Solutions to Linear Algebra Done Right

Third Edition

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- Linear Algebra 2nd Edition by Kenneth M Hoffman, Ray Kunze

Good Linear Algebra textbooks (not complete)

- Introduction to Linear Algebra, Fifth Edition by Gilbert Strang, Solution Manual
- Linear Algebra and Its Applications (5th Edition) by David C. Lay, Steven R. Lay, Judi J. McDonald
- Linear Algebra with Applications 9th Edition by Steven J. Leon
- Linear Algebra 3rd Edition by Serge Lang, Solution Manual
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essay writer reviews · 3 years ago
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Nadim Farhat · 3 years ago
Thank for your sharing the solutions.
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jackkinsella · 3 years ago
By releasing these solutions, you've greatly enhanced an already strong book's educational value. Thanks so much :)
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Nuno Alvares Pereira · 3 years ago
Thanks. I'm going to use this for self-study. I'm no longer having classes at univ, so it will be nice to have some help for learning this out of curiosity. Thanks once again. ;)
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Josué Nazul · 2 years ago
Gracias :)
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Fred · a year ago
I must be very tired, but how does 1.A:2 work? Not even Wolfram Alpha agrees. Nor to the square computation or cube=1...

What am I missing?
https://www.wolframalpha.co...

I got of to a horrible start trying to dust off my linear algebra skills, haha!
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> **Fred** → Fred · a year ago
> Wait a minute... That i is not under the radical sign! xD
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Robert Browning · 2 years ago
First of all, thank you. What a wonderful resource.
Second, I am studying for my qualifying exam and I am using Axler's book. Do you have a TeX file or pdf of all the solutions compiled?
Thanks again.
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Wu Jinyang · 2 years ago
The proof of 7.36 is awful. Took me quite a while to get the point.
In fact, It can be easier once you show that the eigenvectors of R (the square root) is also the eigenvectors of T (the eigenvalue of T is the square of the eigenvalue of R with respect to the same eigenvector), which is quite obvious.
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> **Mohammad Rashidi** Mod → Wu Jinyang · 2 years ago
> Sorry, I don't quite understand your logic. The problem asks to show the square root is unique. Hence there might have many different choices of square roots. For different square roots, the eigenvectors may also be different.
>
> The logic of the proof from the book is quite clear to me.
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>> **Wu Jinyang** → Mohammad Rashidi · 2 years ago
>> Let R be the positive square root of T.
>> Then from the definition of positive operators, R is self-adjoint. Therefore, V can have an orthonormal basis consisting of eigenvectors of R. let them be e1, ..., en.
>> let a1,..., an be the according eigenvalues.
>> Thus Rei = aiei for all i = 1,... n
>> then RRei = ai^2 * ei. since R is a square root of T.
>> T ei = ai^2 * ei. which shows that e1,... en are also eigenvectors of T.
>> Then T have e1,...,en as eigenvectors (orthonormal and serve as a basis at the same time), with a1^2,..., an^2 as eigenvalues.
>>
>> Assume that there exist another positive square root, let's call it R1. let v1,....,vn be a orthonormal basis consisting of eigenvectors. and b1,...bn be the according eigenvalues. By the same logic b1^n,....,bn^2 are eigenvalues of T.
>>
>> Thus the eigenspaces of R and R1 with respect to same eigenvalue is identical. (remember that positive operators have non negative eigenvalues, if b1^2 = a1^2 then we know for sure b1 = a1). and the eigenspace is orthogonal to each other (since they are spanned by distinct orthonormal vectors). Since the direct sum of eigenspace equal V. we can apply that trick like the "orthogonal projection" to show that Rv = R1v for all v of V.
>>
>> Thus we know that R and R1 are identical, only their presentation in the matrix form may be slightly different.
>>
>> What do you think?
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mhmd fthy · 2 years ago
god bless you man , you are awesome
Reply · Share ›

Ross · 2 years ago
This is immensely helpful to those of us who can't afford school and choose to self-study. Thank you SO MUCH.
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Chapter 1 Exercise B →

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Chapter 1 Exercise A

Posted on January 1, 2016 by Linearity

1.Solution: Because $(a+bi)(a-bi)=a^2+b^2$, one has

$$\frac{1}{a+bi}=\frac{a-bi}{a^2+b^2}.$$

Hence

$$c=\frac{a}{a^2+b^2}, d=-\frac{b}{a^2+b^2}.$$

---

2. Solution1:From direct computation, we have

$$\left(\frac{-1+\sqrt{3}i}{2}\right)^2=\frac{-1-\sqrt{3}i}{2},$$

hence

$$\left(\frac{-1+\sqrt{3}i}{2}\right)^3=\frac{-1-\sqrt{3}i}{2}\cdot\frac{-1+\sqrt{3}i}{2}=1.$$

This means $\frac{-1+\sqrt{3}i}{2}$ is a cube root of 1.

Solution2: Note that

$$(a+bi)+(a-bi)=2a$$

and

$$(a+bi)(a-bi)=a^2+b^2,$$

it follows that $\frac{-1+\sqrt{3}i}{2}$ is a root of the quadratic equation $x^2+x+1=0$.
For

$$\frac{-1+\sqrt{3}i}{2}+\frac{-1-\sqrt{3}i}{2}=-1$$

and

$$\frac{-1+\sqrt{3}i}{2}\frac{-1-\sqrt{3}i}{2}=1.$$

Because $x^3-1=(x-1)(x^2+x+1)$, we obtain the conclusion.

---

3. Solution: If we know that $i=e^{\pi i/2}$, then the square roots are

$$e^{\pi i/4} \quad\text{and}\quad e^{(\pi i/2+2\pi i)/2}=e^{5\pi i/4}.$$

Note that for any $x\in\mathbb{R}$, one has $e^{xi}=\cos x+i\sin x$. Then

$$e^{\pi i/4}=\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}=\frac{\sqrt{2}(1+i)}{2}$$

and

$$e^{5\pi i/4}=\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}=\frac{-\sqrt{2}(1+i)}{2}.$$

Hence the roots are $\frac{\sqrt{2}(1+i)}{2}$ and $-\frac{\sqrt{2}(1+i)}{2}$.

Remark: If we don't know this fact, then we should recall how to solve the roots of $x^8-1=0$ or $x^4+1=0$ since $x^2+i=0$ means $x^4+1=0$.

---

4. Solution: Let $\alpha=x+yi$ and $\beta=z+wi$, where $x,y,z,w\in\mathbb{R}$, then

$$\alpha+\beta=(x+yi)+(z+wi)=(x+z)+(y+w)i.$$

Similarly,

$$\beta+\alpha=(z+wi)+(x+yi)=(z+x)+(w+y)i.$$

Because $x+z=z+x$ and $y+w=w+y$, we obtain that $\alpha+\beta=\beta+\alpha$.

---

5. Solution: Let $\alpha=x_1+y_1i, \beta=x_2+y_2i, \lambda=x_3+y_3i$, where x_1,x_2,x_3 and y_1,y_2,y_3 are real numbers. Then

$$\begin{aligned}(\alpha+\beta)+\lambda=&((x_1+x_2)+(y_1+y_2)i)+(x_3+y_3i)\\=&((x_1+x_2)+x_3)+((y_1+y_2)+y_3)i.\end{aligned}$$

Similarly, $\alpha+(\beta+\lambda)=(x_1+(x_2+x_3))+(y_1+(y_2+y_3))i$. Note that

$$(x_1+x_2)+x_3=x_1+(x_2+x_3) \quad\text{and}\quad (y_1+y_2)+y_3=y_1+(y_2+y_3),$$

it follows that $(\alpha+\beta)+\lambda=\alpha+(\beta+\lambda)$.

---

6. Solution: Let $\alpha=x_1+y_1i, \beta=x_2+y_2i, \lambda=x_3+y_3i$, where x_1,x_2,x_3 and y_1,y_2,y_3 are real numbers. Then

$$\begin{aligned}(\alpha\beta)\lambda=&((x_1x_2-y_1y_2)+(x_1y_2+y_1x_2)i)(x_3+y_3i)\\=&((x_1x_2-y_1y_2)x_3-(x_1y_2+y_1x_2)y_3)\\&+((x_1x_2-y_1y_2)x_3+(x_1y_2+y_1x_2)y_3)i.\end{aligned}$$

Similarly, one has

$$\begin{aligned}\alpha(\beta\lambda)=&(x_1+y_1i)((x_2x_3-y_2y_3)+(x_2y_3+y_2x_3)i)\\=&(x_1(x_2x_3-y_2y_3)-y_1(x_2y_3+y_2x_3))\\&+(x_1(x_2y_3-y_2x_3)+y_1(x_2x_3+y_2y_3))i.\end{aligned}$$

It is easy to see

$$(x_1x_2-y_1y_2)x_3-(x_1y_2+y_1x_2)y_3=x_1(x_2x_3-y_2y_3)-y_1(x_2y_3+y_2x_3)$$

and

$$(x_1x_2-y_1y_2)x_3+(x_1y_2+y_1x_2)y_3=x_1(x_2x_3-y_2y_3)-y_1(x_2y_3+y_2x_3),$$

hence we deduce that $(\alpha\beta)\lambda=\alpha(\beta\lambda)$.

---

7. Solution: Let $\alpha=x_1+y_1i, \beta=x_2+y_2i$, where x_1,x_2 and y_1,y_2 are real numbers. If $\alpha+\beta=0$, then

$$0=\alpha+\beta=(x_1+x_2)+(y_1+y_2)i.$$

This means $x_2=-x_1$ and $y_2=-y_1$, which implied uniqueness. If $\beta=-x_1-y_1i$, we also have $\alpha+\beta=0$, which implies existence.

---

8. Solution: We already know the existence in Problem 1. Now let us show the uniqueness, if $\alpha\beta=1$, then

$$\beta=1\cdot\beta=\left(\frac{1}{\alpha}\cdot\alpha\right)\cdot\beta=\frac{1}{\alpha}\cdot(\alpha\cdot\beta)=\frac{1}{\alpha}\cdot 1=\frac{1}{\alpha}.$$

Here the third equality follows from Problem 6.

---

9. Solution: Suppose $\alpha=x_1+y_1i, \beta=x_2+y_2i, \lambda=a+bi$, where x_1,x_2,a and y_1,y_2,b are real numbers. Then

$$\begin{aligned}\lambda(\alpha+\beta)=&(a+bi)((x_1+x_2)+(y_1+y_2)i)\\=&(a(x_1+x_2)-b(y_1+y_2))+(a(y_1+y_2)+b(x_1+x_2))i\\=&[(ax_1-by_1)+(ay_1+bx_1)i]+[(ax_2-by_2)+(ay_2+bx_2)i]\\=&\lambda\alpha+\lambda\beta.\end{aligned}$$

---

10. Solution: Because $(4,-3,1,7)+2x=(5,9,-6,8)$, one has

$$2x=(5,9,-6,8)-(4,-3,1,7)=(1,12,-7,1),$$

hence

$$x=\frac{1}{2}(1,12,-7,1)=\left(\frac{1}{2},6,\frac{-7}{2},\frac{1}{2}\right).$$

---

11. Solution: If such $\lambda\in\mathbb{C}$ exists, then we have

$$\lambda(2-3i)=12-5i \quad\text{and}\quad \lambda(-6+7i)=-32-9i.$$

It follows that

$$(2-3i)(-32-9i)=(-6+7i)(12-5i),$$

this means

$$-91+78i=-37+114i,$$

which is impossible. Hence such $\lambda\in\mathbb{C}$ does not exist.

---

12. Solution: Suppose $x=(x_1,\cdots,x_n)$, $y=(y_1,\cdots,y_n)$ and $z=(z_1,\cdots,z_n)$. Then

$$\begin{aligned}(x+y)+z=&((x_1,\cdots,x_n)+(y_1,\cdots,y_n))+(z_1,\cdots,z_n)\\=&(x_1+y_1,\cdots,x_n+y_n)+(z_1,\cdots,z_n)\\=&((x_1+y_1)+z_1,\cdots,(x_n+y_n)+z_n)\\=&(x_1+(y_1+z_1),\cdots,x_n+(y_n+z_n))\\=&(x_1,\cdots,x_n)+(y_1+z_1,\cdots,y_n+z_n)\\=&(x_1,\cdots,x_n)+((y_1,\cdots,y_n)+(z_1,\cdots,z_n))\\=&x+(y+z).\end{aligned}$$

---

13. Solution: Suppose $x=(x_1,\cdots,x_n)$. Then

$$\begin{aligned}(ab)x=&ab(x_1,\cdots,x_n)=((ab)x_1,\cdots,(ab)x_n)\\=&(a(bx_1),\cdots,a(bx_n))=a(bx_1,\cdots,bx_n)\\=&a(bx).\end{aligned}$$

---

14. Solution: Suppose $x=(x_1,\cdots,x_n)$. Then

$$1x=1(x_1,\cdots,x_n)=(1\cdot x_1,\cdots,1\cdot x_n)=(x_1,\cdots,x_n)=x.$$

---

15. Solution: Suppose $x=(x_1,\cdots,x_n)$ and $y=(y_1,\cdots,y_n)$. Then

$$\begin{aligned}\lambda(x+y)=&\lambda((x_1,\cdots,x_n)+(y_1,\cdots,y_n))\\=&\lambda(x_1+y_1,\cdots,x_n+y_n)=(\lambda(x_1+y_1),\cdots,\lambda(x_n+y_n))\\=&(\lambda x_1,\cdots,\lambda x_n)+(\lambda y_1,\cdots,\lambda y_n)\\=&\lambda x+\lambda y.\end{aligned}$$

---

16. Solution: Suppose $x=(x_1,\cdots,x_n)$. Then

$$\begin{aligned}(a+b)x=&(a+b)(x_1,\cdots,x_n)=((a+b)x_1,\cdots,(a+b)x_n)\\=&(ax_1+bx_1,\cdots,ax_n+bx_n)\\=&(ax_1,\cdots,ax_n)+(bx_1,\cdots,bx_n)\\=&a(x_1,\cdots,x_n)+b(x_1,\cdots,x_n)\\=&ax+bx.\end{aligned}$$

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This entry was posted in Chapter 1 and tagged Exercise A.

Chapter 1 Exercise B →

Rong Ou · 2 years ago
For 3, let a and b be real numbers and (a+bi)^2 = i, then we have a^2-b^2=0 and 2ab=1, b=1/2a, a^2=b^2=1/4a^2, a^4=1/4, a^2=1/2, a=+-1/sqrt(2), b=+-1/sqrt(2), thus the solutions are +-1/sqrt(2)*(1+i).
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dotnet89 · 2 years ago
what a mistake?No.7,x2=-x1 is correct,but,y2=-x1 is wrong...it should be y2=-y1
1 · Reply · Share ›

Mohammad Rashidi Mod → dotnet89 · 2 years ago
Fixed, thank you!
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rage2050a · a year ago
a typo in 15.
for all \lambda\in\mathbb F and all \lambda\in\mathbb F^n
should be
for all \lambda\in\mathbb F and all x,y \in\mathbb F^n
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Chris Gallardo · 2 years ago
N.11 isn't suppose to be -91+78i=-37+114i? On the LH: 2 * -32 = -64 ???
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Mohammad Rashidi Mod → Chris Gallardo · 2 years ago
Fixed, thanks!
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Chapter 1 Exercise B

Posted on January 2, 2016 by Linearity

1. Solution: By definition, we have

$$(-v)+(-(-v))=0 \quad \text{and} \quad v+(-v)=0.$$

This implies both v and $-(-v)$ are additive inverses of $-v$, by the uniqueness of additive inverse, it follows that $-(-v)=v$.

---

2. Solution: If $a\neq 0$, then a has inverse a^{-1} such that $a^{-1}a=1$. Hence

$$v=1\cdot v=(a^{-1}a)v=a^{-1}(av)=a^{-1}\cdot 0=0.$$

Here we use associativity in 1.19 and and 1.30.

---

3. Solution: Let $x=\frac{1}{3}(w-v)$, then

$$v+3x=v+3\cdot\frac{1}{3}(w-v)=v+(w-v)=w.$$

This shows existence. Now we show uniqueness. Suppose, we have another vector x' such that $v+3x'=w$. Then $v+3x'=w$ implies $3x'=w-v$. Similarly, $3x=w-v$. Hence

$$3(x-x')=3x-3x'=(w-v)-(w-v)=0.$$

By Problem 2, it follows that $x-x'=0$. This shows uniqueness.

---

4. Solution: Additive identity: there exists an element $0\in V$ such that $v+0=v$ for all $v\in V$; This means V cannot be empty.

---

5. Solution: If we assume the additive inverse condition, we already showed $0v=0$ for all $v\in V$ in 1.29. Now we assume $0v=0$ for all $v\in V$ and then show additive inverse condition. Since we have $0v=0$ for all $v\in V$, we have

$$v+((-1)v)=1v+((-1)v)=(1+(-1))v=0v=0,$$

this means the existence of additive inverse, i.e. the additive inverse condition.

---

6. Solution: This is not a vector space over $\mathbb{R}$. Consider the distributive properties in 1.19. If this is a vector space over $\mathbb{R}$, we will have

$$\infty=(2+(-1))\infty=2\infty+(-1)\infty=\infty+(-\infty)=0.$$

Hence for any $t\in\mathbb{R}$, one has

$$t=0+t=\infty+t=\infty=0.$$

We get a contradiction since zero vector is unique.

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This entry was posted in Chapter 1 and tagged Exercise B.

← Chapter 1 Exercise A Chapter 1 Exercise C →

MASHIAT MUTMAINNAH · 3 years ago
Hi! Thank you so much for all the solutions, I really appreciate your hard work. Can someone please explain to me the contradiction in the solution Ex 1B question 6?
I just can't seem to understand it. Thank you!
2 Reply · Share ›

Mohammad Rashidi Mod → MASHIAT MUTMAINNAH · 2 years ago
Since zero vector should be unique.
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Abu-Yakitori → Mohammad Rashidi · 2 years ago
Why 0 + t = infinity + t?
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stan_comp_neuro · 3 months ago
I think here gives a better answer for question 2, here you did not address why the a!=0 and V != 0 does not exist https://math.stackexchange....
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Paweł Pod · 10 months ago
An another argument:
8 = 8 + 0 = 8 + (8 + (-8)) = (8 + 8) + (-8) = 8 + (-8) = 8 + ((-8) + (-8)) = (8 + (-8)) + (-8) = 0 + (-8) = (-8), this is the contradiction with the assumption that 8 and (-8) denote distinct objects.
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Maxis Jaisi · 2 years ago
Alternatively, we could just observe that $ t + \infty = \infty$ for every $ t \in \mathbf{R}$ automatically violates uniqueness of additive identity.
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Rong Ou · 2 years ago
For 6 it also violates associativity:

(\infty + \infty) + (-\infty) = \infty + (-\infty) = 0
\infty + (\infty + (-\infty)) = \infty + 0 = \infty
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Chapter 1 Exercise C

Posted on January 3, 2016 by Linearity

1. Solution: (a) $\{(x_1,x_2,x_3)\in\mathbb{F}^3 : x_1+2x_2+3x_3=0\}$ is a subspace of $\mathbb{F}^3$. By 1.34, to show a subset is a subspace, we just need to check Additive identity, Closed under addition and Closed under scalar multiplication.

Additive identity: it is clear that the additive identity $(0,0,0)$ of $\mathbb{F}^3$ is contained in $\{(x_1,x_2,x_3)\in\mathbb{F}^3 : x_1+2x_2+3x_3=0\}$.

Closed under addition: if $(a_1,a_2,a_3),(b_1,b_2,b_3)\in\{(x_1,x_2,x_3)\in\mathbb{F}^3 : x_1+2x_2+3x_3=0\}$, then

$$a_1+2a_2+3a_3=0 \quad\text{and}\quad b_1+2b_2+3b_3=0.$$

Hence

$$(a_1+b_1)+2(a_2+b_2)+3(a_3+b_3)=(a_1+2a_2+3a_3)+(b_1+2b_2+3b_3)=0,$$

this means $(a_1+b_1,a_2+b_2,a_3+b_3)=(a_1,a_2,a_3)+(b_1,b_2,b_3)$ is also in $\{(x_1,x_2,x_3)\in\mathbb{F}^3 : x_1+2x_2+3x_3=0\}$.

Closed under scalar multiplication: if $(a_1,a_2,a_3)\in\{(x_1,x_2,x_3)\in\mathbb{F}^3 : x_1+2x_2+3x_3=0\}$, then $a_1+2a_2+3a_3=0$. For any $\lambda\in\mathbb{F}$, we have

$$\lambda a_1+2(\lambda a_2)+3(\lambda a_3)=\lambda(a_1+2a_2+3a_3)=0.$$

This means

$$\lambda(a_1,a_2,a_3)=(\lambda a_1,\lambda a_2,\lambda a_3)\in\{(x_1,x_2,x_3)\in\mathbb{F}^3 : x_1+2x_2+3x_3=0\}.$$

(b) $\{(x_1,x_2,x_3)\in\mathbb{F}^3 : x_1+2x_2+3x_3=4\}$ is not a subspace of $\mathbb{F}^3$ since $(0,0,0)$ is not in it.

(c) $\{(x_1,x_2,x_3)\in\mathbb{F}^3 : x_1x_2x_3=0\}$ is not a subspace of $\mathbb{F}^3$ since $(1,1,0)$ and $(0,1,1)$ are in it, but the sum $(1,2,1)=(1,1,0)+(0,1,1)$ is not in it.

(d) $\{(x_1,x_2,x_3)\in\mathbb{F}^3 : x_1=5x_3\}$ is a subspace of $\mathbb{F}^3$.

Additive identity: it is clear that the additive identity $(0,0,0)$ of $\mathbb{F}^3$ is contained in $\{(x_1,x_2,x_3)\in\mathbb{F}^3 : x_1=5x_3\}$.

Closed under addition: if $(a_1,a_2,a_3),(b_1,b_2,b_3)\in\{(x_1,x_2,x_3)\in\mathbb{F}^3 : x_1=5x_3\}$, then $a_1=5a_3$ and $b_1=5b_3$. Hence

$$a_1+b_1=5a_3+5b_3=5(a_3+b_3),$$

this means $(a_1+b_1,a_2+b_2,a_3+b_3)=(a_1,a_2,a_3)+(b_1,b_2,b_3)$ is also in $\{(x_1,x_2,x_3)\in\mathbb{F}^3 : x_1=5x_3\}$.

Closed under scalar multiplication: if $(a_1,a_2,a_3)\in\{(x_1,x_2,x_3)\in\mathbb{F}^3 : x_1=5x_3\}$, then $a_1=5a_3$. For any $\lambda\in\mathbb{F}$, we have $\lambda a_1=\lambda(5a_3)=5(\lambda a_3)$. This means

$$\lambda(a_1,a_2,a_3)=(\lambda a_1,\lambda a_2,\lambda a_3)\in\{(x_1,x_2,x_3)\in\mathbb{F}^3 : x_1=5x_3\}.$$

---

2. Solution: (a) If this set is a subspace of $\mathbb{F}^4$, then $(0,0,0,0)\in\mathbb{F}^4$, then $0=5\cdot 0+b$. Hence $b=0$. Follow the same steps of Problem 1 (d), we will see the set is a subspace of $\mathbb{F}^4$ if $b=0$.

(b) (c) and (d) is similar to Problem 3 and 4.

Now let us consider (e). Denote the set of all sequences of complex numbers with limit 0 by A.

Additive identity: it is clear that $(0,0,\cdots)\in A$.

Closed under addition: if $(a_1,a_2,\cdots),(b_1,b_2,\cdots)\in A$, then

$$\lim_{n\to\infty}a_n=0 \quad\text{and}\quad \lim_{n\to\infty}b_n=0.$$

It is easy to see

$$\lim_{n\to\infty}(a_n+b_n)=\lim_{n\to\infty}a_n+\lim_{n\to\infty}b_n=0+0=0.$$

This means $(a_1+b_1,a_2+b_2,\cdots)=(a_1,a_2,\cdots)+(b_1,b_2,\cdots)\in A$.

Closed under scalar multiplication: if $(a_1,a_2,\cdots)\in A$, then

$$\lim_{n\to\infty}a_n=0.$$

For any $\lambda\in\mathbb{C}$, it is easy to see

$$\lim_{n\to\infty}(\lambda a_n)=\lambda\lim_{n\to\infty}a_n=\lambda 0=0.$$

This means $\lambda(a_1,a_2,\cdots)=(\lambda a_1,\lambda a_2,\cdots)\in A$.

---

3. Solution: Denote the set of differentiable real-valued functions f on the interval $(-4,4)$ such that $f'(-1)=3f(2)$ by V.

Additive identity: it is clear that the constant function $f\equiv 0$ is contained in V.

Closed under addition: if $f,g\in V$, then f and g are differentiable real-valued functions. So is $f+g$. Moreover,

$$(f+g)'(-1)=f'(-1)+g'(-1)=3f(2)+3g(2)=3(f(2)+g(2))=3(f+g)(2).$$

This concludes V is closed under addition.

Closed under scalar multiplication: if $f\in V$, for any $\lambda\in\mathbb{R}$, then f is differentiable real-valued functions. So is λf. Moreover,

$$(\lambda f)'(-1)=\lambda f'(-1)=\lambda(3f)(2)=3(\lambda f)(2).$$

This deduces V is closed under scalar multiplication.

---

4. Solution: Denote the set of continuous real-valued functions f on the interval $[0,1]$ such that $\int_0^1 f=b$ by V_b.

If V_b is a subspace of $\mathbb{R}^{[0,1]}$, then for any $f\in V_b$, we have $\int_0^1 f=b$. Because V_b is a subspace of $\mathbb{R}^n$, it follows that $kf\in V_b$ for any $k\in\mathbb{R}$. Hence

$$b=\int_0^1(kf)=k\int_0^1 f=kb,\quad\text{for all } k\in\mathbb{R},$$

this happens if and only if $b=0$.

Now if $b=0$, then for any $f,g\in V_0$ and $\lambda\in\mathbb{R}$. We have that

$$\int_0^1(f+g)=\int_0^1 f+\int_0^1 g=0+0=0$$

and $f+g$ is continuous real-valued functions since f and g are. This deduces $f+g\in V_0$, i.e. V_0 is closed under addition. Similarly,

$$\int_0^1(\lambda f)=\lambda\int_0^1 f=k0=0$$

and λf is continuous real-valued functions since f is. This implies $\lambda f\in V_0$, i.e. V_0 is closed under scalar multiplication. On the other hand, the constant function $f\equiv 0\in V_0$, which is also the additive identity in $\mathbb{R}^{[0,1]}$. Hence V_0 is a subspace of $\mathbb{R}^n$ by 1.34.

---

5. Solution: Note that we consider complex vector space, so if $\mathbb{R}^2$ is a subspace of the complex vector space $\mathbb{C}^2$, then

$$i(1,1)=(i,i)\in\mathbb{R}^2,$$

we get a contradiction. Hence $\mathbb{R}^2$ is not a subspace of the complex vector space $\mathbb{C}^2$.

---

6. Solution: (a) Because $a^3=b^3$ if and only if $a=b$ in $\mathbb{R}$, hence

$$\{(a,b,c)\in\mathbb{R}^3 : a^3=b^3\}=\{(a,b,c)\in\mathbb{R}^3 : a=b\}$$

is obviously a subspace of $\mathbb{R}^3$ by the similar arguments in Problem 1 and Problem 2.

(b) Note that

$$x=\left(1,\frac{-1+\sqrt{3}i}{2},0\right)\in\{(a,b,c)\in\mathbb{C}^3 : a^3=b^3\}$$

and

$$y=\left(1,\frac{-1-\sqrt{3}i}{2},0\right)\in\{(a,b,c)\in\mathbb{C}^3 : a^3=b^3\}.$$

However,

$$x+y=(2,-1,0)\notin\{(a,b,c)\in\mathbb{C}^3 : a^3=b^3\}.$$

This implies $\{(a,b,c)\in\mathbb{C}^3 : a^3=b^3\}$ is not closed under addition, hence not a subspace of $\mathbb{C}^3$.

---

7. Solution: Denote $\{(x,y)\in\mathbb{R}^2 : x,y\in\mathbb{Z}\}$ by U, then U is not empty. If $(x_1,y_1)\in A$ and $(x_2,y_2)\in A$, then $x_1,x_2,y_1,y_2\in\mathbb{Z}$. Hence x_1+x_2 and y_1+y_2 are integers. This means $(x_1+x_2,y_1+y_2)=(x_1,y_1)+(x_2,y_2)\in U$, i.e. U is closed under addition. Similarly, since $(-x_1,-y_1)\in U$, it follows that U is closed under additive inverses. However, U is not closed under scalar multiplication since $(1,1)\in A$ while $\frac{1}{2}(1,1)\notin U$. Hence U is not a subspace of $\mathbb{R}^2$.

---

8. Solution: Denote $\{(x,y)\in\mathbb{R}^2 : x=0 \text{ or } y=0\}$ by U, then U is not empty. If $(x,0)\in U$, then for any $\lambda\in\mathbb{R}$, we have

$$\lambda(x,0)=(\lambda x,0)\in U.$$

Similarly, $\lambda(0,y)\in U$, hence U is closed under scalar multiplication. However, $(1,0),(0,1)\in U$ while $(1,1)=(1,0)+(0,1)\notin U$. This implies U is not closed under addition, hence not a subspace of $\mathbb{R}^2$.

---

9. Solution: Denote the set of periodic functions from $\mathbb{R}$ to $\mathbb{R}$ by S. Then S is not a subspace of $\mathbb{R}^{\mathbb{R}}$. Otherwise, we have $h(x)=\sin\sqrt{2}x+\cos x$ since both $f(x)=\sin\sqrt{2}x$ and $g(x)=\cos x$ are periodic functions from $\mathbb{R}$ to $\mathbb{R}$. Assume there exists a positive number p such that $h(x)=h(x+p)$ for all $x\in\mathbb{R}$, then $1=h(0)=h(p)=h(-p)$. It is equivalent to

$$1=\cos p+\sin\sqrt{2}p=\cos p-\sin\sqrt{2}p,$$

this implies $\sin\sqrt{2}p=0$ and $\cos p=1$. $\cos p=1$ deduces that $p=2k\pi$ where $k\in\mathbb{Z}$. However $\sin\sqrt{2}p=0$ implies $\sqrt{2}p=2l\pi$ where $l\in\mathbb{Z}$. Hence

$$\sqrt{2}=\frac{2l\pi}{2k\pi}=\frac{l}{k}\in\mathbb{Q},$$

which is impossible. Therefore we get the conclusion.

---

10. Solution: Additive identity: by definition $0\in U_1$ and $0\in U_2$, hence $0\in U_1\cap U_2$.

Closed under addition: if $x\in U_1\cap U_2$ and $y\in U_1\cap U_2$, then $x\in U_1$ and $y\in U_1$, hence $x+y\in U_1$ for U_1 is closed under addition. Similarly, $x+y\in U_2$. Therefore $x+y\in U_1\cap U_2$.

Closed under scalar multiplication: if $x\in U_1\cap U_2$, then $x\in U_1$. Then for any $\lambda\in\mathbb{F}$, we have $\lambda x\in U_1$ since U_1 is closed under scalar multiplication. Similarly, $\lambda x\in U_2$. Therefore $\lambda x\in U_1\cap U_2$.

---

11. Solution: Assume U_i are subspaces of V, where $i\in I$. Now we will show $\cap_{i\in I}U_i$ is a subspace of V.

Additive identity: by definition $0\in U_i$ for every $i\in I$, hence $0\in\cap_{i\in I}U_i$.

Closed under addition: if $x\in\cap_{i\in I}U_i$ and $y\in\cap_{i\in I}U_i$, then for any given $i\in I$, we have $x\in U_i$ and $y\in U_i$, hence $x+y\in U_i$ for U_i is closed under addition. Therefore $x+y\in\cap_{i\in I}U_i$.

Closed under scalar multiplication: if $x\in\cap_{i\in I}U_i$, then $x\in U_i$ for any given $i\in I$. Then for any $\lambda\in\mathbb{F}$, we have $\lambda x\in U_i$ since U_i is closed under scalar multiplication. Therefore $\lambda x\in\cap_{i\in I}U_i$.

---

12. Solution: Suppose U and W are two subspaces of V. We argue it by contradiction. If $U\cup W$ is a subspace of V, moreover $U\not\subset W$ and $W\not\subset U$. Consider $u\in U\setminus W$ and $w\in W\setminus U$, then $u+w\in U\cup W$ since $U\cup W$ is a subspace of V. Hence $u+w\in U$ or W. If $u+w\in U$, then $w=(u+w)-u\in U$. We get a contradiction. If $u+w\in W$, then $u=(u+w)-w\in W$. We also get a contradiction. Hence if $U\cup W$ is a subspace of V, we must have $U\subset W$ or $W\subset U$.

If $U\subset W$ or $W\subset U$. Without loss of generality, we can assume $U\subset W$. Then $U\cup W=W$ is obviously a subspace of V.

---

13. See If a field $\mathbb{F}$ is such that $|\mathbb{F}|>n-1|$ why is V a vector space over $\mathbb{F}$ not equal to the union of proper subspaces of V or A question on vector space over an infinite field.

---

14. Solution: It is clear that U and W are subspaces of $\mathbb{F}^4$. Now assume that $(x_1,x_1,y_1,y_1)\in U$ and $(x_2,x_2,x_2,y_2)\in W$, then

$$\begin{aligned}&(x_1,x_1,y_1,y_1)+(x_2,x_2,x_2,y_2)\\ =&(x_1+x_2,x_1+x_2,y_1+x_2,y_1+y_2)\in\{(x,x,y,z) : x,y,z\in\mathbb{F}^4\}.\end{aligned}$$

Hence $U+W\subset\{(x,x,y,z) : x,y,z\in\mathbb{F}^4\}$. For any $x,y,z\in\mathbb{F}$, we have $(0,0,y-x,y-x)\in U$ and $(x,x,x,z+x-y)\in W$. However,

$$(x,x,y,z)=(0,0,y-x,y-x)+(x,x,x,z+x-y)\in U+W,$$

hence $\{(x,x,y,z) : x,y,z\in\mathbb{F}^4\}\subset U+W$. Combining this with previous argument, it follows that $U+W=\{(x,x,y,z) : x,y,z\in\mathbb{F}^4\}$.

---

15. Solution: Because U is a subspace of V, hence closed under addition.Therefore for any $x,y\in U$, we have $x+y\in U$, i.e. $U+U\subset U$. Note that if $x\in U$, then $x=x+0\in U+U$, hence $U\subset U+U$. Combining with these arguments, it follows that $U+U=U$.

---

16. Solution: For $x\in U$ and $y\in W$, because addition in V is commutative, we have $x+y=y+x\in W+U$. This implies $U+W\subset W+U$. Similarly, we have $W+U\subset U+W$. Hence $U+W=W+U$.

---

17. Solution: Note that in V, we have $(x+y)+z=x+(y+z)$. Hence this is similar to Problem 16. Let $x_i\in U_i$, $i=1,2,3$, then

$$(x_1+x_2)+x_3=x_1+(x_2+x_3)\in U_1+(U_2+U_3).$$

Since every element in $(U_1+U_2)+U_3$ can be expressed as the form $(x_1+x_2)+x_3$, it follows that $(U_1+U_2)+U_3\subset U_1+(U_2+U_3)$. Similarly, we also have $U_1+(U_2+U_3)\subset(U_1+U_2)+U_3$. Hence $(U_1+U_2)+U_3=U_1+(U_2+U_3)$.

---

18. Solution: If U is a additive identity, then for any subspace W of V, we have $U+W=W$. This means $U\subset W$ by the similar arguments in Problem 15 and 16. Hence the only possibility is $U=0$, in fact this is the case. Hence 0 is the additive identity. Suppose subspace W of V has additive inverses, then there exists a subspace S of V such that $W+S=0$. This can only happen when $W=0$ since $W\subset W+S$.

---

19. Solution: Here is a counterexample. Let $V=U_1=\{(a,b)\in\mathbb{R}^2 : a,b\in\mathbb{R}\}$, $U_2=\{(a,0)\in\mathbb{R}^2 : a\in\mathbb{R}\}$ and $W=\{(0,b)\in\mathbb{R}^2 : b\in\mathbb{R}\}$. Then it is easy to see that $U_1+W=U_2+W$, but $U_1\neq U_2$.

See comments for the example from Neven Sajko.

---

20. Solution: Take $W=\{(0,z,0,w)\in\mathbb{F}^4 : z,w\in\mathbb{F}\}$. For any $(x,y,z,w)\in\mathbb{F}^4$, we have

$$(x,y,z,w)=(x,x,z,z)+(0,y-x,0,w-z)\in U+W$$

since $(x,x,z,z)\in U$ and $(0,y-x,0,w-z)\in W$. We have $\mathbb{F}$=$U+W$.

Moreover, if $(x,y,z,w)\in U\cap W$, then we must have $x=y$ and $z=w$ since $(x,y,z,w)\in U$.

Similarly, since $(x,y,z,w)\in W$, we have $x=0$ and $z=0$. Therefore, $x=y=0$ and $z=w=0$, hence $(x,y,z,w)=(0,0,0,0)$. It follows that $U\cap W=\{0\}$. Hence $F=U\oplus W$ by 1.45.

---

21. Solution: Since we can only choose the first two coordinates arbitrarily in U, we can take W in which the first two coordinates are zero and the last three are variables.

Let $W=\{(0,0,z,w,s)\in\mathbb{F}^5 : x,y\in\mathbb{F}\}$. If $(x,y,z,w,s)\in U\cap W$, then $x=y=0$ since $(x,y,z,w,s)\in W$. Moreover, we have $z=x+y, w=x-y, s=2x$ since $(x,y,z,w,s)\in U$. Therefore $z=w=s=0$, hence $U\cap W=\{0\}$.

Again, for any $(x,y,z,w,s)\in\mathbb{F}^5$, we have

$$(x,y,z,w,s)=(x,y,x+y,x-y,2x)+(0,0,z-x-y,w-x+y,s-2x)\in U+W$$

since $(x,y,x+y,x-y,2x)\in U$ and $(0,0,z-x-y,w-x+y,s-2x)\in W$. Therefore $\mathbb{F}^5=U+W$.

Since $U\cap W=0$, it follows from 1.45 that $\mathbb{F}^5=U\oplus W$.

---

22. Solution: Let $W_1=\{(0,0,z,0,0)\in\mathbb{F}^5 : x,y\in\mathbb{F}\}$, $W_2=\{(0,0,0,z,0)\in\mathbb{F}^5 : x,y\in\mathbb{F}\}$ and $W_3=\{(0,0,0,0,z)\in\mathbb{F}^5 : x,y\in\mathbb{F}\}$. By the same argument as in Problem 21, we have

$$\mathbb{F}^5=U\oplus W_1\oplus W_2\oplus W_3.$$

---

23. Solution: Here is a counter example. Let $V=\mathbb{R}^2$, $U_1=\{(x,0)\in\mathbb{R}^2 : x\in\mathbb{R}\}$, $U_2=\{(0,y)\in\mathbb{R}^2 : y\in\mathbb{R}\}$ and $W=\{(z,z)\in\mathbb{R}^2 : z\in\mathbb{R}\}$. By the same argument as in Problem 21, one has

$$V=U_1\oplus W \quad\text{and}\quad V=U_2\oplus W,$$

however $U_1\neq U_2$.

This is exactly the example from Neven Sajko.

---

24. Solution: Given any $f\in\mathbb{R}^{\mathbb{R}}$, define $f_e(x)$ to be $\frac{f(x)+f(-x)}{2}$ and define $f_o(x)$ to be $\frac{f(x)-f(-x)}{2}$ for all $x\in\mathbb{R}$. Then $f_e,f_o\in\mathbb{R}^{\mathbb{R}}$. Moreover, for all $x\in\mathbb{R}$, we have

$$f_e(-x)=\frac{f(-x)+f(x)}{2}=\frac{f(x)+f(-x)}{2}=f_e(x)$$

and

$$f_o(-x)=\frac{f(-x)-f(x)}{2}=-\frac{f(x)-f(-x)}{2}=-f_o(x).$$

Hence $f_e\in U_e$ and $f_o\in U_o$. Note that we also have

$$f(x)=\frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2}=f_e(x)+f_o(x),$$

hence $f=f_e+f_o\in U_e+U_o$. Since we can choose f arbitrarily, one has $\mathbb{R}^{\mathbb{R}}=U_e+U_o$.

By 1.45, to show $\mathbb{R}^{\mathbb{R}}=U_e\oplus U_o$, it suffices to prove that $U_e\cap U_o=\{0\}$. Let $f\in U_e\cap U_o$, then $f(x)=f(-x)$ since $f\in U_e$ and $f(x)=-f(-x)$ since $f\in U_o$ for all $x\in\mathbb{R}$. Sum up $f(x)=f(-x)$ and $f(x)=-f(-x)$, we have $f(x)=0$ for all $x\in\mathbb{R}$. Hence $f=0$, which implies $U_e\cap U_o=\{0\}$.

← Chapter 1 Exercise B Chapter 2 Exercise A →

Neven Sajko · 3 years ago
20

W = {(0,x,y,0) : x,y ∈ F}
1 Reply · Share ›

JGer → Neven Sajko · 3 years ago
Can I ask why this is the case? It would appear to me that, using the W you've provided, that U+W=(x,x,y,y), which, doesn't seem to me to be all of F^4.
Reply · Share ›

Eduardo Furihata → JGer · 3 years ago
consider U = {(x1,x1,y1,y1):x1,y1 ∈ F}
and W = {(0,x2,y2,0) : x2,y2 ∈ F}

F^4 = U+W = {(x1, x1+x2, y1+y2, y1) : x1, y1, x2, y2 ∈ F}
1 Reply · Share ›

Abhinav Patil · 2 months ago
For 19: The examples given by @Neven Sajko seem needlessly complicated. Rather than making W the identity function (x = y), make W the entire xy plane. Then U1 (x-axis) and U2 (y-axis) are obviously not equal, but as each is a subset of the plane, U1 + W = W and U2 + W = W which means U1 + W = U2 + W
Reply · Share ›

Hanson Char · a year ago
Exercise 24 means any function in R^R can be decomposed into the sum of two other functions, one odd and one even. Amazing!
Reply · Share ›

Hanson Char · a year ago
Another solution for 8:

$$\{(x,y)\in\mathbb{R}^2 : |x|=|y|\}$$

Reply · Share ›

Brian Nguyen · a year ago
I don't agree #14). It appears as though the second part of your argument where (x,x,x,z+x-y) ∈ W. I do not think this is valid. The (0,0,y-x,y-x) ∈ U is true if you use the closure by addition.
Reply · Share ›

Hanson Char → Brian Nguyen · a year ago
Given W = {(x, x, x, y) ∈ F^4: x, y ∈ F}, for any x, y, z ∈ F^4, (x, x, x, z+x-y) ∈ W.
Reply · Share ›

Brian Nguyen · a year ago
6) b) doesn't seem right. The vectors you chosen for x and y does not satisfy the condition a^3 = b^3; that condition isn't correct, the proof appears wrong and the answer should be that {(a,b,c) C^3: a^3 = b^3} is indeed a subspace of C^3

From the book: Definition of Subspace
A subset U of V is a subspace of V if and only if U satisfies the following three conditions:
1) 0 ∈ U
2) u,w ∈ U
3) a ∈ F, u ∈ U implies au ∈ U

where F can be real or complex vector space.

since you're trying to say that x,y ∈ {(a,b,c), C^3: a^3 = b^3} and the vectors you chose are not part of the set. So, it's not right.
Reply · Share ›

rage2050a · a year ago
In 9, it is written "1 = \cos p + \sin \sqrt2 p = \cos p - \sin \sqrt2 p
this implies $\sin \sqrt2 p = 0$ and $\cos p = 1$"
but it is not necessarily so.
Numerically, p=1.45459662, \sin \sqrt2 p = 0.884061615 (not 0), \cos p = 0.115938388 (not 1), is essentially 1.
Reply · Share ›

Hanson Char → rage2050a · a year ago

$$\begin{aligned}\cos p+\sin\sqrt{2}p&=\cos p-\sin\sqrt{2}p\\ \sin\sqrt{2}p&=-\sin\sqrt{2}p\\ \therefore\sqrt{2}p&=2k\pi\qquad\text{for } k\in\mathbb{Z}\end{aligned}$$

Reply · Share ›

Hanson Char → Hanson Char · a year ago
btw, solution 9 is a very clever construction of a counter example.
Reply · Share ›

Neven Sajko · 3 years ago
24

Space of real functions: R^R.
U_e= {f ∈ R^R : f(-x)=f(x)} (The set of even functions.)
U_o= {f ∈ R^R : f(-x)=-f(x)} (The set of odd functions.)
Show that R^R = U_e ⊕ U_o.

Proof:
Firstly we prove U_e,U_o ≤ R^R.
0 ∈ U_e; ∀x ∈ R 0(x)=0; 0(-x)=0=0(x).
f,g ∈ U_e → (f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x).
g ∈ U_e, λ ∈ R → (λg)(-x) = λg(-x) = λg(x) = (λg)(x).
Similarly U_o ≤ R^R.

We need to prove U_e ∩ U_o = {0}.
f ∈ U_e ∩ U_o → f(x) = f(-x) = -f(x) → f=-f → f=0.
Now we only need to prove U_e+U_o=R^R.
U_e+U_o ⊆ R because U_e,U_o ⊂ R^R and R^R is closed under addition.
U_e+U_o ⊇ R because ∀f ∈ R^R ∀x ∈ R f(x) = ½f(x) + ½f(x) + ½f(-x) -½f(-x) = ½(f(x)+f(-x)) + ½(f(x)-f(-x)) where the first summand is from U_e and the second from U_o.
QED
Reply · Share ›

Neven Sajko · 3 years ago
23

The counter example from 19 still works.
Reply · Share ›

Neven Sajko · 3 years ago
22

Just decompose W from exercise 21 into 3 parts.
Reply · Share ›

Neven Sajko · 3 years ago
21

W = {(0,0,a,b,c) ∈ F⁵ : a,b,c ∈ F}
Reply · Share ›

Neven Sajko · 3 years ago
19

A counter example:

U1= {λ(1,0) : λ ∈ F} (The x axis.)

U2= {λ(0,1) : λ ∈ F} (The y axis.)

W = {λ(1,1) : λ ∈ F} (The x=y line/the identity function graph.)

Obviously U1≠U2, but:

U1+ W = {a(1,0)+b(1,1) : a,b ∈ F}

= {(a(1,0)-b(1,0))+b(1,1) : a,b ∈ F} (Because U1 is closed under addition.)

= {a(1,0)+b(0,1) : a,b ∈ F} (Note that this is the x,y plane.)

= {a(1,0)+(a(0,1)+b(0,1)) : a,b ∈ F}

= {a(1,1)+b(0,1) : a,b ∈ F}

= W + U2 = U2+ W
Reply · Share ›

Chapter 2 Exercise A

Posted on February 1, 2016 by Linearity

1. Suppose v_1, v_2, v_3, v_4 spans V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also spans V.

Solution: We just need to show that v_1, v_2, v_3, v_4 can be expressed as linear combination of $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$. Note that

$$v_1 = (v_1 - v_2) + (v_2 - v_3) + (v_3 - v_4) + v_4,$$

$$v_2 = (v_2 - v_3) + (v_3 - v_4) + v_4,$$

$$v_3 = (v_3 - v_4) + v_4, \quad v_4 = v_4,$$

we will get the conclusion by definition 2.17.

---

2. Verify the assertions in Example 2.18.

Solution: (a) If $v \neq 0$, then $av = 0$ means $a = 0$ by problem 2 in exercise 1B, hence $v \in V$ is linearly independent. Conversely, if $v \in V$ is linearly independent, then $v \neq 0$. Otherwise, we have $1v = v = 0$, i.e. v is linearly dependent. This is a contradiction.

(b) If $v_1 \in V, v_2 \in V$ is linearly independent, then neither vector is a scalar multiple of the other. Otherwise, without loss of generality, we can assume $v_1 = cv_2$, then $1v_1 + (-c)v_2 = 0$. It follows that $v_1 \in V, v_2 \in V$ is linearly dependent. We get a contradiction. Conversely, if $v_1 \in V, v_2 \in V$ is linearly dependent, then there exist a and b such that $av_1 + bv_2 = 0$ and a and b are not both zero. Without loss of generality, we can assume $a \neq 0$, then $av_1 + bv_2 = 0$ means $v_1 = -\frac{b}{a}v_2$. We also get a contradiction.

(c) If there exist $x, y, z, w \in \mathbb{F}$ such that

$$x(1,0,0,0) + y(0,1,0,0) + z(0,0,1,0) = 0,$$

then it means $(x, y, z, 0) = (0,0,0,0)$. Hence $x = y = z = 0$, it follows that the list is linearly independent in $\mathbb{F}^4$.

(d) We just need the sentence before definition 2.12, that is, "Conclusion: the coefficients of a polynomial are uniquely determined by the polynomial". Then use definition 2.17 and the similar method as (c), we can prove this case.

---

3. Find a number t such that

$$(3,1,4), (2,-3,5), (5,9,t)$$

is not linearly independent in $\mathbb{R}^3$.

Solution: Let us consider the following equations

$$3x + 2y = 5, \quad x - 3y = 9. \tag{1}$$

We can get a solution $x = 3, y = -2$. Hence let

$$t = 3 \cdot 4 + (-2) \cdot 5 = 2.$$

Then we have

$$3(3,1,4) + (-2)(2,-3,5) = (5,9,2), \tag{2}$$

this means $t = 2$ is the desired solution.

Remark: From (2), one can know why we get these equations in (1).

---

4. Verify the assertion in the second bullet point in Example 2.20.

Solution: We already see that if $c = 8$, the list is linearly dependent. Now we show that if the list is linearly dependent, then $c = 8$. Suppose there are exist x, y and z such that they are not all zero and

$$x(2,3,1) + y(1,-1,2) + z(7,3,c) = 0. \tag{3}$$

Then we have

$$2x + y + 7z = 0 \quad \text{and} \quad 3x - y + 3z = 0.$$

From these equations, by solving in x and y, we get $x = -2z$ and $y = -3z$. Since x, y, z are not all zero, z is not zero. However, (3) also means $x + 2y + cz = 0$, plugging $x = -2z$ and $y = -3z$, we get

$$-2z + 2(-3)z + cz = 0 \iff (c-8)z = 0.$$

Hence we deduce that $c = 8$ since $z \neq 0$.

Using this method, we can also solve Problem 3.

---

5. (a) Show that if we think of $\mathbb{C}$ as a vector space over $\mathbb{R}$, then the list $(1+i, 1-i)$ is linearly independent.
(b) Show that if we think of $\mathbb{C}$ as a vector space over $\mathbb{C}$, then the list $(1+i, 1-i)$ is linearly dependent.

Solution: (a) Let x and y be in $\mathbb{R}$, then if $x(1+i) + y(1-i) = 0$, we have

$$0 = x(1+i) + y(1-i) = (x+y) + (x-y)i.$$

Hence $x + y = 0$ and $x - y = 0$, it follows that $x = 0$ and $y = 0$. Thus the list $(1+i, 1-i)$ is linearly independent over $\mathbb{R}$.

(b) For this case, note that

$$i(1+i) + 1(1-i) = (i-1) + (1-i) = 0,$$

we conclude the list $(1+i, 1-i)$ is linearly dependent over $\mathbb{C}$.

---

6. Suppose v_1, v_2, v_3, v_4 is linearly independent in V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is also linearly independent in V.

Solution: Suppose there exist numbers x, y, z, w in the field such that

$$x(v_1 - v_2) + y(v_2 - v_3) + z(v_3 - v_4) + wv_4 = 0,$$

then

$$xv_1 + (y-x)v_2 + (z-y)v_3 + (w-z)v_4 = 0.$$

Because v_1, v_2, v_3, v_4 is linearly independent in V, it follows that

$$x = 0, \quad y = x = 0, \quad z - y = 0, \quad w - z = 0.$$

Hence we get $x = y = z = w = 0$. This means the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is also linearly independent in V.

---

7. Prove or give a counterexample: If $v_1, v_2, \cdots, v_m$ is a linearly independent list of vectors in V, then

$$5v_1 - 4v_2, v_2, v_3, \cdots, v_m$$

is linearly independent.

Solution: This is true. For if there exist $a_1, \cdots, a_m \in \mathbb{F}$ such that

$$a_1(5v_1 - 4v_2) + a_2v_2 + \cdots + a_mv_m = 0,$$

we have

$$5a_1v_1 + (a_2 - 4a_1)v_2 + a_3v_3 \cdots + a_mv_m = 0.$$

Because $v_1, v_2, \cdots, v_m$ is a linearly independent, it follows that

$$5a_1 = 0, a_2 - 4a_1 = 0, a_3 = \cdots = a_m = 0.$$

We get $a_1 = a_2 = \cdots = a_m = 0$, hence the list

$$5v_1 - 4v_2, v_2, v_3, \cdots, v_m$$

is linearly independent by Definition 2.17.

---

8. Prove or give a counterexample: If $v_1, v_2, \cdots, v_m$ is a linearly independent list of vectors in V and $\lambda \in \mathbb{F}$ with $\lambda \neq 0$, then $\lambda v_1, \lambda v_2, \cdots, \lambda v_m$ is linearly independent.

Solution: This is true. For if there exist $a_1, \cdots, a_m \in \mathbb{F}$ such that

$$a_1(\lambda v_1) + a_2(\lambda v_2) + \cdots + a_m(\lambda v_m) = 0,$$

we have

$$(a_1\lambda)v_1 + (a_2\lambda)v_2 + \cdots + (a_m\lambda)v_m = 0.$$

Because $v_1, v_2, \cdots, v_m$ is a linearly independent, it follows that

$$a_1\lambda = a_2\lambda = \cdots = a_m\lambda.$$

Since $\lambda \neq 0$, we deduce that $a_1 = a_2 = \cdots = a_m = 0$, hence $\lambda v_1, \lambda v_2, \cdots, \lambda v_m$ is linearly independent.

---

9. Prove or give a counterexample: If $v_1, v_2, \cdots, v_m$ and $w_1, w_2, \cdots, w_m$ are linearly independent lists of vectors in V, then $v_1 + w_1, v_2 + w_2, \cdots, v_m + w_m$ is linearly independent.

Solution: Counterexample: let $w_i = -v_i$, then if $v_1, v_2, \cdots, v_m$ is linearly independent, we have $w_1, w_2, \cdots, w_m$ is linearly independent by Problem 8. However, $v_1 + w_1 = 0, v_2 + w_2 = 0, \cdots, v_m + w_m = 0$ is linearly dependent.

---

10. Suppose $v_1, v_2, \cdots, v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \cdots, v_m + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \cdots, v_m)$.

Solution: Since $v_1 + w, \cdots, v_m + w$ is linearly dependent, there exist $a_1, \cdots, a_m \in \mathbb{F}$, not all 0, such that

$$a_1(v_1 + w) + a_2(v_2 + w) + \cdots + a_m(v_m + w) = 0.$$

Hence we have

$$a_1v_1 + \cdots + a_mv_m + (a_1 + \cdots + a_m)w = 0.$$

If $a_1 + \cdots + a_m = 0$, we get $a_1v_1 + \cdots + a_mv_m = 0$, which will deduce that $a_i \equiv 0$. Hence $a_1 + \cdots + a_m \neq 0$, it follows that

$$w = -\frac{1}{a_1 + \cdots + a_m}(a_1v_1 + \cdots + a_mv_m) \in \operatorname{span}(v_1, \cdots, v_m).$$

---

11. Suppose $v_1, v_2, \cdots, v_m$ is linearly independent in V and $w \in V$. Show that $v_1, v_2, \cdots, v_m, w$ is linearly independent if and only if $w \notin \operatorname{span}(v_1, \cdots, v_m)$.

Solution: It is equivalent to show $v_1, v_2, \cdots, v_m, w$ is linearly dependent if and only if $w \in \operatorname{span}(v_1, \cdots, v_m)$.

If $v_1, v_2, \cdots, v_m, w$ is linearly dependent, there exist $a_1, \cdots, a_m, b \in \mathbb{F}$, not all 0, such that

$$a_1v_1 + a_2v_2 + \cdots + a_mv_m + bw = 0.$$

If $b = 0$, we get $a_1v_1 + \cdots + a_mv_m = 0$, which will deduce that $a_i \equiv 0$. Hence $b \neq 0$, now we can obtain

$$w = -\frac{1}{b}(a_1v_1 + a_2v_2 + \cdots + a_mv_m) \in \operatorname{span}(v_1, \cdots, v_m).$$

Conversely, if $w \in \operatorname{span}(v_1, \cdots, v_m)$, then there exist $a_1, \cdots, a_m \in \mathbb{F}$ such that

$$w = a_1v_1 + a_2v_2 + \cdots + a_mv_m \Longrightarrow a_1v_1 + a_2v_2 + \cdots + a_mv_m - w = 0.$$

This implies $v_1, v_2, \cdots, v_m, w$ is linearly dependent.

---

12. Explain why there does not exist a list of six polynomials that is linearly independent in $\mathcal{P}_4(\mathbb{F})$.

Solution: Note that $1, z, z^2, z^3, z^4$ spans $\mathcal{P}_4(\mathbb{F})$, hence any linearly independent list has no more than 5 polynomials by 2.23.

---

13. Explain why no list of four polynomials spans $\mathcal{P}_4(\mathbb{F})$.

Solution: By the similar process of Problem 2, we can show that $1, z, z^2, z^3, z^4$ is a linearly independent list of $\mathcal{P}_4(\mathbb{F})$. Due to 2.23, no list of four polynomials spans $\mathcal{P}_4(\mathbb{F})$. Otherwise, the length of every linearly independent list of vectors is 5 while the length of some spanning list of vectors is 4.

