda_a^{p,q}(R^n)$中的元素。详见 Besov[1]和Taibleson[2]。

4.6 设$\omega(\delta)(0<\delta)$是一个像前面在§2的2.2.3中定义的那样的正则连续模。对任意闭集$F$与每一个非负整数$k$，除去关于$R_j$的假定应改为

$$|R_j(x,y)|\leqslant M|x-y|^{k-|j|}\omega(|x-y|)$$

以外，我们如同在§2.3中（等式（16）和（17））一样定义空间 Lip$(k+\omega,F)$。则算子$\mathscr{E}_k$给出从 Lip$(k+\omega,F)$到 Lip$(k+\omega,R^n)$ 的一个开拓。

4.7 在§2.3的开拓定理中，假设$\gamma=k$，另外

$$R_j(x,y)=o(|x-y|^{k+|j|})$$

按以下意义成立，对任意$\bar{x}\in F$，及任意$\varepsilon>0$，存在一个$\delta>0$，使得当$|\bar{x}-x|<\delta$，$|\bar{x}-y|<\delta$，及$x,y\in F$，便有$|R_j(x,y)|\leqslant\varepsilon|x-y|^{k-|j|}$. 则$\mathscr{E}_k(f)\in C^k(R^n)$。这实质上是 Whitney[1]型的最初的开拓定理。

4.8 $L_k^p(D)$ 中函数的第一个开拓定理是通过下述论证来证明的，其中 $D$ 是 § 3 中所处理的那种类型的区域。

在这个论证的纲要中，我们只限于特殊Lipschitz区域 $D\subset \boldsymbol{R}^n$ (如同在 § 3.2中关于 $\boldsymbol{R}^{n+1}$ 所定义的一样)。为了我们的目的，作为这个定义的一个结论，我们所需要的一切是存在一个固定的锥体 $\Gamma$，使得对任何 $x\in D$，则 $x+\Gamma\subset D$。在这以后我们固定 $k\geqslant 1$。选取一个支集在 $-\Gamma$ 中的确定的函数 $\varphi$，它具有下述附加的性质：(a) $\varphi$ 具有有界支集；(b) 在原点附近，$\varphi$ 等于一个 $-n+k$ 次齐次的函数；(c) $\varphi$ 在 $\boldsymbol{R}^n-\{0\}$ 中属于 $C^\infty$；(d) $\lim\limits_{\varepsilon\to 0}\int_{S^{n-1}}\varepsilon^{n-k}\times\varphi(\varepsilon x')d\sigma(x')=1/(k-1)!$。(注意由于假定 $\varphi$ 在原点附近的齐次性，故对于很小的 $\varepsilon$，这个积分实际上是常数。)

令

$$\psi(\rho y')=\frac{\partial^k}{\partial\rho^k}(\rho^{n-1}\varphi(\rho y'))\quad (y=\rho y',\ |y|=\rho).$$

则 $\psi$ 在原点附近为零，从而处处属于 $C^\infty$(当然具有有界支集)。令

$$\begin{aligned}\mathfrak{E}(f)(x)=&\int\varphi(\rho y')\rho^{n-1}\frac{\partial^k}{\partial\rho^k}f(x-\rho y')d\rho dy'\\&-\int\psi(\rho y')f(x-\rho y')\rho^{n-1}d\rho dy'.\end{aligned}$$

上式右端第一个积分可以重写为

$$(-1)^k\int_{\boldsymbol{R}^n}\varphi(y)\sum_{|\alpha|=k}\frac{k!}{\alpha!}y^\alpha\frac{\partial^\alpha}{\partial x^\alpha}f(x-y)|y|^{-k}dy,$$

而第二个积分显然等于

$$\int_{\boldsymbol{R}^n}\psi(y)f(x-y)dy.$$

我们注意到 $f$ 与 $\frac{\partial^\alpha f}{\partial x^\alpha}$，$|\alpha|=k$，只是在 $D$ 中给定。为了使这个积分公式定义出 $\mathfrak{E}(f)$，我们设 $f$ 与 $\frac{\partial^\alpha f}{\partial x^\alpha}$ 在 $D$ 外等于零。

可以验证 $\mathfrak{E}(f)$ 确实开拓了 $f$，而且

$$\|\mathfrak{E}(f)\|_{L_k^p(R^n)} \leqslant A_{p,k}\|f\|_{L_k^p(D)}, \quad 1<p<\infty.$$

这后一事实的证明可如下考虑，定义 $\mathfrak{E}(f)$的第一个积分的$k$阶微商实质上引导到第二章§4中所讨论的那种类型的奇异积分算子。这说明了为什么有$1<p<\infty$的限制。进一步的细节，读者可参考Calderón[4]。

关于这个开拓我们作两点进一步的说明。算子$\mathfrak{E}$依赖于特殊的$k$，因此按§3所给的开拓的意义，它不具有普遍性。另一方面它在远离$D$处具有有趣的性质，$\mathfrak{E}(f)$只依赖于$f$在$D$的边界附近的部分。这意味着若$f\in L_k^p(D)$，且$f$在$D$的边界附近为零，则$\mathfrak{E}(f)(x)=0$，当$x\bar{\in}D$。

## 注　释

**节1与节2**　基本文献是Whitney[1]。定理1的特殊形式在Stein[10]中；定理2在Calderón与Zygmund[7]中，也见Glaeser[1]。

**节3**　对定义在带Lipschitz边界的区域内的函数的开拓定理首创于Calderón[4]中。所用的某些思想包含在Sobolev[2]中。Calderón的开拓不适用于极限情形$p=1$，或$p=\infty$，因为它依靠第二章所讨论的奇异积分的$L^p$有界性。对于现在这种形式，见Stein[10]。有关现在这种开拓以及应用于$p=2$的情形，其思想可在Adams，Aronszajn与Smith[1]中找到。

# 第七章　再论调和函数

我们回到调和函数论，对它的某些课题作较深入的研究，特别地，研究在第三章首先讨论过的共轭调和函数概念。主要的发展线索如下：

(a) **非切线收敛概念**　第三章定理 1 保证了 Poisson 积分垂直趋向的边值几乎处处存在，更一般的非切线极限的存在，而这对以后是很基本的，这有可能把经典的 Fatou 定理推广到“局部”的情形。这些是 § 1 要研究的问题。

(b) Lusin **的面积积分**　这在第四章 § 2 已讨论过，但在那里，我们的注意力集中在与之密切相关的 $g$ 与 $g^*$ 函数上。

面积积分的基本作用在于它可以同时用来描述非切线收敛以及 $L^p$ 和 $H^p$ 模收敛。调和函数非切线收敛的特征将在 § 2 给出，而在第八章研究函数可微性时有重要应用。

(c) **共轭调和函数的$H^p$理论**　共轭的概念，早在第三章通过广义 Cauchy-Riemann 方程并与 Riesz 变换相联系引进过。这里，我们将看到，应用它的某种下调和性以及 Lusin 的面积积分，我们可以研究奇异积分算子和乘子变换的各种结果在 $p=1$ 时的类似。想法之一（通过 $g$ 和 $g^*$ 描述这些算子的 $L^p$ 有界性）在第四章 § 3 中用过。

## § 1　非切线收敛与Fatou定理

1.1　我们要用到下面的符号：$\boldsymbol{R}_+^{n+1}$ 是点$(x,y)$的$n+1$维半空间，其中 $y>0$，$x\in\boldsymbol{R}^n$。它的边界$\{(x,0)\}$等同于$\boldsymbol{R}^n$。对任意 $x^0\in\boldsymbol{R}^n$ 和 $\alpha>0$，$\Gamma_\alpha(x^0)$ 表示顶点在 $x^0$ 的（无限）锥：$\Gamma_\alpha(x^0)=$

$\{(x, y) \in \boldsymbol{R}_{+}^{n+1}:\ |x-x^0|<\alpha y\}$。如果 $u(x, y)$ 定义 在 $\boldsymbol{R}_{+}^{n+1}$ 边界点 $(x^0, 0)$ 的附近，且 对每个 $\alpha>0$，当 $(x, y) \in \Gamma_\alpha(x^0)$ 且 $(x, y) \to (x^0, 0)$ 时有 $u(x, y) \to l$，那末就说，$u$ 在 $(x^0, 0)$ 有非切线极限（等于 $l$）。

关于 Poisson 积分非切性收敛的基本结果包含在第三章 定理 1（见第76页）的下述推广中。

**定理1** 设 $f \in L^p(\boldsymbol{R}^n)$, $1 \leqslant p \leqslant \infty$，令 $u(x, y)$ 是它的 Poisson 积分。又设 $\alpha$ 是一固定数，则

(a) $$\sup_{(x, y) \in \Gamma_\alpha(x^0)} |u(x, y)| \leqslant A_\alpha M(f)(x^0),$$

其中 $M(f)$ 是第一章 § 1的极大函数。$A_\alpha$ 与 $f$ 无关。

(b) $$\lim_{\substack{(x, y) \to (x^0, 0) \\ (x, y) \in \Gamma_\alpha(x^0)}} u(x, y) = f(x^0)$$

对几乎所有的 $x^0$，特别地对 $f$ 的 Lebesgue 集的每一点 $x^0$ 成 立。

实际上，这个定理是第三章相应定理的一个比较容易的推论。

为证明(a)，回忆

$$P_y(x) = \frac{c_n y}{(|x|^2 + y^2)^{(n+1)/2}}$$

（第三章命题 5）。由此容易看出

(1) $$P_y(x-t) \leqslant A_\alpha P_y(x), \quad |t|<\alpha y.$$

然而

$$u(x^0-t, y) = \int_{\boldsymbol{R}^n} P_y(x^0-t-z) f(z) dz,$$

因此

$$\begin{aligned}
\sup_{|t|<\alpha y} |u(x^0-t, y)| &= \sup_{(x, y) \in \Gamma_\alpha(x^0)} |u(x, y)| \\
&\leqslant \sup_{|t|<\alpha y} \int_{\boldsymbol{R}^n} P_y(x^0-t-z) |f(z)| dz \\
&\leqslant A_\alpha \sup_y \int_{\boldsymbol{R}^n} P_y(x^0-z) |f(z)| dz
\end{aligned}$$

$$\leqslant A_\alpha M(f)(x^0).$$

从而(a)得证。

显然，简单性质(1)实际上可以表成：若 $P_1(x)=\varphi(x)$，则

$$\sup_{|t|<1}\varphi(x-t)\leqslant A\varphi(x),$$

其中 $A$ 是与 $x$ 无关的某个常数。（从这点注释中，读者不难对逼近 $f*\varphi_\varepsilon$ 写出并证明一个一般的非切线结果，类似于第三章§2.2的定理2。）

为证明(b)，假设 $x^0$ 是 $f$ 的Lebesgue集的一点。这时对任意 $\varepsilon>0$，存在 $\delta>0$，使得

$$\frac{1}{m(B(r))}\int_{B(r)}|f(x^0-z)-f(x^0)|dz<\varepsilon,$$

只要 $r<\delta$，$B(r)$ 表示以原点为中心，$r$ 为半径的球。现在我们令 $g(x)=f(x)-f(x^0)$，当 $|x-x^0|\leqslant\delta$；而 $g(x)=0$，当 $|x-x^0|>\delta$。因此 $M(g)(x^0)<\varepsilon$。然而，我们有

$$u(x^0-t,y)-f(x^0)=\int_{R^n}P_y(z-t)[f(x^0-z)-f(x^0)]dz.$$

这样，当 $|t|<\alpha y$ 时，由(1)

$$|u(x^0-t,y)-f(x^0)|\leqslant A_\alpha\int_{R^n}P_y(z)|f(x^0-z)-f(x^0)|dz$$

$$=A_\alpha\left\{\int_{|z|<\delta}+\int_{|z|\geqslant\delta}\right\}.$$

用第三章定理1（部分(a)），有

$$\int_{|z|<\delta}\leqslant\int_{R^n}P_y(t)|g(x^0-z)|dz\leqslant M(g)(x^0)<\varepsilon,$$

而

$$\int_{|z|\geqslant\delta}P_y(z)|f(x^0-z)-f(x^0)|dz\to 0,\qquad \text{当}y\to 0,$$

这可以直接从

$$\left(\int_{|z|\geqslant\delta}[P_y(z)]^q dz\right)^{1/q}\leqslant A_\delta y\to 0$$

推出，当 $1\leqslant q\leqslant\infty$，在我们这里 $q$ 可以顺次取 $p$ 的共轭指标与1。因此

$$\lim_{|t|<\alpha y}\ \sup_{y\to 0}|u(x^0-t,y)-f(x^0)|\leqslant A_\alpha\varepsilon,$$

这就证明了所要求的在 $f$ 的 Lebesgue 集的每一点非切线 收 敛。根据几乎每一点都是 Lebesgue 集的点（见第一章 § 1.8），我 们就结束了定理的证明。

**1.2 Fatou 定理** 我们首先给出有界调和函数的特征。

**命题1** 设 $u$ 定义在 $\boldsymbol{R}_+^{n+1}$，则 $u$ 是 $\boldsymbol{L}^\infty(\boldsymbol{R}^n)$函 数 的 Poisson 积分，当且仅当 $u$ 调和且有界。

有界函数的 Poisson 积分是（调和）且有界的，这可以 从第三章 § 2 的讨论推出。

为证明其逆，假设 $u$ 在 $\boldsymbol{R}_+^{n+1}$调 和且 $|u|\leqslant M$。对 每 个 整 数 $k$，令 $f_k(x)=u(x,1/k)$，而令 $u_k(x,y)$是 $f_k$ 的 Poisson 积 分。最后，令 $\Delta_k(x,y)=u(x,y+1/k)-u_k(x,y)$。$\Delta_k$ 具有下列显然的性 质。首 先，$\Delta_k$ 在 $\boldsymbol{R}_+^{n+1}$ 调 和；其 次，$|\Delta_k|\leqslant|u(x,y+1/k)|+|u_k(x,y)|\leqslant 2M$，因此 $\Delta_k$ 有界。最后，由第三章 § 2.2的推论（第80页），根 据 $f_k$ 连 续 且 有界，知 $u_k(x,y)$ 在 $\overline{\boldsymbol{R}}_+^{n+1}$ 连 续 且 $u_k(x,0)=f_k(x)$；因此 $\Delta_k$ 在 $\overline{\boldsymbol{R}}_+^{n+1}$连续，并 且 $\Delta_k(x,0)=0$。我们要从这些性质断言 $\Delta_k\equiv 0$。为此只要证 明 $\Delta_k(0,1)=0$；这是由于我们的假设对 $x$ 变量的平移以及同时对 $x$，$y$ 的展缩是不变的，而 $\boldsymbol{R}_+^{n+1}$ 的任何点均可以通过这些变换的乘积 变 为(0, 1)。对 固定的 $\varepsilon>0$，考虑函数

$$U(x,y)=\Delta_k(x,y)+2M\varepsilon\,\mathrm{h}+\varepsilon\left[\prod_{j=1}^n\cosh\left(\frac{\varepsilon\pi}{4n^{1/2}}x_j\right)\right]\cos\left(\frac{\varepsilon\pi}{4}y\right).$$

显然它在 $\boldsymbol{R}_+^{n+1}$ 调和且在 $\overline{\boldsymbol{R}}_+^{n+1}$ 连续。我们限制我们的注意 在 圆柱部分$\sum=\{(x,y):0\leqslant y\leqslant 1/\varepsilon,|x|\leqslant R\}$，其中 $R$ 充分大。在$\sum$的

$y=0$那部分边界上，由 $\Delta_k=0$ 知$U(x,y)\geqslant 0$；当 $y=1/\varepsilon$，由 $|\Delta_k|\leqslant 2M$ 知 $U(x,y)\geqslant 0$。最后，当$R$充分大，再由 $\Delta_k$ 有界，在边界的其余部分仍有$U(x,y)\geqslant 0$。因此用极值原理（见附录C）得到 $U(0,1)\geqslant 0$，这就是说 $\Delta_k(0,1)\geqslant -(2M+1)\varepsilon$；用 $-\Delta_k$ 代替 $\Delta_k$，类似的结论也成立，从而 $\Delta_k\equiv 0$，这就是所要求的。这个结果可以写成

$$u(x,y+1/k)=\int_{\boldsymbol{R}^n}P_y(x-t)f_k(t)dt. \tag{2}$$

现在回忆

$$\|f_k\|_\infty=\|u(x,1/k)\|_\infty\leqslant M.$$

因此用熟知的弱列紧推理，存在 $f\in L^\infty$，$\|f\|_\infty\leqslant M$与一串子序列$\{f_{k'}\}$，使得 $f_{k'}\to f$ 弱收敛，即对任意的 $\varphi\in L^1(\boldsymbol{R}^n)$，

$$\int_{\boldsymbol{R}^n}f_{k'}\varphi dx\to\int_{\boldsymbol{R}^n}f\varphi dx.$$

对固定的$(x,y)\in\boldsymbol{R}_+^{n+1}$，选择 $\varphi(t)=P_y(x-t)$。这时对式(2) 取极限变成

$$u(x,y)=\int_{\boldsymbol{R}^n}P_y(x-t)f(t)dt,$$

这意味着 $u$ 是有界函数 $f$ 的 Poisson 积分。命题因此证完。

把命题与定理 1 结合起来就直接得到经典的 Fatou 定理的 $n$ 维形式。

**定理2** 设 $u$ 在 $\boldsymbol{R}_+^{n+1}$ 调和有界，则 $u$ 在 $\boldsymbol{R}_+^{n+1}$ 的边界 $\boldsymbol{R}^n$几乎处处有非切线极限。

1.2.1 从命题 1 很快导出它本身的推广。

**推论** 设 $u$ 在 $\boldsymbol{R}_+^{n+1}$调和，且 $1\leqslant p\leqslant\infty$。如果

$$\sup_{y>0}\|u(\cdot,y)\|_{L^p(\boldsymbol{R}^n)}<\infty,$$

那末当 $p>1$时，$u$ 是一个 $f\in L^p(\boldsymbol{R}^n)$ 的 Poisson 积分，当 $p=1$ 时，$u$ 是一有限测度的 Poisson 积分，

这个结果曾在第三章 §4.2 不加证明地叙述过。反过来是容易的，包含在那一章的§2.2中。为证明推论，设$p<\infty$。对每个$(x,y)\in \boldsymbol{R}_+^{n+1}$，令$B$为中心在$(x,y)$，半径为$y$的球。由平均值定理

$$|u(x,y)|^p\leqslant\frac{1}{m(B)}\iint_B|u(x',y')|^p dx'dy'.$$

然而$B\subset\{(x',y'): 0<y'<2y\}$，$m(B)=cy^{n+1}$。因此

$$|u(x,y)|^p\leqslant c'y^{-n-1}\int_0^{2y}\int_{\boldsymbol{R}^n}|u(x',y')|^p dx'dy',$$

从而

$$|u(x,y)|\leqslant c''y^{-n/p}.$$

对每个正整数$k$我们便可以应用命题1（与定理2）而得到$u(x,y+1/k)=P_y * f_k$，其中$f_k(x)=u(x,1/k)$。然而由假设$\sup_k\|f_k\|_p<\infty$，故熟知的弱列紧推理可以用。特别地，当$p>1$时，存在$f\in L^p(\boldsymbol{R}^n)$与子序列$\{f_{k'}\}$，使得$f_{k'}\to f$弱收敛。对$p=1$，存在有限测度$d\mu$与子序列$\{f_{k'}\}$，使得$f_{k'}\to d\mu$弱收敛。在上述两种情形下分别推出$u(x,y)=P_y * f$或$u(x,y)=P_y * d\mu$。注意当$p>1$时$\|f\|_p=\sup_{y>0}\|u(\cdot,y)\|_p$，以及$\|d\mu\|=\sup_{y>0}\|u(\cdot,y)\|_1$。

**1.3 一个局部的命题** 我们首先定义非切线有界的合适概念。对任意$\alpha>0$与$h>0$，用$\Gamma_\alpha^h(x^0)$表示**截锥**$\Gamma_\alpha^h(x^0)=\{(x,y)\in\boldsymbol{R}_+^{n+1}: |x-x^0|<\alpha y,\ 0<y<h\}$。设$u$定义在$\boldsymbol{R}_+^{n+1}$，我们说$u$在$x^0$**非切线有界**，如果对某个$\alpha$与$h$

$$\sup|u(x,y)|<\infty,\quad (x,y)\in\Gamma_\alpha^h(x^0).$$

注意，在$x^0$的非切线有界只要求对以$x^0$为顶点的一个截锥而言，而在$x^0$的非切线极限存在（见上面§1.1的叙述）却要求对$x^0$的一切锥而言。

本节的主要结果是Fatou定理（定理2）的下述的局部类似。

**定理3** 假设$u$在$\boldsymbol{R}_+^{n+1}$调和。设$E$是$\boldsymbol{R}^n$的子集而$u$在每个$x^0\in E$非切线有界。则$u$对几乎所有的$x^0\in E$有非切线极限。

1.3.1 **一个预备引理使我们可以把情况标准化。**

**引理** 设 $u$ 在 $\boldsymbol{R}_{+}^{n+1}$ 连续，并且在集 $E\subset\boldsymbol{R}^{n}$ 的每一点非切线有界。则对 $\varepsilon>0$，存在一紧集 $E_1$，满足

(a) $E_1\subset E$，$m(E-E_1)<\varepsilon$；

(b) 对任意 $\alpha>0$，$h>0$，存在界 $M=M(\alpha,h,\varepsilon)$，使得

$$|u(x,y)|\leqslant M,\quad (x,y)\in\Gamma_{\alpha}^{h}(x^0),\quad x^0\in E_1.$$

**证明** 第一步。只考虑 $\alpha$，$h$ 为有理数。这时我们可以找到 $E_0$，满足 $m(E-E_0)$ 充分小（例如说 $m(E-E_0)<\varepsilon/3$），使得

$$|u(x,y)|\leqslant M,\quad (x,y)\in\Gamma_{\alpha}^{h}(x^0),\quad x^0\in E_0,$$

以及对某个固定的 $\alpha$ 与 $h$。我们还可以假设 $E_0$ 是紧集。

第二步。对这样固定的 $E_0$ 以及大的整数 $k$，我们将进一步选取 $E_{00}$，$E_{00}\subset E_0$，满足 $m(E_0-E_{00})<\varepsilon/3$，使得

$$|u(x,y)|\leqslant M',\quad (x,y)\in\Gamma_{k}^{k}(x^0),\quad x^0\in E_{00}.$$

为此我们如下进行。设给定 $\eta<1$ 与 $\varepsilon/3$，这时存在 $\delta>0$ 与子集 $E_{00}$，使得 $m(E_0-E_{00})<\varepsilon/3$，并且当 $x\in E_{00}$ 以及 $r\leqslant\delta$ 时，有

$$\frac{m(B(x,r)\cap E_0)}{m(B(x,r))}\geqslant\eta,$$

这是因为几乎所有 $E_0$ 的点都是全密点。如果我们能证明对充分小的 $\delta$ 有

$$\text{(3)}\qquad \Gamma_{k}^{\delta}(x^0)\subset\bigcup_{x'\in E_0}\Gamma_{\alpha}^{h}(x'),\quad \text{对所有 } x^0\in E_{00},$$

那末我们所要求的在 $E_{00}$ 的有界性就成立了。

为了证明式(3)，为简单起见设 $x^0=0$。这时我们必须考虑锥 $|x|<ky$（其中 $y<\delta$）内的数对 $(x,y)$。固定这样的数对 $(x,y)$。我们要证明，存在 $x'\in E_0$，使得 $|x-x'|<\alpha y$。如果不然，假设对这个 $x$（与 $y$），不存在这样的 $x'$，这时球 $|x-x'|<\alpha y$ 包含在 ${}^cE_0$ 内。把这个半径为 $\alpha y$ 的球与半径为 $ky$（中心在圆点的更大）的球相比较。按假设原点属于 $E_{00}$，而它在大球中的相对测度至多是 $1-\alpha/k$，这就是一个矛盾，因为我们可以选择 $\eta>1-\alpha/k$。

因此对充分小的 $y$，我们可以找到所要求的 $x'$，而式(3)对 $\delta=\delta/k$ 得证。注意到 $u$ 的连续性，显然 $u$ 在离 $\boldsymbol{R}^n$ 有正距离的紧集上是有界的，而我们又可以选取 $E_{00}$ 是紧集。

**第三步**。对每个整数 $k$，我们构造这种子集 $E_{00}$。可数个这样的 $E_{00}$ 的合适交集就给出所要求的集 $E_1$。

1.3.2　对 $\boldsymbol{R}^n$ 的任意紧子集 $E$，我们定义开区域 $\mathscr{R}$ 如下：

$$\mathscr{R}=\bigcup_{x^0\in E}\Gamma_\alpha^h(x^0) \tag{4}$$

(见图4)。很容易看出，引理使我们把定理 3 归结到它的本质部分：设 $u$ 在 $\boldsymbol{R}_+^{n+1}$ 调和且 $|u|\leqslant 1$，当 $(x,y)\in\mathscr{R}$。则对几乎每一个 $x^0\in E$，当 $(x,y)\to(x^0,0)$ 且 $(x,y)\in\mathscr{R}$ 时极限 $\lim u(x,y)$ 存在。这是我们现在就要加以证明的。

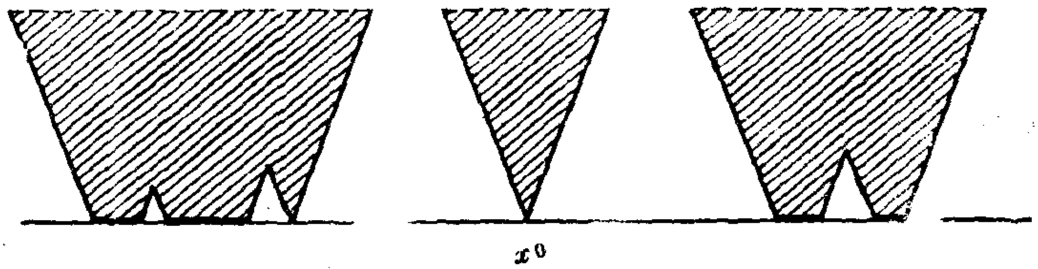

图 4　区域 $\mathscr{R}$，它是锥 $\Gamma_\alpha^h(x^0)$ $(x^0\in E)$ 的并集

1.3.3　对每个 $m>0$，我们定义 $\boldsymbol{R}^n$ 上的函数 $\varphi_m$ 如下：$\varphi_m(x)=u(x,1/m)$，当 $(x,1/m)\in\mathscr{R}$；$\varphi_m=0$，其它。令 $\varphi_m(x,y)$ 是 $\varphi_m$ 的 Poisson 积分，即 $\varphi_m(x,y)=(P_y*\varphi_m)(x)$。定义 $\psi_m$ 如下：

$$u\left(x,y+\frac{1}{m}\right)=\varphi_m(x,y)+\psi_m(x,y). \tag{5}$$

函数序列 $\{\varphi_m(x)\}$ 按 $L^\infty(\boldsymbol{R}^n)$ 模是一致有界的，因此可以找到 $\varphi(x)\in L^\infty$(事实上 $|\varphi(x)|\leqslant 1$)和子序列 $\{\varphi_{m'}\}$，使得 $\varphi_{m'}\to\varphi$ 弱收敛。令 $\varphi(x,y)$ 为 $\varphi$ 的 Poisson 积分。显然在每点 $(x,y)\in\boldsymbol{R}_+^{n+1}$ 有 $\varphi_{m'}(x,y)\to\varphi(x,y)$；还有 $u(x,y+1/m)\to u(x,y)$，从而，$\psi_{m'}(x,y)$ 逐点收敛到 $\psi(x,y)$，并且满足

$$u(x,y)=\varphi(x,y)+\psi(x,y). \tag{5'}$$

函数 $\varphi(x,y)$ 是有界函数的 Poisson 积分，因而它几乎处处非切线收敛。函数 $\psi$ 按构造应该在 $E$ 有零边值，而这正是我们要通过一个不等式来证明的。为此，我们考虑一个辅助的调和函数 $H(x,y)$，在区域 $\mathscr{R}$ 具有下列的性质。我们把 $\mathscr{R}$ 的边界分为两部分：记为 $\partial\mathscr{R}=\mathscr{B}=\mathscr{B}_0\cup\mathscr{B}_+$，其中 $\mathscr{B}_0$ 是区域与超平面 $\boldsymbol{R}^n$ 相交的那部分，而 $\mathscr{B}_+$ 是在 $\boldsymbol{R}^n$ 以上的那部分。也就是说 $\mathscr{B}_0=\{(x,0)\in\overline{\mathscr{R}}-\mathscr{R}\}$，而 $\mathscr{B}_+=\{(x,y)\in\overline{\mathscr{R}}-\mathscr{R},\ y>0\}$。

$H$ 具有下列性质：

(a) $H$ 在 $\boldsymbol{R}_+^{n+1}$ 内调和；

(b) 在 $\boldsymbol{R}_+^{n+1}$ 内有 $H\geqslant 0$（其实只要对 $\mathscr{B}_0$ 说就够了）；

(c) 在 $\mathscr{B}_+$ 上有 $H\geqslant 2$；

(d) 对几乎所有的 $x^0\in E$，当 $(x,y)$ 非切线地趋向 $(x^0,0)$ 时 $H(x,y)\to 0$。

$H$ 的构造是简单的。让 $\chi$ 表示 $E$ 的余集的特征函数。对于即将决定的常数 $c$，记 $H(x,y)=c\{(P_y*\chi)+y\}$。性质 (a) 与 (b) 显然；(d) 是本章定理 1 的直接推论；只有 (c) 需要进一步验证。

现在，边界 $\mathscr{B}_+$ 可以进一步分解为它的“显然”部分，即在 $y=h$ 的部分，以及它的非显然部分，即 $0<y<h$ 的另一部分。对显然部分我们永远可以通过取足够大的 $c(c\geqslant 2/h)$ 使 (c) 成立。剩下来的只要考察边界 $\mathscr{B}_+$ 的非显然部分。应该注意到，并且这在以后的研究中也是重要的，问题中边界的这一部分是由 $y=\dfrac{1}{\alpha}\operatorname{dist}(x,E)$ 给出的 Lipechitz 超曲面的一部分。因此我们可以断言，对每个这样的点 $(x,y)$，$\boldsymbol{R}^n$ 上中心在 $x$ 半径为 $\alpha y$ 的球 $B$ 必然在 $E$ 之外。故

$$(P_y*\chi)(x)=c_n y\int_{\boldsymbol{R}^n}\frac{\chi(t)}{(|x-t|^2+y^2)^{(n+1)/2}}dt$$

$$\geqslant c_n y\int_B\cdots dt=c_n y\int_{|t|<\alpha y}\frac{1\cdot dt}{(|t|^2+y^2)^{(n+1)/2}}=\text{常数}。$$

最后的积分等于常数这一事实，可以通过显然的变量替换看出。

再取 $c$ 充分大，我们便可以验证$H$的一切性质。现在我们来证明，对固定的 $m$，

(6) $$|\psi_m(x,y)|\leqslant H(x,y),\quad 当(x,y)\in\mathscr{R}.$$

注意 $\psi_m$ 与$H$的调和性以及极值原理，如果这不成立，那末存在 $\varepsilon>0$ 和一串收敛到$\mathscr{R}$的边界点的点列$(x_k,y_k)$，使得

$$|\psi_m(x_k,y_k)|\geqslant H(x_k,y_k)+\varepsilon.$$

首先，$u$ 与 $\varphi_m$ 按绝对值都有上界 1，因此，$|\psi_m(x,y)|\leqslant 2$，从而按性质 (c)，$(x_k,y_k)$ 的极限不能在 $\mathscr{B}_+$。然而，$\varphi_m(x,y)$ 是一个连续函数的 Poisson 积分，这函数在包含$E$的一个开集上连续，且它的值等于 $u(x,1/m)$。由定理 1 的(b)，有

$$\lim_k\left\{u\left(x_k,y_k+\frac{1}{m}\right)-\varphi_m(x_k,y_k)\right\}=\lim_k\psi_m(x_k,y_k)=0,$$

这与(b)矛盾。故证得式(6)。再令 $m$ 通过子序列 $\{m'\}$ 趋向$\infty$。结果是

(6′) $$|\psi(x,y)|\leqslant H(x,y).$$

由性质(d)就推出对几乎所有的$(x^0,0)\in E$，当$(x,y)\to(x^0,0)$，$(x,y)\in\mathscr{R}$ 时，$\psi$ 有所要求的极限(为 0)。这就完成了定理 3 的证明。

1.3.4 有趣的是，这个定理与本章的其它结果（特别是下面的定理 5），当用垂直趋向边界代替非切线极限时，类似的结果并不成立。见下面 § 4.12。

## § 2 面积积分

**2.1** 我们刚刚证明的定理表明，对 $R_+^{n+1}$ 的调和函数来说，非切线有界性与非切线极限的存在性几乎处处是等价的。从一般的观点看，这两个性质(有界与极限存在)的差别还不太大。然而，还有另外的条件，几乎处处等价于这两个性质。它具有很不同的特征，是通过 Lusin 引进的某种平方积分来表示的。

我们取定一种典型的截锥$\Gamma_a^h$(即固定$a$与$h$)。当$u$在$\mathbf{R}_+^{n+1}$(或在$\mathbf{R}_+^{n+1}$中包含其边界$\mathbf{R}^n$附近在内的一个子域)给定时，我们定义

(7) $$S(u)(x^0)=\Big(\iint_{\Gamma_a^h(x^0)}|\nabla u|^2y^{1-n}dydx\Big)^{1/2}.$$

(试与第四章§2.3中出现的式子比较一下；见第112页。)

在这里

$$|\nabla u|^2=\left|\frac{\partial u}{\partial y}\right|^2+\sum_{j=1}^n\left|\frac{\partial u}{\partial x_j}\right|^2,$$

而在(7)中写出二重积分是为了强调我们处理的是$n+1$维积分，以区别于以后常见的$n$维积分。名称"面积积分"来自$n=1$的情形。在那里$(S(u)(x^0))^2$表示在解析映照$z\to F(z)$作用下三角形$\Gamma_a^h$在$\mathbf{R}^2$的像域的面积(计算时要考虑重数)，其中$F$的实部是$u$。当$n>1$时，$S$没有这种简单的解释，但我们仍然沿用$n=1$时的术语。非切线收敛可通过面积积分刻划如下。

**定理4** 设$u$在$\mathbf{R}_+^{n+1}$调和，则除了$\mathbf{R}^n$中点$x^0$的一个零测集外，下面两个条件是等价的：

(a) $u$在$x^0$有非切线极限；

(b) $S(u)(x^0)<\infty$。

2.1.1 对定理的叙述需要作一些解释，(a)$\Longrightarrow$(b)的证明表明，除了一个零测集以外，在非切线极限存在的点$x^0$必定有$S(u)(x^0)<\infty$，而同决定截锥形状的参数$a$与$h$的选取无关。

反过来，假设在一已知集合的$x^0$都有$S(u)(x^0)<\infty$，其中截锥形状可以点点不同。但定理的证明表明，$u$几乎在集合的每一点都有非切线极限。

特别地，这就推出，$S$对一类变锥是有限的假定，几乎处处等价于$S$对所有截锥是有限的。(不过，这最后一部分是十分初等的，并且如果愿意的话可以直接证明。)

**2.2**

2.2.1 定理的证明将涉及在§1的1.3.2(第259页)定义的

区域$\mathscr{R}$上用 Green 定理。这区域的边界勉强具有用Green定理所要求的正则性，因而最方便的还是用一族边界十分光滑的区域$\{\mathscr{R}_\varepsilon\}$去逼近$\mathscr{R}$。边界族$\partial\mathscr{R}_\varepsilon$具有某种一致的光滑性，它反映了$\partial\mathscr{R}$起码的光滑程度。

**引理** 存在一族区域$\{\mathscr{R}_\varepsilon\}(\varepsilon>0)$，它具有下列的性质：

(a) $\overline{\mathscr{R}}_\varepsilon\subset\mathscr{R}$，$\mathscr{R}_{\varepsilon_1}\subset\mathscr{R}_{\varepsilon_2}$，当 $\varepsilon_2<\varepsilon_1$；

(b) $\mathscr{R}_\varepsilon\to\mathscr{R}$，当 $\varepsilon\to 0$ $\left(\text{即}\bigcup_\varepsilon\mathscr{R}_\varepsilon=\mathscr{R}\right)$；

(c) 区域的边界$\partial\mathscr{R}_\varepsilon=\mathscr{B}_\varepsilon$是两部分的并：$\mathscr{B}^1_\varepsilon\cup\mathscr{B}^2_\varepsilon$，其中$\mathscr{B}^2_\varepsilon$是超平面 $y=h-\varepsilon$ 的一部分，而，

(d) $\mathscr{B}^1_\varepsilon$ 是超平面 $y=\alpha^{-1}\delta_\varepsilon(x)$的一部分，其中 $\delta_\varepsilon\in C^\infty$，且

$$\left|\frac{\partial\delta_\varepsilon}{\partial x_j}\right|\leqslant 1,\quad j=1,2,\cdots,n.$$

令 $\delta(x)=\operatorname{dist}(x,E)$。显然 $|\delta(x)-\delta(x')|\leqslant|x-x'|$，当 $x,x'\in\boldsymbol{R}^n$。假设 $\varphi\in C^\infty(\boldsymbol{R}^n)$，有紧支集，是正的，且

$$\int_{\boldsymbol{R}^n}\varphi(x)\,dx=1.$$

令$\tilde{\delta}_\eta(x)=\varphi_\eta*\delta$，其中 $\varphi_\eta(x)=\eta^{-n}\varphi(x/\eta)$。显然也有 $\tilde{\delta}_\eta\in C^\infty$，并且

$$\left|\frac{\partial\tilde{\delta}_\eta}{\partial x_j}\right|\leqslant 1,$$

还有 $\tilde{\delta}_\eta\to\delta$一致地成立，当 $\eta\to 0$。由此容易看出，适当选择 $\tilde{\delta}$ 和 $\eta'$ ($\eta'>0$ 是常数)，函数 $\delta_\varepsilon(x)=\tilde{\delta}_\eta(x)+\eta'$ 以及相应的区域 $\mathscr{R}_\varepsilon=\{(x,y):\ \delta_\varepsilon(x)<\alpha y,\ 0<y<h-\varepsilon\}$就满足引理的结论。

2.2.2 对于固定的 $\alpha$ 和 $h$，我们有典型的截锥$\Gamma^h_\alpha=\Gamma^h_\alpha(0)$。以后还需要考虑另一个严格包含它的截锥。我们通过取定$\beta$和$k$来决定它，其中 $\beta>\alpha$，而 $k>h$。显然，$\Gamma^k_\beta\supset\Gamma^h_\alpha$，而且它们边界的唯一公共点是它们公共的顶点。

**引理** 设$u$在$\Gamma^k_\beta$调和。

(a) 若在 $\Gamma_\beta^k$ 有 $|u|\leqslant 1$，则在 $\Gamma_\alpha^h$ 有 $|y\nabla u|\leqslant c$；

(b) 若

$$\iint\limits_{\Gamma_k^\beta}|\nabla u|^2 y^{1-n}dxdy\leqslant 1,$$

则在 $\Gamma_\alpha^h$ 有 $|y\nabla y|\leqslant c$，其中 $c$ 是只依赖于 $\alpha$，$\beta$，$h$，$k$ 及维数 $n$ 的常数。

注意，存在常数 $c_1>0$，具有下面的性质：若 $(x,y)$ 是小的锥 $\Gamma_\alpha^h$ 中的任意一点，则中心在 $(x,y)$，半径为 $c_1 y$ 的球 $B$ 包含在 $\Gamma_\beta^k$ 内。现在用调和函数的中值定理就有下列的不等式（见附录C）：

$$|(\nabla u)(x,y)|\leqslant c_2 r^{-1}\sup_{(x,y)\in B}|u(x,y)|$$

以及

$$|(\nabla u)(x,y)|^2\leqslant c_2 r^{-n-1}\iint\limits_B|\nabla u|^2 dx'dy',$$

其中 $r$ 是 $B$ 的半径 $(=c_1 y)$。

这两个不等式中的第一个直接给出结论(a)。为证明第二个结论，注意对 $(x',y')\in B$，$y'$ 是与 $B$ 的半径可比较的，特别地 $r\geqslant c_3 y'$，从而 $y\geqslant c_4 y'$。因此

$$y^2|\nabla u|^2\leqslant c_4\iint\limits_B|\nabla u|^2 y^{1-n}dx'dy$$

$$\leqslant c_4\iint\limits_{\Gamma_\beta^k}|\nabla u|^2 y^{1-n}dx'dy.$$

从而引理得证。

2.3 (a)$\Longrightarrow$(b)，**正部分的证明** 假设 $\alpha$ 与 $h$ 给定。充分地只要证明，在给定的紧集 $E$ 的几乎每一点 $x^0$，都有

$$\iint\limits_{\Gamma_\alpha^h(x^0)}|\nabla u|^2 y^{1-n}dxdy<\infty,$$

只要

(8) $$\sup_{x^0\in E}\sup_{(x,y)\in\Gamma_\beta^k(x^0)}|u(x,y)|\leqslant 1$$

对某个固定的 $\beta>\alpha$，$k>h$。事实上，假设 $u$ 在一已知集有非切线

极限，我们总可以找出紧子集，它的测度任意接近于已知集，而在它上面式(8)一致成立(在乘上一合适的非零常数之后)。

记$\tilde{\mathscr{R}}=\bigcup\limits_{x^0\in E}\Gamma_\beta^k(x^0)$ $\Big($同样地，$\mathscr{R}=\bigcup\limits_{x^0\in E}\Gamma_\alpha^h(x^0)\Big)$。这样，我们的假设变成在$\tilde{\mathscr{R}}$，$|u|\leqslant 1$。

为了证明$S(u)(x^0)<\infty$几乎处处在$E$成立，充分地只要证明$\int_E S^2(u)(x^0)dx^0<\infty$。这个积分等于

$$\iint\left(\int_E\psi(x^0,x,y)dx^0\right)y^{1-n}|\nabla u(x,y)|^2dxdy,$$

其中$\psi$是集合$\{|x-x^0|<\alpha y,\ 0<y<h\}$的特征函数。然而

$$\int_E\psi(x^0,x,y)dx^0\leqslant\int_{|x^0-x|<\alpha y}dx^0=cy^n,$$

因此我们只要证明

$$\iint_{\mathscr{R}}y|\nabla u(x,y)|^2dxdy<\infty. \tag{9}$$

现在用逼近区域$\mathscr{R}_\varepsilon$代替$\mathscr{R}$，这时式(9)等价于

$$\iint_{\mathscr{R}_\varepsilon}y|\nabla u|^2dxdy\leqslant A<\infty, \tag{9'}$$

其中$A$与$\varepsilon$无关。为估计(9′)，我们对带光滑边界$\partial\mathscr{R}_\varepsilon=\mathscr{B}_\varepsilon$的区域用下述形式的Green公式

$$\iint_{\mathscr{R}_\varepsilon}(A\Delta B-B\Delta A)dxdy=\int_{\mathscr{B}_\varepsilon}\left(A\frac{\partial B}{\partial n_\varepsilon}-B\frac{\partial A}{\partial n_\varepsilon}\right)d\tau_\varepsilon, \tag{10}$$

其中$\dfrac{\partial}{\partial n_\varepsilon}$表示沿外法线的方向微商，$d\tau_\varepsilon$是$\partial\mathscr{R}_\varepsilon$的“面积”元素。取$B=u^2/2$，而$A=y$，这样$\Delta B=|\nabla u|^2$，$\Delta A=0$。由于$\partial\mathscr{R}_\varepsilon\subset\tilde{\mathscr{R}}$，引理中估计式(a)成立，也就是说，

$$\left|\frac{\partial B}{\partial n_\varepsilon}\right|\leqslant|u||\nabla u|,$$

因此，在$\mathscr{B}_\varepsilon$上$\left|\dfrac{A\partial B}{\partial n_\varepsilon}\right|\leqslant c$。类似地，由$\left|\dfrac{\partial y}{\partial n_\varepsilon}\right|\leqslant 1$知$\left|\dfrac{B\partial y}{\partial n_\varepsilon}\right|\leqslant\dfrac{1}{2}$。

总之，式(10)中右边积分中的被积函数是一致有界的，因此

$$\left|\int_{\mathscr{B}_{\varepsilon}}\left(\frac{A\partial B}{\partial n_{\varepsilon}}-\frac{B\partial A}{\partial n_{\varepsilon}}\right)d\tau_{\varepsilon}\right|\leqslant c\int_{\mathscr{B}_{\varepsilon}}d\tau_{\varepsilon}.$$

然而

$$\int_{\mathscr{B}_{\varepsilon}}d\tau_{\varepsilon}=\int_{\mathscr{B}_{\varepsilon}^{1}}d\tau_{\varepsilon}+\int_{\mathscr{B}_{\varepsilon}^{2}}d\tau_{\varepsilon},$$

其中$\mathscr{B}_{\varepsilon}^{2}$是超平面 $y=h-\varepsilon$的一部分，而$\mathscr{B}_{\varepsilon}^{1}$是曲面 $y=\alpha^{-1}\delta_{\varepsilon}(x)$的一部分。在$\mathscr{B}_{\varepsilon}^{2}$ 我们有 $d\tau_{\varepsilon}=dx$，而在$\mathscr{B}_{\varepsilon}^{1}$

$$d\tau_{\varepsilon}=\left(\sqrt{1+\alpha^{-2}\sum_{i=1}^{n}\left(\frac{\partial\delta_{\varepsilon}}{\partial x_{i}}\right)^{2}}\right)dx\leqslant\sqrt{1+\alpha^{-2}n}dx,$$

这是根据§2 的 2.2.1 中引理的性质 (d)。由于$\mathscr{R}$从而$\mathscr{R}_{\varepsilon}$总是包含在一固定的紧集里，因此

$$\int_{\mathscr{B}_{\varepsilon}}d\tau_{\varepsilon}\leqslant\text{常数}<\infty,$$

从而式(9)得证。

2.4 (b)$\Longrightarrow$(a)，**逆部分的证明** 我们可以把我们的假设归结为如下说法。假设对某个$\beta$与$k$（它们可以很小）以及给定的有界集$E_0$，有

$$\iint_{\Gamma_{\beta}^{k}(x^{0})}|\nabla u|^{2}y^{1-n}dxdy\leqslant 1,\qquad x^{0}\in E_{0}.$$

现在选择$E$如下：$E\subset E_0$，$E$是紧的且$m(E_0-E)$小；同时还存在$\eta>0$，使得当$x^0\in E$时有

$$m\{\{|x-x^{0}|<r\}\cap E_{0}\}\geqslant m\{|x-x^{0}|<r\}/2,$$

当$0<r<\eta$。在证明的最后再令$m(E_0-E)\to 0$。

这样选择$E$是可能的，因为$E_0$中几乎所有的点都是$E_0$的全密点。

现假设$\alpha$与$h$固定，满足$\alpha<\beta, h<k$。我们在$\mathscr{R}=\bigcup_{x^{0}\in E}\Gamma_{\alpha}^{h}(x^{0})$上来研究$u$。我们企图把正部分证明的逻辑线索倒过来，而第一个对象就是式(9)的证明。

由于假定了 $E_0$是有界的，我们有

$$\int_{E_0}\left(\iint_{\Gamma_\beta^k}|\nabla u|^2 y^{1-n}dxdy\right)dx^0<\infty. \tag{11}$$

现在我们需要从下界方面来估计

$$\int_{E_0}\Psi(x_0,x,y)dx^0,$$

其中$(x,y)\in\mathscr{R}$， $\Psi$是$\{|x-x^0|<\beta y,0<y<k\}$的特征函数。注意$(x,y)\in\mathscr{R}$意味着存在$z\in E$，使得$|x-z|<\alpha y$且$0<y<h$。因此我们有

$$\int_{E_0}\Psi(x^0,x,y)\,dx^0\geqslant\int_{E_0\cap\{|x^0-z|<(\beta-\alpha)y\}}dx^0.$$

根据假设，对$E$来说$E_0$的相对密度不小于1/2，我们知道上式右端超过$cy^n$，其中$c$是某个正常数。把这代到式(11)就给出了式(9)。但式(9)等价于(9′)，这也就是

$$\left|\int_{\mathscr{B}_\varepsilon}\left(y\frac{\partial u^2}{\partial n_\varepsilon}-u^2\frac{\partial y}{\partial n_\varepsilon}\right)d\tau_\varepsilon\right|\leqslant c<\infty. \tag{12}$$

然而$\mathscr{B}_\varepsilon=\mathscr{B}_\varepsilon^1\cup\mathscr{B}_\varepsilon^2$,而$\mathscr{B}_\varepsilon^2$同集合$\{y=0\}$有严格的正的距离，并且还是紧的，因此式(12)中对应于$\mathscr{B}_\varepsilon^2$的那部分有界，从而有

$$\left|\int_{\mathscr{B}_\varepsilon^1}\left(y\frac{\partial u^2}{\partial n_\varepsilon}-u^2\frac{\partial y}{\partial n_\varepsilon}\right)d\tau_\varepsilon\right|\leqslant c<\infty. \tag{12′}$$

在$\mathscr{B}_\varepsilon^1$，

$$\frac{\partial y}{\partial n_\varepsilon}\leqslant-\alpha\,(\alpha^2+n)^{-1/2}.$$

事实上，$\dfrac{\partial}{\partial n_\varepsilon}$是方程为$F_\varepsilon(x,y)\equiv\alpha y-\delta_\varepsilon(x)=0$的曲面的外法向微商，因此$\dfrac{\partial}{\partial n_\varepsilon}$的方向是由与

$$\left(-\frac{\partial F_\varepsilon}{\partial y},\ \frac{\partial F_\varepsilon}{\partial x_1},\ \cdots,\ \frac{\partial F_\varepsilon}{\partial x_n}\right)=\left(-\alpha,\frac{\partial\delta_\varepsilon}{\partial x_1},\cdots,\frac{\partial\delta_\varepsilon}{\partial x_n}\right)$$

相同方向的单位向量给出的。由于

$$\left|\frac{\partial\delta_\varepsilon}{\partial x_j}\right|\leqslant 1,$$

我们知道

$$\frac{\partial y}{\partial n_\varepsilon}\leqslant -\alpha(\alpha^2+n)^{-1/2}.$$

另外，令

$$\mathscr{I}_\varepsilon^2=\int_{\mathscr{B}_\varepsilon^1}u^2d\tau_\varepsilon.$$

根据我们刚刚说的以及(12′)

$$\mathscr{I}_\varepsilon^2\leqslant c_1\int_{\mathscr{B}_\varepsilon^1}|u|y\left|\frac{\partial u}{\partial n_\varepsilon}\right|d\tau_\varepsilon+c_2.$$

注意$\mathscr{B}_\varepsilon^1\subset\tilde{\mathscr{R}}=\bigcup\limits_{x^0\in E}\Gamma_\beta^h(x^0)$，因此由§2 的 2.2.2中引理的结论(b)，我们知

$$y\left|\frac{\partial u}{\partial n_\varepsilon}\right|\leqslant y|\nabla u|\leqslant c,\qquad 在\mathscr{B}_\varepsilon^1.$$

故

$$\int_{\mathscr{B}_\varepsilon^1}|u|y\left|\frac{\partial u}{\partial n_\varepsilon}\right|d\tau_\varepsilon\leqslant c\int_{\mathscr{B}_\varepsilon^1}|u|d\tau_\varepsilon$$

$$\leqslant c\mathscr{I}_\varepsilon\left(\int_{\mathscr{B}_\varepsilon^1}d\tau_\varepsilon\right)^{1/2}\leqslant c_3\mathscr{I}_\varepsilon.$$

总起来就给出$\mathscr{I}_\varepsilon^2\leqslant c_3\mathscr{I}_\varepsilon+c_4$，从而$\mathscr{I}_\varepsilon$对$\varepsilon$有界，这就是说

(13) $$\int_{\mathscr{B}_\varepsilon^1}u^2d\tau_\varepsilon\leqslant 常数.$$

2.4.1 我们继续逆方向 (b)$\Longrightarrow$(a)的证明。我们寻求用另外的$v$控制函数$u$，而$v$的非切线性质是我们知道的。我们如下进行。曲面$\mathscr{B}_\varepsilon^1$是曲面$y=\alpha^{-1}\delta_\varepsilon(x)$的一部分。令$f_\varepsilon(x)$是定义在$y=0$的函数，它是$u$在$\mathscr{B}_\varepsilon^1$的限制在$y=0$的投影，而在其余地方为0。即

$$f_\varepsilon(x)=u(x,\alpha^{-1}\delta_\varepsilon(x))$$

当$(x,0)$在$\mathscr{B}_\varepsilon^1$之下，而$f_\varepsilon(x)=0$，当$x$在其它的地方。显然由

$d\tau_\varepsilon \geqslant dx$，我们有

$$\int_{R^n} |f_\varepsilon(x)|^2 dx \leqslant \int_{\mathscr{B}_\varepsilon^1} |u(x)|^2 d\tau_\varepsilon \leqslant c.$$

现在记 $v_\varepsilon(x,y)$为$|f_\varepsilon|$的 Poisson 积分，我们来证明存在两个适当的常数 $c_1$与 $c_2$，使得

(14) $$|u(x,y)| \leqslant c_1 v_\varepsilon(x,y) + c_2, \quad (x,y)\in \mathscr{R}_\varepsilon.$$

根据调和函数的极值原理，只要证明上述不等式对$\mathscr{R}_\varepsilon$的边界$\mathscr{B}_\varepsilon$的点成立就够了。已知$\mathscr{B}_\varepsilon = \mathscr{B}_\varepsilon^1 \cup \mathscr{B}_\varepsilon^2$，其中$\mathscr{B}_\varepsilon^2$是超平面$y=h-\varepsilon$上的一部分，它的投影包含在一固定球内。由 $v_\varepsilon \geqslant 0$，我们可以选择 $c_2$充分大（与$\varepsilon$无关）使式(14)在$\mathscr{B}_\varepsilon^2$成立。

剩下来只要研究$(x,y)\in\mathscr{B}_\varepsilon^1$。由于

$$\mathscr{B}_\varepsilon^1 \subset \mathscr{R} = \bigcup_{x^0\in E} \Gamma_\alpha^h(x^0),$$

我们可以找到常数 $c>0$，使得以$(x,y)$为中心，以 $cy$ 为半径的球 $B$ 完全落在 $\bigcup\limits_{x^0\in E} \Gamma_{\alpha^*}^{h^*}(x^0)$内，其中$\alpha<\alpha^*<\beta$，$h<h^*<k$。回忆前面讲过的，对于锥$\Gamma_\beta^k(x^0)$，我们有

$$\iint\limits_{\Gamma_\beta^k(x^0)} |\nabla u|^2 y^{1-n} dxdy \leqslant 1.$$

根据§2的 2.2.2 的引理，对球 $B$ 的一切点有 $y|\nabla u|\leqslant c$。但

$$|u(p_1)-u(p_2)| \leqslant |p_1-p_2| \sup_l |\nabla u|,$$

其中 $l$ 是连结$(p_1,p_2)$的线段；因此当 $p_1=(x,y)$，$p_2$是球 $B$ 中的另外任意一点时，有$|p_1-p_2|\leqslant B$ 的半径$=cy$，从而

(15) $$|u(p_1)-u(p_2)| \leqslant c_2.$$

另外，假设 $S_\varepsilon$是曲面$\mathscr{B}_\varepsilon^1$中包含在球 $B$ 内的那部分。记

$$|S_\varepsilon| = \int_{\mathscr{B}_\varepsilon^1 \cap B} d\tau_\varepsilon.$$

由式(15)显然有

$$|u(p_1)| \leqslant \frac{1}{|S_\varepsilon|}\int_{S_\varepsilon} |u_\varepsilon(p_2)| d\tau_\varepsilon(p_2) + c_2.$$

由 $d\tau_\varepsilon \geqslant dx$ 与 $B$ 的半径是 $cy$，显然 $|S_\varepsilon| \geqslant ay^n$，其中 $a$ 是一合适的常数，$a>0$。回忆 $f_\varepsilon(x)$ 的定义以及 $d\tau_\varepsilon \leqslant c dx$，我们就得到

$$u(p_1) = |u(x,y)| \leqslant b y^{-n} \int_{|z-x|<cy} |f_\varepsilon(z)| dz + c_2.$$

Poisson 核 $P_y$ 满足

$$P_y(z) \geqslant c_1^{-1} b y^{-n}, \qquad |z| < cy$$

（$c_1$是适当的常数）。把这代入上面的式子，注意 $v_\varepsilon$ 是$|f_\varepsilon|$的 Poisson 积分，我们就得到

$$|u(x,y)| \leqslant c_1 v_\varepsilon(x,y) + c_2, \qquad (x,y) \in \mathscr{R}_\varepsilon.$$

根据$\{f_\varepsilon\}$按 $L^2$模有界，我们可选择一子序列$\{f_{\varepsilon'}\}$，它弱收敛到 $L^2(\boldsymbol{R}^n)$的 $f$。用 $v$ 表示 $f$ 的 Poisson 积分；这样当$(x,y) \in \boldsymbol{R}_+^{n+1}$时，$v_{\varepsilon'}(x,y)$就逐点收敛到 $v(x,y)$。最后由 $\mathscr{R}_\varepsilon \to \mathscr{R}$，从式(14)我们就得到

$$(14') \qquad |u(x,y)| \leqslant c_1 v(x,y) + c_2, \qquad (x,y) \in \mathscr{R}.$$

$v$ 是 $L^2$函数的Poisson 积分，由定理 1 它对$\boldsymbol{R}^n$ 几乎所有的点，从而对 $E$ 几乎所有的点非切线有界。由式(14′)，同样的事情对 $u$ 也成立，因此根据定理 3，$u$ 对 $E$ 几乎所有的点有非切线极限。

**2.5 非切线收敛的应用** 设 $u_0, u_1, \cdots, u_n$ 是第三章 §2.3意义下的共轭调和函数系. 为符号简单起见记 $y = x_0$。这样，这 $n+1$ 个函数就满足方程

$$\sum_{j=0}^{n} \frac{\partial u_j}{\partial x_j} = 0,$$

以及

$$\frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j}, \qquad 0 \leqslant k,\ j \leqslant n.$$

我们的目的是要证明下面的定理，它蕴含在刚刚证明的面积积分定理之中。

**定理5** 下面两个命题对任意集合 $E\subset \boldsymbol{R}^n$ 是等价的:

(a) $u_0$ 对 $E$ 几乎所有的点有非切线极限;

(b) $u_1, u_2, \cdots, u_n$ 对 $E$ 几乎所有的点都有非切线极限。

读者可以毫不困难地得到定理 5 的下述推论。

**推论** 设 $H$ 在 $\boldsymbol{R}_+^{n+1}$ 调和, $\dfrac{\partial^k H}{\partial y^k}$ 在 $E\subset \boldsymbol{R}^n$ 几乎每一点有非切线极限。则同样的事情对 $P\left(\dfrac{\partial}{\partial x}\right)H$ 成立, 其中 $P\left(\dfrac{\partial}{\partial x}\right)$ 是

$$\frac{\partial}{\partial x_0},\ \frac{\partial}{\partial x_1},\ \cdots,\ \frac{\partial}{\partial x_n}$$

的 $k$ 次齐次多项式。

由于共轭调和函数可以通过 Riesz 变换解释(像第三章 §2.2 叙述的), 定理 5 就可以用下面的观点来看。它是一个局部性的结果, 描述 Riesz 变换在任意集合上几乎处处的性质; 而整体性的结果, 其中之一由 $L^p(\boldsymbol{R}^n)$ 不等式给出, 已经在第二、三章中研究过了。

情形 $n=1$ 在定理 5 的证明中是例外的, 因为显然有 $|\nabla u_0|^2=|\nabla u_1|^2$, 从而 $S(u_0)(x)=S(u_1)(x)$, 所以在这时, 定理 5 是定理 4 的直接推论。然而, 在一般的情形, $S(u_0)(x)$ 控制 $S(u_j)(x)$ $(j=1,2,\cdots,n)$ 的事实, 需要一个特别的推理, 我们现在就来给出它。

2.5.1 一个引理。

**引理** 设 $\Gamma_\alpha^h$ 与 $\Gamma_\beta^k$ 是一对截锥(为简单起见, 我们假设它们的顶点在原点), $\alpha<\beta$, $h<k$, 从而 $\Gamma_\alpha^h$ 严格包含在 $\Gamma_\beta^k$ 中。如果

$$\iint\limits_{\Gamma_\beta^k}\left|\frac{\partial u}{\partial y}\right|^2 y^{1-n}dxdy<\infty,$$

那末

$$\iint_{\Gamma_a^h}\left|\frac{\partial u}{\partial x_j}\right|^2 y^{1-n}dxdy<\infty,\qquad j=1,\cdots,n.$$

我们证明如下。用$\rho$表示小锥$\Gamma_a^h$内的任意单位向量，即$\rho$是任意单位向量，使得对某些$s>0, s\rho\in\Gamma_a^h$。令$s_0$为满足$s\rho\in\Gamma_a^h$的上确界。显然$h\leqslant s_0\leqslant h^*$，其中$h^*$是只依赖于$\Gamma_a^h$的常数。对任意定义在$\Gamma_a^h$的函数$U$，我们用$U_\rho$表示它在由$\rho$决定的射线上的限制；更准确些是指$U_\rho(s)=U(\rho s)$。我们来证明

$$\int_0^h s\left|\frac{\partial u_\rho(s)}{\partial x_j}\right|^2 ds\leqslant A<\infty,$$