---

14. Prove that V is infinite-dimensional if and only if there is a sequence $v_1, v_2, \cdots$ of vectors in V such that $v_1, v_2, \cdots, v_m$ is linearly independent for every positive integer m.

Solution: If there is a sequence $v_1, v_2, \cdots$ of vectors in V such that $v_1, v_2, \cdots, v_m$ is linearly independent for every positive integer m, then V is obviously infinite-dimensional.

If V is infinite-dimensional, then V cannot be spanned by finitely many vectors. Now we obtain such a sequence $v_1, v_2, \cdots$ of vectors in V by induction. Let $v_1 \neq 0$ is a vector in V. Since V is infinite-dimensional, there must exist some $v_2 \in V$ such that $v_2 \notin \operatorname{span}\{v_1\}$. Similarly, if $v_1, v_2, \cdots, v_m$ is linearly independent, then there must exist some $v_{m+1} \in V$ such that $v_{m+1} \notin \operatorname{span}\{v_1, \cdots, v_m\}$. Since V is infinite-dimensional, we can always do this process. Hence we will get a sequence $v_1, v_2, \cdots$ of vectors in V. By 2.21, we deduce that $v_1, v_2, \cdots, v_m$ is linearly independent for every positive integer m.

---

15. Prove that $\mathbb{F}^\infty$ is infinite-dimensional.

Solution: Let $e_i = (0, \cdots, 0, 1, 0, \cdots)$ be the vector that has 1 in the ith-component and 0 in other components. Then we can easily check that $e_1, e_2, \cdots, e_m$ is linearly independent for every positive integer m. Now by Problem 14, we conclude that $\mathbb{F}^\infty$ is infinite-dimensional.

---

16. Prove that the real vector space of all continuous real-valued functions on the interval $[0,1]$ is infinite-dimensional.

Solution: Define $f_n = \begin{cases} x - 1/n, & x \geqslant 1/n; \\ 0, & x \in [0, 1/n]. \end{cases}$, then f_n is continuous on the interval $[0,1]$. We will show that $f_{n+1} \notin \operatorname{span}\{f_1, \cdots, f_n\}$, then by the argument in Problem 14 we can conclude that the real vector space of all continuous real-valued functions on the interval $[0,1]$ is infinite-dimensional. For if $f_{n+1} \in \operatorname{span}\{f_1, \cdots, f_n\}$, then

$$f_{n+1}(x) = a_1f_1(x) + \cdots + a_nf_n(x).$$

Note that $f_1(1/n) = \cdots = f_n(1/n) = 0$, but $f_{n+1}(1/n) = 1/(n(n+1)) \neq 0$, we get a contradiction.

---

17. Suppose $p_0, p_1, \cdots, p_m$ are polynomials in $\mathcal{P}_m(\mathbb{F})$ such that $p_j(2) = 0$ for each j. Prove that $p_0, p_1, \cdots, p_m$ is not linearly independent in $\mathcal{P}_m(\mathbb{F})$.

Solution: If $p_0, p_1, \cdots, p_m$ is linearly independent in $\mathcal{P}_m(\mathbb{F})$, then consider $z, p_0, p_1, \cdots, p_m$. By 2.23, we know that length of linearly independent list $\leqslant$ length of spanning list. Because $1, z, \cdots, z^m$ is a spanning list of $\mathcal{P}_m(\mathbb{F})$, it follows that every linearly independent list in $\mathcal{P}_m(\mathbb{F})$ has length no more than $m+1$. Now we obtain that $z, p_0, p_1, \cdots, p_m$ is linearly dependent(of length $m+2$), then by Problem 11, there exist $a_i \in \mathbb{F}$such that

$$z = a_0p_0(z) + a_1p_1(z) + \cdots + a_{m+1}p_{m+1}(z). \tag{4}$$

Note that $p_j(2) = 0$ for each j, we deduce that $2 = 0$ from (4). Hence we get a contradiction.

← Chapter 1 Exercise C | Chapter 2 Exercise B →

Nuno Alvares Pereira · 3 years ago
for 16), much easier is to think about the space P_m (polinomials up to degree m). it's easy to find lin. indep. vectors which span P_m, and then see that, for all positive integers, P_{m+1} is not spanned by vectors of smaller degree... P_m is a strict subset for V for all m.

Hence by 14) it must infinite dimensional.
3 · Reply · Share ›

Hanson Char → Nuno Alvares Pereira · a year ago
Nice!
Reply · Share ›

Eric · 2 years ago
Does question 17 need all of that work? The degree 0 polynomial (i.e., the constant polynomial) being equal to 0 is proof enough of linear dependence. "Every list of vectors in V containing the 0 vector is linearly dependent."
1 · Reply · Share ›

Mohammad Rashidi Mod → Eric · 2 years ago
But no one tells you that p_j has degree 0 for some j=0,...,m.
Reply · Share ›

Elena Orins · 7 months ago
I do not understand the solution for 11. How does b go from equaling 0 to no equaling 0? How do they get -1/b? Also, why did they decide to make w to be in the span (v1, ... , vm) ? Can someone please explain
Reply · Share ›

Lionel Ngouh · a year ago
I don't get #9. Anyone with a better explanation
Reply · Share ›

Atharva Chingre → Lionel Ngouh · 5 months ago
[Note: Please correct me if my thinking/mathematical logic is wrong]

Explanation 1: To find a counterexample for v1+w1, v2+w2, ... vm+wm is linearly independent, we have to find a1(v1+w1) + a2(v2+w2) + ... am(vm+w2) = 0 [eq.1], where not all a1, a2, ... am are 0 according to definition 2.17.
The easiest way to do this would be to take any one term, aj(vj+wj), from the left hand side of [eq.1] and make it equal to 0, such that aj is not equal to 0. Therefore take vj+wj=0, i.e. vj=(-wj). Now, aj can be any value not equal to zero, which still satisfies [eq.1], making this list linearly dependent. Since both lists v1, v2, v3.... vm and the list -v1, -v2, -v3.... -vm [i.e. the list w1, w2, w3.... wm] are independent (see problem 8, where lamda=-1), and their combination, [eq.1] is dependent, we have a counterexample.

Explanation 2: Since the list v1, v2, v3.... vm of length of m and the list w1, w2, w3.... wm of length m are linearly independent, then any list greater than the length m is linearly dependent [2.23] (all lists in this sentence are over V). Simplifying [eq.1], we get a1v1+a1w1 + a2v2+a2w2 + ...amvm+amwm. This could be seen as list v1, v2, ...vm, w1, w2, ...wm, with length 2m. Since 2m>m, this list is linearly dependent.
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Hanson Char · a year ago
Apparently the solution to 2.c is incorrect. There isn't a vector (0,0,0,1) in the question, which provides a list of only 3 vectors in F^4.

(c) If there exist $x, y, z, w \in \mathbb{F}$ such that

$$x(1,0,0,0) + y(0,1,0,0) + z(0,0,1,0) + w(0,0,0,1) = 0,$$

then it means $(x, y, z, w) = (0,0,0,0)$. Hence $x = y = z = w = 0$, it follows that the list is linearly independent in $\mathbb{F}^4$.

Reply · Share ›

Hanson Char · a year ago
Question 5 is cool!
Reply · Share ›

Rich Yu · 3 years ago
Another way to see problem 10)

Let us suppose the w is not in the span (v1,...,vm)

This implies that no linear combination a1v1+....amvm = w.

Now the condition for a linearly dependence means that for some numbers a1,..., am in the field

a1 (v1+w) + ... am (vm+w) = 0 such that not all ai = 0.

Consider v = a1v1+....amvm. Then any v+w cannot = 0 because no linear combination of vectors in (v1,...,vm) = w.

If v =/= w in every case, then all ai must equal 0 for a1 (v1+w) + ... am (vm+w) = 0, since (vi + w) =/=0.
Therefore, if the list of vectors (v1+w),..., (vm+w) is linearly dependent, w must in the span of (v1,...,vm).

If w is not in the span of (v1,...,vm) then (v1+w),..., (vm+w) is independent since all ai must =0.
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Chapter 2 Exercise C →

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Chapter 2 Exercise B

Posted on February 2, 2016 by Linearity

1. Find all vector spaces that have exactly one basis.

Solution: The only vector spaces is $\{0\}$. For if there is a nonzero vector v in a basis, then we can get a new basis by changing v to $2v$.

Here, we just consider the fields $\mathbb{R}$ and $\mathbb{C}$, hence $v \neq 2v$. For finite fields such as $\mathbb{F}_2$, there may be other solutions.

---

2. Verify all the assertions in Example 2.28.

We have already known how to check linear independence. So, we just verify they are a spanning list. The process is easy and tedious, so I omit them.

---

3. (a) Let U be the subspace of $\mathbb{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \text{~and~} x_3 = 7x_4\}.$$

Find a basis of U. (b) Extend the basis in part (a) to a basis of $\mathbb{R}^5$.(c) Find a subspace W of $\mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$.

Solution: (a) $(3,1,0,0,0)$, $(0,0,7,1,0)$ and $(0,0,0,0,1)$.(b) $(3,1,0,0,0)$, $(0,0,7,1,0)$, $(0,0,0,0,1)$, $(1,0,0,0,0)$ and $(0,0,1,0,0)$.(c) $W = \operatorname{span}\{(1,0,0,0,0),(0,0,1,0,0)\}$ by (b).

---

4. (a) Let U be the subspace of $\mathbb{C}^5$ defined by

$$U = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : 6z_1 = z_2 \text{~and~} z_3 + 2z_4 + 3z_5 = 0\}.$$

Find a basis of U. (b) Extend the basis in part (a) to a basis of $\mathbb{C}^5$. (c) Find a subspace W of $\mathbb{C}^5$ such that $\mathbb{C}^5 = U \oplus W$.

Solution: (a) $(1,6,0,0,0)$, $(0,0,2,-1,0)$ and $(0,0,3,0,-1)$. (b) $(1,6,0,0,0)$, $(0,0,2,-1,0)$, $(0,0,3,0,-1)$, $(1,0,0,0,0)$ and $(0,0,1,0,0)$. (c) $W = \operatorname{span}\{(1,0,0,0,0),(0,0,1,0,0)\}$ by (b).

Here I consider the vector space is over $\mathbb{C}$.

---

5. Prove or disprove: there exists a basis p_0, p_1, p_2, p_3 of $\mathcal{P}_3(F)$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2.

Solution: Because $1, x, x^2, x^3$ is a basis of $\mathcal{P}_3(F)$, hence

$$1 + x^3, x + x^3, x^2 + x^3, x^3$$

is also a basis of $\mathcal{P}_3(F)$. However none of the polynomials $1 + x^3, x + x^3, x^2 + x^3, x^3$ has degree 2.

Here I use a fact that v_1, v_2, v_3, v_4 is a basis of V, then

$$v_1 + v_4, v_2 + v_4, v_3 + v_4, v_4$$

is also a basis of V. Proof of this is similar to Problem 6.

---

6. Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of V.

Solution: First, we need show that $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is linear independent. Assume that

$$0 = a(v_1 + v_2) + b(v_2 + v_3) + c(v_3 + v_4) + dv_4,$$

then $av_1 + (a+b)v_2 + (b+c)v_3 + (c+d)v_4 = 0$. Note that v_1, v_2, v_3, v_4 is a basis of V, it follows that $a = 0$, $a + b = 0$, $b + c = 0$ and $c + d = 0$. Then $a = b = c = d = 0$, this means $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is linear independent.

Now, note that

$$v_3 = (v_3 + v_4) - v_4, \quad v_2 = (v_2 + v_3) - (v_3 + v_4) + v_4$$

and

$$v_1 = (v_1 + v_2) - (v_2 + v_3) + (v_3 + v_4) - v_4,$$

we can conclude that v_1, v_2, v_3, v_4 can be expressed as linear combinations of $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$. Hence all vectors that can be expressed as linear combinations of v_1, v_2, v_3, v_4 can also be linearly expressed by $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$, i.e. $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ spans V.

Above all,

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of V.

---

7. Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis of U.

Solution: Counterexample: let $V = \mathbb{R}^4$, $v_1 = (1,0,0,0)$, $v_2 = (0,1,0,0)$, $v_3 = (0,0,1,1)$, $v_4 = (0,0,0,1)$ and

$$U = \{(x, y, z, 0) | x, y, z \in \mathbb{R}\}.$$

Then all the conditions are satisfied, but v_1, v_2 is not a basis of U since $(0,0,1,0)$ can not be linearly expressed by v_1, v_2.

---

8. Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, \cdots, u_m$ is a basis of U and $w_1, \cdots, w_n$ is a basis of W. Prove that

$$u_1, \cdots, u_m, w_1, \cdots, w_n$$

is a basis of V.

Solution: First, we show that $u_1, \cdots, u_m, w_1, \cdots, w_n$ is linearly independent. If there exist $a_1, \cdots, a_m \in \mathbb{F}$ and $b_1, \cdots, b_n \in \mathbb{F}$ such that

$$a_1u_1 + \cdots + a_mu_m + b_1w_1 + \cdots + b_nw_n = 0.$$

Then

$$a_1u_1 + \cdots + a_mu_m = -(b_1w_1 + \cdots + b_nw_n) \in U \cap W,$$

it follows that

$$a_1u_1 + \cdots + a_mu_m = 0, \quad b_1w_1 + \cdots + b_nw_n = 0$$

since $V = U \oplus W$ implies $U \cap W = \{0\}$. However, note that $u_1, \cdots, u_m$ is a basis of U and $w_1, \cdots, w_n$ is a basis of W, it follows that $a_1 = \cdots = a_m = 0$ and $b_1 = \cdots = b_n = 0$. Hence $u_1, \cdots, u_m, w_1, \cdots, w_n$ is linearly independent.

Now, it suffices to verify that $u_1, \cdots, u_m, w_1, \cdots, w_n$ spans V. For any $v \in V$, there exist $u \in U$ and $w \in W$ such that $v = u + w$ since $V = U \oplus W$. Note that $u_1, \cdots, u_m$ is a basis of U and $w_1, \cdots, w_n$ is a basis of W, it follows that there exist $a_1, \cdots, a_m \in \mathbb{F}$ and $b_1, \cdots, b_n \in \mathbb{F}$ such that

$$u = a_1u_1 + \cdots + a_mu_m,$$

$$w = b_1w_1 + \cdots + b_nw_n.$$

Hence

$$v = u + w = a_1u_1 + \cdots + a_mu_m + b_1w_1 + \cdots + b_nw_n,$$

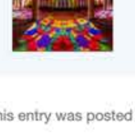
which means $u_1, \cdots, u_m, w_1, \cdots, w_n$ spans V.

Above all,

$$u_1, \cdots, u_m, w_1, \cdots, w_n$$

is a basis of V.

Remark: For any $v \in U \cap W$, if I can show that $v = 0$. Then $U \cap W = 0$, since we choose v arbitrarily. I have always skipped this step. For instance, for any $v \in V$, if I can show that $v \in W$, then $V \subset W$. The argument is similar.



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This entry was posted in Chapter 2 and tagged Exercise B.

← Chapter 2 Exercise A

Chapter 2 Exercise C →

samrao1997 · 3 years ago
In 5 I think the last vector of the basis should be x^3 not x^2 because that has degree 2....
2 · Reply · Share ›

Mohammad Rashidi Mod → samrao1997 · 2 years ago
Thank you!
Reply · Share ›

汪哲 → samrao1997 · 2 years ago
+1
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Chapter 2 Exercise C

Posted on February 3, 2016 by Linearity

1. Solution: Let $u_1, u_2, \cdots, u_n$ be a basis of U. Thus $n = \dim U = \dim V$. Hence $u_1, u_2, \cdots, u_n$ is a linearly independent list of vectors in V with length $\dim V$. By 2.39, $u_1, u_2, \cdots, u_n$ is a basis of V. In particular, any vector in V can be written as a linear combination of $u_1, u_2, \cdots, u_n$. As $u_i \in U$, it follows that $V \subset U$. This means that $U = V$.

---

2. Solution: The dimension of a subspace U of $\mathbb{R}^2$ can only be 0,1,2. If $\dim U = 0$, then $U = \{0\}$. If $\dim U = 2$, then $U = \mathbb{R}^2$ by problem 1. If $\dim U = 1$, then for any nonzero $x \in U$, it follows that

$$U = \{kx : k \in \mathbb{R}\},$$

which it is the line through x and the origin.

---

3. Solution: It is similar to Problem 2. If $\dim U = 2$, there exist two linearly independent $x, y \in \mathbb{R}^3$. Then

$$U = \{k_1x + k_2y : k_1 \in \mathbb{R}, k_2 \in \mathbb{R}\},$$

which it is the plane through x, y and the origin.

---

4. Solution: (a) A basis of U is $x-6, x^2-6x, x^3-6x^2$ and x^4-6x^3. Of course, $x-6, x^2-6x, x^3-6x^2$ and x^4-6x^3 is linearly independent since they has different degrees (It is easy to check). Moreover, if $p(6)=0$, then $p(x)$ is divided by $x-6$, hence

$$\begin{aligned} p(x) =&(x-6)(k_3x^3+k_2x^2+k_1x+k_0)\\ =&k_3(x^4-6x^3)+k_2(x^3-6x^2)+k_1(x^2-6x)+k_0(x-6) \end{aligned}$$

is a linear combination of $x-6, x^2-6x, x^3-6x^2$ and x^4-6x^3.

(b) Of course, $1, x-6, x^2-6x, x^3-6x^2$ and x^4-6x^3 is a basis of $\mathcal{P}_4(\mathbb{F})$.

(c) Let $W = \{c : c \in \mathbb{F}\}$, then $\mathcal{P}_4(\mathbb{F}) = U \oplus W$ by (b).

---

5. Solution: (a) For polynomial $f(x) = ax^4+bx^3+cx^2+dx+e$, consider the condition when $f''(6)=0$. Then you will get a linear equation about a, b, c, d, e. Find a basis of the solution space of this linear equation. Then substitute it into $f(x) = ax^4+bx^3+cx^2+dx+e$, you will get a basis of U(why?). I skip the details and give a example of basis. $1, x, x^3-18x^2, x^4-12x^3$ is a basis of U.

(b) Of course, $1, x, x^2, x^3-18x^2$ and x^4-12x^3 is a basis of $\mathcal{P}_4(\mathbb{R})$.

(c) Let $W = \{cx^2 : c \in \mathbb{R}\}$, then $\mathcal{P}_4(\mathbb{R}) = U \oplus W$ by (b).

---

6. Solution: (a) For polynomial $f(x) = ax^4+bx^3+cx^2+dx+e$, consider the condition when $f(2)=f(5)$. Then you will get a linear equation about a, b, c, d, e. The dimension of the solution space of this linear equation is 4, so is U(why?). Thus we only have to give 4 linearly independent polynomials in $\mathcal{P}_4(\mathbb{F})$ such that each of them attain the same value at $x=2$ and $x=5$. A good example is $1, x^2-7x+10, x(x^2-7x+10)$ and $x^2(x^2-7x+10)$.

(b) Of course, $1, x, x^2-7x+10, x(x^2-7x+10)$ and $x^2(x^2-7x+10)$ is a basis of $\mathcal{P}_4(\mathbb{F})$.

(c) Let $W = \{cx : c \in \mathbb{F}\}$, then $\mathcal{P}_4(\mathbb{F}) = U \oplus W$ by (b).

---

7. Solution: (a) For polynomial $f(x) = ax^4+bx^3+cx^2+dx+e$, consider the condition when $f(2)=f(5)=f(6)$. Then you will get 2 linear equations about a, b, c, d, e. The dimension of the solution space of this linear equation is 3, so is U(why?). Thus we only have to give 3 linearly independent polynomials in $\mathcal{P}_4(\mathbb{F})$ such that each of them attain the same value at $x=2$, $x=5$ and $x=6$. A good example is 1, $(x-2)(x-5)(x-6)$ and $x(x-2)(x-5)(x-6)$.

(b) Of course, $1, x, x^2, (x-2)(x-5)(x-6)$ and $x(x-2)(x-5)(x-6)$ is a basis of $\mathcal{P}_4(\mathbb{F})$.

(c) Let $W = \{cx+dx^2 : c \in \mathbb{F}, d \in \mathbb{F}\}$, then $\mathcal{P}_4(\mathbb{F}) = U \oplus W$ by (b).

---

8. Solution: (a) For polynomial $f(x) = ax^4+bx^3+cx^2+dx+e$, consider the condition when $\int_{-1}^1 f = 0$. Then you will get a linear equation about a, b, c, d, e, which is $a/5+d/3+e=0$. Find a basis of the solution space of this linear equation. Then substitute it into $f(x) = ax^4+bx^3+cx^2+dx+e$, you will get a basis of U(why?). I skip the details and give a example of basis. $x, 3x^2-1, x^3$ and $5x^4-1$ is a basis of U.

(b) Of course, $1, x, 3x^2-1, x^3$ and $5x^4-1$ is a basis of $\mathcal{P}_4(\mathbb{R})$.

(c) Let $W = \{c : c \in \mathbb{R}\}$, then $\mathcal{P}_4(\mathbb{R}) = U \oplus W$ by (b).

---

9. Solution: Note that

$$v_2 - v_1 = (v_2+w) - (v_1+w),$$

it follows that $v_2 - v_1 \in \operatorname{span}(v_1+w, \cdots, v_m+w)$. Similarly, $v_i - v_1 \in \operatorname{span}(v_1+w, \cdots, v_m+w)$ for all $2 \leqslant i \leqslant m$.

Actually, $v_2-v_1, \cdots, v_m-v_1$ is linearly independent since $v_1, \cdots, v_m$ is linearly independent in V. (It is easy to prove, see examples in Exercise 2.A and 2.B). By 2.33, it follows that

$$\dim \operatorname{span}(v_1+w, \cdots, v_m+w) \geqslant m-1.$$

---

10. Solution: Because p_0 has degree 0, we have $\operatorname{span}(p_0) = \operatorname{span}(1)$. If we assume that

$$\operatorname{span}(p_0, p_1, \cdots, p_i) = \operatorname{span}(1, x, \cdots, x^i).$$

Then by assumption, it is trivial that

$$\operatorname{span}(p_0, p_1, \cdots, p_i, p_{i+1}) \subset \operatorname{span}(1, x, \cdots, x^i, x^{i+1}).$$

On the other hand, p_{i+1} has degree $i+1$, hence it can be written as

$$p_{i+1} = a_{i+1}x^{i+1} + f_{i+1}(x),$$

where $a_{i+1} \neq 0$ and $\deg f_{i+1}(x) \leqslant i$. Then

$$x^{i+1} = \frac{1}{a_{i+1}}(p_{i+1} - f_{i+1}(x)) \in \operatorname{span}(1, x, \cdots, x^i, p_{i+1}).$$

Note that $\operatorname{span}(p_0, p_1, \cdots, p_i) = \operatorname{span}(1, x, \cdots, x^i)$, we conclude

$$\operatorname{span}(1, x, \cdots, x^i, p_{i+1}) = \operatorname{span}(p_0, p_1, \cdots, p_i, p_{i+1}).$$

Thus

$$x^{i+1} \in \operatorname{span}(p_0, p_1, \cdots, p_i, p_{i+1}),$$

then

$$\operatorname{span}(1, x, \cdots, x^i, x^{i+1}) \subset \operatorname{span}(p_0, p_1, \cdots, p_i, p_{i+1}).$$

By induction, we have

$$\operatorname{span}(p_0, p_1, \cdots, p_i) = \operatorname{span}(1, x, \cdots, x^i).$$

for all $0 \leqslant i \leqslant m$. In particular,

$$\operatorname{span}(p_0, p_1, \cdots, p_m) = \operatorname{span}(1, x, \cdots, x^m)$$

means $p_0, p_1, \cdots, p_m$ is a basis of $\mathcal{P}(\mathbb{F})$. Because $p_0, p_1, \cdots, p_m$ is a spanning list of $\mathcal{P}(\mathbb{F})$ with the same length as the dimension of $\mathcal{P}_m(\mathbb{F})$ (2.42).

---

11. Solution: By 2.43, we have

$$\dim(U \cap W) = \dim U + \dim W - \dim(U+W) = 8 - \dim(\mathbb{R}^8) = 0.$$

Hence $U \cap W = \{0\}$, combining with $U + W = \mathbb{R}^8$, it follows that $\mathbb{R}^8 = U \oplus W$.

---

12. Solution: By 2.43, we have

$$\dim(U \cap W) = \dim U + \dim W - \dim(U+W) = 10 - \dim(U+W).$$

Note that $U+W$ is a subspace of $\mathbb{R}^9$, it follows that $\dim(U+W) \leqslant 9$ (by 2.38). Hence $\dim(U \cap W) \geqslant 1$, i.e. $U \cap W \neq \{0\}$.

---

13. Solution: By 2.43, we have

$$\dim(U \cap W) = \dim U + \dim W - \dim(U+W) = 8 - \dim(U+W) \geqslant 8 - \dim(\mathbb{C}^6) = 2.$$

Hence there exists $e_1, e_2 \in U \cap W$ such that e_1 and e_2 are linearly independent. Then neither of e_1 or e_2 is a scalar multiple of the other.

---

14. Solution: Choose a basis $\mathcal{W}_i$ of U_i, then by definition of direct sum, $U_1 + \cdots + U_m$ can be spanned by the union of $\mathcal{W}_1, \cdots, \mathcal{W}_m$. From 2.31, we conclude

$$\dim(U_1 + \cdots + U_m) \leqslant \dim U_1 + \cdots + \dim U_m,$$

since the cardinality of the gather of $\mathcal{W}_1, \cdots, \mathcal{W}_m$ is no more than $\dim U_1 + \cdots + \dim U_m$. In particular, $U_1 + \cdots + U_m$ is finite-dimensional.

---

15. Solution: Let $(v_1, \cdots, v_n)$ be a basis of V. For each j, let U_j equal $\operatorname{span}(v_j)$; in other words, $U_j = \{av_j : a \in \mathbb{F}\}$. It is easy to see that $\dim U_j = 1$ for all $j = 1, \cdots, n$. Because $(v_1, \cdots, v_n)$ is a basis of V, each vector in V can be written uniquely in the form

$$a_1v_1 + \cdots + a_nv_n,$$

where $a_1, \cdots, a_n \in \mathbb{F}$. By definition of direct sum, this means that $V = U_1 \oplus \cdots \oplus U_n$.

---

16. Solution: Since $U_1 + \cdots + U_m$ is a direct sum, it follows that

$$U_1 \oplus \cdots \oplus U_m = U_1 + \cdots + U_m.$$

Hence $U_1 \oplus \cdots \oplus U_m$ is finite dimensional by Problem 14. Now we use induction on m to show

$$\dim U_1 \oplus \cdots \oplus U_m = \dim U_1 + \cdots + \dim U_m.$$

By 2.43, for $m = 2$, we have

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2) = \dim U_1 + \dim U_2$$

since $U_1 \cap U_2 = 0$ as $U_1 + U_2$ is a direct sum.

Suppose the equality is true for $m-1$. Now consider the case m, if $U_1 + \cdots + U_m$ is a direct sum, then the only way to write 0 as a sum $u_1 + \cdots + u_m$, where each u_j is in U_j, is by taking each u_j equal to 0. Therefore the only way to write 0 as a sum $u_1 + \cdots + u_{m-1}$, where each u_j is in U_j, is by taking each u_j equal to 0. It follows that $U_1 + \cdots + U_{m-1}$ is a direct sum, hence

$$\dim U_1 \oplus \cdots \oplus U_{m-1} = \dim U_1 + \cdots + \dim U_{m-1}.$$

On the other hand, let $W = U_1 \oplus \cdots \oplus U_{m-1}$, then $U_1 \oplus \cdots \oplus U_m = W + U_m$. Suppose $0 = x + y$, where $x = x_1 + \cdots + x_{m-1} \in W$ and $y \in u_m$, where each x_j is in U_j, it follows from 1.44 that $x_i = 0$ and $y = 0$. Hence $W + U_m$ is a direct sum again by 1.44. Therefore by the inductive assumption, we have

$$\dim U_1 \oplus \cdots \oplus U_m = \dim(W + U_m) = \dim W + \dim U_m = \dim U_1 + \cdots + \dim U_{m-1} + \dim U_m.$$

---

17. Solution: To give a counterexample, let $V = \mathbb{R}^2$, and let

$$U_1 = \{(x, 0) : x \in \mathbb{R}\},$$

$$U_2 = \{(0, y) : y \in \mathbb{R}\},$$

$$U_3 = \{(x, x) : x \in \mathbb{R}\}.$$

Then $U_1 + U_2 + U_3 = \mathbb{R}^2$, so $\dim(U_1 + U_2 + U_3) = 2$. However,

$$\dim U_1 = \dim U_2 = \dim U_3 = 1$$

and

$$\dim(U_1 \cap U_2) = \dim(U_2 \cap U_3) = \dim(U_3 \cap U_1) = \dim(U_1 \cap U_2 \cap U_3) = 0.$$

Thus in this case our guess would reduce to the formula $2 = 3$, which is obviously false.

About Linearity
This website is supposed to help you study Linear Algebras. Please only read these solutions after thinking about the problems carefully. Do not just copy these solutions.

This entry was posted in Chapter 2 and tagged Exercise C.

← Chapter 2 Exercise B   Chapter 3 Exercise A →

Max Zarazov · 2 years ago
#10
no two polynomials are of the same degree => linearly independent
there are m+1 of those => size is the same as dimension of P_m
linearly independent + right size => basis
#14
induction proof is also simple
#17
it's only true iff the following holds:
(U1 + U2) ∩ U3 = (U1 ∩ U2) + (U1 ∩ U3)
which is true iff U1 ⊆ U3 and U2 ⊆ U3
1 · Reply · Share ›

Diego Ramos · 2 years ago
#14
Hi. Your solution of excercise #14 is using dimension of V, but the excercise do not mention nothing about dimV.
Reply · Share ›

Mohammad Rashidi Mod → Diego Ramos · 2 years ago
Yes, you are correct. Thank you.
Reply · Share ›

MetaMusic · 2 years ago
You don't need to use induction for 16; more insight ensues if you use another method
Reply · Share ›

Eric · 2 years ago
The solution to 15 seems incorrect. We are supposed to prove that ONE-DIMENSIONAL subspaces exist, and you don't seem to mention the dimension. I think you are supposed to apply 2.34 recursively: start with some one-dimensional subspace of V called U1. We know from 2.34 that there is some corresponding subspace to U1, let's call it W1, such that U1 ++ W1 = V (where ++ is direct sum).

Then, we replace V with W1 and do the same thing. Pick a one-dimensional subspace of W1 called U2 and apply 2.34 to get some corresponding subspace of W1 and call it W2. U2 ++ W2 = W1. Now we have U1 ++ U2 ++ W2 = U.

Repeat this process until you have Un-1 ++ Wn-1. Wn-1 will be our Un.
Reply · Share ›

Mohammad Rashidi Mod → Eric · 2 years ago
It is correct since U_j is obviously one-dimensional.
Reply · Share ›

Nuno Alvares Pereira · 3 years ago
The solution to 16 is incomplete because the text is wrong. It's an equality, not an inequality.
Reply · Share ›

Chapter 3 Exercise A

Posted on March 1, 2016 by Linearity

1. Suppose $b, c \in \mathbb{R}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz).$$

Show that T is linear if and only if $b = c = 0$.

Solution: If T is linear, then

$$(0, 0) = T(0, 0, 0) = (b, 0)$$

by 3.11, hence $b = 0$. We also have

$$T(1, 1, 1) = T(1, 1, 0) + T(0, 0, 1),$$

it is equivalent to

$$(1 + b, 6 + c) = (b - 2, 6) + (3 + b, 0) = (1 + 2b, 6).$$

Thus $6 + c = 6$ implies $c = 0$.

Conversely, if $b = c = 0$, T is obviously linear. See 3.4 or Problem 3.

You may consider a linear map only has linear terms, hence the terms like xyz is impossible.

---

2. Suppose $b, c \in \mathbb{R}$. Define $T : \mathcal{P}(\mathbb{R}) \to \mathbb{R}^2$ by

$$Tp = \left(3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^{2} x^3 p(x)dx + c \sin p(0)\right).$$

Show that T is linear if and only if $b = c = 0$.

Solution: We show that if $b = c = 0$, then T is linear. Let $f, g \in \mathcal{P}(\mathbb{R})$, then we have

$$(f + g)(4) = f(4) + g(4)$$

and

$$(f + g)'(4) = f'(4) + g'(4).$$

It is easy to check. Moreover, by linearity of integration, one has

$$\int_{-1}^{2} x^3 (f + g)(x)dx = \int_{-1}^{2} x^3 (f(x) + g(x))dx = \int_{-1}^{2} x^3 f(x)dx + \int_{-1}^{2} x^3 g(x)dx.$$

By the above, it follows that

$$\begin{aligned} T(f + g) =&\left(3(f + g)(4) + 5(f + g)'(6), \int_{-1}^{2} x^3 (f + g)(x)dx\right) \\ =&\left(3f(4) + 5f'(6), \int_{-1}^{2} x^3 f(x)dx\right) + \left(3g(4) + 5g'(6), \int_{-1}^{2} x^3 g(x)dx\right) \\ =&Tf + Tg. \end{aligned}$$

Similarly, we can check that $T(\lambda f) = \lambda Tf$ for any $\lambda \in \mathbb{R}$ and $f \in \mathcal{P}(\mathbb{R})$. Conversely, denote the linear map above by S. Then by Problem 5, it follows that $T - S$ is a linear map. That means

$$(T - S)p = (bp(1)p(2), c \sin p(0))$$

is linear. Consider $f(x) = \pi/2$ and $g(x) = \pi/2$, then $f, g \in \mathcal{P}(\mathbb{R})$. We have

$$(T - S)(f + g) = (b\pi^2, c \sin \pi) = (b\pi^2, 0)$$

and

$$(T - S)f + (T - S)g = (b\pi^2/2, c) + (b\pi^2/2, c) = (b\pi^2/2, 2c).$$

Thus, we should have

$$(b\pi^2, 0) = (b\pi^2/2, 2c).$$

It follows that $b = c = 0$.

We do not really need Problem 5, we need them just for simplifying computation.

---

3. Suppose $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$. Show that there exist scalars $A_{j,k} \in \mathbb{F}$ for $j = 1, \cdots, m$ and $k = 1, \cdots, n$ such that

$$T(x_1, \cdots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \cdots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

for every $(x_1, \cdots, x_n) \in \mathbb{F}^n$. [The exercise above shows that T has the form promised in the last item of Example 3.4.]

Solution: If we denote

$$\begin{aligned} T(1, 0, \cdots, 0) &= (A_{1,1}, \cdots, A_{m,1}), \\ T(0, 1, \cdots, 0) &= (A_{1,2}, \cdots, A_{m,2}), \\ &\cdots\cdots \end{aligned}$$

and

$$T(0, 0, \cdots, 0, 1) = (A_{1,n}, \cdots, A_{m,n}).$$

Note that $(1, 0, \cdots, 0), (0, 1, \cdots, 0), \cdots$ and $(0, \cdots, 0, 1)$ is a basis of $\mathbb{F}^n$, then by the proof of 3.5 we conclude that

$$T(x_1, \cdots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \cdots, A_{m,1}x_1 + \cdots + A_{m,n}x_n).$$

---

4. Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \cdots, v_m$ is a list of vectors in V such that $Tv_1, \cdots, Tv_m$ is a linearly independent list in W. Prove that $v_1, \cdots, v_m$ is linearly independent.

Solution: Suppose there are $a_1, \cdots, a_m \in \mathbb{F}$ such that

$$0 = a_1v_1 + \cdots + a_mv_m,$$

then

$$0 = T(a_1v_1 + \cdots + a_mv_m) = a_1Tv_1 + \cdots + a_mTv_m.$$

Note that $Tv_1, \cdots, Tv_m$ is linearly independent, it follows that

$$a_1 = \cdots = a_m = 0,$$

thus $v_1, \cdots, v_m$ is linearly independent.

---

5. Prove the assertions in 3.7.

Solution: It suffices to show that $T + S$ and λT are linear maps provided T and S are linear maps, where $\lambda \in \mathbb{F}$. Then $\mathcal{L}(V, W)$ is closed under addition and scalar multiplication, hence is a vector space.

For any $u, v \in V$, we have

$$\begin{aligned} (T + S)(u + v) =&T(u + v) + S(u + v) = Tu + Tv + Su + Sv \\ =&(Tu + Su) + (Tv + Sv) = (T + S)u + (T + S)v. \end{aligned}$$

The first and last equality hold by the definitions in 3.6, the second equality holds since T and S are linear maps. Similarly, for $\eta \in \mathbb{F}$,

$$\begin{aligned} (T + S)(\eta u) =&T(\eta u) + S(\eta u) = \eta Tu + \eta Su \\ =&\eta(Tu + Su) = \eta(T + S)u. \end{aligned}$$

Combining these arguments, it follows that $T + S$ is a linear map. Again, for any $u, v \in V$, we have

$$\begin{aligned} (\lambda T)(u + v) =&\lambda(T(u + v)) = \lambda(Tu + Tv) \\ =&\lambda(Tu) + \lambda(Tv) = (\lambda T)u + (\lambda T)v. \end{aligned}$$

The first and last equality hold by the definitions in 3.6, the second equality holds since T is a linear map. Similarly, for $\eta \in \mathbb{F}$,

$$\begin{aligned} (\lambda T)(\eta u) =&\lambda(T(\eta u)) = \lambda(\eta T(u)) \\ =&\lambda\eta(Tu) = \eta(\lambda Tu) = \eta(\lambda T)u. \end{aligned}$$

Combining these arguments, it follows that λT is a linear map.

---

6. Prove the assertions in 3.9.

Solution: Associativity: by definition, for any $x \in V$, we have

$$((T_1T_2)T_3)x = (T_1T_2)(T_3x) = T_1(T_2(T_3x)),$$

while

$$(T_1(T_2T_3))x = T_1((T_2T_3)x) = T_1(T_2(T_3x)).$$

Hence $((T_1T_2)T_3)x = (T_1(T_2T_3))x$ for any $x \in V$, therefore $(T_1T_2)T_3 = T_1(T_2T_3)$.

Identity: for any $x \in V$, we have

$$(TI)x = T(Ix) = Tx$$

while

$$(IT)x = I(Tx) = Tx.$$

Hence $(IT)x = (TI)x = Tx$ for any $x \in V$, therefore $IT = TI = T$.

Distributive properties: we only show $(S_1 + S_2)T = S_1T + S_2T$. For any $u \in U$, we have

$$((S_1 + S_2)T)u = (S_1 + S_2)(Tu) = S_1(Tu) + S_2(Tu),$$

while

$$(S_1T + S_2T)u = (S_1T)u + (S_2T)u = S_1(Tu) + S_2(Tu).$$

Hence $((S_1 + S_2)T)u = (S_1T + S_2T)u$ for any $u \in U$, therefore $(S_1 + S_2)T = S_1T + S_2T$. Similarly, we can show $S(T_1 + T_2) = ST_1 + ST_2$.

---

7. Show that every linear map from a 1-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V, V)$, then there exists $\lambda \in \mathbb{F}$ such that $Tv = \lambda v$ for all $v \in V$.

Solution: See Linear Algebra Done Right Solution Manual Chapter 3 Problem 1.

---

8. Give an example of a function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\varphi(av) = av$$

for all $a \in \mathbb{R}$ and all $v \in \mathbb{R}^2$ but φ is not linear.[*The exercise above and the next exercise show that neither homogeneity nor additivity alone is enough to imply that a function is a linear map.*]

Solution: See Linear Algebra Done Right Solution Manual Chapter 3 Problem 2.

---

9. Give an example of a function $\varphi : \mathbb{C} \to \mathbb{C}$ such that

$$\varphi(w + z) = \varphi(w) + \varphi(z)$$

for all $w, z \in \mathbb{C}$ but φ is not linear. (Here $\mathbb{C}$ is thought of as a complex vector space.) [There also exists a function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{R}$ satisfies the additivity condition above but vp is not linear. However, showing the existence of such a function involves considerably more advanced tools.]

Solution: Consider the function defined by $\varphi(a + bi) = a$, where $a, b \in \mathbb{R}$. If $w = \alpha_1 + \beta_1 i$ and $z = \alpha_2 + \beta_2 i$, where $\alpha_1, \alpha_2, \beta_1$ and β_2 are real numbers, then we have

$$\varphi(w + z) = \varphi(\alpha_1 + \beta_1 i + \alpha_2 + \beta_2 i) = \alpha_1 + \alpha_2 = \varphi(w) + \varphi(z).$$

However,

$$i\varphi(1) = i \neq \varphi(i \cdot 1) = 0,$$

hence φ is not linear over $\mathbb{C}$.

For $\varphi : \mathbb{R} \to \mathbb{R}$ case, please see On sort-of-linear functions.

---

10. Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq 0$ (which means that $Su \neq 0$ for some $u \in U$). Define $T : V \to W$ by

$$Tv = \begin{cases} Sv, & \text{if } v \in U, \\ 0, & \text{if } v \in V \text{ and } v \notin U. \end{cases}$$

Prove that T is not a linear map on V.

Solution: Note that $U \neq V$ and $S \neq 0$, we can choose $u \in U$ such that $Su \neq 0$ and $v \in V$ but $v \notin U$, then $u + v \notin U$. Otherwise

$$v = (u + v) - u \in U$$

will yield a contradiction. Hence $T(u + v) = 0$ by definition. On the other hand, $Tu + Tv = Su \neq 0$. It follows that $T(u + v) \neq Tu + Tv$, hence T is not a linear map on V.

---

11. Suppose V is finite-dimensional. Prove that every linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

Solution: Let $u_1, \cdots, u_m$ be a basis of U, then we can extend it to a basis of V by 2.33. That is $u_1, \cdots, u_m, v_{m+1}, \cdots, v_n$ is a basis of V. Define $T \in \mathcal{L}(V, W)$ as below

$$Tu_i = Su_i, \quad Tv_j = 0, \quad 1 \leq i \leq m, m + 1 \leq j \leq n.$$

The existence of T is guaranteed by 3.5(unique). Then for any $u = a_1u_1 + \cdots + a_mu_m$, $a_i \in \mathbb{F}$, we have

$$\begin{aligned} Tu =&T(a_1u_1 + \cdots + a_mu_m) \\ =&a_1Tu_1 + \cdots + a_mTu_m \\ =&a_1Su_1 + \cdots + a_mSu_m \\ =&S(a_1u_1 + \cdots + a_mu_m) = Su. \end{aligned}$$

---

12. Suppose V is finite-dimensional with $\dim V > 0$, and suppose W is infinite-dimensional. Prove that $\mathcal{L}(V, W)$ is infinite-dimensional.

Solution: Here, we use Problem 14 of Exercises 2.A. Hence there is a sequence $W_1, W_2, \cdots$ of vectors in W such that $W_1, W_2, \cdots, W_m$ is linearly independent for every positive integer m. Consider $T_i \in \mathcal{L}(V, W)$ such that $T_i(v_1) = w_i$, where $v_1, v_2, \cdots, v_n$ is a basis of V. The existence of T_i are guaranteed by 3.5(not unique). Then we show that $T_1, \cdots, T_m$ is linearly independent for every positive integer m. Suppose there are $a_1, \cdots, a_m \in \mathbb{F}$ such that

$$a_1T_1 + \cdots + a_mT_m = 0.$$

Then we have $(a_1T_1 + \cdots + a_mT_m)(v_1) = 0$, i.e.

$$a_1w_1 + \cdots + a_mw_m = 0.$$

Since $W_1, W_2, \cdots, W_m$ is linearly independent, it follows that $a_1 = \cdots = a_m = 0$. Thus $T_1, \cdots, T_m$ is linearly independent. Again by Problem 14 of Exercises 2.A, it follows that $\mathcal{L}(V, W)$ is infinite-dimensional.

---

13. Suppose $v_1, v_2, \cdots, v_m$ is a linearly dependent list of vectors in V. Suppose also that $W \neq \{0\}$. Prove that there exist $w_1, w_2, \cdots, w_m \in W$ such that no $T \in \mathcal{L}(V, W)$ satisfies $Tv_k = w_k$ for each $k = 1, \cdots, m$.

Solution: Because $v_1, v_2, \cdots, v_m$ is a linearly dependent, there exist $a_1, \cdots, a_m \in \mathbb{F}$ such that

$$a_1v_1 + \cdots + a_mv_m = 0$$

and some $a_i \neq 0$(this i is fixed). Then let $w_i \neq 0$ while $w_j = 0$ if $j \neq i$, where $w_1, w_2, \cdots, w_m \in W$. We will show there is no $T \in \mathcal{L}(V, W)$ satisfies $Tv_k = w_k$ for each $k = 1, \cdots, m$. Otherwise, we have

$$0 = T(a_1v_1 + \cdots + a_mv_m) = a_1w_1 + \cdots + a_mw_m = a_iw_i.$$

Notice that $a_i \neq 0$ and $w_i \neq 0$ by our choice, we get a contradiction.

---

14. Suppose V is finite-dimensional with $\dim V \geq 2$. Prove that there exist $S, T \in \mathcal{L}(V, V)$ such that $ST \neq TS$.

Solution: Let $e_1, \cdots, e_n$ be a basis of V, define $T \in \mathcal{L}(V, V)$ such that

$$Te_1 = e_2, \quad Te_2 = e_1, \quad Te_i = e_i, \quad \text{for} \quad i \geq 2,$$

and $S \in \mathcal{L}(V, V)$ such that

$$Se_1 = e_1, \quad Se_2 = 2e_2, \quad Se_i = e_i, \quad \text{for} \quad i \geq 2.$$

The existences of S and T are guaranteed by 3.5(unique). Then

$$STe_1 = Se_2 = 2e_2, \quad TSe_1 = Te_1 = e_2.$$

Hence $STe_1 \neq TSe_1$, it implies $ST \neq TS$.

Hanson Char · a year ago

Perhaps the last step in Ex.3 can be illustrated a little more with:

$$\begin{aligned} T\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} &= T\left(x_1\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}\right) \\ &= x_1 \cdot T\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \cdot T\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \cdot T\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \\ &= x_1\begin{pmatrix} A_{1,1} \\ A_{2,1} \\ \vdots \\ A_{m,1} \end{pmatrix} + x_2\begin{pmatrix} A_{1,2} \\ A_{2,2} \\ \vdots \\ A_{m,2} \end{pmatrix} + \cdots + x_n\begin{pmatrix} A_{1,n} \\ A_{2,n} \\ \vdots \\ A_{m,n} \end{pmatrix} \\ \therefore T\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} &= \begin{pmatrix} A_{1,1}x_1 + A_{1,2}x_2 + \cdots + A_{1,n}x_n \\ A_{2,1}x_1 + A_{2,2}x_2 + \cdots + A_{2,n}x_n \\ \vdots \\ A_{m,1}x_1 + A_{m,2}x_2 + \cdots + A_{m,n}x_n \end{pmatrix} \end{aligned}$$

□

Reply · Share ›

Hanson Char · a year ago

Here is an alternative proof of Ex.7:

By 3.5, there exists a unique linear map T such that T(1) = λ, where 1 is the basis of the 1-dimensional vector space V.

So, for all v in F, v·T(1) = v·λ = λv, while, v·T(1) = T(v·1) = Tv, by homogeneity (3.2).

∴ Tv = λv □

Reply · Share ›

Hanson Char · a year ago

Ex. 2 typo.

$$(T - S)f + (T - S)g = (b\pi^2/2, c) + (b\pi^2/2, c) = (b\pi^2/2, 2c).$$

It should be (b*pi^2 / 4, c), not (b*pi^2 / 2, c)

Reply · Share ›

Jared Couzens · 2 years ago

Number 8 has a typo. The second line says "phi(av) = av" when it should say "phi(av)=aphi(v).

Reply · Share ›

korewayume · 2 years ago

Program 2.
the f and g are polynomials which is not linear map, so how we have "(f+g)(4)=f(4)+g(4)" ?

Reply · Share ›

Mohammad Rashidi Mod → korewayume · 2 years ago

The definition of f+g.

Reply · Share ›

Elliott Mestas · 2 years ago

could a shorter proof to 13 use w1, ..., wm linearly independent vectors and a proof by contradiction using exercise 4?

Reply · Share ›

Mohammad Rashidi Mod → Elliott Mestas · 2 years ago

What if dim W< m? Then w1, ..., wm is always linearly dependent.

Reply · Share ›

Elliott Mestas → Mohammad Rashidi · 2 years ago

Ah right, I hadn't thought of that

Reply · Share ›

Nuno Alvares Pereira · 3 years ago

The solution to 11 is wrong. It's an example of a function which we were asked to prove in 10) that it's not a linear function.

Reply · Share ›

Kristian Georgiev → Nuno Alvares Pereira · 3 years ago

It is correct. The construction in 11 does not imply that T of every element not in U is zero. Try it yourself and see (sorry for not using displaymath, I am writing from a smartphone)

Reply · Share ›

Вадим Родимин → Kristian Georgiev · a year ago

It isn't correct. See https://mrashidi568blog.fil... Comment on Chapter 3, problem 3

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Chapter 3 Exercise B

Posted on March 2, 2016 by Linearity

1. Give an example of a linear map T such that $\dim \text{null}T = 3$ and $\dim \text{range}T = 2$.

Solution: Assume V is 5-dimensional vector space with a basis $e_1, \cdots, e_5$. Define $T \in \mathcal{L}(V, V)$ by

$$Te_1 = e_1, Te_2 = e_2, Te_3 = Te_4 = Te_5 = 0.$$

Then $\text{null}T = \text{span}(e_3, e_4, e_5)$, hence $\dim \text{null}T = 3$. Similarly, $\text{range}T = \text{span}(e_1, e_2)$, hence $\dim \text{range}T = 2$.

---

2. Suppose V is a vector space and $S, T \in \mathcal{L}(V, V)$ are such that

$$\text{range}S \subset \text{null}T.$$

Prove that $(ST)^2 = 0$.

Solution: Since $\text{range}S \subset \text{null}T$, it follows that $TSv = 0$ for any $v \in V$. Hence for any $u \in V$,

$$(ST)^2u = S[(TS)Tu] = S0 = 0,$$

i.e. $(ST)^2 = 0$.

---

3. Suppose $v_1, \cdots, v_m$ is a list of vectors in V. Define $T \in \mathcal{L}(\mathbb{F}^m, V)$ by

$$T(z_1, \cdots, z_m) = z_1v_1 + \cdots + z_mv_m.$$

(a) What property of T corresponds to $v_1, \cdots, v_m$ spanning V ?(b) What property of T corresponds to $v_1, \cdots, v_m$ being linearly independent?

Solution: (a) Note that $\text{range}T = \text{span}(v_1, \cdots, v_m)$, hence if $v_1, \cdots, v_m$ spans V, it follows that T is surjective.

(b) Note that $z_1v_1 + \cdots + z_mv_m = 0$ if and only if $(z_1, \cdots, z_m) = (0, \cdots, 0)$ since $v_1, \cdots, v_m$ is linearly independent. Hence $\text{null}T = \{(0, \cdots, 0)\}$, then T is injective.

---

4. Show that

$$\{T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4) : \dim \text{null}T > 2\}$$

is not a subspace of $L(\mathbb{R}^5, \mathbb{R}^4)$.

Solution: Let $e_1, \cdots, e_5$ be a basis of $\mathbb{R}^5$ and f_1, f_2, f_3, f_4 be a basis of $\mathbb{R}^4$. Define S_1 and S_2 by

$$S_1e_i = 0, \quad S_1e_4 = f_1, \quad S_1e_5 = f_2, \quad \text{for~}i = 1, 2, 3;$$

$$S_2e_i = 0, \quad S_2e_3 = f_3 \quad S_2e_5 = f_4, \quad \text{for~}i = 1, 2, 4.$$

Then it is obvious that $S_1, S_2 \in \{T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4) : \dim \text{null}T > 2\}$. However,

$$(S_1 + S_2)e_1 = 0, \quad (S_1 + S_2)e_2 = 0$$

and

$$(S_1 + S_2)e_3 = f_3, \quad (S_1 + S_2)e_4 = f_1, \quad (S_1 + S_2)e_5 = f_2 + f_4.$$

Then you can check that $\dim \text{null}(S_1 + S_2) = 2$. Hence $\{T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4) : \dim \text{null}T > 2\}$ is not closed under addition, this implies it is not a subspace of $L(\mathbb{R}^5, \mathbb{R}^4)$.

---

5. Give an example of a linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that

$$\text{range}T = \text{null}T.$$

Solution: Let e_1, e_2, e_3, e_4 be a basis of $\mathbb{R}^4$. Define $T \in \mathcal{L}(\mathbb{R}^4, \mathbb{R}^4)$ by

$$Te_1 = e_3, Te_2 = e_4, Te_3 = Te_4 = 0.$$

Then $\text{null}T = \text{span}(e_3, e_4)$, and $\text{range}T = \text{span}(e_3, e_4)$. Hence $\text{range}T = \text{null}T$.

---

6. Prove that there does not exist a linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that

$$\text{range}T = \text{null}T.$$

Solution: By 3.22, we know that

$$\dim \text{range}T + \dim \text{null}T = \dim(\mathbb{R}^5) = 5.$$

If $\text{range}T = \text{null}T$, we will get that $\text{range}T = \text{null}T = 2.5$. This is impossible since dimension is an integer.

---

7. Suppose V and W are finite-dimensional with $2 \leq \dim V \leq \dim W$. Show that $\{T \in \mathcal{L}(V, W) : T\text{~is not injective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

Solution: Let $v_1, \cdots, v_n$ be a basis of V and $w_1, \cdots, w_m$ be a basis of W, then we have $2 \leq n \leq m$. Define $T_1, T_2 \in \mathcal{L}(V, W)$ by

$$T_1v_1 = 0, T_1v_i = w_i, i = 2, \cdots, n$$

and

$$T_2v_1 = w_1, T_2v_2 = 0, T_2v_i = w_i, i = 3, \cdots, n.$$

Then $T_1, T_2 \in \{T \in \mathcal{L}(V, W) : T\text{~is not injective}\}$. However, we have

$$(T_1 + T_2)v_1 = w_1, (T_1 + T_2)v_2 = w_2, (T_1 + T_2)v_i = 2w_i, i = 3, \cdots, n.$$

Then by Problem 3 (b), it follows that $T_1 + T_2$ is injective. Hence $\{T \in \mathcal{L}(V, W) : T\text{~is not injective}\}$ is not closed under addition, which implies it is not a subspace of $\mathcal{L}(V, W)$.

---

8. Suppose V and W are finite-dimensional with $\dim V \geq \dim W \geq 2$. Show that $\{T \in \mathcal{L}(V, W) : T\text{~is not surjective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

Solution: Let $v_1, \cdots, v_n$ be a basis of V and $w_1, \cdots, w_m$ be a basis of W, then we have $n \geq m \geq 2$. Define $T_1, T_2 \in \mathcal{L}(V, W)$ by

$$T_1v_1 = 0, T_1v_i = w_i, T_1v_j = 0, i = 2, \cdots, m; j = m + 1, \cdots, n$$

and

$$T_2v_1 = w_1, T_2v_2 = 0, T_2v_i = w_i, T_2v_j = 0, i = 3, \cdots, m; j = m + 1, \cdots, n.$$

Then $T_1, T_2 \in \{T \in \mathcal{L}(V, W) : T\text{~is not surjective}\}$. However, we have

$$(T_1 + T_2)v_1 = w_1, (T_1 + T_2)v_2 = w_2, (T_1 + T_2)v_i = 2w_i, i = 3, \cdots, m;$$

Then by Problem 3 (a), it follows that $T_1 + T_2$ is surjective. Hence $\{T \in \mathcal{L}(V, W) : T\text{~is not surjective}\}$ is not closed under addition, which implies it is not a subspace of $\mathcal{L}(V, W)$.

---

9. Suppose $T \in \mathcal{L}(V, W)$ is injective and $v_1, \cdots, v_n$ is linearly independent in V. Prove that $Tv_1, \cdots, Tv_n$ is linearly independent in W.

Solution: See Linear Algebra Done Right Solution Manual Chapter 3 Problem 5.

---

10. Suppose $v_1, \cdots, v_n$ spans V and $T \in \mathcal{L}(V, W)$. Prove that the list $Tv_1, \cdots, Tv_n$ spans $\text{range}T$.

Solution: Note that $v_1, \cdots, v_n$ spans V, any $v \in V$ can be written as a linear combination of $v_1, \cdots, v_n$. That is there are $a_1, \cdots, a_n \in \mathbb{F}$ such that

$$v = a_1v_1 + \cdots + a_nv_n.$$

Since $T \in \mathcal{L}(V, W)$, it follows that

$$Tv = a_1Tv_1 + \cdots + a_nT_n.$$

Hence $\text{range}T \subset \text{span}(Tv_1, \cdots, Tv_n)$. On the other hand $Tv_1, \cdots, Tv_n$ are contained in $\text{range}T$. By the definition of span, we conclude that $Tv_1, \cdots, Tv_n$ spans $\text{range}T$.

See a similar problem in Linear Algebra Done Right Solution Manual Chapter 3 Problem 7.

---

11. Suppose $S_1, \cdots, S_n$ are injective linear maps such that $S_1S_2 \cdots S_n$ makes sense. Prove that $S_1S_2 \cdots S_n$ is injective.

Solution: See Linear Algebra Done Right Solution Manual Chapter 3 Problem 6.

---

12. Suppose that V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap \text{null} = \{0\}$ and $\text{range}T = \{Tu : u \in U\}$.

Solution: See Linear Algebra Done Right Solution Manual Chapter 3 Problem 8.

---

13. Suppose T is a linear map from $\mathbb{F}^4$ to $\mathbb{F}^2$ such that

$$\text{null}T = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_1 = 5x_2 \text{ and} x_3 = 7x_4\}.$$

Prove that T is surjective.