其中$A$同问题中的方向$\rho$无关。对所有适当的$\rho$进行积分就给出我们的结论。

记

$$(6)\qquad \frac{\partial u}{\partial x_j}=-\int_y^h\frac{\partial^2 u(x,\tau)}{\partial x_j\partial y}d\tau+\frac{\partial u(x,h)}{\partial x_j}.$$

上式右边第二项由于一致有界是无关紧要的(它表示$\frac{\partial u}{\partial x_j}$在严格离开大锥$\Gamma_\beta^k$边界的点上取的值)。因此我们只需研究右边的积分。为此，我们用下面的估计，它是平均值定理的推论(附录C)：

$$(17)\qquad \left|\frac{\partial^2 u(x,\tau)}{\partial x_j\partial y}\right|^2\leqslant\frac{cr^{-2}}{m(B)}\iint_B\left|\frac{\partial u}{\partial y}\right|^2dx'dy',$$

其中$B$是中心为$(x,\tau)$，半径为$r$的球。现在我们选取常数$c_1$具有这样的性质，若$(x,\tau)\in\Gamma_a^h$，则半径为$r=c_1\tau$的球$B$完全包含在大锥$\Gamma_\beta^k$中。这样(17)就变成

$$(17')\qquad \left|\frac{\partial^2 u(x,\tau)}{\partial x_j\partial y}\right|\leqslant c_2\tau^{-(n+3)/2}\left(\iint_B\left|\frac{\partial u}{\partial y}\right|^2dx'dy'\right)^{1/2}.$$

我们称$S_\tau$为大锥$\Gamma_\beta^k$的“层”，它由下式给出：

$$S_\tau=\{(x,y):|x|<\beta y,\ \tau-c_1\tau<y<\tau+c_1\tau\}$$

(见图5)．由$B\subset\Gamma_\beta^k$显然知$B\subset S_\tau$。因此如果我们记

$$\mathscr{I}_\tau = \iint\limits_{S_\tau} \left|\frac{\partial u}{\partial y}\right|^2 dx'dy',$$

那末不等式(17′) 变成

$$(17'') \qquad \left|\frac{\partial^2 u(x,\tau)}{\partial x_j \partial y}\right| \leqslant c_2 \tau^{-(n+3)/2} \mathscr{I}_\tau^{1/2}.$$

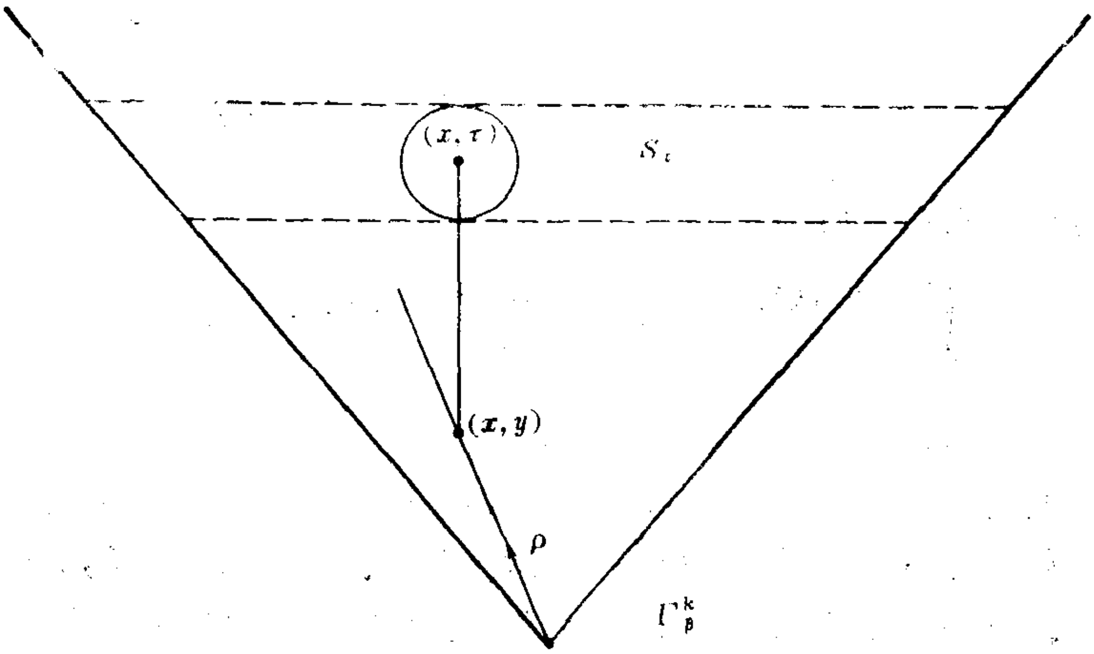


图5 § 2的2.5.1引理的证明情况

我们记 $\theta$ 为 $\rho$ 与 $y$ 轴构成的角度。由于这个方向包含在锥 $|x| < ay$，$y > 0$ 中，因此 $\cos\theta \geqslant a_0 = (1+a^2)^{-1/2} > 0$。根据这一点我们可以把式(17″)代入(16)中，结果就是

$$\left|\frac{\partial u_\rho(s)}{\partial x_j}\right| \leqslant c_3 \int_{sa_0}^{h} \tau^{-(n+3)/2} \mathscr{I}_\tau^{1/2} d\tau + c_4.$$

现在用 Hardy 不等式(见附录 A)我们就得到

$$\int_0^h s\left|\frac{\partial u_\rho(s)}{\partial x_j}\right|^2 ds \leqslant c_5 \int_0^h \tau^{-n} \mathscr{I}_\tau d\tau + c'.$$

然而

$$\int_0^h \tau^{-n} \mathscr{I}_\tau d\tau = \int_0^h \tau^{-n} \left\{\iint\limits_{S_\tau} \left|\frac{\partial u}{\partial y}\right|^2 dx'dy'\right\} d\tau$$

$$\leqslant\iint\limits_{\Gamma_\beta^k}\left|\frac{\partial u}{\partial y}\right|^2\left\{\int\tau^{-n}\chi(\tau,x',y')d\tau\right\}dx'dy',$$

其中$\chi$是$\Gamma_\beta^k$的层$S_\tau$的特征函数。但

$$\int\tau^{-n}\chi(\tau,x',y)d\tau\leqslant\int\limits_{\tau-c_1\tau<y<\tau+c_1\tau}\tau^{-n}d\tau$$

$$=\int_{y/(1+c_1)}^{y/(1-c_1)}\tau^{-n}d\tau=c'y^{-n+1}.$$

把这代入上式就给出

$$\int_0^h s\left|\frac{\partial u_\rho(s)}{\partial x_j}\right|^2 ds\leqslant c\iint\limits_{\Gamma_\beta^k}\left|\frac{\partial u}{\partial y}\right|^2 y^{1-n}dxdy+c,$$

最后对$\rho$进行积分就完成了引理的证明。

2.5.2 为了后面引用，我们叙述两个密切相关的不等式。这里用$\Gamma_\alpha$表示无穷锥$\Gamma_\alpha=\{(x,y):|x|<\alpha y\}$。我们还考虑$\Gamma_\beta$，其中$\beta>\alpha$。对每一个正的$k$，用$|\nabla^k u|^2$表示$u$的$k$阶梯度的平方，也就是$k$阶偏微商的正定二次型，它可以递推地定义为

$$|\nabla^k u|^2=\sum_{j=0}^{n}\left|\nabla^{k-1}\frac{\partial u}{\partial x_j}\right|^2\quad(x_0=y).$$

**引理1** 设$u$在$\Gamma_\beta$调和，则对每个$k\geqslant1$有

$$\iint\limits_{\Gamma_\alpha}|\nabla^k u|^2 y^{2k-n-1}\ dxdy\leqslant c_{\alpha,\beta,k}\iint\limits_{\Gamma_\beta}|\nabla u|^2 y^{1-n}dxdy.$$

**引理2** 设$u$在$\Gamma_\beta$调和且当$y\to\infty$，$(x,y)\in\Gamma_\beta$时$|\nabla u|\to0$。则对每个$k\geqslant1$

$$\iint\limits_{\Gamma_\beta}|\nabla u|^2 y^{1-n}dxdy\leqslant c_{\beta,k}\iint\limits_{\Gamma_\beta}|\nabla^k u|^2 y^{2k-n-1}dxdy.$$

为证明引理1，我们用不等式

$$|\nabla^k u(x,\tau)|^2 \leqslant c_k \frac{r^{-2(k-1)}}{m(B)} \iint_B |\nabla u|^2 dx' dy',$$

其中 $B$ 是中心在$(x,\tau)$，半径为 $r=c_1\tau$ 的球；剩下来的就像 § 2的 2.5.1一样。

为了证明引理 2，我们注意，显然对任意 $\Gamma_\beta$ 的真子锥，当 $y\to\infty$时 $|\nabla u|\to 0$蕴含了 $|\nabla^k u|\to 0$。现在，若 $k>1$，则

$$\frac{\partial u(x,y)}{\partial x_j} = \frac{-1}{(k-2)!}\int_y^\infty \frac{\partial^k u(x,\tau)}{\partial y^{k-1}\partial x_j}(y-\tau)^{k-2} d\tau.$$

用带齐次核的积分算子的不等式(见附录 A)，就得到

$$\int_{|x|/\beta}^\infty \left|\frac{\partial u(x,y)}{\partial x_j}\right|^2 y^{1-n} dy \leqslant c_k \int_{|x|/\beta}^\infty \left|\frac{\partial^k u(x,y)}{\partial y^{k-1}\partial x_j}\right|^2 y^{2k-n-1} dy,$$

再对 $x$ 积分就给出所要求的对$\frac{\partial u}{\partial x_j}$的结果。对$\frac{\partial u}{\partial y}$可进行类似的推理。

§ 2 的 2.5.1的引理以及2.5.2的那些引理还告诉我们，谈及面积积分的基本性质时，我们还可以像用一阶微商那样用高阶微商。然而，重要的是要注意，我们不能用 0 阶微商。事实上甚至当 $u$ 是常数时，对 $k=0$的面积积分的推广，它的值等于无穷。

## § 3 $H^p$空间论的应用

我们将进一步研究由广义 Cauchy-Riemann 方程

$$\text{(18)} \qquad \sum_{j=0}^n \frac{\partial u_j}{\partial x_j}=0 \quad 与 \quad \frac{\partial u_j}{\partial x_k}=\frac{\partial u_k}{\partial x_j}, \qquad x_0=y$$

给出的共轭概念。工具之一是关于下调和性的一个结果，它包含在下面的引理中。

**3.1 梯度的下调和性** 记 $F=(u_0,u_1,\cdots,u_n)$，其中 $u_j$ 在

某个开集内满足 Cauchy-Riemann 方程(18)。(注意，这蕴含了在这集合的每个点的一个邻域，向量 $F$ 是这邻域的某个调和函数的梯度。)为了某些技术上的理由，认为 $u_j$ 在一个固定的有穷维的实内积空间(即 Hilbert 空间)里取值，将是很方便的。我们用 $|\cdot|$ 表示这个空间的模，用 $u_i\cdot u_i'$ 表示 $u_i$ 与 $u_i'$ 的内积，还有

$$|F|=\left(\sum_{j=0}^{n}|u_j|^2\right)^{1/2},\qquad \Delta=\sum_{j=0}^{n}\frac{\partial^2}{\partial x_j^2}.$$

**引理** 假设在一个点 $|F|>0$。若 $q\geqslant(n-1)/n$，则在这个点 $\Delta(|F|^q)\geqslant 0$。更精确些，若 $q>(n-1)/n$，则

(19) $$C_q|F|^{q-2}|\nabla F|^2\geqslant\Delta(|F|^q)\geqslant c_q|F|^{q-2}|\nabla F|^2,$$

其中 $C_q$ 与 $c_q$ 是两个正常数①。

为了搞清楚这引理的意义，我们指出，结论 $\Delta|F|^q\geqslant 0$（当 $q\geqslant 1$）只是每个 $u_j$ 是调和这一事实的一个推论，而与 $u_j$ 的微分方程（18）无关。这个引理的特殊用处是它对某些 $q<1$ 成立。$q$ 的范围 $q\geqslant(n-1)/n$ 是最好的可能，这一点下面将有例子说明。

不等式（19）还可以同下面的两个较简单的恒等式相比较。第一个是

(20) $$\Delta u^q=q(q-1)u^{q-2}|\nabla u|^2,$$

只要 $u$ 是调和的且 $u>0$。我们在第四章 § 2已经用过这个恒等式。

第二个是，当 $n=1$ 而 $F=u_0-iu_1$ 是 $x_0+ix_1$ 的解析函数时，

(21) $$\Delta|F|^q=\frac{q^2}{2}|F|^{q-2}|\nabla F|^2=q^2|F|^{q-2}|F'|^2.$$

3.1.3 为了证明引理，我们用符号 $F\cdot G=u_0\cdot v_0+u_1\cdot v_1$

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① 证明将表明，我们可以取 $C_q=q(q-1)$，$c_q=q$，当 $q\geqslant 2$；而 $C_q=q$，$c_q=q\left[1+(q-2)\dfrac{n}{n+1}\right]$，当 $q\leqslant 2$。条件 $q\geqslant\dfrac{n-1}{n}$ 实际上就是条件 $c_q\geqslant 0$。注意，还有 $C_2=c_2=2$。

$+\cdots+u_n\cdot v_n$；其中 $F=(u_0,u_1,\cdots,u_n)$，$G=(v_0,v_1,\cdots,v_n)$；还有 $F_{x_j}=\dfrac{\partial F}{\partial x_j}$；这时 $|\nabla F|^2=\sum_{j=0}^{n}|F_{x_j}|^2$。注意

$$\frac{\partial}{\partial x_j}(F\cdot G)=F_{x_j}\cdot G+F\cdot G_{x_j},$$

就有

$$\frac{\partial}{\partial x_j}|F|^q=\frac{\partial}{\partial x_j}(F\cdot F)^{q/2}=q|F|^{q-2}(F_{x_j}\cdot F).$$

因此

$$\frac{\partial^2}{\partial x_j^2}|F|^q=q(q-2)|F|^{q-4}(F_{x_j}\cdot F)^2$$
$$+q|F|^{q-2}\{|F_{x_j}|^2+F_{x_jx_j}\cdot F\}.$$

关于 $j$ 求和并注意到

$$F_{x_0x_0}+F_{x_1x_1}+\cdots+F_{x_nx_n}=0,$$

我们得到

$$\Delta|F|^q=q|F|^{q-4}\Big\{(q-2)\sum_j(F_{x_j}\cdot F)^2$$
$$+|F|^2\sum_j|F_{x_j}|^2\Big\}.$$

问题只是比较

$$(q-2)\sum_j(F_{x_j}\cdot F)^2+|F|^2\sum_j|F_{x_j}|^2 \tag{22}$$

与 $|F|^2|\nabla F|^2=|F|^2\sum_j|F_{x_j}|^2$。

用Schwarz不等式 $\sum_j(F_{x_j}\cdot F)^2\leqslant|F|^2\sum_j|F_{x_j}|^2$，式(22)被下式控制：

$$|F|^2|\nabla F^2|,\qquad q\leqslant 2;$$
$$(q-1)|F|^2|\nabla F|^2,\qquad q\geqslant 2.$$

类似地，当 $q\geqslant2$ 时可以简单地得到 $\Delta|F|^q$ 的下界，而我们还没

有用到 Cauchy-Riemann 方程的全部作用。现在假设 $q<2$。这时，问题的中心是要得到 Schwarz 不等式的下述改进

$$(23)\qquad \sum_i (F_{x_i}\cdot F)^2 \leqslant \frac{n}{n+1}|F|^2\sum_i |F_{x_i}|^2,$$

这是我们稍后要证明的。假设(23)成立，这时式(22)有下界

$$|F|^2\sum_i |F_{x_i}|^2\left\{1+(q-2)\left(\frac{n}{n+1}\right)\right\}.$$

把这代到前面的式子里，就得到

$$\Delta|F|^q \geqslant c_q|F|^{q-2}|\nabla F|^2,$$

其中

$$c_q = q\left\{1+(q-2)\frac{n}{n+1}\right\},\qquad q\leqslant 2.$$

不等式(23)用到了 Cauchy-Riemann 方程的全部，一旦我们得到了它，引理也就证明完毕。

3.1.2 首先我们注意，只需对函数 $u_0, u_1, \cdots, u_n$ 是实值的特殊情形，即所考虑的 Hilbert 空间是一维的，证明式(23)就够了。过渡到所考虑的一般情况，只需引入任意一组标准正交基。

令 $\mathscr{M}=(m_{jk})$ 是(实)元素 $m_{jk}$ 的 $(n+1)\times(n+1)$ 矩阵，对这种矩阵我们考虑两种模。一个就是通常的模 $\|\mathscr{M}\|=\sup|\mathscr{M}(F)|$，其中上确界是对所有长度为 1 的向量 $F$ 取的。另一个是 Hilbert-Schmidt 模 $|||\mathscr{M}|||$，由 $|||\mathscr{M}|||^2=\sum_{j,k}|m_{jk}|^2$ 定义。用 Schwarz 不等式显然有 $\|\mathscr{M}\|\leqslant|||\mathscr{M}|||$。现在假设矩阵 $\mathscr{M}$ 对称并且迹为零。这时我们可以把这不等式加强到

$$(23')\qquad \|\mathscr{M}\|^2 \leqslant \frac{n}{n+1}|||\mathscr{M}|||^2.$$

为了证明(23′)，我们注意，这两种模是正交不变的，而在 $n+1$ 维欧氏空间中选择合适的基以后，我们可以假设对称矩阵 $\mathscr{M}$ 是

对角的，其对角线元素为 $\lambda_0,\lambda_1,\cdots,\lambda_n$。由 $\mathscr{M}$ 的迹为零，知

$$\sum_{j=0}^{n}\lambda_j=0.$$

因此

$$\lambda_{j_0}=-\sum_{j\neq j_0}\lambda_j;$$

用 Schwarz 不等式就有

$$\lambda_{j_0}^2\leqslant n\sum_{j\neq j_0}\lambda_j^2,$$

因此

$$\sup_j\lambda_j^2=\lambda_{j_0}^2\leqslant\frac{n}{n+1}\sum_j\lambda_j^2.$$

这就是对角矩阵时的(23′)，而根据刚才说的，式(23′)对任意对称且迹为零的矩阵也成立。对 $F=(u_0,u_1,\cdots,u_n)$，令

$$m_{jk}=\frac{\partial u_j}{\partial x_k}.$$

这时广义 Cauchy-Riemann 方程 (18) 恰好表明矩阵 $\mathscr{M}=(m_{jk})$ 是对称的且迹为零。容易看出，(23′)蕴含了(23)，引理证毕。

**3.2 $H^p$ 空间；特别地 $H^1$ 空间** 类似于经典理论，我们如下定义 Hardy 空间 $H^p$ 的 $n$ 维情形。假设 $F=(u_0,u_1,\cdots,u_n)$ 在 $\boldsymbol{R}_+^{n+1}$ 满足广义 Cauchy-Riemann 方程(18)，对 $p>0$，如果

$$\sup_{y>0}\left(\int_{\boldsymbol{R}^n}|F(x,y)|^p\,dx\right)^{1/p}<\infty. \tag{24}$$

我们就说 $F\in H^p$，并用 $\|F\|_p$ 记上面出现的量。当 $p\geqslant1$ 时它是一个模。

由于 §3.1 的引理，当 $p>(n-1)/n$ 时，$H^p$ 空间的某些经典理论($n=1$)可以推广到 $n$ 维情形，而对我们来说，最有兴趣的是它包括了 $p=1$ 的情形。为更好理解后者的含意，我们首先考虑 $H^p$ 空间，$1<p<\infty$。

假设 $F\in H^p$，$1<p<\infty$。这时，由§1的1.2.1的推论，存在 $f_0,f_1,\cdots,f_n$，每一个都属于 $L^p(\boldsymbol{R}^n)$，使得 $u_j(x,y)$ 是 $f_j$ 的 Poisson 积分，$j=0,1,\cdots,n$。再从第三章的§4.4，$f_j=R_j(f_0)$，其中 $R_1$，$R_2,\cdots,R_n$ 是 Riesz 变换。反过来，假设 $f_0\in L^p(\boldsymbol{R}^n)$，$1<p<\infty$，令 $f_j=R_j(f_0)$，$u_j(x,y)$ 是 $f_j$ 的Poisson 积分，$j=0,\cdots,n$。则 $F=(u_0,u_1,\cdots,u_n)\in H^p$；进一步

$$\|f_0\|_p\leqslant\|F\|_p\leqslant A_p\|f_0\|_p.$$

综合起来，当 $1<p<\infty$时，空间 $H^p$ 本质上等价于空间 $L^p(\boldsymbol{R}^n)$。

对 $p=1$，这种等价性不再成立，但我们可以把 $H^1$ 空间看成 $L^1(\boldsymbol{R}^n)$ 的替代物。我们的目的是要表明，这时有许多对 $L^1$ 不成立的结果，对 $H^1$ 却有一个肯定的命题。因此，$H^1$ 的定理在某种意义下可以想像为出现在第一、二章的包含弱(1,1)型定理的类似与补充。我们第一个这类定理就是研究极大函数的。

**定理6** 设 $F\in H^1$，则 $\lim\limits_{y\to 0}F(x,y)=F(x)$几乎处处与按 $L^1$模存在，并且

$$\int_{\boldsymbol{R}^n}\sup_{y>0}|F(x,y)|\,dx\leqslant A\sup_{y>0}\int_{\boldsymbol{R}^n}|F(x,y)|\,dx=A\|F\|_1.$$

在证明这个定理以前，我们提出若干推论，它们将较明白地说明这定理包含些什么。

设 $d\mu_0$ 是$\boldsymbol{R}^n$ 上的有限测度。我们将说，测度 $d\mu_j$ 是它的 Riesz 变换$R_j(d\mu_0)$，如果等式

$$\hat{\mu}_j(x)=i\frac{x_j}{|x|}\hat{\mu}_0(x) \tag{25}$$

成立，其中 $\hat{\mu}_0$ 与$\hat{\mu}_j$ 分别表示测度 $d\mu_0$ 与 $d\mu_j$ 的 Fourier 变换。自然，这同第三章由等式(8)给出的通常定义是相容的（见第70页）。

作为这定义的一种特殊情形，我们可以假设 $d\mu_0=f_0dx$，其中 $f_0\in L^1(\boldsymbol{R}^n)$。类似地，我们将说 $R_j(f_0)\in L^1(\boldsymbol{R}^n)$，如果存在

$\hat{f}_j\in L^1(\boldsymbol{R}^n)$，使得

$$\hat{f}_j(x)=i\frac{x_j}{|x|}\hat{f}_0(x).$$

**推论1** 假设 $d\mu_0$ 是有限测度以及所有它的 Riesz 变换 $R_j(d\mu)$ 也是有限测度，

$$R_j(d\mu_0)=d\mu_j,\quad j=1,\cdots,n.$$

则存在 $L^1(\boldsymbol{R}^n)$的函数 $f_0,f_1,\cdots,f_n$，使得

$$d\mu_j=f_j\,dx,\quad j=0,\cdots,n.$$

空间 $H^1$ 实质上同构于 $L^1(\boldsymbol{R}^n)$中函数 $f_0$ 构成的空间，其中 $f_0$ 具有性质

$$R_j(f_0)\in L^1(\boldsymbol{R}^n),\quad j=1,\cdots,n.$$

$H^1$模也就等价于

$$\|f_0\|_1+\sum_{j=1}^{n}\|R_j(f_0)\|_1.$$

这个 $f_0$ 的空间，实际上是由 $H^1$ 中 $F$ 的边值的“实部”构成的。这就使我们把这个 Banach 空间作为 $H^1$ 来引用，只要这样做不会与原来定义的 $F$ 组成的空间引起混乱。

**推论2** 假设 $f_0\in L^1(\boldsymbol{R}^n)$且 $R_j(f_0)\in L^1(\boldsymbol{R}^n)$，$R_j(f_0)=f_j$，$j=1,\cdots,n$。则

$$\sum_{j=0}^{n}\int_{\boldsymbol{R}^n}\sup_{y>0}|u_j(x,y)|\,dx\leqslant A\sum_{j=0}^{n}\|f_j\|_1.$$

3.2.1 我们来证明定理 6 及其推论。让我们从这样的假设开始，就像前面做过的，设 $F=(u_0,\cdots,u_n)$，每个 $u_j$ 取值在一固定的有限维 Hilbert 空间；我们记这内积空间为 $V_1$，我们还需要另外的有限维 Hilbert 空间 $V_2$，并考虑它们的直和 $V=V_1\oplus V_2$；$V_1$ 与 $V_2$ 在 $V$ 中是正交补。

下面我们寻找 $\boldsymbol{R}_+^{n+1}$ 中一个固定的函数，$\Phi(x,y)=(v_0(x,y),v_1(x,y),\cdots,v_n(x,y))$，使得 $v_j$ 在 $V_2$ 取值，并且

(a) $v_j$ 满足 Cauchy-Riemann 方程(18);

(b) $|\Phi(x,y)| = c|(x,y+1)|^{-n-1}$

$$= c(|x|^2+(y+1)^2)^{-(n+1)/2}.$$

为做到这点，用 $V_2$ 表示标准的 $n+1$ 维坐标空间; $H(x,y)$ 是调和函数 $|(x,y+1)|^{-n+1} = (|x|^2+(y+1)^2)^{-(n-1)/2}$①. 对每个 $j$, $0 \leqslant j \leqslant n$, 令

$$v_j(x,y) = \left(\frac{\partial^2 H}{\partial x_j \partial x_k}\right)_{k=0}^{n}.$$

这时
$$|\Phi(x,y)|^2 = \sum_{j=0}^{n}\sum_{k=0}^{n}\left|\frac{\partial^2 H}{\partial x_j \partial x_k}\right|^2.$$

由此容易验证(a)与(b)，其中$c^2 = (n^2-1)(n^2-n)$.

下面对每个$\varepsilon > 0$，我们用

(26) $$F_\varepsilon(x,y) = F(x,y+\varepsilon) + \varepsilon\Phi(x,y)$$

定义 $F_\varepsilon$. 如果记 $F_\varepsilon(x,y) = (u_0^\varepsilon(x,y), \cdots, u_n^\varepsilon(x,y))$, 那末

$$u_j^\varepsilon(x,y) = u_j(x,y+\varepsilon) + \varepsilon v_j(x,y),$$

并且 $u_j^\varepsilon$ 取值于 $V_1 \oplus V_2 = V$. 注意, $F_\varepsilon$ 的分量 $u_j^\varepsilon$ 满足 Cauchy-Riemann 方程，并且 $F_\varepsilon$ 在 $\bar{\boldsymbol{R}}_+^{n+1}$ 连续.

由于对$F$的假设并注意到本章§1的1.2.1的推论，我们可以断言每个 $u_j(x,y)$ 是有限测度的 Poisson 积分. 因此，容易看出，对任意固定的$\varepsilon > 0$，每个 $u_j(x,y+\varepsilon)$ 都趋向于零，当 $|(x,y)| \to \infty$, $(x,y) \in \bar{\boldsymbol{R}}_+^{n+1}$. 同样的事实对 $\Phi$ 的分量也成立(由性质(b))，故

$$|F_\varepsilon(x,y)| \to 0, \quad 当 |(x,y)| \to \infty, \ (x,y) \in \bar{\boldsymbol{R}}_+^{n+1}.$$

此外，还有

$$|F_\varepsilon(x,y)|^2 = |F(x,y+\varepsilon)|^2 + \varepsilon^2|\Phi(x,y)|^2 > 0,$$

这是因为 $V_1$ 和 $V_2$ 在$V$是正交的. 这样, $|F_\varepsilon|^q$ 是光滑的，并且我们可以对 $q=(n-1)/n$ 应用§3.1的引理，得到 $\Delta(|F_\varepsilon|^q) \geqslant 0$

① 在此以及本章的后一部分，我们都假设 $n > 1$. 对 $n=1$, 推理需作某些细小的修改.

处处成立。令

$$g_\varepsilon(x)=|F_\varepsilon(x,0)|^q=(|F(x,\varepsilon)|^2+\varepsilon^2|\Phi(x,0)|^2)^{q/2}.$$

我们就有

$$\int_{\mathbf{R}^n}(g_\varepsilon(x))^p dx=\int_{\mathbf{R}^n}|F_\varepsilon(x,0)|dx\leqslant\|F\|_1+\varepsilon\|\Phi\|_1, \tag{27}$$

其中 $p=1/q$，而这很重要，因为这是 $p>1$ 的一个结果。让 $g_\varepsilon(x,y)$ 表示 $g_\varepsilon$ 的 Poisson 积分。我们来证明

$$|F_\varepsilon(x,y)|^q\leqslant g_\varepsilon(x,y), \qquad (x,y)\in\bar{\mathbf{R}}^{n+1}_+. \tag{28}$$

为验证式(28)，注意 $F_\varepsilon$ 与 $g_\varepsilon$ 都在 $\bar{\mathbf{R}}^{n+1}_+$ 连续，并且 $F_\varepsilon$ 在无穷远为零。我们有 $\Delta(|F_\varepsilon|^q-g_\varepsilon)\geqslant0$，因此从附录 C 的极值原理知，只要在边界 $y=0$ 验证(27)，而在那里等号是成立的。故(28)得证。

现在我们在 $\{g_\varepsilon(x)\}(\varepsilon\to0)$ 中选取一个子序列，它弱收敛到 $L^p(\mathbf{R}^n)$ 的函数 $g$。由式(27)有 $\|g\|_p^p\leqslant\|F\|_1$。用 $g(x,y)$ 表示 $g$ 的 Poisson 积分，从(28)就推出

$$|F(x,y)|^q\leqslant g(x,y). \tag{28$'$}$$

然而，用 Poisson 积分的极值原理（见第三章，第76页），我们有

$$\sup_{y>0}|F(x,y)|^q\leqslant\sup_{y>0}|g(x,y)|\leqslant M(g)(x),$$

从而

$$\int_{\mathbf{R}^n}\sup_{y>0}|F(x,y)|dx\leqslant\int_{\mathbf{R}^n}(M(g)(x))^p dx$$

$$\leqslant A_p^p\int_{\mathbf{R}^n}(g(x))^p dx\leqslant A_p^p\|F\|_1.$$

这就证明了定理 6 的主要结论。极限 $\lim\limits_{y\to0}F(x,y)$ 也就几乎处处存在，这是因为有限测度的 Poisson 积分具有这种性质（见第三章 § 4.1)。几乎处处收敛也可以借助于式（28′）证明，因为式（28′）表明，$F(x,y)$ 几乎处处非切线有界，于是就可用本章的定理 3 。

最后，按 $L^1$ 模收敛是几乎处处收敛和刚刚证明的极大不等式的一个推论，这不等式表明，$|F(x,y)|$被一固定的可积函数控制。

为了证明推论1，我们只需要注意，若 $u_j(x,y)$ 是 $d\mu_j$ 的 Poisson 积分，而 $d\mu_j=R_j(d\mu_0)$，则$F=(u_0,\cdots,u_n)$满足 Cauchy-Riemann 方程。还有$\sup\limits_{y>0}\|u_j(x,y)\|_1=\|d\mu_j\|<\infty$。因此 $F\in H^1$。令 $f_j(x)=\lim\limits_{y\to 0}u_j(x,y)$，这个极限的存在性(按 $L^1$ 模)是由定理保证的。设 $\hat{\mu}_j$ 和 $\hat{f}_j$ 分别是 $d\mu_j$ 与 $f_j$ 的 Fourier 变换，我们就有 $(\mu_j(x,y))^\wedge=\hat{\mu}_j(x)e^{-2\pi|x|y}$。因此 $\hat{\mu}_j(x)e^{-2\pi|x|y}\to\hat{f}_j(x)$，当 $y\to 0$；从而 $\hat{\mu}_j(x)=\hat{f}_j(x)$，故 $d\mu_j=f_jdx$。推论2可以从推论1推出，只要注意，当 $d\mu_j=f_j(x)dx$时，$\|d\mu_j\|=\|f_j\|_1$。

**3.3 面积积分与 $H^1$** 当 $f\in L^p(R^n)$时，在第四章以及部分地在本章，我们研究过三个辅助函数，它们是

$$g(f)(x),\quad S(f)(x)\quad 和\quad g_\lambda^*(f)(x).$$

它们定义如下。设 $u(x,y)$是 $f$ 的 Poisson 积分，则

$$g(f)(x)=\left(\int_0^\infty y|\nabla u(x,y)|^2dy\right)^{1/2}$$

与

$$g_\lambda^*(f)(x)=\left(\int_0\int_{R^n}|\nabla u(x-t,y)|^2\left(\frac{y}{|t|+y}\right)^{n\lambda}y^{1-n}dtdy\right)^{1/2}.$$

为定义面积积分 $S$，我们需要固定一个包含在它的定义中的锥。在以后，把这个锥取定为正圆锥是十分方便的。这样，$S$ 的定义是

$$S(f)(x)=\left(\iint_{|t|<y}|\nabla u(x-t,\ y)|^2y^{1-n}dt\,dy\right)^{1/2}.$$

在第四章我们得到过 $g(f)(x)\leqslant cS(f)(x)\leqslant c_\lambda g_\lambda^*(f)(x)$，并在定理1与2中证明了

$$B_p\|f\|_p\leqslant\|g(f)\|_p\leqslant c_\lambda\|g_\lambda^*(f)\|_p\leqslant A_{p,\lambda}\|f\|_p,$$

只要 $1<p<\infty$ 与 $\lambda\geqslant 2$。

我们的目的是要把这些事实在 $p=1$ 的推广通过 $H^1$ 空间描

述出来，并在下面§4.3中应用它。

设 $F\in H^p$，$1\leqslant p$，与上述情形类似，我们定义 $S(F)(x)$ 与 $g_\lambda^*(F)(x)$ 如下：

$$S(F)(x)=\left(\iint\limits_{|t|<y}|\nabla F(x-t,y)|^2y^{1-n}dtdy\right)^{1/2},$$

$$g_\lambda^*(F)(x)=\left(\iint|\nabla F(x-t,y)|^2\left(\frac{y}{|t|+y}\right)^{n\lambda}y^{1-n}dtdy\right)^{1/2}.$$

自然仍有 $S(F)(x)\leqslant c_\lambda g_\lambda^*(F)(x)$。我们的结果如下：

**定理7** 若 $F\in H^p$，$1\leqslant p<\infty$，则 $\|g_\lambda^*(F)\|_p\leqslant A_{p,\lambda}\|F\|_p$，只要 $p>2/\lambda$。特别地，$\|S(F)\|_p\leqslant A_p\|F\|_p$。

**定理8** 设 $F\in H^p$，$1\leqslant p<\infty$，则 $B_p\|F\|_p\leqslant\|S(F)\|_p$。

可以证明，这类结果对 $p>(n-1)/n$ 也成立(见下面§4.9)，但在这里的讲述中，定理的叙述与证明，对一般的情形来说已是相当典型的了。对这些定理，我们有兴趣的只是 $p=1$ 的情形，因为定理中 $p>1$ 那部分已包含在第四章的结果中。因此在后面的推理中我们只限于 $p=1$ 的情形。

3.3.1 **定理7的证明**。为了证明这个以及接下去的定理，把我们的注意力限制在我们下面就要定义的 $H^p$ 的一个合适的子类，将是方便的。

设 $H_0^p$ 是由所有这样的 $F$ 组成的：

(a) $F\in H^p$；

(b) $F$ 在 $\bar{\boldsymbol{R}}_+^{n+1}$ 连续，并且还是速降的，即对每个 $x_1,\cdots,x_n$，$y$ 的多项式 $p$，$pF$ 在 $\bar{\boldsymbol{R}}_+^{n+1}$ 有界；

(c) 性质(b)还对 $F$ 的关于 $x_1,\cdots,x_n$，$y$ 的任意阶偏微商成立。

对我们的目的来说最重要的是下面的事实。

**引理** $H_0^p$ 在 $H^p$ 是稠密的，$1\leqslant p<\infty$。

当 $1<p<\infty$ 时，引理可以通过一个直接的极限过程来证明，这是因为 Riesz 变换 $R_j$ 是 $L^p$ 连续的，$p=1$ 时没有连续性，使

得这个基本的特殊情形变得复杂，但正是这种情形是我们现在感兴趣的。然而，为了直接进行定理 7 与 8 的证明，我们把引理的证明推迟到下面 § 3的3.3.3。

现在假设引理是对的，我们来证明

(29) $$\|g_\lambda^*(F)\|_1 \leqslant A_\lambda \|F\|_1, \quad \lambda > 2, \ F \in H_0^1.$$

类似于 § 3 的 3.2.1 中的证明，我们不妨在我们的问题中用 $F+\varepsilon\Phi$ 代替 $F$。只要回忆 $\Phi$ 的分量与 $F$ 的分量是正交的，他们也就满足Cauchy-Riemann 方程，并且

$$|\Phi(x,y)| = c|(x,y+1)|^{-n-1}.$$

(注意，以后会用到 $|\nabla\Phi| = c'|(x,y+1)|^{-n-2}$。)

引进摄动 $\varepsilon\Phi$ 的主要目的自然是消除 $F$ 的零点；事实上我们有 $|F+\varepsilon\Phi|^2 = F^2 + \varepsilon^2|\Phi|^2 > 0$。因此我们要证的是用 $F+\varepsilon\Phi$ 代替 $F$ 以后的式 (29)。

我们的推理是修改在第四章 § 2，特别是在 § 2的2.5.2中给出的 $p>1$ 情况的证明。让我们平行而又较为详细地叙述如下。

引理 1 (见第109页)将用不等式

(30) $$c_1|F|^{-1}|\nabla F|^2 \geqslant \Delta|F| \geqslant c_1|F|^{-1}|\nabla F|^2$$

代替，它只是本章 § 3.1的引理的特殊情形。

引理 2 (见第109页) 将作微小的改动如下。设 $G$ 在 $\overline{\boldsymbol{R}}_+^{n+1}$ 连续，$G$ 在 $\boldsymbol{R}_+^{n+1}$ 属于 $C^2$ 类，$y\Delta G \in L^1(\boldsymbol{R}_+^{n+1}, dydx)$，$|G(x,y)| \leqslant |(x,y)|^{-n-\varepsilon}$，而

$$|\nabla G(x,y)| \leqslant A|(x,y)|^{-n-1-\varepsilon}, \text{ 其中 } \varepsilon > 0,$$

则

(31) $$\iint_{\boldsymbol{R}_+^{n+1}} y\Delta G\, dxdy = \int_{\boldsymbol{R}^n} G(x,0)dx.$$

最后，对极大引理 3，以及它的变形不等式(24)(见第116页)，我们将有

*对任意固定的 $\mu$，$\mu$ 充分接近于 1，$\mu < 1$，*

(32) $$|F(x-t,y)| \leqslant (1+|t|/y)^{n/\mu} F_\mu^*(x),$$

其中$\int_{R^n} F_\mu^*(x)dx \leqslant A_\mu \|F\|_1$。事实上，用第四章的不等式(24)，我们有

$$|g(x-t,y)| \leqslant A\left(1+\frac{|t|}{y}\right)^n M(g)(x),$$

其中 $g(x,y)$是任意函数 $g(x)$ 的 Poisson 积分。现在借助控制式(28′)，即$|F(x,y)|^q \leqslant g(x,y)$，其中 $g(x)=|F(x,0)|^q$，并选 $q=\mu<1$，这样

$$F(x-t,y)=A^{1/\mu}(1+|t|/y)^{n/\mu}M^{1/\mu}(g).$$

取

$$F_\mu^*(x)=A^{1/\mu}M^{1/\mu}(g)(x),$$

只要回忆，若 $p=1/q$，则$\|g\|_p^p \leqslant \|F\|_1$，这就证明了式 (32)。

现在对固定的 $\lambda$，$\lambda>2$，我们可以找到 $\lambda'>1$ 和 $\mu<1$，$\mu$ 充分接近于 1 ，使得 $\lambda'=\lambda-1/\mu$。令

$$I^*(x)=\iint_{R_+^{n+1}} y^{1-n}\left(\frac{y}{y+|t|}\right)^{\lambda' n} \Delta G(x-t,y)dt\,dy,$$

其中 $G(x,y)=|F(x,y)+\varepsilon\Phi(x,y)|$。这时我们有（像在第四章§2的2.5.2那样）

$$\begin{aligned}\int_{R^n} I^*(x)dx &= c_{\lambda'}\iint_{R_+^{n+1}} y\Delta G(t,y)dtdy\\ &= c_{\lambda'}\int_{R^n} G(t,0)dt\\ &= c_{\lambda'}\int_{R^n}|F(x,0)+\varepsilon\Phi(x,0)|dx,\end{aligned}$$

为此只需验证$G$满足导出式(31)的那些要求。事实上，由$F\in H_0^1$，知它以及它所有的偏微商都是速降的；而

$|\Phi(x,y)|=c|(x,y+1)|^{-n-1}$ 与 $|\nabla\Phi|=c'|(x,y+1)|^{-n-2}$。

因此，用 $F+\varepsilon\Phi$ 代替$F$，由式(30)我们有

$$\Delta|F+\varepsilon\Phi|\leqslant A|(x,y+1)|^{-n-3}.$$

这就证明了 $y\Delta G\in L^1(\boldsymbol{R}^{n+1}_+,\ dydx)$。类似地

$$|G(x,y)|\leqslant A|(x,y+1)|^{-n-1}$$

并且
$$|\nabla G(x,y)|\leqslant A|(x,y+1)|^{-n-2}$$

接着，由式（30）又有

$$g^*_\lambda(F+\varepsilon\Phi)^2(x)\leqslant c_1^{-1}\iint_{\boldsymbol{R}^{n+1}_+} y^{1-n}\left(\frac{y}{y+|t|}\right)^{\lambda n} G\Delta G dt dy.$$

由（32），上式右端被 $c_1^{-1}I^*(x)F^*_{\mu\varepsilon}(x)$所控制；其中

$$\sup_{t,y}\left(\frac{y}{y+|t|}\right)^{n/\mu} G(x-t,y)=F^*_{\mu,\varepsilon}(x),$$

并且

$$\|F^*_{\mu,\varepsilon}\|_1\leqslant A_\mu\|F+\varepsilon\Phi\|_1.$$

由 Schwarz 不等式便得

$$\int_{\boldsymbol{R}^n} g^*_\lambda(F+\varepsilon\Phi)(x)dx\leqslant c_1^{-1/2}c_\lambda^{1/2}\|F+\varepsilon\Phi\|_1.$$

令 $\varepsilon\to0$ 便证得我们所要求的不等式（29），从而证明了定理 7。

3.3.2 **定理8的证明**。除了算子 $S$ 以外，我们还要考虑一族变形，它们对任意 $q(q>(n-1)/n)$定义。令

$$\mathfrak{S}_q(F)(x)=\left(\iint_{\Gamma(x)} y^{1-n}\Delta|F(y,t)|^q dydt\right)^{1/q},$$

其中 $\Gamma(x)$是我们的基本锥

$$\Gamma(x)=\{(t,y):\ |x-t|<y\}.$$

$\mathfrak{S}_q(F)$是有定义的，因为根据§3.1的引理，

$$\Delta|F|^q\geqslant0,\qquad \text{当}q\geqslant\frac{n-1}{n}.$$

关于 $\mathfrak{S}_q$，我们注意

(a)
$$\mathfrak{S}_2(F)=\sqrt{2}\,S(F),$$

这是因为由§3.1的引理知 $\Delta|F|^2=2|\nabla F|^2$。另外，$\mathfrak{S}_q$对$q$满足某种凸性，即

$$(\beta)\qquad \mathfrak{S}_q(F)\leqslant c(\mathfrak{S}_{q_0}(F))^{1-\theta}(\mathfrak{S}_{q_1}(F))^{\theta},$$

其中 $q_j>(n-1)/n$，而$1/q=(1-\theta)/q_0+\theta/q_1$，$0\leqslant\theta\leqslant1$，$c$是只依赖于$q_0,q_1$与$\theta$的常数，同$F$无关。不等式$(\beta)$是Hölder不等式与§3.1引理的推论。事实上，这引理说明，$\Delta|F|^q$与$|F|^{q-2}|\nabla F|^2$是可比较的，因此$\mathfrak{S}_q(F)$在性质上类似于$|F|$的$q$次模。

在做了这些准备以后，我们回到定理的证明，为了证明不等式

$$(33)\qquad B\|F\|_1\leqslant\|S(F)\|_1,\qquad F\in H^1,$$

像在定理7证明中所做的那样。我们可以用$F+\varepsilon\Phi$代替$F$，其中$F\in H_0^1$，而$\Phi$同上。为了简化符号，我们仍用$F$表示这个新函数。记$G(x,y)=|F(x,y)|$，再一次用等式

$$\iint_{R_+^{n+1}} y\Delta G dxdy=\int_{R^n}G(x,0)dx,$$

我们曾经说明过这是完全合法的。这样

$$\int_{R^n}\mathfrak{S}_1(F)dx=\int_{R^n}\left(\iint_{\Gamma(x)}\Delta(|F(t,y)|)y^{1-n}dydt\right)dx$$

$$=c\iint_{R_+^{n+1}}y\Delta|F(x,y)|dxdy=c\int_{R^n}|F(x,0)|dx.$$

因此 $\|F\|_1=c^{-1}\|\mathfrak{S}_1(F)\|_1$。由$(\beta)$并用Hölder不等式，我们有

$$(34)\qquad \|F\|_1\leqslant c'\|(\mathfrak{S}_2(F))^{\theta'}(\mathfrak{S}_\eta(F))^{1-\theta'}\|_1$$
$$\leqslant c'\|\mathfrak{S}_2(F)\|_1^{\theta'}\|\mathfrak{S}_\eta(F)\|_1^{1-\theta'},$$

其中$\eta$是任意但固定的指数，满足$(n-1)/n<\eta<1$，而$\theta'$由此选定。

如果我们能够证明不等式

$$(35)\qquad \|\mathfrak{S}_\eta(F)\|_1\leqslant c\|F\|_1,$$

那末把这式代回(34)就给出$\|F\|_1\leqslant c'\|\mathfrak{S}_2(F)\|_1$，这就是(33)。

因此我们把注意力转到（35）。

$\mathfrak{S}_\eta(F)\in L^1(\boldsymbol{R}^n)$是否成立，等价于$\mathfrak{S}_\eta(F)^\eta\in L^{1/\eta}(\boldsymbol{R}^n)$是否成立，并且由于$\eta<1$，指数$1/\eta$是大于1的。设 $r$ 是$1/\eta$ 的共轭指标，即$1/r+\eta=1$。我们需要做的就是估计

$$\sup_\varphi\int_{\boldsymbol{R}^n}\mathfrak{S}_\eta(F)^\eta(x)\varphi(x)dx,$$

其中 $\varphi$ 取遍 $L^r(\boldsymbol{R}^n)$的一个合适的稠密子集，满足$\|\varphi\|_r\leqslant1$。

在我们这里，取 $\varphi$ 是$\boldsymbol{R}^n$的非负 $C^\infty$函数，每个都具有紧支集，使得$\|\varphi\|_r\leqslant1$。现在

$$\mathfrak{S}_\eta(F)^\eta(x)=\iint_{\Gamma(x)}\Delta(|F(t,y)|^\eta)y^{1-n}dtdy$$

$$=\iint_{\boldsymbol{R}_+^{n+1}}\psi(x,t,y)\Delta(|F|^\eta(t,y))y^{1-n}dtdy,$$

其中 $\psi(x,t,y)$是锥 $\Gamma(x)=\{|x-t|<y\}$的特征函数。若 $P_y(x)$是 Poisson 核，则 $\psi(x,t,y)y^{-n}\leqslant cP_y(x)$，因此

$$\int_{\boldsymbol{R}^n}\mathfrak{S}_\eta(F)^\eta(x)\varphi(x)dx\tag{36}$$

$$\leqslant c\iint_{\boldsymbol{R}_+^{n+1}}\varphi(x,y)\Delta(|F(x,y)|^\eta)ydxdy,$$

其中 $\varphi(x,y)$是 $\varphi$ 的 Poisson 积分，即 $\varphi(x,y)=(P_y*\varphi)(x)$。另外，我们注意下面的微分恒等式：若 $\Delta A=0$，则

$$A\Delta B=\Delta(AB)-2\sum_{i=0}^n\frac{\partial A}{\partial x_i}\frac{\partial B}{\partial x_i}.$$

令 $A=\varphi(x,y)$，　$B=|F(x,y)|^\eta$　$(y=x_0)$，

则（36）的右边被

$$\iint_{\boldsymbol{R}_+^{n+1}}\Delta(\varphi|F|^\eta)ydydx+2\iint_{\boldsymbol{R}_+^{n+1}}|\nabla|F|^\eta||\nabla\varphi|ydydx$$

的常数倍所控制。上式第一个积分可以用（31）来计算，由于对$F$的假设，（31）的应用是允许的。因此它的值是

$$\int_{R^n}|F(x,0)|^{\eta}\varphi(x,0)dx,$$

而这又被

$$\|F\|_1^{\eta}\|\varphi\|_r\leqslant\|F\|_1^{\eta}$$

所控制。第二个积分等于常数乘上

$$(37)\qquad \int_{R^n}\left\{\iint_{\Gamma(x)}|\nabla|F(t,y)|^{\eta}|\cdot|\nabla\varphi(t,y)|y^{1-n}dtdy\right\}dx.$$

现在，$|\nabla|F|^{\eta}|\leqslant$常数$\cdot|F|^{\eta-1}|\nabla F|$。应用极大不等式（式（32），见第286页）还有

$$\sup_{(t,y)\in\Gamma(x)}|F(t,y)|\leqslant cF^*(x),$$

其中
$$\int_{R^n}F^*(x)dx\leqslant A\|F\|_1.$$

最后$\Delta|F|^{\eta}\geqslant$常数$\cdot|F|^{\eta-2}|\nabla F|^2$。综合起来，用Schwarz不等式，我们看到式（37）被

$$\int_{R^n}(F^*(x))^{\eta/2}(\mathfrak{S}_{\eta}(F))^{\eta/2}S(\varphi)dx$$

的常数倍所控制。

我们把形如

$$\int_{R^n}A_1A_2A_3dx\leqslant\|A_1\|_{p_1}\|A_2\|_{p_2}\|A_3\|_{p_3}$$

的Hölder不等式用到上面的积分，其中$1/p_1+1/p_2+1/p_3=1$，而$p_1=p_2=2/\eta$，$p_3=r$（回忆$1/r+\eta=1$）。这样积分就被

$$\|F^*\|_1^{\eta/2}\|\mathfrak{S}_{\eta}(F)\|_1^{\eta/2}\|S(\varphi)\|_r$$

所控制，从而被$\|F\|_1^{\eta/2}\|\mathfrak{S}_{\eta}(F)\|_1^{\eta/2}$所控制；这两步都是对的，因为用第三章的结果有$\|S(\varphi)\|_r\leqslant A\|\varphi\|_r\leqslant A$（在这里$r>1$），以及曾经指出过的$\|F^*\|_1\leqslant A\|F\|_1$。

总之，我们得到下面的

$$\|\mathfrak{S}_\eta(F)\|_1^\eta=\sup_\varphi\int_{\mathbf{R}^n}\mathfrak{S}_\eta(F)^\eta\varphi dx$$

$$\leqslant\|F\|_1^\eta+A\|F\|_1^{\eta/2}\|\mathfrak{S}_\eta(F)\|_1^{\eta/2}.$$

这蕴含了不等式（35），从而也蕴含了不等式（33），这就是当$F$取特殊形式$F+\varepsilon\Phi$，$F\in H_0^1$时所要求的结论。通过极限过程过渡到一般的$F\in H^1$就是常规的了。

3.3.3 稠密性引理的证明。现在我们处理在前面§3的3.3.1叙述了而未加证明的引理。我们来证明$H_0^1$在$H^1$是稠密的。

假设$f\in L^1(\mathbf{R}^n)$，并且$f$具有性质：$R_j(f)=f_j$均属于$L^1(\mathbf{R}^n)$（这假设自然意味着每一个$i\frac{x_j}{|x|}\hat{f}(x)$都是$L^1$函数的Fourier变换；注意，作为一个推论，就有$\hat{f}(0)=\hat{f}_j(0)=0$，$j=1,2,\cdots,n$）。

引理的证明将分为两步。首先，我们将看到，对每个上述的$f$，可以找到序列$\{f^{(k)}\}_k\in L^1$，使得$\hat{f}^{(k)}$的支集是紧的且与原点有正的距离，还使得按$L^1$模$f^{(k)}\to f$与$R_j(f^{(k)})\to R_j(f)$，当$k\to\infty$。

让我们选定在$\mathbf{R}^n$有紧支集的$C^\infty$函数$\Phi$，满足$\Phi(x)=1$，当$|x|\leqslant 1$。对每一个$\delta>0$，用$(T_\delta(f))^\wedge(x)=\Phi(x/\delta)\hat{f}(x)$来定义$L^1(\mathbf{R}^n)$上的变换$T_\delta$。显然

$$T_\delta(f)(x)=\delta^n\int_{\mathbf{R}^n}f(x-y)\varphi(\delta y)dy,$$

其中$\hat{\varphi}=\Phi$。还注意到$\|T_\delta(f)\|_1\leqslant A\|f\|_1$，其中$A$与$\delta,f$无关。

考虑$T_N(I-T_\varepsilon)f$。当$N\to\infty$与$\varepsilon\to 0$时，它按$L^1(\mathbf{R}^n)$模收敛到$f$，只要$f$属于$L^1(\mathbf{R}^n)$的闭子空间$L_0^1$，它由其Fourier变换在原点消失的函数组成。为了证明这点，注意到算子$T_N(I-T_\varepsilon)$的一致有界性，只需对这子空间的一个稠密子集验证它就可以

了。一个这样的合适子集是由其 Fourier 变换的支集是紧的且同原点有正距离的 $L^1(\boldsymbol{R}^n)$函数组成。对这样的 $f$ 显然$T_N(I-T_\varepsilon)f=f$，只要$N$充分大而 $\varepsilon$ 充分小。这样的 $f$ 在闭子空间 $L_0^1$是稠密的这一事实，可以用直接计算证明，或借助于刻划 $L^1(\boldsymbol{R}^n)$极大理想的 Wiener 定理。无论如何，我们令 $T_k(I-T_{1/k})f=f^{(k)}$，则按 $L^1$模 $f^{(k)}\to f$。还有 $R_j(f^{(k)})=T_k(I-T_{1/k})R_j(f)$，因此 $R_j(f^{(k)})\to R_j(f)$，从而完成了第一步。

现在我们可以假设 $f\in L^1(\boldsymbol{R}^n)$，并且 $\hat{f}$ 的支集$K$是紧的且与原点分离。考虑由 $\hat{f}*k^n\psi(kx)$给出的 $\hat{f}$ 的一个标准正则化，其中 $\psi$ 属于 $C^\infty$，有紧支集，且

$$\int_{\boldsymbol{R}^n}\psi dx=1.$$

注意 $\hat{f}*k^n\psi(kx)$属于 $C^\infty$，且当 $k$ 充分大时它们有共同的支集，它包含在一个同原点有正距离的紧集 $K'$ 中。若 $\Psi^\wedge(x)=\psi(x)$，则 $\Psi(0)=1$，并且 $\hat{f}*k^n\psi(kx)$ 是 $f(x)\Psi(x/k)=f_k$ 的Fourier 变换，显然还有 $f_k$ 按$L^1(\boldsymbol{R}^n)$ 模收敛到$f$。我们希望还有 $R_j(f_k)\to R_j(f)$。事实上，对于每一个上述类型的紧集 $K'$，存在 $C^\infty$ 函数 $m_j(x)$，使得 $m_j(x)=ix_j/|x|$，当 $x\in K'$。设 $M_j$ 是由$\hat{M}_j(x)=m_j(x)$ 决定的 $L^1$函数。这样

$$M_j*f_k=R_j(f_k).$$

$R_j(f_k)$按 $L^1(\boldsymbol{R}^n)$模收敛到 $R_j(f)$就显然了。然而，$H^1$中其边界值为$(f_k,R_1(f_k),\cdots,R_n(f_k))$的元素一定属于$H_0^1$。这就证明了引理。

把引理的本质记录如下将是有用的。考虑 Banach 空间

$$\{f\in L^1(\boldsymbol{R}^n):\ R_j(f)\in L^1(\boldsymbol{R}^n),\ j=1,\cdots,n\},$$

其模为$\|f\|=\|f\|_1+\sum_{j=1}^{n}\|R_j(f)\|_1$。我们在上面已证明了：

**推论** 其 Fourier 变换属于$C^\infty$且具有严格与原点分离的紧支集的 $f$ 的全体，在整个 Banach 空间是稠密的。

让我们称 $H^1$中由$F$构成的子空间为 $H_{00}^1$。自然，我们有

$H^1_{00}\subset H^1_0\subset H^1$，并且我们证明了 $H^1_{00}$ 在 $H^1$ 是稠密的。

**3.4 $H^1$的乘子变换** 在前面对函数 $S(F)$ 与 $g^*_\lambda(F)$ 作的研究以后，我们来得出某些结果，它们表明上述努力是令人满意的。我们要证明，在第二、三、四章中研究的很多奇异积分算子（更广一些，乘子变换），都可以推广为 $H^1$的有界算子。

我们需要一个定义。设 $m(x)$ 是定义在 $\boldsymbol{R}^n$ 的函数。我们假设，对 $F\in H^1$ 可以找到 $H^1$的另外一个元素 $\tilde{F}$ 具有这样的性质，若记

$$\lim_{y\to 0} F(x,y)=F(x,0)=(f_0(x),f_1(x),\cdots,f_n(x)),$$

而 $\tilde{F}(x,0)=(\tilde{f}_0(x),\cdots,\tilde{f}_n(x))$，则

$$(\tilde{f}_j)^\wedge(x)=m(x)\hat{f}_j(x),\qquad j=0,1,\cdots,n。\tag{38}$$

函数 $m$ 就通过 $\tilde{F}=T_m(F)$ 定义了一个 $H^1$ 到自身的变换 $T_m$。若 $T_m$ 在 $H^1$是有界的①，则称 $m$ 为 $H^1$乘子。还可以用另外一种说法。我们称 $m$ 是 $H^1$乘子，如果存在常数 $A$ 具有下列的性质：若 $f_0,f_1,\cdots,f_n\in L^1(\boldsymbol{R}^n)$ 且 $f_j=R_j(f_0)$，则存在 $L^1(\boldsymbol{R}^n)$ 的函数 $\tilde{f}_0,\tilde{f}_1,\cdots,\tilde{f}_n$，它们由式（38）决定，使得

$$\sum_{j=0}^n\|\tilde{f}_j\|_1\leqslant A\sum_{j=0}^n\|f_j\|_1。$$

我们的定理如下。

**定理9** 假设 $m(x)$ 在 $\boldsymbol{R}^n$ 原点以外属于 $C^{(n+1)}$。若对每个微分单项式 $\left(\frac{\partial}{\partial x}\right)^\alpha$ 有

$$\sup_{0<R<\infty} R^{2|\alpha|-n}\int_{R<|x|<2R}\left|\frac{\partial^\alpha m(x)}{\partial x^\alpha}\right|^2 dx\leqslant B,\tag{39}$$

其中 $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ 而 $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n\leqslant n+1$，则 $m$ 是 $H^1$乘子，即

① $T_m$在$H^1$有界的假设，严格地说是不必要的，因为，它可以从闭图定理以及 $T_m$ 在整个 $H^1$ 有定义推出。

(40) $$\|T_m(F)\|_1 \leqslant A\|F\|_1.$$

满足定理条件的算子有下列一些：

(a) 算子

$$f \to \lim_{\varepsilon\to 0}\int_{|y|>\varepsilon} \frac{\Omega(y)}{|y|^n} f(x-y)dy,$$

其中假设 $\Omega$ 是充分光滑的 0 次齐次函数，在单位球的平均值为零。这种算子来自第二章 § 4，以及第三章定理 6 （见第94页）所描述的某类算子。自然，后一类包含了 Riesz 变换及由它们所生成的代数。

(b) 来自第四章 $L^p$乘子定理（定理 3 及其推论，见第121页）的乘子的重要子类。

附带说明，证明将显示出$A \leqslant CB$，其中$C$是某个绝对常数；因此定理将直接引出它本身的一个小小的推广，其中条件 $C^{(n+1)}$ 可以放松为包含 $L^2$可微性的某个条件。然而，我们将不追求这种精确。

3.4.1 定理是下述引理的直接推论。

**引理** 假设 $F$ 属于稠密子空间 $H^1_{00}$，则 $T_m(F) \in H^1$，并且

(41) $$S(T_m(F))(x) \leqslant A' g^*_\lambda(F)(x),$$

其中 $\lambda = (2n+2)/n$。

证明引理用的方法同第四章 § 3.2相应引理所用的差不多是一样的。

首先由于 $F \in H^1_{00}$，每个$\hat{f}_j$属于 $C^\infty$，且有与原点分离的紧支集。因此，根据$m$在原点以外属于$C^{(n+1)}$，$m(x)\hat{f}_j(x)$就属于 $C^{(n+1)}$ 且有紧支集，因而它又是 $L^1$函数的 Fourier 变换。按定义 (38)，$T_m(F) \in H^1$。