Solution: See Linear Algebra Done Right Solution Manual Chapter 3 Problem 9.

---

14. Suppose U is a 3-dimensional subspace of $\mathbb{R}^8$ and that T is a linear map from $\mathbb{R}^8$ to $\mathbb{R}^5$ such that $\text{null}T = U$. Prove that T is surjective.

Solution: By 3.22, we have

$$\dim \text{null}T + \dim \text{range}T = \dim(\mathbb{R}^8) = 8.$$

Note that $\text{null}T = U$ and $\dim U = 3$, it follows that

$$\dim \text{range}T = 8 - \dim \text{null}T = 8 - 3 = 5 = \dim(\mathbb{R}^5).$$

Therefore T is surjective by Problem 1 of Exercises 2.C.

---

15. Prove that there does not exist a linear map from $\mathbb{F}^5$ to $\mathbb{F}^2$ whose null space equals

$$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 : x_1 = 3x_2 \text{and} x_3 = x_4 = x_5\}.$$

Solution: Solution: See Linear Algebra Done Right Solution Manual Chapter 3 Problem 10.

---

16. Suppose there exists a linear map on V whose null space and range are both finite-dimensional. Prove that V is finite-dimensional.

Solution: Solution: See Linear Algebra Done Right Solution Manual Chapter 3 Problem 11.

---

17. Suppose V and W are both finite-dimensional. Prove that there exists an injective linear map from V to W if and only if $\dim V \leq \dim W$.

Solution: By 3.22, it follows that for any injective $T \in \mathcal{L}(V, W)$, we have

$$\dim V = \dim \text{null}T + \dim \text{range}T = \dim \text{range}T \leq \dim W.$$

Hence there exists an injective linear map from V to W, then $\dim V \leq \dim W$.

If $n = \dim V \leq \dim W = m$, then let $v_1, \cdots, v_n$ and $w_1, \cdots, w_m$ be the bases of V and W, respectively. Define $T \in \mathcal{L}(V, W)$ such that

$$Tv_i = w_i, \quad i = 1, \cdots, n.$$

Here we use $n \leq m$. Similar to Problem 3(b), we can show that T is injective.

---

18. Suppose V and W are both finite-dimensional. Prove that there exists an surjective linear map from V to W if and only if $\dim V \geq \dim W$.

Solution: See Linear Algebra Done Right Solution Manual Chapter 3 Problem 12. It is similar to Problem 17.

---

19. Suppose V and W are finite-dimensional and that U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\text{null}T = U$ if and only if $\dim U \geq \dim V - \dim W$.

Solution: See Linear Algebra Done Right Solution Manual Chapter 3 Problem 13.

---

20. Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V.

Solution: See Linear Algebra Done Right Solution Manual Chapter 3 Problem 14.

---

21. Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that TS is the identity map on V.

Solution: See Linear Algebra Done Right Solution Manual Chapter 3 Problem 15.

---

22. Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \text{null}ST \leq \dim \text{null}S + \dim \text{null}T.$$

Solution: See Linear Algebra Done Right Solution Manual Chapter 3 Problem 16.

---

23. Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \text{range}ST \leq \min\{\dim \text{range}S, \dim \text{range}T\}.$$

Solution: It is obvious that $\text{range}ST \subset \text{range}S$, hence $\dim \text{range}ST \leq \dim \text{range}S$. Let $u_1, \cdots, u_m$ be a basis of $\text{range}T$, then by Problem 10, we have

$$\text{range}ST = \text{span}(Su_1, \cdots, Su_m).$$

Hence $\dim \text{range}ST \leq m = \dim \text{range}T$. Thus

$$\dim \text{range}ST \leq \min\{\dim \text{range}S, \dim \text{range}T\}.$$

---

24. Suppose W is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\text{null}T_1 \subset \text{null}T_2$ if and only if there exists $S \in \mathcal{L}(W, W)$ such that $T_2 = ST_1$.

Solution: If we assume $\text{null}T_1 \subset \text{null}T_2$. Let $u_1, \cdots, u_m$ be a basis of $\text{null}T_1$, then it can be extended to a basis $u_1, \cdots, u_m, v_1, \cdots, v_n$ of V. Then from the process of the proof of 3.22, we know that

$$T_1v_1, \cdots, T_1v_n$$

is linear independent. Hence we can extend it to a basis $T_1v_1, \cdots, T_1v_n, \nu_1, \cdots, \nu_l$ of W. Now define $S \in \mathcal{L}(W, W)$ such that

$$ST_1v_i = T_2v_i, \quad S\nu_j = 0, \quad i = 1, \cdots, n; j = 1, \cdots, l.$$

Then $ST_1v_i = T_2v_i$ and $ST_1u_j = 0 = T_2u_j$ since $\text{null}T_1 \subset \text{null}T_2$, hence $ST_1 = T_2$ by uniqueness in 3.5.

If there exists $S \in \mathcal{L}(W, W)$ such that $T_2 = ST_1$, then for any $\mu \in \text{null}T_1$, we have

$$T_2\mu = ST_1\mu = S0 = 0.$$

Hence $\mu \in \text{null}T_2$, it follows that $\text{null}T_1 \subset \text{null}T_2$.

---

25. Suppose W is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\text{range}T_1 \subset \text{range}T_2$ if and only if there exists $S \in \mathcal{L}(V, V)$ such that $T_1 = T_2S$.

Solution: If we assume $\text{range}T_1 \subset \text{range}T_2$. Let $u_1, \cdots, u_m$ be a basis of V, then we can find $v_1, \cdots, v_m \in V$ such that $T_1u_i = T_2v_i$ for $i = 1, \cdots, m$ since $\text{range}T_1 \subset \text{range}T_2$. Define $S \in \mathcal{L}(V, V)$ by $Su_i = v_i$, then we have

$$T_1u_i = T_2v_i = T_2Su_i, \quad i = 1, \cdots, m,$$

hence $T_1 = T_2S$ by uniqueness in 3.5.

If there exists $S \in \mathcal{L}(V, V)$ such that $T_1 = T_2S$, then for any $\mu \in V$, we have

$$T_1\mu = T_2S\mu \in \text{range}T_2.$$

Hence $\text{range}T_1 \subset \text{range}T_2$.

---

26. Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ is such that $\deg Dp = \deg p - 1$ for every nonconstant polynomial $p \in \mathcal{P}(\mathbb{R})$. Prove that D is surjective. [The notation D is used above to remind you of the differentiation map that sends a polynomial p to p'. Without knowing the formula for the derivative of a polynomial (except that it reduces the degree by 1), you can use the exercise above to show that for every polynomial $q \in \mathcal{P}(\mathbb{R})$, there exists a polynomial $p \in \mathcal{P}(\mathbb{R})$ such that $p' = q$.]

Solution: Consider Problem 10 of Exercise 2C. Note that we have

$$\deg Dx^n = n - 1,$$

it follows that for any $j \in \mathbb{N}$, there is a polynomial Dx^{j+1} with degree j. Note that $\text{range}D$ is a subspace, hence

$$\text{span}(Dx, Dx^2, \cdots) \subset \text{range}D.$$

Moreover, by Problem 10 of Exercise 2C, we have

$$\text{span}(Dx, Dx^2, Dx^3, \cdots) = \text{span}(1, x, x^2, \cdots).$$

Hence $\text{span}(1, x, \cdots) \subset \text{range}D$, namely $\text{range}D = \mathcal{P}(\mathbb{R})$.

---

27. Suppose $p \in \mathcal{P}(\mathbb{R})$. Prove that there exists a polynomial $q \in \mathcal{P}(\mathbb{R})$ such that $5q'' + 3q' = p$. [This exercise can be done without linear algebra, but it's more fun to do it using linear algebra.]

Solution: Denote the differentiation map by D, then $5D^2 + 3D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ is of the type in Problem 26. Hence $5D^2 + 3D$ is surjective, thus there exists $q \in \mathcal{P}(\mathbb{R})$ such that $(5D^2 + 3D)q = p$. Note that $(5D^2 + 3D)q = 5q'' + 3q'$, the proof completes.

---

28. Suppose $T \in \mathcal{L}(V, W)$ and $w_1, \cdots, w_m$ is a basis of range T. Prove that there exist $\varphi_1, \cdots, \varphi_m \in \mathcal{L}(V, \mathbb{F})$ such that

$$Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m$$

for every $v \in V$.

Solution: For any $v \in V$, there is a uniquely linear combination

$$Tv = a_1w_1 + \cdots + a_mw_m,$$

where $a_1, \cdots, a_m \in \mathbb{F}$, as $w_1, \cdots, w_m$ is a basis of range T. Denote a_i by $\varphi_i(v)$. Now we check that $\varphi_i \in \mathcal{L}(V, \mathbb{F})$, $i = 1, \cdots, m$. For $u, v \in V$, we have

$$Tu = \varphi_1(u)w_1 + \cdots + \varphi_m(u)w_m$$

and

$$Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m.$$

Similarly, we also have

$$T(u + v) = \varphi_1(u + v)w_1 + \cdots + \varphi_m(u + v)w_m.$$

Since $T \in \mathcal{L}(V, W)$, it follows that

$$\begin{aligned}&\varphi_1(u + v)w_1 + \cdots + \varphi_m(u + v)w_m \\ =&T(u + v) = Tu + Tv \\ =&\varphi_1(u)w_1 + \cdots + \varphi_m(u)w_m + \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m \\ =&(\varphi_1(u) + \varphi_1(v))w_1 + \cdots + (\varphi_m(u) + \varphi_m(v))w_m.\end{aligned}$$

As $w_1, \cdots, w_m$ is a basis of range T, it follows that

$$\varphi_i(u + v) = \varphi_i(u) + \varphi_i(v), i = 1, \cdots, m.$$

Similarly, by consider $T(\lambda u) = \lambda Tu$ for $\lambda \in \mathbb{F}$, it follows that

$$\varphi_i(\lambda u) = \lambda\varphi_i(u), i = 1, \cdots, m.$$

Hence $\varphi_i \in \mathcal{L}(V, \mathbb{F})$.

---

29. Suppose $\varphi \in \mathcal{L}(V, \mathbb{F})$. Suppose $u \in V$ is not in null φ. Prove that

$$V = \text{null}\varphi \oplus \{au : a \in \mathbb{F}\}.$$

Solution: See Linear Algebra Done Right Solution Manual Chapter 3 Problem 4.

---

30. Suppose φ_1 and φ_2 are linear maps from V to $\mathbb{F}$ that have the same null space. Show that there exists a constant $c \in \mathbb{F}$ such that $\varphi_1 = c\varphi_2$.

Solution: If $\text{null}\varphi_1 = \text{null}\varphi_2 = V$, then $\varphi_1 = \varphi_2 = 0$.

If $\text{null}\varphi_1 = \text{null}\varphi_2 \neq V$, then there is a $u \in V$ is not in null φ_2, namely $\varphi_2(u) \neq 0$. By problem 29, we have

$$V = \text{null}\varphi_1 \oplus \{au : a \in \mathbb{F}\}.$$

Hence for any $v \in V$, v can be written as $w + a_vu$ where $w \in \text{null}\varphi_1$. Then $\varphi_1(w) = \varphi_2(w) = 0$, we have

$$\begin{aligned}\frac{\varphi_1(u)}{\varphi_2(u)}\varphi_2(v) =&\frac{\varphi_1(u)}{\varphi_2(u)}\varphi_2(w + a_vu) = \frac{\varphi_1(u)}{\varphi_2(u)}a_v\varphi_2(u) \\ =&a_v\varphi_1(u) = \varphi_1(w + a_vu) \\ =&\varphi_1(v),\end{aligned}$$

i.e. $\varphi_1 = \frac{\varphi_1(u)}{\varphi_2(u)}\varphi_2$. Let $c = \frac{\varphi_1(u)}{\varphi_2(u)}$, we complete the proof.

---

31. Give an example of two linear maps T_1 and T_2 from $\mathbb{R}^5$ to $\mathbb{R}^2$ that have the same null space but are such that T_1 is not a scalar multiple of T_2.

Solution: Let e_1, e_2, e_3, e_4, e_5 and f_1, f_2 be a basis of $\mathbb{R}^5$ and $\mathbb{R}^2$, respectively. Define $T_1 \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^2)$ by

$$T_1e_1 = f_1, T_1e_2 = f_2, T_1e_3 = T_1e_4 = T_1e_5 = 0.$$

Define $T_2 \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^2)$ by

$$T_2e_1 = f_1, T_2e_2 = 2f_2, T_2e_3 = T_2e_4 = T_2e_5 = 0.$$

Then

$$\text{null}T_1 = \text{null}T_2 = \text{span}(e_3, e_4, e_5)$$

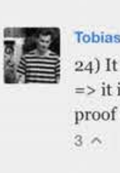
and T_1 is not a scalar multiple of T_2. Otherwise, if $T_1 = cT_2$, then

$$f_1 = T_1e_1 = cT_2e_1 = cf_1.$$

We get $c = 1$. However, from

$$f_2 = T_1e_2 = cT_2e_2 = 2cf_2,$$

we get $c = 1/2$.



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This entry was posted in Chapter 3 and tagged Exercise B.

← Chapter 3 Exercise A Chapter 3 Exercise C →

Tobias Pielok · 2 years ago
24) It is not stated that V is finite-dimensional => null T1 doesn't have to be finite-dimensional => it is possible that there is no basis for null T1. But I think it is possible to construct a general proof with the range, which is finite dimensional.
3 · Reply · Share ›

Célio Passos → Tobias Pielok · 2 years ago
You can use the same idea from the solution to Exercise 4 in 3.D.
Reply · Share ›

Sam Tay → Célio Passos · 7 months ago
Do you think this is a mistake in the book? Or is there a proof the author had in mind that only uses the results up to 3.B.24? I've been stuck on this one for a while.
Reply · Share ›

Show more replies

Sam Tay · 7 months ago
The definition of S is too loose in Problem 25. In particular, your version is not guaranteed to be well defined (consider when some of those u_i, v_i are in null spaces). You'll find that you need to have a basis of null T2, extend it to a basis of V, and map S(u_i) to v where v is in the span of the extension list, and has no part in the null space of T2, because that choice of v is unique. In particular, there may be many more such v where T1(u_i) = T2(v), but only one of them is in the span of the extension list.

Edit: Come to think of it, I guess it doesn't matter that you pick the unique v in that span. So long as you choose one of those v_i, the proof holds. So never mind.
Reply · Share ›

MetaMusic · 2 years ago
You assumed V is a finite-dimensional vector space for #24 and you wrote W instead of V for #25
Reply · Share ›

Elliott Mestas · 2 years ago
I believe you can shorten the proofs to both 17 and 18 (proving the "=>" direction) by using the contrapositive statements of 3.23 and 3.24
Reply · Share ›

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Chapter 3 Exercise C

Posted on March 3, 2016 by Linearity

1. Solution: Suppose for some basis $v_1, \cdots, v_n$ of V and some basis $w_1, \cdots, w_m$ of W, the matrix of T has at most $\dim \operatorname{range} T - 1$ nonzero entries. Then there are at most $\dim \operatorname{range} T - 1$ nonzero vectors in $Tv_1, \cdots, Tv_n$. Note that $\operatorname{range} T = \operatorname{span}(Tv_1, \cdots, Tv_n)$, it follows that

$$\dim \operatorname{range} T \leq \dim \operatorname{range} T - 1.$$

We get a contradiction, hence completing the proof.

---

2. Solution: The basis of $\mathcal{P}(\mathbb{R}^3)$ is $x^3, x^2, x, 1$. The corresponding basis of $\mathcal{P}(\mathbb{R}^2)$ is $3x^2, 2x, 1$ (with the indicated order).(Check it!)

---

3. Solution: We use the notation and the proof of 3.22. Extend $Tv_1, \cdots, Tv_n$ to a basis of W as $Tv_1, \cdots, Tv_n, \mu_1, \cdots, \mu_s$. Then with respect to the basis $v_1, \cdots, v_n, u_1, \cdots, u_m$ of V and the basis $Tv_1, \cdots, Tv_n, \mu_1, \cdots, \mu_s$ of W, all entries of $\mathcal{M}(T)$ are 0 except that the entries in row j, column j, equal 1 for $1 \leq j \leq \dim \operatorname{range} T$.

---

4. Solution: If $Tv_1 = 0$, then any basis $w_1, \cdots, w_n$ of W will satisfy the desired conditions. If $Tv_1 \neq 0$, then any basis $w_1, \cdots, w_n$ of W such that $w_1 = Tv_1$ will satisfy the desired conditions.

---

5. Solution: Let $\nu_1, \cdots, \nu_m$ be a basis of V, denote the first row of $\mathcal{M}(T)$ with respect to the bases $\nu_1, \cdots, \nu_m$ and $w_1, \cdots, w_n$ by $(a_1, \cdots, a_m)$. If $(a_1, \cdots, a_m) = 0$, then we can choose $v_i = \nu_i, i = 1, \cdots, m$. If $(a_1, \cdots, a_m) \neq 0$, suppose $a_i \neq 0$. Then let

$$v_1 = \frac{\nu_i}{a_i}, v_j = \nu_{j-1} - a_{j-1}v_1, v_k = \nu_k - a_k v_1$$

for $j = 2, \cdots, i$, $k = i + 1, \cdots, m$. Then you can check that $v_1, \cdots, v_m$ satisfies the desired conditions.

---

6. Solution: Suppose there exist a basis $v_1, \cdots, v_m$ of V and a basis $w_1, \cdots, w_n$ of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ equal 1. Then

$$Tv_i = w_1 + \cdots + w_n, \quad i = 1, \cdots, m.$$

Hence $\operatorname{range} T = \operatorname{span}(w_1 + \cdots + w_n)$, it follows that $\dim \operatorname{range} T = 1$.

Conversely, if $\dim \operatorname{range} T = 1$, then $\dim \operatorname{null} T = \dim V - 1$. Let $\nu_1, \nu_2, \cdots, \nu_m$ be a basis of V such that $\nu_2, \cdots, \nu_m \in \operatorname{null} T$. Note that $T\nu_1 \neq 0$, hence we can extend it to a basis of W as $T\nu_1, w_2, \cdots, w_n$. Let $w_1 = T\nu_1 - w_2 - \cdots - w_n$ and $v_1 = \nu_1$, $v_i = \nu_i + \nu_1$ for $i = 2, \cdots, m$.

$$T(v_1) = T(v_i) = w_1 + w_2 + \cdots + w_n, \quad i = 2, \cdots, m.$$

It is obvious that $v_1, \cdots, v_m$ is a basis of V and $w_1, \cdots, w_n$ is a basis of W. Now we can directly check that all entries of $\mathcal{M}(T)$ with respect to these bases equal 1.

---

7. Solution: Given a basis $v_1, \cdots, v_m$ of V and a basis $w_1, \cdots, w_n$ of W, denote v and $\mathcal{M}(S)$ with respect to these bases by A and B, respectively. Then we have

$$Tv_j = \sum_{k=1}^{n} A_{k,j} w_k$$

and

$$Sv_j = \sum_{k=1}^{n} B_{k,j} w_k.$$

Hence

$$(T+S)v_j = Tv_j + Sv_j = \sum_{k=1}^{n} (A_{k,j} + B_{k,j}) w_k,$$

it follows that the entries in row k, column j of $\mathcal{M}(T+S)$ with respect to these bases are $A_{k,j} + B_{k,j}$. By 3.35, we deduce that $\mathcal{M}(T+S) = \mathcal{M}(T) + \mathcal{M}(S)$.

---

8. Verify 3.38.

Solution: It is almost the same as the previous Exercise.

---

9. Prove 3.52.

Solution: It is almost the same as Problem 11. Just consider the entries.

---

10. Suppose A is an m-by-n matrix and C is an n-by-p matrix. Prove that

$$(AC)_{j,\cdot} = A_{j,\cdot} C$$

for $1 \leq j \leq m$. In other words, show that row j of AC equals (row j of A) times C.

Solution: It is almost the same as Problem 11. Just consider the entries.

These exercises are tedious. I prefer solving other interesting exercises… If you have problems regarding to them, please make a comment.

---

11. Solution: By 3.41, we have

$$(aC)_{1,k} = \sum_{i=1}^{n} a_i C_{i,k}.$$

It is obvious that $(a_i C_{i,\cdot})_{1,k} = a_i C_{i,k}$. Hence

$$(aC)_{1,k} = (a_1 C_{1,\cdot})_{1,k} + \cdots + (a_n C_{n,\cdot})_{1,k} = (a_1 C_{1,\cdot} + \cdots + a_n C_{n,\cdot})_{1,k}$$

by 3.35. Thus we deduce that $aC = a_1 C_{1,\cdot} + \cdots + a_n C_{n,\cdot}$.

---

12. Solution: Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, then we have

$$AC = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

and

$$CA = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}.$$

Hence $AC \neq CA$.

---

13. Solution: See Linear Algebra Done Right Solution Manual Chapter 3 Problem 17.

---

14. Solution: See Linear Algebra Done Right Solution Manual Chapter 3 Problem 18.

---

15. Solution: Note that $AAA = (AA)A$, denote $AA = B$, then by definition (3.41)

$$B_{j,r} = \sum_{p=1}^{n} A_{j,p} A_{p,r}. \tag{1}$$

Similarly, the entry in row j, column k, of A^3 is

$$\sum_{r=1}^{n} B_{j,r} A_{r,k}.$$

Hence by (1), we have

$$\sum_{r=1}^{n} B_{j,r} A_{r,k} = \sum_{r=1}^{n} \sum_{p=1}^{n} A_{j,p} A_{p,r} A_{r,k} = \sum_{p=1}^{n} \sum_{r=1}^{n} A_{j,p} A_{p,r} A_{r,k}.$$

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This entry was posted in Chapter 3 and tagged Exercise C.

← Chapter 3 Exercise B | Chapter 3 Exercise D →

AliD · 9 months ago

Hello,

I think there is a minor issue in your solution to 3.C.3 since T(v1),...,T(vn) is not guaranteed to be linearly independent (since T might not be injective). I think you should instead find a basis of null T, then extend this to a basis of V. Then T is injective on the basis vectors not in null T, so use your solution on these vectors.

^ v · Reply · Share ›

Mohammad Rashidi Mod → AliD · 9 months ago

There are no issues. It's said we are using the notation as in the proof of 3.22.

^ v · Reply · Share ›

Stat student · a year ago

Hi Mohammad,

I'm having a little trouble understanding the solution of 3.C.1. Could you enlighten me on why the the matrix of T has at most dimRange T - 1 nonzero entries?

^ v · Reply · Share ›

Math student · 2 years ago

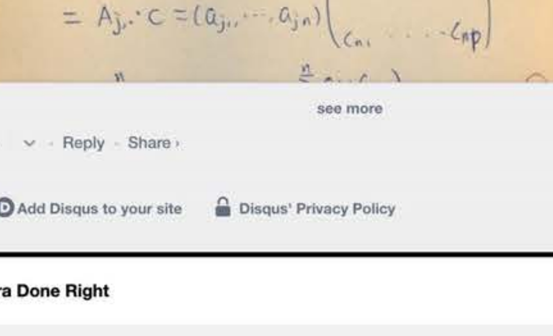
Hi Mohammad,

Thank you for your work, I'm a big fan of this page. I'm stuck as to to solve for 3.C.10; would appreciate your insight.

^ v · Reply · Share ›

Mohammad Rashidi Mod → Math student · 2 years ago

See the following picture. If you have trouble with that, just write down the matrix explicitly. Then just write down all matrices explicitly and compare them. However, this is not a good solution. It might be better to use another solution for which one can see the structure behind it. I don't know how to make this clear in English.



see more

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Chapter 3 Exercise D

Posted on March 4, 2016 by Linearity

1. Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ / is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

Solution: See Linear Algebra Done Right Solution Manual Chapter 3 Problem 22. It is almost the same.

---

2. Suppose V is finite-dimensional and $\dim V > 1$. Prove that the set of noninvertible operators on V is not a subspace of $\mathcal{L}(V)$.

Solution: See Linear Algebra Done Right Solution Manual Chapter 3 Problem 25.

---

3. Suppose V is finite-dimensional, U is a subspace of V, and $S \in \mathcal{L}(U, V)$. Prove there exists an invertible operator $T \in \mathcal{L}(V)$ such that $Tu = Su$ for every $u \in U$ if and only if S is injective.

Solution: If there exists an invertible operator $T \in \mathcal{L}(V)$ such that $Tu = Su$ for every $u \in U$, then S is injective since T is injective.

If S is injective. Assume $u_1, \cdots, u_m$ is a basis of U, we can extend it to a basis of V as $u_1, \cdots, u_m, v_{m+1}, \cdots, v_n$. Since S is injective, by Problem 9 of Exercises 3B, we have $Su_1, \cdots, Su_m$ is linearly independent in V. Hence we can extend it to a basis of V as $Su_1, \cdots, Su_m, w_{m+1}, \cdots, w_n$. Define $T \in \mathcal{L}(V)$ as below

$$Tu_i = Su_i, \quad Tv_j = w_j, \quad 1 \leq i \leq m, m+1 \leq j \leq n.$$

The existence of T is guaranteed by 3.5(unique). Then for any $u = a_1u_1 + \cdots + a_mu_m$, $a_i \in \mathbb{F}$, we have

$$\begin{aligned} Tu =& T(a_1u_1 + \cdots + a_mu_m) \\ =& a_1Tu_1 + \cdots + a_mTu_m \\ =& a_1Su_1 + \cdots + a_mSu_m \\ =& S(a_1u_1 + \cdots + a_mu_m) = Su. \end{aligned}$$

Moreover, T is surjective by Problem 10 of Exercises 3B hence invertible by 3.69.

Compare this problem with Problem 11 of Exercises 3A.

---

4. Suppose W is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that null T_1 = null T_2 if and only if there exists an invertible operator $S \in \mathcal{L}(W)$ such that $T_1 = ST_2$.

Solution: If we assume $\text{null}T_1 = \text{null}T_2$. Since W is finite-dimensional, so is range T_2. Let $w_1, \cdots, w_n$ be a basis of rangeT_2, then there exist $v_1, \cdots, v_n \in V$ such that

$$T_2v_i = w_i, \quad i = 1, \cdots, n.$$

Now we will show that $V = \text{null}T_2 \oplus \text{span}(v_1, \cdots, v_n)$. For any $v \in V$, we

$$T_2v = a_1w_1 + \cdots + a_nw_n$$

for some $a_1, \cdots, a_n \in \mathbb{F}$. Hence

$$T_2(v - a_1v_1 - \cdots - a_nv_n) = 0,$$

namely

$$v = (v - a_1v_1 - \cdots - a_nv_n) + (a_1v_1 + \cdots + a_nv_n),$$

this implies $V = \text{null}T_2 + \text{span}(v_1, \cdots, v_n)$. Moreover, if $a_1v_1 + \cdots + a_nv_n \in \text{null}T_2$, then we have

$$T_2(a_1v_1 + \cdots + a_nv_n) = a_1w_1 + \cdots + a_nw_n = 0.$$

Note that $w_1, \cdots, w_n$ is linearly independent, it follows $a_1 = \cdots = a_n = 0$. Thus we have

$$V = \text{null}T_2 \oplus \text{span}(v_1, \cdots, v_n).$$

Similarly, $T_1v_1, \cdots, T_1v_n$ is linearly independent. For if $a_1T_1v_1 + \cdots + a_nT_1v_n = 0$, we have

$$T_1(a_1v_1 + \cdots + a_nv_n) = 0.$$

Note that $\text{null}T_1 = \text{null}T_2$, it follows that

$$0 = T_2(a_1v_1 + \cdots + a_nv_n) = a_1w_1 + \cdots + a_nw_n.$$

Thus $a_1 = \cdots = a_n = 0$. Now extend $w_1, \cdots, w_n$ to a basis of W as $w_1, \cdots, w_n, e_1, \cdots, e_m$ and $T_1v_1, \cdots, T_1v_n$ to a basis of W as $T_1v_1, \cdots, T_1v_n, f_1, \cdots, f_m$. Define $S \in \mathcal{L}(W)$ by

$$Sw_i = T_1v_i, Se_j = f_j, i = 1, \cdots, n; j = 1, \cdots, m.$$

Note that

$$V = \text{null}T_2 \oplus \text{span}(v_1, \cdots, v_n),$$

any $v \in V$ can be expressed as

$$v = v_{\text{null}} + a_1v_1 + \cdots + a_nv_n,$$

where $v_{\text{null}} \in \text{null}T_2 = \text{null}T_1$ and $a_1, \cdots, a_n \in \mathbb{F}$. Hence we have

$$\begin{aligned} ST_2(v) =& ST_2(v_{\text{null}} + a_1v_1 + \cdots + a_nv_n) \\ =& ST_2(a_1v_1 + \cdots + a_nv_n) \\ =& S(a_1w_1 + \cdots + a_nw_n) \\ =& a_1T_1v_1 + \cdots + a_nT_1v_n \\ =& T_1(a_1v_1 + \cdots + a_nv_n) \\ =& T_1(v_{\text{null}} + a_1v_1 + \cdots + a_nv_n) = T_1(v) \end{aligned}$$

namely $ST_2 = T_1$. Moreover, S is surjective by Problem 10 of Exercises 3B hence invertible by 3.69.

If there exists an invertible operator $S \in \mathcal{L}(W)$ such that $ST_2 = T_1$, then for any $\mu \in \text{null}T_1$, we have

$$ST_2\mu = T_1\mu = 0.$$

As S is invertible, we have $T_2\mu = 0$. Hence $\mu \in \text{null}T_2$, it follows that $\text{null}T_1 \subset \text{null}T_2$. Similarly, consider $T_2 = S^{-1}T_1$, $\text{null}T_2 \subset \text{null}T_1$. Thus we conclude null T_1= null T_2.

Compare this problem with Problem 24 of Exercises 3B.

---

5. Suppose V is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that range T_1 = range T_2 if and only if there exists an invertible operator $S \in \mathcal{L}(V)$ such that $T_1 = T_2S$.

Solution: If we assume $\text{range}T_1 = \text{range}T_2$. Let $u_1, \cdots, u_m$ be a basis of null T_1, then we can extend it to a basis of V as $u_1, \cdots, u_m, w_1, \cdots, w_n$. Then range T_1 is $\text{span}(T_1w_1, \cdots, T_1w_n)$ and $T_1w_1, \cdots, T_1w_n$ is linearly independent. There exist $v_1, \cdots, v_n \in V$ such that $T_1w_i = T_2v_i$ for $i = 1, \cdots, n$ since $\text{range}T_1 = \text{range}T_2$. As $T_1w_1, \cdots, T_1w_n$ is linearly independent, it follows that $v_1, \cdots, v_n$ is linearly independent by Problem 4 of Exercises 3A. Note that range T_1 = range T_2 implies null T_1 and null T_2 have the same dimension. Let $\zeta_1, \cdots, \zeta_m$ be a basis of null T_2, then $\zeta_1, \cdots, \zeta_m, v_1, \cdots, v_n$ is a basis of V by the proof of 3.22. Define $S \in \mathcal{L}(V)$ by $Su_i = \zeta_i$ and $Sw_j = v_j$, then we have

$$T_1w_j = T_2v_j = T_2Sw_j, \quad j = 1, \cdots, n,$$

and

$$T_1u_i = 0 = T_2\zeta_i = T_2Su_i, \quad i = 1, \cdots, m,$$

hence $T_1 = T_2S$ by uniqueness in 3.5. Moreover, S is surjective by Problem 10 of Exercises 3B hence invertible by 3.69.

If there exists an invertible operator $S \in \mathcal{L}(V)$ such that $T_1 = T_2S$, then for any $\mu \in V$, we have

$$T_1\mu = T_2S\mu \in \text{range}T_2.$$

Hence $\text{range}T_1 \subset \text{range}T_2$. As S is invertible, we have $T_2 = T_1S^{-1}$. Similarly, we conclude $\text{range}T_1 \subset \text{range}T_2$. Thus range T_1 = range T_2.

Compare this problem with Problem 25 of Exercises 3B.

---

6. Suppose V and W are finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that there exist invertible operators $R \in \mathcal{L}(V)$ and $S \in \mathcal{L}(W)$ such that $T_1 = ST_2R$ if and only if dim null T_1 = dim null T_2.

Solution: If there exist invertible operators $R \in \mathcal{L}(V)$ and $S \in \mathcal{L}(W)$ such that $T_1 = ST_2R$, then $S^{-1}T_1 = T_2R$. Hence null T_1= null T_2R by Problem 4. Note that we have range T_2R= range T_2 by Problem 5, it follows that

$$\dim \text{null}T_2R = \dim V - \dim \text{range}T_2R = \dim V - \dim \text{range}T_2 = \dim \text{null}T_2.$$

Hence $\dim \text{null}T_1 = \dim \text{null}T_2$.

Conversely, if dim null T_1 = dim null T_2. Let $u_1, \cdots, u_m$ be a basis of null T_1, then we can extend it to a basis of V as $u_1, \cdots, u_m, w_1, \cdots, w_n$. Let $v_1, \cdots, v_m$ be a basis of null T_2, then we can extend it to a basis of V as $v_1, \cdots, v_m$, $\zeta_1, \cdots, \zeta_n$. Then $T_1w_1, \cdots, T_1w_n$ is linearly independent in W, hence we can extend it to a basis of V as T_1w_1, $\cdots, T_1w_n, \alpha_1, \cdots, \alpha_l$. Similarly, $T_2\zeta_1, \cdots, T_2\zeta_n$ is linearly independent in W, hence we can extend it to a basis of V as $T_2\zeta_1, \cdots, T_2\zeta_n, \beta_1, \cdots, \beta_l$. Define $R \in \mathcal{L}(V)$ by

$$Ru_i = v_i, Rw_j = \zeta_j, i = 1, \cdots, m; j = 1, \cdots, n.$$

Define $S \in \mathcal{L}(W)$ by

$$ST_2\zeta_j = T_1w_j, S\beta_k = \alpha_k, j = 1, \cdots, n; k = 1, \cdots, l.$$

Since S and T map basis to basis, hence S and T are invertible(surjective). Moreover, it is easy to check $T_1u_i = 0 = ST_2Ru_i$ and

$$T_1w_j = ST_2\zeta_j = ST_2Rw_j,$$

hence $T_1 = ST_2R$.

Compare this problem with Problem 4 and Problem 5.

---

7. Suppose V and W are finite-dimensional. Let $v \in V$. Let

$$E = \{T \in \mathcal{L}(V, W) : Tv = 0\}.$$

(a) Show that E is a subspace of $\mathcal{L}(V, W)$. (b) Suppose $v \neq 0$. What is $\dim E$?

Solution: a) Let $T, S \in E$, then

$$(T + S)v = Tv + Sv = 0 + 0 = 0,$$

hence $T + S \in E$, namely E is closed under addition. For every $\lambda \in \mathbb{F}$,

$$(\lambda T)v = \lambda(Tv) = \lambda 0 = 0,$$

hence $\lambda T \in E$, namely E is closed under scalar multiplication. Therefore E is a subspace of $\mathcal{L}(V, W)$.

(b) Since $v \neq 0$, we can extend it to a basis of V, namely $v, v_2, \cdots, v_n$. Let $w_1, \cdots, w_m$ be a basis of W. Under these bases, we have a isometric between $\mathcal{L}(V, W)$ and $\mathbb{F}^{m,n}$ by 3.60. Moreover, $Tv = 0$ if and only if the first column vector of $\mathcal{M}(T)$ is zero. Hence $\dim E$ is exactly all matrices in $\mathbb{F}^{m,n}$ such the first column vector is zero. Now , it is easily seen that

$$\dim E = m(n-1) = \dim W(\dim V - 1).$$

Or it is equivalent to the dimension all linear map from $\text{span}(v_2, \cdots, v_n)$ to W. Then we can use 3.61.

---

8. Suppose V is finite-dimensional and $T : V \to W$ is a surjective linear map of V onto W. Prove that there is a subspace U of V such that $T|_U$ is an isomorphism of U onto W. (Here $T|_U$ means the function T restricted to U. In other words, $T|_U$ is the function whose domain is U, with $T|_U$ defined by $T|_U(u) = Tu$ for every $u \in U$.)

Solution: Let $w_1, \cdots, w_n$ be a basis of W. Since T is surjective, there exist $v_1, \cdots, v_n$ such that $Tv_i = w_i$. Moreover, by Problem 4 of Exercises 3A, it follows that $v_1, \cdots, v_n$ is linearly independent. Consider $U = \text{span}(v_1, \cdots, v_n)$, then $T|_U$ maps a basis of U to a basis of W. Hence $T|_U$ is an isomorphism of U onto W.

---

9. Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if both S and T are invertible.

Solution: See Linear Algebra Done Right Solution Manual Chapter 3 Problem 22.

---

10. Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$.

Solution: See Linear Algebra Done Right Solution Manual Chapter 3 Problem 23.

---

11. Suppose V is finite-dimensional and $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show that T is invertible and that $T^{-1} = US$.

Solution: By Problem 9, we have TU is invertible. Again by Problem 9, we have T is invertible. Multiply both side of $STU = I$ by S^{-1} on left, we get $TU = S^{-1}$. Multiply both side of $TU = S^{-1}$ by S on right, we have $TUS = I$. Multiply both side of $TUS = I$ by T^{-1} on left, we get $US = T^{-1}$.

---

12. Show that the result in the previous exercise can fail without the hypothesis that V is finite-dimensional.

Solution: Consider $V = \mathbb{C}^\infty$. Define T, S, U by

$$T(z_1, z_2, z_3, \cdots) = (0, z_1, z_2, z_3, \cdots)$$

$$S(z_1, z_2, z_3, \cdots) = (z_2, z_3, z_4, \cdots)$$

and $U = I$. Then $STU = I$. However T is not surjective.

---

13. Suppose V is a finite-dimensional vector space and $R, S, T \in \mathcal{L}(V)$ are such that RST is surjective. Prove that S is injective.

Solution: Since V is finite-dimensional, the surjectivity of RST implies RST is invertible. Hence S is invertible by Problem 11.

---

14. Suppose $v_1, \cdots, v_n$ is a basis of V. Prove that the map $T : V \to \mathbb{F}^{n,1}$ defined by

$$Tv = \mathcal{M}(v)$$

is an isomorphism of V onto $\mathbb{F}^{n,1}$, here $\mathcal{M}(v)$ is the matrix of $v \in V$ with respect to the basis $v_1, \cdots, v_n$.

Solution: See Linear Algebra Done Right Solution Manual Chapter 3 Problem 20.

---

15. Prove that every linear map from $\mathbb{F}^{n,1}$ to $\mathbb{F}^{m,1}$ is given by a matrix multiplication. In other words, prove that if $T \in \mathcal{L}(\mathbb{F}^{n,1}, \mathbb{F}^{m,1})$, then there exists an m-by-n matrix A such that $Tx = Ax$ for every $x \in \mathbb{F}^{n,1}$.

Solution: See Linear Algebra Done Right Solution Manual Chapter 3 Problem 21.

---

16. Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.

Solution: See Linear Algebra Done Right Solution Manual Chapter 3 Problem 24.

---

17. Suppose V is finite-dimensional and E is a subspace of $\mathcal{L}(V)$ such that $ST \in E$ and $TS \in E$ for all $S \in \mathcal{L}(V)$ and all $T \in E$. Prove that $E = \{0\}$ or $E = \mathcal{L}(V)$.

Solution: Let $e_1, \cdots, e_n$ be a basis of V. Suppose $E \neq \{0\}$, then there is nonzero $T \in E$. This implies there is some $s \in \{1, \cdots, n\}$ such that $Te_s \neq 0$. Let

$$Te_s = a_1e_1 + \cdots + a_ne_n, \quad a_j \in \mathbb{F}, j = 1, \cdots, n.$$

Since $Te_s \neq 0$, there is some $t \in \{1, \cdots, n\}$ such that $a_t \neq 0$. Define $E_{ij} \in \mathcal{L}(V)$ by $E_{ij}e_k = \delta_{ik}e_j$, where $\delta_{ik} = 0$ if $i \neq k$ and $\delta_{ii} = 0$ for every i.

By definition, for every $i \in \{1, \cdots, n\}$, we have

$$E_{ti}TE_{is}e_j = a_t\delta_{ij}e_i, \quad j = 1, \cdots, n.$$

By assumption, $E_{ti}TE_{is} \in E$. Note that E is a subspace of $\mathcal{L}(V)$. It follows that

$$(E_{t1}TE_{1s} + \cdots + E_{tn}TE_{ns})e_j = a_te_j, \quad j = 1, \cdots, n,$$

and $(E_{t1}TE_{1s} + \cdots + E_{tn}TE_{ns}) \in E$. This implies $a_jI \in E$, hence $I \in E$. Therefore for any $S \in \mathcal{L}(V)$, $S = SI \in E$, namely $E = \mathcal{L}(V)$.

---

18. Show that V and $\mathcal{L}(\mathbb{F}, V)$ are isomorphic vector spaces.

Solution: For given $v \in V$, define $\varphi_v : \mathbb{F} \to V$ by $\varphi_v(\lambda) = \lambda v$. Then $\varphi_v \in \mathcal{L}(\mathbb{F}, V)$ (check it). Hence we can define $\varphi : V \to \mathcal{L}(\mathbb{F}, V)$ by $v \mapsto \varphi_v$. It suffices to show φ is an isomorphic from V to $\mathcal{L}(\mathbb{F}, V)$. First, we should check φ is linear. For every $v_1, v_2 \in V$ and $\lambda, a \in \mathbb{F}$, we have

$$\begin{aligned} \varphi_{v_1+\lambda v_2}(a) =& a(v_1 + \lambda v_2) = av_1 + \lambda av_2 \\ =& \varphi_{v_1}(a) + (\lambda\varphi_{v_2})(a) = (\varphi_{v_1} + \lambda\varphi_{v_2})(a). \end{aligned}$$

Hence φ is linear. Then $\varphi_v \equiv 0$ imples $av = 0$ for every $a \in \mathbb{F}$, thus $v = 0$. We conclude injectivity. For any $f \in \mathcal{L}(\mathbb{F}, V)$, if $f(1) = v$ then $f(\lambda) = \lambda v = \varphi_v(\lambda)$. Hence every $f \in \mathcal{L}(\mathbb{F}, V)$ can be express as $\varphi_{f(1)}$, namely φ is surjective.

---

19. Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is such that T is injective and $\deg Tp \leq \deg p$ for every nonzero polynomial $p \in \mathcal{P}(\mathbb{R})$.

(a) Prove that T is surjective.

(b) Prove that $\deg Tp = \deg p$ for every nonzero $p \in \mathcal{P}(\mathbb{R})$.

Solution: (a) Since $\deg Tp \leq \deg p$ for every nonzero polynomial $p \in \mathcal{P}(\mathbb{R})$, we have $T|_{\mathcal{P}_n(\mathbb{R})} : \mathcal{P}_n(\mathbb{R}) \to \mathcal{P}_n(\mathbb{R})$ for any $n \in \mathbb{N}^+$. Note that T is injective, hence $T|_{\mathcal{P}_n(\mathbb{R})} : \mathcal{P}_n(\mathbb{R}) \to \mathcal{P}_n(\mathbb{R})$ is injective. Note that $\mathcal{P}_n(\mathbb{R})$ is finite-dimensional, we conclude $T|_{\mathcal{P}_n(\mathbb{R})} : \mathcal{P}_n(\mathbb{R}) \to \mathcal{P}_n(\mathbb{R})$ is surjective by 3.56. Hence T is surjective as any polynomial must be contained in some $\mathcal{P}_n(\mathbb{R})$.

(b) We argue it by induction on $\deg p$. It is true for $\deg = 0$. Suppose it is true for n. As $T|_{\mathcal{P}_n(\mathbb{R})} : \mathcal{P}_n(\mathbb{R}) \to \mathcal{P}_n(\mathbb{R})$ is surjective, every polynomial p with degree $\leq n$ can be attained by Tq for some $q \in \mathcal{P}_n(\mathbb{R})$. Moreover, $T|_{\mathcal{P}_{n+1}(\mathbb{R})} : \mathcal{P}_{n+1}(\mathbb{R}) \to \mathcal{P}_{n+1}(\mathbb{R})$ is surjective, every polynomial p with degree $\leq n+1$ can be attained by Tq for some $q \in \mathcal{P}_{n+1}(\mathbb{R})$. If there exist $p \in \mathcal{P}_{n+1}(\mathbb{R})$ with degree $n+1$ such that $\deg Tp < n+1$. Then there exist some $q \in \mathcal{P}_n(\mathbb{R})$ such that $Tq = Tp$ since $T|_{\mathcal{P}_n(\mathbb{R})} : \mathcal{P}_n(\mathbb{R}) \to \mathcal{P}_n(\mathbb{R})$ is surjective. Hence $Tp = Tq$. Note that T is injective and $p \neq q$ (different degrees), we get a contradiction. Hence $\deg Tp = \deg p$ for every nonzero $p \in \mathcal{P}_{n+1}(\mathbb{R})$ with degree $n+1$. The proof is complete.

---

20. Suppose n is a positive integer and $A_{i,j} \in \mathbb{F}$ for $i, j = 1, \cdots, n$. Prove that the following are equivalent (note that in both parts below, the number of equations equals the number of variables):

(a) The trivial solution $x_1 = \cdots = x_n = 0$ is the only solution to the homogeneous system of equations

$$\sum_{k=1}^{n} A_{1,k}x_k = 0$$

$$\cdots$$

$$\sum_{k=1}^{n} A_{n,k}x_k = 0.$$

(b) For every $c_1, \cdots, c_n \in \mathbb{F}$, there exists a solution to the system of equations

$$\sum_{k=1}^{n} A_{1,k}x_k = c_1$$

$$\cdots$$

$$\sum_{k=1}^{n} A_{n,k}x_k = c_n.$$

Solution: See Linear Algebra Done Right Solution Manual Chapter 3 Problem 26.

About Linearity

This website is supposed to help you study Linear Algebras. Please only read these solutions after thinking about the problems carefully. Do not just copy these solutions.

This entry was posted in Chapter 3 and tagged Exercise D.

← Chapter 3 Exercise C Chapter 3 Exercise E →

Eli Bashwinger · 7 months ago

For problem 18, why not just use the fact that L(F,V) and V are both vectors spaces over the same field and of the same dimension? Using these facts, it follows immediately from theorems 3.61 and 3.59 that L(F,V) and V are isomorphic.

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Eli Bashwinger → Eli Bashwinger · 7 months ago

Ah! Perhaps you didn't use those two theorems because the problem statement does not specify whether V is finite dimensional. Is that why?

Reply · Share ›

zagortenay333 · 8 months ago

Seriously, by what line of reasoning did you produce the solution for exercise 4, and how long did it take you to do it?

It just seems impossible to come up with that.. Baahhh, so frustrating...

Is this pattern of not being sure whether a space V is finite-dim and, therefore, decomposing it as a direct sum of a span and a kernel a recurring thing? Did you see it somewhere else or did you just produce it out of thin air for the purposes of the exercise?

Reply · Share ›

Sam Tay → zagortenay333 · 6 months ago

I actually ran into this problem in 3.B.24 as well, and have since had to use this fact regularly in many problems. I think it may have been an oversight in the author's exposition, as it would have been a good stepping stone problem to cite for these.

Reply · Share ›

Célio Passos · 2 years ago

I have another proof for the converse in exercise 16.

The problem says that if v is in V, then S(Tv) = T(Sv) for every S in L(V). If v is in null S, then T(Sv) = T(0) = 0 = S(Tv) which implies Tv is in null S. Let S in L(V) such that v is a basis for null S (one can easily show a map like this exists for every v). We have that Tv is also in null S, but every vector in null S is a scalar multiple of v, thus Tv = cv = cIv for some scalar c. We but need to show that this c is the same for any other vector in V.

Let u = dv, where d is a scalar. Because T is linear, we have that Tu = dTv = dcv = cu, hence c is the same for all multiples of v. Let w in V such that w is not a scalar multiple of v, that is v and w are linearly independent, and Tw = aw, where a is a scalar. We have that T(v + w) = b(v + w) for some scalar b, but also T(v + w) = Tv + Tw = cv + aw. Thus bv + bw = cv + aw, which implies (b - c)v + (b - a)w = 0. Because v and w are linearly independent we have that c = a = b. The same can be done for every vector that is not a multiple of v, hence the scalar is the same for all vectors.

Therefore T is a scalar multiple of the identity.

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Elliott Mestas · 2 years ago

For exercise 17, I think I have a shorter proof:

Consider an invertible map in E, call it T (it must exist by exercise 2). Then T T^-1 = I is in E by definition.

But if I is in E, then S*I = S is in E for all S in L(V). Therefore, E = L(V).

Reply · Share ›

Mohammad Rashidi Mod → Elliott Mestas · 2 years ago

You cannot conclude the existence of T from exercise 2.

Reply · Share ›

statssigsushi · 4 years ago

For problem 3,can you just define T by T(ui) = S(ui) for bases of U and T(vi) = vi for the extended part of basis of V and prove that only {0} is in null T?

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Chapter 3 Exercise E

Posted on March 5, 2016 by Linearity

Exercises 1,2 and 4. For Problem 2, please also see Carson Rogers's comment.

---

4. Solution: For any $f \in \mathcal{L}(V_1 \times \cdots \times V_m, W)$ and given $i \in \{1, \cdots, m\}$, define $f_i : V_i \to W$ by the rule:

$$f_i(v_i) = f(0, \cdots, 0, v_i, 0, \cdots, 0),$$

where v_i sits in the i-th slot. One can check that $f_i \in \mathcal{L}(V_i, W)$.

Define $\varphi : \mathcal{L}(V_1 \times \cdots \times V_m, W) \to \mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ by

$$\varphi(f) = (f_1, \cdots, f_m),$$

where $f_1, \cdots, f_m$ are defined in the previous paragraph. Now we are going to check that φ is an isomorphism between $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ by construct an inverse map.

Define $\psi : \mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W) \to \mathcal{L}(V_1 \times \cdots \times V_m, W)$ by

$$\psi(f_1, \cdots, f_m)(v_1, \cdots, v_m) = f_1(v_1) + \cdots + f_m(v_m).$$

Please check that ψ and φ are linear. It is also not hard to see that $\psi \circ \varphi = Id$, and $\varphi \circ \psi = Id$. Hence we are done.

---

7. Solution: Note that $v + U = x + W$, hence $v = x + w_1$, where $w_1 \in W$. It follows that $v - x \in W$. Hence for any $u \in U$, we have

$$v + u = x + w_2$$

for some $w_2 \in W$ since $v + U = x + W$. Hence we conclude that

$$u = (x - v) + w_2 = -w_1 + w_2 \in W,$$

it follows that $U \subset W$ for u is chosen arbitrarily. Similarly, we deduce that $W \subset U$. Thus $U = W$.

---

8. Solution: If A is an affine subset of V, then there exist a vector $a \in V$ and a subspace U of V such that $A = a + U$. Then any $v, w \in A$ can be written as $v = a + u_1$ and $w = a + u_2$ for some $u_1, u_2 \in U$. Hence

$$\lambda v + (1 - \lambda)w = a + [\lambda u_1 + (1 - \lambda)u_2] \in a + U = A.$$

Conversely, since A is nonempty, let $a \in A$. We will show that

$$A - a = \{x - a : x \in A\}$$

is a subspace of V. For $x - a \in A - a$ and $\lambda \in \mathbb{F}$ where $x \in A$, then

$$\lambda x + (1 - \lambda)a \in A \Longrightarrow \lambda(x - a) = \lambda x + (1 - \lambda)a - a \in A - a.$$

This implies $A - a$ is closed under scalar multiplication. For $x - a$ and $y - a \in A - a$, where $x, y \in A$. We have

$$\frac{x}{2} + \frac{y}{2} - a \in A - a.$$

Note that $A - a$ is closed under scalar multiplication, it follows that

$$(x - a) + (y - a) = 2\left(\frac{x}{2} + \frac{y}{2} - a\right) \in A - a.$$

That is $A - a$ is closed under addition. Hence $A - a$ is a subspace of V. Note that $A = a + (A - a)$, it follows that A is an affine subset of V.

---

9. Solution: Suppose $A_1 \cap A_2 \neq \varnothing$, then for any $x, y \in A_1 \cap A_2$ and $\lambda \in \mathbb{F}$, we have

$$\lambda x + (1 - \lambda)y \in A_1$$

and

$$\lambda x + (1 - \lambda)y \in A_2$$

by Problem 8 since A_1 and A_2 are affine subsets of V. Hence

$$\lambda x + (1 - \lambda)y \in A_1 \cap A_2.$$

Again by Problem 8, we deduce that $A_1 \cap A_2$ is an affine subset of V.

---

10. Solution: It is the same as Problem 9.

---

11. Solution: (a) For $v = \lambda_1 v_1 + \cdots + \lambda_m v_m \in A$ and $w = \eta_1 v_1 + \cdots + \eta_m v_m \in A$,where $\lambda_1, \cdots, \lambda_m \in \mathbb{F}$, $\lambda_1 + \cdots + \lambda_m = 1$ and $\eta_1, \cdots, \eta_m \in \mathbb{F}, \eta_1 + \cdots + \eta_m = 1$. For any $\lambda \in \mathbb{F}$, we have

$$\lambda v + (1 - \lambda)w = \sum_{i=1}^{m} (\lambda\lambda_i + (1 - \lambda)\eta_i)v_i.$$

Note that

$$\sum_{i=1}^{m} (\lambda\lambda_i + (1 - \lambda)\eta_i) = \lambda \sum_{i=1}^{m} \lambda_i + (1 - \lambda) \sum_{i=1}^{n} \eta_i = \lambda + (1 - \lambda) = 1,$$

we deduce that $\lambda v + (1 - \lambda)w \in A$. Hence A is an affine subset of V by Problem 8.

(b) We will use induction to show that for any affine subset W of V that contains $v_1, \cdots, v_m k \leq m$, then if $\lambda_1 + \cdots + \lambda_k = 1$, we have

$$\sum_{j=1}^{k} \lambda_j v_j \in W.$$

It is obvious for $k = 1$ and $k = 2$ by Problem 8. Suppose this is true for k, then we will show it for $k + 1$ ($k + 1 \leq m$). Assume $\lambda_1 + \cdots + \lambda_{k+1} = 1$. If $\lambda_{k+1} = 1$, then

$$\sum_{j=1}^{k+1} \lambda_j v_j = v_{k+1} \in W.$$

If $\lambda_{k+1} \neq 1$, then

$$\frac{1}{1 - \lambda_{k+1}}(\lambda_1 + \cdots + \lambda_k) = 1.$$

Hence by assumption, we deduce that

$$\frac{1}{1 - \lambda_{k+1}}(\lambda_1 v_1 + \cdots + \lambda_k v_k) \in W.$$

By Problem 8, we have

$$(1 - a_{k+1})\left(\frac{1}{1 - \lambda_{k+1}}(\lambda_1 v_1 + \cdots + \lambda_k v_k)\right) + a_{k+1}v_{k+1} \in W,$$

namely

$$\lambda_1 v_1 + \cdots + \lambda_{k+1}v_{k+1} \in W.$$

By mathematical induction, we conclude if $\lambda_1, \cdots, \lambda_m \in \mathbb{F}, \lambda_1 + \cdots + \lambda_m = 1$, then

$$\lambda_1 v_1 + \cdots + \lambda_m v_m \in W,$$

thus $A \subset W$.

(c) Note that if $\lambda_1 + \cdots + \lambda_m = 1$, we have

$$\lambda_1 v_1 + \cdots + \lambda_m v_m = v_1 + \lambda_2(v_2 - v_1) + \cdots + \lambda_m(v_m - v_1).$$

Hence $A \subset v_1 + \operatorname{span}(v_2 - v_1, \cdots, v_m - v_1)$. Similarly, for any

$$v \in v_1 + \operatorname{span}(v_2 - v_1, \cdots, v_m - v_1),$$

v can be written as

$$v_1 + \sum_{i=2}^{m} \lambda_i(v_i - v_1) = (1 - \lambda_2 - \cdots - \lambda_m)v_1 + \sum_{i=2}^{m} \lambda_i v_i$$

for some $\lambda_2, \cdots, \lambda_m \in \mathbb{F}$. Note that

$$(1 - \lambda_2 - \cdots - \lambda_m) + \sum_{i=2}^{m} \lambda_i = 1,$$

we deduce that $v_1 + \operatorname{span}(v_2 - v_1, \cdots, v_m - v_1) \subset A$. Hence

$$A = v_1 + \operatorname{span}(v_2 - v_1, \cdots, v_m - v_1).$$

Let $v = v_1$ and $U = \operatorname{span}(v_2 - v_1, \cdots, v_m - v_1)$, then $\dim U \leq m - 1$.

---

12. Solution: Since V/U is finite-dimensional, we can suppose $v_1 + U, \cdots, v_n + U$ is a basis of V/U. Then for any $v \in V$, there exist a unique list of $\lambda_1, \cdots, \lambda_n \in \mathbb{F}$ such that

$$v + U = \sum_{i=1}^{n} \lambda_i(v_i + U).$$

Then

$$v - \sum_{i=1}^{n} \lambda_i v_i \in U.$$

Define $\varphi : V \to U \times V/U$ by

$$\varphi(v) = \left(v - \sum_{i=1}^{n} \lambda_i v_i, \sum_{i=1}^{n} \lambda_i(v_i + U)\right).$$

We first check that φ is linear. For any $v, w \in V$, we

$$v + U = \sum_{i=1}^{n} \lambda_i(v_i + U)$$

and

$$w + U = \sum_{i=1}^{n} \eta_i(v_i + U).$$

for some $\lambda_1, \cdots, \lambda_n \in \mathbb{F}$ and $\eta_1, \cdots, \eta_n \in \mathbb{F}$. Then for any $\lambda \in \mathbb{F}$, we have

$$(v + \lambda w) + U = (v + U) + \lambda(w + U) = \sum_{i=1}^{n} (\lambda_i + \lambda\eta_i)(v_i + U).$$

Hence

$$\varphi(v + \lambda w) = \left(v + \lambda w - \sum_{i=1}^{n} (\lambda_i + \lambda\eta_i)v_i, \sum_{i=1}^{n} (\lambda_i + \lambda\eta_i)(v_i + U)\right),$$

and

$$\varphi(v) = \left(v - \sum_{i=1}^{n} \lambda_i v_i, \sum_{i=1}^{n} \lambda_i(v_i + U)\right),$$

and

$$\varphi(w) = \left(w - \sum_{i=1}^{n} \eta_i v_i, \sum_{i=1}^{n} \eta_i(v_i + U)\right).$$

Therefore we have (check it)

$$\varphi(v) + \lambda\varphi(w) = \varphi(v + \lambda w),$$

namely φ is linear.