让我们用

$$M(x,y) = \int_{\mathbf{R}^n} e^{-2\pi i x\cdot t} e^{-2\pi |t| y} m(t)dt$$

定义调和函数 $M(x,y)$，$y>0$。这样，沿着第四章§3.3的推理，我们有

$$\tilde{F}(x,y)=T_m(F)(x,y)$$

$$=\int_{R^n}M(t,y_1)F(x-t,y_2)dt,\quad y=y_1+y_2,$$

因此

$$|\nabla^{k+1}\tilde{F}(x,y)|\leqslant\int_{R^n}|\nabla^k M(t,y/2)||\nabla F(x-t,y/2)|dt, \tag{42}$$

其中 $\nabla^k$ 表示 $k$ 阶梯度。

条件（39）可以通过 $M$ 来解释。根据第四章用的同样理由，我们得到

$$|\nabla^k M(t,y)|\leqslant B'y^{-n-k}, \tag{43}$$

$$\int_{R^n}|t|^{2k}|\nabla^k M(t,y)|^2dt\leqslant B'y^{-n},\quad k=n+1. \tag{43'}$$

为了证明不等式（41），根据平移不变性只要考虑原点就可以了。把式（43）与（43′）代入（42）并用Schwarz不等式，就得到（对 $k=n+1$）

$$|\nabla^{k+1}\tilde{F}(x,y)|^2\leqslant Ay^{-n-2k}\int_{|t|<2y}|\nabla F(x-t,y/2)|^2dt$$

$$+Ay^{-n}\int_{|t|\geqslant 2y}|\nabla F(x-t,y/2)|^2|t|^{-2k}dt$$

$$=I_1(x,y)+I_2(x,y).$$

这样

$$\iint_{|x|<y}|\nabla^{k+1}\tilde{F}(x,y)|^2y^{2k-n+1}dxdy$$

$$\leqslant\sum_{j=1}^{2}\iint_{|x|<y}I_j(x,y)y^{2k-n+1}dxdy.$$

为了估计

$$\iint_{|x|<y} I_1(x,y)y^{2k-n+1}dxdy,$$

注意 $|t|\leqslant 2y$ 和 $|x|\leqslant y$ 蕴含了 $|x-t|\leqslant 3y$。简单的计算就表明，这个积分被

$$\iint_{|x'|\leqslant 6y} |\nabla F(x',y)|^2 y^{1-n}dx'dy$$

的常数倍所控制。

类似地，积分

$$\iint_{|x|\leqslant y} I_2(x,y)y^{2k-n+1}dxdy$$

被

$$\iint_{|x'|\geqslant 2y} |\nabla F(x',y)|^2 y^{1-n}\left(\frac{y}{|x'|}\right)^{2k}dx'dy$$

的常数倍所控制。

这两个结果自然都被 $(g_\lambda^*(F)(0))^2$ 的常数倍所控制，其中 $\lambda=2k/n$。由于 $k=n+1$，我们有 $\lambda=(2n+2)/n$。综合起来我们就验证了

$$\iint_{|x|\leqslant y} |\nabla^{k+1}\tilde{F}(x,y)|^2 y^{2k-n+1}dxdy\leqslant A'[g_\lambda^*(F)(0)]^2.$$

我们现在借助本章 §2的 2.5.2 的引理 2(见第274页)。作为一个结果我们得到

$$S(\tilde{F})(0)=S(T_m(F))(0)\leqslant Ag_\lambda^*(F)(0),$$

在平移到任意 $x$ 之后，这就是式（41），引理得证。

定理 9 的证明可以如下完成。定理 7 与 8 直接表明，当 $F\in H_{00}^1$ 时，$\|T_m(F)\|_1\leqslant A\|F\|_1$，而 $A$ 与 $F$ 无关。

定义在 $H_{00}^1$ 的有界算子 $T_m$ 有一个抽象的延拓，成为整个 $H^1$ 的有界算子。通过极限的推理显然可以看出，这个延拓满足性质（38）。定理 9 因此完全得证。

## §4 进一步的结果

4.1 当用 $\boldsymbol{R}^{n+1}$ 的单位球 $B^{n+1}$（其边界是单位球面 $S^n$）代替上半平面 $\boldsymbol{R}_+^{n+1}$ 时，类似于§1的结果成立。令 $\mathscr{P}(x,y)$ 是球的 Poisson 核

$$\mathscr{P}(x,y)=c_n\frac{1-|x|^2}{|x-y|^{n+1}} \quad (|x|<1,\ |y|=1),$$

而 $d\sigma(y)$ 是 $S^n$ 的 Lebesgue 测度。对每个 $f\in L^p(S^n,d\sigma)$，它的 Poisson 积分是

$$u(x)=\int_{S^n}\mathscr{P}(x,y)f(y)d\sigma(y).$$

我们有下述的结果。假设 $u$ 在 $B^{n+1}$ 调和。

(a) $u$ 是 $L^p(1<p\leqslant\infty)$ 函数的 Poisson 积分，当且仅当

$$\sup_{0<r<1}\left(\int_{S^n}|u(ry)|^p d\sigma(y)\right)^{1/p}<\infty.$$

(b) $u$ 是 $S^n$ 上有限测度的 Poisson 积分，当且仅当

$$\sup_{0<r<1}\int_{S^n}|u(ry)|d\sigma(y)<\infty.$$

(c) $u$ 是 $S^n$ 有限正测度的 Poisson 积分，当且仅当 $u\geqslant 0$ 在 $B^{n+1}$。

在上述三点中的任一条件下，$u$（适当定义的）非切线极限几乎处处在 $S^n$ 存在。

4.2 假设 $u(x,y)$ 在 $\boldsymbol{R}_+^{n+1}$ 调和。则 $u(x,y)\geqslant 0$ 当且仅当它具有形式

$$u(x,y)=\int_{R^n}P_y(x-t)d\mu(t)+ay,\qquad a\geqslant 0,$$

其中 $d\mu$ 是非负 Borel 测度，满足

$$\int_{R^n}\frac{d\mu(t)}{(1+|t|^2)^{(n+1)/2}}<\infty$$ （提示：用§4.1的(c)）。

4.3 非切线有界蕴含了在几乎所有的点非切线极限存在的事实（定理3），可以在几个方向进行推广 .

(a) 充分地只要假设在所考虑的点，给定的调和函数是非切线下有界的（Carleson[1]）.

(b) 这些结果可以推广到具有 Lipschitz 边界的区域（R. Hunt 和 Wheeden[1]）.

4.4 设 $d\mu$ 是 $\boldsymbol{R}_+^{n+1}$ 上具有下述性质的任意非负测度，对 $\boldsymbol{R}_+^{n+1}$ 上任意与边界 $\boldsymbol{R}^n$ 相贴的方体 $Q$，有

$$\mu(Q)\leqslant c(\operatorname{diam}(Q))^n.$$

令 $u(x,y)$ 是 $L^p(\boldsymbol{R}^n)$ 函数的 Poisson 积分，则

$$\left(\iint_{R_+^{n+1}}|u(x,y)|^p d\mu\right)^{1/p}\leqslant cA_p\left(\int_{R^n}|f(x)|^p dx\right)^{1/p},$$

$$1<p\leqslant\infty.$$

见 Carleson[2]；还可见 Hörmander[4].

这个结果实际上是通常的极大定理(第一章定理1)的推论.这可以说明如下.设 $\Phi(x,y)$ 与 $\varphi(x)$ 分别是 $\boldsymbol{R}_+^{n+1}$ 与 $\boldsymbol{R}^n$ 的非负函数，通过非切线不等式 $\sup\limits_{|x-x'|<y}\Phi(x',y)\leqslant\varphi(x)$ 联系起来.则对每个 $\alpha$ 有

$$\mu\{(x,y):\ \Phi>\alpha\}\leqslant cm\{x:\ \varphi>\alpha\},$$

并且作为一个结果还有

$$\iint_{R_+^{n+1}}\Phi^p d\mu\leqslant c\int_{R^n}\varphi^p(x)dx.$$

一旦有了这点，我们只需取

$$\Phi(x,y)=|u(x,y)|,\qquad \varphi(x)=AM(f)(x).$$

非切线不等式 $\sup\limits_{|x-x'|<y}\Phi(x',y)\leqslant\varphi(x)$ 就包含在定理1中，而结果就从 $M(f)$ 的 $L^p$ 不等式推出.

4.5 存在另一种类型的 Poisson 积分的极大不等式。它具有 $g_\lambda^*$ 函数的某些性质，即存在依赖于 $\lambda$ 的临界 $L^p$ 类。用

$$\mathscr{M}_\lambda(f)(x)=\sup_{y>0}\left(\int_{\mathbf{R}^n}|u(x-t,y)|^2y^{-n}\left(\frac{y}{|t|+y}\right)^{n\lambda}dt\right)^{1/2}$$

定义 $\mathscr{M}_\lambda$。

注意 $\mathscr{M}_\lambda(f)(x)\geqslant c_\lambda M(f)(x)$，当 $f\geqslant 0$。令 $1<\lambda\leqslant 2$。

(a) 若 $p=2/\lambda$，则变换 $f\to\mathscr{M}_\lambda(f)$ 是弱 $(p,p)$ 型的；

(b) 若 $p>2/\lambda$，则

$$\|\mathscr{M}_\lambda(f)\|_p\leqslant A_{p,\lambda}\|f\|_p;$$

(c) 若 $p<2/\lambda$，则存在 $f\in L^p(\mathbf{R}^n)$ 使得 $\mathscr{M}_\lambda(f)=\infty$ 处处成立。见 Stein[4]。

4.6 设 $H$ 是调和函数

$$\frac{|(x,y)|^{-n+1}}{-n+1},\qquad n>1$$

（其中 $|(x,y)|=(y^2+x_1^2+\cdots+x_n^2)^{1/2}$）。

令
$$F=\nabla H=\left(\frac{\partial H}{\partial y},\ \frac{\partial H}{\partial x_1},\cdots,\frac{\partial H}{\partial x_n}\right).$$

则 $|F|=|(x,y)|^{-n}$，而 $\Delta|F|^q=nq[nq-n+1]|(x,y)|^{-nq-2}$。

因此当 $q>0$ 时，仅当 $q\geqslant\dfrac{n-1}{n}$ 有 $\Delta|F|^q\geqslant 0$。

4.7 在 $H^1$ 理论中分数次积分的 $L^p$ 不等式（见第四章§1.2）对 $p=1$ 是成立的。若 $f\in L^1(\mathbf{R}^n)$ 并且 $R_j(f)\in L^1(\mathbf{R}^n)$, $j=1,\cdots,n$, 则

$$I_\alpha(f)(x)=\frac{1}{\gamma(\alpha)}\int_{\mathbf{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy\in L^q(\mathbf{R}^n),$$

并且 $\|I_\alpha(f)\|_q\leqslant A_\alpha\left(\|f\|_1+\sum_{j=1}^n\|R_j(f)\|_1\right)$，其中 $1/q=1-\alpha/n$，而 $0<\alpha<n$。见 Stein 与 Weiss[2]。

4.8　假设 $f \geqslant 0$，$f \in L^1(\boldsymbol{R}^n)$。定义 $R_j(f)$ 为

$$\lim_{\varepsilon \to 0} c_n \int_{|y| \geqslant \varepsilon} \frac{y_j}{|y|^{n+1}} f(x-y) dy,$$

我们已知它是几乎处处存在的(见第二章定理 4)。

假设每个 $R_j(f)$ 在任意紧集上可积，则 $|f|\log(2+|f|)$ 在任意紧集上也可积。(提示：用本章 § 3.2中定理 6 的推论2，再加上第一章的 § 5.2的(c)。详情见Stein[12].)

4.9　定理6至9可以推广到 $H^p$，$p>(n-1)/n$，其方法与这里 $p=1$所给出的完全相同。

(a)　若 $F \in H^p$，则 $\lim\limits_{y \to 0} F(x,y)=F(x)$ 几乎处处存在，

$$\int_{\boldsymbol{R}^n} |F(x,y)-F(x)|^p dx \to 0, \qquad \text{当} y \to 0,$$

并且

$$\int_{\boldsymbol{R}^n} \sup_{y>0} |F(x,y)|^p dx \leqslant A_p^p \|F\|_p^p.$$

见 Stein 与 Weiss[2]。

(b)　$B_p\|F\|_p \leqslant \|S(F)\|_p \leqslant A_p\|F\|_p$。见 Calderon[6]，Segovia[1]，还有 Gasper[1]。

(c)　$\|g_\lambda^*(F)\|_p \leqslant A_{p,\lambda}\|F\|_p$，当 $p>2/\lambda$。

(d)　$m$ 是 $H^p$的乘子，如果 $|m(x)| \leqslant B$，且

$$\sup_{0<R<\infty} R^{2|\alpha|-n} \int_{R<|x|<2R} \left|\frac{\partial^\alpha m(x)}{\partial x^\alpha}\right|^2 dx \leqslant B,$$

其中 $|\alpha| \leqslant k$，而 $k \geqslant n/p$ $(p \leqslant 2)$。对(c)与(d)，见 Stein[9]。

4.10　设 $a(x)$定义在 $R^1$上，并且 $|a(x)-a(y)| \leqslant M|x-y|$，$x,y \in R^1$。

令

$$T_\varepsilon(f)(x) = \int_{|x-y| \geqslant \varepsilon} \left\{\frac{a(x)-a(y)}{(x-y)^2}\right\} f(y) dy,$$

则$\|T_\varepsilon(f)\|_p\leqslant A_p\|f\|_p$，$1<p<\infty$，$A_p$与$\varepsilon$无关。这个结果可以作为定理8的应用而证明。详情以及这类进一步的结果见Calderón[6]。

4.11 令$u(x,y)$在$\boldsymbol{R}_+^{n+1}$调和。在单个点$x^0$,下面三个条件，除了(b)当然蕴含(a)外，都可以是独立的：

(a) $u$在$x^0$非切线有界；

(b) $u$在$x^0$有非切线极限；

(c) $$\iint\limits_{\Gamma(x^0)}|\nabla u|^2y^{1-n}dxdy<\infty.$$

为了理解这点，取$n=1$，考虑三个函数$e^{-i/z}$，$e^{i\gamma\log z}$和$(\log z)^{1-\delta}$，其中$\gamma$是实数，$1/2<\delta<1$；每个函数在$(x,y)\in\boldsymbol{R}_+^2$都是$z=x+iy$的单值解析函数。令$x^0=0$。对第一个函数我们有(b)(与(a))但沒有(c)；对第二个我们有(a)但沒有(b)与(c)；对第三个我们有(c)但沒有(b)与(a)。

4.12 当用垂直趋向边界代替非切线趋向时，本章的许多结果并不成立。这可以从下面的定理看出。

**定理** 设$E$是$\boldsymbol{R}^1$的第一纲集（可以是全测度的）。若$\Phi(x,y)$是$\boldsymbol{R}_+^2$的任意连续函数，则存在$\boldsymbol{R}^2$的解析函数$F(z),z=x+iy$，使得$\lim\limits_{y\to0}\{F(x+iy)-\Phi(x,y)\}=0$，对每个$x\in E$（见Bagemhil与Seidel[1]）。

(a) 借助于这个定理，选择适当的$\Phi$，当$y\to0$时它有界但振动，我们看到§1.3的定理3，对垂直趋向的类似是不成立的。类似地选择$\Phi$，再加上它是纯实値的，则§2.5中的定理5的类似也是否定的。

(b) 对固定的$\Psi(y)$，令$\Phi(x,y)=\Psi'(y)$，并假设$f(z)$是$\boldsymbol{R}_+^2$的解析函数,满足$f'(x+iy)-\Psi'(y)\to0$，当$y\to0$，$x\in E$。这样，显然$f(x+iy)-\Psi(y)$有极限，当$y\to0$，$x\in E$。当选择$\Psi(y)=ye^{i/y}$时，我们看出，当$y\to0$时$f(x+iy)$可以几乎处处有极限，

但$\int_0^c y|f'(x+iy)|^2dy=\infty$几乎处处。反过来，只要选择$\psi(y)=(\log 1/y)^\delta$ $(0<\delta<1/2)$，当$y\leqslant 1$，我们就看出

$$\int_0^c y|f'(x+iy)|^2dy$$

可以几乎处处有极限，却沒有$\lim\limits_{y\to 0}f(x+iy)$几乎处处存在。这就说明了（§2中）面积定理的类似也是不成立的。

## 注释

**节1** Fatou定理的局部说法（当$n=1$）是由Privalov 用复方法证明的。见Zygmund[8，第XIV章]。本书叙述的一般形式和证明来自Calderón[1]。

**节2** 面积积分在$n=1$时是Lusin引入的。在这种情形，定理4是Marcinkiewicz，Zygmund 和Spencer 给出的（见Zygmund[8，第XIV章]）。一般情形见Stein[5]，但单方向的蕴含关系较早由Calderón[2]证明过。研究共轭调和函数的非切线收敛的定理5（在第八章中它将是一个关键的工具），在$n=1$时应追溯到Plessner。用复方法的推理可在Zygmund[8，第XIV章]中找到。一般情形以及用来证明它的§2中2.5.1的引理，见Stein[5]。

**节3** $H^p$ 空间论可以在，例如Zygmund[8，第VII章]，K. Hoffman[1]中找到。$n$维的实理论开始于Stein 与 G.Weiss[2]，基于§3.1引理的一个变形。定理7是在Stein[9]中宣告的。定理8的推理是Segovia[1]给出的原来证明的简化。对导致定理8的思路以及本章所提供的推理也可见Zygmund[8，第XIV章]，而比较细致的见Calderón[6]。

把许多奇异积分定理推广到$H^1$的定理9，见Stein[9]。用推广方程组（18）的方法推广$H^p$空间论，见Calderón 与 Zygmund[8]，Stein 与 G.Weiss[2]以及《富里叶分析》第VI章。

# 第八章 函数的微分

在本章中，我们将把在本书中发展起来的各种技巧，一起用来研究多元函数的可微性质。与我们的方法相协调，我们并不追求最广的一般性，但是我们将尽可能选出这个尚未达到成熟的理论的显著特征。我们将涉及以下的问题。

(a) 什么条件能保证函数几乎处处有微商？这是我们在此要考虑的中心问题的特殊情形。

(b) 相对于一已知的 $\boldsymbol{R}^n$ 的可测子集 $E$，对函数加什么样的条件，能保证这个函数在 $E$ 的几乎所有的点是可微的？

对第二个问题我们采纳下面的观点。我们先寻找问题的一个合适的**整体**的类似。结果通常都叙述成一组包含函数空间的模的等式或不等式在整个 $\boldsymbol{R}^n$ 处处（或几乎处处）成立。往往是更深刻的派生的**局部**说法，是用类似的条件与结论给出的，但它们只涉及到一任意的可测集 $E$。

我们在第七章（§1.2中）曾经碰到过一对这种整体与局部的例子，那就是研究调和函数边值的Fatou定理的两个说法。

本章的目的是要给出关于可微性条件的三个局部的定理。它们的整体的类似如下：Lipschitz函数几乎处处可微；奇异积分变换在 $L^q(\boldsymbol{R}^n)$ 的有界性，$1<q<\infty$；通过 $g$ 函数或 Lusin 积分的 $L^q$ 类来刻划函数的 $L^q$ 类的特征。

除了已经提到的整体结果外，配合的主要的技巧有下面几点：

(a) 可微的通常定义的推广；

(b) 把函数分解为“好”部与“坏”部的基本引理。在某种意义下，它可以看作第二章中用来研究奇异积分的对应的整体分解的一个局部的类似；

(c) 一种**对称化**的推理，它使我们有可能从对称的假设得到非对称的结论。

(a)和(b)的思想在§1与§2叙述，而(c)在§4中详述。

我们还得加上一个**学究式**的评注。自然，我们这里研究的课题，原则上不能排除不可测集进入我们推理的某一步这种可能性。由于这是一个很难对付的地方，我们不妨向前走，而在后面§3的3.1.1中再回过头来作较详细的讨论，但让我们在这里作下面的约定。凡以后写出的"函数"与"集合"，将意味着是Lebesgue可测函数与可测集，除非有明确的相反的说明。

## §1 逐点可微的几个概念

1.1 假设$f$在$R^n$中集合$E$的一个开邻域定义。我们说$f$在$x^0$有**通常的微商**（或说是**可微的**），如果存在线性函数$\Lambda=\Lambda_{x^0}$，使得

(1) $$f(x^0+y)=f(x^0)+\Lambda(y)+o(|y|),\qquad |y|\to 0,$$

或者换句话说，

$$\sup_{|y|<r}\frac{|f(x^0+y)-f(x^0)-\Lambda(y)|}{|y|}$$

当$r$趋向于零时趋向于零。

另一方面，设$f$（例如说）是局部可积的，我们可以像第五章§2.1那样定义它的**弱意义**的（在广义函数意义下的）一阶偏微商（简称一阶弱偏微商），例如说

$$\frac{\partial f}{\partial x_k}=f_k,\qquad k=1,\cdots,n,$$

$f_k$局部可积，如果

(2) $$\int_{R^n}f\frac{\partial\varphi}{\partial x_k}dx=-\int_{R^n}f_k\varphi dx$$

对每个支集是紧的并严格包含在$f$定义域内的$\varphi$成立。

出现的第一个重要问题是，能否从第二个定义推论出，$f$ 对几乎所有的 $x^0$ 有逐点微商(1)？

在一维的情形，回答自然是肯定的，因为每个这样的函数是在$\boldsymbol{R}^1$ 局部绝对连续的（见第五章§6.1）。然而，当维数大于1时，情况就不同了。第五章§6.3的例子表明，$\dfrac{\partial f}{\partial x_k}$可以在弱意义下存在，且属于$L^p$，对$p\leqslant n$；同时 $f$ 却可能在**每一点** $x^0$没有(1)意义下的微商。这就推动我们放宽微商定义中的要求，只要保持 $f$ 具有我们所期望的那一类性质。我们说 $f$ 在 $x^0$ 有 $L^q$ 微商，$1\leqslant q<\infty$，如果

$$(3)\qquad \left(h^{-n}\int_{|y|<h}|f(x^0+y)-f(x^0)-\Lambda(y)|^q dy\right)^{1/q}$$

$$=o(h),\quad h\to 0.$$

显然这是原来定义(1)的推广。

下面的结果说明刚刚引入的概念的意义。我们假设$n>1$。

**定理1** 假设 $f$ 是在开集 $\Omega$ 内给定的局部可积函数，在那里 $\dfrac{\partial f}{\partial x_j}$，$j=1,\cdots,n$ 按弱意义存在，并且 $f$ 与$\dfrac{\partial f}{\partial x_j}$局部$L^p(\boldsymbol{R}^n)$可积，$j=1,\cdots,n$。

(a) 若$n<p$，则 $f$ 对几乎所有的 $x^0$ 有通常的微商（按意义(1)），只要 $f$ 在一零测集上作适当的修改。

(b) 若$1<p<n$，则 $f$ 对几乎所有的 $x^0$ 有 $L^q$ 微商，其中$1/q=1/p-1/n$。

**1.2 定理1的证明** 只要乘上一个光滑的紧支集函数，我们可以设 $f$ 是紧支集的，并且 $f$ 与$\dfrac{\partial f}{\partial x_j}$都属于$L^p(\boldsymbol{R}^n)$，其中后者是按广义函数意义说的；这就是说$f\in L_1^p(\boldsymbol{R}^n)$。我们还可以假设$p<\infty$，因为$p=\infty$时的结果是$n<p<\infty$时的推论。

当 $f$ 属于$\mathscr{D}$时，我们有恒等式

$$(4) \qquad f = I_1\left(\sum_{j=1}^{n} R_j\left(\frac{\partial f}{\partial x_j}\right)\right)$$

（见第五章§2.3）。

现在，此处的 $f$ 属于$L_1^p(R^n)$，因此它可以是$\mathscr{D}$中元素序列$\{f_m\}$按该空间的模的极限（见第五章§2.1）；由于每一个$\frac{\partial f}{\partial x_j}$也属于$L^{p'}(R^n)$，对任意 $p'\leqslant p$，逼近推理的证明还表明

$$\frac{\partial f_m}{\partial x_j}\to\frac{\partial f}{\partial x_j}, \qquad \text{在} L^{p'}(R^n).$$

回忆$R_j$是$L^{p'}(R^n)$到自身连续的，又当$1<p'<\infty$时，$I_1$是从$L^{p'}(R^n)$到$L^{q'}(R^n)$连续的，只要$1/q'=1/p'-1/n$，$1<p'<n$（见第五章§1.2）。总起来知恒等式（4）对我们的 $f\in L_1^p(R^n)$成立。为了我们的目的，它可以改述如下。

$$(5) \qquad f(x)=\int_{R^n}\frac{g(y)}{|x-y|^{n-1}}dy,$$

其中$g\in L^{p'}(R^n)$，$1<p'\leqslant p$。现在，我们已有

$$g(x)=\frac{1}{\gamma(1)}\sum_{j=1}^{n}\left(R_j\frac{\partial f}{\partial x_j}\right)(x).$$

自然，积分(5)对几乎所有的 $x^0$ 是绝对收敛的(这可从取$p'<n$看出)。修改 $f$，使它在积分绝对收敛的地方取积分值，而在其余的 $x$ 点（如果它们存在）我们可以任意定义 $f$。

另外，对$g\in L^p(R^n)$，下面两个性质对几乎一切$x^0$成立：

$$(6) \qquad \frac{1}{r^n}\int_{|y|\leqslant r}|g(x^0-y)-g(x^0)|^p dy\to 0, \quad \text{当} r\to 0.$$

$$(7) \qquad \lim_{\varepsilon\to 0}\int_{|y|\geqslant\varepsilon}\frac{y_j}{|y|^{n+1}}g(x^0-y)dy \text{ 存在}.$$

性质(6)仅仅是第一章§5.7结果的重述；性质(7)是第二章§4.5的奇异积分几乎处处存在定理的一个特殊情形。

我们来证明，在使(6)与(7)同时成立的点$x^0$，$f$的合适的微商存在。

为了简化符号，我们假设 $x^0=0$与$g(x^0)=0$。这是可以实现的，只要首先通过一个合适的平移，然后用$g(x)-g(x^0)\varphi(x)$代替$g(x)$，其中$\varphi$是具有紧支集的光滑函数，且满足$\varphi(x^0)=1$。显然，用$g-g(x^0)\varphi$代替$g$所得的结果蕴含了对$g$所要求的结果。

在做了这些准备以后，我们可以把假设归结为

$$(6') \qquad \frac{1}{r^n}\int_{|y|<r}|g(y)|^p dy \to 0, \quad r\to 0$$

以及

$$(7') \qquad a_j=\lim_{\varepsilon\to 0}(n-1)\int_{|y|>\varepsilon} g(y)\frac{y_j}{|y|^{n+1}}dy \text{ 存在。}$$

我们令$\Lambda(x)=\sum_{j=1}^n a_j x_j$，其中$x=(x_1,\cdots,x_n)$。注意由于多次指出过的在无穷的估计，即$g\in L^{p'}(\boldsymbol{R}^n)$，$1<p'\leqslant p$，以及在原点附近的控制（6′），表示$f(0)$的积分$\int_{\boldsymbol{R}^n}\frac{g(y)}{|y|^{n-1}}dy$绝对收敛。

我们的目的是找出

$$f(x)-f(0)-\sum_{j=1}^n a_j x_j \quad (\text{当}\ |x|\to 0)$$

是很小的合适估计。根据式（5）我们有

$$\begin{aligned} f(x)-f(0)=&\int_{|y|<2|x|} g(y)|x-y|^{-n+1}dy \\ &-\int_{|y|<2|x|} g(y)|y|^{-n+1}dy \\ &+\int_{|y|>2|x|} g(y)\{|x-y|^{-n+1}-|y|^{-n+1}\}dy \end{aligned}$$

$$= A_x - B_x + C_x.$$

**1.2.1 积分$C_x$的研究。**我们考察

$$C_x = \int_{|y|>2|x|} g(y)\{|x-y|^{-n+1} - |y|^{-n+1}\}dy;$$

它将是差$f(x)-f(0)$的主项。我们要证明

$$|x|\to 0, \quad C_x = \sum_{j=1}^{n} a_j x_j + o(|x|).$$

事实上，把 $|x-y|^{-n+1}$ 用 Taylor 展开为 $x_1, \cdots, x_n$ 的幂，就有

$$|x-y|^{-n+1} - |y|^{-n+1} = (-n+1)\sum_{j=1}^{n} x_j \frac{y_j}{|y|^{n+1}} + R(x,y),$$

其中

$$|R(x,y)| \leqslant A\frac{|x|^2}{|y|^{n+1}}, \qquad 当|y| \geqslant 2|x|.$$

因此

$$C_x = (-n+1)\sum_{j=1}^{n} x_j \int_{|y|\geqslant 2|x|} g(y)\frac{y_j}{|y|^{n+1}}dy + R_x,$$

其中

$$|R_x| \leqslant A|x|^2 \int_{|y|\geqslant 2|x|} \frac{|g(y)|}{|y|^{n+1}}dy.$$

对任意$\delta>0$，我们有

$$|x|\int_{|y|>2|x|} \frac{|g(y)|dy}{|y|^{n+1}}$$

$$= |x|\int_{\delta\geqslant|y|>2|x|} \frac{|g(y)|}{|y|^{n+1}}dy + |x|\int_{|y|>\delta} \frac{|g(y)|dy}{|y|^{n+1}}.$$

上式右端第一个积分被 $\delta^{-n}\int_{|y|<\delta}|g(y)|dy$ 的常数倍所控制，由式（6′）知它随着$\delta$趋向于零。当$\delta$固定时，第二个积分显然随着$|x|$趋向于零。总之，我们证明了

(8) $$C_x=\sum_{j=1}^{n}a_jx_j+o(|x|),\quad 当|x|\to 0。$$

1.2.2　当 $n<p<\infty$ 时 $A_x$ 的估计。让我们考察

$$A_x=\int_{|y|<2|x|}g(y)|x-y|^{-n+1}dy.$$

对常数 $p$ 和 $r$，$1/p+1/r=1$，应用 Hölder 不等式；由于$n<p$等价于$(n-1)r<n$，$|y|^{-n+1}$在$L^r(\boldsymbol{R}^n)$局部可积。因此，由于在这个积分中$|x-y|\leqslant 3|x|$，我们有

$$A_x\leqslant\left(\int_{|y|<2|x|}|g(y)|^pdy\right)^{1/p}\left(\int_{|y|<3|x|}y^{(-n+1)r}dy\right)^{1/r}$$

$$=o(|x|^{n/p})\times\{C|x|^{(n+(-n+1)r)/r}\}$$
$$=o(|x|).$$

因此

(9) $$A_x=o(|x|),\quad 当|x|\to 0。$$

1.2.3　当$1<p<n$时$A_x$的估计。对每个$n>0$，令$\chi_h(y)$表示球$|y|\leqslant h$的特征函数。这样，假设$|x|\leqslant h$，我们就有

$$|A_x|=\left|\int_{|y|<2|x|}\frac{g(y)}{|x-y|^{n-1}}dy\right|$$

$$\leqslant\int_{R^n}\frac{|g(y)|\chi_{2h}(y)}{|x-y|^{n-1}}dy.$$

因此，由第五章§1.2的分数次积分的$L^p$定理，就得到

$$\|\chi_hA_x\|_q\leqslant A\|g\chi_{2h}\|_p,\ 其中 1/q=1/p-1/n。$$

也就是说

$$\int_{|x|<h}|A_x|^qdx\leqslant A^q\left(\int_{|y|<2h}|g(y)|^pdy\right)^{q/p}$$

$$=o(h^{n\bar{q}/p})=o(h^{n+\bar{q}}),$$

这里用了性质（6′）．故

(10) $$\left(\frac{1}{h^n}\int_{|x|<h}|A_x|^q dx\right)^{1/q}=o(h),\quad 当h\to 0.$$

1.2.4 **$B_x$的估计．**由于

$$B_x=\int_{|y|<2|x|}\frac{g(y)}{|y|^{n-1}}dy,$$

根据性质（6′）我们有

$$|B_x|\leqslant\sum_{j=-1}^{\infty}\int_{|x|/2<2^j|y|<|x|}$$

$$\leqslant\sum_{j=-1}^{\infty}(2^{-j-1}|x|)^{-n+1}o((2^{-j}|x|)^n).$$

因此

$$B_x=o(|x|)\sum_{j=-1}^{\infty}2^{-j},$$

从而

(11) $$B_x=o(|x|),\quad 当|x|\to 0.$$

综合(8),(9),(10)与(11)，我们就证明了，当$n<p$时

$$f(x)-f(0)-\sum_{j=1}^{n}a_jx_j=o(|x|),$$

而当 $1<p<n$ 时

$$\left(h^{-n}\int_{|x|<h}\left|f(x)-f(0)-\sum_{j=1}^{n}a_jx_j\right|^q dx\right)^{1/q}=o(h).$$

**定理得证．**

1.2.5 在$p=1$的情形，定理仍然成立，虽然证明必须修改．这点以及别的推广将在下面§6中叙述，

## §2 函数的分解

**2.1 调和微商** 我们现在来研究我们要用到的主要技巧之一，就是把函数分解成"好的"部分与"坏的"部分。由于我们本质上要借助调和函数论，因此重要的是在从这理论中导出可微概念时有可能实现这个分解。我们提到的这个思想事实上是较§1给出的定义更为一般的，我们将叙述如下。

设 $f$ 是定义在开集 $\Omega$ 内的局部可积函数。对固定的 $x^0\in\Omega$，我们修改 $f$ 在包含 $x^0$ 的一个有界开集外的值，使它等于零。对这样得到的 $f$（它现在属于 $L^1(R^n)$），我们取它的 Poisson 积分 $u(x,y)=P_y*f$。我们说 $f$ 在 $x^0$ 有**调和微商**，如果

$$u,\ \frac{\partial u}{\partial x_1},\ \frac{\partial u}{\partial x_2},\ \cdots,\ \frac{\partial u}{\partial x_n}$$

在 $x^0$ 有非切线极限。这时，我们定义

$$f(x^0)=\lim_{(x,y)\to(x^0,0)} u(x,y)$$

与

$$\left.\frac{\partial f}{\partial x_j}\right|_{x^0}=\lim_{(x,y)\to(x^0,0)}\frac{\partial u(x,y)}{\partial x_j},$$

其中变点 $(x,y)$ 非切线地趋向于 $(x^0,0)$。

关于这个定义，我们需要作以下几点解释。

(a) 调和微商概念的定义，在这样的意义下是没有问题的，即它与在远离 $x^0$ 的什么地方修改 $f$ 无关。特别地容易验证，如果 $f$ 属于 $L^1(R^n)$并在 $x^0$ 的一个邻域恒为零，那末 $u$ 与它对 $x_j$ 的偏微商在 $x^0$ 有非切线极限零。

(b) $f$ 在 $x^0$ 有通常意义的微商或更一般有 $L^q$ 微商（定义见式(3)），则 $f$ 在 $x^0$ 有调和微商，并且这些微商值都相等。为说明这点，充分地只要假设 $f$ 有 $L^1$ 微商。这时，对合适的常数 $A_0$，$a_1,\cdots,a_n$，有

$$f(x^0+y)=A_0+\sum_{j=1}^{n}a_jy_j+\varepsilon(y),$$

$$\int_{|y|<r}|\varepsilon(y)|\,dy=o(r^{n+1}),\quad 当 r\to 0。$$

注意，由此直接推出，当取 $A_0$ 为 $f(x^0)$ 时，$x^0$ 就是 $f$ 的Lebesgue 点，再由第七章定理 1 （见第253页）推出，$u$ 在 $x^0$ 有非切线极限 $A$。另外，

$$\frac{\partial u(x,y)}{\partial x_j}=\int_{R^n}P_y^j(t)f(x-t)\,dt,$$

其中

$$P_y^j(t)=\frac{\partial}{\partial t_j}P_y(t),\quad j=1,\cdots,n。$$

这样

$$\begin{aligned}\frac{\partial u}{\partial x_j}(x^0+h,y)&=\int P_y^j(t+h)f(x^0-t)\,dt\\&=\int P_y^j(t+h)A_0\,dt+\sum_{k=1}^{n}\int P_y^j(t+h)a_kt_k\,dt\\&\quad+\int P_y^j(t+h)\varepsilon(t)\,dt。\end{aligned}$$

上式第一项为零，而和号中每项等于 $a_j$。这只要通过一个简单的推理把对 $P_y(t)$ 求微商分别转移到对 $A_0$ 与 $\sum_k a_kt_k$ 上。剩下的只有最后一项；为此，我们对某个 $a$ 取 $|h|<ay$ （这是非切线条件）并改写积分为

$$(12)\quad \int_{|t|<y}P_y^j(t+h)\varepsilon(t)\,dt+\int_{1>|t|>y}P_y^j(t+h)\varepsilon(t)\,dt$$
$$+\int_{|t|>1}P_y^j(t+h)\varepsilon(t)\,dt。$$

由于

$$P_y(t)=\frac{cy}{(|t|^2+y^2)^{(n+1)/2}},$$

显然当$|h|<\alpha y$时，

$$\frac{\partial P_y(t+h)}{\partial t_j}$$

分别被两个函数控制：第一个是$cy^{-n-1}$，第二个是$cy/|t|^{n+2}$。我们把第一个估计代入式（12）的第一个积分，就得到它被

$$y^{-n-1}\int_{|t|<y}|\varepsilon(t)|dt$$

的常数倍控制，而这是随着$y$趋向于零的。类似地，第二个积分也被

$$y\int_{1>|t|>y}\frac{|\varepsilon(t)|}{|t|^{n+2}}dt\leqslant\sum y\int_{2^{k+1}y>|t|>2^k y}\frac{|\varepsilon(t)|}{|t|^{n+2}}dt$$

的常数倍所控制，其中求和越过所有满足$2^{k+1}y\leqslant 2$的非负整数。由$\varepsilon$的假定知，这个和中的每一项是

$$y(2^k y)^{-n-2}o((2^{k+1}y)^{n+1})=2^{-k}o(1);$$

所以这个积分随着$y$趋向于零。第三个积分显然随着$y$趋向于零，因为$\varepsilon(t)$是若干项的和，这些项被$c|t|$和$L^1(\boldsymbol{R}^n)$函数所控制。这就证明了所要求的。

(c) 关于调和微商的最后一点解释。我们的定义对$\frac{\partial}{\partial y}u(x,y)$的性质沒有任何要求。事实上，$\frac{\partial u}{\partial y}$在已知点$x^0$的非切线极限的存在并非在$x^0$通常可微的推论。然而，如果已有在$x^0$的正测度集合的可微性（通常的，$L^q$的，或调和的），我们就可以断言，对几乎所有这样的$x^0$，$\frac{\partial u}{\partial y}$的非切线极限存在。我们很快就回到这个比较深刻的事实。

2.2 **分解** 我们的定理如下：

**定理 2** 设 $f$ 是一已知的局部可积函数，对有限测度集 $E$ 中的每一点 $x^0$，$f$ 有调和微商，则对任意的 $\varepsilon>0$，总可以找到紧集 $F$，使得 $F\subset E$，$m(E-F)<\varepsilon$，并且还有 $f=g+b$，其中

(a) $g\in L_1^\infty(\boldsymbol{R}^n)$；

(b) $b$ 在 $F$ 上为零。

注意，$g$ 是“好”函数。根据定理 1，它几乎处处有通常的微商。（按定义它在 $\boldsymbol{R}^n$ 也有本性有界的一阶弱微商。）虽然 $b$ 是“坏”函数，但它有有用的补救特征，这就是在集 $F$ 上为零。

我们现在来证明定理 2。如果需要，只要考虑 $E$ 的有界子集（为简单起见，我们仍称这子集为 $E$），我们可以假设 $f$ 在这个集合的一个邻域外为零；因此 $f\in L^1(\boldsymbol{R}^n)$。设 $u(x,y)$ 是 $f$ 的 Poisson 积分，根据我们的假设，

$$\frac{\partial u}{\partial x_1},\ \cdots,\ \frac{\partial u}{\partial x_n}$$

在 $E$ 的每一点有非切线极限。这样，从第七章的定理 5（第 270 页）推出，$\frac{\partial u}{\partial y}$ 几乎对 $E$ 的所有点 $x^0$ 有非切线极限。我们又知道 $u$ 几乎对 $E$ 的所有点 $x^0$（特别地在 Lebesgue 集的每一点）也有非切线极限。让我们固定参数 $\alpha$ 与 $h$，并且考虑截锥 $\Gamma_\alpha^h(x^0)=\{(x,y):|x-x^0|<\alpha y,\ 0<y<h\}$。这样，根据前面说的以及第七章 §1 中 1.3.1 的一致化引理（见第 258 页），我们可以断言，总可以找到紧集 $F$，使得 $F\subset E$，$m(E-F)<\varepsilon$，而且

$$u,\ \frac{\partial u}{\partial y}\ 与\ \frac{\partial u}{\partial x_1},\ \cdots,\ \frac{\partial u}{\partial x_n}$$

在 $\mathscr{R}=\bigcup\limits_{x^0\in F}\Gamma_\alpha^h(x^0)$ 都一致有界。我们还可以假设，$u$ 在每一点 $x^0\in F$ 有非切线极限 $f(x^0)$。现在，代替截锥考虑无穷锥 $\Gamma_\alpha(x^0)=\{(x,y):|x-x^0|<\alpha y\}$。由于 $u$ 是 $L^1$ 函数的 Poisson 积分，因此

$$u, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}$$

在半平面 $y \geqslant h$ 也是一致有界的，加上在 $\mathscr{R}$ 上的有界性就知它们在

$$\tilde{\mathscr{R}} = \bigcup_{x^0 \in F} \Gamma_\alpha(x^0)$$

也是有界的。

现在，让我们仔细地考察 $\tilde{\mathscr{R}}$。$\tilde{\mathscr{R}}$ 是 $R_+^{n+1}$ 的开子集，其边界由超曲面 $y = \alpha^{-1}\delta(x)$ 给定，其中 $\delta(x) = \operatorname{dist}(x, F)$，即 $x$ 到 $F$ 的距离。由于 $\delta(x)$ 显然满足条件 $|\delta(x) - \delta(x^1)| \leqslant |x - x^1|$，我们知道，按第六章 §3.2的术语，$\tilde{\mathscr{R}}$ 是一特殊的 Lipschitz 区域，因此 §3.2的延拓定理可以应用限制在 $\tilde{\mathscr{R}}$ 上的 $u$，更准确些，用 $U(x, y)$ 表示 $u$ 在 $\tilde{\mathscr{R}}$ 的限制。自然 $U$ 在 $\tilde{\mathscr{R}}$ 就是 $C^\infty$ 的，而 $U$ 与它的一阶偏微商在 $\tilde{\mathscr{R}}$ 有界；这样 $U \in L_1^\infty(\tilde{\mathscr{R}})$。设 $\mathfrak{G}(U)$ 表示 $U$ 到整个 $R_+^{n+1}$ 的延拓。因此就有 $\mathfrak{G}(U) \in L_1^\infty(R_+^{n+1})$。最后，令 $g$ 是 $\mathfrak{G}(U)$ 在 $R^n$ 的限制。由于 $L_1^\infty(R_+^{n+1})$ 函数的限制还在 $L_1^\infty(R^n)$（这是第五章 §6.2(a)的一个直接推论），我们知道 $g \in L_1^\infty(R^n)$。

因为 $u$ 在 $R^{n+1}$ 的每一个点 $(x^0, 0)$ 有非切线极限 $f(x^0)$，其中 $x^0 \in F$，所以 $U$ 也如此；而由于 $\mathfrak{G}(U)$ 连续，因此对每一点 $x^0 \in F$ 有 $\mathfrak{G}(U)(x^0, 0) = f(x^0)$。故对它在 $R^n = \{(x, 0)\}$ 的限制 $g$，同样的事也是对的；即对每个 $x^0 \in F$ 有 $g(x^0) = f(x^0)$。记 $b(x) = f(x) - g(x)$，我们便得到所要求的结论。

## §3 可微的特征

**3.1 差商的有界性** 这些技巧的第一个应用是证明一个定理，它刻划出几乎处处可微的概念。

读者一定会注意到下面叙述的定理与 Fatou 定理的局部形式（第七章定理 3，第257页）十分相像。应用关于调和函数的这个 Fatou 定理，是下面证明的主要思想。

**定理3** 设$f$在一已知集合$E$的开邻域定义，则$f$在$E$几乎处处可微（通常意义），当且仅当

(13) $$f(x^0+y)-f(x^0)=O(|y|),\quad \text{当}|y|\to 0,$$

对几乎所有的$x^0\in E$成立。自然，无需假设在上面"$O$"中出现的常数对$x^0$是一致的。

显然，在$x^0$可微蕴含了式（13），因此只要证明反过来那部分。

如果我们只考虑$E$的子集$E_0$，在其中$f$有界并且式（13）成立，那末，显然在这个子集的一个开邻域之外$f$也是有界的，并且我们可以修改$f$使它在这个邻域之外为0。另外，还可以取$E_0$是有界的，并且它的合适的邻域也是有界的。这样，我们只要证明，修改后的函数几乎处处在$E_0$可微。为了简化符号我们用$E$代替$E_0$；仍用$f$记修改以后的$f$。因此，这个$f\in L^1(\mathbf{R}^n)$。我们用$u$表示它的 Poisson 积分。就像§2.1中(b)的推理那样，我们知道在每个使式（13）成立的点$x^0$，$\dfrac{\partial u}{\partial x_j}$在$x^0$非切线有界。这只需用"$O$"代替"$o$"，重复第314页的推理。这样，用第七章的定理1与3，$u$与$\dfrac{\partial u}{\partial x_1}$，$\dfrac{\partial u}{\partial x_2}$，…，$\dfrac{\partial u}{\partial x_n}$的非切线极限在$E$几乎处处存在；因此，$f$在$E$几乎处处有调和微商。现在我们可以借助上面§2.2的分解定理了。对每个$\varepsilon>0$，存在紧集$F$，$F\subset E$，$m(E-F)<\varepsilon$，以及分解$f=g+b$，在$F$上$f=g$。由于$g\in L_1^\infty(\mathbf{R}^n)$，§1.1的定理1就表明，$g$在$\mathbf{R}^n$几乎处处有通常的微商，从而只要证明$b$对几乎所有的$x\in F$是可微的。然而，$b$在$F$的所有点都等于零，因此代替式（13）我们有

(13′) $$b(x^0+y)=O(|y|),\quad \text{当}|y|\to 0,$$

对几乎所有$x^0\in F$。

现在我们希望把式（13′）做成一致的。对每个整数$k$，我们令$F_k$为

$$F_k=\{x^0: |b(x^0+y)|\leqslant k|y|,\ |y|\leqslant 1/k\}.$$

显然，$\bigcup_{k=1}^{\infty}F_k$ 包含了所有使式（13′）成立的点，因而我们只要对 $x^0\in F_k$ 进行考虑。

3.1.1 现在我们不得不插入几句话，**因为我**们碰到了推理中不大愉快的地方；这就是，由于集合 $F_k$ 是用连续统势这么多的不等式给出的，它不一定是 Lebesgue 可测的。为了避开这个困难，我们可以从一开始就假设 $f$ 是处处连续的，但这个假设对我们的问题来说是很人为的。

我们代之以考察命题"$E$的几乎每一点都是$E$的全密点"，它对非可测集来说，只要作适当的修饰仍然是成立的。事实上，对任意的不一定是可测的集合$E$，用$m_e(E)$表示它的外测度，按定义$m_e(E)=\inf m(F)$，$E\subset F$，其中$F$取遍所有的可测集。显然，对任意集合$E$，存在可测集$\tilde{E}$，使得$m_e(E)=m(\tilde{E})$，并且更一般的对任意的可测集$F$，$m_e(E\cap F)=m(\tilde{E}\cap F)$。

其次，我们说 $x^0$ 是$E$的**外全密点**，如果

$$\frac{m_e(E\cap B(x^0,r))}{m(B(x^0,r))}\to 1,\qquad 当r\to 0,$$

其中$B(x^0,r)$是以$x^0$为中心，$r$为半径的球。显然，$x^0$是$E$的外全密点，当且仅当它是$\tilde{E}$的全密点。因此，除了$E$的一个 Lebesgue 测度为零的子集外，$E$的点都是$E$的外全密点。

3.1.2 为了完成定理的证明，我们只需证明，在$F_k$的每一外全密点$x^0$，当$|y|\to 0$时，有$b(x^0+y)=o(|y|)$。因为这时$b$在$F_k$的这点可微；再对$k$取并，除了$F$中一测度为零的集合外，$b$是可微的。

现在，推论仅仅是重复第一章命题2（在§2.2中，第13页）的证明中已经做过了的事情。对$F_k$的一个外全密点$x^0$与给定的$\varepsilon>0$，我们考虑中心在$x^0+y$，半径为$\varepsilon|y|$的"小"球；还考虑中心在$x^0$，半径为$|y|+\varepsilon|y|$的"大"球，根据全密点的意义，只

要$|y|$足够小，存在$z\in F_k$属于小球。这就说明，在到点$x^0+y$的距离为$\varepsilon|y|$的范围内，我们可以找到$F_k$的点。由$F_k$的定义知$|b(x^0+y)|\leqslant k\varepsilon|y|$，而从$\varepsilon$的任意性推出$b(x^0+y)=o(|y|)$，当$|y|\to 0$。

**3.2 分解的改进** 在叙述另一个应用以前我们需要改进§2.2的分解定理。设$1\leqslant q<\infty$，并设$f(x)$在集合$E$的每一点有$L^q$微商(见§1.1的定义)，因而$f$在$E$的每一点有调和微商。令$\varepsilon>0$。根据§2.2的定理，可以找到$F$，$F\subset E$，$m(E-F)<\varepsilon$以及$f=g+h$，使得$g\in L_1^\infty(\boldsymbol{R}^n)$，并且对$x^0\in F$有$b(x^0)=0$。我们还有更强一些的结果。

**推论（定理2的）** 对几乎所有的$x^0\in F$

$$\int_{|y|<1}\frac{|b(x^0+y)|}{|y|^{n+1}}dy<\infty. \tag{14}$$

**证明** 从$g\in L_1^\infty(\boldsymbol{R}^n)$知$g$几乎处处有通常的（因而有$L^q$）微商；特别地有对几乎所有$x^0\in F$，有

$$\frac{1}{r^n}\int_{|y|<r}|b(x^0+y)|^q dy=O(r^q),\qquad 当\ r\to 0. \tag{15}$$

在作容易的修改（与定理3证明中所做的同一种类型）后，我们还可以假设$f\in L^q(\boldsymbol{R}^n)$。在作进一步的显然的修改后，还可假设$b\in L^q(\boldsymbol{R}^n)$。

令$F_k$是由

$$F_k=\left\{x^0:\ \frac{1}{r^n}\int_{|y|<r}|b(x^0+y)|^q dy\leqslant kr^q,\ 0<r\leqslant 1/k\right\}$$

给出的闭集。（注意这时$F_k$必然是可测的。）

我们来证明，当用$\delta(x)$表示$x$到闭集$F_k$的距离时，就有不等式

$$\int_{|y|<1}\frac{|b(x^0+y)|}{|y|^{n+1}}dy \tag{16}$$

$$\leqslant c\int_{|y|<1}\frac{\delta(x^0+y)dy}{|y|^{n+1}}, \qquad x^0\in F,$$

由此我们的推论便可由第一章§2.3（第14页）给出的 Marcinkiewicz 积分的有限性定理推出。

为此，根据第六章的定理 1，把 $F_k$ 的余集写成“不重叠”的方体$\{Q_j\}$的并，其中 $Q_j$ 的直径同它到 $F_k$ 的距离是可比较的。

这样，我们有

$$\int_{|y|<1}\frac{b(x^0+y)dy}{|y|^{n+1}}=\int_{\mathscr{C}F_k\cap\{|x^0-y|<1\}}\frac{b(y)dy}{|x^0-y|^{n+1}}$$

$$=\sum_j\int_{Q_j\cap\{|x^0-y|<1\}}\frac{b(y)dy}{|x^0-y|^{n+1}}.$$

如果 $x^0\in F_k$，那末，粗略地说，$\dfrac{1}{|x^0-y|^{n+1}}$ 当 $y$ 在 $Q_j$ 中时，与 $y$ 是无关的；即

$$\sup_{y\in Q_j}\frac{1}{|x^0-y|^{n+1}}\leqslant c_1\inf_{y\in Q_j}\frac{1}{|x^0-y|^{n+1}}.$$

另外，每个 $Q_j$ 都包含在以 $F_k$ 的某点为中心的球 $B_j$ 中，其中 $B_j$ 的半径与 $Q_j$ 的直径是可比较的。这样，由 $F_k$ 的定义推出

$$\int_{Q_j}|b(y)|^q dy\leqslant c_2 m(Q_j)^{1+q/n},$$

因此由 Hölder 不等式

$$\int_{Q_j}|b(y)|dy \qquad m(Q_j)^{1+1/n}.$$

最后，$Q_j$ 的直径和 $Q_j$ 中任一点到 $F_k$ 的距离是可比较的。故

$$\int_{Q_j}|b(y)|dy\leqslant c_4 m(Q_j)\delta(y), \qquad y\in Q_j,$$

总起来就是

$$\int_{Q_j \cap \{|x^0-y|<1\}} \frac{|b(y)|dy}{|x^0-y|^{n+1}}$$

$$\leqslant c \int_{Q_j \cap \{|x^0-y|<1\}} \frac{\delta(y)}{|x^0-y|^{n+1}}dy.$$

对 $j$ 求和就得到(16)，从而证得推理。

**注** 上述推理还证明了

$$(17) \qquad \int_{|y|<1} \frac{|b(x^0+y)|^q dy}{|y|^{n+q}} < \infty, \text{ 对几乎所有的 } x^0 \in F.$$

因为按推论一样的理由，积分（17）被

$$\int_{|y|<1} \frac{(\delta(x^0+y))^q}{|y|^{n+q}}dy$$

控制。

在使(17)成立的每个点 $x^0$，显然还有结论

$$(18) \qquad \frac{1}{r^n}\int_{|y|<r} |b(x^0+y)|^q dy = o(r^q), \qquad r \to 0.$$

这表明了，不仅 $b$ 在 $F$ 上为零，而且对 $F$ 的几乎所有的点，$b$ 的一阶 $L^q$ 微商也是零。

**3.3 奇异积分作用下的不变性** 下面我们考虑涉及奇异积分算子 $L^q(R^n)$ 有界性的某些事实的局部类似。我们将证明，逐点 $L^q$ 微商的概念，在合适的奇异积分变换的作用下，按几乎处处的意义说是稳定的。

我们处理的是下面一类算子，其核满足

(a) $K$ 在原点之外属于 $C^1$ 类；

(b) $|K(x)| \leqslant A/|x|^n$，$|\nabla K(x)| \leqslant A/|x|^{n+1}$，$x \neq 0$；

(c) 若 $T_\varepsilon(f) = \int_{|y|>\varepsilon} K(y)f(x-y)dy$，则对某个固定的

$q$，当 $f \in L^q(\boldsymbol{R}^n)$ 时，有

$$\|T_\varepsilon(f)\|_q \leqslant A_q \|f\|_q,$$

其中 $A_q$ 与 $\varepsilon$ 无关。我们还假设，当 $\varepsilon \to 0$ 时，$T_\varepsilon(f)$ 按 $L^q$ 模收敛到极限 $T(f)$。

这种变换的例子是

$$K(x) = \frac{\Omega(x)}{|x|^n},$$

其中 $\Omega$ 是 0 次齐次，在单位球面属于 $C^1$ 类，并且它在这球面的平均值为零。自然，这类变换包含了 Riesz 变换和第三章的高阶 Riesz 变换。

**定理4** 设 $1<q<\infty$，假设 $f \in L^q(\boldsymbol{R}^n)$，$f$ 在 $E$ 的每一点有 $L^q$ 微商。如果 $T$ 是上述类中的奇异积分变换，那末在 $E$ 的几乎所有的点 $T(f)$ 有 $L^q$ 微商。

自然要问，在奇异积分算子作用下，通常的微商概念是否也是稳定的。然而，甚至在 $n=1$ 时，这也是不对的，见下面 §6.8。

3.3.1 **证明。** 首先，我们注意，若 $f$ 在一已知点 $x^0$ 的固定邻域中为零，则容易看出，$T(f)$（在一零测度的集合上进行适当的修改后）在这邻域的每一点，特别地在 $x^0$，都有通常的微商。这说明，$T(f)$ 在 $x^0$ 可微的问题，仅仅依赖于 $f$ 在 $x^0$ 附近的性质。因此不妨假设 $E$ 是有界的。用分解定理，记 $f=g+b$，其中 $g \in L_1^\infty(\boldsymbol{R}^n)$，在 $F$ 有 $b=0$，$F \subset E$，$m(E-F)$ 很小。容易看出，我们可以修改 $g$ 与 $b$，使得 $g$ 有紧支集而不改变所叙述的任何性质。这样 $g$ 还属于 $L_1^q(\boldsymbol{R}^n)$。由 $T$ 是 $L^q$ 的有界算子，因此 $T(g)$ 也属于 $L_1^q(\boldsymbol{R}^n)$。事实上，一个 $L^q$ 函数是否属于 $L_1^q$ 的问题，完全是由这函数的 $L^q$ 连续模决定的（见第178页）。显然，这对 $T$ 是不变的，因为 $T$ 是 $L^q$ 有界并且是平移不变的。

现在，我们知道 $T(g) \in L_1^q(\boldsymbol{R}^n)$，由本章的定理 1 我们得

到，$T(g)$在 $\boldsymbol{R}^n$ 几乎处处有 $L^q$ 微商.因此只要考虑 $T(b)$ 并证明它在 $F$ 几乎所有的点有 $L^q$ 微商。我们来证明，在每个使

$$
(19)\qquad \begin{aligned}
&\frac{1}{r^n}\int_{|y|<r}|b(x^0+y)|^q dy = o(r^q), \qquad r\to 0,\\
&\int_{|y|<1}\frac{|b(x^0+y)|}{y^{n+1}}dy<\infty
\end{aligned}
$$

同时成立的点 $x^0$，$T(b)$的确有 $L^q$ 微商。

由§3.2的讨论，我们知道，对 $F$ 中几乎所有的 $x^0\in F$ 式(19)是成立的。根据 $b\in L^q(\boldsymbol{R}^n)$，以及式(19)中积分的有限性，我们又知道

$$
(20)\qquad \int_{\boldsymbol{R}^n}\frac{|b(x^0+y)|}{|y|^n}dy<\infty \text{ 与 } \int_{\boldsymbol{R}^n}\frac{|b(x^0+y)|}{|y|^{n+1}}dy<\infty.
$$

为了符号简单起见，我们取 $x^0=0$。让我们暂时固定正数 $r$ . 这样

$$
\begin{aligned}
T(b)(x) &= \int K(x-y)b(y)dy\\
&= \int_{|x-y|>2r} K(x-y)b(y)dy\\
&\quad + \lim_{\varepsilon\to 0}\int_{\varepsilon<|x-y|<2r} K(x-y)b(y)dy\\
&= I_r^1(x)+I_r^2(x).
\end{aligned}
$$

用 $I_r^1(x)$ 表示的积分是绝对收敛的；根据假设 $I_r^2$ 按 $L^q$ 模收敛，并且 $I_r^2=(T-T_{2r})(b)$。让我们更仔细地观察 $I_r^2$。若限制 $|x|\leqslant r$，则 $I_r^2$ 中的被积函数只包含满足 $|y|\leqslant 3r$ 的 $y$ 。这样我们可以用 $b(y)\chi_{3r}(y)$代替 $b(y)$，其中 $\chi_{3r}$ 是球 $|y|\leqslant 3r$ 的特征函数。因此

$$
\int_{|x|<r}|I_r^2(x)|^q dx\leqslant\int_{\boldsymbol{R}^n}|(T-T_{2r})(b\chi_{3r})|^q dx
$$

$$\leqslant A_q^q\int_{R^n}|b(y)\chi_{3r}(y)|^q dy$$

$$=A_q^q\int_{|y|<3r}|b(y)|^q dy=o(r^{n+q}).$$

这里我们用了 $T_\varepsilon$ 按 $L^q(R^n)$ 模的一致有界性，以及式 (19) 中的第一个性质($x^0=0$)。从而成功地证明了

$$\text{(21)}\qquad \int_{|x|<r}|I_r^2(x)|^q dx=o(r^{n+q}),\quad r\to 0.$$

为了研究 $I_r^1$，我们用 Taylar 展开式如下：

$$K(x-y)=K(-y)+(x,\nabla K(-y))+\varepsilon(r)|x|/|y|^{n+1}.$$

现在 $|x|\leqslant r$，$|x-y|\geqslant r$，而 $\varepsilon(r)$ 随 $r$ 趋向于零。把这代入定义 $I_r^1$ 的积分中去，我们有

$$\text{(22)}\qquad I_r^1(x)=\int_{|x-y|\geqslant 2r}K(-y)b(y)\,dy$$

$$+\int_{|x-y|\geqslant 2r}(x,\nabla K(-y))b(y)dy$$

$$+\varepsilon'(r)|x|\int\frac{|b(y)|}{|y|^{n+1}}dy.$$

上式右端第一个积分等于

$$\int_{R^n}K(-y)b(y)dy+O\left(\int_{|y|<3r}\frac{|b(y)|}{|y|^n}dy\right).$$

由 (20) 的第一个不等式($x^0=0$)推出

$$\int_{R^n}K(-y)b(y)dy$$

绝对收敛。从(19)我们还容易得到

$$\int_{|y|<3r}\frac{|b(y)|}{|y|^n}dy=o(r),\qquad \text{当 } r\to 0.$$

式 (22) 中右端的第二个积分可以用同样方法处理。它等于

$$\sum_{j=1}^{n} x_j \int_{R^n} \frac{\partial K(-y)}{\partial y_j} b(y) dy + o(r), \qquad \text{当 } r \to 0,$$