Injectivity: if $\varphi(v) = 0$, then $\lambda_1 = \cdots = \lambda_n = 0$(defined as above) and

$$0 = v - \sum_{i=1}^{n} \lambda_i v_i = v.$$

Surjectivity: For $\left(u, \sum_{i=1}^{n} \xi(v_i + U)\right) \in U \times V/U$, it is easy to see

$$\varphi\left(u + \sum_{i=1}^{n} \xi v_i\right) = \left(u, \sum_{i=1}^{n} \xi(v_i + U)\right).$$

Hence φ is an isomorphism, namely V is isomorphic to $U \times V/U$.

---

13. Solution: For any $v \in V$, since $v_1 + U, \cdots, v_m + U$ is a basis of V/U, there exist $\lambda_1, \cdots, \lambda_m \in \mathbb{F}$ such that

$$v + U = \sum_{i=1}^{m} \lambda_i(v_i + U).$$

This implies

$$v - \sum_{i=1}^{m} \lambda_i v_i \in U.$$

Note that $u_1, \cdots, u_n$ is a basis of U, it follows that

$$v - \sum_{i=1}^{m} \lambda_i v_i = \sum_{j=1}^{n} \eta_j u_j$$

for some $\eta_1, \cdots, \eta_n \in \mathbb{F}$. Hence

$$v = \sum_{i=1}^{m} \lambda_i v_i + \sum_{j=1}^{n} \eta_j u_j,$$

this implies $V = \operatorname{span}(v_1, \cdots, v_m, u_1, \cdots, u_n)$ since v is chosen arbitrarily. Hence it suffices to show that $v_1, \cdots, v_m, u_1, \cdots, u_n$ is linearly independent. Suppose for some $\lambda_1, \cdots, \lambda_m \in \mathbb{F}$ and $\eta_1, \cdots, \eta_n \in \mathbb{F}$, we have

$$\sum_{i=1}^{m} \lambda_i v_i + \sum_{j=1}^{n} \eta_j u_j = 0.$$

Then

$$\sum_{i=1}^{m} \lambda_i(v_i + U) = 0,$$

hence $\lambda_1 = \cdots = \lambda_m = 0$ since $v_1 + U, \cdots, v_m + U$ is a basis of V/U. It follows that

$$\sum_{j=1}^{n} \eta_j u_j = 0.$$

Note that $u_1, \cdots, u_n$ is a basis of U, we obtain that $\eta_1 = \cdots = \eta_n = 0$. Hence $v_1, \cdots, v_m, u_1, \cdots, u_n$ is linearly independent.

---

15. Solution: Note that $\varphi : V \to \mathbb{F}$ and $\varphi \neq 0$, there exists $v \in V$ such that $\varphi(v) \neq 0$. Then for any $a \in \mathbb{F}$, we have

$$\varphi\left(\frac{av}{\varphi(v)}\right) = \frac{a}{\varphi(v)}\varphi(v) = a.$$

Therefore $\operatorname{range} \varphi = \mathbb{F}$.

By 3.91(d), we have $V/\operatorname{null} \varphi \cong \mathbb{F}$. Hence

$$\dim V/\operatorname{null} \varphi = \dim \mathbb{F} = 1.$$

---

16. Solution: Since $\dim V/U = 1$, there exists a vector $v \notin U$ such that $v + U$ is a basis of V/U. Define $\theta : V/U \to \mathbb{F}$ by

$$\theta(kv + U) = k,$$

then $\theta \in \mathcal{L}(V/U, \mathbb{F})$.

Let π be the quotient map $\pi : V \to V/U$. Let $\varphi = \theta \circ \pi$, then $\varphi \in \mathcal{L}(V, \mathbb{F})$. Now we check that $\operatorname{null} \varphi = U$.

Let $w \in V$ such that $\varphi(w) = 0$, i.e. $\theta(\pi(w)) = 0$. By the definition of θ, we have

$$w + U = 0v + U.$$

Hence $w \in U$, which implies that $\operatorname{null} \varphi \subset U$.

On the other hand, let $u \in U$, then $\pi(u) = u + U = 0 + U$. Hence $\theta \circ \pi(u) = 0$, which implies that $U \subset \operatorname{null} \varphi$.

Therefore $\operatorname{null} \varphi = U$. (Please note that such φ is not unique.)

---

17. Solution: Since V/U is finite-dimensional, let $n = \dim(V/U)$, then there exist $w_1, \cdots, w_n \in V$ such that $w_1 + U, \cdots, w_n + U$ form a basis of V/U. It is easy to see that $w_1, \cdots, w_n$ is linearly independent otherwise $w_1 + U, \cdots, w_n + U$ would be linearly dependent. Consider the subspace W of V spanned by $w_1, \cdots, w_n$, then we have $\dim W = n = \dim(V/U)$.

Now we are going to show that $V = U \oplus W$. For any $v \in V$, since $w_1 + U, \cdots, w_n + U$ is a basis of V/U, there exist $k_1, \cdots, k_n \in \mathbb{F}$ such that

$$v + U = \sum_{i=1}^{n} k_i(w_i + U).$$

This implies that

$$v - \sum_{i=1}^{n} k_i w_i \in U.$$

Therefore consider

$$v = \left(v - \sum_{i=1}^{n} k_i w_i\right) + \sum_{i=1}^{n} k_i w_i,$$

it tells us that $v \in U + W$. Since v is chosen arbitrarily, we conclude that $V \subset U + W$. To show that $V = U \oplus W$, it suffices to show that $U \cap W = 0$.

Suppose $v \in U \cap W$, then there exist $k_1, \cdots, k_n \in \mathbb{F}$ such that

$$v = \sum_{i=1}^{n} k_i w_i.$$

Hence we have $0 = \pi(v) = \sum_{i=1}^{n} k_i(w_i + U)$, where π is the canonical quotient map from V onto V/U. However, note that $w_1 + U, \cdots, w_n + U$ is a basis of V/U we have $k_1 = \cdots = k_n = 0$. Thus

$$v = \sum_{i=1}^{n} k_i w_i = 0.$$

This shows that $U \cap W = 0$.

---

18. Solution: Suppose there exists $S \in \mathcal{L}(V/U, W)$ such that $T = S \circ \pi$. Since for any $u \in U$ we have $\pi u = 0$, hence

$$Tu = (S \circ \pi)u = S(\pi u) = S0 = 0.$$

Hence $u \in \operatorname{null} T$. Note that this is true for any $u \in U$, we conclude that $U \subset \operatorname{null} T$.

Conversely, suppose $U \subset \operatorname{null} T$. Define a map $S : V/U \to W$ by $S(v + U) = Tv$. We have to check that this map is well-defined (since there may exist different $v_1, v_2 \in V$ such that $v_1 + U = v_2 + U$ and we need to check that $Tv_1 = Tv_2$).

For $v_1, v_2 \in V$ such that $v_1 + U = v_2 + U$, we have $v_1 - v_2 \in U$. Note that $U \subset \operatorname{null} T$, it follows that $T(v_1 - v_2) = 0$, hence we have $Tv_1 = Tv_2$. This implies that S is well-defined.

Note that $T \in \mathcal{L}(V, W)$, it is easy to see that $S \in \mathcal{L}(V/U, W)$. By definition, we have

$$S \circ \pi(v) = S(v + U) = Tv$$

for any $v \in V$. This implies that $T = S \circ \pi$.

---

19. Solution: For any given finite set A, denote by $|A|$ the number of elements in A. Let $S_i, i = 1, \cdots, n$, be finite sets. The union $\bigcup_{i=1}^{n} S_i$ is a disjoint union if and only if

$$|S_1| + \cdots + |S_n| = \left|\bigcup_{i=1}^{n} S_i\right|.$$

---

20. Solution: (a) To show that Γ is linear, we need to show that for any $S_1, S_2 \in \mathcal{L}(V/U, W)$ and $k_1, k_2 \in \mathbb{F}$ we have

$$\Gamma(k_1 S_1 + k_2 S_2) = k_1\Gamma(S_1) + k_2\Gamma(S_2).$$

This follows directly since

$$\begin{aligned}\Gamma(k_1 S_1 + k_2 S_2) =&(k_1 S_1 + k_2 S_2) \circ \pi\\ =&k_1 S_1 \circ \pi + k_2 S_2 \circ \pi\\ =&k_1\Gamma(S_1) + k_2\Gamma(S_2).\end{aligned}$$

(b) To show that Γ is injective, one has to show that if $\Gamma(S) = 0$ then $S = 0$. If $\Gamma(S) = 0$, it implies that $S \circ \pi = 0$. Hence $(S \circ \pi)v = 0$ for all $v \in V$. Thus $S(\pi(v)) = S(v + U) = 0$ for all $v \in V$. We conclude that $S = 0$, which shows Γ is injective.

(c) By Problem 18, we have

$$\begin{aligned}T \in \operatorname{range} \Gamma &\iff \text{there exists } S \in \mathcal{L}(V/U, W) \text{ s.t. } T = S \circ \pi\\ &\iff U \subset \operatorname{null} T \quad \text{by Problem 18}\\ &\iff T \in \mathcal{L}(V, W) : Tu = 0 \text{ for every } u \in U.\end{aligned}$$

This completes the proof.

About Linearity
This website is supposed to help you study Linear Algebras. Please only read these solutions after thinking about the problems carefully. Do not just copy these solutions.

This entry was posted in Chapter 3 and tagged Exercise E.

← Chapter 3 Exercise D Chapter 3 Exercise F →

Carson Rogers · 2 years ago
The solution to problem 2 is wrong: the sum V_1+...+V_m is not defined unless all of the V_i are subspaces of a single larger space, which they are not assumed to be. Of course, each V_i is naturally isomorphic to a subspace of the product, but once you observe that, you have one sentence left to finish the solution.
61 · Reply · Share ›

Mohammad Rashidi Mod → Carson Rogers · 2 years ago
Yes, you are correct. Thank you!
Reply · Share ›

Wu Jinyang → Mohammad Rashidi · 2 years ago
Also for question 11 (c). at the part "V can be written as", you miss a "v1" on the right side of the equation.
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Ed Haarmann IV · 4 years ago
Where are the solutions for 1,...,6 and 14,...,20?
3 · Reply · Share ›

Jacqueline Garcia → Ed Haarmann IV · 4 years ago
Did you find the solutions?
Reply · Share ›

Edgar Baquero → Jacqueline Garcia · 2 years ago
I need the 5th
Reply · Share ›

Mahdi Dibaiee · a year ago
How do you use distributive property on function composition in 20? Function composition is not distributive, is it in this case? If so, what implies it?
1 · Reply · Share ›

AliD · 8 months ago
For question 15 (b),
Define vectors,
v1 = { 1 , 2 , 3 , 4 , 5 , 6 , ...}
v2 = { 1^2 , 2^2 , 3^2 , 4^2 , 5^2 , ...}
v3 = { 1^3 , 2^3 , 3^3 , 4^3 , 5^3 , ...}
v4 = { 1^4 , 2^4 , 3^4 , 4^4 , 5^4 , ...}
vn = { 1^n , 2^n , 3^n , 4^n , 5^n , ...}
These are all clearly vectors not in U.

Now, take some positive integer n. We just have to show that v1 + U, ... , vn + U is a linearly independent list, since this will imply that F-inf / U is infinite dimensional.

Suppose a1*(v1 + U) + ... + an*(vn + U) = U
then (a1*v1 + ... + an*vn) + U = U
so a1*v1 + ... + an*vn must be in U.

Now, in order for this vector to be in U it can only be non-zero for finitely many indices. Therefore, there exists an integer N such that for all integers M > N, the M-th index is zero. I.E. a1*M + a2*M^2 + ... + an*M^n = 0
But, polynomials of degree n have at most n roots, and this poynomial has a infinite number of roots. Therefore our polynomial must be the zero polynomial, that is a1 = ... = an. We have now showed out list is linearly independent so our proof is done.
Please let me know if you think there are any issues with this work or if I didn't explain something well. Thank you
Reply · Share ›

AliD → AliD · 8 months ago
sorry this is problem 14 (b)
Reply · Share ›

Lucas Ferrari Soto · a year ago
I'm guessing 10) is supposed to go by induction and thus holds for finite collections of affine subsets... is the statement true for infinite collections (the exercise doesn't specify)? how would I go about proving it?
Also, I think 8) can be easily modified to not require that there's a 2 in the underlying field.
Reply · Share ›

Mahdi Dibaiee · a year ago
Why not just show that the two sets have equal dimensions, since two vector spaces are isomorphic to each other if and only if they have the same dimension in problem 4 and 5 and 6?
Reply · Share ›

Diego Ramos · 2 years ago
Problem 16# may be solved in a lettle simpler way by stating that theta is simply any isomorfism from V/U onto F.
Reply · Share ›

This comment was deleted.

Christina DeMarzo → Guest · 2 years ago
Anyway you could send me the solutions for 5 and 6? Those still haven't been posted.
1 · Reply · Share ›

This comment was deleted.

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李子鉴 → Guest · 2 years ago
Could you?
Reply · Share ›

This comment was deleted.

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Elliott Mestas · 2 years ago
On the 6th line of the correction of 11.c), you are missing v1 on the RHS

Also, for 11.b), I think I have a shorter proof:
Suppose there is an affine subset A' of V that contains v_1, ..., v_m. Then A' = v + U for some v in V and some U subspace of V.
v_1 = v + u_1, ..., v_m = v + u_m for some u_1, ..., u_m in U.
Consider any lambda_1, ..., lambda_m in F s.t. lambda_1 + ... + lambda_m = 1.
Then:
lambda_1 (v + u_1) + ... + lambda_m (v + u_m)
= v (lambda_1 + ... + lambda_m) + (u_1 lambda_1 + ... + u_m lambda_m)
= v + something in U
Therefore it is in A'.
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Mohammad Rashidi Mod → Elliott Mestas · 2 years ago
Thank you. As for (b), are you using the result from (c)?
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Elliott Mestas → Mohammad Rashidi · 2 years ago
No, I am just using the definition of the affine subset and the definition of A

I have some answers for the other exercises in this section, would it be helpful for me to share them?
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Marcel Ackermann · 2 years ago
"is a basis of V" should be "is a basis of V/U"
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Mohammad Rashidi Mod → Marcel Ackermann · 2 years ago
For which problem? Thanks.
Reply · Share ›

Marcel Ackermann → Mohammad Rashidi · 2 years ago
3E12, first sentence of solution
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Nuno Alvares Pereira · 3 years ago
for 12), a much simpler solution uses just 3.76, 3.89, 3.73, and 3.59 . ;)
Reply · Share ›

Jinming Yu → Nuno Alvares Pereira · 3 years ago
but you dont know if V and U are finite dimensional
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Nuno Alvares Pereira → Jinming Yu · 3 years ago
You're right... =D
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Chapter 3 Exercise F

Posted on March 6, 2016 by Linearity

1. Solution: For any $\varphi \in \mathcal{L}(V, \mathbb{F})$, if $\dim \operatorname{range}\varphi = 0$, then φ is the zero map. If $\dim \operatorname{range}\varphi = 1$, then φ is surjective since $\dim \mathbb{F} = 1$. Moreover, $\dim \operatorname{range}\varphi \leqslant \dim \mathbb{F} = 1$. Hence, that is all the possible cases.

---

2. Solution: Let $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{L}(\mathbb{R}^{[0,1]}, \mathbb{F})$ defined by

$$\varphi_1(f) = f(0), \quad \varphi_2(f) = f(0.5), \quad \varphi_3(f) = f(1).$$

Please check that $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{L}(\mathbb{R}^{[0,1]}, \mathbb{F})$ and they are different from each other.

---

3. Solution: Extend v to a basis of V and use 3.96.

---

4. Solution: Let $u_1, \cdots, u_m$ be a basis of U, since $U \neq V$ we can extend it to a basis of V as $u_1, \cdots, u_m, u_{m+1}, \cdots, v_{m+n}$, where $n \geqslant 1$. Hence we can define $\varphi \in V'$ by

$$\varphi(u_i) = \begin{cases} 0, & \text{if } i \neq m+1; \\ 1, & \text{if } i = m+1. \end{cases}$$

Then $\varphi \in V'$ and $\varphi(u) = 0$ for every $u \in U$ but $\varphi \neq 0$.

---

5. Solution: Define $P_i \in \mathcal{L}(V_i, V_1 \times \cdots \times V_m)$ by

$$P_i(x) = (0, \cdots, 0, x, 0, \cdots, 0)$$

with x in the i-th component. Define $\varphi \in \mathcal{L}((V_1 \times \cdots \times V_m)', V_1' \times \cdots \times V_m')$ by

$$\varphi(f) = (P_1'f, \cdots, P_m'f).$$

Now let us check that φ is an isomorphism.

Injectivity: suppose $(P_1'f, \cdots, P_m'f) = 0$, that is for any $(x_1, \cdots, x_m) \in V_1 \times \cdots \times V_m$, we have

$$P_i'f(x_i) = 0 \Longrightarrow f(0, \cdots, x_i, \cdots, 0) = 0$$

by the definition of P_i and dual map. This implies

$$f(x_1, \cdots, x_m) = \sum_{i=1}^{m} f(0, \cdots, x_i, \cdots, 0) = 0,$$

namely $f = 0$. Thus φ is injective. Here $(0, \cdots, x_i, \cdots, 0)$ means the i-th component is x_i and all other components are zero.

Surjectivity: for any $(f_1, \cdots, f_m) \in V_1' \times \cdots \times V_m'$, define $f \in (V_1 \times \cdots \times V_m)'$ by

$$f(x_1, \cdots, x_m) = \sum_{i=1}^{m} f_i(x_i).$$

Then we can easily check that $\varphi f = (f_1, \cdots, f_m)$.

By the arguments above, it follows that $(V_1 \times \cdots \times V_m)'$ and $V_1' \times \cdots \times V_m'$ are isomorphic.

---

6. Solution: (a) If $v_1, \cdots, v_m$ spans V, then $\Gamma(\varphi) = 0$ implies

$$\varphi(v_1) = \cdots = \varphi(v_m) = 0.$$

Hence $\varphi = 0$ since $v_1, \cdots, v_m$ spans V. Specifically, for any $v \in V$, we can write

$$v = \sum_{i=1}^{m} k_i v_i, \quad k_i \in \mathbb{F}.$$

Thus

$$\varphi(v) = \varphi\left(\sum_{i=1}^{m} k_i v_i\right) = \sum_{i=1}^{m} k_i \varphi(v_i) = 0.$$

This implies $\varphi = 0$. We conclude Γ is injective.

If Γ is injective and $\operatorname{span}(v_1, \cdots, v_m) \neq V$, then by Problem 4, there exists a $\varphi \in V'$ such that

$$\varphi(\operatorname{span}(v_1, \cdots, v_m)) = 0$$

and $\varphi \neq 0$. This implies Γ is not injective. We get a contradiction. Hence $v_1, \cdots, v_m$ spans V.

(b) If $v_1, \cdots, v_m$ is linearly independent, then for any $(f_1, \cdots, f_m) \in \mathbb{F}^m$, there exists a $\varphi \in V'$ such that

$$\varphi(v_i) = f_i, \quad i = 1, \cdots, m.$$

This is easy to show by extending $v_1, \cdots, v_m$ to a basis of V and using 3.5. Then by definition of Γ, we have

$$\Gamma(\varphi) = (f_1, \cdots, f_m).$$

This implies Γ is surjective.

If Γ is surjective, suppose $v_1, \cdots, v_m$ is linearly dependent. Then there exist $k_1, \cdots, k_m \in \mathbb{F}$ such that

$$k_1 v_1 + \cdots + k_m v_m = 0$$

and some k_i is nonzero. Let $k_i \neq 0$, then v_i can be written as a linear combination of $v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n$. Hence, $(0, \cdots, 0, 1, 0, \cdots, 0)$ is not in $\operatorname{range}\Gamma$, where 1 is on the i-th component. Otherwise, we have $\varphi \in V'$ such that $\Gamma(\varphi) = (0, \cdots, 0, 1, 0, \cdots, 0)$. Then

$$\varphi(v_j) = 0, \varphi(v_i) = 1, j = 1, \cdots, i-1, i+1, \cdots, m.$$

This implies $\varphi(v) = 0$ if v is a linear combination of $v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n$. Thus $\varphi(v_i) = 0$ by our previous argument. However, we also have $\varphi(v_i) = 1$. Therefore this can not happen, namely Γ is not surjective. That means that the assumption that $v_1, \cdots, v_m$ is linearly dependent can never happen. Hence $v_1, \cdots, v_m$ is linearly independent.

---

7. Solution: By calculating them directly, we have

$$\varphi_j(x^i) = \delta_{i,j},$$

where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$. Note that the dual basis of one given basis is unique(if exist). Hence we have the dual basis of the basis $1, x, \cdots, x_m$ of $\mathcal{P}_m(\mathbb{R})$ is $\varphi_0, \varphi_1, \cdots, \varphi_m$.

---

8. Solution: (a) This is easy, see Problem 10 of Exercise 2C.

(b) The dual basis of the basis $1, x-5, \cdots, (x-5)_m$ of $\mathcal{P}_m(\mathbb{R})$ is $\varphi_0, \varphi_1, \cdots, \varphi_m$, where $\varphi_j(p) = \frac{p^{(j)}(5)}{j!}$. Here $p^{(j)}$ denotes the j^{th} derivative of p, with the understanding that the 0^{th} derivative of p is p. The proof is similar to Problem 7.

---

9. Solution: Note $v_1, \cdots, v_n$ is a basis of V and $\varphi_1, \cdots, \varphi_n$ is the corresponding dual basis of V', we have

$$(\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n)(v_1) = \psi(v_1).$$

Similarly, we also have

$$(\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n)(v_i) = \psi(v_i).$$

Hence

$$\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n,$$

as they coincide at a basis of V.

---

10. Solution: (a) $(S+T)' = S' + T'$ for all $S, T \in \mathcal{L}(V, W)$. For each $\varphi \in W'$, we have

$$\begin{aligned}(S+T)'(\varphi)(x) &= \varphi((S+T)x) = \varphi(Sx + Tx) = \varphi(Sx) + \varphi(Tx) \\ &= S'(\varphi)(x) + T'(\varphi)(x) = (S' + T')(\varphi)(x)\end{aligned}$$

for all $x \in W$. The first and forth equality hold by the definition of dual map (3.99). The other ones hold by 3.6. Hence $(S+T)'(\varphi) = (S'+T')(\varphi)$ for each $\varphi \in W'$, namely $(S+T)' = S' + T'$.

(b) $(\lambda T)' = \lambda T'$ for all $\lambda \in \mathbb{F}$ and all $T \in \mathcal{L}(V, W)$. For each $\varphi \in W'$, we have

$$\begin{aligned}(\lambda T)'(\varphi)(x) &= \varphi((\lambda T)x) = \varphi(\lambda Tx) = \lambda\varphi(Tx) \\ &= \lambda T'(\varphi)(x) = (\lambda T')(\varphi)(x)\end{aligned}$$

for all $x \in W$. Here we also use 3.6 and 3.99. Similarly, we conclude $(\lambda T)' = \lambda T'$.

---

15. Solution: If $T = 0$, then for any $f \in W'$ and any $v \in V$, we have

$$(T'f)v = f(Tv) = f(0) = 0.$$

Therefore $T'f = 0$ for all $f \in W'$ and hence $T' = 0$.

Conversely, suppose $T' = 0$, we are going to show that $T = 0$ by contradiction. We assume that $T \neq 0$, then there exists $v \in V$ such that $Tv \neq 0$. Since W is finite, it follows from Problem 3 that there exists $\varphi \in W'$ such that $\varphi(Tv) \neq 0$. Note that $(T'\varphi)v = \varphi(Tv) \neq 0$, which contradicts with the assumption that $T' = 0$. Hence $T = 0$.

---

16. Solution: Let $\Gamma : \mathcal{L}(V, W) \to \mathcal{L}(W', V')$ defined by

$$\Gamma(T) = T'.$$

By 3.60, we have $\dim \mathcal{L}(V, W) = \dim \mathcal{L}(W', V')$. Hence, by 3.69, it suffices to show Γ is injective. Suppose $\Gamma(S) = 0$ for some $S \in \mathcal{L}(V, W)$, that is $S' = 0$. Hence for any $\varphi \in W'$ and $v \in V$, we have

$$S'(\varphi)(v) = \varphi(Sv) = 0.$$

By Problem 3, this can only happen when $Sv = 0$. Hence $Sv = 0$ for all $v \in V$. Thus $S = 0$. We conclude Γ is injective.

---

17. Solution: Note that

$$\varphi(u) = 0 \text{ for all } u \in U \iff U \subset \operatorname{null}\varphi.$$

---

18. Solution: By Problem 17, $U^0 = V'$ if and only if $U \subset \operatorname{null}\varphi$ for all $\varphi \in V'$. Note that by Problem 3, $v \in \operatorname{null}\varphi$ for all $\varphi \in V'$ if and only if $v = 0$. This implies $U^0 = V'$ if and only if $U = \{0\}$.

Other solution: by 3.106, we have

$$\dim \operatorname{span}(U) + \dim U^0 = \dim V.$$

Hence

$$\dim U^0 = \dim V' \iff \dim \operatorname{span}(U) = 0$$

since $\dim V' = \dim V$.

---

19. Solution: By 3.106, we have

$$\dim U + \dim U^0 = \dim V.$$

Hence

$$\dim U = \dim V \iff \dim U^0 = 0.$$

That is $U = V$ if and only if $U^0 = \{0\}$.

---

20. Solution: If $\varphi \in W^0$, then $\varphi(w) = 0$ for all $w \in W$. As $U \subset W$, we also have $\varphi(u) = 0$ for all $u \in W$, hence $\varphi \in U^0$. Since φ is chosen arbitrarily, we deduce that $W^0 \subset U^0$.

---

21. Solution: Since $W^0 \subset U^0$, it follows from Problem 22 that

$$(U+W)^0 = U^0 \cap W^0 = W^0.$$

Note that V is finite-dimensional, by 3.106 we have

$$\dim(U+W)^0 = \dim V - \dim(U+W), \quad \dim W^0 = \dim V - \dim W.$$

Therefore, we have $\dim(U+W) = \dim W$. As $W \subset U + W$ and $\dim(U+W) = \dim W$, we conclude that $U + W = W$, which implies that $U \subset W$.

---

22. Solution: Note that $U \subset U + W$ and $W \subset U + W$, it follows from Problem 20 that $(U+W)^0 \subset U^0$ and $(U+W)^0 \subset W^0$. Therefore, $(U+W)^0 \subset U^0 \cap W^0$.

On the other hand, for any given $f \in U^0 capW^0$, we have $f(u) = 0$ and $f(w) = 0$ for any $u \in U$ and any $w \in W$. Therefore,

$$f(u+w) = f(u) + f(w) = 0$$

for any $u \in U$ and any $w \in W$. Note that every vector $x \in U + W$ can be written in the form of $u + w$, where $u \in U$ and $w \in W$. Therefore, we prove that $f(x) = 0$ for all $x \in U + W$. This implies that $f \in (U+W)^0$, hence we have $U^0 \cap W^0 \subset (U+W)^0$.

Therefore, $(U+W)^0 = U^0 \cap W^0$.

---

23. Solution: Note that $U \cap W \subset U$ and $U \cap W \subset W$, it follows from Problem 20 that $U^0 \subset (U \cap W)^0$ and $W^0 \subset (U \cap W)^0$. Hence $U^0 + W^0 \subset (U \cap W)^0$.

On the other hand, since V is finite-dimensional, it follows from 3.106 that

$$\begin{aligned}
\dim(U^0 + W^0) &= \dim U^0 + \dim W^0 - \dim(U^0 \cap W^0) \\
\text{by Problem 22 and 3.106} \quad &= \dim V - \dim U + \dim V - \dim W - \dim((U+W)^0) \\
\text{by 3.106} \quad &= \dim V - \dim U + \dim V - \dim W - \dim V + \dim(U+W) \\
&= \dim V - \dim U - \dim W + (\dim U + \dim W - \dim(U \cap W)) \\
\text{by 3.106} \quad &= \dim V - \dim(U \cap W) = \dim((U \cap W)^0).
\end{aligned}$$

Since $\dim(U^0 + W^0) = \dim \dim((U \cap W)^0)$ and $U^0 + W^0 \subset (U \cap W)^0$, they must equal. Therefore, $U^0 + W^0 = (U \cap W)^0$.

---

34. Solution: (a) Given $k_1, k_2 \in \mathbb{F}$ and $v_1, v_2 \in V$. For any $\varphi \in V'$, we have

$$\begin{aligned}(\Lambda(k_1 v_1 + k_2 v_2))(\varphi) &= \varphi(k_1 v_1 + k_2 v_2) \\ &= k_1\varphi(v_1) + k_2\varphi(v_2) \\ &= k_1(\Lambda v_1)(\varphi) + k_2(\Lambda v_2)(\varphi) \\ &= (k_1\Lambda v_1 + k_2\Lambda v_2)(\varphi).\end{aligned}$$

Since this is true for any φ, it follows that

$$\Lambda(k_1 v_1 + k_2 v_2) = k_1\Lambda v_1 + k_2\Lambda v_2.$$

Hence Λ is a linear map from V to V''.

(b) For any given $v \in V$, $(T'' \circ \lambda)v = T''(\Lambda v)$ and $(\Lambda \circ T)v = \Lambda(Tv)$ are elements of V''. To show they are equal, it suffices to show that for any $f \in V'$ we have

$$(T''(\Lambda v))f = (\Lambda(Tv))f.$$

To see this, we have

$$\begin{aligned}
& (T''(\Lambda v))f \\
\text{by the definition of dual map, 3.99} \quad &= (\Lambda v)(T'f) \\
\text{by the definition of } \Lambda \quad &= (T'f)v \\
\text{by the definition of dual map, 3.99} \quad &= f(Tv).
\end{aligned}$$

On the other hand, by the definition of Λ, we also have

$$(\Lambda(Tv))f = f(Tv).$$

Hence we have $T''(\Lambda v) = \Lambda(Tv)$, therefore

$$(T'' \circ \lambda)v = (\Lambda \circ T)v.$$

As the vector v is chosen arbitrarily, we prove that $T'' \circ \lambda = \Lambda \circ T$.

(c) Since V is finite-dimensional, by 3.95, we have $\dim V = \dim V' = \dim V''$. Hence it suffices to show that Λ is injective. Suppose $\Lambda v = 0$, then for any $f \in V'$ we have

$$(\Lambda v)f = f(v) = 0.$$

Let $U = \{v\}$ as in Problem 18, by our assumption we have $U^0 = V'$, hence it follows from Problem 18 that $U = \{0\}$. Therefore $v = 0$, which implies that Λ is injective.

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This entry was posted in Chapter 3 and tagged Exercise F.

← Chapter 3 Exercise E | Chapter 4 Exercise →

Kal · 2 years ago

Do you happen to have the solutions to 30-33?

1 · Reply · Share ›

Sam Tay · 5 months ago

I'm curious what others have to say about problem 36(b) if anyone here has completed it. Given that i: U -> V is the inclusion map, i.e. i(u) = u, we are asked to show that "If V is finite-dimensional, then range i' = U'." That is, i': V' -> U' is surjective. I don't see why the antecedent that V is finite-dimensional is necessary. Where is the pitfall in the logic that, given any f in U', for all u in U, i'(f(u)) = f(i(u)) = f(u), hence i'(f) = f and thus f is in range i'? It seems too simple and doesn't use the fact that V is finite-dimensional..

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LyToucan · 9 months ago

I have a better solution of exercise 15.

Here's a list of logically equivalent statements that prove 15:

1) T=0

2) (range T)^0 = W'

by 3.107a this is equivalent to

3) null T' = W'

4) T' = 0

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Aoden · 2 years ago

I believe that you have an error in your solution to the first problem. You wrote dim range V. V is a vector space, not a function, hence it does not have a range. I believe you meant dim range ϕ.

Also, for the 5th problem, I believe the proof could be much shorter if you show that the dimensions of the two vector spaces are equal, since two vector spaces are isomorphic if and only if they have the same dimension.

Thank you so much for taking the time to put these solutions up. They have been very helpful.

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Mohammad Rashidi Mod → Aoden · 2 years ago

Fixed, thanks!

Reply · Share ›

Wu Jinyang · 2 years ago

Hi. can you post the solution to question 34(B). It took me almost the whole afternoon yielding no results. I have some trouble understanding A, the linear transformation that maps V to V''.

According to the book (which is something like this)

$\Psi(x)(\varphi) = \varphi(x), \quad x \in V, \varphi \in V'.$

wouldn't A maps V to P. Since phi (x) would equal to a number

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Mohammad Rashidi Mod → Wu Jinyang · 2 years ago

The map Λ maps x (in V) to Λx, not φ(x). The element Λx, as an element of V'', acts on V', which gives us a number. I updated Problem 34.

Reply · Share ›

Wu Jinyang · 2 years ago

Also for solution to question 6, line 4. The expression of v is missing some parts

Reply · Share ›

Mohammad Rashidi Mod → Wu Jinyang · 2 years ago

Thanks!

Reply · Share ›

Wu Jinyang · 2 years ago

The other solution to question 18 is wrong. Since you don't know if U is a subspace or not, "dim U" doesn't make any sense. 3.106 specify U to be a subspace of V.

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Mohammad Rashidi Mod → Wu Jinyang · 2 years ago

It does not matter, since we can substitute U by span(U).

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Marcel Ackermann · 2 years ago

Solution to 3F12:

$I'(\phi)=\phi \circ I$ (definition of dual map)

$\phi \circ I(v) = \phi(v) \forall v in V$, which is the definition of the identity map.

Reply · Share ›

Supreeth Ravish · 3 years ago

Thank you so much, whoever did this.

Reply · Share ›

Gus · 4 years ago

I think 6. can be done directly by proving that dim span(v1, ..., vm) = dim range gamma (assume for simplicity v1, ..., vn is a basis of the span, and map v using gamma(coordinate_functional(v)), where coordinate_functional(v)(vi) is the i-th coefficient of vi in v. This is obviously invertible as for some v gives you the list of coordinates plus some garbage)... Using that equality both (a) and (b) seem easy. If I'm not mistaken somewhere.

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Chapter 4 Exercise

Posted on April 1, 2016 by Linearity

1. Empty

---

2. Solution: False. Consider $1=(z^m+1)+(-z^m)\notin\{0\}\cup\{p\in\mathcal{P}(\mathbb{F}):\deg p=m\}$. Note that

$$(z^m+1)\in\{0\}\cup\{p\in\mathcal{P}(\mathbb{F}):\deg p=m\}$$

and

$$-z^m\in\{0\}\cup\{p\in\mathcal{P}(\mathbb{F}):\deg p=m\},$$

it follows that $\{0\}\cup\{p\in\mathcal{P}(\mathbb{F}):\deg p=m\}$ is not closed under addition. Hence it is not a subspace of $\mathcal{P}(\mathbb{F})$.

---

3. Solution: False. Consider $z=(z^2+z)+(-z^2)\notin\{0\}\cup\{p\in\mathcal{P}(\mathbb{F}):\deg p\text{~is even}\}$. Note that

$$(z^2+z)\in\{0\}\cup\{p\in\mathcal{P}(\mathbb{F}):\deg p\text{~is even}\}$$

and

$$-z^2\in\{0\}\cup\{p\in\mathcal{P}(\mathbb{F}):\deg p\text{~is even}\},$$

it follows that $\{0\}\cup\{p\in\mathcal{P}(\mathbb{F}):\deg p\text{~is even}\}$ is not closed under addition. Hence it is not a subspace of $\mathcal{P}(\mathbb{F})$.

---

4. Solution: Define $p\in\mathcal{P}(\mathbb{F})$ by

$$p(z)=(z-\lambda_1)^{n-m+1}(z-\lambda_2)\cdots(z-\lambda_m).$$

Then p is a polynomial of degree n such that $0=p(\lambda_1)=\cdots=p(\lambda_m)$ and such that p has no other zeros.

---

5. Solution: See Linear Algebra Done Right Solution Manual Chapter 4 Problem 2.

---

6. Solution: See Linear Algebra Done Right Solution Manual Chapter 4 Problem 4.

---

7. Solution: See Linear Algebra Done Right Solution Manual Chapter 4 Problem 5.

---

8. Solution: First we show that T is a linear map. Then we show $Tp\in\mathcal{P}(\mathbb{R})$ for a basis of $p\in\mathcal{P}(\mathbb{R})$, then by linearity of T, we have $Tp\in\mathcal{P}(\mathbb{R})$ for every polynomial $p\in\mathcal{P}(\mathbb{R})$. For any $\lambda\in\mathbb{R}$ and $p,q\in\mathcal{P}(\mathbb{R})$, we have

$$\begin{aligned}T(p+q)=&\frac{(p+q)-(p+q)(3)}{x-3}=\frac{(p+q)-p(3)-q(3)}{x-3}\\=&\frac{p-p(3)}{x-3}+\frac{q-q(3)}{x-3}=Tp+Tq,\end{aligned}$$

if $x\neq 3$. Similarly,

$$T(\lambda p)=\frac{(\lambda p)-(\lambda p)(3)}{x-3}=\frac{\lambda p-\lambda p(3)}{x-3}=\lambda\frac{p-p(3)}{x-3}=\lambda Tp.$$

If $x=3$, then T is a composition of the differentiation map and evaluation map. Both of them are linear, hence T is also linear. We can show it directly

$$T(\lambda p+q)=(\lambda p+q)'(3)=(\lambda p'+q')(3)=\lambda p'(3)+q'(3)=\lambda Tp+Tq.$$

Therefore T is a linear map. Let us consider Tx^n for $n\in\mathbb{N}^+$, if $x\neq 3$,

$$T(x^n)=\frac{x^n-3^n}{x-3}=x^{n-1}+3x^{n-2}+\cdots+3^kx^{n-1-k}+\cdots+3^{n-1}\in\mathcal{P}(\mathbb{R}).$$

Moreover, if $x=3$, we have $T(x^n)=3^{n-1}n$. Note that when $x=3$, it is true that

$$x^{n-1}+3x^{n-2}+\cdots+3^kx^{n-1-k}+\cdots+3^{n-1}=3^{n-1}n.$$

We get

$$T(x^n)=x^{n-1}+3x^{n-2}+\cdots+3^kx^{n-1-k}+\cdots+3^{n-1}\in\mathcal{P}(\mathbb{R})$$

for $x\in\mathbb{R}$. Similarly, we can show $T(1)=0\in\mathcal{P}(\mathbb{R})$.

Since any polynomial of $\mathcal{P}(\mathbb{R})$ is a linear combination of $1,x,x^2,\cdots$, it follows that $Tp\in\mathcal{P}(\mathbb{R})$ for every polynomial $p\in\mathcal{P}(\mathbb{R})$.

I am not sure the textbook indicates that $1,x,x^2,\cdots$ is a basis of $\mathcal{P}(\mathbb{R})$, so I use some easier arguments such as any polynomial of $\mathcal{P}(\mathbb{R})$ is a linear combination of $1,x,x^2,\cdots$.

---

9. Solution: If $f(z)=a_nz^n+\cdots+a_1z+a_0$, where $a_n,\cdots,a_0\in\mathbb{C}$, then

$$\overline{f(\overline{z})}=\overline{a_n}z^n+\cdots+\overline{a_1}z+\overline{a_0}.$$

That implies $\overline{f(\overline{z})}$ is a polynomial. As the product of polynomials is a polynomial as well, we conclude q is a polynomial.

Now let us show q has only real coefficients. Denote $q(z)$ by

$$q(z)=\mu_{2n}z^{2n}+\cdots+\mu_1z+\mu_0.$$

Note that $\overline{q(\overline{z})}=\overline{f(\overline{z})\overline{f(z)}}=\overline{f(\overline{z})}f(z)=q(z)$, it follows

$$\overline{\mu_{2n}}z^{2n}+\cdots+\overline{\mu_1}z+\overline{\mu_0}=\mu_{2n}z^{2n}+\cdots+\mu_1z+\mu_0.$$

Hence $\overline{\mu_k}=\mu_k$, i.e $\mu_k\in\mathbb{R}$, for $k=0,\cdots,2n$.

Here you can also compute the coefficients μ_k in terms of a_i and show $\mu_k=\overline{\mu_k}$ or use some calculus methods.

---

10. Note that $x_0,x_1,\cdots,x_m$ are distinct, we can define the polynomial

$$f(x)=\sum_{j=0}^{m}\frac{(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_m)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_m)}p(x_j).$$

Then it is obvious that $f(x)\in\mathcal{P}_m(\mathbb{C})$. Moreover, since x_j and $p(x_j),j=0,1,\cdots,m$, are real, it follows that the coefficients of $f(x)$ are real. Hence it suffices to show that $p(x)=f(x)$.

By plugging $x=x_i$ into $f(x)$, we have $f(x_i)=p(x_i)$ since all summands except one are zero (see the link in the remark below),

$$\frac{(x_i-x_0)(x-x_1)\cdots(x_i-x_{j-1})(x_i-x_{j+1})\cdots(x_i-x_m)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_m)}=\delta_{ij},$$

where $\delta_{ij}=0$ if $i\neq j$ and $\delta_{ij}=1$ if $i=j$.

This implies that $f-p$ has $m+1$ distinct zeros. Since $f-p\in\mathcal{P}_m(\mathbb{C})$, it follows from 4.12 that the degree of $f-p$ can not be nonnegative. Hence $f-p$ is the zero polynomial, thus completing the proof.

The polynomial used here is called the Lagrange Interpolating Polynomial . Please see the following link for more detail.

---

11. By the division algorithm of polynomials in 4.8, we know that for every polynomial $f\in\mathcal{P}(\mathbb{F})$ there exist unique polynomials q and r such that

$$f=pq+r,\quad\text{and}\quad\deg r<\deg p.$$

This implies that $\mathcal{P}(\mathbb{F})=U\oplus\mathcal{P}_{\deg p-1}(\mathbb{F})$.

Therefore

$$\mathcal{P}(\mathbb{F})/U\cong\mathcal{P}_{\deg p-1}(\mathbb{F}).$$

It follows that

$$\dim\mathcal{P}(\mathbb{F})/U=\dim\mathcal{P}_{\deg p-1}(\mathbb{F})=\deg p.$$

Moreover, a basis of $\mathcal{P}(\mathbb{F})/U$ is $1,x,x^2,\cdots,x^{\deg p-1}$.

Here I used the fact that if $V=U\oplus W$, then $V/U\cong W$. Please try to prove as the following alternative solution.

Another solution (explains the solution above more explicitly).

For any given polynomial $f\in\mathcal{P}(\mathbb{F})$, let $r(f)$ be the reminder of f divided by q. Note that $\deg r(f)<\deg q$we have a map $r:\mathcal{P}(\mathbb{F})\to\mathcal{P}_{\deg p-1}(\mathbb{F})$. One can check this is a linear map. Moreover, $\operatorname{Null}r=U$. By taking polynomials in $\mathcal{P}_{\deg p-1}(\mathbb{F})$, we have that $\operatorname{range}r=\mathcal{P}_{\deg p-1}(\mathbb{F})$.

By 3.91(d), we have that

$$\mathcal{P}(\mathbb{F})/U=\mathcal{P}(\mathbb{F})/\operatorname{Null}r\cong\operatorname{range}r=\mathcal{P}_{\deg p-1}(\mathbb{F}).$$

Now the problem is solved similarly.

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This entry was posted in Chapter 4.

← Chapter 3 Exercise F Chapter 5 Exercise A →

Michael D. Nguyen · 7 months ago
The another altenative solution for 10 is by deriving the result of 5.
There is a polynomial p of real coefficients with m+1 distinct values in Pm(R) and p must also be in Pm(C). And since p must be uniquely defined in Pm(C), we have arrived at what is needed to be proven.
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bystandling · 2 years ago
Did you choose not to provide a solution to #11? I found one, though it involved proving P(F)/U isomorphic to P_(m-1) (F) by defining a linear transformation T(p)=r where r is the remainder defined by the division algorithm. I was wondering if there was a simpler method.
Reply · Share ›

Mohammad Rashidi Mod → bystandling · 2 years ago
That is the essential part of this problem.
Reply · Share ›

Chapter 5 Exercise A

Posted on May 1, 2016 by Linearity

1. Solution: (a) For any $u \in U$, then $Tu = 0 \in U$ since $U \subset \text{null}T$, hence U is invariant under T.

(b) For any $u \in U$, then $Tu \in \text{range}T \subset U$, hence U is invariant under T.

---

2. Solution: See Linear Algebra Done Right Solution Manual Chapter 5 Problem 4. Let $\lambda = 0$.

---

3. Solution: For any $u \in \text{range}S$, there exists $v \in V$ such that $Sv = u$, hence

$$Tu = TSv = STv \in \text{range}S.$$

Therefore range S is invariant under T.

---

4. Solution: See Linear Algebra Done Right Solution Manual Chapter 5 Problem 1.

---

5. Solution: See Linear Algebra Done Right Solution Manual Chapter 5 Problem 2.

---

6. Solution: See Linear Algebra Done Right Solution Manual Chapter 5 Problem 3.

---

7. Solution: Let (x, y) be an eigenvector of T corresponding to eigenvalue λ, then we have

$$T(x, y) = \lambda(x, y),$$

i.e., $(\lambda x, \lambda y) = (-3y, x)$. Hence we have $\lambda x = -3y$ and $\lambda y = x$, it follows that $\lambda^2 xy = -3xy$. If $xy \neq 0$, then $\lambda^2 = -3$, this is impossible.

If $x = 0$, then $y = 0$ by $\lambda x = -3y$. However (x, y) is an eigenvector, hence $(x, y) \neq (0, 0)$. We get a contradiction.

If $y = 0$, then $x = 0$ by $\lambda y = x$. Similarly, we get a contradiction.

Hence no such eigenvectors exist, namely T has no eigenvalues.

---

8. Solution: See Linear Algebra Done Right Solution Manual Chapter 5 Problem 5.

---

9. Solution: See Linear Algebra Done Right Solution Manual Chapter 5 Problem 6.

---

10. Solution: (a) Suppose $v = (v_1, \cdots, v_n)$ is a eigenvector of T corresponding to eigenvalue λ. Then we have $Tv = \lambda v$, hence

$$(v_1, 2v_2, \cdots, v_n) = (\lambda v_1, \lambda v_2, \cdots, \lambda v_n). \tag{1}$$

As $v \neq 0$ by definition of eigenvectors, there is some $i \in \{1, 2, \cdots, n\}$ such that $v_i \neq 0$. Note that we have $iv_i = \lambda v_i$ by (1), it implies $\lambda = i$. For $\lambda = i$, it is easy to solve (1). We can conclude the corresponding eigenvectors are of the form $(0, \cdots, 0, a, 0, \cdots, 0)$, $a \in \mathbb{F}$ with a in i-th component. Similarly, all eigenvalues of T are $1, 2, \cdots, n$. All eigenvectors with respect to i are of the form $(0, \cdots, 0, a, 0, \cdots, 0)$, $a \in \mathbb{F}$ with a in i-th component.

(b) Suppose W is an invariant subspace of T. Assume $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ with 1 in i-th component. Then $e_1, \cdots, e_n$ is a basis of $\mathbb{F}^n$ and e_i is an eigenvector of T corresponding to i. If $a_1e_1 + \cdots + a_ke_k \in W$ with $a_1 \cdots a_k \neq 0$, we will show $\text{span}(e_1, e_2, \cdots, e_k) \subset W$. Note that $a_1e_1 + \cdots + a_ke_k \in W$ and W is invariant with respect to T, it follows that

$$T(a_1e_1 + \cdots + a_ke_k) = a_1e_1 + \cdots + ka_ke_k \in W.$$

Hence

$$k(a_1e_1 + \cdots + a_ke_k) - (a_1e_1 + \cdots + ka_ke_k) = (k-1)a_1e_1 + \cdots + a_{k-1}e_{k-1} \in W,$$

and the coefficients are nonzero. Inductively, we will get some

$$\lambda_1e_1 + \cdots + \lambda_ie_i \in W$$

for $\lambda_1 \cdots \lambda_i \neq 0$ for any $i \leqslant k$($\lambda_1, \cdots, \lambda_i$ change as i changes). In particular, $\mu_1e_1 \in W$ and $\mu_1 \neq 0$. Hence $e_1 \in W$. Then, consider $\eta_1e_1 + \eta_2e_2 \in W$ where $\eta_1\eta_2 \neq 0$, we will get $e_2 \in W$. Inductively, we can show that $\{e_1, e_2, \cdots, e_k\} \subset W$. Hence $\text{span}(e_1, e_2, \cdots, e_k) \subset W$. Similarly, if $a_{i_1}e_{i_1} + \cdots + a_{i_k}e_{i_k} \in W$ with $a_{i_1} \cdots a_{i_k} \neq 0$ and all $\{a_{i_j}\}$ distinct, then $\text{span}(e_{i_1}, \cdots, e_{i_k}) \subset W$. Now let us consider the general form of W. Suppose $W \cap \{e_1, \cdots, e_n\} = \{e_{i_1}, \cdots, e_{i_k}\}$, then we will show $\text{span}(e_{i_1}, \cdots, e_{i_k}) = W$. It is obvious that $\text{span}(e_{i_1}, \cdots, e_{i_k}) \subset W$. If there some $w \in W$ but $w \notin \text{span}(e_{i_1}, \cdots, e_{i_k})$. Then w can be written as

$$w = b_1e_1 + \cdots + b_ne_n, \quad b_1, \cdots, b_n \in \mathbb{F}$$

such that there is some $s \notin \{i_1, \cdots, i_k\}$ and $b_s \neq 0$. By previous argument, we have $e_s \in W$. This contradicts with $W \cap \{e_1, \cdots, e_n\} = \{e_{i_1}, \cdots, e_{i_k}\}$. Hence we show that $\text{span}(e_{i_1}, \cdots, e_{i_k}) = W$. Moreover, all invariant subspaces of T have this form.

I am not satisfied with this solution. **!**

---

11. Solution: Suppose λ is an eigenvalue of T with an eigenvector q, then

$$q' = Tq = \lambda q.$$

Note that in general $\deg p' < \deg p$(because we consider $\deg 0 = -\infty$). If $\lambda \neq 0$, then $\deg \lambda q > \deg q'$. We get a contradiction. If $\lambda = 0$, then $q = c$ for nonzero $c \in \mathbb{R}$. Hence the only eigenvalue of T is zero with nonzero constant polynomials as eigenvectors.

---

12. Solution: Suppose λ is an eigenvalue of T with an eigenvector q. Let $q = a_nx^n + \cdots + a_1x + a_0$ such that $a_n \neq 0$, then

$$\lambda q = Tq = xq',$$

namely

$$\lambda a_nx^n + \cdots + \lambda a_1x + \lambda a_0 = na_nx^n + \cdots + 2a_2x^2 + a_1x.$$

Since $a_n \neq 0$, it follows that $\lambda = n$ by considering the leading coefficient. Then we have $a_0 = a_1 = \cdots = a_{n-1} = 0$, hence $q = a_nx^n$. Hence all eigenvalues of T are $0, 1, 2, \cdots$ and all eigenvectors correspond to m is λx^m such that $m \in \mathbb{N}$, $\lambda \neq 0$ and $\lambda \in \mathbb{R}$.

---

13. Solution: Let $\alpha_i \in \mathbb{F}$ such that

$$|\alpha_i - \lambda| = \frac{1}{1000 + i}, \quad i = 1, \cdots, \dim V + 1.$$

These α_i exist and are different from each other since $F = \mathbb{R}$ or $\mathbb{C}$. Note that each operator on V has at most $\dim V$ distinct eigenvalues by 5.13. Hence there exists some $i \in \{1, 2, \cdots, \dim V + 1\}$ such that α_i is not an eigenvalue of T. Then by 5.6, $T - \alpha_iI$ is invertible.

---

14. Solution: Note that any $v \in V$ can be written uniquely as $u + w$ for $u \in U$ and $w \in W$ since $V = U \oplus W$. It follow that this P is well-defined. Maybe you also need to check$P \in \mathcal{L}(V)$. Now let us consider the eigenvalues of P. By consider $v \neq 0$, if there exists $\lambda \in \mathbb{F}$ such that $Pv = \lambda v$. Write $v = u + w$ for $u \in U$ and $w \in W$, then u and w can not be both zero. Hence by definition of P, we have

$$Pv = u, \quad \lambda v = \lambda u + \lambda w.$$

It follows that $u = \lambda u + \lambda w$, namely $(\lambda - 1)u + \lambda w = 0$. Note that $V = U \oplus W$, it follows that $(\lambda - 1)u = \lambda w = 0$. If $u \neq 0$, then $\lambda = 1$. Hence $w = 0$ and the corresponding eigenvectors are nonzero vectors $v \in U$. If $w \neq 0$, then $\lambda = 0$. Hence $u = 0$ and the corresponding eigenvectors are nonzero vectors $v \in W$.

---

15. Solution: (a) Suppose λ is an eigenvalue of T, then there exists a nonzero vector $v \in V$ such that $Tv = \lambda v$. Hence

$$S^{-1}TS(S^{-1}v) = S^{-1}Tv = S^{-1}(\lambda v) = \lambda S^{-1}v.$$

Note that $S^{-1}v \neq 0$ as S^{-1} is invertible, hence λ is an eigenvalue of $S^{-1}TS$, namely every eigenvalue of T is an eigenvalue of $S^{-1}TS$. Similarly, note that $S(S^{-1}TS)S^{-1} = T$, we have every eigenvalue of $S^{-1}TS$ is an eigenvalue of T. Hence T and $S^{-1}TS$ have the same eigenvalues.

(b) From the process of (a), one can easily deduce that v is an eigenvector of T if and only if $S^{-1}v$ is an eigenvector of $S^{-1}TS$.

---

16. Solution: Although this problem is true for infinite-dimensional vector space, I will just consider finite-dimension case since we are considering the matrix of T(otherwise, it would be a infinite matrix). Suppose the matrix of T with respect to basis $e_1, \cdots, e_n$ of V contains only real entries. Then

$$Te_j = A_{1,j}e_1 + \cdots + A_{n,j}e_n,$$

where $A_{i,j} \in \mathbb{R}$ for all $i, j = 1, 2, \cdots, n$. Let

$$v = k_1e_1 + \cdots + k_ne_n$$

be a eigenvector with respect to λ, where $k_i \in \mathbb{C}$, $i = 1, \cdots, n$. Then we have

$$Tv = \lambda v,$$

namely

$$\lambda \sum_{i=1}^{n} k_ie_i = \sum_{i=1}^{n} k_iTe_i = \sum_{i=1}^{n}\sum_{j=1}^{n} k_iA_{j,i}e_j. \tag{2}$$

Consider the complex conjugation of (2), we have

$$\overline{\lambda} \sum_{i=1}^{n} \overline{k_i}e_i = \sum_{i=1}^{n} \overline{k_i}Te_i = \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{k_i}A_{j,i}e_j \tag{3}$$

since $A_{i,j} \in \mathbb{R}$ for all $i, j = 1, 2, \cdots, n$. (why? consider components)Note that (3) implies

$$T(\overline{k_1}e_1 + \cdots + \overline{k_n}e_n) = \overline{\lambda} \sum_{i=1}^{n} \overline{k_i}e_i. \tag{4}$$

Since $v = k_1e_1 + \cdots + k_ne_n \neq 0$, it follows that not all k_i is zero, so is $\overline{k_i}$. Hence $\overline{k_1}e_1 + \cdots + \overline{k_n}e_n \neq 0$, hence (4) tell us $\overline{\lambda}$ is an eigenvalue of T.

---

17. Solution: See Linear Algebra Done Right Solution Manual Chapter 5 Problem 23.

---

18. Solution: Suppose λ is an eigenvalue of T and one corresponding eigenvector is $(w_1, w_2, \cdots)$. Then not all of w_i is zero. Moreover, we have

$$(0, w_1, w_2, \cdots) = T(w_1, w_2, \cdots) = \lambda(w_1, w_2, \cdots).$$

If $\lambda = 0$, then

$$(0, w_1, w_2, \cdots) = 0$$

implies $w_i \equiv 0$ for any $i \in \mathbb{N}^+$. We get a contradiction. If $\lambda \neq 0$. Consider the first component, we have $0 = \lambda w_1$, hence $w_1 = 0$. Then consider the second component, we have $\lambda w_2 = w_1 = 0$, hence $w_2 = 0$. By induction, one can easily deduce that $w_i \equiv 0$ for any $i \in \mathbb{N}^+$. We get a contradiction as well. Hence T has no eigenvalues.

---

19. Solution: See Linear Algebra Done Right Solution Manual Chapter 5 Problem 7.

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20. Solution: See Linear Algebra Done Right Solution Manual Chapter 5 Problem 8.

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21. Solution: See Linear Algebra Done Right Solution Manual Chapter 5 Problem 10. (b) is almost proved there.