这些积分收敛，是根据（20）中的第二个不等式．按同样的理由，(22)中右端的第三个积分也有限。综合起来

(23) $$I_r^1(x) = A_0 + \sum_{j=1}^{n} x_j A_j + o(|x|), \qquad \text{当 } x \to 0,$$

其中

$$A_0 = \int_{R^n} K(-y) b(y) dy,$$

$$A_j = \int_{R^n} \frac{\partial K(-y)}{\partial y_j} b(y) dy, \qquad j = 1, \cdots, n.$$

联系(21)，我们看到，(23)包含了我们所要求的结果。

## §4 对称化原理

我们首先从比较一般然而是难懂的形式考察对称化的思想。然后做几个评注并给出若干有助于搞清其意义的解释。

**4.1 一般定理** 我们来研究函数 $U(x,y)$，$(x,y) \in \boldsymbol{R}^n \times \boldsymbol{R}_+^1 = \boldsymbol{R}_+^{n+1}$；$U$ 在 $n+1$ 维的上半空间可测，并且为简单起见（由于只牵涉到 $y=0$附近的性质），我们假设$U$在 $y \geqslant h$ 等于零，其中 $h>0$ 固定。还假设$U$在 $\boldsymbol{R}_+^{n+1}$ 的任何与边界 $\boldsymbol{R}^n$ 有正距离的有界子集上是平方可积的。

**定理5** 假设已知集合 $E \subset \boldsymbol{R}^n$，对任意 $x^0 \in E$ 有下列两条件成立：

(24) $$\int_0^\infty y |U(x^0, y)|^2 dy < \infty;$$

(25) $$\iint_{|t|<y} y^{1-n} |U(x^0+t, y) + U(x^0-t, y)|^2 dt dy < \infty.$$

则对几乎所有的 $x^0 \in E$，有

$$(26)\qquad \iint_{|t|<y} y^{1-n}|U(x^0+t,y)|^2 dt dy<\infty.$$

根据通常的推理，我们可以假设（代替$E$考虑一个较小的子集$F$），$F$本身是紧集，积分（24）与（25）对$x^0$属于$F$是一致有界的。这时，只要证明式（26）对$F$中几乎所有的$x^0$成立。我们的假设就是

$$(24')\qquad \int_0^\infty y|U(x^0,y)|^2 dy\leqslant M,\qquad 当\ x^0\in F,$$

$$(25')\qquad \iint_{|t|<y} y^{1-n}|U(x^0+t,y)+U(x^0-t,y)|^2 dt dy\leqslant M,$$

$$当\ x^0\in F,$$

对不等式(25′)在$F$上取积分，作变量替换$x^0+t=u$与$x^0-t=v$，就有

$$\iint_{(u+v)/2\in F} du dv \int_{|u-v|<2y} |U(u,y)+U(v,y)|^2 y^{1-n} dy<\infty.$$

如果通过限制变量$v$落在$F$中来缩小积分区域，那末就一定有

$$(26')\qquad \iint_{(u+v)/2\in F,\ v\in F} du dv \int_{|u-v|<2y} |U(u,y)+U(v,y)|^2 y^{1-n} dy<\infty.$$

另外，在不等式（24′)中对$x^0$在$F$上进行积分，并作合适的变换，即用$v$代替$x^0$，就有

$$(27)\qquad \int_{v\in F}\int_0^\infty |U(v,y)|^2 y dy dv<\infty,$$

现在，令

$$(28)\quad \begin{cases} I = \displaystyle\iint_{(u+v)/2\in F,\ v\in F} dudv \int_{|u-v|<2y} |U(v,y)|^2 y^{1-n} dy \\ \mathscr{I} = \displaystyle\iint_{(u+v)/2\in F,\ v\in F} dudv \int_{|u-v|<2y} |U(u,y)|^2 y^{1-n} dy. \end{cases}$$

把限制$(u+v)/2\in F$ 去掉，可以得到 $I$ 的控制

$$I \leqslant \int_{\mathbf{R}^n} du \int_{v\in F} dv \int_{|u-v|<2y} |U(v,y)|^2 y^{1-n} dy$$

$$= \int_{v\in F}\int_0^\infty |U(v,y)|^2 \left\{\iint_{|u-v|<2y} du\right\} dv y^{1-n} dy.$$

由于花括弧中的积分等于 $cy^n$，从(27)我们就得到 $I<\infty$。注意到(26)就有 $\mathscr{I}<\infty$。

显然，$\mathscr{I}$ 可以写成二重积分

$$(29)\qquad \mathscr{I} = \int_{\mathbf{R}^n}\int_0^\infty |U(u,y)|^2 \sigma(u,y) y^{1-n} dudy,$$

其中
$$\sigma(u,y) = \int_{\substack{(u+v)/2\in F,\ v\in F \\ |u-v|<2y}} dv.$$

这就是说，对固定的$y>0$及 $u\in\mathbf{R}^n$，$\sigma(u,y)$表示 $\mathbf{R}^n$中 $v$ 的集合的测度，其中 $v$ 在以 $u$ 为中心，$2y$ 为半径的球内，并且限制 $v\in F$，$(u+v)/2\in F$。

证明的转折点是要证明，如果$u^0$是 $F$ 的全密点，那末

$$\sigma(u,y) \sim m(B(u,2y)) = c_1 y^n.$$

只要变点$(u,y)$非切线地趋向于$(u^0,0)$。

为了符号简单起见，假设 $u^0=0$ 是 $F$ 的全密点。在 $u^0$这点，非切线趋向意味着限制$|u|<y$，令 $\chi$ 表示集合 $F$ 的特征函数，而

$\tilde{\chi}=1-\chi$ 是其余集的特征函数。这时

$$\sigma(u,y)=\int_{|u-v|<2y}\chi(v)\chi((u+v)/2)dv$$
$$=\int_{|u-v|<2y}dv-\int_{|u-v|<2y}\tilde{\chi}(v)dv$$
$$-\int_{|u-v|<2y}\tilde{\chi}((u+v)/2)dv$$
$$+\int_{|u-v|<2y}\tilde{\chi}(v)\tilde{\chi}((u+v)/2)dudv.$$

上式右边的第一个积分当然等于 $m(B(u,2y))=c_1y^n$；因此只需证明后面的三个积分每个都是 $o(y^n)$。注意 $|u|\leqslant y$，这时第二个积分就被

$$\int_{|v|<3y}\tilde{\chi}(v)dv=m({}^cF\cap B(0,3y))$$

所控制，它等于 $o(y^n)$，这是因为 0 是 $F$ 的全密点。第三个积分可以改写成 $\int_{|u-x|<y}\tilde{\chi}(x)dx$，它被 $\int_{|x|<2y}\tilde{\chi}(x)dx$ 所控制，因此也是 $o(y^n)$。第四个积分必定被第二个积分所控制，也就是 $o(y^n)$。故 $\sigma(u,y)=c_1y^n+o(y^n)$，当 $y\to 0$ 与 $|u|<y$，从而所要求的 $\sigma(u,y)\sim c_1y^n$ 得证。

这就推出，存在 $F$ 的闭子集 $F_0$，满足 $m(F-F_0)$ 固定但任意小，使得只要 $|u-u^0|<y$，$0<y<c_3$，就有 $\sigma(u,y)\geqslant c_2y^n$，其中 $c_2,c_3$ 是合适的正常数。现在令

$$\mathscr{R}=\bigcup_{u^0\in F_0}\Gamma_1^{c_3}(u^0),$$

其中 $\Gamma_1^{c_3}(u^0)$ 是截锥

$$\Gamma_1^{c_3}(u^0)=\{(u,y):|u-u^0|<y,\ 0<y<c_3\}.$$

式(29)是有限的以及我们对 $\sigma$ 已经证明了的结果就蕴含了

$$\iint_{\mathscr{R}}|U(u,y)|^2ydudy$$

是有限的。

然而，用我们在第六章做过的一个十分简单的推理（见第265页），最后积分的有限蕴含了对几乎所有的 $u^0 \in F_0$，积分

$$\iint_{\substack{|u-u^0|<y\\ y<c_3}} |U(u,y)|^2 y^{1-n}dudy = \iint_{\substack{|t|<y\\ y<c_3}} |U(u^0+t,y)|^2 y^{1-n}dtdy$$

是有限的。因此，整个积分

$$\iint_{|t|<y} |U(u^0+t,y)|^2 y^{1-n}dtdy$$

收敛，这是因为按假设 $U$ 是局部平方可积的。由于 $F_0$ 和 $F$ 只差一个测度任意小的集合，定理的证明也就完成了。

### 4.2 注

4.2.1 第一组的评注有点琐碎。用普通的推理（或近似于证明中的验证），可以看出，假设式(25)可以代之以较弱的假设

$$\iint_{|t|<\alpha y} y^{1-n}|U(x^0+t,y)+U(x^0-t,y)|^2dtdy<\infty,$$

其中决定开锥的常数 $\alpha$ 可以随点 $x^0$ 而不同。我们还可以得到(表面上看更强的)结论：

$$\iint_{|t|<\beta y} y^{1-n}|U(x^0+t,y)|^2dtdy<\infty$$

对每一个 $\beta$，其中 $x^0$ 几乎取遍 $E$ 中所有的点。

定理还有一个简单(而直接)的变形，就是和 $U(x^0+t,y)+U(x^0-t,y)$ 可以被另一种组合 $U(x^0+t,y)-U(x^0-t,y)$ 代替；还有，式(24)，(25)与(26)中的平方也可以分别用 $y|U(x^0,y)|^p$，$y^{1-n}|U(x^0+t,y)+U(x^0-t,y)|^p$ 以及 $y^{1-n}|U(x^0+t,y)|^p$ 代替，其中 $p<\infty$。$p=\infty$ 的变形在后面另外叙述。

4.2.2 假设对集合 $E$ 中的每个 $x^0$，

$$\sup_{y>0} |U(x^0,y)|<\infty$$

而
$$\sup_{|t|<y} |U(x^0+t,y)+U(x^0-t,y)|<\infty.$$

这时对$E$中几乎所有的$x^0$，$\sup\limits_{|t|<y}|U(x^0+t,y)|$也有限.这个命题的证明类似于定理5，在概念上甚至更简单。读者会毫不困难地完成推理的细节。这里可能涉及不可测集的地方，可像§3的3.1.1那样处理。

4.2.3　§4的4.2.2中命题的一个推论，是值得单独研究的。设$f$在$x^0$附近定义。考虑很有意义的条件。

(30)　　$$f(x^0+t)+f(x^0-t)-2f(x^0)=O(|t|),$$

$$\text{当 } t\to 0.$$

当$f$在$x^0$有通常的微商时，这条件当然满足，但反过来不对。而且，可以证明，存在$f$，对所有$x^0$一致满足式(30)，但在任意$x^0$沒有通常的微商。

然而，虽然式(30)本身不蕴含可微性，但他却起着“Taube型条件”的作用，即允许人们从一种形式的可微性提高到另一种形式。我们叙述一个这样的结果。

**推论**　假设$f$在一已知集合$E$的每一点$x^0$有调和微商(特别地只要$f$对每个$x^0\in E$是$L^q$可微的)。又假设式(30)对每个$x^0\in E$成立，则$f$对几乎所有的$x^0\in E$，有通常的微商。

证明是我们曾经做过的事的一个简单应用。由§2.2的分解定理，可以写$f=g+b$，其中$g$是几乎处处通常可微的，而$b$在$F$为零，$F\subset E$，$m(E-F)$很小。只要证明$b$在$F$几乎处处有通常的微商。由于$g$几乎处处可微，它就几乎处处满足式(30)，因此$b$就在$F$几乎处处满足条件(30)。决定性的条件是$b$在$F$为零。这样，条件变成

$$b(x^0+t)+b(x^0-t)=O(|t|),\qquad \text{当 } t\to 0.$$

现在令$U(x,y)=b(x)/y$。由$b$为零就得到，对几乎所有的$x^0\in F$有

$$\sup_{y>0}|U(x^0,y)|=0,$$

而且对几乎所有的$x^0\in F$,

$$\sup_{|t|<y}|U(x^0+t,y)+U(x^0-t,y)|<\infty.$$

用§4的4.2.2中的命题，便得到

$$\sup_{|t|=y}|U(x^0+t,y)|<\infty,$$

它就是

$$\sup_{t\neq 0}\frac{|b(x^0+t)|}{|t|}<\infty,$$

即

$$b(x^0+t)=O(|t|),\quad 当\ t\to 0$$

仍对几乎所有的 $x^0\in F$ 成立。§3.3的定理3使我们断定，$b$ 在 $F$ 几乎处处有通常的微商。

## §5 可微的另一个特征

**5.1** 我们要证明的定理如下。

**定理6** 假设 $f\in L^2(\boldsymbol{R}^n)$，则对几乎所有的 $x^0\in\boldsymbol{R}^n$，下面两个条件是等价的：

(a) $f$ 在 $x^0$ 有 $L^2$ 微商；

(b) $$\int_{\boldsymbol{R}^n}\frac{|f(x^0+t)+f(x^0-t)-2f(x^0)|^2dt}{|t|^{n+2}}<\infty.$$

这个定理有如下的一个推论。

**推论** 假设 $f$ 定义在 $E$ 的一个开邻域上，则 $f$ 对几乎所有的 $x^0\in E$ 有通常微商的充分必要条件是对几乎所有的 $x^0\in E$，下列两个条件成立：

(a) $f(x^0+t)+f(x^0-t)-2f(x^0)=O(|t|)$，当 $t\to 0$；

(b) $$\int_{|t|<\delta}\frac{|f(x^0+t)+f(x^0-t)-2f(x^0)|^2dt}{|t|^{n+2}}<\infty,$$

其中 $\delta$ 是充分小的正数(依赖于$x^0$)。

**5.2** 我们需要一个事实，它可以看成是我们定理的一个“整

体的”类似(整体与局部的合适概念，在本章的开头说明中讨论过)。

**命题** 假设 $f\in L_1^2(R^n)$，则

$$\int_{R^n}\int_{R^n}\frac{|f(x+t)+f(x-t)-2f(x)|^2}{|t|^{n+2}}dxdt \tag{31}$$
$$=a_n\int_{R^n}\sum_{j=1}^n\left|\frac{\partial f}{\partial x_j}\right|^2dx.$$

证明最好的办法是用 Fourier 变换。令

$$\hat{f}(y)=\int_{R^n}f(x)e^{2\pi ix\cdot y}dx,$$
$$f(x)=\int_{R^n}\hat{f}(y)e^{-2\pi ix\cdot y}dy,$$

这是在 $L^2$意义下说的。用 Plancherel 定理以及 Sobolev 空间 $L_1^2(R^n)$的性质，我们就有

$$\int_{R^n}\left|\frac{\partial f}{\partial x_j}\right|^2dx=4\pi^2\int_{R^n}|x_j|^2|\hat{f}(x)|^2dx.$$

还有

$$\int_{R^n}|f(x+t)+f(x-t)-2f(x)|^2dx$$
$$=4\int_{R^n}|\hat{f}(x)|^2|1-\cos 2\pi x\cdot t|^2dx.$$

因此

$$\int_{|t|>\varepsilon}\int_{R^n}|f(x+t)+f(x-t)-2f(x)|^2dx\frac{dt}{|t|^{n+2}}$$
$$=\int_{R^n}|\hat{f}(x)|^2\mathscr{I}_\varepsilon(x)dx,$$

其中

$$\mathscr{I}_\varepsilon(x)=4\int_{|t|>\varepsilon}\frac{|1-\cos 2\pi x\cdot t|^2}{|t|^{n+2}}dt.$$

显然，当$\varepsilon\to 0$时，$\mathscr{I}_\varepsilon(x)$单调上升趋向于

$$\mathscr{I}(x)=4\int_{R^n}\frac{|1-\cos 2\pi x\cdot t|^2}{|t|^{n+2}}dt.$$

它对$x$来说，显然是径向的并且为二次齐次。因此

$$\mathscr{I}(x)=b_n|x|^2,$$

其中
$$b_n=4\int_{R^n}\frac{|1-\cos 2\pi t|^2}{|t|^{n+2}}dt.$$

故(31)的左边是

$$b_n\int_{R^n}|x|^2|\hat{f}(x)|^2dx,$$

而右边是

$$a_n 4\pi^2\int_{R^n}|x|^2|\hat{f}(x)|^2dx.$$

只要取$a_n=b_n/4\pi$，命题便得到证明。读者可把这命题与第五章§3.5的命题5比较一下。

**5.3** 一个迅速的反映是，为了证明定理及其推论，不妨假设$f$在一有界集外为零。我们就对$f$作这样的假设。假设$f$在集$E$的每一个点有$L^2$微商。由于这蕴含了$f$在每点有调和微商，从§2.2的定理2我们可以分解$f=g+b$，其中$g\in L_1^\infty(R^n)$，而在$F$有$b=0$，$F\subset E$，$m(E-F)$很小。我们不妨假设$g$(因而$b$)在一有界集外也为零。由于$g$有有界支集，且$g\in L_1^\infty(R^n)$，用上面的命题也就有

$$\int_{R^n}\frac{|g(x^0+t)+g(x^0-t)-2g(x^0)|^2}{|t|^{n+2}}dt<\infty \tag{32}$$

对几乎所有的$x^0\in R^n$。

进一步，$g$在$R^n$中几乎所有的点有通常微商(见定理1)，因此$b$对$E$中从而对$F$中几乎所有的点有$L^2$微商。由于$b$在$F$为零，由§3.2的(17)我们有

$$\int_{|t|<1} \frac{|b(x^0+t)|^2}{|t|^{n+2}}dt<\infty$$

对$F$中几乎所有的$x^0$成立。又由于永远有$b\in L^2(\boldsymbol{R}^n)$，我们看出，对这样的$x^0$，

$$\int_{\boldsymbol{R}^n} \frac{|b(x^0+t)|^2}{|t|^{n+2}}dt<\infty. \tag{33}$$

把这同$b(x^0)=0$以及式(32)联系起来，我们就得到定理的结论(b)。

**5.4** 现在我们来证明定理6的反过来那部分。设

$$u(x,y)=P_y*f$$

是$f$的Poisson积分，并考虑$\frac{\partial^2 u}{\partial y^2}$。为此我们需要$\frac{\partial^2 P_y}{\partial y^2}$的一个合适的估计。由于

$$\frac{\partial^2 P_y}{\partial y^2}=-\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}P_y(x),$$

只要注意

$$\left|\frac{\partial^2}{\partial x_j^2}P_y(x)\right|\leqslant y^{-n-2}\psi\left(\frac{x}{y}\right),$$

其中$\psi(x)\leqslant A(1+|x|)^{-n-3}$。我们还指出

$$\int_{\boldsymbol{R}^n}\frac{\partial^2}{\partial y^2}P_y(x)dx=0,$$

以及决定性的，$\frac{\partial^2}{\partial y^2}P_y(x)$对$x$是径向的，因此是偶函数。注意到这些事实，便有

$$\frac{\partial^2 u}{\partial y^2}=\frac{1}{2}\int_{\boldsymbol{R}^n}\frac{\partial^2}{\partial y^2}P_y(t)[f(x+t)+f(x-t)-2f(x)]dt, \tag{34}$$

从而

$$(35)\qquad \left|\frac{\partial^2 u}{\partial y^2}\right| \leqslant A' y^{-n-2}\int_{|t|<y}|\Delta_t|dt$$

$$+A' y\int_{|t|>y}\frac{|\Delta_t|}{|t|^{n+3}}dt = I_1(y)+I_2(y),$$

其中我们记 $\Delta_t = f(x+t)+f(x-t)-2f(x)$。我们通过 $I_1$和 $I_2$的类似积分来估计

$$\int_0^\infty y\left|\frac{\partial^2 u}{\partial y^2}\right|^2 dy.$$

用 Schwarz 不等式

$$|I_1(y)|^2 \leqslant (A')^2 y^{-2n-4}\left(\int_{|t|<y}\frac{|\Delta_t|^2}{|t|^{n+1}}dt\right)$$

$$\times\left(\int_{|t|<y}|t|^{n+1}dt\right)$$

$$= By^{-3}\int_{|t|<y}\frac{|\Delta_t|^2}{|t|^{n+1}}dt.$$

因此

$$\int_0^\infty y|I_1(y)|^2 dy \leqslant B\int_0^\infty y^{-2}\int_{|t|<y}\frac{|\Delta_t|^2}{|t|^{n+1}}dt dy$$

$$= B\int_{R^n}\frac{|\Delta_t|^2}{|t|^{n+2}}dt.$$

对 $I_2$有类似的推理，合起来就得到

$$(36)\qquad \int_0^\infty y\left|\frac{\partial^2 u}{\partial y^2}\right|^2 dy$$

$$\leqslant B'\int_{R^n}\frac{|f(x+t)+f(x-t)-2f(x)|^2}{|t|^{n+2}}dt.$$

注意到$\dfrac{\partial^2 P}{\partial y^2}$是偶函数，我们还可以修改(34)，并把它写成更广的形式

$$(34') \quad \frac{\partial^2}{\partial y^2}u(x+\tau,y)+\frac{\partial^2}{\partial y^2}u(x-\tau,y)$$

$$=\int_{R^n}\frac{\partial^2}{\partial y^2}P_y(t+\tau)[f(x+t)+f(x-t)-2f(x)]dt,$$

其中 $\tau$ 是任意的。现在，如果 $|\tau|\leqslant y$，那末我们就可以对 $\frac{\partial^2}{\partial y^2}P_y(t+\tau)$ 得到类似的估计，从而

$$(35') \qquad \left|\frac{\partial^2}{\partial y^2}u(x+\tau,y)+\frac{\partial^2}{\partial y^2}u(x-\tau,y)\right|$$

$$\leqslant A\{I_1(y)+I_2(y)\}, \qquad \text{当} |\tau|\leqslant y.$$

用同样的推理，代替(36)我们就得到

$$(36') \quad \iint_{|\tau|<y}\left|\frac{\partial^2}{\partial y^2}u(x+\tau,y)+\frac{\partial^2}{\partial y^2}u(x-\tau,y)\right|^2 y^{1-n}d\tau dy$$

$$\leqslant B\int_{R^n}\frac{|f(x+t)+f(x-t)-2f(x)|^2}{|t|^{n+2}}dt.$$

现在，我们对使定理 6 的条件(b)成立的每一点运用 § 4.1的对称化原理。这里我们令

$$U(x,y)=\frac{\partial^2}{\partial y^2}u(x,y),$$

而当$y>1$时，$U=0$。结果就是，对几乎所有这样的 $x^0$

$$(37) \qquad \iint_{|\tau|<y,\ y<2}\left|\frac{\partial^2}{\partial y^2}u(x^0+\tau)\right|^2 y^{1-n}d\tau dy<\infty,$$

根据上一章的理论(见 § 2.5，特别是第271页)，$\frac{\partial u}{\partial y}$也就对几乎所有这样的 $x^0$有非切线极限；而最后，由于第七章的 § 2.5，$u$ 也就对几乎所有使条件(b)成立的点有调和微商。

我们现在可以分解 $f$ 为$g+b$，其中 $g$ 属于 $L_1^\infty(\boldsymbol{R}^n)$，并且根据上面所说就知道，代替 $f$，对 $g$ 来说积分的条件几乎处处成

立，从而对 $b$ 来说在几乎所有合适的点也成立；但 $b$ 在这些点为零，总起来

$$(38)\qquad \int_{R^n}\frac{|b(x^0+t)+b(x^0-t)|^2}{|t|^{n+2}}dt<\infty,$$

$$b(x^0)=0,\ x^0\in F,$$

其中 $F\subset E$，而 $m(E-F)$ 很小。

现在再一次应用对称化原理，这时取 $U(x,y)=y^{-2}b(x)$，当 $y<1$。事实上就有

$$\int_0^\infty y|U(x^0,y)|^2dy=0,\qquad x^0\in F$$

而

$$\iint_{|t|<y} y^{1-n}|U(x^0+t,y)+U(x^0-t,y)|^2dtdy<\infty,$$

这是(38)的一个简单推论。结果就是对 $F$ 中几乎所有的 $x^0$，有

$$\int_{|t|<1}\frac{|b(x^0+t)|^2}{|t|^{n+2}}dt<\infty,$$

而在这样的点，显然

$$\int_{|t|\leqslant r}|b(x^0+t)|^2dt=o(r^{n+2}),\qquad \text{当} r\to 0,$$

特别地这就意味着，$b$ 有 $L^2$ 微商。反过来这部分也就证明了。推论是定理与§4的4.2.3中推论的一个直接结果。

## §6 进一步的结果

6.1 本章的大多数结果对高阶微商都有类似的命题。合适的定义如下。设 $k$ 是整数，$k\geqslant 1$，我们说 $f$ 在 $x^0$ 有 **$k$ 阶通常微商**，如果存在阶数 $\leqslant k$ 的 $y$ 的多项式 $P_{x^0}(y)$，使得

$$f(x^0+y)-P_{x^0}(y)=o(|y|^k),\quad y\to 0.$$

类似地，我们说 $f$ 有 **$k$ 阶 $L^q$ 微商**，如果

$$h^{-n}\int_{|y|<h}|f(x^0+y)-P_{x^0}(y)|^q dy=o(h^{kq}),\ h\to 0.$$

最后，如果我们假设 $f$ 在 $x^0$附近可积，并且在 $x^0$ 的一个邻域之外为零，

$$\left\{\frac{\partial^\alpha}{\partial x^\alpha}u(x,y)\right\}_{|\alpha|<k}$$

在$x^0$都有非切线极限，那末称 $f$ 在 $x^0$有 $k$ 阶调和微商。

定理1的推广是下述命题：若$1<p\leqslant\infty$，$f\in L_k^p(R^n)$，则当$p>n/k$时，$f$ 在$R^n$几乎处处有 $k$ 阶通常微商；当$p<n/k$时，$f$ 在几乎所有的点有 $k$ 阶 $L^q$微商，其中$1/q=1/p-k/n$。定理2的推广是分解 $f=g+b$，其中 $g\in L_k^\infty(R^n)$，$b$ 在 $F$ 为零，$F\subset E$，$m(E-F)$很小，而 $f$ 假设在 $E$ 的每一点有 $k$ 阶调和微商。为了推广定理3，只要假设对每个$x^0\in E$，存在阶数$\leqslant k-1$的 $y$ 的多项式$\bar{P}_{x^0}(y)$，使得

$$f(x+y)-\bar{P}_{x^0}(y)=O(|y|^k),\qquad 当\ y\to 0,$$

则 $f$ 在几乎所有的 $x^0\in E$ 有 $k$ 阶通常微商。

对定理4中核$K(x)$的条件(b)，必须扩充为

$$|\nabla^r K(x)|\leqslant A/|x|^{n+r},\qquad 0\leqslant r\leqslant k;$$

这样修改后，问题中的奇异积分也几乎处处保持 $k$ 阶 $L^q$可微性。

最后，若 $f\in L^2(R^n)$，则刻划 $k$ 阶 $L^2$微商存在的条件由

$$\int_{R^n}\frac{|\Delta_x^{k\,0}(t)|^2}{|t|^{n+2k}}dt$$

的几乎处处有限给出，其中 $f(x^0+t)-P_{x^0}(t)=R_{x^0}(t)$，而

$$\Delta_x^{k\,0}(t)=R_{x^0}(t)+(-1)^{k-1}R_{x^0}(-t),$$

上述这些见 Calderón 与 Zygmund[7]，Stein 与 Zygmund[1]，以及Stein[8]。

**6.2** 定理1的结果对 $p=1$ 也成立，但要求一个不同的推理(试比较第五章§2.5中的不等式与同一章§2.3的等式(18))。上述一个进一步的推论是下面的定理。若 $f$ 在 $R^n$ 是Tonelli 有界变差的，则对几乎所有的点 $f$ 有 $L^q$ 微商，其中

$$q=n/(n-1).$$

见 Calderón 与 Zygmund[6]。

6.3 （§2.2的）分解定理可以有某个更好的形式；只要我们假设 $f$ 是 $L^q$ 可微的。我们写出一阶微商的结果。设 $f\in L^q(\boldsymbol{R}^n)$，$1\leqslant q$，并且对每个 $x^0\in F$，$F$ 是紧集，有

$$h^{-n}\int_{|y|<h}|f(x^0+y)-f(x^0)|^q dy\leqslant Ah^q,\qquad 0<h<\infty,$$

其中 $A$ 与 $x^0$ 无关。则 $f=g+b$，其中 $g$ 连续可微，并且它以及它的微商都有界；$b$ 在 $F$ 为零。对高阶微商的一般说法以及更多的细节，见 Calderón 与 Zygmund[8]。

6.4 鉴于在§3 的 3.1.1中讨论过的可测性困难，叙述下面的定理可能是令人感兴趣的。设 $f$ 在 $\boldsymbol{R}^n$ 是 Lebesgue 可测的，则 $f$ 有通常微商的点集是 Lebesgue 可测的（见Haslam-Jones[1]）。

集合 $E=\left\{x:\ \lim\limits_{h\to 0}\sup\left|\dfrac{f(x+h)-f(x)}{h}\right|<\infty\right\}$ 是Lebesgue可测这一点，可以这样看：对每个整数 $k$，令

$$E_k=\{x:\ |f(x+h)-f(x)|\leqslant k|h|,\ \text{对所有的}|h|<1/k\}.$$

则 $E=\bigcup E_k$；还由于我们取了开球 $|h|<1/k$，因此 $f|_{E_k}$ 是连续的而且 $E_k$ 是闭的①。

6.5 假设 $k$ 是奇数。我们说 $f$ 在 $x^0$ 有 $k$ 阶 $L^q$ 对称微商，如果存在多项式 $P_{x^0}(y)$，对 $y$ 的阶数 $\leqslant k$，使得

$$h^{-n}\int_{|y|<h}|f(x^0+y)-f(x^0-y)-P_{x^0}(y)|^q dy=o(h^{kq}),$$

当 $y\to 0$。当 $k$ 是偶数时，我们有相同的定义，这只要用 $f(x^0+y)+f(x^0-y)$ 代替 $f(x^0+y)-f(x^0-y)$。

**定理** 假设对每个 $x^0\in E$，$f$ 有 $k$ 阶 $L^q$ 对称微商，则对 $E$ 中几乎所有的 $x^0$，$f$ 有 $k$ 阶 $L^q$ 微商。

对 $n=1$，$k=1$，以及通常的可微概念，这定理应追溯回

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① 在这里我得到过 H.Federer 的帮助。

Khintchine[1]。对一维的形式见 M. Weiss[1]。一般的情形可用本章的方法得到。例如设 $k=1$。令 $u$ 是 $f$ 的 Poisson 积分并考虑

$$\frac{\partial u}{\partial x_j}=\frac{\partial P_y}{\partial x_j}*f.$$

由于核 $\frac{\partial P}{\partial x_j}$ 是奇的，我们可以写出

$$\frac{\partial}{\partial x_j}u(x+\tau,y)+\frac{\partial}{\partial x_j}u(x-\tau,y)$$

$$=\int_{\mathbf{R}^n}\frac{\partial}{\partial x_j}P_y(t+\tau)[f(x+t)-f(x-t)]dt,$$

而我们的假设表明，

$$\frac{\partial}{\partial x_j}u(x^0+\tau,y)+\frac{\partial}{\partial x_j}u(x^0-\tau,y)$$

对 $x^0\in E$ 当 $|\tau|\leqslant y$ 与 $y\to 0$ 时是有极限的，因此由 § 4的4.2.2中对称化定理知，$\frac{\partial u}{\partial x_j}$ 对几乎所有的 $x^0\in E$，在 $x^0$ 是非切线有界的。故 $f$ 在 $E$ 几乎处处有一阶调和微商，从而用分解引理把问题归结为 $f$ 在 $E$ 为零的特殊情形。

更一般地，当 $k$ 是偶数时，考虑

$$\frac{\partial^k}{\partial y^k}u(x+\tau,y)+\frac{\partial^k}{\partial y^k}u(x-\tau,y);$$

当 $k$ 是奇数时，取

$$\frac{\partial^k}{\partial y^{k-1}\partial x_j}u(x+\tau,y)+\frac{\partial^k}{\partial y^{k-1}\partial x_j}u(x-\tau,y),$$

$$j=1,\cdots,n;$$

应用第七章的结果表明，$f$ 在 $E$ 几乎处处有 $k$ 阶调和微商，这就把问题归结为 $f$ 在 $E$ 为零的特殊情形。对这个十分容易的特殊情形，可以应用 § 4 用过的那种推理；例如见 Stein 与 Zygmund

[1,引理14]。

6.6 设$f$在每个$x^0\in E$有$L^q$微商，则$f$的一阶$L^q$偏微商对几乎所有的$x^0\in E$存在。特别地，若$x^0\in E$有

$$\frac{1}{h^n}\int_{|y|<h}|f(x^0+y)-f(x^0)-\sum a_{x^0}^j y_j|^q dy=o(h^q),\quad h\to 0,$$

则当$e_j$是单位向量$(0,\cdots,1,0,\cdots,0)$时，对几乎所有的$x^0\in E$，有

$$\frac{1}{h}\int_{|y_j|<h}|f(x^0+e_jy_j)-f(x^0)-a_{x^0}^j y_j|^q dy_j=o(h^q).$$

有关的结果见 M.Weiss[2]与[3]。

6.7 设$2\leqslant q<\infty$，$f\in L^q(\boldsymbol{R}^n)$，则$f$对几乎所有的$x^0\in E$有一阶$L^q$微商，当且仅当下列两个条件对几乎所有的$x^0\in E$成立：

(a) $$\int_{\boldsymbol{R}^n}\frac{|f(x^0+t)+f(x^0-t)-2f(x^0)|^q}{|t|^{n+q}}dt<\infty;$$

(b) $$\int_{\boldsymbol{R}^n}\frac{|f(x^0+t)+f(x^0-t)-2f(x^0)|^2}{|t|^{n+2}}dt<\infty.$$

见 Stein 与 Zygmund[1]；还可见Wheeden[1], Neugebauer[1]。

6.8 下述例子表明，§3.3的定理4不能推广到通常微商（即$q=\infty$）。我们考虑$\boldsymbol{R}^1$的情形，$T$是 Hilbert 变换。

首先考虑定义在$\boldsymbol{R}^1$的函数$F_0(x)$，具有性质：

$$F_0(x)=\left(\log\frac{1}{|x|}\right)^{1-\varepsilon},\quad 当|x|<\frac{1}{2}\ (0<\varepsilon<1);$$

$F_0(x)$在一紧集外为零，并且在原点外是光滑的；还有$F_0(x)\geqslant 0$，$x\in\boldsymbol{R}^1$。用$\tilde{F}_0(x)$表示$F_0$的 Hilbert 变换。不难看出$\tilde{F}_0$（在一零测集上作适当修改后）是在$\boldsymbol{R}^1$绝对连续的，并且$\frac{d\tilde{F}_0}{dx}\in L^1(\boldsymbol{R})$。记$F(x)=\sum 2^{-k}F_0(x+r_k)$，其中$r_1,r_2,r_3,\cdots$是有理数。这时$\tilde{F}=$

$\sum 2^{-k}\tilde{F}_0(x+r_k)$是$F$的 Hilbert 变换，并且是绝对连续的，$\frac{d\tilde{F}}{dx}\in L^1(\boldsymbol{R}^1)$。因此$\tilde{F}$对几乎所有的$x$有通常微商，但由于$F$在每一点附近是无界的，因此它处处沒有通常微商。

对$F_0$的周期类似，我们取$f_0$:

$$f_0\sim\sum_{n>1}\frac{\cos nx}{n(\log n)^{\varepsilon}},\qquad \tilde{f}_0(x)\sim\sum_{n>1}\frac{\sin nx}{n(\log n)^{\varepsilon}}.$$

关于这些级数的研究见 Zygmund[8,第V章]。

## 注　释

**节1**　在一已知点的$L^p$可微概念最早在 Calderón 与 Zygmund[1]中得到系统研究。定理1的(b)部分属于他们，涉及通常微商的(a)部分却是更早的；例如见 Cesari[1]，读者还可以阅读 Federer[1]，其中包括本章有关材料的各种专题。

**节2**　分解定理的思想，涉及到单变量的通常微商，最早发表在 Marcinkiewicz[1]。这个基本技巧在 Calderón 与 Zygmund[7]中被推广到了多元；这里基于调和函数理论的叙述，比起原来的方法有了很多本质的改进。这是属于 Zygmund 与作者的，并且概述在作者的综合报告 Stein[8]中。

**节3**　定理3是 Denjoy，Rademacher 和 Stepanov 的著名定理；见 Saks[2,第Ⅸ章]。

这里给出的证明，当然不是标准的，它基于调和微商的概念。对§3.2以及定理4的变形，见 Calderón 与 Zygmund[7]。

**节5**　定理6及其推论的原来证明是 Stein 与 Zygmund[1]的；还可见 Wheeden[1]。这里给出的推理也属于 Zygmund 与作者，并概述在 Stein[8]中。

# 附　录

## A. 若干不等式

在这里我们汇总前面系统地用过的若干不等式。更多的细节可在 Zygmund[1]第 I 章 与 Hardy, Littlewood, Polya[1]中 找到。

**A.1** 积分的 Minkowski 不等式实际上说的是，积分的模不大于对应的模的积分。在 $L^p$ 空间的情形，可以用明白的形式改述如下。设$1\leqslant p<\infty$，则

$$\left(\int_{\mathscr{Y}}\left(\int_{\mathscr{X}}|F(x,y)|dx\right)^p dy\right)^{1/p}\leqslant\int_{\mathscr{X}}\left(\int_{\mathscr{Y}}|F(x,y)|^p dy\right)^{1/p}dx.$$

这里 $F(x,y)$在 $\sigma$ 有限的乘积测度空间$\mathscr{X}\times\mathscr{Y}$是可测的；$dx$ 与 $dy$ 分别表示$\mathscr{X}$与$\mathscr{Y}$上的测度。

**A.2** 关于卷积的 Young 不等式如下：设 $h=f*g$，则

$$\|h\|_q\leqslant\|f\|_p\|g\|_r,$$

其中$1\leqslant p,q,r\leqslant\infty$，满足$1/q=1/p+1/r-1$。

两个值得指出的特殊情形是，一个是 $r=1$，这时 $p=q$；另一个是 $r$ 为 $p$ 的共轭指标（即$1/p+1/r=1$），这 时 $q=\infty$。在后一种情形，可以证明 $h$ 是连续的。

**A.3** 下面是有广泛应用的一个一般的积分不等式。

设

$$T(f)(x)=\int_0^\infty K(x,y)f(y)dy,$$

其中假设$K$是$-1$次齐次的，即 $K(\lambda x,\lambda y)=\lambda^{-1}K(x,y)$，当 $\lambda>0$。如果还假设

$$\int_0^\infty |K(1,y)|y^{-1/p}dy = A_K < \infty,$$

那末

$$\|T(f)\|_p \leqslant A_K \|f\|_p,$$

其中模$\|\cdot\|_p$是指$L^p((0,\infty);dx)$中的模，$1\leqslant p\leqslant\infty$。

为了证明上式，记

$$T(f)(x) = \int_0^\infty K(1,y)f(yx)dy,$$

并用积分的 Minkowski 不等式。

一个有兴趣的特殊情形是 Hilbert 积分，这时

$$K(x,y) = 1/(x+y)。$$

**A.4** A.3 的其它的有用例子是 Hardy 的不等式对：

$$\left(\int_0^\infty\left(\int_0^x f(y)dy\right)^p x^{-r-1}dx\right)^{1/p} \leqslant \frac{p}{r}\left(\int_0^\infty (yf(y))^p y^{-r-1}dy\right)^{1/p},$$

$$\left(\int_0^\infty\left(\int_x^\infty f(y)dy\right)^p x^{r-1}dx\right)^{1/p} \leqslant \frac{p}{r}\left(\int_0^\infty (yf(y))^p y^{r-1}dy\right)^{1/p},$$

其中 $f\geqslant 0$，$p\geqslant 1$与 $r>0$。

## B. Marcinkiewicz内插定理

**B.1** 我们现在推广第一章 § 4 的定理。我们假设 $p_0, p_1, q_0, q_1$是给定的指数，满足 $1\leqslant p_i\leqslant q_i\leqslant\infty$，$p_0<p_1$，而 $q_0\neq q_1$。$T$ 是定义在 $L^{p_0}(\boldsymbol{R}^n)+L^{p_1}(\boldsymbol{R}^n)$上的次可加变换。我们回忆$T$是弱$(p_i, q_i)$型的定义。这是指存在常数 $A_i$，使得对每个 $f\in L^{p_i}(\boldsymbol{R}^n)$

$$m\{x:\ |T(f)(x)|>\alpha\}\leqslant\left(\frac{A_i\|f\|_{p_i}}{\alpha}\right)^{q_i},\qquad \text{对一切 } \alpha>0。$$

当 $q_i=\infty$时，这是指$\|T(f)\|_{q_i}\leqslant A_i\|f\|_{p_i}$。

**定理** 假设$T$同时是弱$(p_0, q_0)$型与弱$(p_1, q_1)$型的，如果

$$\frac{1}{p}=(1-\theta)\frac{1}{p_0}+\theta\frac{1}{p_1}, \quad \frac{1}{q}=(1-\theta)\frac{1}{q_0}+\theta\frac{1}{q_1},$$

$$0<\theta<1.$$

那末$T$是$(p,q)$型的，也就是

$$\|T(f)\|_q\leqslant A\|f\|_p, \quad f\in L^p(\boldsymbol{R}^n),$$

其中$A=A(A_i,p_i,q_i,\theta)$，但与$T$及$f$无关。

从后面给出的证明容易看出，定理可以推广到下面的情形：首先，定义$L^{p_i}(\boldsymbol{R}^n)$的测度空间$\boldsymbol{R}^n$可以代之以一般的测度空间（而且出现在$T$的定义域中的测度空间不一定与出现在$T$的值域中的相同）。其次，次可加条件可以代之以更一般的条件

$$|T(f_1+f_2)(x)|\leqslant K\{|T(f_1)(x)|+|T(f_2)(x)|\}.$$

定理的一个不太显然的推广可以通过 Lorentz 空间给出，这种空间是把通常的$L^p$与弱型空间统一与推广了。关于 Marcinkiewicz 内插定理这种更一般形式的推论，见《富里叶分析》第V章。

**B.2** 假设$h$是$\boldsymbol{R}^n$上的可测函数。我们曾经用过它的分布函数$\lambda(\alpha)$的概念，其定义为$\lambda(\alpha)=m\{x:|h(x)|>\alpha\}$，其中$m$是$\boldsymbol{R}^n$的 Lebesgue 测度。为了证明上面的定理，我们考虑$h$的非增重排函数，它在别的地方也是很有用的。这就是函数$h^*$，它定义在$(0,\infty)$上，同$h$有相同分布函数，只是它在$(0,\infty)$是非降的。$h^*$可以定义为$h^*(t)=\inf\{\alpha,\lambda(\alpha)\leqslant t\}$。$h^*$与$\lambda$都是非负的非增函数，而且是右连续的。$h^*$与$h$有相同的分布函数，是由于使$h^*(t)>\alpha$的集合就是区间$0\leqslant t<\lambda(\alpha)$，它的测度自然是$\lambda(\alpha)$。因此

$$\|h^*\|_p=\left(\int_0^\infty|h^*(t)|^p dt\right)^{1/p}=\left(\int_{\boldsymbol{R}^n}|h(x)|^p dx\right)^{1/p}=\|h\|_p.$$

**B.3** 我们还要用到某些定义在$(0,\infty)$上的函数的积分不等式。

首先是附录A.4给出的 Hardy 不等式对。其次还需要

$$\left(\int_0^\infty [t^{1/p}h(t)]^{q_2}\frac{dt}{t}\right)^{1/q_2}\leqslant A\left(\int_0^\infty [t^{1/p}h(t)]^{q_1}\frac{dt}{t}\right)^{1/q_1},$$

只要 $h$ 是 $(0,\infty)$ 的非负非增函数，$0<p\leqslant\infty$，而 $q_1\leqslant q_2\leqslant\infty$。$A=A(p,q_1,q_2)$ 与 $h$ 无关。为了证明这个不等式，假设右边的积分等于1。只在 $t/2$ 与 $t$ 之间进行积分（同时注意 $h$ 在这个区间中大于等于 $h(t)$），我们得到 $\sup\limits_{0<t} t^{1/p}h(t)\leqslant A_1$。这就是 $q_2=\infty$ 时所要证的结果。对 $q_1\leqslant q_2<\infty$ 的一般结果，可如下推出

$$\int_0^\infty [t^{1/p}h(t)]^{q_2}\frac{dt}{t}\leqslant\sup_t[t^{1/p}h(t)]^{q_2-q_1}\int_0^\infty [t^{1/p}h(t)]^{q_1}\frac{dt}{t}.$$

**B.4** 我们现在来证明 Marcinkiewicz 定理。用 $\sigma$ 表示 $\boldsymbol{R}^2$ 上连接点 $(1/p_0,1/q_0)$ 与 $(1/p_1,1/q_1)$ 的线段的斜率。由于 $(1/p,1/q)$ 在线段上，我们有

$$\sigma=\frac{1/q_0-1/q}{1/p_0-1/p}=\frac{1/q-1/q_1}{1/p-1/p_1}.$$

对任意 $t>0$，我们分解任意函数 $f\in L^p(\boldsymbol{R}^n)$ 如下：

$$f=f^t+f_t,$$

其中 $f^t(x)=f(x)$，如果 $|f(x)|>f^*(t^\sigma)$；而 $f^t(x)=0$ 在其它地方；$f_t=f-f^t$。

容易推出

$$(f^t)^*(y)\leqslant f^*(y),\qquad 0\leqslant y\leqslant t^\sigma;$$
$$(f^t)^*(y)=0,\qquad y>t^\sigma.$$

而

$$(f_t)^*(y)\leqslant f^*(t^\sigma),\qquad y\leqslant t^\sigma,$$
$$(f_t)^*(y)\leqslant f^*(y),\qquad y>t^\sigma.$$

现在，当 $f=f_1+f_2$，容易验证

$$(T(f))^*(t)\leqslant(T(f_1))^*(t/2)+(T(f_2))^*(t/2).$$

关于 $T$ 是弱 $(p_0,q_0)$ 型的假定，蕴含了当 $f\in L^{p_0}(\boldsymbol{R}^n)$，就有

$$(T(f))^*(t)\leqslant A_0t^{-1/q_0}\|f\|_{p_0}.$$

类似地，若 $f\in L^{p_1}(R^n)$，则 $(T(f))^*(t)\leqslant A_1 t^{-1/q_1}\|f\|_{p_1}$。我们现在用分解 $f=f^t+f_t$，$f\in L^p(R^n)$，由于 $p_0<p<p_1$，$f^t\in L^{p_0}(R^n)$ 而 $f_t\in L^{p_1}(R^n)$。把这代入上式就给出

$$(T(f))^*(t)\leqslant A_0\left(\frac{2}{t}\right)^{1/q_0}\|f^t\|_{p_0}+A_1\left(\frac{2}{t}\right)^{1/q_1}\|f_t\|_{p_1}.$$

然而 $\|T(f)\|_q=\|(T(f))^*\|_q$，根据 B.3的不等式后者被

$$\int_0^\infty (t^{1/q}(T(f))^*(t))^p\frac{dt}{t}$$

的常数倍所控制，因为 $p\leqslant q$，这是由于 $p_i\leqslant q_i$。

对 $(T(f))^*$ 应用上面的估计，容易把控制 $\|T(f)\|_q$ 的积分归结为估计

(1)
$$\left\{\int_0^\infty[t^{(1/q)-1/q_0}\|f^t\|_{p_0}]^p\frac{dt}{t}\right\}^{1/p}$$
$$+\left\{\int_0^\infty[t^{(1/q)-1/q_1}\|f_t\|_{p_1}]^p\frac{dt}{t}\right\}^{1/p}.$$

注意到 $(f^t)^*(y)\leqslant f^*(y)$，当 $y\leqslant t^\sigma$；而 $(f^t)^*(y)=0$，当 $y>t^\sigma$，我们有

$$\begin{aligned}\|f^t\|_{p_0}&=\left(\int_0^\infty(y^{1/p_0}(f^t)^*(y))^{p_0}\frac{dy}{y}\right)^{1/p_0}\\&\leqslant c\int_0^\infty y^{1/p_0}(f^t)^*(y)\frac{dy}{y}\\&\leqslant c\int_0^{t^\sigma}y^{1/p_0}f^*(y)\frac{dy}{y}.\end{aligned}$$

把 $\|f^t\|_{p_0}$ 的这个估计代入上面式(1)中的第一个括号里。在作变量替换($t^\sigma\to t$)以后，应用第一个 Hardy 不等式(见附录 A.4)，我们看到，(1)的第一项被 $\|f\|_p$ 的常数倍控制。对第二项进行类似的论证便完成了定理的证明。

## C. 调和函数的某些初等性质①

**C.1** 极值原理的一个有用形式可叙述如下。假设$u$是有界区域$\mathscr{R}$内的$C^2$类实值函数，并设$u$在$\overline{\mathscr{R}}$连续。又设在$\mathscr{R}$内有$\Delta(u)\geqslant 0$。如果在$\mathscr{R}$的边界上$u\leqslant 0$，那末在整个$\mathscr{R}$都有$u\leqslant 0$。

为了证明这个结果，在$\mathscr{R}$内作更强的假设$\Delta u>0$是方便的。我们可以这样做，只要代替$u$，考虑$u+\varepsilon|x|^2-\delta$，其中$\varepsilon>0$，$\delta>0$，而$\varepsilon$与$\delta$同时都很小。现在我们设在$\mathscr{R}$内$\Delta u>0$。假设在$\mathscr{R}$内$u\leqslant 0$这个结论不成立，那末$u$在某个点$x^0\in\mathscr{R}$达到正的极大。由于$(\Delta u)(x^0)>0$，必定对某个$j$有

$$\frac{\partial^2 u}{\partial x_j^2}(x^0)>0.$$

由极值性质知

$$\frac{\partial u}{\partial x_j}(x^0)=0,$$

因而根据 Taylor 定理有

$$u(x^0+\xi e_j)-u(x^0)=\frac{\xi^2}{2}\frac{\partial^2 u}{\partial x_j^2}(x^0)+o(\xi^2),$$

其中$e_j$是沿$x_j$方向的单位向量而$\xi$很小。这与$u(x^0)$是$u$在$\mathscr{R}$的极大值是矛盾的。

**C.2** 假设$u$在$\mathscr{R}$调和，则

$$u(x^0)=\frac{1}{\omega_{n-1}}\int_{S^{n-1}}u(x^0+y'r)d\sigma(y'),$$

其中$d\sigma(y')$是$R^n$单位球面的面积元素；$\omega_{n-1}$是球的面积；$r$充分小，使得以$x^0$为中心，$r$为半径的球整个包含在$\mathscr{R}$内。这是调和函数的**平均值性质**。大家知道，这个事实可以从下面的 Green 公式推出

① 调和函数初等性质的讨论，还可以参看《富里叶分析》第Ⅱ章。

$$\int_{\mathscr{D}}(u\Delta v - v\Delta u)dx = \int_{\partial\mathscr{D}}\left(\frac{u\partial v}{\partial n} - \frac{v\partial u}{\partial n}\right)d\tau,$$

其中$\mathscr{D}$是以 $x^0$ 为中心，以 $r$ 为半径的球与以 $\varepsilon$ 为半径的球之间所包含的区域（$\varepsilon$ 很小）；$\partial\mathscr{D}$包括两个球的球面，$d\tau$ 是 $\partial\mathscr{D}$ 的面积元素，$\partial/\partial n$ 是外法向微商。取 $v(x) = |x - x^0|^{-n+2} - r^{-n+2}$，则在$\mathscr{D}$有 $\Delta v = 0$。令 $\varepsilon \to 0$，并项就给出所要的结果。

**C.3** 平均值性质与正则化过程直接导出调和函数的一种不同的有用估计。设 $\varphi$ 是 $\boldsymbol{R}^n$ 中固定的 $C^\infty$ 的径向函数，支在单位球内，并且是规一化了的，即

$$\int_{\boldsymbol{R}^n}\varphi(x)dx = 1.$$

通过积分我们得到平均值性质的一个直接推论

$$u(x^0) = \int_{\boldsymbol{R}^n}u(x^0 - y)\varphi_r(y)dy,$$

其中 $\varphi_r(y) = r^{-n}\varphi(y/r)$。

现在我们像以前那样，假设 $x_0$到$\mathscr{R}$的边界的距离超过 $r$。上式可以改写为

$$u(x^0) = \int_{\boldsymbol{R}^n}u(y)\varphi_r(x^0 - y)dy,$$

这样

$$\frac{\partial^\alpha}{\partial x^\alpha}u(x_0) = \int_{\boldsymbol{R}^n}u(y)\left(\frac{\partial}{\partial x^0}\right)^\alpha\varphi_r(x^0 - y)dy.$$

由此并用 Schwarz 不等式，我们得到 $n$ 元调和函数的下述不等式

$$\left|\frac{\partial^\alpha}{\partial x^\alpha}u(x^0)\right| \leqslant A_\alpha r^{-\frac{n}{2}-|\alpha|}\left(\int_{B_r}|u(y)|^2dy\right)^{1/2},$$

其中 $B_r$ 表示以 $x^0$为中心，$r$ 为半径的球。

**C.4** 我们现在来证明第三章§3 的3.1.5所作的断言，这就

是，若 $f$ 有球调和展开

$$f(x) \sim \sum_k Y_k(x),$$

而

$$\int_{S^{n-1}} |Y_k(x)|^2 d\sigma(x) = O(k^{-N}), \qquad k \to \infty \text{对每个} N,$$

则可以在一零测集上修改 $f$ 使之成为 $S^{n-1}$ 的连续函数。为了证明这点，只要验证不等式

$$(*) \qquad \sup_{|x|=1} \left| \frac{\partial^\alpha Y_k(x)}{\partial x^\alpha} \right| \leqslant A'_\alpha k^{(n/2+|\alpha|)} \times \left( \int_{S^{n-1}} |Y_k(x)|^2 d\sigma(x) \right)^{1/2}.$$

让我们规一化 $Y_k$，即设

$$\int_{S^{n-1}} |Y_k(x)|^2 d\sigma(x) = 1.$$

$Y_k$ 当然是零次齐次的。如果我们令 $P_k(x) = |x|^k Y_k(x)$，那末 $P_k$ 是 $k$ 阶球体调和函数。现在

$$\int_{|x| \leqslant 1+\varepsilon} |P_k(x)|^2 dx = \left( \int_{S^{n-1}} |Y_k(x)|^2 d\sigma(x) \right) \int_0^{1+\varepsilon} r^{2k+n-1} dr$$

$$\leqslant (1+\varepsilon)^{2k+n}.$$

我们用 C.3 的最后一个不等式，取 $u = P_k$，$x^0$ 是 $S^{n-1}$ 上的任意点，$B_r$ 是以 $x^0$ 为中心，$r$ 为半径的球。结果就是

$$\left| \frac{\partial^\alpha}{\partial x^\alpha} P_k(x^0) \right| \leqslant A_\alpha \varepsilon^{-n/2-|\alpha|} (1+\varepsilon)^{k+n/2}.$$

$\varepsilon$ 是任意的，我们因此可以取它等于 $1/k$，这样 $(1+\varepsilon)^{k+n/2} \leqslant$ 常数，从而我们有

$$(**)\qquad \sup_{|x|=1}\left|\frac{\partial^{\alpha}}{\partial x^{\alpha}}P_k(x)\right|\leqslant A''_{\alpha}k^{n/2+|\alpha|}.$$

最后，注意 $Y_k(x)=|x|^{-k}P_k(x)$，再用 Leibnitz 公式，就可由 (**)推得(*)。

## D．关于 Rademacher 函数的不等式

我们的目的是给出第五章 § 5.2关于 Rademacher 函数不等式的证明。还可参阅 Zygmund[8]第V章。

**D.1** 设 $\mu, a_0, a_1, \cdots, a_N$ 是实数。由于 Rademacher 函数是互相独立变化的，我们有

$$\int_0^1 \exp \mu\sum_{m=0}^{N} a_m r_m(t)\,dt=\prod_{m=0}^{N}\int_0^1 \exp \mu a_m r_m(t)\,dt.$$

然而注意它们的定义，

$$\int_0^1 e^{\mu a_m r_m(t)}dt=\cosh \mu a_m.$$

现在应用简单的不等式$\cosh x\leqslant e^{x^2}$，我们得到

$$\int_0^1 e^{\mu F(t)}dt\leqslant \exp \mu^2\sum_m a_m^2,$$

其中 $F(t)=\sum_m a_m r_m(t)$。

**D.2** 让我们做规一化假设 $\sum_{m=0}^{N}a_m^2=1$。由于 $e^{\mu|F|}\leqslant e^{\mu F}+e^{-\mu F}$，我们有

$$\int_0^1 e^{\mu|F(t)|}\,dt\leqslant 2e^{\mu^2}.$$

设 $\lambda(\alpha)=m\{t:|F(t)|>\alpha\}$是$|F|$的分布函数。在上述不等式取 $\mu=\alpha/2$，我们得到 $\lambda(\alpha)\leqslant 2e^{(\alpha/2)^2-(\alpha/2)\alpha}$，因此

$$\lambda(a) \leqslant 2e^{-a^2/4}.$$

由此直接推出

$$\left(\int_0^1 |F(t)|^p dt\right)^{1/p} = \|F\|_p \leqslant A_p \leqslant A p^{1/2}, \qquad p<\infty,$$

从而一般地有

$$\|F(t)\|_p \leqslant A_p \left(\sum_{m=0}^{N} |a_m|^2\right)^{1/2}, \qquad p<\infty.$$

**D.3** 我们现在把最后一个不等式推广到多元。两个变量的情形，对于用归纳法来证明一般情形，已经是够典型的了。

我们可以局限于 $p>2$ 的情形，因为在 $p\leqslant 2$ 时，所要证的不等式是 Hölder 不等式与 Redemacher 函数正交性的一个简单推论。

我们有

$$F(t_1, t_2) = \sum_0^N \sum_0^N a_{m_1 m_2} r_{m_1}(t_1) r_{m_2}(t_2) = \sum_0^N F_{m_1}(t_2) r_{m_1}(t_1).$$

现在，用刚刚证明过的

$$\int_0^1 |F(t_1, t_2)|^p dt_1 \leqslant A_p^p \left(\sum_{m_1} |F_{m_1}(t_2)|^2\right)^{p/2}.$$

对 $t_2$ 积分，并对函数序列用下面的事实

$$\int_0^1 \left(\sum_{m_1} |F_{m_1}(t_2)|^2\right)^{p/2} dt_2 \leqslant \left(\sum_{m_1} \left(\int_0^1 |F_{m_1}(t_2)|^p dt_2\right)^{2/p}\right)^{p/2}.$$

这只不过是对函数 $|F_{m_1}(t_2)|^2$ 与指标 $p/2$ 用 Minkowski 不等式(见附录 A.1) 而已。

然而 $F_{m_1}(t_2) = \sum_{m_2} a_{m_1 m_2} r_{m_2}(t_2)$，因此刚刚证明的结果表明

$$\left(\int_0^1 |F_{m_1}(t_2)|^p dt_2\right)^{2/p} \leqslant A_p^2 \sum_{m_2} a_{m_1 m_2}^2.$$

把它代入上面的式子就得到

$$\int_0^1\int_0^1 |F(t_1,t_2)|^p dt_1 dt_2 \leqslant A_p^{\prime p}\left(\sum_{m_2}\sum_{m_1} a_{m_1 m_2}^2\right)^{p/2}$$

它就是所要证的不等式

$$\|F\|_p \leqslant A_p' \|F\|_2, \qquad p<\infty.$$

**D.4** 反过来的不等式

$$\|F\|_2 \leqslant B_p \|F\|_p, \qquad p>0$$

是直接不等式的简单推论。

事实上，对任意 $p>0$（这里我们可假设 $p<2$），定义指标 $r>2$，使得指标 2 是 $p$ 与 $r$ 的中项（在合适的意义下），也就是说，取

$$\frac{1}{2}=\frac{1}{2}\left(\frac{1}{p}+\frac{1}{r}\right).$$

用 Hölder 不等式

$$\|F\|_2 \leqslant \|F\|_p^{1/2}\|F\|_r^{1/2}.$$

我们已知 $\|F\|_r \leqslant A_r' \|F\|_2$，$r>2$。因此，我们得到

$$\|F\|_2 \leqslant (A_r')^2 \|F\|_p,$$

这就是所要证的反过来的不等式。

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# 名 词 索 引

## 二 画



## 三 画



## 四 画



## 五 画



## 六 画



## 七 画



## 八 画



## 九 画



## 十 画



## 十 一 画

## 十二画



## 十三画



## 十四画



## 以拉丁字母开始的名词