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22. Solution: Note that we have

$$T(v + w) = Tv + Tw = 3w + 3v = 3(v + w),$$

and

$$T(v - w) = Tv - Tw = 3w - 3v = -3(v - w).$$

If $v - w$ or $v + w$ is nonzero, then 3 or -3 is an eigenvalue of T. In fact if $v - w = 0$ and $v + w = 0$, it is easy to see $v = w = 0$. It contradicts with $v \neq 0$ and $w \neq 0$.

---

23. Solution: See Linear Algebra Done Right Solution Manual Chapter 5 Problem 11.

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24. Solution: (a) (a) If the sum of the entries in each row of A equals 1, then one can easily deduce that

$$T\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Hence 1 is an eigenvalue of T with $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ as a corresponding eigenvector.

(b) This problem is interesting. It is simple by considering determinant. However it is complicated here. We just need to show that $T - I$ is not invertible. It suffices to show $T - I$ is not surjective by 5.6. Note that we have

$$(T - I)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} A_{1,i}x_i - x_1 \\ \vdots \\ \sum_{i=1}^{n} A_{n,i}x_i - x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \tag{5}$$

where $A_{i,j}$ is the (i, j)-component of A. Moreover, we have

$$1 = \sum_{i=1}^{n} A_{i,j} \quad j = 1, \cdots, n.$$

Hence

$$\begin{aligned} y_1 + \cdots + y_n &= \sum_{j=1}^{n}\sum_{i=1}^{n} A_{j,i}x_i - \sum_{j=1}^{n} x_j \\ &= \sum_{i=1}^{n} x_i \sum_{j=1}^{n} A_{j,i} - \sum_{j=1}^{n} x_j \\ &= \sum_{i=1}^{n} x_i - \sum_{j=1}^{n} x_j = 0. \end{aligned}$$

By (5) and previous equation, it follows that

$$\text{range}(T - I) \subset \{(x_1, \cdots, x_n)^T \in \mathbb{F}^n : x_1 + \cdots + x_n = 0\},$$

where $(x_1, \cdots, x_n)^T$ means $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. It follows that $T - I$ is not surjective, hence completing the proof.

---

25. Solution: Let the eigenvalues corresponding to u, v are λ_1, λ_2 respectively, then we have

$$Tu = \lambda_1u, \quad Tv = \lambda_2v.$$

If the eigenvalue corresponding to $u + v$ is λ, we have

$$\lambda(u + v) = T(u + v) = Tu + Tv = \lambda_1u + \lambda_2v.$$

It follows that $(\lambda - \lambda_1)u + (\lambda - \lambda_2)v = 0$. If $\lambda_1 \neq \lambda_2$, then $\lambda - \lambda_1$ and $\lambda - \lambda_2$ can not be both zero. Hence u, v is not linearly independent. By 5.10, it follows that u and v correspond to the same eigenvalue. Hence $\lambda_1 = \lambda_2$.

---

26. Solution: See Linear Algebra Done Right Solution Manual Chapter 5 Problem 12.

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27. Solution: See Linear Algebra Done Right Solution Manual Chapter 5 Problem 13.

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28. Solution: For any nonzero vector $v \in V$, extend it to a basis of V as $v = v_1, v_2, \cdots, v_n$. Then

$$Tv_1 = \sum_{k=1}^{n} \lambda_kv_k.$$

Consider $U = \text{span}(v_1, v_2)$, as U is invariant under T by assumption. It follows that $Tv_1 \in U$. Hence $\lambda_3 = \cdots = \lambda_n = 0$. Similarly, consider $U = \text{span}(v_1, v_3)$ (note that $\dim V \geq 3$), we will conclude $\lambda_2 = \lambda_4 = \cdots = \lambda_n = 0$. Hence $\lambda_2 = \cdots = \lambda_n = 0$. This means v_1 is an eigenvector of T. That is every nonzero vector in V is an eigenvector of T since v is chosen arbitrarily. By Problem 26, we deduce that T is a scalar multiple of the identity operator.

---

29. Solution: See Linear Algebra Done Right Solution Manual Chapter 5 Problem 9.

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30. Solution: Note that T has at most $\dim(\mathbb{R}^3) = 3$ eigenvalues (by 5.13) and 4, 5, and $\sqrt{7}$ are eigenvalues of T, it follows that 9 is not an eigenvalues of T. Hence $(T - 9I)$ is surjective (by 5.6). Thus there exists $x \in \mathbb{R}^3$ such that $(T - 9I)x = (4, 5, \sqrt{7})$, namely $Tx - 9x = (4, 5, \sqrt{7})$.

---

31. Solution: If there exists $T \in \mathcal{L}(V)$ such that $v_1, \cdots, v_m$ are eigenvectors of T corresponding to distinct eigenvalues. Then $v_1, \cdots, v_m$ is linearly independent by 5.10.

Conversely, if $v_1, \cdots, v_m$ is linearly independent. Then we can extend it to a basis of V as $v_1, \cdots, v_m, v_{m+1}, \cdots, v_n$. Define $T \in \mathcal{L}(V)$ by

$$Tv_i = iv_i, \quad i = 1, \cdots, n.$$

Then $v_1, \cdots, v_m$ are eigenvectors of T corresponding to eigenvalues $1, \cdots, m$, respectively.

---

32. Solution: Let $V = \text{span}(e^{\lambda_1x}, \cdots, e^{\lambda_nx})$, and define an operator $T \in \mathcal{L}(V)$ by $Tf = f'$(You should check $T \in \mathcal{L}(V)$). Then consider

$$Te^{\lambda_ix} = \lambda_ie^{\lambda_ix}.$$

Hence λ_i is an eigenvalue of T with an corresponding eigenvector e^{λ_ix}. As $\lambda_1, \cdots, \lambda_n$ is a list of distinct real numbers, by 5.10, it follows that $e^{\lambda_1x}, \cdots, e^{\lambda_nx}$ is linearly independent.

---

33. Solution: By definition, for any $x + \text{range}T \in V/(\text{range}T)$, we have

$$T/(\text{range}T)(x + \text{range}T) = Tx + \text{range}T.$$

Note that $Tx \in \text{range}T$, it follows that $T/(\text{range}T)(x + \text{range}T) = 0$. Since $x + \text{range}T$ is choosed arbitrarily, we conclude that $T/(\text{range}T) = 0$.

---

34. Solution: By definition, for any $x + \text{null}T \in V/(\text{null}T)$, we have

$$T/(\text{null}T)(x + \text{null}T) = Tx + \text{null}T.$$

Hence $T/(\text{null}T)$ is injective if and only if

$$Tx \in \text{null}T \iff x \in \text{null}T.$$

Note that

$$Tx \in \text{null}T \iff x \in \text{null}T$$

is equivalent to $\text{null}T \cap \text{range}T = \{0\}$. Because if we assume $Tx \in \text{null}T \iff x \in \text{null}T$, then for any $v \in \text{null}T \cap \text{range}T$, then there exists $u \in V$ such that $Tu = v$, hence $Tu \in \text{null}T$ implies $u \in \text{null}T$. That is $v = Tu = 0$. The other direction is also true. Hence the proof is completing.

---

35. Solution: Suppose $\lambda \in \mathbb{F}$ is an eigenvalue of T/U, we need to show λ is an eigenvalue of T. There exists a nonzero $x + U \in V/U$(i.e. $x \notin U$) such that

$$(T/U)(x + U) = \lambda(x + U) \Longrightarrow Tx - \lambda x \in U.$$

If λ is an eigenvalue of $T|_U$, then we are done. If λ is not an eigenvalue of T, then $T|_U - \lambda I : U \to U$ is invertible by 5.6 (here use $\dim V < \infty$). Hence there exists a $y \in U$ such that

$$(T|_U - \lambda I)y = Tx - \lambda x \Longrightarrow Ty - \lambda y = Tx - \lambda x$$

since $Tx - \lambda x \in U$. Hence we have

$$T(x - y) = \lambda(x - y),$$

and $x - y \neq 0$ since $x \notin U$ and $y \in U$. It follows that λ is an eigenvalue of T.

---

36. Solution: In Problem 32, we showed $1 = e^{0x}, e^x, e^{2x}, \cdots$ is linearly independent in the vector space of real-valued functions on R. Consider $V = \text{span}(1, e^x, e^{2x}, \cdots)$ and $U = \text{span}(e^x, e^{2x}, \cdots)$, then U and V are subspaces of the vector space of real-valued functions on R. Define $T \in \mathcal{L}(V)$ by $T(f) = e^xf$. Please check $T \in \mathcal{L}(V)$ and U is invariant under T. Consider T/U, we have

$$(T/U)(1 + U) = e^x + U = 0.$$

Since $1 \notin U$, it follows that 0 is an eigenvalue of T/U. However 0 is not an eigenvalue of T. Otherwise suppose there exists a nonzero $f \in V$ such that $Tf = 0$, then we have $e^xf = 0$. Hence $f = 0$ since $e^x \neq 0$ for any $x \in \mathbb{R}$. We get a contradiction.

About Linearity

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This entry was posted in Chapter 5 and tagged Exercise A.

← Chapter 4 Exercise Chapter 5 Exercise B →

Sam Tay · 4 months ago

Just a note on 35, a simpler proof: given an eigenvalue c of T/U, note that T/U - cI is not surjective by 5.6 so there exists w + U in V/U such that (T/U - cI)(v + U) is not equal to w + U for all v in V. Equivalently (Tv - cv) + U is not equal to w + U for all v in V. Hence Tv - cv is not equal to w for all v in V and it follows quickly that T - cI is not surjective in V, so that c is an eigenvalue of T.

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Ling Min Hao · a year ago

For Question 16, there is a simpler method. Suppose lambda is the eigenvalue of T. Then Tv=lambda.v . By taking conjugate at both side, this is solved.

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Ling Min Hao · a year ago

For Question 10, since each eigenvector can form an invariant subspaces, we have n invariant subspaces. By question 4, we know the sum of invariant subspaces is still invariant. There are 2^n such possibilities (including zero subspace). Until here, we know it AT LEAST have 2^n invariant subspace. But we know from (a), the eigenvectors actually form basis of F^n.This means F^n=U_1+...+U_m (each U_j represents an eigenspace) and hence it can maximally have 2^n subspace. Therefore, there are 2^n invariant subspace in T where each invariant subspace, say W is actually an element of span(u_1,...,u_n).

To make it clear, one of the possibility of invariant subspace could be the invariant subspace U_1+U_2, where it can be written as a_1u_1+b_1u_2+0u_3+...+0u_n which is an element of span(u_1,...,u_n).

(I might skip a few logical steps here, hope you can understand what I meant as it's hard to type things without Latex.)

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Chapter 5 Exercise B

Posted on May 2, 2016 by Linearity

1. Suppose $T\in\mathcal{L}(V)$ and there exists a positive integer n such that $T^n=0$.

(a) Prove that $I-T$ is invertible and that
$$(I-T)^{-1}=I+T+\cdots+T^{n-1}.$$

(b) Explain how you would guess the formula above.

Solution: (a) Note that
$$(I-T)(I+T+\cdots+T^{n-1})=I-T^n=I$$
and
$$(I+T+\cdots+T^{n-1})(I-T)=I-T^n=I,$$
(in fact we just need to check only one) it follows that $I-T$ is invertible and
$$(I-T)^{-1}=I+T+\cdots+T^{n-1}.$$

(b) From the familiar formula
$$1-x^n=(1-x)(1+x+\cdots+x^{n-1}).$$

---

2. Suppose $T\in\mathcal{L}(V)$ and $(T-2I)(T-3I)(T-4I)=0$. Suppose λ is an eigenvalue of T. Prove that $\lambda=2$ or $\lambda=3$ or $\lambda=4$.

Solution: Let v be an eigenvector of T corresponding to λ, then we have $Tv=\lambda v$. Similarly, we have
$$T^2v=T(\lambda v)=\lambda^2v \quad T^3v=T(\lambda^2v)=\lambda^3v,$$
and $T^nv=\lambda^nv$ for $n\in\mathbb{N}^+$. This implies for any polynomial p, we have $p(T)v=p(\lambda)v$. Hence
$$0=(T-2I)(T-3I)(T-4I)v=(\lambda-2)(\lambda-3)(\lambda-4)v.$$
As $v\neq 0$, it follows that
$$(\lambda-2)(\lambda-3)(\lambda-4)=0.$$
Thus $\lambda=2$ or $\lambda=3$ or $\lambda=4$.

---

3. Suppose $T\in\mathcal{L}(V)$ and $T^2=I$ and -1 is not an eigenvalue of T. Prove that $T=I$.

Solution: Note that for any $v\in V$, we have
$$v=\frac{1}{2}(v-Tv)+\frac{1}{2}(v+Tv). \tag{1}$$
Since T^2-I, it follows that
$$(T+I)\left(\frac{1}{2}(v-Tv)\right)=\frac{1}{2}(I-T^2)v=0.$$
Hence $\frac{1}{2}(v-Tv)\in\operatorname{null}(T+I)$.Similarly we have
$$\frac{1}{2}(v+Tv)\in\operatorname{null}(T-I).$$
By (1), it follows that $V=\operatorname{null}(T-I)+\operatorname{null}(T+I)$. However, -1 is not an eigenvalue of T. Hence $\operatorname{null}(T+I)=\{0\}$. Thus $V=\operatorname{null}(T-I)$. This implies $T=I$, since for any $v\in V$ we have $Tv-v=(T-I)v=0$.

---

4. Suppose $T\in\mathcal{L}(V)$ and $P^2=P$. Prove that $V=\operatorname{null}P\oplus\operatorname{range}P$.

Solution: Note that for any $v\in V$, we have
$$v=Pv+(v-Pv). \tag{2}$$
It is clear that $Pv\in\operatorname{range}P$. Since $P^2=P$, it follows that $P(v-Pv)=(P-P^2)v=0$. Thus $v-Pv\in\operatorname{null}P$. By (2), we have $V=\operatorname{null}P+\operatorname{range}P$. Suppose $v\in\operatorname{null}P\cap\operatorname{range}P$, then there exists $u\in V$ such that $Pu=v$. Moreover $Pv=0$. Hence
$$0=Pv=P(Pu)=P^2u=Pu=v.$$
This implies $\operatorname{null}P\cap\operatorname{range}P=\{0\}$. Therefore $V=\operatorname{null}P\oplus\operatorname{range}P$.

---

5. Suppose $S,T\in\mathcal{L}(V)$ and S is invertible. Suppose $p\in\mathcal{P}(\mathbf{F})$ is a polynomial. Prove that
$$p(STS^{-1})=Sp(T)S^{-1}.$$

Solution: See Linear Algebra Done Right Solution Manual Chapter 5 Problem 14.

---

6. Suppose $T\in\mathcal{L}(V)$ and U is a subspace of V invariant under T. Prove that U is invariant under $p(T)$ for every polynomial $p\in\mathcal{P}(\mathbf{F})$.

Solution: Note that $TU\subset U$, one can easily deduce $T^nU\subset U$ for any $n\in\mathbb{N}^+$ by induction. Hence $\lambda T^nU\subset U$ for any $\lambda\in\mathbf{F}$ since U is a vector space. If we assume for any $p\in\mathcal{P}(\mathbf{F})$ with $\deg p\leq n-1$, U is invariant under $p(T)$. Then we will show U is invariant under $q(T)$ for every polynomial $q\in\mathcal{P}(\mathbf{F})$ with $\deg q=n$. Let $q=\sum_{k=0}^n a_kx^k$, then
$$\begin{aligned}q(T)U&=\left(\sum_{k=0}^n a_kT^k\right)U=\left(\sum_{k=0}^{n-1} a_kT^k\right)U+a_nT^nU\\&\subset U+U=U.\end{aligned}$$
Here $U+U=U$ by Problem 15 of Exercise 1C. Hence U is invariant under $q(T)$ for every polynomial $q\in\mathcal{P}(\mathbf{F})$ with $\deg q=n$. By induction, we conclude U is invariant under $p(T)$ for every polynomial $p\in\mathcal{P}(\mathbf{F})$.

---

7. Suppose $T\in\mathcal{L}(V)$. Prove that 9 is an eigenvalue of T^2 if and only if 3 or -3 is an eigenvalue of T.

Solution: By problem 2, we have $T^2v=\lambda^2v$ if v be an eigenvector of T corresponding to λ. Hence 3 or -3 is an eigenvalue of T, then 9 is an eigenvalue of T^2 since $3^2=9$ and $(-3)^2=9$. Conversely, if 9 is an eigenvalue of T^2. It follows that T^2-9I is not injective, namely $(T-3I)(T+3I)$ is not injective. By Problem 11 of Exercise 3B, we have $T-3I$ or $T+3I$ is not injective. Hence we conclude 3 or -3 is an eigenvalue of T. (My apologies for using 5.6, it is only true for finite-dimensional V.)

---

8. Give an example of $T\in\mathcal{L}(\mathbf{R}^2)$ such that $T^4=-I$.

Solution: Denote $T\in\mathcal{L}(\mathbf{R}^2)$ by
$$T(x,y)=\left(\frac{\sqrt{2}}{2}x-\frac{\sqrt{2}}{2}y,\frac{\sqrt{2}}{2}x+\frac{\sqrt{2}}{2}y\right).$$
You can directly check that $T^4=-I$.

Here I use a fact that
$$\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}^n=\begin{pmatrix}\cos n\theta & \sin n\theta\\ -\sin n\theta & \cos n\theta\end{pmatrix}$$

---

9. Suppose V is finite-dimensional, $T\in\mathcal{L}(V)$, and $v\in V$ with $v\neq 0$. Let p be a nonzero polynomial of smallest degree such that $p(T)v=0$. Prove that every zero of p is an eigenvalue of T.

Solution: By definition, an eigenvalue of T must be contained in $\mathbf{F}$, hence we should assume that every zero of p is in $\mathbf{F}$. Let λ be a zero of p, then by 4.11 we have $p(z)=(z-\lambda)q(z)$, where $q(z)\in\mathcal{P}(\mathbf{F})$. Suppose λ is not an eigenvalue of T, then $T-\lambda I$ is injective. Hence
$$0=p(T)v=(T-\lambda I)q(T)v$$
implies $q(T)v=0$. However $\deg q<\deg p$ and q is nonzero(otherwise p is zero). This contradicts with the choice of p. Thus every zero of p is an eigenvalue of T.

---

10. Suppose $T\in\mathcal{L}(V)$ and v is an eigenvector of T with eigenvalue λ. Suppose $p\in\mathcal{P}(\mathbf{F})$. Prove that $p(T)v=p(\lambda)v$.

Solution: By the proof of Problem 2, it follows that $T^nv=\lambda^nv$. Hence for $p\in\mathcal{P}(\mathbf{F})$, suppose
$$p=\sum_{n=0}^k a_nx^n.$$
Then
$$p(T)v=\left(\sum_{n=0}^k a_nT^n\right)v=\sum_{n=0}^k a_nT^nv=\sum_{n=0}^k a_n\lambda^nv=p(\lambda)v.$$

---

11. Suppose $\mathbf{F}=\mathbf{C}$, $T\in\mathcal{L}(V)$, $p\in\mathcal{P}(\mathbf{C})$ is a polynomial, and $\alpha\in\mathbf{C}$. Prove that α is an eigenvalue of $p(T)$ if and only if $\alpha=p(\lambda)$ for some eigenvalue λ of T.

Solution: See Linear Algebra Done Right Solution Manual Chapter 5 Problem 15.

---

12. Show that the result in the previous exercise does not hold if $\mathbf{C}$ is replaced with $\mathbf{R}$.

Solution: See Linear Algebra Done Right Solution Manual Chapter 5 Problem 16.

---

13. Suppose W is a complex vector space and $T\in\mathcal{L}(W)$ has no eigenvalues. Prove that every subspace of W invariant under T is either $\{0\}$ or infinite-dimensional.

Solution: Suppose a subspace U of W invariant under T and $\dim U<\infty$. If $U\neq\{0\}$, then by 5.21, $T|_U$ has an eigenvalue with an eigenvector $v(\neq 0)$. That is $T|_U(v)=\lambda v$, namely $Tv=\lambda v$. Note that $v\neq 0$, we conclude T has an eigenvalue λ. We get a contradiction since $T\in\mathcal{L}(W)$ has no eigenvalues. Hence every subspace of W invariant under T is either $\{0\}$ or infinite-dimensional.

---

14. Give an example of an operator whose matrix with respect to some basis contains only 0's on the diagonal, but the operator is invertible. [The exercise above and the exercise below show that 5.30 fails without the hypothesis that an upper-triangular matrix is under consideration.]

Solution: See Linear Algebra Done Right Solution Manual Chapter 5 Problem 18.

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15. Give an example of an operator whose matrix with respect to some basis contains only nonzero numbers on the diagonal, but the operator is not invertible.

Solution: See Linear Algebra Done Right Solution Manual Chapter 5 Problem 19.

---

16. Rewrite the proof of 5.21 using the linear map that sends $p\in\mathcal{P}_n(\mathbf{C})$ to $(p(T))v\in V$ (and use 3.23).

Solution: Define $\varphi:\mathcal{P}_n(\mathbf{C})\to V$ by $\varphi(p)=p(T)v$, then φ is a linear map(check it). Note that $\dim\mathcal{P}_n(\mathbf{C})=n+1$ and $\dim V=n$, it follows that φ is not injective by 3.23. Hence there exists a nonzero $p\in\mathcal{P}_n(\mathbf{C})$ such that $p(T)v=0$. The remained is the same as 5.21.

---

17. Rewrite the proof of 5.21 using the linear map that sends $p\in\mathcal{P}_{n^2}(\mathbf{C})$ to $p(T)\in\mathcal{L}(V)$ (and use 3.23).

Solution: Define $\varphi:\mathcal{P}_{n^2}(\mathbf{C})\to\mathcal{L}(V)$ by $\varphi(p)=p(T)$, then φ is a linear map(check it). Note that $\dim\mathcal{P}_{n^2}(\mathbf{C})=n^2+1$ and $\dim\mathcal{L}(V)=n^2$, it follows that φ is not injective by 3.23. Hence there exists a nonzero $p\in\mathcal{P}_{n^2}(\mathbf{C})$ such that $p(T)=0$. This implies $p(T)v=0$. The remained is the same as 5.21.

---

18. Suppose V is a finite-dimensional complex vector space and $T\in\mathcal{L}(V)$. Define a function $f:\mathbf{C}\to\mathbf{R}$ by
$$f(\lambda)=\dim\operatorname{range}(T-\lambda I).$$
Prove that f is not a continuous function.

Solution: Let λ_0 be an eigenvalue of T, then $T-\lambda_0I$ is not surjective by 5.6. Hence $\dim\operatorname{range}(T-\lambda_0I)<\dim V$. Note that T has only finitely many eigenvalues, there exists a sequence of number λ_n such that
$$\lim_{n\to\infty}\lambda_n=\lambda_0$$
and λ_n are not eigenvalues of T. Then by 5.6, $T-\lambda_nI$ is surjective. Hence
$$\dim\operatorname{range}(T-\lambda_nI)=\dim V.$$
This implies $\lambda_n\to\lambda_0$, but
$$f(\lambda_0)\neq\lim_{n\to\infty}f(\lambda_n).$$
Thus f is not a continuous function.

---

19. Suppose V is finite-dimensional with $\dim V>1$ and $T\in\mathcal{L}(V)$. Prove that
$$\{p(T):p\in\mathcal{P}(\mathbf{F})\}\neq\mathcal{L}(V).$$

Solution: Note that $Tp(T)=p(T)T$, if $\{p(T):p\in\mathcal{P}(\mathbf{F})\}=\mathcal{L}(V)$, then $ST=TS$ for every $S\in\mathcal{L}(V)$. By Problem 16 of Exercises 3D, it follows that T is a scalar multiple of the identity. Suppose $T=\lambda I$, then
$$\{p(T):p\in\mathcal{P}(\mathbf{F})\}=\{p(\lambda)I:p\in\mathcal{P}(\mathbf{F})\}=\{\mu I:\mu\in\mathbf{F}\}.$$
Since $\dim V>1$, we have
$$\dim\mathcal{L}(V)=(\dim V)^2$$
and
$$\dim\{p(T):p\in\mathcal{P}(\mathbf{F})\}=1.$$
Hence
$$\{p(T):p\in\mathcal{P}(\mathbf{F})\}\neq\mathcal{L}(V).$$

---

20. Suppose V is a finite-dimensional complex vector space and $T\in\mathcal{L}(V)$. Prove that T has an invariant subspace of dimension k for each $k=1,\cdots,\dim V$.

Solution: See Linear Algebra Done Right Solution Manual Chapter 5 Problem 17.

About Linearity

This website is supposed to help you study Linear Algebras. Please only read these solutions after thinking about the problems carefully. Do not just copy these solutions.

This entry was posted in Chapter 5 and tagged Exercise B.

← Chapter 5 Exercise A | Chapter 5 Exercise C →

ToeJam · 3 years ago

In problem 19, the vector space under consideration is L(V), so we can't use the result in problem 3.D.
The result pertains to operators on the vector space, so we could use it for an operator W belonging to L(L(V))- not the elements of the vector space itself.

Or, put another way, this is similar to saying that u belonging to a vector space U is such that uv = vu for all v belonging to U. Correct me if I'm wrong, but I don't think you can use result 3.D in this context.
Also, T is an arbitrary operator on L(V), so it certainly isn't necessarily a scalar multiple of identity operator- and the result Tp(T) = p(T)T holds for any T belonging to L(V).

Edit: I'd also like to say- thank you for this website! It's been of huge help while working through Sheldon Axler's book.

1 · Reply · Share ›

Mohammad Rashidi Mod → ToeJam · 2 years ago

I don't think you are correct. P(T) belongs to L(V), T also belongs to L(V).

Reply · Share ›

Ying Kit Hui · a year ago

For problem 19, I think in the following way. (I must admit that your solution is better.) For F=C, then by 5.27, there is a basis of V to which T has an upper triangular matrix. Denote this basis be v(1), ... , v(n). Note that n>1, since dimV>1. Thus span(v(1), ... , v(j)), is invariant under p(T) for any p. For any linear map S such that S maps v(1) to v(2), based on the fact that span(v(1)) is not invariant under S, we can conclude that S is not in the set containing all p(T). Thus the statement is proved.
For F=R, if T has eigenvalue, then proceeds with similar argument as above. However, if T has no eigenvalues, how can I proceed with this approach? Thx.

Reply · Share ›

Michael D. Nguyen → Ying Kit Hui · 3 months ago

If T has no eigenvalues then it is trivially that the two sets are not equal. Because L(V) have some linear maps that have eigenvalues and the p(T) eigenvalues must always has the form p(lambda) for lambda being T's eigenvalue.

Reply · Share ›

Masacroso · 2 years ago

For exercise 7: I think that you dont need to use 5.6, observe that if (T-L)v=0 for non-zero v then, regardless of the dimension of V, L is an eigenvalue of T. And is clear that T^2-9I=(T-3I)(T+3I)=(T+3I)(T-3I), hence or -3 or 3 is the eigenvalue of v, or -3 is the eigenvalue of (T-3I)v, or 3 is the eigenvalue of (T+3I)v.

And it seems that the exercise 11 is not correctly formulated. See here:

https://math.stackexchange....

for a discussion about the matter.

Reply · Share ›

Mohammad Rashidi Mod → Masacroso · 2 years ago

Yes, thank you.

Reply · Share ›

Marcel Ackermann · 2 years ago

In 5B3: "Since T²−I, it follows that" should be "Since T²−I=0, it follows that"

Reply · Share ›

Chapter 5 Exercise C

Posted on May 3, 2016 by Linearity

1. Solution: It is not said V is finite-dimensional, but I will do it by assuming $\dim V < \infty$.

If T is invertible, then $\text{null}T = 0$ and $\text{range}T = V$ since T is bijective and surjective. Hence $V = \text{null}T \oplus \text{range}T$.

If T is not invertible, let $0, \lambda_1, \cdots, \lambda_m$ be all eigenvalues of T, where $\lambda_i \neq 0$ for $i = 1, \cdots, m$. Then by 5.41(d), we have

$$V = E(0,T) \oplus E(\lambda_1,T) \oplus \cdots \oplus E(\lambda_m,T). \tag{1}$$

By definition, it follows $E(0,T) = \text{null}T$. Moreover, for any $v_i \in E(\lambda_i, T)$,

$$T\left(\frac{1}{\lambda_i}v_i\right) = \frac{1}{\lambda_i}Tv_i = v_i.$$

This implies $E(\lambda_i,T) \subset \text{range}T$. Therefore

$$E(\lambda_1,T) \oplus \cdots \oplus E(\lambda_m,T) \subset \text{range}T. \tag{2}$$

On the other hand, any $v \in V$ can be written as

$$v = v_0 + v_1 + \cdots + v_m,$$

where $v_0 \in E(0,T)$ and $v_i \in E(\lambda_i,T)$ for $i = 1, \cdots, m$. Hence

$$T(v) = T(v_0 + v_1 + \cdots + v_m) = \lambda_1 v_1 + \cdots + \lambda_m v_m \in E(\lambda_1,T) \oplus \cdots \oplus E(\lambda_m,T).$$

This implies

$$\text{range}T \subset E(\lambda_1,T) \oplus \cdots \oplus E(\lambda_m,T). \tag{3}$$

By (2) and (3), we have

$$E(\lambda_1,T) \oplus \cdots \oplus E(\lambda_m,T) = \text{range}T. \tag{4}$$

Combining (1) and (4), it follows that $V = \text{null}T \oplus \text{range}T$.

If we can show something like (1) for infinite-dimensional vector spaces, then we can deduce this problem for infinite-dimensional case by using similar arguments.

---

3. Solution: (a) $\Longrightarrow$ (b): It is obvious.

(b) $\Longrightarrow$ (c): By 3.22, we have

$$\dim V = \dim \text{null}T + \dim \text{range}T. \tag{5}$$

Note that $V = \text{null}T + \text{range}T$ and 2.43, we have

$$\dim V = \dim \text{null}T + \dim \text{range}T - \dim(\text{null}T \cap \text{range}T). \tag{6}$$

By (5) and (6), we have $\dim(\text{null}T \cap \text{range}T) = 0$. Hence $\text{null}T \cap \text{range}T = \{0\}$.

(c) $\Longrightarrow$ (a): Again by 2.43 and 3.22, we have

$$\dim(\text{null}T + \text{range}T) = \dim \text{null}T + \dim \text{range}T - \dim(\text{null}T \cap \text{range}T).$$

and

$$\dim V = \dim \text{null}T + \dim \text{range}T.$$

As $\dim(\text{null}T \cap \text{range}T) = 0$, it follows that

$$\dim(\text{null}T + \text{range}T) = \dim V.$$

Hence $\text{null}T + \text{range}T = V$, thus $\text{null}T \oplus \text{range}T = V$ since $\text{null}T \cap \text{range}T = \{0\}$.

5.Solution: If T is diagonalizable, so is $T - \lambda I$. Hence by problem 1, we have

$$V = \text{null}(T - \lambda I) \oplus \text{range}(T - \lambda I)$$

for every $\lambda \in \mathbb{C}$.

Lemma 1: Let U, W, S be subspaces of V, if $V = U \oplus W$ and $U \subset S$, then $S = U \oplus (W \cap S)$.

Proof of Lemma 1: For any $s \in S$, s can be written as $u + w$, where $u \in U$ and $w \in W$, since $V = U \oplus W$. As $U \subset S$, it follows that $u \in S$. Hence $w = s - u \in S$, namely $w \in s \cap W$. This implies $S = U + (W \cap S)$. Note that $U \cap W = \{0\}$, we have $S = U \oplus (W \cap S)$.

Lemma 2: For $a, b \in \mathbb{C}$ such that $a \neq b$, we have $\text{null}(T - aI) \subset \text{range}(T - bI)$.

Proof of Lemma 2: For any $v \in \text{null}(T - aI)$, we have $Tv = av$. Hence

$$(T - bI)\left(\frac{1}{a-b}v\right) = v,$$

namely $v \in \text{range}(T - bI)$. Thus $\text{null}(T - aI) \subset \text{range}(T - bI)$.

Proof of Problem: Conversely, since V is finite-dimensional, T has only finitely many eigenvalues. Suppose $\lambda_1, \cdots, \lambda_m$ are all distinct eigenvalues of T. Note that we have

$$V = \text{null}(T - \lambda_1 I) \oplus \text{range}(T - \lambda_1 I)$$

and $\text{null}(T - \lambda_2 I) \subset \text{range}(T - \lambda_1 I)$ (by Lemma 2), we have

$$\text{range}(T - \lambda_1 I) = \text{null}(T - \lambda_2 I) \oplus \text{range}(T - \lambda_1 I) \cap \text{range}(T - \lambda_2 I)$$

by Lemma 1. Similarly, we also have

$$\text{null}(T - \lambda_3 I) \subset \text{range}(T - \lambda_1 I) \cap \text{range}(T - \lambda_2 I).$$

By using Lemma 1 and Lemma 2 inductively, we have

$$\begin{aligned} V =& \text{null}(T - \lambda_1 I) \oplus \cdots \oplus \text{null}(T - \lambda_m I) \oplus \\ & (\text{range}(T - \lambda_1 I) \cap \cdots \cap \text{range}(T - \lambda_m I)). \end{aligned}$$

If $\text{range}(T - \lambda_1 I) \cap \cdots \cap \text{range}(T - \lambda_m I) = \{0\}$, we showed

$$V = \text{null}(T - \lambda_1 I) \oplus \cdots \oplus \text{null}(T - \lambda_m I).$$

Hence T is diagonalizable. If $\Gamma = \text{range}(T - \lambda_1 I) \cap \cdots \cap \text{range}(T - \lambda_m I) \neq \{0\}$, then note that $(T - \lambda_i I)T = T(T - \lambda_i I)$, we have $\text{range}(T - \lambda_i I)$ is invariant under T for all $i = 1, \cdots, m$ by Problem 3 of Exercises 5A. Hence Γ is invariant under T by Problem 5 of Exercises 5A. Consider $T|_\Gamma$, it has an eigenvalue $\lambda \in \mathbb{C}$ with a corresponding eigenvector μ by 5.21. Hence λ is also an eigenvalue of T. Suppose $\lambda = \lambda_i$ for some $i \in \{1, \cdots, m\}$. Then $\mu \in \text{null}(T - \lambda_i I)$, $\mu \in \Gamma$ and $\mu \neq 0$, this contradicts with

$$\begin{aligned} V =& \text{null}(T - \lambda_1 I) \oplus \cdots \oplus \text{null}(T - \lambda_m I) \oplus \\ & (\text{range}(T - \lambda_1 I) \cap \cdots \cap \text{range}(T - \lambda_m I)). \end{aligned}$$

Hence $\Gamma = 0$ and therefore T is diagonalizable.

---

6. Solution: Since $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues, then T is diagonalizable by 5.44. Let $v_1, \cdots, v_{\dim V}$ be the basis of V defined in the proof of 5.44, then $v_1, \cdots, v_{\dim V}$ are eigenvectors of T. As $S \in \mathcal{L}(V)$ has the same eigenvectors as T, $v_1, \cdots, v_{\dim V}$ are eigenvectors of S. Hence there exists $\lambda_1, \cdots, \lambda_{\dim V} \in \mathbb{F}$ and $\theta_1, \cdots, \theta_{\dim V} \in \mathbb{F}$ such that

$$Tv_i = \lambda_i v_i \text{ and } Sv_i = \theta_i v_i, \quad i = 1, \cdots, \dim V.$$

Hence we have

$$STv_i = S(\lambda_i v_i) = \lambda_i S v_i = \lambda_i \theta_i v_i, \quad i = 1, \cdots, \dim V$$

and

$$TSv_i = T(\theta_i v_i) = \theta_i T v_i = \theta_i \lambda_i v_i, \quad i = 1, \cdots, \dim V.$$

It follows that $STv_i = TSv_i$ for $i = 1, \cdots, \dim V$. Note that $v_1, \cdots, v_{\dim V}$ is a basis of V, we deduce that $ST = TS$.

---

8. Solution: Suppose $T - 2I$ and $T - 6I$ are not invertible, then 2 and 6 are eigenvalues of T. Note that λ is an eigenvalue of T if and only if $E(\lambda, T) \neq \{0\}$. Hence $\dim E(2,T) \geq 1$ and $\dim E(6,T) \geq 1$. By 5.38, we have

$$4 + 1 + 1 \leq \dim E(8,T) + \dim E(2,T) + \dim E(6,T) \leq \dim(\mathbb{F}^5) = 5.$$

This is impossible. Hence $T - 2I$ or $T - 6I$ is invertible.

---

9. Solution: For every $\lambda \in \mathbb{F}$ with $\lambda \neq 0$, let $v \in E(\lambda, T)$. Then we have $Tv = \lambda v$. Note that T is invertible and $\lambda \neq 0$, it follows that $\frac{1}{\lambda}v = T^{-1}v$. Hence $v \in E(1/\lambda, T^{-1})$, we conclude $E(\lambda,T) \subset E(1/\lambda, T^{-1})$. By symmetry, we also have $E(1/\lambda, T^{-1}) \subset E(\lambda, T)$. To sum up, we deduce $E(\lambda,T) = E(1/\lambda, T^{-1})$ for every $\lambda \in \mathbb{F}$ with $\lambda \neq 0$.

---

12. Solution: Note that R and T has three eigenvalues and $\dim(\mathbb{F}^3) = 3$. By 5.44, we have R and T are diagonalizable. Hence there exist bases e_1, e_2, e_3 and ξ_1, ξ_2, ξ_3 such that

$$Te_1 = 2e_1, Te_2 = 6e_2, Te_3 = 7e_3$$

and

$$R\xi_1 = 2\xi_1, R\xi_2 = 6\xi_2, R\xi_3 = 7\xi_3.$$

Define $S \in \mathcal{L}(\mathbb{F}^3)$ by

$$S\xi_i = e_i, \quad i = 1,2,3.$$

Then we have S is invertible and $S^{-1}e_i = \xi_i$. Moreover,

$$S^{-1}TS\xi_1 = S^{-1}Te_1 = S^{-1}(2e_1) = 2\xi_1 = R\xi_1.$$

Similarly $S^{-1}TS\xi_2 = R\xi_2$ and $S^{-1}TS\xi_3 = R\xi_3$. Hence $R = S^{-1}TS$ as they coincide in the basis ξ_1, ξ_2, ξ_3.

---

13. Solution: Let e_1, e_2, e_3, e_4 be a basis of $\mathbb{F}^4$, define $R, T \in \mathcal{L}(\mathbb{F}^4)$ by

$$Re_1 = 2e_1, Re_2 = 2e_2, Re_3 = 6e_3, Re_4 = 7e_4$$

and

$$Te_1 = 2e_1, Te_2 = 2e_2 + e_1, Te_3 = 6e_3, Te_4 = 7e_4.$$

Then R is diagonalizable. In fact T is not diagonalizable since $\dim E(2,T) = 1$, $\dim(6,T) = 1$ and $\dim(7,T) = 1$ imply

$$\dim(2,T) + \dim(6,T) + \dim(7,T) < \dim(\mathbb{F}^4).$$

If there exist an invertible operator $S \in \mathcal{L}(\mathbb{F}^4)$ such that $R = S^{-1}TS$, $\iff SRS^{-1} = T$, then Se_1, Se_2, Se_3, Se_4 is a basis of $\mathbb{F}^4$. Moreover,

$$T(Se_1) = SRS^{-1}(Se_1) = SRe_1 = S(2e_1) = 2Se_1.$$

Similarly,

$$T(Se_2) = 2Se_2, T(Se_3) = 6Se_3, T(Se_4) = 7Se_4.$$

This implies T is diagonalizable. Thus we get a contradiction. Hence there does not exist an invertible operator $S \in \mathcal{L}(\mathbb{F}^4)$ such that $R = S^{-1}TS$.

---

14. Solution: Let $T \in \mathcal{L}(\mathbb{C})$ defined by

$$Te_1 = 6e_1, Te_2 = 6e_2 + e_1, Te_3 = 7e_3,$$

where e_1, e_2, e_3 is a basis of $\mathbb{C}^3$. Then for any nonzero $\alpha \in \mathbb{C}^3$, write α by $k_1e_1 + k_2e_2 + k_3e_3$, if there exists $\lambda \in \mathbb{C}$ such that $T\alpha = \lambda\alpha$, then we have

$$\lambda(k_1e_1 + k_2e_2 + k_3e_3) = T\alpha = (6k_1 + k_2)e_1 + 6k_2e_2 + 7k_3e_3. \tag{7}$$

If $k_3 \neq 0$, then we have $\lambda k_3 = 7k_3$ by the previous equation. Hence $\lambda = 7$. If $k_3 = 0$, we have

$$(6 - \lambda)k_2 = 0 \text{ and } (6 - \lambda)k_1 = -k_2.$$

Note that $\alpha \neq 0$, it follows that k_1 or k_2 is not zero. If $k_2 \neq 0$, then $\lambda = 6$. If $k_1 \neq 0$, then

$$(6 - \lambda)^2 k_1 = (6 - \lambda)(-k_2) = -(6 - \lambda)k_2 = 0.$$

Thus $\lambda = 6$.

By above, all the eigenvalues of T are 6 and 7. Moreover, $\dim E(6,T) = 1$ and $\dim E(7,T) = 1$ by solving (7). That imples

$$2 = \dim E(6,T) + \dim E(7,T) < \dim \mathbb{C}^3 = 3.$$

Thus T is not diagonalizable.

---

15. Solution: Since T does not have a diagonal matrix with respect to any basis of $\mathbb{C}^3$, T is not diagonalizable. Hence 8 is not an eigenvalue of T, otherwise T has 3 eigenvalues hence diagonalizable by 5.44. This implies $T - 8I$ is surjective by 5.6. Hence there exists $(x,y,z) \in \mathbb{C}^3$ such that

$$(T - 8I)(x,y,z) = (17, \sqrt{5}, 2\pi),$$

namely

$$T(x,y,z) = (17 + 8x, \sqrt{5} + 8y, 2\pi + 8z).$$

---

16. Solution: (a) We show this part by induction. Note that

$$T(0,1) = (1,1) = (F_1, F_2).$$

Hence it is true for $n = 1$. Suppose we have $T^n(0,1) = (F_n, F_{n+1})$ then

$$\begin{aligned} T^{n+1}(0,1) =& T(T^n(0,1)) = T(F_n, F_{n+1}) \\ =& (F_{n+1}, F_n + F_{n+1}) = (F_{n+1}, F_{n+2}). \end{aligned}$$

Hence if it is true for n, so is the case for $n + 1$. Thus we get the conclusion by induction.

(b) and (c) We need solve this equation

$$T(x,y) = \lambda(x,y),$$

where $(x,y) \neq (0,0)$ and $\lambda \in \mathbb{R}$. By definition of T, it is equivalent to

$$\begin{cases} \lambda x = y, \\ \lambda y = x + y. \end{cases}$$

If $y = 0$, then $x = 0$ by the second equation. Hence $y \neq 0$, it follows that $x \neq 0$ and $\lambda \neq 0$. By the first equation, we have $x/y = 1/\lambda$. By the second one we have $\lambda = x/y + 1$. Hence we have

$$\lambda = \frac{1}{\lambda} + 1,$$

the solutions for this equation are

$$\lambda = \frac{1 \pm \sqrt{5}}{2}.$$

By $x/y = 1/\lambda$, we have the eigenvectors corresponding to $\frac{1 \pm \sqrt{5}}{2}$ are $\left(1, \frac{1 \pm \sqrt{5}}{2}\right)$ respectively.

(d) Denote $\left(1, \frac{1+\sqrt{5}}{2}\right)$ and $\left(1, \frac{1-\sqrt{5}}{2}\right)$ by e_1 and e_2 respectively. Then we have

$$(0,1) = \frac{1}{\sqrt{5}}(e_1 - e_2).$$

It follows that

$$\begin{aligned} T^n(0,1) =& T^n\left(\frac{1}{\sqrt{5}}(e_1 - e_2)\right) = \frac{1}{\sqrt{5}}(T^ne_1 - T^ne_2) \\ =& \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n e_1 - \left(\frac{1-\sqrt{5}}{2}\right)^n e_2\right]. \end{aligned}$$

By (a) and comparing the first component, we deduce that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]. \tag{8}$$

(e) Note that $\sqrt{5} \geqslant 2$, we have

$$\frac{1}{\sqrt{5}}\left|\frac{1-\sqrt{5}}{2}\right|^n = \frac{1}{\sqrt{5}}\left|\frac{2}{1+\sqrt{5}}\right|^n \leq \frac{1}{2} \times \frac{2}{3} < \frac{1}{2}. \tag{9}$$

Moreover, $F_n \in \mathbb{Z}$ is easily shown by induction. Combining (8) and (9), it follows that

$$\left|\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - F_n\right| = \frac{1}{\sqrt{5}}\left|\frac{1-\sqrt{5}}{2}\right|^n < \frac{1}{2}$$

By $F_n \in \mathbb{Z}$, we deduce that the Fibonacci number F_n is the integer that is closest to

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

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This entry was posted in Chapter 5 and tagged Exercise C.

← Chapter 5 Exercise B | Chapter 6 Exercise A →

Michael D. Nguyen · 3 months ago
For question number for, an example would be the differentiate operator D over vector space P(F).
null D = {1} and range D = P(F).
So (b) holds but (a) and (c) don't.
Reply · Share ›

Tim Nguyen · a year ago
For question 1, since T is diagonalizable, doesn't that imply V is finite-dimensional?
Or is there a way to represent an operator in infinite-dimension as a matrix? The definition of a matrix of an operator (5.22) assumes basis of V as v_1,...v_n. Thank you!
Reply · Share ›

Wu Jinyang · 2 years ago
Hi. I wonder if you can post the answer to question number 2.
the converse of the statement seems to hold.
Currently, I am stuck on trying to prove all invertible T can be diagonalized. (which is very unlikely, or the book should have mentioned it).

Silly me, T(x,y) = (x,x+y) is probably the easiest example,
null T = (0,0) with range T = V and the only eigenvalue is 1 with eigenvector (0,1)
Still, I wonder if you can give me some advice on solving this type of question. It always took me forever searching for a counter example.

By the way, for question 1. equation 4. it should be an equal sign instead of a subset sign.
Reply · Share ›

Michael D. Nguyen → Wu Jinyang · 3 months ago
Another example would be T(x,y)=(y,x). This is another invertible map and has no eigenvalues. And thus not diagonalizable.
Reply · Share ›

Mohammad Rashidi Mod → Wu Jinyang · 2 years ago
Yes, your example is a counterexample. A diagonalizable operator (on finite-dimensional space) whoes eigenvalues are all 1 must be identity operator. Not all invertible operators are diagonalizable.

It is hard to give advice on that. But usually, it might be a counterexample.

Thank you for correcting me.
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Batuhan Eroğlu · 2 years ago
The solution for the 14th problem has an error. You state that there are two possiblities, namely k_1 =! 0 and k_2 =! 0. However (6−λ)k2=0 and (6−λ)k1=−k2 forces k_2 to be zero eitherway.
So the only possibility is that k_1 =! 0 and k_2 =0
Reply · Share ›

Mohammad Rashidi Mod → Batuhan Eroğlu · 2 years ago
I used "or", not "and".
Reply · Share ›

Mickey Mike · 2 years ago
Mild error:
For 5.C.1 equation (4) should be an equals sign, not a subset sign
Reply · Share ›

Mohammad Rashidi Mod → Mickey Mike · 2 years ago
Thanks!
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← Chapter 5 Exercise C Chapter 6 Exercise B →

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Chapter 6 Exercise A

Posted on June 1, 2016 by Linearity

2. Solution: It does not satisfy definiteness. For the function takes $(0,1,0),(0,1,0)$ to 0, but $(0,1,0)\neq 0$.

---

4. Solution: (a) Note that V is a real inner product space, we have $\langle u,v\rangle=\langle v,u\rangle$. Hence

$$\begin{aligned}\langle u+v,u-v\rangle&=\langle u,u\rangle-\langle u,v\rangle+\langle v,u\rangle-\langle v,v\rangle\\&=\langle u,u\rangle-\langle v,v\rangle=\|u\|^2-\|v\|^2.\end{aligned}$$

(b) By (a).

(c) See the picture in Page 174 and note $\|u\|=\|v\|$ for a rhombus, then use (b).

---

5. Solution: Suppose V is finite-dimensional here (I am not sure whether it is true for infinite-dimensional case). Hence we just need to show $T-\sqrt{2}I$ is injective. Suppose $u\in\operatorname{null}(T-\sqrt{2}I)$, then

$$Tu=\sqrt{2}u\Longrightarrow\|Tu\|=\sqrt{2}\|u\|.$$

As $\|Tv\|\le\|v\|$ for every $v\in V$, it follows that $\|u\|=0$, hence $u=0$. That implies $T-\sqrt{2}I$ is injective.

---

6. Solution: If $\langle u,v\rangle=0$, then

$$\|u+av\|^2=\|u\|^2+\|av\|^2\ge\|u\|^2$$

by 6.13.

If $\|u\|\le\|u+av\|$ for all $a\in\mathbb{F}$, this implies

$$\|u+av\|^2-\|u\|=|a|^2\|v\|+a\langle v,u\rangle+\bar{a}\langle u,v\rangle\ge 0.$$

If $v=0$, then $\langle u,v\rangle=0$. If $v\neq0$, plug $a=-\langle u,v\rangle/\|v\|^2$ into the previous equation, we obtain

$$-\frac{|\langle u,v\rangle|^2}{\|v\|^2}\ge 0.$$

Hence $\langle u,v\rangle=0$.

---

7. Solution: If $\|av+bu\|=\|au+bv\|$ for all $a,b\in\mathbb{R}$, by setting $a=1$ and $b=0$, we have $\|u\|=\|v\|$.

Conversely, suppose $\|u\|=\|v\|$. For all $a,b\in\mathbb{R}$, we have

$$\begin{aligned}\|av+bu\|^2&=\langle av+bu,av+bu\rangle\\&=a^2\|u\|^2+ab(\langle u,v\rangle+\langle v,u\rangle)+b^2\|v\|^2\end{aligned}$$

and

$$\begin{aligned}\|au+bv\|^2&=\langle au+bv,au+bv\rangle\\&=a^2\|v\|^2+ab(\langle u,v\rangle+\langle v,u\rangle)+b^2\|u\|^2.\end{aligned}$$

Hence if $\|v\|=\|u\|$, we have

$$a^2\|u\|^2+b^2\|v\|^2=a^2\|v\|^2+b^2\|u\|^2.$$

Therefore $\|av+bu\|^2=\|au+bv\|^2$, i.e. $\|av+bu\|=\|au+bv\|$.

---

8. Solution: Consider $\|u-v\|^2$, we have

$$\begin{aligned}\|u-v\|^2=&\langle u-v,u-v\rangle=\langle u,u\rangle-\langle u,v\rangle-\langle v,u\rangle+\langle v,v\rangle\\=&\|u\|^2-\langle u,v\rangle-\overline{\langle u,v\rangle}+\|v\|^2=0,\end{aligned}$$

hence $u-v=0$ by definiteness. That is $u=v$.

---

9. Solution: By 6.15, we have $|\langle u,v\rangle|\leqslant\|u\|\|v\|$. Since $\|u\|\leqslant$ and $\|v\|\leqslant$, we also have

$$0\leqslant1-\|u\|\|v\|\leqslant1-|\langle u,v\rangle|.$$

To show $\sqrt{1-\|u\|^2}\sqrt{1-\|v\|^2}\leqslant1-|\langle u,v\rangle|$, it suffices to show that

$$\sqrt{1-\|u\|^2}\sqrt{1-\|v\|^2}\leqslant1-\|u\|\|v\|.$$

Since $0\leqslant1-\|u\|\|v\|$, by squaring both sides, we only need to show

$$(1-\|u\|^2)(1-\|v\|^2)\leqslant(1-\|u\|\|v\|)^2,$$

which amounts to show

$$(\|u\|-\|v\|)^2\geqslant0.$$

This completes the proof.

---

10. Solution: Let $v=(x,y)$ and $u=z(1,3)$, where $x,y,z\in\mathbb{R}$. Note that v is orthogonal to $(1,3)$, we have

$$(x,y)\cdot(1,3)=x+3y=0.$$

It follows that $v=x(-3,1)$. Since $(1,2)=u+v$, we obtain

$$x(-3,1)+z(1,3)=(z-3x,x+3z)=(1,2).$$

We can solve this equation and get $x=-1/10$ and $z=7/10$. Hence $u=(7/10,21/10)$ and $v=(3/10,-1/10)$.

---

11. Solution: Consider Example 6.17 (a), we have

$$\begin{aligned}&(a+b+c+d)\left(\frac1a+\frac1b+\frac1c+\frac1d\right)\\\geqslant&\left(\sqrt{a\times\frac1a}+\sqrt{b\times\frac1b}+\sqrt{c\times\frac1c}+\sqrt{d\times\frac1d}\right)^2\\=&4^2=16.\end{aligned}$$

---

12. Solution: In Example 6.17 a), let $y_i=1$.

---

15. Solution: Consider Example 6.17 (a). Let $x_j=\sqrt{j|a_j|}$ and $y_j=\sqrt{\frac{|b_j|}{j}}$ and note that

$$|a_1b_1+\cdots+a_nb_n|\leqslant\sum_{j=1}^n|a_jb_j|.$$

---

16. Solution: Note that (.....After I finished this, I found that is exactly 6.22...)

$$\begin{aligned}&\|u+v\|^2+\|u-v\|^2\\=&\langle u+v,u+v\rangle+\langle u-v,u-v\rangle\\=&\langle u,u\rangle+\langle u,v\rangle+\langle v,u\rangle+\langle v,v\rangle+\langle u,u\rangle-\langle u,v\rangle-\langle v,u\rangle+\langle v,v\rangle\\=&2\langle u,u\rangle+2\langle v,v\rangle=2\|u\|^2+2\|v\|^2,\end{aligned}$$

it follows that

$$2\times3^2+2\|v\|^2=4^2+6^2.$$

Hence $\|v\|=\sqrt{17}$.

---

17. Solution: By 6.22, if there is such an inner product on $\mathbb{R}^2$, then we must have

$$\|\alpha-\beta\|^2+\|\alpha+\beta\|^2=2(\|\alpha\|^2+\|\beta\|^2).$$

Let $\alpha=(1,0)$ and $\beta=(0,1)$, we will get a counterexample.

---

19. Solution: See it here Exercise 1 or See Linear Algebra Done Right Solution Manual Chapter 6 Problem 6.

---

20. Solution: See it here Exercise 1 or See Linear Algebra Done Right Solution Manual Chapter 6 Problem 7.

---

21. Solution: See Linear Algebra Done Right Solution Manual Chapter 6 Problem 8.

---

22. Solution: It follows directly from Problem 12.

---

24. Solution: Positivity: $\langle u,u\rangle_1=\langle Su,Su\rangle\ge0$ for all $u\in V$.

Definiteness: $0=\langle u,u\rangle_1=\langle Su,Su\rangle$, hence $Su=0$. As S is injective, it follows that $u=0$.

Additivity in first slot:

$$\begin{aligned}\langle u+v,w\rangle_1=&\langle S(u+v),Sw\rangle=\langle Su+Sv,Sw\rangle\\=&\langle Su,Sw\rangle+\langle Sv,Sw\rangle=\langle u,w\rangle_1+\langle v,w\rangle_1.\end{aligned}$$

Homogeneity in first slot:

$$\begin{aligned}\langle\lambda u,w\rangle_1=&\langle S(\lambda u),Sw\rangle=\langle\lambda Su,Sw\rangle\\=&\lambda\langle Su,Sw\rangle=\lambda\langle u,w\rangle_1.\end{aligned}$$

Conjugate symmetry: $\langle u,v\rangle_1=\langle Su,Sv\rangle=\overline{\langle Sv,Su\rangle}=\overline{\langle v,u\rangle_1}$.

---

25. Solution: Note that $S\in\mathcal{L}(V)$ is not injective, there exists a nonzero $u\in V$ such that $Su=0$. Now we have $\langle u,u\rangle_1=\langle Su,Su\rangle=0$, this implies that $\langle u,v\rangle_1$ do not satisfy definiteness.

---

27. Solution: Let $a=(w-u)/2$ and $b=(w-v)/2$, by 6.22, we have

$$\|a-b\|^2+\|a+b\|^2=2\|a\|^2+2\|b\|^2.$$

Plug $a=(w-u)/2$ and $b=(w-v)/2$ into the expression above, we get

$$\left\|w-\frac12(u+v)\right\|^2=\frac{\|w-u\|^2+\|w-v\|^2}{2}-\frac{\|u-v\|^2}{4}.$$

---

28. Solution: Suppose there are two such vectors in C. Denote them by ξ and μ ($\xi\neq\mu$), then we have

$$\|w-\xi\|\le\|w-\mu\|\text{ and }\|w-\mu\|\le\|w-\xi\|$$

by the choice of ξ and μ. Hence $\|w-\xi\|=\|w-\mu\|$. By the previous exercise, we have

$$\left\|w-\frac12(\xi+\mu)\right\|^2=\frac{\|w-\xi\|^2+\|w-\mu\|^2}{2}-\frac{\|\xi-\mu\|^2}{4}<\|w-\xi\|^2.$$

This contradicts with the choice of ξ. Hence there is at most one $u\in C$ such that

$$\|w-u\|\le\|w-v\|\quad\text{for all }v\in C.$$

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This entry was posted in Chapter 6 and tagged Exercise A.

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Mathily · 2 years ago
I posted a solution to #3 at my blog: https://todayiproved.blogsp...
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Marcel Ackermann · 2 years ago
Solution to 6A31:
The triangle can be extended to a parallelogram (rotation mirror at the intersection of d and c). Thus we can apply the parallelogram equality: $2 (\Vert \vec{a} \Vert^2 + \Vert \vec{b} \Vert^2) = \Vert 2\vec{d} \Vert^2 + \Vert \vec{c} \Vert^2 \Leftrightarrow \Vert \vec{a} \Vert^2 + \Vert \vec{b} \Vert^2 = \frac{1}{2} \Vert \vec{c} \Vert^2 + \Vert \vec{d} \Vert^2$
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Marcel Ackermann · 2 years ago
Does anybody have a proof for 6A5 for the infinite dimensional case? (eg. showing surjectivity)
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Brian Lubeck → Marcel Ackermann · 3 months ago
Per the errata: Page 175, Exercise 5: Assume that V is finite-dimensional.
http://linear.axler.net/LAD...
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Dalton Burke · 2 years ago
For problem 27, the given substitution does not work. Try a = w/2 - u/2 and b = w/2 - v/2 instead.
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Mohammad Rashidi Mod → Dalton Burke · 2 years ago
Yes, you are correct. Thank you.
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Chapter 6 Exercise B

Posted on June 2, 2016 by Linearity

1. Solution:

(a) One can easily check that each of the four vectors has norm $\sin^2\theta+\cos^2\theta$, which equals 1. Moreover, we have

$$\begin{aligned}\langle(\cos\theta,\sin\theta),(-\sin\theta,\cos\theta)\rangle&=-\cos\theta\sin\theta+\sin\theta\cos\theta=0\\ \langle(\cos\theta,\sin\theta),(\sin\theta,-\cos\theta)\rangle&=\cos\theta\sin\theta-\sin\theta\cos\theta=0,\end{aligned}$$

which shows that they are orthogonal.

(b) Clearly, for any v and u in $\mathbb{R}^2$ with $||v||=||u||=1$, we can write $v=(\cos\theta,\sin\theta)$ and $v=(\cos\alpha,\sin\alpha)$ for some angles θ and α. If v,u is an orthonormal basis, then we must have

$$0=\langle v,u\rangle=\langle(\cos\theta,\sin\theta),(\cos\alpha,\sin\alpha)\rangle=\cos\theta\cos\alpha+\sin\theta\sin\alpha=\cos(\theta-\alpha).$$

One solution is to take choose θ and α such that $\theta-\alpha=\frac{\pi}{2}$. Then

$$\begin{aligned}(\cos\alpha,\sin\alpha)&=(\cos(\theta+\frac{\pi}{2}),\sin(\theta+\frac{\pi}{2}))\\&=(\cos\theta\cos\frac{\pi}{2}-\sin\theta\sin\frac{\pi}{2},\sin\theta\cos\frac{\pi}{2}+\sin\frac{\pi}{2}\cos\theta)\\&=(-\sin\theta,\cos\theta).\end{aligned}$$

Which shows that v,u is of the first form given in part (a).

---

2. Solution: If $v\in\operatorname{span}(e_1,\cdots,e_m)$, then $e_1,\cdots,e_m$ is an orthonormal basis of $\operatorname{span}(e_1,\cdots,e_m)$ by 6.26. By 6.30, it follows that

$$\|v\|^2=|\langle v,e_1\rangle|^2+\cdots+|\langle v,e_m\rangle|^2.$$

If $\|v\|^2=|\langle v,e_1\rangle|^2+\cdots+|\langle v,e_m\rangle|^2$, we denote

$$\xi=v-(\langle v,e_1\rangle e_1+\cdots+\langle v,e_m\rangle e_m).$$

It is easily seen that

$$\langle\xi,e_i\rangle=\langle v,e_i\rangle-\langle v,e_i\rangle=0$$

for $i=1,\cdots,m$. This implies

$$\langle\xi,e_1\rangle e_1+\cdots+\langle v,e_m\rangle e_m\rangle=0.$$

By 6.13, we have

$$\begin{aligned}\|v\|^2&=\|\xi\|^2+\|\langle v,e_1\rangle e_1+\cdots+\langle v,e_m\rangle e_m\|^2\\&=\|\xi\|^2+|\langle v,e_1\rangle|^2+\cdots+|\langle v,e_m\rangle|^2.\end{aligned}$$

It follows that $\|\xi\|^2=0$, hence $\xi=0$. Thus $v=\langle v,e_1\rangle e_1+\cdots+\langle v,e_m\rangle e_m$, namely $v\in\operatorname{span}(e_1,\cdots,e_m)$.

---

3. Solution: Applying the Gram-Schmidt Procedure to the given basis, we get the following basis

$$(1,0,0),\ \frac{1}{\sqrt{2}}(0,1,1),\ \frac{1}{\sqrt{2}}(0,-1,1).$$

As in the proof of 6.37, we see that the matrix of T with respect to this basis is upper triangular.

---

4. Solution: See Linear Algebra Done Right Solution Manual Chapter 6 Problem 9.

---

5. Solution: Applying the Gram-Schmidt Procedure, we get the following basis

$$1,\ 2\sqrt{3}(x-\frac{1}{2}),\ 6\sqrt{5}(x^2-x+\frac{1}{6}).$$

---

6. Solution: Let D denote the differential operator. Note that D is already upper-triangular with respect to the standard basis of $\mathcal{P}_2\mathbb{R}$. Therefore, by the same reasoning used in the proof of 6.37, $\mathcal{M}(D)$ is upper-triangular with respect to the basis found in Exercise 5.

---

7. Solution:Defining $\varphi(p)=p(\frac{1}{2})$ and $\langle p,q\rangle=\int_0^1p(x)q(x)\,dx$ and using the formula from 6.43 together with the basis found in Exercise 5, we find that

$$q(x)=-15x^2+15x-\frac{3}{2}.$$

---

8. Solution: Using the orthonormal basis found in Exercise 5 and the formula in 6.43, we get $q(x)=\frac{-24}{\pi^2}\left(x-\frac{1}{2}\right)$.

---

9. Solution: Suppose $v_1,\dots,v_m$ is a lienarly dependent list in V. Let k be the smallest integer such that $v_k\in\operatorname{span}(v_1,\dots,v_{k-1})$. Then $v_1,\dots,v_{k-1}$ is linearly independent and we can apply the Gram-Schmidt Procedure to produce an orthonormal list $e_1,\dots,e_{k-1}$ whose span is the same. Therefore $v_k\in\operatorname{span}(e_1,\dots,e_{k-1})$ and, by 6.30,

$$v_k=\langle v_k,e_1\rangle e_1+\cdots+\langle v_k,e_{k-1}\rangle e_{k-1}.$$

But the right hand side is exactly what we subtract from v_k when calculating e_k, hence the Gram-Schmidt Procedure cannot continue because we can't divide by 0. If, however, you discard v_k (and every other vector to which happens the same thing), you end up producing an orthonormal basis whose span equals $\operatorname{span}(v_1,\dots,v_m)$.

---

10. Solution: Just apply the Gram-Schmidt Procedure once on $v_1,\dots,v_m$ to get the orthonormal an $e_1,\dots,e_m$. Note that, if we switch the order of this list without relabeling the vectors, we still have $\operatorname{span}(v_1,\dots,v_j)=\operatorname{span}(e_1,\dots,e_j)$ for all $j\in\{1,\dots,m\}$, because the vectors themselves remain the same. There 2^m such permutations.

---

11. Solution: Let $w\in V$. Define $\varphi(v)=\langle v,w\rangle_1$ and $\psi(v)=\langle v,w\rangle_2$. Since $\varphi(v)=0$ if and only if $\psi(v)=0$, it follows that $\operatorname{null}\varphi=\operatorname{null}\psi$. By Theorem 1 in Chapter 3 notes we have

$$\operatorname{span}(\varphi)=(\operatorname{null}\varphi)^0=(\operatorname{null}\psi)^0=\operatorname{span}(\psi).$$

Thus $\varphi=c\psi$ for some $c\in\mathbb{F}$. Hence, for each fixed w we have $\langle v,w\rangle_1=c\langle v,w\rangle_2$ for every $v\in V$. Chosing $v=w$ now implies that c is real and positive. Fix $w_1,w_2\in V$ and let $c_1,c_2\in\mathbb{F}$ such that

$$\begin{aligned}\langle v,w_1\rangle_1&=c_1\langle v,w_1\rangle_2\\ \langle v,w_2\rangle_1&=c_2\langle v,w_2\rangle_2.\end{aligned}$$

Pluging $v=w_2$ in the first equation and $v=w_1$ in the second yields

$$\begin{aligned}\langle w_2,w_1\rangle_1&=c_1\langle w_2,w_1\rangle_2\\ \langle w_1,w_2\rangle_1&=c_2\langle w_1,w_2\rangle_2.\end{aligned}$$

Then

$$c_1\langle w_2,w_1\rangle_2=\langle w_2,w_1\rangle_1=\overline{\langle w_1,w_2\rangle_1}=\overline{c_2\langle w_1,w_2\rangle_2}=\bar{c_2}\langle w_2,w_1\rangle_2.$$

Hence $c_1=\bar{c_2}$. Because both are real, it follows that $c_1=c_2$. Therefore, the constant is the same for all $v,w\in V$.

---

14. Solution: Since $e_1,\cdots,e_n$ is an orthonormal basis of V, we have $\dim V=n$. To show that $v_1,\cdots,v_n$ is a basis of V, it suffices to show that $v_1,\cdots,v_n$ is linearly independent. We prove it by contradiction. Suppose $v_1,\cdots,v_n$ is linearly dependent, then there exist $a_1,\cdots,a_n\in\mathbb{F}$ such that $a_k\neq 0$ for some $k\in\{1,\cdots,n\}$ and

$$\sum_{i=1}^n a_iv_i=0.$$

On one hand, by 6.25, we have

$$\Big\|\sum_{i=1}^n a_i(e_i-v_i)\Big\|^2=\Big\|\sum_{i=1}^n a_ie_i\Big\|^2=\sum_{i=1}^n|a_i|^2.$$

On the other hand, we also have

$$\begin{aligned}\Big\|\sum_{i=1}^n a_i(e_i-v_i)\Big\|^2&=\Big\langle\sum_{i=1}^n a_i(e_i-v_i),\sum_{j=1}^n a_j(e_j-v_j)\Big\rangle\\&=\sum_{i=1}^n\sum_{j=1}^n\Big\langle a_i(e_i-v_i),a_j(e_j-v_j)\Big\rangle\\&\leqslant\Big|\sum_{i=1}^n\sum_{j=1}^n\Big\langle a_i(e_i-v_i),a_j(e_j-v_j)\Big\rangle\Big|\\&\leqslant\sum_{i=1}^n\sum_{j=1}^n\Big|\Big\langle a_i(e_i-v_i),a_j(e_j-v_j)\Big\rangle\Big|\\ \text{by 6.15}\quad&\leqslant\sum_{i=1}^n\sum_{j=1}^n\|a_i(e_i-v_i)\|\|a_j(e_j-v_j)\|\\&=\sum_{i=1}^n\sum_{j=1}^n|a_i||a_j|\|e_i-v_i\|\|e_j-v_j\|\\ \text{by assumption and } a_k\neq 0\quad&<\sum_{i=1}^n\sum_{j=1}^n\frac{1}{n}|a_i||a_j|=\frac{1}{n}\Big(\sum_{i=1}^n|a_i|\Big)^2\\ \text{by Problem 6.A.12}\quad&\leqslant\sum_{i=1}^n|a_i|^2.\end{aligned}$$

Hence we get

$$\sum_{i=1}^n|a_i|^2<\sum_{i=1}^n|a_i|^2,$$

which is impossible, hence completing the proof.

---

15. Solution: Suppose there exists g such that $\varphi(f)=\langle f,g\rangle$ for all $f\in C_{\mathbb{R}}[-1,1]$. We would like to show a contradiction.
For any positive integer n and integer $-n\leqslant i\leqslant n-1$, define

$$f_{n,i}(x)=\begin{cases}4n^2(x-i/n), & \text{if } x\in[i/n,i/n+1/(2n)]\\ 4n^2((i+1)/n-x), & \text{if } x\in[i/n+1/(2n),(i+1)/n]\\ 0, & \text{otherwise },\end{cases}$$

then $f_{n,i}(x)\in C_{\mathbb{R}}[-1,1]$ and $f_{n,i}(0)=0$.
Given any $\epsilon>0$, since $g\in C_{\mathbb{R}}[-1,1]$, by the fact that a continuous function on a closed interval is uniformally continuous, there exists N such that for any $n\geqslant N$, we have

$$|g(x)-g(y)|\leqslant\epsilon \tag{1}$$

if $|x-y|\leqslant 1/n$.
Note that

$$\int_{-1}^1 f_{n,i}(x)dx=\int_{i/n}^{(i+1)/n}f_{n,i}(x)dx=1, \tag{2}$$

for any $y\in[i/n,(i+1)/n]$ we have

$$\begin{aligned}&\Big|g(y)-\int_{-1}^1 f_{n,i}(x)g(x)dx\Big|\\&=\Big|\int_{i/n}^{(i+1)/n}f_{n,i}(x)\,(g(y)-g(x))\,dx\Big|\\&\leqslant\int_{i/n}^{(i+1)/n}f_{n,i}(x)\,|g(y)-g(x)|\,dx\\ \text{by (1) and (2)}\quad&\leqslant\int_{i/n}^{(i+1)/n}f_{n,i}(x)\epsilon dx=\epsilon.\end{aligned}$$

On the other hand, we also have

$$0=f_{n,i}(0)=\varphi(f_{n,i})=\langle f_{n,i},g\rangle=\int_{-1}^1 f_{n,i}(x)g(x)dx.$$

Hence we have

$$|g(y)|=|g(y)-f_{n,i}(0)|\leqslant\epsilon$$

for any $y\in[i/n,(i+1)/n]$. Thus $|g(x)|\leqslant\epsilon$ by taking all $-n\leqslant i\leqslant n-1$ with $n\geqslant N$.
Since ϵ is chosen arbitrarily, we have $g(x)\equiv 0$. Hence $\varphi f\equiv 0$ for all $f\in C_{\mathbb{R}}[-1,1]$, which is impossible. Therefore the proof is complete.

---

17. Solution:

(a) For additivity, suppose $u_1,u_2\in V$. Then, for $v\in V$, we have

$$(\Phi(u_1+u_2))(v)=\langle v,u_1+u_2\rangle=\langle v,u_1\rangle+\langle v,u_2\rangle=(\Phi u_1)(v)+(\Phi u_2)(v).$$

For homogeneity, suppose $u\in V$ and $c\in\mathbb{R}$. Then, for $v\in V$, we have

$$(\Phi(cu))(v)=\langle v,cu\rangle=c\langle v,u\rangle=c(\Phi u)(v).$$

(b) If $\mathbb{F}=\mathbb{C}$, then the homogeneity property of linear maps is not satisfied, because we would have $(\Phi(cu))(v)=\bar{c}(\Phi u)(v)$, but $c=\bar{c}$ if and only if c is a real number.

(c) This is the same as the second part in the proof of 6.42. Suppose there are u_1 and u_2 in V such that $\Psi u_1=\Psi u_2$. Then

$$0=(\Psi u_1-\Psi u_2)(v)=(\Psi(u_1-u_2))(v)=\langle v,u_1-u_2\rangle$$

for all $v\in V$. Choosing $v=u_1-u_2$ shows that $u_1-u_2=0$ and thus $u_1=u_2$.

(d) From (c), we get that $\dim\operatorname{null}\Phi=0$. Thus, from 3.22, we have

$$\dim V=\dim\operatorname{null}\Phi+\dim\operatorname{range}\Phi=\dim\operatorname{range}\Phi.$$

However, $\dim V=\dim V'$. This shows that Φ also surjective. Hence Φ is invertible, that is, an isomorphism from V to V'.

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Chapter 6 Exercise C

Posted on June 3, 2016 by Linearity

1. Solution: Suppose $w \in \{v_1, \dots, v_m\}^\perp$. Let $v =\in \operatorname{span}(v_1, \dots, v_m)$. We have that

$$v = a_1 v_1 + \dots a_m v_m$$

for some $a_1, \dots, a_m \in \mathbb{F}$. Moreover

$$\langle v, w\rangle = \langle a_1 v_1 + \dots a_m v_m, w\rangle = a_1\langle v_1, w\rangle + \cdots + a_m\langle v_m, w\rangle = 0.$$

Thus $w \in (\operatorname{span}(v_1, \dots, v_m))^\perp$ and so $\{v_1, \dots, v_m\}^\perp \subset (\operatorname{span}(v_1, \dots, v_m))^\perp$.
Now suppose $w \in (\operatorname{span}(v_1, \dots, v_m))^\perp$. Since each v_j is in $\operatorname{span}(v_1, \dots, v_m)$, it follows that w is orthogonal to each v_j. Therefore $w \in \{v_1, \dots, v_m\}^\perp$ and thus $(\operatorname{span}(v_1, \dots, v_m))^\perp \subset \{v_1, \dots, v_m\}^\perp$.

---

2. Solution: Suppose $U^\perp = \{0\}$. Since $V = U \oplus U^\perp$ (by 6.47), it follows that $U = V$.

Conversely, suppose $U = V$. Then $U^\perp = V^\perp = \{0\}$.

---

3. Solution: By 6.31, we have

$$\operatorname{span}(e_1, \cdots e_m) = \operatorname{span}(u_1, \cdots, u_m) = U.$$

Note that $e_1, \cdots e_m$ is an orthonormal list, it follows that $e_1, \cdots e_m$, is an orthonormal basis of U. By 6.47, we have $V = U \oplus U^\perp$. As $e_1, \cdots e_m, f_1, \cdots, f_n$ is an orthonormal list, it follows that

$$\operatorname{span}(f_1, \cdots, f_n) \subset U^\perp$$

by definition. Oh the other hand, $V = U \oplus U^\perp$ implies

$$\dim V = \dim U + \dim U^\perp \Longrightarrow m + n = m + \dim U^\perp.$$

This means $\dim U^\perp = n$. Note that

$$\dim \operatorname{span}(f_1, \cdots, f_n) = n$$

and

$$\operatorname{span}(f_1, \cdots, f_n) \subset U^\perp,$$

we conclude

$$\operatorname{span}(f_1, \cdots, f_n) = U^\perp.$$

Thus $f_1, \cdots, f_n$ is an orthonormal basis of $U^\perp$.

---

5. Solution: By 6.47, we have $V = U \oplus U^\perp$. For any $v \in V$, v can be written as $v = u + w$, where $u \in U$ and $w \in U^\perp$. By the definition of orthogonal projection, we have $P_U(v) = u$. By 6.51, we have $(U^\perp)^\perp = U$, hence $P_{U^\perp}(v) = w$ as $w \in U^\perp$ and $v \in U = (U^\perp)^\perp$. Hence

$$P_{U^\perp}(v) = w = (u + w) - u = I(v) - P_U(v) = (I - P_U)(v).$$

As v is chosen arbitrarily, it follows that $P_{U^\perp} = I - P_U$.

---

6. Solution: Suppose $P_U P_W = 0$. For any $w \in W$, we have $P_U P_W w = P_U w = 0$. It follows that $w \in U^\perp$ by Proposition 6.55 (c). Therefore $\langle w, u\rangle$ for all $u \in U$. Note that $w \in W$ is chosen arbitrarily, one has $\langle u, w\rangle = 0$ for all $u \in U$ and all $w \in W$.

Conversely, if $\langle u, w\rangle = 0$ for all $u \in U$ and all $w \in W$, then $W \subset U^\perp$. Now again by Proposition 6.55 (c), $P_U W = 0$, therefore $P_U P_W = 0$ since for all $x \in V$ we have $P_W x \in W$.

---

7. Solution: Define $U = \operatorname{range} P$. Suppose $u \in U$. There exists $v \in V$ such that $Pv = u$. Then

$$u = Pv = P^2 v = P(Pv) = Pu.$$

We've shown that $Pu = u$ for any $u \in U$. Let $v \in V$. By Exercise 4 in section 5B, we can write $v = u + w$ for some $u \in U$ and $w \in \operatorname{null} P$. Note that $\operatorname{null} P \subset U^\perp$. Thus

$$Pv = P(u + w) = Pu = u = P_U(u + w) = P_U v.$$

Therefore $P = P_U$.

---

8. Solution: By Exercise 5B.4, we have

$$V = \operatorname{null}P \oplus \operatorname{range}P.$$

Hence for any $u \in \operatorname{null}P$ and $w \in \operatorname{range}P$. By assumption, we have

$$\|w\| = \|P(\lambda u + w)\| \le \|\lambda u + w\|$$

for any $\lambda \in \mathbb{F}$. By Exercise 6A.6, it follows that $\langle u, w\rangle = 0$. As u, w are chosen arbitrarily, we deduce that $\operatorname{null}P \perp \operatorname{range}P$. Now we can choose $U = \operatorname{null}P$ and it is obvious that $P = P_U$ now.

Solution: See Linear Algebra Done Right Solution Manual Chapter 6 Problem 18.

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9. Solution: If U is invariant under T. Then for any $v \in V$, we can express it as $v = u + w$, where $u \in U$ and $w \in U^\perp$. Since U is invariant under T, it follows that $Tu \in U$. By definition of orthogonal projection, we have

$$P_U T P_U(v) = P_U T(u) = T(u)$$

and $TP_U(v) = Tu$. Hence $P_U T P_U(v) = TP_U(v)$ for any $v \in V$, i.e. $P_U T P_U = TP_U$.
Suppose $P_U T P_U = TP_U$, then for any $u \in U$, we have $P_U(u) = u$. Hence

$$P_U T P_U(u) = P_U(Tu)$$

and $TP_U(u) = Tu$. That implies $P_U(Tu) = Tu$, hence $Tu \in U$ by 6.55 (d). Therefore U is invariant under T.
Solution: See Linear Algebra Done Right Solution Manual Chapter 6 Problem 19.

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10. Solution: See Linear Algebra Done Right Solution Manual Chapter 6 Problem 20.

---

11. Solution: See Linear Algebra Done Right Solution Manual Chapter 6 Problem 21.

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12. Solution: See Linear Algebra Done Right Solution Manual Chapter 6 Problem 22.

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13. Solution: See Linear Algebra Done Right Solution Manual Chapter 6 Problem 23.

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14. Solution: (a) Let $\varphi \in U^\perp$, by continuity it suffices to show that $\varphi(x) = 0$ for all $x \in (-1, 0) \cup (0, 1)$.
Suppose $\varphi(x_0) = \xi \neq 0$ for some $x_0 \in (-1, 0) \cup (0, 1)$. Then there exist a $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset (-1, 0) \cup (0, 1)$ and $\varphi(x) \ge \xi/2$ for all $x \in [x_0 - \delta, x_0 + \delta]$. Define $f \in C_{\mathbb{R}}[-1, 1]$ by

$$f(x) = \begin{cases} 0, & x \in [-1, x_0 - \delta] \cup [x_0 + \delta, 1]; \\ (x - x_0 + \delta)/\delta, & x \in [x_0 - \delta, x_0]; \\ (-x + x_0 + \delta)/\delta, & x \in [x_0, x_0 + \delta]. \end{cases}$$

Then $f \in U$ and $f(x) \ge 1/2$ for $x \in [x_0 - \delta/2, x_0 + \delta/2]$. Hence

$$\langle f, \varphi\rangle = \int_{-1}^{1} f(x)\varphi(x)dx \ge \int_{x_0-\delta/2}^{x_0+\delta/2} f(x)\varphi(x)dx \ge \delta\xi/4 > 0.$$

Hence we get a contradiction, thus completing the proof.

(b) Use U as a counterexample.

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This entry was posted in Chapter 6 and tagged Exercise C.

← Chapter 6 Exercise B   Chapter 7 Exercise A →

Chapter 7 Exercise A

Posted on July 1, 2016 by Linearity

1. Solution: By definition, we have

$$\begin{aligned}\langle (z_1,\dots,z_n), T^*(w_1,\dots,w_n)\rangle &= \langle T(z_1,\dots,z_n),(w_1,\dots,w_n)\rangle\\ &= z_2\overline{w_1}+\cdots+z_n\overline{w_{n-1}} = \langle (z_1,\dots,z_n),(w_2,\dots,w_n,0)\rangle.\end{aligned}$$

Therefore $T^*(w_1,\dots,w_n)=(w_2,\dots,w_n,0)$. Or $T^*(z_1,\dots,z_n)=(z_2,\dots,z_n,0)$.
See also Linear Algebra Done Right Solution Manual Chapter 6 Problem 27.

2. Solution: (This solution works for $\dim V<\infty$, I am not sure whether V is finite dimensional or not)Note that $(T^*)^*=T$, it suffices to show λ is an eigenvalue of T then $\bar\lambda$ is an eigenvalue of T^*. Let v be a eigenvectors of T corresponding to λ, then $Tv=\lambda v$. We have

$$0=\langle (T-\lambda I)v,w\rangle=\langle v,(T^*-\bar\lambda I)w\rangle \tag{1}$$

for all $w\in V$. If $\bar\lambda$ is not an eigenvalue of T^*, then $T^*-\bar\lambda I$ is surjective by 3.6. It follows that there exists some $\xi\in V$ such that $(T^*-\bar\lambda I)\xi=v$. By (1), we have

$$0=\langle v,(T^*-\bar\lambda I)\xi\rangle=\langle v,v\rangle.$$

But $v\neq 0$ since v is a eigenvectors, hence we get the contradiction.
Solution: See Linear Algebra Done Right Solution Manual Chapter 6 Problem 28.

3. Solution: Let $u\in U$ and $w\in U^\perp$, then we have

$$\langle Tu,w\rangle=\langle u,T^*w\rangle. \tag{2}$$

If U is invariant under T, then $Tu\in U$ for all $u\in U$. Hence for a fixed $w\in U^\perp$, we have

$$0=\langle Tu,w\rangle=\langle u,T^*w\rangle$$

for all $u\in U$. This implies $T^*w\in U^\perp$. As w is chosen arbitrarily, we conclude $U^\perp$ is invariant under T^*. The other direction is similar.
Solution: See Linear Algebra Done Right Solution Manual Chapter 6 Problem 29.

4. Solution: See Linear Algebra Done Right Solution Manual Chapter 6 Problem 30.

5. Solution: We have

$$\begin{aligned}\dim\operatorname{null} T^* &= \dim(\operatorname{range} T)^\perp\\ &= \dim W-\dim\operatorname{range} T\\ &= \dim W+\dim\operatorname{null} T-\dim V\end{aligned}$$

where the first line follows from 7.7 (a), the second from 6.50 and the third from 3.22. We also have

$$\begin{aligned}\dim\operatorname{range} T^* &= \dim(\operatorname{null} T)^\perp\\ &= \dim V-\dim\operatorname{null} T\\ &= \dim\operatorname{range} T\end{aligned}$$

where the first line follows from 7.7 (b), the second from 6.50 and the third from 3.22.

6. Solution: (a) If T were self-adjoint, we would have

$$\langle Tp,q\rangle=\langle p,T^*q\rangle=\langle p,Tq\rangle.$$

However, let $p(x)=a_0+a_1x+a_2x^2$ and $q(x)=b_0+b_1x+b_2x^2$. We have

$$\begin{aligned}\langle Tp,q\rangle &= \langle a_1x,q\rangle\\ &= a_1\int_0^1 b_0x+b_1x^2+b_2x^3\,dx\\ &= a_1\left(\frac{b_0}{2}x^2+\frac{b_1}{3}x^3+\frac{b_2}{4}x^4\right)\Big|_0^1\\ &= a_1\left(\frac{b_0}{2}+\frac{b_1}{3}+\frac{b_2}{4}\right).\end{aligned}$$

Similarly

$$\langle p,Tq\rangle=\langle p,b_1x\rangle=b_1\left(\frac{a_0}{2}+\frac{a_1}{3}+\frac{a_2}{4}\right).$$

Thus, taking $a_1=0$ and $b_1,a_0>0$ clearly shows $\langle Tp,q\rangle\neq\langle p,Tq\rangle$.
(b) 7.10 requires the chosen basis to be orthonormal.

7. Solution: We have $(ST)^*=T^*S^*$ by 7.6 (e). Hence ST is self-adjoint if and only if

$$T^*S^*=ST.$$

Note that $S,T\in\mathcal{L}(V)$ are self-adjoint, we have $S^*=S$ and $T^*=T$. Therefore ST is self-adjoint if and only if $T^*S^*=ST$, which is equivalent to $ST=TS$.

8. Solution: See Linear Algebra Done Right Solution Manual Chapter 7 Problem 3 (a).

9. Solution: See Linear Algebra Done Right Solution Manual Chapter 7 Problem 3 (b).

10. Solution: See Linear Algebra Done Right Solution Manual Chapter 7 Problem 5.

11. Solution: See Linear Algebra Done Right Solution Manual Chapter 7 Problem 4.

12. Solution: Let u be a unit eigenvector (i.e. $\|u\|=1$) of T corresponding to eigenvalue 3, then $Tu=3u$. Let w be a unit eigenvector (i.e. $\|w\|=1$) of T corresponding to eigenvalue 4, then $Tw=4w$. By 7.22, we have $\langle u,w\rangle=0$. Let $v=au+bw$, then it follows from 6.25 that

$$\|v\|^2=\|au+bw\|^2=|a|^2+|b|^2$$

and

$$\|Tv\|^2=\|Tau+Tbw\|^2=\|3au+4bw\|^2=9|a|^2+16|b|^2.$$

Hence we need to choose a and b such that

$$|a|^2+|b|^2=2,\quad 9|a|^2+16|b|^2=25.$$

A simple solution is $a=1$ and $b=1$. Hence $v=u+w$ satisfies the requirement.

13. Solution: Define $T\in\mathcal{L}(\mathbf{C}^4)$ by

$$T(z_1,z_2,z_3,z_4)=(z_4,z_1,z_2,z_3).$$

We have

$$\begin{aligned}\langle (z_1,z_2,z_3,z_4),T^*(x_1,x_2,x_3,x_4)\rangle &= \langle T(z_1,z_2,z_3,z_4),(x_1,x_2,x_3,x_4)\rangle\\ &= \langle (z_4,z_1,z_2,z_3),(x_1,x_2,x_3,x_4)\rangle\\ &= z_4\overline{x_1}+z_1\overline{x_2}+z_2\overline{x_3}+z_3\overline{x_4}\\ &= \langle (z_1,z_2,z_3,z_4),(x_2,x_3,x_4,x_1)\rangle.\end{aligned}$$

Thus $T^*(z_1,z_2,z_3,z_4)=(z_2,z_3,z_4,z_1)$. Note that T is normal (T^*T and TT^* equal the identity), however $T\neq T^*$.

14. Solution: Since v and w are eigenvectors corresponding to distinct eigenvalues, by 7.22 they are orthogonal. Thus

$$\begin{aligned}\|T(v+w)\|^2 &= \|Tv+Tw\|^2\\ &= \|3v+4w\|^2\\ &= \|3v\|^2+\|4w\|^2\\ &= 9\|v\|^2+16\|w\|^2\\ &= 100,\end{aligned}$$

where the third line follows from the Pythagorean Theorem.

15. Solution: Let $w_1,w_2\in V$. We have

$$\begin{aligned}\langle w_1,T^*w_2\rangle &= \langle Tw_1,w_2\rangle\\ &= \langle\langle w_1,u\rangle x,w_2\rangle\\ &= \langle w_1,u\rangle\langle x,w_2\rangle\\ &= \langle w_1,\overline{\langle x,w_2\rangle}u\rangle\\ &= \langle w_1,\langle w_2,x\rangle u\rangle\end{aligned}$$

Hence $T^*v=\langle v,x\rangle u$.

(a) Suppose T is selft-adjoint. Then

$$\langle v,u\rangle x-\langle v,x\rangle u=Tv-T^*v=0,$$

for all $v\in V$. We can assume u and x are non-zero (otherwise there is nothing to prove). Taking $v=u$ forces $\langle v,u\rangle\neq 0$, showing that x and u are linearly dependent.

Conversely, suppose x and u are linearly dependent. We can assume x and u are non-zero, otherwise T would equal 0, which already is self-adjoint. Then $u=cx$, for some non-zero $c\in\mathbb{R}$. Thus

$$\begin{aligned}Tv &= \langle v,u\rangle x\\ &= \langle v,cx\rangle\frac{1}{c}u\\ &= \langle v,x\rangle u\\ &= T^*v.\end{aligned}$$

Therefore $T=T^*$.

(b) Again, we can assume u and x are both non-zero in both directions of the proof.
We have

$$\begin{aligned}\langle\langle v,u\rangle x,x\rangle u &= T^*(\langle v,u\rangle x)\\ &= T^*Tv\\ &= TT^*v\\ &= T(\langle v,x\rangle u)\\ &= \langle\langle v,x\rangle u,u\rangle x.\end{aligned}$$

Taking $v=u$ ensures $\langle\langle v,u\rangle x,x\rangle\neq 0$, showing that u and x are linearly dependent.

Conversely, suppose x and u are linearly dependent. Then $u=cx$ for some non-zero $c\in\mathbb{F}$. Then

$$\begin{aligned}&= TT^*v\\ &= T(\langle v,x\rangle u)\\ &= \langle\langle v,x\rangle u,u\rangle x\\ &= \langle\langle v,x\rangle x,cx\rangle cx\\ &= \langle\langle v,cx\rangle x,x\rangle cx\\ &= \langle\langle v,u\rangle x,x\rangle u\\ &= T^*(\langle v,u\rangle x)\\ &= T^*Tv.\end{aligned}$$

Hence $TT^*=T^*T$.

16. Solution: See Linear Algebra Done Right Solution Manual Chapter 7 Problem 6.

17. Solution: See Linear Algebra Done Right Solution Manual Chapter 7 Problem 7.

18. Solution: We give a counterexample. Let $V=\mathbb{R}^2$ and T defined by

$$Te_1=e_1+e_2, Te_2=-e_1-e_2$$

where e_1,e_2 is the standard basis of $\mathbb{R}^2$. Its matrix with respect to the same basis is

$$\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}.$$

Taking the transpose, we see that T^* is defined by

$$T^*e_1=e_1-e_2, T^*e_2=e_1-e_2.$$

Note that $\|Te_1\|=\|T^*e_1\|$ and $\|Te_2\|=\|T^*e_2\|$. However $\mathcal{M}(T^*)\mathcal{M}(T)\neq\mathcal{M}(T)\mathcal{M}(T^*)$, thus T is not normal.

19. Solution: We saw in Exercise 16 that $\operatorname{null} T=\operatorname{null} T^*$ (for T normal). Thus $(z_1,z_2,z_3)\in\operatorname{null} T$ and we have

$$\begin{aligned}0 &= \langle T^*(z_1,z_2,z_3),(1,1,1)\rangle\\ &= \langle (z_1,z_2,z_3),T(1,1,1)\rangle\\ &= \langle (z_1,z_2,z_3),(2,2,2)\rangle\\ &= 2z_1+2z_2+3z_3.\end{aligned}$$

Dividing by 2 yields the desired result.

20. Solution: Let $v\in V$ and $w\in W$. Then

$$\begin{aligned}((\Phi_V\circ T^*)(w))(v) &= (\Phi_V(T^*w))(v)\\ &= \langle v,T^*v\rangle\\ &= \langle Tv,w\rangle.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}((T'\circ\Phi_W)(w))(v) &= (T'\circ\Phi_W(w))(v)\\ &= (\Phi_W(w)\circ T)(v)\\ &= (\Phi_W(w))(Tv)\\ &= \langle Tv,w\rangle.\end{aligned}$$

Therefore $\Phi_V\circ T^*=T'\circ\Phi_W$.

21. Solution:

(a) Let $e_j=\dfrac{\cos jx}{\sqrt{\pi}}$ and $f_j=\dfrac{\sin jx}{\sqrt{\pi}}$. By Exercise 4 in section 6B, $\frac{1}{\sqrt{2\pi}},e_1,\dots,e_n,f_1,\dots,f_n$ is an orthonormal basis of V. Note that $De_j=-jf_j$ and $Df_j=je_j$. Then, for any $v,w\in V$, we have

$$\begin{aligned}\langle v,D^*w\rangle &= \langle Dv,w\rangle\\ &= \left\langle D\left(\left\langle v,\frac{1}{\sqrt{2\pi}}\right\rangle\frac{1}{\sqrt{2\pi}}+\sum_{j=1}^n(\langle v,e_j\rangle e_j+\langle v,f_j\rangle f_j)\right),w\right\rangle\\ &= \left\langle\sum_{j=1}^n(-j\langle v,e_j\rangle f_j+j\langle v,f_j\rangle e_j),\left\langle w,\frac{1}{\sqrt{2\pi}}\right\rangle\frac{1}{\sqrt{2\pi}}+\sum_{j=1}^n(\langle w,e_j\rangle e_j+\langle w,f_j\rangle f_j)\right\rangle\\ &= \sum_{j=1}(-j\langle v,e_j\rangle\langle w,f_j\rangle+j\langle v,f_j\rangle\langle w,e_j\rangle)\\ &= \left\langle\left\langle v,\frac{1}{\sqrt{2\pi}}\right\rangle\frac{1}{\sqrt{2\pi}}+\sum_{j=1}^n(\langle v,e_j\rangle e_j+\langle v,f_j\rangle f_j),\sum_{j=1}^n(-j\langle w,f_j\rangle e_j+j\langle w,e_j\rangle f_j)\right\rangle\\ &= \langle v,-Dw\rangle.\end{aligned}$$

Hence $D^*=-D$. Obviously D is normal but not self-adjoint.

(b) Note that $T=D^2$. Thus $T^*=(DD)^*=D^*D^*=(-D)(-D)=D^2=T$.

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Wu Jinyang · 2 years ago
question 12

4th line and 6th line, you are missing two squares on the left side of the equation
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Mohammad Rashidi Mod → Wu Jinyang · 2 years ago
Fixed, thanks!
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Wu Jinyang → Mohammad Rashidi · 2 years ago
By the way, Can you provide an alternate proof for 7.7 (b): range T* = (null T)^
(let ^ denote the orthogonal complement, unable to type that thing).
I really don't like the proof book provides. but I am unable to come up with my own proof.
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Show more replies

Marcel Ackermann · 2 years ago
Solution to 7A6:
a) self adjoint means $<tv,w>=<v,tw> \forall v,w \in V$. Thus $\int_0^1 v_1 x (w_0+w_1 x + w_2 x^2)dx = \int_0^1 (v_0+v_1 x + v_2 x^2) w_1 x dx$, which is obviously not true. Trivial counterexample $0 != x^3+C \forall x$

b) Theorem 7.10 requires an orthonormal basis, $(1,x,x^2)$ isn't orthonormal.
Reply · Share ›

Matt Lundy · 3 years ago
Regarding your solution to number 2, your comment on the questionable dimension of V, there is a standing assumption in this chapter that everything is finite dimensional (7.1).
Reply · Share ›

Mohammad Rashidi Mod → Matt Lundy · 3 years ago
Thank you! I haven't read this new edition carefully.
Reply · Share ›

Chapter 7 Exercise B

Posted on July 2, 2016 by Linearity

1. Solution: It is true. Consider the standard orthonormal basis e_1, e_2, e_3 of $\mathbb{R}^3$.

Define $T \in \mathcal{L}(\mathbb{R}^3)$ by the rule:

$$Te_1 = e_1, \quad Te_2 = 2e_2 + e_1, \quad Te_3 = 3e_3.$$

Since we have

$$\langle Te_1, e_2 \rangle = \langle e_1, e_2 \rangle = 0,$$

$$\langle e_1, Te_2 \rangle = \langle e_1, e_1 + 2e_2 \rangle = 1,$$

it follows that T is not self-adjoint.

On the other hand, it is easy to see that $e_1, e_2 + e_1, e_3$ is a basis consisting eigenvectors of T.

> *Actually, T is also not normal. Indeed, one can get $T^*e_1 = e_1 + e_2$, then*
>
> $$\|Te_1\| = 1, \quad \|T^*e_1\| = \|e_1 + e_2\| = 2.$$
>
> *Hence by 7.20, T is not normal.*

---

2. Solution: Since T is self-adjoint, it follows from 7.23 that there exists an orthonormal basis $e_1, \dots, e_n$ of V. Note that 2 and 3 are the ony eigenvalues of T, hence for any given $i \in \{1, \cdots, n\}$ we have

$$(T - 2I)e_i = 0 \quad \text{or} \quad (T - 3I)e_i = 0.$$

Note that

$$T^2 - 5T + 6I = (T - 2I)(T - 3I) = (T - 3I)(T - 2I),$$

one has

$$(T^2 - 5T + 6I)e_i = (T - 2I)(T - 3I)e_i = (T - 3I)(T - 2I)e_i.$$

Because $(T - 2I)e_i = 0$ or $(T - 3I)e_i = 0$ for any given $i \in \{1, \cdots, n\}$, we have

$$(T^2 - 5T + 6I)e_i = 0$$

for any given $i \in \{1, \cdots, n\}$. Therefore $T^2 - 5T + 6I = 0$.

---

3. Solution: Let e_1, e_2, e_3 be the standard basis of $\mathbb{C}^3$, define $T \in \mathcal{L}(\mathbb{C}^3)$ by the rule:

$$Te_1 = 2e_1, \quad Te_2 = 2e_2 + e_1, \quad Te_3 = 3e_3.$$

The matrix of T with respect to the basis e_1, e_2, e_3 is

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

This is an upper-triangular matrix. By 5.32, the eigenvalues of T are 2 and 3.

On the other hand,

$$(T^2 - 5T + 6I)e_2 = (T - 3I)(T - 2I)e_2 = (T - 3I)e_1 = -e_1 \neq 0.$$

Hence $T^2 - 5T + 6I \neq 0$.

---

4. Solution: Suppose T is normal, by 7.22 we have all pairs of eigenvectors corresponding to distinct eigenvalues of T are orthogonal. By 7.24 and 5.41, we have

$$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T),$$

where $\lambda_1, \cdots, \lambda_m$ denote the distinct eigenvalues of T.

Suppose all pairs of eigenvectors corresponding to distinct eigenvalues of T are orthogonal and

$$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T),$$

where $\lambda_1, \cdots, \lambda_m$ denote the distinct eigenvalues of T, then V has an orthonormal basis consisting of eigenvectors of T. In fact, this basis can be chosen as the union of orthonormal basis of $E(\lambda_i, T), i = 1, \cdots, m$. Again by 7.24, we have T is normal.

---

5. Solution: Suppose T is self-adjoint, v and u are eigenvectors of T corresponding to eigenvalues λ and ξ respectively, where $\lambda \neq \xi$. Then T is self-adjoint

$$\langle Tv, u \rangle = \langle v, Tu \rangle,$$

hence

$$\lambda \langle v, u \rangle = \xi \langle v, u \rangle.$$

This implies $\langle v, u \rangle = 0$ since $\lambda \neq \xi$. By 7.29 and 5.41, we have

$$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T),$$

where $\lambda_1, \cdots, \lambda_m$ denote the distinct eigenvalues of T.

Suppose all pairs of eigenvectors corresponding to distinct eigenvalues of T are orthogonal and

$$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T),$$

where $\lambda_1, \cdots, \lambda_m$ denote the distinct eigenvalues of T, then V has an orthonormal basis consisting of eigenvectors of T. In fact, this basis can be chosen as the union of orthonormal basis of $E(\lambda_i, T), i = 1, \cdots, m$. Again by 7.29, we have T is self-adjoint.

---

6. Solution: See Linear Algebra Done Right Solution Manual Chapter 7 Problem 9.

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7. Solution: See Linear Algebra Done Right Solution Manual Chapter 7 Problem 10.

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8. Solution: Consider $T \in \mathcal{L}(\mathbb{C}^3)$. Let $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. Define T such that

$$Te_1 = 0, \quad Te_2 = e_1, \quad Te_3 = e_2.$$

Then we have $T^3e_1 = 0$, $T^3e_2 = T^2e_1 = 0$ and $T^3e_3 = T^2e_2 = Te_1 = 0$, i.e. $T^3 = 0$. It follows that $T^9 = T^8$. On the other hand, we have $T^2e_2 = Te_1 = 0$ and $Te_2 = e_1$, it follows that $T^2 \neq T$.

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9. Solution: See Linear Algebra Done Right Solution Manual Chapter 7 Problem 11.

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10. Solution: See Linear Algebra Done Right Solution Manual Chapter 7 Problem 12.

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11. Solution: See Linear Algebra Done Right Solution Manual Chapter 7 Problem 13.

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12. Solution: See Linear Algebra Done Right Solution Manual Chapter 7 Problem 14.

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14. Solution: See Linear Algebra Done Right Solution Manual Chapter 7 Problem 15.

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15. Solution: Let the entry be x. Then by definition we have

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & x \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & x \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & x \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & x \end{pmatrix}.$$

Consider the (1,3) entry of the two products, we get $x = 1$.

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Marcel Ackermann · 2 years ago
7B1) True. Example: Let (1,0,0), (1,1,0), (1,1,1) be a basis of R3. Let T be defined by T(1,0,0)=(1,0,0), T(1,1,0)=(2,2,0), T(1,1,1)=(3,3,3). By construction there is a basis consisting of eigenvectors of T. And T is not self-adjoint: $<t(1,0,0), (1,1,0)=""> = 1$ while $<(1,0,0), T(1,1,0)> = 2$.
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Mohammad Rashidi Mod → Marcel Ackermann · 2 years ago
Thank you
1 Reply · Share ›

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Chapter 7 Exercise C

Posted on July 3, 2016 by Linearity

1. Solution: We give a counterexample. Define $T \in \mathcal{L}(\mathbb{R}^2)$ by

$$Te_1 = e_1$$
$$Te_2 = -e_2$$

where e_1, e_2 is the standard basis of $\mathbb{R}^2$. The matrix of T with respect to this same basis is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which equals its transpose, therefore T is self-adjoint. Moreover, the basis $\frac{1}{\sqrt{2}}(e_1+e_2), \frac{1}{\sqrt{2}}(e_1-e_2)$ is orthonormal and

$$\left\langle T\left(\frac{1}{\sqrt{2}}(e_1+e_2)\right), \frac{1}{\sqrt{2}}(e_1+e_2)\right\rangle = 0$$
$$\left\langle T\left(\frac{1}{\sqrt{2}}(e_1-e_2)\right), \frac{1}{\sqrt{2}}(e_1-e_2)\right\rangle = 0,$$

but T is not positive because

$$\langle Te_2, e_2\rangle = \langle -e_2, e_2\rangle = -1.$$

---

2. Solution: Note that T is a positive operator on V, we have

$$\langle T(v-w), v-w\rangle \geq 0. \qquad (1)$$

On the other hand,

$$Tv = w \quad \text{and} \quad Tw = v$$

imply that $T(v-w) = w - v$, hence

$$\langle T(v-w), v-w\rangle = -\langle v-w, v-w\rangle \leq 0. \qquad (2)$$

Therefore $\langle v-w, v-w\rangle = 0$ by (1) and (2), i.e. $v = w$.

---

3. Solution: For all $u \in U$, we have

$$\langle T|_U u, u\rangle = \langle Tu, u\rangle = \langle u, Tu\rangle = \langle u, T|_U u\rangle.$$

Thus, $T|_U$ is self-adjoint. Furthermore,

$$\langle T|_U u, u\rangle = \langle Tu, u\rangle \geq 0,$$

which shows that $T|_U$ is positive.

---

4. Solution: By 7.6 (c) and (e), we have

$$(TT^*)* = (T^*) * T^* = TT^*, \quad (T^*T)^* = T^*(T^*)^* = T^*T.$$

Hence both TT^* and T^*T are self-adjoint.

On the other hand, for any $v \in V$, we have

$$\langle T^*Tv, v\rangle = \langle Tv, (T^*)^*v\rangle = \langle Tv, Tv\rangle \geqslant 0.$$

Hence T^*T is a positive operator.

Similarly, for any $w \in W$, we have

$$\langle TT^*v, v\rangle = \langle T^*v, T^*v\rangle \geqslant 0.$$

Hence TT^* is a positive operator.

---

5. Solution: Suppose T and S are positive operators on V, then $T^* = T$ and $S^* = S$. Therefore, we have

$$(T+S)^* = T^* + S^* = T + S.$$

Hence $T + S$ is self-adjoint.

Again since T and S are positive operators on V,, for any $v \in V$, $\langle Tv, v\rangle \geqslant 0$ and $\langle Sv, v\rangle \geqslant 0$. Thus we have

$$\langle (T+S)v, v\rangle = \langle Tv, v\rangle + \langle Sv, v\rangle \geqslant 0.$$

Therefore $T + S$ is a positive operator.

---

6. Solution: Since T is positive, it follows from 7.35 (a) $\iff$ (d) that there exists a self-adjoint operator S such that $S^2 = T$. For any positive integer k, we have $(S^k)^* = (S^*)^k = S^k$ by 7.6 (e) and the equality $S = S^*$. Hence S^k is self-adjoint.

Note that we also have $(S^k)^2 = (S^2)^k = T^k$, hence T^k has a self-adjoint square root. It follows from 7.35 (a) $\iff$ (d) again that T^k is positive.

---

7. Solution: Suppose $\langle Tv, v\rangle > 0$ for every $v \in V$ with $v \neq 0$. If T is not invertible, there must exist a nonzero $u \in V$ such that $Tu = 0$, hence $\langle Tu, u\rangle = 0$ for $u \neq 0$. Therefore we get a contradiction, which in turn implies that T is invertible.

Conversely, suppose T is invertible. Since T is positive, it follows from 7.35 (a) $\iff$ (d) that there exists a self-adjoint operator S such that $S^2 = T$. Because T is injective, so is S. Hence for every $v \in V$ with $v \neq 0$, we have $Sv \neq 0$. Moreover, since S is self-adjoint (and $Sv \neq 0$), we have

$$\langle Tv, v\rangle = \langle S^2v, v\rangle = \langle Sv, Sv\rangle > 0.$$

---

8. Solution: If $\langle \cdot, \cdot\rangle_T$ is an inner product on V, then for any $v \in V$ we have

$$\langle Tv, v\rangle = \langle v, v\rangle_T \geqslant 0.$$

Hence T is positive. Moreover, for any nonzero $v \in V$ we have

$$\langle Tv, v\rangle = \langle v, v\rangle_T > 0.$$

It follows from Problem 7 that T is invertible.

Conversely, suppose that T is an invertible positive operator. We show that $\langle \cdot, \cdot\rangle_T$ is an inner product on V by checking definition 6.3.

Positivity: Note that T is positive, we have

$$\langle v, v\rangle_T = \langle Tv, v\rangle \geqslant 0.$$

Definiteness: If $v = 0$, then

$$\langle v, v\rangle_T = \langle Tv, v\rangle \geqslant 0.$$

If $\langle v, v\rangle_T = 0$, since T is invertible positive operator on V, it follows from Problem 7 that $v = 0$.

Additivity, homogeneity, and conjugate symmetry can be checked directly with out any difficulties.

---

9. Solution: Let e_1, e_2 be an orthonormal basis of $\mathbb{F}^2$ and $\theta \in [0, 2\pi)$, define $T_\theta \in \mathcal{L}(\mathbb{F}^2)$ by

$$T_\theta e_1 = \cos\theta e_1 + \sin\theta e_2, \quad T_\theta e_2 = \sin\theta e_1 - \cos\theta e_2.$$

Note that

$$\cos\theta = \langle T_\theta e_1, e_1\rangle = \langle e_1, (T_\theta) * e_1\rangle,$$
$$\sin\theta = \langle T_\theta e_2, e_1\rangle = \langle e_2, (T_\theta)^* e_1\rangle,$$

we have $(T_\theta)^* e_1 = \cos\theta e_1 + \sin\theta e_2$. Similarly, note that

$$\sin\theta = \langle T_\theta e_1, e_2\rangle = \langle e_1, (T_\theta)^* e_2\rangle,$$
$$-\cos\theta = \langle T_\theta e_2, e_2\rangle = \langle e_2, (T_\theta)^* e_2\rangle,$$

we have $(T_\theta)^* e_2 = \sin\theta e_1 - \cos\theta e_2$. Hence $T_\theta = (T_\theta)^*$, which implies that T_θ is self-adjoint. Also note that

$$\begin{aligned}(T_\theta)^2 e_1 &= T_\theta(\cos\theta e_1 + \sin\theta e_2)\\ &= \cos\theta(\cos\theta e_1 + \sin\theta e_2) + \sin\theta(\sin\theta e_1 - \cos\theta e_2)\\ &= (\cos^2\theta + \sin^2\theta)e_1 = e_1,\end{aligned}$$

$$\begin{aligned}(T_\theta)^2 e_2 &= T_\theta(\sin\theta e_1 - \cos\theta e_2)\\ &= \sin\theta(\cos\theta e_1 + \sin\theta e_2) - \cos\theta(\sin\theta e_1 - \cos\theta e_2)\\ &= (\cos^2\theta + \sin^2\theta)e_2 = e_2,\end{aligned}$$

we have $(T_\theta)^2 = \mathrm{id}$. Therefore we have infinitely many self-adjoint operators as the square root of id.

The construction comes from the following idea. Let S be self-adjoint such that $S^2 = \mathrm{id}$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the matrix with respect to an orthonormal basis of $\mathbb{F}^2$. Since S is self-adjoint, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}.$$

Hence $a, d \in \mathbb{R}$ and $b = \bar{c}$. If $S^2 = \mathrm{id}$, then we have $a^2 + b\bar{b} = 1$, $ab + bd = 0$, and $d^2 + b\bar{b} = 1$. Hence we can take $a = -d$ and $b = c$. Then $a^2 + b^2 = 1$. Take $a = \cos\theta$ and $b = \sin\theta$, we get the construction.

---

10. Solution: "(a)⟹(b)" If S is an isometry, so is S^* by 7.42 (g) $\iff$ (a). It follows from 7.42 (a) $\iff$ (b) that

$$\langle S^*u, S^*v\rangle = \langle u, v\rangle$$

for all $u, v \in V$.

"(b)⟹(c)" If $e_1, \dots, e_m$ is an orthonormal basis of V, then we have $\langle e_i, e_j\rangle = \delta_{ij}$. Since

$$\langle S^*u, S^*v\rangle = \langle u, v\rangle$$

for all $u, v \in V$, we have

$$\langle S^*e_i, S^*e_j\rangle = \langle e_i, e_j\rangle = \delta_{ij}.$$

Hence $S^*e_1, \dots, S^*e_m$ is an orthonormal basis of V.

"(c)⟹(d)" is trivial.

"(d)⟹(a)" It follows from 7.42 (a) $\iff$ (d) that S^* is an isometry. So is S by 7.42 (g) $\iff$ (a) since $(S^*)^* = S$ from 7.6 (c).

---

11. Solution: Let e_1, e_2, e_3 and f_1, f_2, f_3 be orthonormal bases of $\mathbb{F}^3$ consisting of eigenvectors of T_1 and T_2, respectively, corresponding to the eigenvalues 2, 5, 7.

Define S by

$$Se_j = f_j$$

for $j = 1, 2, 3$. One easily checks that S is an isometry (using the Pythagorean Theorem). Then, because $S^{-1} = S^*$ (by 7.42), we have $S^*f_j = e_j$. Thus

$$T_1e_1 = 2e_1 = S^*(2f_1) = S^*(T_2f_1) = S^*T_2Se_1.$$

Similarly $T_1e_2 = S^*T_2Se_2$ and $T_1e_3 = S^*T_2Se_3$. Therefore $T_1 = S^*T_2S$.

---

12. Solution: Let e_1, e_2, e_3, e_4 denote an orthonormal basis of $\mathbb{F}^4$. Define $T_1, T_2 \in \mathcal{L}(\mathbb{F}^4)$ by

$$T_1e_1 = 2e_1$$
$$T_1e_2 = 2e_2$$
$$T_1e_3 = 5e_3$$
$$T_1e_4 = 7e_4$$

$$T_2e_1 = 2e_1$$
$$T_2e_2 = 5e_2$$
$$T_2e_3 = 5e_3$$
$$T_2e_4 = 7e_4.$$

Then both T_1 and T_2 are self-adjoint (the matrices equal their transposes) and 2, 5, 7 are their eigenvalues. Suppose by contradiction that S is an isometry on V such that $T_1 = S^*T_2S$. Let $v \in V$ be the vector that S maps to e_2. Then

$$T_1v = S^*T_2Sv = S^*T_2e_2 = 5S^*e_2 = 5v.$$

Therefore $v \in E(T_1, 5) = \mathrm{span}(e_3)$. Let also $w \in V$ be the vector that S maps to e_3. Note that v, w is linearly independent, because e_2, e_3 is linearly independent. Then

$$T_1w = S^*T_2Sw = S^*T_2e_3 = 5S^*e_3 = 5w.$$

Therefore $w \in E(T_1, 5) = \mathrm{span}(e_3)$. But this is a contradiction, because we can't have a linearly independent list of length 2, v, w, in a 1-dimensional vector space, $\mathrm{span}(e_3)$. Hence, there does not exist such S. Notice that it wasn't necessary to require S to be an isometry, we just needed to suppose, by contradiction, the existence of an invertible S such that $T_1 = S^{-1}T_2S$. This S does not exist. Since the desired isometry must satisfy the same property (because the adjoint of an isometry equals its inverse), it follows that there cannot exist such isometry. The key idea here is that the eigenspaces of T_1 and T_2 don't fit.

---

13. Solution: It is false. Let $e_1, \dots, e_n$ be an orthonormal basis of V and define $Se_j = e_1$ for $j = 1, \dots, n$. Then $||Se_j|| = 1$ for each e_j, but obviously S is not invertible, therefore S is not an isometry (7.42 requires isometries to be invertible).

---

14. Solution: In the exercise, T was already shown to be self-adjoint. So $-T$ is also self-adjoint. Note that

$$1, \cos x, \cos 2x, \dots, \cos nx, \sin x, \sin 2x, \dots, \sin nx$$

is a basis of V consisting of eigenvectors of $-T$ whose corresponding eigenvalues are all nonnegative. Thus by 7.35 $-T$ is positive.

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This entry was posted in Chapter 7 and tagged Exercise C.

← Chapter 7 Exercise B   Chapter 7 Exercise D →

Adol · 2 years ago
#13 is in the solutions manual you refer us to commonly; it's in chapter 7 #21.
1 ^ v · Reply · Share ›

Matt Lundy · 3 years ago
What are your goals for this site? I would be happy to contribute some solutions if you are interested.
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Hunter → Matt Lundy · 3 years ago
Please do this! there are a lot of missing solutions and this site has helped me tremendously!
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Marcel Ackermann · a year ago
7C11) Let u_2, u_5, u_7 the corresponding eigenvectors of T_1, and v_2, v_5, v_7 the ones of T_2. Define S as S u_i = v_i. We want to show that T_1 u_i = S* T_2 S u_i. So: T_1 u_i = S* T_2 S u_i <=> i u_i = S* T_2 v_i <=> i u_i = S* i v_i <=> u_i = S* v_i. According to 7.42d S is an isometry iff there is an orthonormal basis e_i of V s.t. S e_i is orthonormal, which we have. So S^-1 = S*.
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Marcel Ackermann · 2 years ago
7C6) Alternative:
for k even: <t^k v,v=""> = <t^{k 2}v,="" t^{k="" 2}v=""> \geq 0 (T is positive, so it's self adjoint, the unequality is by positivity of inner product) => T^k is positive
for k odd: <t^k v,v=""> = <tt^{k-1} v,v=""> = <tt^{(k-1) 2}="" v,t^{(k-1)="" 2}="" v=""> = <t v',="" v'=""> \geq 0 (because T is positive) => T^k is positive
^ v · Reply · Share ›

Célio Passos · 2 years ago
In the forward direction of Exercise 8, you have show that T is self-adjoint before claiming it is positive.
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Marcel Ackermann · 2 years ago
7C3) by definition $T|_U$ is an operator on U.
Is it positive? Eg. is $<t|_u v,="" v=""> \geq 0$? Clearly yes, because T is (even more elements than the restriction fulfill the property).
^ v · Reply · Share ›

Mohammad Rashidi Mod → Marcel Ackermann · 2 years ago
Thanks
^ v · Reply · Share ›

Marcel Ackermann · 2 years ago
7C1) Counter example: Let $T(x_1, x_2) = (-x_2, -x_1)$. It is self-adjoint: $<t(v_1, v_2),="" (w_1,="" w_2)=""> = <(-v_1, -v_2), (w_1, w_2)> = -v_1 w_1 - v_2 w_2 = <(v_1, v_2), T(w_1, w_2)>$ and $<t(e_j), e_j="">=0$ for the standard basis. Yet, T is not a positive operator: $<t(1,1),(1,1)>=-2$.
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Chapter 7 Exercise D

Posted on July 4, 2016 by Linearity

1. Solution. A quick calculation shows that $T^*Tv = \|x\|^2\langle v,u\rangle u$ for every $v \in V$. The map $R \in \mathcal{L}(V)$ defined by

$$Rv = \frac{\|x\|}{\|u\|}\langle v,u\rangle u$$

is a square root of T^*T. Moreover, it is easy to check that $\langle Rv, v\rangle \geq 0$ for all $v \in V$. We but need to prove that R is self-adjoint. Let $e_1, \dots, e_n$ be an orthonormal basis of V. We can write

$$u = a_1e_1 + \cdots + a_ne_n$$

for some $a_1, \dots, a_n \in \mathbf{F}$. Note that

$$Re_j = \frac{\|x\|}{\|u\|}\langle e_j,u\rangle u = \frac{\|x\|}{\|u\|}(\overline{a_j}a_1e_1 + \cdots + \overline{a_j}a_ne_n).$$

Therefore, the matrix of R with respect to the basis $e_1, \dots, e_n$ has entries defined by

$$\mathcal{M}(R)_{i,j} = \frac{\|x\|}{\|u\|}a_i\overline{a_j}.$$

Thus $\mathcal{M}(R)_{j,i} = \overline{\mathcal{M}(R)_{i,j}}$, that is, $\mathcal{M}(R) = \mathcal{M}(R^*)$. Hence R is self-adjoint.

2. Solution. Define $T \in \mathcal{L}(\mathbf{C}^2)$ by

$$Te_1 = 0$$
$$Te_2 = 5e_1$$

where e_1, e_2 is the standard basis of $\mathbf{C}^2$. Then the matrix of T with respect to this basis is

$$\mathcal{M}(T) = \begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix},$$

which is upper triangular. Thus 0 is the only eigenvalue of T. We have

$$\mathcal{M}(T^*T) = \mathcal{M}(T^*)\mathcal{M}(T) = \begin{pmatrix} 0 & 0 \\ 5 & 0 \end{pmatrix}\begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 25 \end{pmatrix}.$$

Hence the eigenvalues of T^*T are 0 and 25. By 7.52, the singular values of T are 0 and 5.

3. Solution. By the Polar Decomposition (7.45), there exists an isometry $S \in \mathcal{L}(V)$ such that $T^* = S\sqrt{TT^*}$. Taking the adjoint of each side, we get

$$T = (S\sqrt{TT^*})^* = (\sqrt{TT^*})^*S^* = \sqrt{TT^*}S^*,$$

where the last equality follows because $\sqrt{TT^*}$ is self-adjoint. This yields the desired result, because S^* is also an isometry.

4. Solution. Let $v \in V$ be an eigenvector of $\sqrt{T^*T}$ with $\|v\| = 1$ corresponding to s and let $S \in \mathcal{L}(V)$ be an isometry such that $T = S\sqrt{T^*T}$. Then

$$\|Tv\| = \|S\sqrt{T^*T}v\| = \|\sqrt{T^*T}v\| = |s|\,\|v\| = s,$$

where the last equality follows because $\sqrt{T^*T}$ is positive.

5. Solution. We have

$$\mathcal{M}(T^*T) = \mathcal{M}(T^*)\mathcal{M}(T) = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 16 \end{pmatrix}.$$

Therefore, the singular values of T are 1 and 4.

6. Solution. We will use the orthonormal basis $\sqrt{\frac{1}{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3})$ of $\mathcal{P}_2(\mathbf{R})$, which was found in the example.

We have

$$D\sqrt{\tfrac{1}{2}} = 0$$
$$D\sqrt{\tfrac{3}{2}}x = \sqrt{3}\left(\sqrt{\tfrac{1}{2}}\right)$$
$$D\sqrt{\tfrac{45}{8}}\left(x^2 - \tfrac{1}{3}\right) = \sqrt{15}\left(\sqrt{\tfrac{3}{2}}x\right)$$

Therefore

$$\mathcal{M}(D) = \begin{pmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{15} \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus

$$\mathcal{M}(D^*D) = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & \sqrt{15} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{15} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix}.$$

Thus, the singular values of D are $0, \sqrt{3}, \sqrt{15}$.

7. Solution. A quick calculation, like the ones in the previous exercises, shows that

$$T^*T(z_1, z_2, z_3) = (4z_1, 9z_2, z_3).$$

Thus

$$\sqrt{T^*T}(z_1, z_2, z_3) = (2z_1, 3z_2, z_3).$$

Define $S \in \mathcal{L}(\mathbf{F}^3)$ by

$$S(z_1, z_2, z_3) = (z_3, z_1, z_2).$$

S is clearly an isometry (it maps the standard basis, which is orthonormal, to a permutation of the standard basis) and we have $T = S\sqrt{T^*T}$.

8. Solution. Let $S' \in \mathcal{L}(V)$ be an isometry such that $T = S'\sqrt{T^*T}$. Then

$$\begin{aligned} 0 &= \|Tv\|^2 - \|Tv\|^2 \\ &= \|SRv\|^2 - \|S'\sqrt{T^*T}v\|^2 \\ &= \|Rv\|^2 - \|\sqrt{T^*T}v\|^2 \\ &= \langle R^*Rv, v\rangle - \langle T^*Tv, v\rangle \\ &= \langle (R^*R - T^*T)v, v\rangle \\ &= \langle (R^2 - T^*T)v, v\rangle \end{aligned}$$

where the last line follows because R is self-adjoint. 7.16 now implies that $R^2 = T^*T$. Since R is positive and the positive square root of T^*T is unique (by 7.36), it follows that $R = \sqrt{T^*T}$.

9. Solution. Consider the proof of 7.45. T being invertible implies $\dim(\operatorname{range} T)^\perp = 0$, thus $S_2 = 0$ and so $S = S_1$. Clearly, S_1 is unique, so S must also be unique. Conversely, if S is unique then S_2 is unique, so we must have that $m = 0$, because otherwise we could set $S = S_1 - S_2$ and it would still be an isometry satisfying $T = S\sqrt{T^*T}$. This implies that $m = 0$, because from the definition we see that S_2 must be invertible for any positive integer m. Thus $\dim(\operatorname{range} T)^\perp = 0$ and so T is invertible.

10. Solution. Suppose λ is an eigenvalue of T and v a corresponding eigenvector. Then

$$T^*Tv = T^2v = \lambda^2v = |\lambda|^2v,$$

where the last equality follows because the eigenvalues of self-adjoint operators are real (see 7.13). Therefore, the eigenvectors of T are also eigenvectors of T^*T with corresponding eigenvalues squared. Since T has a basis consisting of eigenvectors, so does T^*T and thus all eigenvalues of T^*T are squares of the absolute values of eigenvalues of T. 7.52 now implies that the singular values of T are the absolute values of the eigenvalues of T.

11. Solution: It follows from 7.45 that $T = S\sqrt{T^*T}$ for an isometry $S \in \mathcal{L}(V)$. Note that $\sqrt{T^*T}$ is self-adjoint and $S^{-1} = S^*$ (7.42 (e), (f)), we have

$$TT^* = S\sqrt{T^*T}(S\sqrt{T^*T})^* = S\sqrt{T^*T}\sqrt{T^*T}S^* = S(T^*T)S^{-1}.$$

It follows from Problem 15 of Exercise 5A that TT^* and T^*T have the same eigenvalues. Moreover, each eigenvalue has the same multiplicity.
Since TT^* and T^*T are positive operators, see Problem 4 of Exercise 7C, they have nonnegative eigenvalues. Also the singular value of T are the nonnegative square roots of the eigenvalues of T^*T while the singular value of T are the nonnegative square roots of the eigenvalues of TT^*, see 7.52. We conclude that T and T^* have the same singular values. Moreover, each eigenvalue has the same multiplicity.

See: Wu Jinyang's Comment. Basically, matrices or operators related in the way of Problem 15 of Exercise 5A can be considered as the "same". They have almost the same structure. It is called they are similar to each other.

12. Solution. As a counterexample, take the linear map T from Exercise 5. We have

$$\mathcal{M}((T^2)^*T^2) = \mathcal{M}(T^*)^2\mathcal{M}(T)^2 = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} = \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}.$$

Therefore, the singular values of T^2 are $4, 4$. However, the singular values of T are $1, 4$.

13. Solution. Suppose T is invertible. Then

$$\operatorname{null} T^* = (\operatorname{range} T)^\perp = V^\perp = \{0\},$$

where the first equality follows from 7.7 and the second because T is surjective. This shows that T^* is also invertible. Therefore 0 is not eigenvalue of T^*T and so, by 7.52, it cannot be a singular value of T.

Conversely, suppose 0 is not a singular value of T. Then $T^*Tv \neq 0$ for all non-zero $v \in V$. This implies that $Tv \neq 0$ for all non-zero $v \in V$. Thus T is invertible.

14. Solution. First we will prove that $\operatorname{range} T = \operatorname{range} T^*T$. From Exercise 5 in section 7A we see that $\operatorname{range} T = \operatorname{range} T^*$.

Suppose $w \in \operatorname{range} T$. Then $w \in \operatorname{range} T^*$. Thus $w = T^*v$ for some $v \in V$. We can write $v = v' + v''$ for some $v' \in \operatorname{null} T^*$ and $v'' \in (\operatorname{null} T^*)^\perp$. Thus $w = T^*v''$. But 7.7 shows that $(\operatorname{null} T^*)^\perp = \operatorname{range} T$. Therefore $v'' = Tu$ for some $u \in V$ and so $w = T^*Tu \in \operatorname{range} T^*T$. Hence $\operatorname{range} T \subset \operatorname{range} T^*T$.
The inclusion in the other direction is easy. We have

$$\operatorname{range} T^*T \subset \operatorname{range} T^* = \operatorname{range} T.$$

Therefore $\operatorname{range} T = \operatorname{range} T^*T$.

Since $\sqrt{T^*T}$ is diagonalizable (because it is self-adjoint), it follows that the number of nonzero singular values of T equals the dimension of $\operatorname{range} \sqrt{T^*T}$. Note that $\operatorname{range} T^*T = \operatorname{range} \sqrt{T^*T}$. Therefore $\dim \operatorname{range} \sqrt{T^*T} = \dim \operatorname{range} T$, completing the proof.

15. Solution. The forward direction is obvious, because if S is an isometry, then $\sqrt{S^*S}$ equals the identity, whose eigenvalues equal 1.
Suppose all singular values of S equal 1. This implies that $\sqrt{S^*S}$, and therefore so does S^*S, equals the identity. 7.42 now implies that S is an isometry.

16. Solution. Let $S_1, S_2 \in \mathcal{L}(V)$ be isometries such that $T_1 = S_1\sqrt{T_1^*T_1}$ and $T_2 = S_2\sqrt{T_2^*T_2}$ and let $e_1, \dots, e_n$ and $f_1, \dots, f_n$ be orthonormal basis of V consisting of eigenvectors of T_1 and T_2, respectively, corresponding to the singular values $s_1, \dots, s_n$. Defin $S \in \mathcal{L}(V)$ by

$$Se_j = f_j$$

for each $j = 1, \dots, n$. Then S is also an isometry and we have

$$\begin{aligned} \sqrt{T_1^*T_1}e_j &= s_je_j \\ &= S^*(s_jf_j) \\ &= S^*\sqrt{T_2^*T_2}f_j \\ &= S^*\sqrt{T_2^*T_2}Se_j. \end{aligned}$$

So $\sqrt{T_1^*T_1} = S^*\sqrt{T_2^*T_2}S$. Therefore

$$T_1 = S_1\sqrt{T_1^*T_1} = S_1S^*\sqrt{T_2^*T_2}S = S_1S^*S_2^*T_2S,$$

where the last equality follows by multiplying both sides of the equation $T_2 = S_2\sqrt{T_2^*T_2}$ by S_2^*. This gives the desired result, because the product of isometries is also an isometry.

17. Solution.

(a) We have $Te_j = s_jf_j$. This implies that the matrix of T if respect to the bases $e_1, \dots, e_n$ and $f_1, \dots, f_n$ is the diagonal matrix whose diagonal entries are the singular values of T. By 7.10, we have $T^*f_j = s_je_j$ for each j. If we replace v with f_j in the right hand side of the desired result we get the same thing, therefore

$$T^*v = s_1\langle v, f_1\rangle e_1 + \cdots + s_n\langle v, f_n\rangle e_n$$

by uniqueness of linear maps (see 3.5).

(b) Just apply the previous item to the formula given of the singular value decomposition of T.

(c) Note that the e_j's are eigenvectors of T^*T with corresponding eigenvalue s_j^2. Thus $\sqrt{T^*T}e_j = s_je_j$. Plugging e_j in the place of v in the right hand side yields the same thing, so the result holds by uniqueness of linear maps again.

(d) The given formula satisfies $TT^{-1} = I$ and $T^{-1}T = I$ and it is well defined, because from Exercise 13 we see that none of s_j's are 0.

18. Solution.

(a) Use the same notation from the Singular Value Decompositon theorem (7.52). We have

$$\begin{aligned} \|Tv\|^2 &= |s_1\langle v, e_1\rangle|^2 + \cdots + |s_n\langle v, e_n\rangle|^2 \\ &= |s_1|^2|\langle v, e_1\rangle|^2 + \cdots + |s_n|^2|\langle v, e_n\rangle|^2 \\ &\leq |s|^2|\langle v, e_1\rangle|^2 + \cdots + |s|^2|\langle v, e_n\rangle|^2 \\ &= |s|^2|\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_n\rangle|^2 \\ &= s^2\|v\|^2 \end{aligned}$$

for all $v \in V$, where the first line follows from the Pythagorean Theorem (6.13) and last because s is a positive real value. Taking the square root of both sides shows that $\|Tv\| \leq s\|v\|$ for all $v \in V$. The proof that $\hat{s}\|v\| \leq \|Tv\|$ is almost the same.

(b) Let v be an eigenvector of T corresponding to λ with $\|v\| = 1$. Then

$$|\lambda| = |\lambda|\,\|v\| = \|Tv\| \leq s\|v\| = s.$$

Similarly, we have $\hat{s} \leq |\lambda|$.

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This entry was posted in Chapter 7 and tagged Exercise D.

← Chapter 7 Exercise C | Chapter 8 Exercise A →

Marcel Ackermann · a year ago
7D6) T=[[0,1,0],[0,0,2],[0,0,0]], sqrt(T*T)=[[1,0,0],[0,2,0],[0,0,0]], so the singular values are 0, 1, 2.
Reply · Share ›

Marcel Ackermann · 2 years ago
7D10) \sqrt(T*T)=\sqrt(TT) (because T is self-adjoint). By spectral theorem T has a diagonal matrix, so \sqrt(TT)= \sqrt(T') with T' the diagonal elements squared. Then diag(|d_1|, ..., |d_n|) is a positive square root. The singular values are the eigenvalues of this, so it has the absolute values of T has eigenvalues.
Reply · Share ›

Marcel Ackermann · 2 years ago
7D7) T^{*}(z_1,z_2,z_3)=(2z_2, 3z_3, z_1)
T^*T(z_1,z_2,z_3)=(4z_1,9z_2, z_3)
\sqrt(T^*T)(z_1,z_2,z_3) = (2z_1,3z_2,z_3)
This is a positive square root as can easily be seen.
so: S(z_1,z_2,z_3)=(z_3,z_1,z_2)
Reply · Share ›

Wu Jinyang · 2 years ago
Hi, I am stuck at question 11.
I checked the solution manual of the second edition and it is not included there.
Can you upload the answer to this question?

I tried to use 7.52 to prove it
since T*T ej = aj ej
TT* (Tej) = aj Tej
yet Tej can equal to zero, making it not a proper eigenvector.

Am I at least heading the correct direction?
Reply · Share ›

Mohammad Rashidi Mod → Wu Jinyang · 2 years ago
Updated.
Reply · Share ›

Wu Jinyang → Mohammad Rashidi · 2 years ago
That was fast!

Just one trivial thing. To some extend, the proof is still not complete.
Two vectors can have same eigenvalues with the number of the same eigenvalue different.

For example let V have 2,2,2,3 as eigenvalues. let W have 2,2,3,3 as eigenvalues. their eigenvalues are the same: 2,3. If you get my point.

5A.15 not only show that they have the same eigenvalue, but also shows that the dimension of the eigenspace of the two operators with respect to the same eigenvalue is the same. So you might want to slightly rephrase line 4 to let it contain this meaning.
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Show more replies

Marcel Ackermann · 2 years ago
7D1) At first find T*: $T*(w)=<w,x> u$ because $<tu,w>=<<v,u>x,w>=<v,u><x,w>$ and $<v,t*w>=<v,<w,x>u>=<v,u><x,w>$. Let $Rv=\frac{||x||}{||u||}<v,u>u$. We need to show that 1) $T*Tv=RRv \forall v$ and 2) $<rv,v> \geq 0 \forall v$.
1) $T*Tv=T*(<v,u>x)=<<v,u>x,x>u=<v,u><x,x>u$ and $RRv=\frac{||x||}{||u||}<\frac{||x||}{||u||}<v,u>u,u>u=<x,x><,u,u>u$.
2) $<rv,v>=<\frac{||x||}{||u||}<v,u>u,v> \geq <v,u><u,v> \geq 0$.
Reply · Share ›

Célio Passos → Marcel Ackermann · 2 years ago
You still need to show that R is self-adjoint.
Reply · Share ›

Scarlet Remilia · 2 years ago
I can't open exercise 8
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Chapter 8 Exercise A

Posted on August 1, 2016 by Linearity

1. Solution: Since

$$T^2(w,z) = T(z,0) = (0,0),$$

it follows that $G(0,T) = V$. Therefore every vector in $\mathbb{C}^2$ is a generalized eigenvector of T.

---

2. Solution: The eigenvalues of T are i and $-i$. Since $\mathbb{C}^2$ has dimension 2, the generalized eigenspaces are the eigenspaces themselves.

---

3. Solution: We will prove $\operatorname{null}(T-\lambda I)^n = \operatorname{null}\left(T^{-1}-\frac{1}{\lambda}I\right)^n$ for all nonnegative integers n by induction on n.

It is easy to check that $\operatorname{null}(T-\lambda I) = \operatorname{null}\left(T^{-1}-\frac{1}{\lambda}I\right)$ (see Exercise 9 in section 5C). Let $n > 1$ and assume the result holds for all nonnegative integers less than n.

Suppose $v \in \operatorname{null}(T-\lambda I)^n$. Then

$$(T-\lambda I)v \in \operatorname{null}(T-\lambda I)^{n-1}.$$

By the induction hypothesis

$$(T-\lambda I)v \in \operatorname{null}\left(T^{-1}-\frac{1}{\lambda}I\right)^{n-1}.$$

Thus

$$0 = \left(T^{-1}-\frac{1}{\lambda}I\right)^{n-1}(T-\lambda I)v = (T-\lambda I)\left(T^{-1}-\frac{1}{\lambda}I\right)^{n-1}v,$$

where the second equality follows from Theorem 1 below.

Theorem 1. Suppose $T \in \mathcal{L}(V)$ is invertible and $p, q \in \mathcal{P}(\mathbb{F})$. Then $p(T^{-1})q(T) = q(T)p(T^{-1})$.

Proof. The key idea used here is that T commutes with T^{-1}, even when raised to different powers. Suppose $p(z) = \sum_{j=0}^m a_j z^j$ and $q(z) = \sum_{k=0}^n b_k z^k$ for $z \in \mathbb{F}$. Then

$$\begin{aligned} p(T)\,q(T^{-1}) &= \left(\sum_{j=0}^m a_j T^j\right)\left(\sum_{k=0}^n b_k (T^{-1})^k\right) \\ &= \sum_{j=0}^m \sum_{k=0}^n a_j b_k T^j (T^{-1})^k \\ &= \sum_{j=0}^m \sum_{k=0}^n b_k a_j (T^{-1})^k T^j \\ &= \sum_{k=0}^n \sum_{j=0}^m b_k a_j (T^{-1})^k T^j \\ &= \left(\sum_{k=0}^n b_k (T^{-1})^k\right)\left(\sum_{j=0}^m a_j T^j\right) \\ &= q(T^{-1})\,p(T) \end{aligned}$$

Therefore

$$\left(T^{-1}-\frac{1}{\lambda}I\right)^{n-1}v \in \operatorname{null}(T-\lambda I).$$

But

$$\operatorname{null}(T-\lambda I) = \operatorname{null}\left(T^{-1}-\frac{1}{\lambda}I\right).$$

Hence

$$\left(T^{-1}-\frac{1}{\lambda}I\right)^{n-1}v \in \operatorname{null}\left(T^{-1}-\frac{1}{\lambda}I\right)$$

and so

$$0 = \left(T^{-1}-\frac{1}{\lambda}I\right)\left(T^{-1}-\frac{1}{\lambda}I\right)^{n-1}v = \left(T^{-1}-\frac{1}{\lambda}I\right)^n v,$$

which shows that $v \in \operatorname{null}(T^{-1}-\frac{1}{\lambda}I)$. Therefore

$$\operatorname{null}(T-\lambda I)^n \subset \operatorname{null}\left(T^{-1}-\frac{1}{\lambda}I\right)^n.$$

To prove the inclusion in the other direction, it suffices to repeat the same thing replacing $(T-\lambda I)$ with $(T^{-1}-\frac{1}{\lambda}I)$ and vice versa.

Now, by 8.11, we have

$$G(\lambda,T) = \operatorname{null}(T-\lambda I)^{\dim V} = \operatorname{null}\left(T^{-1}-\frac{1}{\lambda}I\right)^{\dim V} = G\left(\frac{1}{\lambda}, T^{-1}\right).$$

---

4. Solution: Suppose $v \in G(\alpha,T) \cap G(\beta,T)$ and suppose by contradiction that $v \neq 0$. Then v, v are generalized eigenvectors corresponding to distinct generalized eigenvalues of T. Now 8.13 implies that v, v is linearly independent, which is clearly a contradiction. Therefore v must be 0.

---

5. Solution: Let $a_0, a_1, \ldots, a_{m-1} \in \mathbb{F}$ such that

$$0 = a_0 v + a_1 Tv + \cdots + a_{m-1}T^{m-1}v$$

Applying T^{m-1} to both sides of the equation above yields

$$0 = a_0 T^{m-1}v,$$

which shows that $a_0 = 0$. Therefore

$$0 = a_1 Tv + \cdots + a_{m-1}T^{m-1}v.$$

Applying T^{m-2} yields

$$0 = a_1 T^{m-1}v,$$

which shows that $a_1 = 0$. Continuing in this fashion, we see that $a_0 = a_1 = \cdots = a_m = 0$. Thus $v, Tv, T^2v, \ldots, T^{m-1}v$ is linearly independent.

---

6. Solution: Suppose by contradiction that $S \in \mathcal{L}(\mathbb{C}^3)$ is a square root of T. Note that $V = \operatorname{null} T^3$. We have

$$\begin{aligned} V = \operatorname{null} T^3 &= \operatorname{null} R^6 \\ = \operatorname{null} R^3 &= \operatorname{null} RT \\ \subset \operatorname{null} R^2T &= \operatorname{null} T^2, \end{aligned}$$

where the third line follows by 8.4. But this a contradiction, since $T^2(z_1,z_2,z_3) = (z_3,0,0)$, we see that $\operatorname{null} T^2 = \{(0,0,z) : z \in \mathbb{C}\}$, then we can't have $V \subset \operatorname{null} T^2$.

---

7. Solution: This follows directly from 8.19 and 5.32.

---

8. Solution: False. Let $V = \mathbb{C}^2$. Define $S, T \in \mathcal{L}(\mathbb{C})$ by

$$\begin{aligned} S(z_1,z_2) &= (0,z_1) \\ T(z_1,z_2) &= (z_2,0). \end{aligned}$$

Both S and T are nilpotent, however $S+T$ is not (its square equals the identity).

---

9. Solution: We have

$$\operatorname{null}(TS)^{\dim V} = \operatorname{null} TS(TS)^{\dim V} = \operatorname{null} T(ST)^{\dim V}S = V,$$

where the first equality follows from 8.4 and the third because $(ST)^{\dim V} = 0$ (by 8.18). Thus $(TS)^{\dim V} = 0$ and so TS is nilpotent.

---

10. Solution: If T is not nilpotent, then $\dim \operatorname{null} T^n < n$ and , by the same reasoning used in 8.4, it follows that $\operatorname{null} T^{n-1} = \operatorname{null} T^n$. Thus, by 8.5, we have

$$V = \operatorname{null} T^{n-1} + \operatorname{range} T^n.$$

Since $\operatorname{range} T^n \subset \operatorname{range} T^{n-1}$, we must also have

$$V = \operatorname{null} T^{n-1} + \operatorname{range} T^{n-1}.$$

Then, by the Fundamental Theorem of Linear Maps (3.22),

$$\dim(\operatorname{null} T^{n-1} + \operatorname{range} T^{n-1}) = \dim V = \dim \operatorname{null} T^{n-1} + \dim \operatorname{range} T^{n-1}.$$

3.78 now implies that $\operatorname{null} T^{n-1} + \operatorname{range} T^{n-1}$ is a direct sum.

---

12. Solution: Suppose $v_1, \ldots, v_n$ is such basis. Then $Nv_1 = 0$, because the the first column of the matrix has 0 in all its entries. The definition of matrix of linear map shows that $Nv_2 \in \operatorname{span}(v_1)$. But this implies that $N^2v_2 = 0$. Similarly, $Nv_3 \in \operatorname{span}(v_1, v_2)$, so $N^3v_3 = 0$.

Continuing like this, we see that $N^j v_j = 0$, for each $j = 1, \ldots, n$. Therefore $N^n = 0$ and so N is nilpotent.

---

13. Solution: It is easy when $\mathbb{F} = \mathbb{C}$, because then V has a basis consisting of eigenvectors of N and for each vector v in this basis we have $0 = N^{\dim V}v = \lambda^{\dim V}v$ for the corresponding eigenvalue λ, which implies that $\lambda = 0$.

More generally, without restricting $\mathbb{F}$ to $\mathbb{C}$, we will prove $N^{\dim V-1} = 0$ and this fact can be used to show $N^{\dim V-2} = 0$, which then can be used to show... and so on until N^1.

Let $\mathcal{N} = N^{\dim V-1}$. Note that $\mathcal{N}$ is also normal and that $\mathcal{N}^2 = 0$. Then, for all $v \in V$,

$$||\mathcal{N}^*\mathcal{N}v||^2 = ||\mathcal{N}\mathcal{N}v|| = 0,$$

where the first equality comes from 7.20. Thus $\mathcal{N}^*\mathcal{N} = 0$. Therefore

$$||\mathcal{N}v||^2 = \langle \mathcal{N}v, \mathcal{N}v\rangle = \langle v, \mathcal{N}^*\mathcal{N}v\rangle = 0,$$

which shows that $\mathcal{N} = 0$.

---

14. Solution: This follows directly from 8.19 and 6.37.

---

15. Solution: By the same reasoning used in the proof of 8.4, it follows that $\dim \operatorname{null} N^{\dim V} \geq \dim V$. Thus $\dim \operatorname{null} N^{\dim V} = \dim V$ and so N is nilpotent. We have $\dim V + 1$ null spaces each of different dimension. Since the sequence

$$\dim \operatorname{null} N^0, \dim \operatorname{null} N^1, \ldots, \dim \operatorname{null} N^{\dim V}$$

must be sorted in strictly increasing order, the only way this can fit is if $\dim \operatorname{null} N^j = j$ for each j.

---

16. Solution: Obviously $V = \operatorname{range} T^0 = \operatorname{range} I$. Let k be a nonnegative integer. Suppose $v \in \operatorname{range} T^{k+1}$. Then $v = T^{k+1}u$ for some $u \in V$. But then $v = T^k(Tu)$. This implies that $v \in \operatorname{range} T^k$.

---

17. Solution: By the Fundamental Theorem of Linear Maps (3.22), we have

$$\dim \operatorname{null} T^m + \dim \operatorname{range} T^m = \dim \operatorname{null} T^{m+1} + \dim \operatorname{range} T^{m+1},$$

which implies that $\dim \operatorname{null} T^m = \dim \operatorname{null} T^{m+1}$ (because $\dim \operatorname{range} T^m = \dim \operatorname{range} T^{m+1}$). Thus, by 8.3, for all $k > m$, we have

$$\dim \operatorname{null} T^m = \dim \operatorname{null} T^{m+k}.$$

Applying the Fundamental Theorem of Linear Maps again to T^m and T^{m+k} we see that

$$\dim \operatorname{range} T^m = \dim \operatorname{range} T^{m+k}.$$

Since $\operatorname{range} T^{m+k} \subset \operatorname{range} T^m$, it follows that $\operatorname{range} T^{m+k} = \operatorname{range} T^m$.

---

18. Solution: This follows directly from the previous exercise and 8.4.

---

19. Solution: This is just a matter of realizing that $\operatorname{null} T^m \subset \operatorname{null} T^{m+1}$ and $\operatorname{range} T^{m+1} \subset \operatorname{range} T^m$ and applying the Fundamental Theorem of Linear Maps.

---

20. Solution: By Exercise 19, $\operatorname{null} T^4 \neq \operatorname{null} T^5$. By Exercise 15, this implies that T is nilpotent.

---

21. Solution: Let $W = \mathbb{F}^\infty \times \mathbb{F}^\infty$ and define $T \in \mathcal{L}(W)$ by

$$T\big((x_1,x_2,x_3,\ldots),(y_1,y_2,y_3,\ldots)\big)=\big((x_2,x_3,\ldots),(0,y_1,y_2,y_3,\ldots)\big),$$

that is, T applies the backward shift operator (call it B) on the first slot and forward shift operator (call it F) on the second slot. Thus, for each positive integer k, we have

$$\operatorname{null} B^k = \{(x_1,x_2,x_3,\ldots) \in \mathbb{F}^\infty : x_j = 0 \text{ for all } j > k\}$$

and

$$\operatorname{range} F^k = \{(x_1,x_2,x_3,\ldots) \in \mathbb{F}^\infty : x_1 = x_2 = \cdots = x_k = 0\}.$$

Moreover $\operatorname{range} B^k = \mathbb{F}^\infty$ and $\operatorname{null} F^k = \{0\}$. Note that $\operatorname{null} B^k \subsetneq \operatorname{null} B^{k+1}$ and $\operatorname{range} F^k \supsetneq \operatorname{range} F^{k+1}$. Thus

$$\operatorname{null} T^k = \{(x,0) \in \mathbb{F}^\infty \times \mathbb{F}^\infty : x \in \operatorname{null} B^k\}$$

and

$$\operatorname{range} T^k = \{(x,y) \in \mathbb{F}^\infty \times \mathbb{F}^\infty : y \in \operatorname{range} T^k\}.$$

Hence $\operatorname{null} T^k \subsetneq \operatorname{null} T^{k+1}$ and $\operatorname{range} T^k \supsetneq \operatorname{range} T^{k+1}$.

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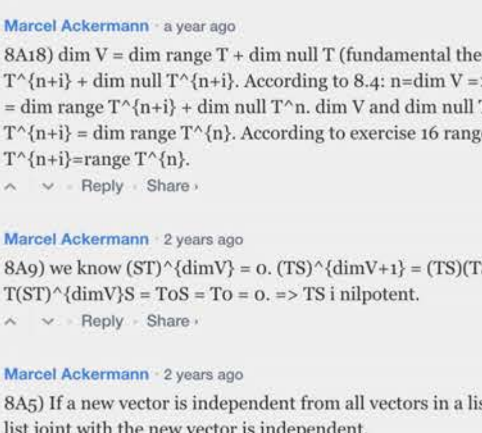
← Chapter 7 Exercise D   Chapter 8 Exercise B →

Chayan Ghosh · 5 months ago
8.A problem 3



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Marcel Ackermann · a year ago
8A18) dim V = dim range T + dim null T (fundamental theorem of LA), so dim V = dim range T^{n+i} + dim null T^{n+i}. According to 8.4: n=dim V => null T^n = null T^{n+i}, so: dim V = dim range T^{n+i} + dim null T^n. dim V and dim null T^n are constant so dim range T^{n+i} = dim range T^{n}. According to exercise 16 range T^{n+i} are subsets, so range T^{n+i}=range T^{n}.
Reply · Share ›

Marcel Ackermann · 2 years ago
8A9) we know (ST)^{dimV} = 0. (TS)^{dimV+1} = (TS)(TS)...(TS) = T(ST)(ST)...(ST)S = T(ST)^{dimV}S = T0S = T0 = 0. => TS i nilpotent.
Reply · Share ›

Marcel Ackermann · 2 years ago
8A5) If a new vector is independent from all vectors in a list and the list is independent then the list joint with the new vector is independent.
By 8.2. and 8.3.: {0}=null T^0 \subset null T^1 ... \subset null T^{m-1} \subset V
Tv \in null T^{m-1}, v \not\in null T^{m-1}, by the previous statement that means that Tv and v are linearly independent because else we would have a contradiction (vector spaces closed by scalar multiplication).
This generalizes to all vectors in the form of the exercises: T^{m-k} \not\in null T^{k-1} but T^{m-k} \in null T^k.
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Marcel Ackermann · 2 years ago
8A1) $G(0,T)={(w,z): w,z \in \mathbb{C}}$, because $T^2(w,z)=(0,0)$
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Chapter 8 Exercise B

Posted on August 2, 2016 by Linearity

1. Solution: By 8.21 (a), $V=G(0,N)$. Since $G(0,N)=\operatorname{null} N^{\dim V}$ (see 8.11), it follows that $N^{\dim V}=0$ and so N is nilpotent.

---

2. Solution: Define $T\in\mathcal{L}(\mathbb{R}^3)$ by

$$T(x,y,z)=(0,-z,x).$$

That is, T squashes vectors onto the yz plane and rotates them counterclockwise by $\pi/2$ radians. So all eigenvectors of T are contained in the x-axis and correspond to the eigenvalue 0. T obviously is not nilpotent.

---

3. Solution: Suppose λ is an eigenvalue of T and $v\in V$ a corresponding eigenvalue. S is surject, so there exists $u\in V$ such that $Su=v$. Then

$$S^{-1}TSu=S^{-1}Tv=\lambda S^{-1}v=\lambda u,$$

which shows that λ is an eigenvalue of $S^{-1}TS$. Hence every eigenvalue of T is an eigenvalue of $S^{-1}TS$. We will prove these eigenvalues have the same multiplicity and it will follow that $S^{-1}TS$ cannot have other eigenvalues (by 8.26).

Suppose $\lambda_1,\dots,\lambda_m$ are the distinct eigenvalues of T. Fix $k\in\{1,\dots,m\}$. Let $v_1,\dots,v_d$ be a basis of $G(\lambda_k,T)$. There exist $u_1,\dots,u_d\in V$ such that $Su_j=v_j$ for each $j=1,\dots,d$. It easy to check that the u's are linearly independent. We have

$$G(\lambda_k,S^{-1}TS)=\operatorname{null}(S^{-1}TS-\lambda_kI)^{\dim V}=\operatorname{null} S^{-1}(T-\lambda_kI)^{\dim V}S,$$

where the first equality comes from 8.11 and the second from Exercise 5 in section 5B. For each j, we have

$$S^{-1}(T-\lambda_kI)^{\dim V}Su_j=S^{-1}(T-\lambda_kI)^{\dim V}v_j=0,$$

where the second equality follows because $v_j\in G(\lambda_k,T)$. This shows that $u_1,\dots,u_d\in G(\lambda_k,S^{-1}TS)$. Hence

$$\dim G(\lambda_k,S^{-1}TS)\geq d=\dim G(\lambda_k,T). \tag{1}$$

By 8.26, we must have

$$\dim G(\lambda_1,T)+\cdots+\dim G(\lambda_m,T)=\dim V \tag{2}$$

and

$$\dim G(\lambda_1,S^{-1}TS)+\cdots+\dim G(\lambda_m,S^{-1}TS)\leq\dim V. \tag{3}$$

(1) and (2) imply that (3) is only possible if

$$\dim G(\lambda_k,S^{-1}TS)=\dim G(\lambda_k,T).$$

Hence, their multiplicieties are the same and $S^{-1}TS$ cannot have other generalized eigenspaces (the ones shown here already eat up the dimension of V).

---

4. Solution: By the same reasoning used in the proof of 8.4, it follows that $\dim\operatorname{null} T^{n-1}\geq n-1$. But $\operatorname{null} T^{n-1}\subset\operatorname{null} T^n=G(0,T)$ (see 8.2 and 8.11). Thus $\dim G(0,T)\geq n-1$ and 0 is an eigenvalue of T.

If $\dim G(0,T)=n$, 8.26 shows that 0 is the only eigenvalue of T. If $\dim G(0,T)=n-1$, there is only space for one more eigenvalue with multiplicty 1.

---

5. Solution: Every eigenvector is also a generalized eigenvector, so in the forward direction we can make a similar argument to that of Exercise 3, where the dimensions only fit if each eigenspace equals its corresponding generalized one (because the former is a subset of the latter). The other direction is obvious from 8.23.

---

6. Solution: The formula for N doesn't really matter here. We only care about the dimension of $\mathbb{F}^5$, which is 5. Using the same reasoning from the proof of 8.31, and because $N^j=0$ for $j\geq 5$, we have

$$\begin{aligned}I+N&=(I+a_1N+a_2N^2+a_3N^3+a_4N^4)\\&=I+2a_1N+(2a_2+a_1^2)N^2+(2a_3+2a_1a_2)N^3+(2a_4+2a_1a_3+a_2^2)N^4\end{aligned}$$

for some $a_1,a_2,a_3,a_4\mathbb{F}$.

Choose

$$a_1=\frac{1}{2},\quad a_2=\frac{-1}{8},\quad a_3=\frac{1}{16}\quad a_4=\frac{-5}{128}$$

and the terms on the second line will collapse to $N+I$. Hence

$$I+\frac{1}{2}N-\frac{1}{8}N^2+\frac{1}{16}N^3-\frac{5}{128}N^4$$

is a square root of $N+I$.

---

7. Solution: One can use the same strategy as in the proof of 8.31 to show that $I+N$ has a cube root for any nilpotent $N\in\mathcal{L}(V)$ and the rest of the proof will be same as the proof of 8.33.

---

8. Solution: If 0 is not an eigenvalue of T, then T^j is injective and surjective for all integers j, which gives the desired result (take $j=n-2$).

Suppose 0 is an eigenvalue of T. Since 3 and 8 are also eigenvalues of T, by 8.26 the multiplicity of 0, namely $\dim G(0,T)$ which equals $\dim\operatorname{null} T^n$, is at most $n-2$. By the same reasoning used in the proof of 8.4, we have $\operatorname{null} T^{n-2}=\operatorname{null} T^n$ (because the $\operatorname{null} T^{n-2}\subset\operatorname{null} T^n$). Exercise 19 of section 8A implies that $\operatorname{range} T^{n-2}=\operatorname{range} T$. Now 8.5 completes the proof.

---

9. Solution: Keep in mind that when we mention the size of an n-by-n matrix here we mean n and not n times n.

Let V be vector space whose dimension equals the size A (or B, since they're the same). Choose a basis of V and define $S,T\in\mathcal{L}(V)$ such that $\mathcal{M}(S)=A$ and $\mathcal{M}(T)=B$. Then $\mathcal{M}(ST)=AB$.

Let d_j equal the size of A_j (or B_j, because they're the same). Consider the list consisting of the first d_1 vectors in the chosen basis. A and B show that the span of these vectors are invariant under S and T.

Similarly, the span of the next d_2 vectors after this list is also invariant under T. Continuing in this fashion, we see that there are m distinct lists of consecutive vectors, with no intersections, in the chosen basis whose spans are invariant under S and T.

Let $U_1,\dots,U_m$ denote such spans. Clearly $\mathcal{M}(S|_{U_j})=A_j$ and $\mathcal{M}(T|_{U_j})=B_j$ for each j. Hence $\mathcal{M}(S|_{U_j}T|_{U_j})=A_jB_j$ and so it easy to see that $\mathcal{M}(ST)$ (which equals AB) has the desired form.

---

10. Solution: Let $v_1,\dots,v_n$ denote a basis of V consisting of generalized eigenvectors of T (which exists by 8.23). Define $\langle\cdot,\cdot\rangle:V\times V\to\mathbb{C}$ by

$$\langle a_1v_1+\cdots+a_nv_n,b_1v_1+\cdots+b_nv_n\rangle=a_1\overline{b_1}+\cdots+a_n\overline{b_n},$$

where the a's and b's are complex numbers. You can check that $\langle\cdot,\cdot\rangle$ is a well defined inner product on V. Thus $v_1,\dots,v_n$ is an orthonormal basis of V. Moreover, the generalized eigenspaces of T are orthogonal to each other. This implies that, if $v\in G(\beta,T)$, then

$$P_{G(\alpha,T)}v=\begin{cases}v, & \text{if } \alpha=\beta\\ 0, & \text{if } \alpha\neq\beta\end{cases} \tag{$*$}$$

where $P_{G(\alpha,T)}$ is the orthogonal projection of V onto $G(\alpha,T)$.

Let $\lambda_1,\dots,\lambda_m$ denote the distinct eigenvalues of T. We have

$$T=T|_{G(\lambda_1,T)}P_{G(\lambda_1,T)}+\cdots+T|_{G(\lambda_m,T)}P_{G(\lambda_m,T)}.$$

For each $j=1,\dots,m$, we can write $T|_{G(\lambda_j,T)}=\lambda_jI+N_j$ where N_j is a nilpotent operator under which $G(\lambda_j,T)$ is invariant (see 8.21 (c)). Therefore

$$\begin{aligned}T&=(\lambda_1I+N_1)P_{G(\lambda_1,T)}+\cdots+(\lambda_mI+N_m)P_{G(\lambda_m,T)}\\&=\underbrace{\lambda_1P_{G(\lambda_1,T)}+\cdots+\lambda_mP_{G(\lambda_m,T)}}_{(4)}+\underbrace{N_1P_{G(\lambda_1,T)}+\cdots+N_mP_{G(\lambda_m,T)}}_{(5)}.\end{aligned}$$

Fix $k\in\{1,\dots,n\}$. Then $v_k\in G(\lambda_j,T)$ for some $j\in\{1,\dots m\}$. ($*$) shows that (4) maps v_k to λ_jv_k. Hence $v_1,\dots,v_n$ is a basis of eigenvectors of (4) and so (4) is diagonalizable. ($*$) also shows that (5) maps v_k to N_jv_k. But $G(\lambda_j,T)$ is invariant under N_j, so ($*$) actually implies that (5) raised to the power of $\dim V$ maps v_k to $N_j^{\dim V}v_k$ which equals 0. Therefore (5) is nilpotent. It is easy to see that (4) and (5) commute (they map v_k to $\lambda_jN_jv_k$, no matter the order), which completes the proof.

---

11. Solution: Suppose T has an upper-triangular matrix with respect to the basis $v_1,\dots,v_n$. Suppose also that λ appears on the j-th diagonal entry of $\mathcal{M}(T)$. Then

$$Tv_j=a_1v_1+\cdots+a_{j-1}v_{j-1}+\lambda v_j$$

for some $a_1,\dots,a_{j-1}\in\mathbb{F}$, and so

$$(T-\lambda I)v_j=a_1v_1+\cdots+a_{j-1}v_{j-1}\in\operatorname{span}(v_1,\dots,v_{j-1}).$$

We have

$$(T-\lambda I)v_{j-1}=c_1v_1+\cdots+(c_{j-1}-\lambda)v_{j-1}$$

for some $c_1,\dots,c_{j-1}\in\mathbb{F}$. If $c_{j-1}-\lambda=0$, then $(T-\lambda I)^2v_j\in\operatorname{span}(v_1,\dots,v_{j-2})$. If $c_{j-1}-\lambda\neq0$, then

$$(T-\lambda I)\left(v_j-\frac{a_{j-1}}{c_{j-1}-\lambda}v_{j-1}\right)\in\operatorname{span}(v_1,\dots,v_{j-2}).$$

We go on, either squaring by squaring $(T-\lambda I)$ or subtracting a vector $u\in\operatorname{span}(v_1,\dots,v_{j-1})$ from v_j in the argument of $(T-\lambda I)$, and we will have $(T-\lambda I)^2(v_j-u)$ in the span of the first $j-3$ vectors of the basis, then in span of the first $n-4$, an so on until it will be in the span of an empty list, that is, $\{0\}$. This means that $v_j-u\in G(\lambda,T)$ for some $u\in\operatorname{span}(v_1,\dots,v_{j-1})$.

Let $\nu_1,\dots,\nu_d$ denote the vectors of the chosen basis of V that correspond to the columns of $\mathcal{M}(T)$ in which λ appears and in the order that they appear on the basis. We can repeat the previous process and find $u_1,\dots,u_d$ such

$$\nu_1-u_1,\dots,\nu_d-u_d\in G(\lambda,T), \tag{6}$$

where each u_k is in the span of the basis vectors that come before ν_k. We claim this list is linearly independent. To see this, fix $k\in\{1,\dots,d\}$. Suppose ν_k is the j-th basis vector, i.e. $\nu_k=v_j$. This means that

$$\operatorname{span}(\nu_1-u_1,\dots,\nu_{k-1}-u_{k-1})\subset\operatorname{span}(v_1,\dots,v_{j-1}).$$

Therefore, we can't have $\nu_k-u_k\in\operatorname{span}(\nu_1-u_1,\dots,\nu_{k-1}-u_{k-1})$, because that would imply that

$$v_j\in\operatorname{span}(v_1,\dots,v_{j-1}),$$

since $u_k\in\operatorname{span}(v_1,\dots,v_{j-1})$. This argument can be repeated for each k. Therefore no vector in (6) is in the span of the previous ones. It follows that the list in (6) is linearly independent.

Hence $\dim G(\lambda,T)\geq d$. By 8.26, the dimension of $G(\lambda,T)$ cannot be greater d, because we have $\dim V$ diagonal entries and each one adds at least 1 to the dimension of some generalized eigenspace. Thus $\dim G(\lambda,T)=d$, completing the proof.

← Chapter 8 Exercise A Chapter 8 Exercise C →

Chapter 8 Exercise C

Posted on August 3, 2016 by Linearity

1. Solution: Because

$$4 = \dim \mathbb{C}^4 = \dim G(3,T) + \dim G(5,T) + \dim G(8,T),$$

it follows that the multiplicities of the eigenvalues of T are at most 2. Thus $p(z) = (z-3)^2(z-5)^2(z-8)^2$ is a polynomial multiple of the characteristic polynomial. Therefore, we must have $p(T) = 0$.

---

2. Solution: We have

$$1 \le \dim G(5,T), \dim G(6,T) \le n-1.$$

Hence $p(z) = (z-5I)^{n-1}(z-6I)^{n-1}$ is a polynomial multiple of the characteristic polynomial. Therefore, $p(T) = 0$.

---

3. Solution: Define $T \in \mathcal{L}(\mathbb{C}^4)$ by

$$T(z_1, z_2, z_3, z_4) = (7z_1, 7z_2, 8z_3, 8z_4).$$

Since $E(7,T) \subset G(7,T)$, $E(8,T) \subset G(8,T)$ and $\mathbb{C}^4 = E(7,T) \oplus E(8,T)$, it follows that $E(7,T) = G(7,T)$ and $E(8,T) = G(8,T)$. We have $\dim E(7,T) = \dim E(8,T) = 2$. Hence, the characteristic polynomial of T is

$$(z-7)^2(z-8)^2.$$

---

4. Solution:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Let

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}.$$

Then

$$\begin{aligned}(A-5I)^2(A-I) &= \begin{pmatrix} -4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \\ &= \begin{pmatrix} -4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.\end{aligned}$$

Define $T \in \mathcal{L}(\mathbb{C}^4)$ by

$$T(z_1, z_2, z_3, z_4) = (z_1 + z_2, 5z_2 + z_3, 5z_3, 5z_4).$$

Then $\mathcal{M}(T) = A$ and the eigenvalues of T are thus 1 and 5 (the entries on the diagonal). Now 8.36 implies that the minimal polynomial of T is a polynomial multiple of $(z-5)(z-1)$. The previous work shows that $(T-5I)(T-I) \ne 0$ and $(T-5I)^2(T-I) = 0$. Hence $(z-5)^2(z-1)$ is the minimal polynomial of T. By Exercise 11 in section 8B, the multiplicity of 1 is 1 and of 5 is 3. Thereby the characteristic polynomial of T is $(z-1)(z-5)^3$.

---

5. Solution: Let

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Then

$$\begin{aligned}(A-I)^2(A-3)A &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}^2 \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.\end{aligned}$$

Define $T \in \mathcal{L}(\mathbb{C}^4)$ by

$$T(z_1, z_2, z_3, z_4) = (0, z_2 + z_3, z_3, 3z_4).$$

Then $\mathcal{M}(T) = A$. The rest of this exercise is almost the same as the previous one.

---

6. Solution: Define $T \in \mathcal{L}(\mathbb{C}^4)$ by

$$T(z_1, z_2, z_3, z_4) = (0, z_1, z_2, 3z_4).$$

Then, the standard basis of $\mathbb{C}^4$ consists of eigenvectors of T corresponding to the eigenvalues 0, 1, 1, 3. Applying $T(T-I)(T-3)$ to each of these basis vectors shows that $T(T-I)(T-3) = 0$. Hence $z(z-1)(z-3)$ is the minimal polynomial of T. We have

$$\mathcal{M}(T) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Thus, by Exercise 11 in section 8B, the characteristic polynomial of T is $z(z-1)^2(z-3)$.

---

7. Solution: By Exercise 4 in section 5B, we have

$$V = \operatorname{null} P \oplus \operatorname{range} P. \tag{1}$$

It is easy to check that if $v \in \operatorname{range} P$ then $Pv = v$. Thus

$$\operatorname{null} P \subset G(0,T) \text{ and } \operatorname{range} P \subset G(1,T).$$

(1) and 8.26 then imply that these inclusions are actually equalities, which gives the desired result.

---

8. Solution: Let p denote the minimal polynomial of T. We have

$$\begin{aligned} T \text{ is invertible} &\iff 0 \text{ is not an eigenvalue of } T \\ &\iff p(0) \ne 0 \\ &\iff \text{the consant term of } p \text{ is nonzero,} \end{aligned}$$

where the equivalence between the first and second lines follows from 8.49.

---

9. Solution: Since

$$4 + 5T - 6T^2 - 7T^3 + 2T^4 + T^5 = 0,$$

multiplying both sides by T^{-5} we get

$$4T^{-5} + 5T^{-4} - 6T^{-3} - 7T^{-2} + 2T^{-1} + I = 0.$$

Hence $4z^5 + 5z^4 - 6z^3 - 7z^2 + 2z + 1$ is a polynomial multiple of the minimal polynomial of T^{-1} (by 8.46). As it turns out, this is actually the minimal polynomial of T^{-1} (with the coefficients multiplied by 4). To see this, suppose by contradiction that it is not. Hence, the minimal polynomial of T has degree at most 4. This means that

$$a_0 I + a_1 T^{-1} + a_2 T^{-2} + a_3 T^{-3} + a_4 T^{-4} = 0$$

for some $a_0, a_1, a_2, a_3, a_4 \in \mathbb{F}$, not all equal 0. Multiplying both sides of the equation above by T^4, we get

$$a_0 T^4 + a_1 T^3 + a_2 T^2 + a_3 T + a_4 I = 0,$$

which is a contradiction, because the minimal polynomial of T has degree 5.

---

10. Solution: Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T and $d_1, \ldots, d_m$ their corresponding multiplicities. Then

$$p(z) = (z-\lambda_1)^{d_1} \cdots (z-\lambda_m)^{d_m}.$$

The eigenvalues of T^{-1} are $\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_m}$ and by Exercise 3 in section 8A they have multiplicities $d_1, \ldots, d_m$. Thus

$$\begin{aligned} q(z) &= \left(z - \frac{1}{\lambda_1}\right)^{d_1} \cdots \left(z - \frac{1}{\lambda_m}\right)^{d_m} \\ &= z^{d_1}\left(1 - \frac{1}{\lambda_1 z}\right)^{d_1} \cdots z^{d_m}\left(1 - \frac{1}{\lambda_m z}\right)^{d_m} \\ &= z^{d_1 + \cdots + d_m}\left(1 - \frac{1}{\lambda_1 z}\right)^{d_1} \cdots \left(1 - \frac{1}{\lambda_m z}\right)^{d_m} \\ &= z^{\dim V}\left(1 - \frac{1}{\lambda_1 z}\right)^{d_1} \cdots \left(1 - \frac{1}{\lambda_m z}\right)^{d_m} \\ &= z^{\dim V}\frac{1}{\lambda_1^{d_1}}\left(\lambda_1 - \frac{1}{z}\right)^{d_1} \cdots \frac{1}{\lambda_m^{d_m}}\left(\lambda_m - \frac{1}{z}\right)^{d_m} \\ &= z^{\dim V}\frac{1}{\lambda_1^{d_1} \cdots \lambda_m^{d_m}}\left(\lambda_1 - \frac{1}{z}\right)^{d_1} \cdots \left(\lambda_m - \frac{1}{z}\right)^{d_m} \\ &= z^{\dim V}\frac{(-1)^{d_1 + \cdots + d_m}}{\lambda_1^{d_1} \cdots \lambda_m^{d_m}}\left(\frac{1}{z} - \lambda_1\right)^{d_1} \cdots \left(\frac{1}{z} - \lambda_m\right)^{d_m} \\ &= z^{\dim V}\frac{(-1)^{d_1 + \cdots + d_m}}{\lambda_1^{d_1} \cdots \lambda_m^{d_m}} p\left(\frac{1}{z}\right) \\ &= z^{\dim V}\frac{1}{p(0)} p\left(\frac{1}{z}\right). \end{aligned}$$

---

11. Solution: Let $z^m + \cdots + a_1 z + a_0$ be the minimal polynomial of T. Then

$$a_0 I = -T^m - \ldots - a_1 T.$$

Multiplying both sides of the equation above by $\frac{1}{a_0}T^{-1}$ gives the desired result (note that a_0 is nonzero by Exercise 8).

---

12. Solution: Suppose V has a basis consisting of eigenvectors of T. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T. Then, it is easy to see that

$$(T - \lambda_1 I) \cdots (T - \lambda_m I) = 0$$

by applying the left side of the equation to each of the basis vectors, because the parentheses commute. By 8.46, $(z-\lambda_1) \cdots (z-\lambda_m)$ is a polynomial multiple of the minimal polynomial of T. This polynomial has no repeated zeros. Hence the minimal polynomial has no repeated zeros.

Conversely, suppose the minimal polynomial of T, call it p, has no repeated zeros. By 8.23, T has a basis of generalized eigenvectors. Let v be one vector in this basis. Then $v \in G(\lambda, T)$ for some eigenvalue λ of T. By 8.49, λ is zero of p. Thus, we can write $p(z) = (z-\lambda)q(z)$ for some polynomial q with $q(\lambda) \ne 0$. We have

$$0 = p(T)v = q(T)(T-\lambda)v.$$

Note that $(T-\alpha)\hat{v} \ne 0$ for all nonzero $\hat{v} \in G(\lambda, T)$ and all $\alpha \in \mathbb{F}$ with $\alpha \ne \lambda$. This implies that $(T-\lambda)v = 0$, since $G(\lambda, T)$ is invariant under every polynomial of T and so $(T - \lambda I)v \ne 0$ implies $q(T)(T - \lambda I) \ne 0$ (because we can factor $q(T)$ and λ is not a zero of q). Therefore v is an eigenvector of T and the basis of V consisting of generalized eigenvectors of T actually consists of eigenvectors of T.

---

13. Solution: If $\mathbb{F} = \mathbb{C}$, this follows directly from the previous exercise from the Complex Spectral Theorem (7.24).

---

14. Solution: As you can see in the solution to Exercise 10, the constant term in the characteristic polynomial, which equals $p(0)$, is is 1 or -1 times the product of the eigenvalues of S raised to the their respective multiplicities. The eigenvalues of S all have absolute value 1 (see 7.43 (b)), hence $|p(0)| = 1$.

---

15. Solution:

(a) Just repeat the proof of 8.40 replacing I with v, T^j with $T^j v$ and n^2 with n.

(b) This is essentially the same as the proof of 8.46.
Let q denote the minimal polynomial of T. Then we also have $q(T)v = 0$. Furthermore, $\deg q \ge \deg p$. By the Division Algorithm for Polynomials (4.8), there exist $s, r \in \mathcal{P}(\mathbb{F})$ such that

$$q = sp + r$$

and $\deg r < \deg p$. This implies that

$$0 = q(T)v = s(T)p(T)v = r(T)v = r(T)v.$$

By the same reasoning used in the proof, it follows that $r = 0$.

---

16. Solution: We have

$$a_0 I + a_1 T + a_2 T^2 + \cdots + a_{m-1}T^{m-1} + T^m = 0.$$

Taking the adjoint of it side yields

$$\overline{a_0} I + \overline{a_1} T^* + \overline{a_2}(T^*)^2 + \cdots + \overline{a_{m-1}}(T^*)^{m-1} + (T^*)^m = 0$$

and we see that the minimal polynomial of T^* is

$$\overline{a_0} + \overline{a_1} z + \overline{a_2} z^2 + \cdots + \overline{a_{m-1}} z^{m-1} + z^m.$$

To see this, suppose by contradiction this is not the minimal polynomial of T^*. Let p denote the minimal polynomial T. Then $\deg p < m$. Because $p(T^*) = 0$, taking the adjoing of each side as we did above shows that $\overline{p}(T) = 0$, where $\overline{p}$ equals p with conjugated coefficients. But $\deg \overline{p} < m$, which is a contradiction because the minimal polynomial of T has degree m.

---

17. Solution: The characteristic polynomial of T, call it q, has degree $\dim V$ (see 8.36) and is a polynomial multiple of the minimal polynomial of T (see 8.48), call it p. Because q is a multiple of q, it follows that $q = ps$ for some polynomial s and, because $\deg q = \deg p$, it follows that $\deg s = 0$. Hence s is a constant. Since they're both monic, $s = 1$ and they should be equal.

---

19. Solution: This follows directly from Exercise 11 in section 8B.

---

20. Solution: This is a bit obvious if you realize that

$$\dim G(\lambda, T) = \dim G(\lambda, T|_{V_1}) + \cdots + G(\lambda, T|_{V_m}) \tag{2}$$

for every $\lambda \in \mathbb{F}$. This is true because the subspaces on the right side of the equation are contained in $G(\lambda, T)$ and because when we write each V_j as a direct sum of generalized eigenspaces of $T|_{V_j}$ in the equation

$$V = V_1 \oplus \cdots \oplus V_m$$

the only way the dimensions will fit is if (2) is true.

This entry was posted in Chapter 8 and tagged Exercise C.

← Chapter 8 Exercise B   Chapter 8 Exercise D →

Chayan Ghosh · 5 months ago
Can anyone post a solution for 8C problem 20?
Reply · Share ›

Marcel Ackermann · a year ago
8C18) How do the eigenvalues look like for this operator?
Tv=\lambda v
=>
-a_0=\lambda v_1
v_1 -a_1 v_n = \lambda v_2
v_2 - a_2 v_n = \lambda v_3
...
subsituting yields: [0] a_0 + a_1 \lambda + a_2 \lambda^2 + ... + a_{n-1} \lambda^{n-1}=0
all eigenvalues fulfill this equation, every polynomial in C has a factorization with \lambda_i the eigenvalues, so we know that [0] is the characteristic polynomial.
The minimal polynomial depends on the a_i values.
Reply · Share ›

Marcel Ackermann · a year ago
8C1) Because the multiplicities of the eigenvalues sum up to the dimensionality of the operator, we know that the multiplicity of one of the eigenvalues has to be two. So there are three possibilities for the minimal polynomial: q_1(z)=(z-3)^2(z-5)(z-8) (etc.). Applying T to these q yields q(T)=0 according to the Cayley-Hamilton theorem (8.37). Multiplying 0 with any polynomial yields the 0 polynomial.
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← Chapter 8 Exercise C   Chapter 9 Exercise A →

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Chapter 8 Exercise D

Posted on August 4, 2016 by Linearity

1. Solution: By Exercise 11 in section 8B the characteristic polynomial is z^4 and by 8.46 this is a polynomial multiple of the minimal polynomial. Since $N^3 \neq 0$, it follows that the minimal polynomial of N is z^4.

---

2. Solution: Similarly, the characteristic polynomial is z^6 and a quick computation shows that the matrix of N equals 0 when raised to the third power, hence the minimal polynomial of N is z^3.

---

3. Solution: Let $\mathcal{M}(N)$ denote the matrix of N with respect to some Jordan basis for N. Then $\mathcal{M}(N)$ is a block diagonal matrix of the form

$$\mathcal{M}(N) = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_p \end{pmatrix}.$$

Because N is nilpotent, 0 is the only eigenvalue of N (see Exercise 7 in section 8A). Hence the diagonal entries of $\mathcal{M}(N)$ are all 0 and each A_j has the following form

$$A_j = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}.$$

Thus every string of consecutive 1's corresponds to one of these blocks and its length is the same as the length of the side of the block minus 1. It is easy to see that if A_j is n-by-n, then $A_j^{n-1} \neq 0$ and $A_j^n = 0$ (think of it as the matrix of an operator on a n dimensional vector space with respect some basis, each basis vector is mapped to the previous one, except the first one obviously, and to send the last one to 0 we have to apply the operator n times). Exercise 9 in section 8B now implies that $\mathcal{M}(N)^m \neq 0$ and $\mathcal{M}(N)^{m+1} = 0$. Hence the minimal polynomial of N is z^{m+1}.

---

4. Solution: The difference is that the order of the blocks in the diagonal is reversed and the 1's appear on the line below the diagonal.

---

5. Solution: By Exercise 9 in section 8B we just need to square each block on the diagonal of the Jordan form of T. In other words, the matrix of T^2 is a block diagonal matrix where each block has the following form

$$\begin{pmatrix} \lambda_j & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{pmatrix}^2 = \begin{pmatrix} \lambda_j^2 & 2\lambda_j & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 2\lambda_j \\ 0 & & & \lambda_j^2 \end{pmatrix}.$$

---

6. Solution: No vector in the span of

$$N^{m_1-1}v_1, \dots, Nv_1, v_1, \dots, N^{m_n-1}v_n, \dots, Nv_n, v_n$$

is in $\operatorname{null} N$, because applying N to any such vector we get a linear of combination of

$$N^{m_1}v_1, \dots, Nv_1, \dots, N^{m_n}v_n, \dots, Nv_n,$$

which is linearly independent. The vectors above are all in $\operatorname{range} N$, therefore, by the Fundamental Theorem of Linear Maps (3.22), the dimension of $\operatorname{null} N$ is at most the dimension of V minus the dimension of the span of the vectors above, that is, at most n. Since $N^{m_1}v_1, \dots, N^{m_n} \in \operatorname{null} N$ is linearly independent and has length n, it must be a basis of $\operatorname{null} N$.

---

7. Solution: Let $\lambda_1, \dots, \lambda_m$ denote the distinct zeros of p and q. Then

$$p(z) = (z-\lambda_1)^{d_1} \cdots (z-\lambda_m)^{d_m}$$

and

$$q(z) = (z-\lambda_1)^{h_1} \cdots (z-\lambda_m)^{h_m}$$

for some positive integers $d_1, \dots, d_m, h_1, \dots, h_m$, where $h_j \geq d_j$ for each j (because q is a polynomial multiple of p). Let A equal the following block diagonal matrix

$$A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{pmatrix}$$

where each A_j is the h_j-by-h_j matrix defiend by

$$A_j = \begin{pmatrix} \lambda_j & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & \ddots & 1 & & & \\ & & & \lambda_j & 0 & & \\ & & & & \lambda_j & \ddots & \\ & & & & & \ddots & 0 \\ & & & & & & \lambda_j \end{pmatrix}$$

where the 1's appear up to the d_j-th column and the 0's fill the rest. Note that A is a $\deg q$-by-$\deg q$ matrix. Define $T \in \mathcal{L}(\mathbf{C}^{\deg q})$ such that the matrix of T with respect to the standard basis is A. Then each λ_j is an eigenvalue of T, because A is upper triangular and λ_j appears on the diagonal of A. Moreover, the multiplicity of λ_j as an eigenvalue of T is h_j, by Exercise 11 in section 8B. Hence, the characteristic polynomial of T is q.

It is easy to see that $(A_j-\lambda_j I)^{d_j-1} \neq 0$ but $(A_j-\lambda_j I)^{d_j} = 0$ (we can use a reasoning similar to that of Exercise 3 to show this). This, together with Exercise 9 from section 8B, shows that the j-th block is nonzero in $(A-\lambda_j I)^{d_j-1}$ and is zero in $(A-\lambda_j I)^{d_j}$ (pay attention to this fact). It follows that

$$(A-\lambda_1 I)^{d_1} \cdots (A-\lambda_m I)^{d_m} = 0.$$

Hence $p(T) = 0$. We claim now p is the minimal polynomial of T. To see this, suppose that it is not. Then we can subtract 1 from one of the exponents in the equation above and it will still hold. Thus, we can write

$$0 = p'(T)(T-\lambda_j)^{d_j-1}$$

for some $j \in \{1, \dots, m\}$ and some polynomial p', with $p'(\lambda_j) \neq 0$. Let $v \in G(\lambda_j, T)$ such that $(T-\lambda_j)^{d_j-1}v \neq 0$ (this v exists due to the fact we pinpointed before). We have

$$0 = p'(T)(T-\lambda_j)^{d_j-1}v,$$

but this is contradiction because $(T-\lambda_j)^{d_j-1}v \in G(\lambda_j, T)$ and λ_j is not an zero of p'.

---

8. Solution: Suppose there does not exist a direct sum decomposition of V into two proper subspaces invariant under T. Therefore, the block diagonal matrix of T with respect to some Jordan basis for T only has one block (the span of the basis vectors that correspond to each block is invariant under T). Therefore T only has one eigenvalue, call it λ. Exercise 3 now implies that $(z-\lambda)^{\dim V}$ is minimal polynomial of T.

For the other direction, we will prove the contrapositive. Suppose we can decompose V into two proper subspaces invariant under T. If T has more than one eigenvalue the result is obvious. Assume T has only one eigenvalue, call it λ. Then $V = G(\lambda, T)$ and there are proper subspaces U_1, U_2 of $G(\lambda, T)$ invariant under T such that $G(\lambda, T) = U_1 \oplus U_2$ with $1 \leq \dim U_1, \dim U_2 < \dim V$. We have that $(z-\lambda)^{\dim U_1}$ and $(z-\lambda)^{\dim U_2}$ are the characteristic polynomials of $T|_{U_1}$ and $T|_{U_2}$. Thus $(T-\lambda)^{\max\{\dim U_1, \dim U_2\}} = 0$ and so the minimal polynomial of T cannot be $z^{\dim V}$.

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This entry was posted in Chapter 8 and tagged Exercise D.

← Chapter 8 Exercise C   Chapter 9 Exercise A →

Marcel Ackermann · a year ago

8D3) It's sufficient to look at the blocks of the block diagonal matrix independently, because when multiplying the block don't interact with each other. We know that for nilpotent operators the eigenvalues are all 0. So lets look at only one upper-triangular matrix with the diagonal 0 and the values of the diagonal 0. What happens if you multiply that matrix with itself? The diagonal gets moved one up. Let m be the longest consecutive number of 1s, then it takes m+1 steps until all ones are "moved out". The characteristic polynomial of a nilpotent operator is z^dimV, so we can confirm that the minimal polynomial is of the right form.

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Marcel Ackermann · a year ago

8D1) eigenvalue is 0 (with multiplicity of 4) according to 5.32: so the characteristic polynomial is (z-0)^4=z^4. The minimal polynomial is z^4, because N^4=0 (and no smaller multiplicity).

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Chapter 9 Exercise A

Posted on September 1, 2016 by Linearity

1. Solution: $V_{\mathbf{C}}$ is clearly closed under addition. We can write each complex number in the form $a+bi$ for some $a, b \in \mathbf{R}$ and we have

$$(a+bi)(u+iv) = (au-bv) + i(av+bu) = (au-bv, av+bu) \in V \times V = V_{\mathbf{C}},$$

where the first equality follows from the definition and the second because the vectors inside both parentheses are in V. Thus $V_{\mathbf{C}}$ is closed under complex scalar multiplication. If 0 is the additive identity on V, then $0+i0$ is the additive identity on $V_{\mathbf{C}}$ because

$$(u+iv) + (0+i0) = (u+0) + i(v+0) = u+iv.$$

One easily checks that the rest of the properties listed in 1.19 are satisfied by $V_{\mathbf{C}}$. Therefore $V_{\mathbf{C}}$ is a complex vector space.

---

2. Solution: Let $u_1, u_2, v_1, v_2 \in V$. Then

$$\begin{aligned} T_{\mathbf{C}}\big((u_1+iv_1)+(u_2+iv_2)\big) &= T_{\mathbf{C}}\big((u_1+u_2)+i(v_1+v_2)\big) \\ &= T(u_1+u_2) + iT(v_1+v_2) \\ &= (Tu_1+iTv_1) + (Tu_2+iTv_2) \\ &= T_{\mathbf{C}}(u_1+iv_1) + T_{\mathbf{C}}(u_2+iv_2). \end{aligned}$$

Hence $T_{\mathbf{C}}$ satisfies the additivity property of linear maps. Now let $u, v \in V$ and $a, b \in \mathbf{R}$. Then

$$\begin{aligned} T_{\mathbf{C}}\big((a+bi)(u+iv)\big) &= T_{\mathbf{C}}\big((au-bv)+i(av+bu)\big) \\ &= T(au-bv) + iT(av+bu) \\ &= (aTu-bTv) + i(aTv+bTu) \\ &= (a+bi)(Tu+iTv) \\ &= (a+bi)T_{\mathbf{C}}(u+iv) \end{aligned}$$

where the fourth line follows from the definition of complex scalar multiplication on $V_{\mathbf{C}}$. Hence $T_{\mathbf{C}}$ satisfies the homogeneity property of linear maps. Therefore $T_{\mathbf{C}}$ is a linear map.

---

3. Solution: The forward direction is obvious, because $\mathbf{R} \subset \mathbf{C}$. For the other direction we can just repeat the same argument used in the proof of 9.4 to show the linear independence of $v_1, \dots, v_n$.

---

4. Solution: Suppose $v_1, \dots, v_m$ spans $V_{\mathbf{C}}$. Let $v \in V$. Then $v+i0 \in V_{\mathbf{C}}$ and we can write

$$v+i0 = \lambda_1 v_1 + \cdots + \lambda_m v_m = (\operatorname{Re}\lambda_1 v_1 + \cdots + \operatorname{Re}\lambda_m v_m) + i(\operatorname{Im}\lambda_1 v_1 + \cdots + \operatorname{Im}\lambda_m v_m)$$

for some $\lambda_1, \dots, \lambda_m \in \mathbf{C}$. The equation above implies that

$$\operatorname{Re}\lambda_1 v_1 + \cdots + \operatorname{Re}\lambda_m v_m = v$$

Therefore $v \in \operatorname{span}(v_1, \dots, v_m)$. Hence $v_1, \dots, v_m$ spans V.
Conversely, suppose $v_1, \dots, v_m$ spans V. Then we can reduce this list to a basis of V. But a basis of V is also a basis $V_{\mathbf{C}}$. Therefore we can reduce $v_1, \dots, v_m$ to a basis of $V_{\mathbf{C}}$ and this implies that it spans $V_{\mathbf{C}}$.

---

5. Solution: Suppose $u, v \in V$. Then

$$\begin{aligned} (S+T)_{\mathbf{C}}(u+iv) &= (S+T)u + i(S+T)v \\ &= (Su+iSv) + (Tu+iTv) \\ &= S_{\mathbf{C}}(u+iv) + T_{\mathbf{C}}(u+iv). \end{aligned}$$

Thus $(S+T)_{\mathbf{C}} = S_{\mathbf{C}} + T_{\mathbf{C}}$. Now suppose $\lambda \in \mathbf{R}$. Then

$$(\lambda T)_{\mathbf{C}}(u+iv) = \lambda Tu + \lambda iTv = \lambda(Tu+iTv) = \lambda T_{\mathbf{C}}(u+iv).$$

Therefore $(\lambda T)_{\mathbf{C}} = \lambda T_{\mathbf{C}}$.

---

6. Solution: Suppose $T_{\mathbf{C}}$ is invertible. Then, because $T_{\mathbf{C}}$ is surjective, for every $w \in V$ there exist $u, v \in V$ such that $T_{\mathbf{C}}(u+iv) = w+i0$. This means that

$$Tu + iTv = w + i0.$$

Thus $Tu = w$. Hence T is surjective and therefore invertible.

Conversely, suppose T is invertible. Let $u, v \in V$. By surjectivity of T, there exist $\hat{u}, \hat{v} \in V$ such that $T\hat{u} = u$ and $T\hat{v} = v$. Thus

$$T_{\mathbf{C}}(\hat{u} + i\hat{v}) = T\hat{u} + iT\hat{v} = u + iv.$$

Since u and v were arbitrary, it follows that $T_{\mathbf{C}}$ is surjective and therefore invertible.

---

7. Solution: Suppose $N_{\mathbf{C}}$ is nilpotent. By 8.18, for any $v \in V$, we have

$$0 + i0 = (N_{\mathbf{C}})^{\dim V}(v+i0) = N^{\dim V}v + i0$$

where the second equality follows from 9.9. This implies that $N^{\dim V}v = 0$. Thus N is nilpotent.
The other direction is obvious from 9.9 and the definitions.

---

8. Solution: If $T_{\mathbf{C}}$ had a nonreal eigenvalue λ, then by 9.11 and 9.16 it would have 4 eigenvalues, namely $5, 7, \lambda, \bar{\lambda}$, which is a contradiction because the dimension of $(\mathbf{R}^3)_{\mathbf{C}}$ equals 3 (see 9.4 (b)) and $T_{\mathbf{C}}$ has at most 3 eigenvalues (see 5.13).

---

9. Solution: Suppose by contradiction that $T \in \mathcal{L}(\mathbf{R}^7)$ is such that T^2+T+I is nilpotent. Then by Exercise 7 $(T^2+T+I)_{\mathbf{C}}$ is also nilpotent and its minimal polynomial is of the form z^j for some positive integer j (because 0 is its only eigenvalue, see Exercise 7 in section 8A and 8.49). We have

$$z^2+z+1 = \left(z - \frac{-1+i\sqrt{3}}{2}\right)\left(z - \frac{-1-i\sqrt{3}}{2}\right).$$

Define $p \in \mathcal{P}(\mathbf{R})$ by

$$p(z) = (z^2+z+1)^j.$$

Then p has no real roots and $p(T_{\mathbf{C}}) = 0$. This is a contradiction, because p must be a polynomial multiple of the minimal polynomial of $T_{\mathbf{C}}$ (by 8.46) and the minimal polynomial of $T_{\mathbf{C}}$ has at least one real root (see 9.19, 9.11 and 8.49).

---

10. Solution: Choose $\lambda \in \mathbf{C}$ such that $\lambda^2+\lambda+1 = 0$ and define $T \in \mathcal{L}(\mathbf{C}^7)$ by

$$Tv = \lambda v$$

for all $v \in \mathbf{C}^7$. Then

$$(T^2+T+I)v = T^2v + Tv + Iv = (\lambda^2+\lambda+1)v = 0.$$

Therefore T^2+T+I is nilpotent.

---

11. Solution: From the definitions, it is easy to see that

$$(T_{\mathbf{C}}^2 + bT_{\mathbf{C}} + cI) = 0.$$

Therefore z^2+bz+c is polynomial multiple of the minimal polynomial of $T_{\mathbf{C}}$ (see 8.46). Note that, because this polynomial has real coefficients, it either has two real roots or two nonreal roots. Thus T has an eigenvalue if and only if $T_{\mathbf{C}}$ has a real root (see 9.10), which happens if and only if z^2+bz+c has a real root (by the previous reasoning and 8.49), which happens if and only if $b^2 \geq 4c$.

---

12. Solution: By Exercise 3 in section 8D, the minimal polynomial of $T^2+bT+cI$ is of the form z^m for some positive integer m. Let the p be the polynomial defined by

$$p(z) = (z^2+bz+c)^m.$$

Then p has no real roots (because z^2+bz+c does not) and $p(T_{\mathbf{C}}) = 0$. Thus 8.46 and 8.49 imply that $T_{\mathbf{C}}$ has no real eigenvalues. Now 9.10 implies that T has no eigenvalues.

---

13. Solution: We have

$$z^2+bz+c = (z-\lambda)(z-\bar{\lambda})$$

for some nonreal $\lambda \in \mathbf{C}$. Suppose $v \in \operatorname{null}(T_{\mathbf{C}}^2+bT_{\mathbf{C}}+cI)^j$. Then

$$0 = (T_{\mathbf{C}}^2+bT_{\mathbf{C}}+cI)^j v = (T_{\mathbf{C}}-\lambda I)^j(T_{\mathbf{C}}-\bar{\lambda}I)^j v.$$

It is easy to check that v is of the form $v = u+w$ where $u \in G(\lambda, T_{\mathbf{C}})$ and $w \in G(\bar{\lambda}, T_{\mathbf{C}})$. We have

$$0 = (T_{\mathbf{C}}-\lambda I)^j(T_{\mathbf{C}}-\bar{\lambda}I)^j v = (T_{\mathbf{C}}-\lambda I)^j(T_{\mathbf{C}}-\bar{\lambda}I)u + (T_{\mathbf{C}}-\lambda I)^j(T_{\mathbf{C}}-\bar{\lambda}I)^j w.$$

Since each generalized eigenspace is invariant under $T_{\mathbf{C}}$ and every vector in V (the left side of the equation) can be written uniquely as linear combination of generalized eigenvectors that correspond to distinct eigenvalues, the equation above implies that

$$(T_{\mathbf{C}}-\lambda I)^j(T_{\mathbf{C}}-\bar{\lambda}I)^j u = 0 \text{ and } (T_{\mathbf{C}}-\lambda I)^j(T_{\mathbf{C}}-\bar{\lambda}I)^j w = 0.$$

Hence $u \in \operatorname{null}(T_{\mathbf{C}}-\lambda I)^j$ and $w \in \operatorname{null}(T_{\mathbf{C}}-\bar{\lambda}I)^j$. Therefore

$$\operatorname{null}(T_{\mathbf{C}}^2+bT_{\mathbf{C}}+cI)^j \subset \operatorname{null}(T_{\mathbf{C}}-\lambda I)^j \oplus \operatorname{null}(T_{\mathbf{C}}-\bar{\lambda}I)^j.$$

The inclusion in the other direction is obvious. Thus

$$\operatorname{null}(T_{\mathbf{C}}^2+bT_{\mathbf{C}}+cI)^j = \operatorname{null}(T_{\mathbf{C}}-\lambda I)^j \oplus \operatorname{null}(T_{\mathbf{C}}-\bar{\lambda}I)^j,$$

which gives us that

$$\dim\operatorname{null}(T_{\mathbf{C}}^2+bT_{\mathbf{C}}+cI)^j = \dim\operatorname{null}(T_{\mathbf{C}}-\lambda I)^j + \dim\operatorname{null}(T_{\mathbf{C}}-\bar{\lambda}I)^j.$$

By 9.12, the two dimensions on the right side of the equation above are equal. Therefore the left side is even. One easily checks that

$$(\operatorname{null}(T^2+bT+cI))_{\mathbf{C}} = \operatorname{null}(T_{\mathbf{C}}^2+bT_{\mathbf{C}}+Ic).$$

Now 9.4 (b) yields the desired result.

---

14. Solution: Because it is nilpotent, the minimal polynomial of $T_{\mathbf{C}}^2+T_{\mathbf{C}}+I$ is of the form z^m for some positive integer m. We have

$$z^2+z+1 = (z-\lambda)(z-\bar{\lambda})$$

for some nonreal $\lambda \in \mathbf{C}$. Thus

$$0 = (T_{\mathbf{C}}^2+T_{\mathbf{C}}+I)^m = (T_{\mathbf{C}}-\lambda)^m(T_{\mathbf{C}}-\bar{\lambda})^m.$$

This, together with 9.16, implies that the eigenvalues of $T_{\mathbf{C}}$ are λ and $\bar{\lambda}$, with equal multiplicities, namely 4. The characteristic polynomial of $T_{\mathbf{C}}$, and of T as well by definition, is therefore

$$(z-\lambda)^4(z-\bar{\lambda})^4.$$

which equals $(z^2+z+1)^4$. By the Cayley-Hamilton Theorem (9.24), it follows that

$$(T^2+T+I)^4 = 0.$$

---

15. Solution: Suppose U is a subspace of V invariant under T. Because $T|_U$ doesn't have an eigenvalue, 9.19 implies that U has even dimension.

---

16. Solution: Suppose T is such that $T^2 = -I$. Then clearly T does not have an eigenvalue. Then 9.19 implies that $\dim V$ is even.
Conversely, suppose V has even dimension. Let $v_1, \dots, v_n, u_1, \dots, u_n$ be a basis of V. Define $T \in \mathcal{L}(V)$ by

$$Tv_j = -u_j, Tu_j = v_j$$

for each $j = 1, \dots, n$. We have

$$T^2v_j = -Tu_j = -v_j \text{ and } T^2u_j = Tv_j = -u_j.$$

Thus $T^2 = -I$.

---

17. Solution:
(a) Obviously V closed under addition and complex scalar multiplication. One can easily check that the other properties in 1.19 are satisfied.
(b) Let n be the integer such that $\dim V = 2n$ and consider the following process.

- Step 1: Choose a nonzero $v_1 \in V$. Then v_1, Tv_1 is linearly independent in V as real vector space because v_1 is not an eigenvector of T (because T has no eigenvectors). Set $U_1 = \operatorname{span}(v_1, Tv_1)$. Then $\dim U_1 = 2$ and U_1 is invariant under T.
- Step j: If $j = n+1$, stop the process. We have that

  $$\dim U_{j-1} = 2(j-1) \leq 2n-2 < \dim V,$$

  U_{j-1} is invariant under T and

  $$v_1, Tv_1, \dots, v_{j-1}, Tv_{j-1}$$

  is a basis of U_{j-1}. Hence there exists a nonzero $w \in V$ such that $w \notin U_{j-1}$. Since T is surjective, there exists $v_j \in V$ such that $Tv_j = w$. Thus $v_j \notin U_{j-1}$, because U_{j-1} being invariant under T would imply $w \in U_{j-1}$. Moreover, the list

  $$v_1, Tv_1, \dots, v_j, Tv_j \tag{1}$$

  is linearly independent. To see this, let $a_1, \dots, a_j, c_1, \dots, c_j \in \mathbf{R}$ such that

  $$a_1v_1 + c_1Tv_1 + \cdots + a_jv_j + c_jTv_j = 0$$

  We already know that v_j is not in the span of the previous vectors, so $c_j = 0$ implies $a_j = 0$ which implies that the rest of the a's and c's is 0. Assume by contradiction $c_j \neq 0$. We can write the equation above as

  $$a_1v_1 + c_1Tv_1 + \cdots + a_{j-1}v_{j-1} + c_{j-1}Tv_{j-1} = -a_jv_j - c_jTv_j. \tag{2}$$

  Applying T to both sides we get

  $$a_1Tv_1 - c_1v_1 + \cdots + a_{j-1}Tv_{j-1} - c_{j-1}v_{j-1} = -a_jTv_j + c_jv_j. \tag{3}$$

  Multiplying (3) by $\frac{a_j}{c_j}$ and summing with (2) shows us that

  $$\left(\frac{-a_j^2}{c_j} - c_j\right)Tv_j \in U_{j-1}.$$

  Hence the parentheses above equals 0. This means that $-a_j^2 = c_j^2$, which is only possible if $a_j = c_j = 0$ and contradicts our assumption that $c_j \neq 0$. Therefore the list in (1) is linearly independent. Set

  $$U_j = \operatorname{span}(v_1, Tv_1, \dots, v_j, Tv_j).$$

  Then $\dim U_j = 2j$ and U_j is invariant under T.

At the end of the process, we will have constructed a subspace U_n of V which has a basis

$$v_1, Tv_1, \dots, v_n, Tv_n$$

and dimension $2n$. Thus $U_n = V$ and the above list is a basis of V. Clearly $v_1, \dots, v_n$ spans V as a complex vector space. Futhermore, it is linearly independent in V as a complex vector space, because if a (complex) linear combination of it equals 0, then a linear combination of the list above equals 0, which implies that the coefficients are 0. Therefore V as a complex vector space has dimension n, completing the proof.

---

18. Solution: The proof that (a) ⇒ (b) below is basically a rewrite of the proof 5.27 with a few tweakings. Suppose (a) holds. We will prove (b) by induction on $\dim V$. First, if $\dim V = 1$, then the matrix of T with respect to any basis only has one entry and thus (b) trivially holds. Assume that $\dim V > 1$ and that (a) implies (b) for all vector spaces of lower dimension.
Let $\lambda \in \mathbf{R}$ be an eigenvalue of T, which exists by (a) and 9.11. Define

$$U = \operatorname{range}(T-\lambda I).$$

Because $\dim\operatorname{null}(T-\lambda I) \geq 1$, the Fundamental Theorem of Linear Maps implies that $\dim U < \dim V$. Futhermore, U is invariant under T and so $U_{\mathbf{C}}$ is invariant under $(T|_U)_{\mathbf{C}}$. All the eigenvalues of $(T|_U)_{\mathbf{C}}$ are real. To see this, suppose $\lambda \in \mathbf{C}$ is an eigenvalue of $(T|_U)_{\mathbf{C}}$ and $u_1 + iu_2$ a corresponding eigenvector for some $u_1, u_2 \in U$. Then

$$\lambda(u_1+iu_2) = (T|_U)_{\mathbf{C}}(u_1+iu_2) = (T|_U)u_1 + i(T|_U)u_2 = Tu_1 + iTu_2 = T_{\mathbf{C}}(u_1+iu_2),$$

which implies that λ is an eigenvalue of $T_{\mathbf{C}}$ and thus must be real. By the induction hypothesis there exists a basis $u_1, \dots, u_m$ of U with respect to which the matrix of $T|_U$ is upper triangular. From 5.26, we get that

$$Tu_j = (T|_U)u_j \in \operatorname{span}(u_1, \dots, u_j). \tag{4}$$

Extend it to a basis $u_1, \dots, u_m, v_1, \dots, v_n$ of V. For each k we have

$$Tv_k = (T-\lambda I)v_k + \lambda v_k.$$

The definition of U shows that $(T-\lambda I)v_k \in U$. Hence, the equation above shows that

$$Tv_k \in \operatorname{span}(u_1, \dots, u_m, v_1, \dots, v_k). \tag{5}$$

Using (4), (5) and 5.26, we conclude that the matrix of T with respect to basis $u_1, \dots, u_m, v_1, \dots, v_n$ of V is upper triangular.

9.7 and 5.32 show that (b) implies (a). Hence the equivalence between (a) and (b) is established. Now it suffices to show that (a) and (c) are equivalent.
Suppose (a) holds. Let $\lambda \in \mathbf{R}$ be an eigenvalue of $T_{\mathbf{C}}$ and let d denote its multiplicity. Let $u_1+iv_1, \dots, u_d+iv_d$ be a basis of $G(\lambda, T_{\mathbf{C}})$. For each $j = 1, \dots, d$, we have

$$(T-\lambda)^{\dim V}u_j + i(T-\lambda)^{\dim V}v_j = ((T-\lambda)^{\dim V})_{\mathbf{C}}(u_j+iv_j) = (T_{\mathbf{C}}-\lambda)^{\dim V}(u_j+iv_j) = 0.$$

This implies that $u_1, v_1, \dots, u_d, v_d \in G(\lambda, T)$. Since $G(\lambda, T_{\mathbf{C}})$ is contained in the complex span of this list, it follows that the dimension of this complex span is at least d. 9.4 (b) now implies that the dimension of the real span of $u_1, v_1, \dots, u_d, v_d$ is at least d. Hence $\dim G(\lambda, T) \geq d$. Since the sum of multiplicities of the generalized eigenspaces of T is at most $\dim V$, it follows that the only way the dimensions will fit is if $\dim G(\lambda, T) = d$.
Thus, summing all the generalized eigenspaces of T we get a subspace of V with dimension $\dim V$, namely V. Taking a basis of generalized eigenspace and putting this bases together gives a basis consisting of generalized eigenvectors of V. Thus (c) holds.
Now suppose (c) holds. Then $T_{\mathbf{C}}$ has real eigenvalues, whose sum of multiplicities already equal $\dim V$. Hence $T_{\mathbf{C}}$ cannot have a nonreal eigenvalue. Thus (a) holds.

---

19. Solution: By the same reasoning used in the proof 8.4, it follows that $\dim\operatorname{null} T^{n-1} \geq n-1$. Thus, the multiplicty of 0 as an eigenvalue of T, and $T_{\mathbf{C}}$ as well, is at least $n-1$. Hence $T_{\mathbf{C}}$ has at most another eigenvalue. Since nonreal eigenvalues of $T_{\mathbf{C}}$ come in pairs (see 9.16), this other eigenvalue of $T_{\mathbf{C}}$ must be real.

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Solutions to Linear Algebra Done Right

Third Edition

Chapter 9 Exercise B

Posted on September 2, 2016 by Linearity

1. Solution: Choose an orthonormal basis of $\mathbb{R}^3$ that puts the matrix of S in the form given by 9.36. Since $\mathcal{M}(S)$ is a 3-by-3 matrix, one of the diagonal blocks is a 1-by-1 matrix containing 1 or -1. Hence $Sx = x$ or $Sx = -x$ for some nonzero vector x in the chosen basis. Applying S again in the two cases gives $S^2x = x$, as desired.
Geometrically speaking, an isometry on $\mathbb{R}^3$ is a rotation about an axis, perhaps with a reflexion through a plane orthogonal to the axis. Hence an isometry on $\mathbb{R}^3$ either sends the vectors in the axis to themselves or to their reflexion. If it's the first case, we already have what we wanted to prove, if it's the second, applying the isometry again sends the reflexions of the vectors back to the vectors themselves.

---

2. Solution: This basically the same as the previous exercise. Every operator on an odd-dimensional vector space has n-by-n matrix for some odd integer n, hence one of the diagonal blocks is a 1-by-1 matrix containing 1 or -1. The basis vector that corresponds to column where this vector appears is an eigenvector corresponding to 1 or -1.

---

3. Solution: For $u_1, u_2, v_1, v_2 \in V$ we have

$$\begin{aligned}
&\langle (u_1 + iv_1) + (u_1 + iv_2), x + iy\rangle \\
&= \langle (u_1 + u_2) + i(v_1 + v_2), x + iy\rangle \\
&= \langle u_1 + u_2, x\rangle + \langle v_1 + v_2, y\rangle + \big(\langle v_1 + v_2, x\rangle - \langle u_1 + u_2, y\rangle\big)i \\
&= \langle u_1, x\rangle + \langle u_2, x\rangle + \langle v_1, y\rangle + \langle v_2, x\rangle + \big(\langle v_1, x\rangle - \langle u_1, y\rangle\big)i + \big(\langle v_2, x\rangle - \langle u_2, y\rangle\big)i \\
&= \langle u_1 + iv_1, x + iy\rangle + \langle u_2 + iv_2, x + iy\rangle.
\end{aligned}$$

Therefore it satisfies additivity in the first slot. For $a, b \in \mathbb{R}$, we have

$$\begin{aligned}
\langle (a + bi)(u + iv), x + iy\rangle &= \langle (au - bv) + i(av + bu), x + iy\rangle \\
&= \langle au - bv, x\rangle + \langle av + bu, y\rangle + \big(\langle av + bu, x\rangle - \langle au - bv, y\rangle\big)i \\
&= \langle au, x\rangle - \langle bv, x\rangle + \langle av, y\rangle + \langle bu, y\rangle + \big(\langle av, x\rangle - \langle au, y\rangle\big)i + (\langle bu, x\rangle + \langle bv, y\rangle)i \\
&= \langle au, x\rangle + \langle av, y\rangle + \big(\langle av, x\rangle - \langle au, y\rangle\big)i - \langle bv, x\rangle + \langle bu, y\rangle + (\langle bu, x\rangle + \langle bv, y\rangle)i \\
&= a\langle u + iv, x + iy\rangle + bi\big(i\langle v, x\rangle - i\langle u, y\rangle + \langle u, x\rangle + \langle v, y\rangle\big) \\
&= a\langle u + iv, x + iy\rangle + bi\langle u + iv, x + iy\rangle \\
&= (a + bi)\langle u + iv, x + iy\rangle
\end{aligned}$$

Hence homogeneity in the first slot is satisfied. For positivity, we have

$$\langle u + iv, u + iv\rangle = \langle u, u\rangle + \langle v, v\rangle + \big(\langle v, u\rangle - \langle u, v\rangle\big)i = \langle u, u\rangle + \langle v, v\rangle \geq 0,$$

where the second equality follows because V is a real inner product space. The equation above also displays definiteness, because the left side equals 0 if and only if $u = v = 0$, in other words, if $u + iv = 0$. For conjugate symmetry, we have

$$\begin{aligned}
\langle u + iv, x + iy\rangle &= \overline{\langle u, x\rangle + \langle v, y\rangle - (\langle v, x\rangle - \langle u, y\rangle)i} \\
&= \overline{\langle x, u\rangle + \langle y, v\rangle - (\langle x, v\rangle - \langle y, u\rangle)i} \\
&= \overline{\langle x + iy, u + iv\rangle}.
\end{aligned}$$

---

4. Solution: Choose an orthonormal basis of V. Then this is also a basis of $V_{\mathbf{C}}$, see (9.4). We have

$$\mathcal{M}(T_{\mathbf{C}}) = \mathcal{M}(T) = \mathcal{M}(T^*) = \mathcal{M}((T^*)_{\mathbf{C}}) = \mathcal{M}((T_{\mathbf{C}})^*),$$

where the first and third equalities follow from 9.7, the second from 7.10 and the fourth from 9.30 (c) (take $U = V$). This implies that $T_{\mathbf{C}} = (T_{\mathbf{C}})^*$.

---

5. Solution: Suppose V is a real inner product space and $T \in \mathcal{L}(V)$ is self-adjoint. Then $T_{\mathbf{C}}$ is a self-adjoint operator on the complex inner product space $V_{\mathbf{C}}$ defined in Exercise 3. By the Complex Spectral Theorem, there is an orthonormal basis $e_1 + if_1, \dots, e_n + if_n$ of $V_{\mathbf{C}}$ consisting of eigenvectors of $T_{\mathbf{C}}$. By 7.13, the eigenvalues of $T_{\mathbf{C}}$ are all real. Thus, there are $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that

$$Te_j + iTf_j = T_{\mathbf{C}}(e_j + if_j) = \lambda_j e_j + i\lambda_j Tf_j$$

for each j. The equation above shows that the e's and f's are eigenvectors of T. Fix $j \in \{1, \dots, n\}$. Let $e_1{}' + if_1{}', \dots, e'_{d(\lambda_j)} + if'_{d(\lambda_j)}$ denote the basis vectors that correspond to λ_j, where $d(\lambda_j) = \dim E(\lambda_j, T_{\mathbf{C}})$. Then the list

$$e_1{}', f_1{}', \dots, e'_{d(\lambda_j)}, f'_{d(\lambda_j)} \in E(\lambda_j, T)$$

is in $E(\lambda_j, T)$. Since $E(\lambda, T_{\mathbf{C}})$ is contained in the complex span of the list above, it follows that the dimension of the compelx span of the list above is at least $d(\lambda_j)$. Thus, by 9.4, the dimension of the real span of the list above is at least $d(\lambda_j)$. Hence $\dim E(\lambda_j, T) \geq d(\lambda_j)$. Because

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_n, T) \leq \dim V$$

and

$$d(\lambda_1) + \cdots + d(\lambda_n) = \dim V$$

we must have $\dim E(\lambda_j, T) = d(\lambda_j)$ for each j. The sum of eigenspaces corresponding to different eigenvalues is a direct sum. The equation above thus shows that the sum of the eigenspaces of T has the same dimension as V. This means that

$$E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_n, T) = V.$$

For each eigenspace of T, we can select an orthonormal basis. Putting these bases together, we get a basis of V (by the equation above) consisting of eigenvectors of T. By 7.22, this basis is orthonormal.

---

6. Solution: Define $T \in \mathcal{L}(\mathbb{R}^2)$ by

$$T(x, y) = (x + y, y).$$

Then $\operatorname{span}((1, 0))$ is invariant under T, however, its orthogonal complement, namely $\operatorname{span}((0, 1))$, is not, because

$$T(0, 1) = (1, 1) \notin \operatorname{span}((0, 1)).$$

---

7. Solution: We have

$$\mathcal{M}(T_1 \cdots T_m) = \mathcal{M}(T_1) \cdots \mathcal{M}(T_m) = \mathcal{M}(T),$$

where the second equality follows from Exercise 9 in section 8B. The left and right side of the equation above imply that $T_1 \cdots T_m = T$.

---

8. Solution: Define

$$e_j = \frac{\cos jx}{\sqrt{\pi}} \text{ and } f_j = \frac{\sin jx}{\sqrt{\pi}}$$

for each $j = 1, \dots, n$. By Exercise 4 in section 6B, the list $\frac{1}{\sqrt{2\pi}}, f_1, e_1, \dots, f_n, e_n$ is an orthonormal basis of V. Note that $D\frac{1}{\sqrt{2\pi}} = 0$, $Df_j = je_j$ and $De_j = -jf_j$. Hence, the matrix of D with respect to this basis has the desired form, where the first block is the 1-by-1 matrix $\begin{pmatrix} 0 \end{pmatrix}$ and others are of the form

$$\begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}$$

for some $j \in \{1, \dots, n\}$.

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This entry was posted in Chapter 9 and tagged Exercise B.

← Chapter 9 Exercise A Chapter 10 Exercise A →

Chapter 10 Exercise A

Posted on October 1, 2016 by Linearity

1. Solution: If T is invertible, then there exists $S \in \mathcal{L}(V)$ such that $TS = ST = I$. Then it follows from 10.4 that

$$\mathcal{M}(S,(v_1,\cdots,v_n))\mathcal{M}(T,(v_1,\cdots,v_n)) = \mathcal{M}(ST,(v_1,\cdots,v_n)) = I$$

$$\mathcal{M}(T,(v_1,\cdots,v_n))\mathcal{M}(S,(v_1,\cdots,v_n)) = \mathcal{M}(TS,(v_1,\cdots,v_n)) = I.$$

Hence $\mathcal{M}(T,(v_1,\cdots,v_n))$ is invertible.

If $\mathcal{M}(T,(v_1,\cdots,v_n))$ is invertible, then there exists $B \in \mathbb{F}^{n,n}$ such that

$$B\mathcal{M}(T,(v_1,\cdots,v_n)) = \mathcal{M}(T,(v_1,\cdots,v_n))B.$$

Note that by 3.60, we have an isomorphism between $\mathcal{L}(V)$ and $\mathbb{F}^{(n,n)}$ by taking $W = V$ and $w_i = v_i$ for all i, we can choose $S \in \mathcal{L}(V)$ such that $B = \mathcal{M}(S,(v_1,\cdots,v_n))$. Again by 10.4, we have

$$\mathcal{M}(ST,(v_1,\cdots,v_n)) = B\mathcal{M}(T,(v_1,\cdots,v_n)) = I,$$

$$\mathcal{M}(TS,(v_1,\cdots,v_n)) = \mathcal{M}(T,(v_1,\cdots,v_n))B = I.$$

Therefore $ST = I$ and $TS = I$, hence T is invertible.

See also Linear Algebra Done Right Solution Manual Chapter 10 Problem 1.

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2. Solution: Let V be $\mathbb{F}^n$, where n is the number of rows of A. Fix a basis $v_1,\cdots,v_n$ of V. By 3.60, we choose $T,S \in \mathcal{L}(V)$ such that

$$\mathcal{M}(T,(v_1,\cdots,v_n)) = A \quad \text{and} \quad \mathcal{M}(S,(v_1,\cdots,v_n)) = B.$$

Since $AB = I$, it follows from 10.4 that

$$\mathcal{M}(TS,(v_1,\cdots,v_n)) = \mathcal{M}(T,(v_1,\cdots,v_n))\mathcal{M}(S,(v_1,\cdots,v_n)) = AB = I,$$

hence $TS = I$. By Problem 10 of Exercise 3D, we have $ST = I$. Again by 10.4, we have

$$BA = \mathcal{M}(S,(v_1,\cdots,v_n))\mathcal{M}(T,(v_1,\cdots,v_n)) = \mathcal{M}(ST,(v_1,\cdots,v_n)) = I.$$

See also Linear Algebra Done Right Solution Manual Chapter 10 Problem 2.

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3. Solution: See Linear Algebra Done Right Solution Manual Chapter 10 Problem 3.

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4. Solution: See Linear Algebra Done Right Solution Manual Chapter 10 Problem 4.

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5. Solution: Let $V = \mathbb{C}^n$, where n is the number of the rows of B. Choose a basis $v_1,\cdots,v_n$ of V. Let $T \in \mathcal{L}(V)$ so that

$$\mathcal{M}(T,(v_1,\cdots,v_n)) = B.$$

By 8.29, there exists another basis $w_1,\cdots,w_n$ of V such that $\mathcal{M}(T,(w_1,\cdots,w_n))$ is an upper-triangular matrix.

Let $A = \mathcal{M}(T,(w,\cdots,w_n),(v_1,\cdots,v_n))$, then A is intertible. It follows from 10.7 that

$$\mathcal{M}(T,(w,\cdots,w_n) = A^{-1}\mathcal{M}(T,(v_1,\cdots,v_n))A = A^{-1}BA.$$

Since $\mathcal{M}(T,(w_1,\cdots,w_n))$ is an upper-triangular matrix, so is $A^{-1}BA$.

See also Linear Algebra Done Right Solution Manual Chapter 10 Problem 5.

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6. Solution: Let $V = \mathbb{R}^2$, then $e_1 = (1,0)$, $e_2 = (0,1)$ is a basis of V. Let T be the unique operator in $\mathcal{L}(V)$ such that $Te_1 = e_2$ and $Te_2 = -e_1$. Then

$$T^2e_1 = Te_2 = -e_1 \quad \text{and} \quad T^2e_2 = -Te_1 = -e_2.$$

Hence $\mathcal{M}(T^2,(e_1,e_2)) = -I$. In particular, trace $T^2 = -2 < 0$.

See also Linear Algebra Done Right Solution Manual Chapter 10 Problem 6.

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7. Solution: See Linear Algebra Done Right Solution Manual Chapter 10 Problem 7.

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8. Solution: See Linear Algebra Done Right Solution Manual Chapter 10 Problem 8.

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9. Solution: See Linear Algebra Done Right Solution Manual Chapter 10 Problem 9.

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10. Solution: Choose an orthonormal basis $e_1,\cdots,e_n$ of V. By 7.10, we have

$$\mathcal{M}(T^*,(e_1,\cdots,e_n)) = \overline{\mathcal{M}(T,(e_1,\cdots,e_n))}^T.$$

By the definition of the trace of a matrix, we have trace A^T = trace A for any square matrix A. Therefore, by 10.16, we have

$$\begin{aligned}\text{trace } T^* &= \text{trace}(\mathcal{M}(T^*,(e_1,\cdots,e_n)))\\ &= \text{trace}(\overline{\mathcal{M}(T,(e_1,\cdots,e_n))}^T)\\ &= \text{trace}(\overline{\mathcal{M}(T,(e_1,\cdots,e_n))})\\ &= \overline{\text{trace}(\mathcal{M}(T,(e_1,\cdots,e_n)))}\\ &= \overline{\text{trace } T}.\end{aligned}$$

Here we use the fact that trace $\overline{A} = \overline{\text{trace } A}$ for any square matrix A. Why? Prove it.

---

11. Solution: By 7.35, we know that T is self-adjoint and all the eigenvalues of T are nonnegative. By the Spectral Theorem, T has a diagonal matrix with respect to some orthonormal basis of V. Fix this orthonormal basis of V. Then trace $T = \text{trace}(\mathcal{M}(T)) = 0$.

Note that $\mathcal{M}(T)$ is a diagonal matrix with the eigenvalues of T on the diagonal entries and the fact that all the eigenvalues of T are nonnegative, it follows from $\text{trace}(\mathcal{M}(T)) = 0$ that all the eigenvalues of T are zero. Hence $\mathcal{M}(T) = 0$, which implies $T = 0$.

---

12. Suppose V is an inner product space and $P,Q \in \mathcal{L}(V)$ are orthogonal projections. Prove that $\text{trace}(PQ) \geq 0$.

---

13. Solution: Since $\text{trace}(T)$ is the sum of all eigenvalues and the sum of the diagonal entries of $\mathcal{M}(T)$. Hence the third eigenvalue
of T is

$$51 + (-40) + 1 - (-48) - 24 = 36.$$

See also Linear Algebra Done Right Solution Manual Chapter 10 Problem 12.

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14. Solution: Choose a basis of V. We have $\text{trace}(cT) = \text{trace}(\mathcal{M}(cT))$ and $\text{trace}(T) = \text{trace}\mathcal{M}(T)$ by 10.16. Note that by 3.38, we have $\mathcal{M}(cT) = c\mathcal{M}(T)$. Therefore, we have

$$\text{trace}(\mathcal{M}(cT)) = \text{trace}(c\mathcal{M}(T)) = c\text{trace}(\mathcal{M}(T)).$$

Hence

$$\text{trace}(cT) = \text{trace}(\mathcal{M}(cT)) = c\text{trace}(\mathcal{M}(T)) = c\text{trace}(T).$$

---

15. Solution: Choose a basis of V. By 10.4, 10.14 and 10.16, we have

$$\begin{aligned}& \text{trace}(ST)\\ \text{by } 10.16 \quad &= \text{trace}(\mathcal{M}(ST))\\ \text{by } 10.4 \quad &= \text{trace}(\mathcal{M}(S)\mathcal{M}(T))\\ \text{by } 10.14 \quad &= \text{trace}(\mathcal{M}(T)\mathcal{M}(S))\\ \text{by } 10.4 \quad &= \text{trace}(\mathcal{M}(ST))\\ \text{by } 10.16 \quad &= \text{trace}(TS).\end{aligned}$$

---

16. Solution: Let $V = \mathbb{F}^2$. Fix a basis v_1, v_2 of V. Let $S = T$, then $\text{trace}(ST) = \text{trace}(I) = 2$ and

$$\text{trace}(S) = \text{trace}(T) = 2.$$

However

$$\text{trace}(ST) = 2 \neq 4 = \text{trace}(S)\text{trace}(T).$$

See also Linear Algebra Done Right Solution Manual Chapter 10 Problem 14.

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17. Solution: We assume that $\dim V = n$. If $T \neq 0$, then $\text{Ker}(T) \neq V$. Let $v_{m+1},\cdots,v_n$ be a basis of $\text{Ker}(T)$, then $m \geq 1$. Extend $v_{m+1},\cdots,v_n$ to a basis $v_1,\ldots,v_n$ of V. By the proof of 3.22, we have $Tv_1,\cdots,Tv_m$ is linearly independent in V. Hence we can extend $Tv_1,\cdots,Tv_m$ to a basis $Tv_1,\cdots,Tv_m,w_{m+1},\cdots,w_m$. Hence there exists $S \in \mathcal{L}(V)$ such that $S(Tv_1) = v_1$ (since $m \geq 1$, this is possible), $S(Tv_i) = 0$ for all $i = 2,\cdots,m$ and $Sw_j = 0$ for all $j = m+1,\cdots,n$. Then we have

$$STv_1 = v_1, \quad STv_i = 0, \quad STv_j = S0 = 0, \quad i = 2,\cdots,m, \quad j = m+1,\ldots,n.$$

Thus $1, 0$ are all the eigenvalues of ST. Moreover, the eigenvalue 1 has multiplicity one while the eigenvalue 0 has multiplicities $n-1$. Since trace is the sum of eigenvalues with mltiplicity, we have $\text{trace}(ST) = 1$ and we get a contradiction. Therefore $T = 0$.

See also Linear Algebra Done Right Solution Manual Chapter 10 Problem 15.

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18. Solution: See Linear Algebra Done Right Solution Manual Chapter 10 Problem 16.

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19. Solution: See Linear Algebra Done Right Solution Manual Chapter 10 Problem 18.

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20. Solution: See Linear Algebra Done Right Solution Manual Chapter 10 Problem 17.

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21. Solution: See Linear Algebra Done Right Solution Manual Chapter 10 Problem 19.

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This entry was posted in Chapter 10 and tagged Exercise A.

Chapter 10 Exercise B

Posted on October 2, 2016 by Linearity

1. Solution. Because $T_{\mathbb{C}}$ has no real eigenvalues, if λ is an eigenvalue of $T_{\mathbb{C}}$, then $\bar{\lambda}$ is also an eigenvalue of $T_{\mathbb{C}}$ with equal multiplicity. The determinant of $T_{\mathbb{C}}$, which equals the determinant of T, is thus a product of terms of the form $\lambda\bar{\lambda} = |\lambda|^2$, which are all greater than 0, because 0 is not eigenvalue of T. Hence $\det T > 0$.

---

2. Solution. By the previous exercise, T has at least one eigenvalue α, which is negative because $\det T < 0$. Suppose by contradiction that α is the only eigenvalue of T. Then this can't be the only eigenvalue of $T_{\mathbb{C}}$, otherwise the $\det T$ would equal $\lambda^{\dim V}$ which is positive (because $\dim V$ is even). The other eigenvalues of $T_{\mathbb{C}}$ are nonreal, so they come in pairs and won't change sign of $\det T$. It follows that $\det T$ is a product of absolute values times λ raised to its multiplicity. Thus the multiplicity of λ must be odd. This is a contradiction, because the sum of the multiplicities of the other eigenvalues of $T_{\mathbb{C}}$ is even (because they come in pairs) and the sum of all multiplicies must equal $\dim V$, which is even. Hence, our assumption that λ was the only eigenvalue of T is false.

---

3. Solution. (a) The characteristic polynomial of T is

$$(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_n).$$

Expanding this product we see that the coefficient of z^{n-2} is the sum of the products of all pairs of distinct eigenvalues. In other words, it equals

$$\lambda_1\sum_{j=2}^{n}\lambda_j + \lambda_2\sum_{j=3}^{n}\lambda_j + \cdots + \lambda_{n-1}\sum_{j=n}^{n}\lambda_j.$$

(b) It equals the sum of the products of the eigenvalues of T with one term missing:

$$\sum_{j=1}^{n}\prod_{k\neq j}(-1)^{n-1}\lambda_k.$$

More succinctly, assuming 0 is not an eigenvalue of T:

$$\sum_{j=1}^{n}(-1)^{n-1}\frac{\det T}{\lambda_j}.$$

---

4. Solution. This is follows easily from the definitions. If λ is an eigenvalue of T, or $T_{\mathbb{C}}$, then $c\lambda$ is an eigenvalue of cT, or $(cT)_{\mathbb{C}}$, with equal multiplicity. Let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of $T_{\mathbb{C}}$, repeated according to its multiplicity. Then $c\lambda_1, \ldots, c\lambda_n$ are the eigenvalues of $(cT)_{\mathbb{C}}$ and we have

$$\det(cT) = (c\lambda_1)\cdots(c\lambda_n) = c^{\dim V}\lambda_1\cdots\lambda_n = c^{\dim V}\det T.$$

---

5. Solution We give a counterexample. Let I be the identity operator on $\mathbb{R}^2$. Then

$$\det(I-I) = 0 \neq 2 = \det(I) + \det(-I).$$

---

7. Solution. From 10.16, we have

$$\operatorname{trace} S = \operatorname{trace} A = \operatorname{trace} T$$

and, from 10.42, we have

$$\det S = \det A = \det T.$$

---

8. Solution. Suppose $\mathbb{F} = \mathbb{C}$. Choose an orthonormal basis of V with respect to which the matrix of T is upper triangular. Then, by 10.35, the determinant of $\mathcal{M}(T)$ equals the product of the entries on the diagonal. By the same reasoning used in 10.35, the determinant of $\mathcal{M}(T^*)$ is also the product of the entries on diagonal. The diagonal entries of $\mathcal{M}(T^*)$ are the conjugates of the diagonal entries of $\mathcal{M}(T)$. Hence $\det \mathcal{M}(T^*) = \overline{\det \mathcal{M}(T)}$. Now 10.42 implies that $\det T^* = \overline{\det T}$.
From 10.44, we have

$$(\det\sqrt{T^*T})^2 = \det(\sqrt{T^*T}\sqrt{T^*T}) = \det(T^*T) = \det T^*\det T = \overline{\det T}\det T = |\det T|^2.$$

Taking the square root of each side we get $\det\sqrt{T^*T} = |\det T|$.

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