Educasheer

Educating Kashmir

PROBABILITY AND STATISTICS

ASSIGNMENT 1

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ASSIGNMENT 01
PROBABILITY AND STATISTICS.

A.1) Proving Basic Theorms

Proving The following Theorms

1) $A \subseteq B,\ B \subseteq C \Rightarrow A \subseteq C$.

2) $A \subseteq B,\ B \subseteq A. \Rightarrow A = B$

$\vee \rightarrow$ Inclusive OR.
$\wedge \rightarrow$ And

Proof: 1) $A \subseteq B,\ B \subseteq C \Rightarrow A \subseteq C$

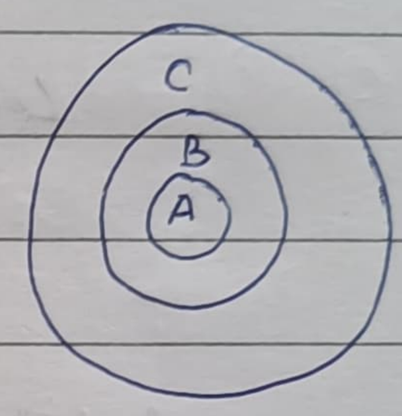
Let $x \in A$

By Using definition of Subset We Can Say that

$x \in B,\ \forall x \in A$ (Eq-01)

Similarly

$x \in C,\ \forall x \in B$ (Eq-02)



Now By Combining Equation 01 and 02. We get

$x \in C\ \forall x \in A$

Clearly We Can See All Elements that belong to Set A also belongs to Set C

$\therefore A \subseteq C$

2) $A \subseteq B,\ B \subseteq A \rightarrow A = B$

$A \subseteq B \Rightarrow \forall x \in A,\ x \in B$ (Every Element of A is also in B)

$B \subseteq A \Rightarrow \forall y \in B,\ y \in A$ (Every Element of B is also in A)

Combining Above Two ~~Results~~ Statements

- Every Element in A is also in B
- Every Element in B is also in A

This Implies A and B Contains Exactly the Same Elements (i.e $A = B$).

A.2) Properties of general Set

~~Proof~~: a) Commutative properties:

a.1) $(A \cup B) = (B \cup A)$

Proof: Let $x \in (A \cup B)$

$\Rightarrow x \in A \vee x \in B$

If $x \in A$, Then $x \in (B \cup A)$

and if $x \in B$, Then $x \in (B \cup A)$

This Shows that Every Element in $(A \cup B)$ is also in $(B \cup A)$

Let $y \in (B \cup A)$

$\Rightarrow y \in B \vee y \in A$

If $y \in B$, Then $y \in (A \cup B)$

And if $y \in A$, Then $y \in (A \cup B)$

This Shows that Every Element in $(B \cup A)$ is also in $(A \cup B)$

By Combining both the results We Conclude that $(A \cup B) = (B \cup A)$

a.2) $(A \cap B) = (B \cap A)$

Proof: Let $x \in (A \cap B)$.

$\Rightarrow x \in A \wedge x \in B$

Since x is in A and B, it is also in B and A, Which means $x \in (B \cap A)$

Let $y \in (B \cap A)$

$\Rightarrow y \in B \wedge y \in A$

Since y is in B and B, it is also in A and B,

Which means $y \in (A \cap B)$

Combining both the results, we conclude that $(A \cap B) = (B \cap A)$

B) Associative Properties

B.1) $(A \cup B) \cup C = A \cup (B \cup C)$

Proof: Let $x \in (A \cup B) \cup C$

$\Rightarrow x \in (A \cup B) \vee x \in C$

if $x \in (A \cup B)$ Then $x \in A$ or $x \in B$

- if $x \in A$ Then $x \in A \cup (B \cup C)$
- if $x \in B$ Then $x \in A \cup (B \cup C)$

if $x \in C$ Then $x \in (A \cup B) \cup C$.

Let $y \in A \cup (B \cup C)$

$\Rightarrow y \in A \vee y \in (B \cup C)$

if $y \in A$, Then $y \in (A \cup B) \cup C$.

if $y \in (B \cup C)$ Then $y \in B \vee y \in C$.

- if $y \in B$ Then $y \in (A \cup B) \cup C$
- if $y \in C$ Then $y \in (A \cup B) \cup C$

This Shows that Every Element in $(A \cup B) \cup C$ is also in $A \cup (B \cup C)$

B.2) $(A \cap B) \cap C = A \cap (B \cap C)$

Proof: Let $x \in (A \cap B) \cap C$

$\Rightarrow x \in (A \cap B) \wedge x \in C$.

1. Since $x \in (A \cap B) \Rightarrow x \in A \wedge x \in B$

- ~~Since $x \in (B \cap C)$~~
- Since x is in B and C, it is also in $B \cap C$.
- Since x is in A and $(B \cap C)$, it is in $A \cap (B \cap C)$

Let $y \in A \cap (B \cap C)$

$\Rightarrow \quad y \in A \wedge y \in (B \cap C)$

1. Since y is in $(B \cap C)$, it is both in B and C
   - Since y is in A and B it is also in $A \cap B$
   - Since y is in $(A \cap B)$ and C it is also in $(A \cap B) \cap C$

This shows that every element in $(A \cap B) \cap C$ is also in $A \cap (B \cap C)$

By combining both results we conclude that

$$(A \cap B) \cap C = A \cap (B \cap C)$$

c) DISTRIBUTIVE LAW:

C.2) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof:

Let $x \in A \cup (B \cap C)$

$\Rightarrow \quad x \in A \vee x \in (B \cap C)$

if $x \in A$ Then $x \in (A \cup B) \wedge x \in (A \cup C)$

- Since $x \in (A \cup B)$ and $(A \cup C)$ it is also in $(A \cup B) \cap (A \cup C)$

if $x \in (B \cap C)$, Then $x \in B \wedge x \in C$.

- Since $x \in B$ and C, it is also in $(A \cup B) \cap (A \cup C)$
- Since x is in $A \cup B$ and $A \cup C$ it is also in $(A \cup B) \cap (A \cup C)$

Let $y \in (A \cup B) \cap (A \cup C)$

$\Rightarrow \quad y \in (A \cup B) \wedge y \in (A \cup C)$

Since $y \in (A \cup B)$, it is either in A or B

if $y \in A$, Then $y \in A \cup (B \cap C)$

and if $y \in B$, Then $y \in A \cup (B \cap C)$

Since $y \in (A \cup C)$, it is either in A or C.

if $y \in A$, Then $y \in A \cup (B \cap C)$

and, if $y \in C$. Then, $y \in A \cup (B \cap C)$

This Shows that Every Element in $A \cup (B \cap C)$ is also in $(A \cup B) \cap (A \cup C)$.

Combining both the results, we Conclude that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

C.2) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof:

Let $x \in A \cap (B \cup C)$

$\Rightarrow \quad x \in A \wedge x \in (B \cup C)$

$\Rightarrow \quad x \in A \wedge (x \in B \vee x \in C)$

Since x is in A and $B \cup C$, it must be in A and in B, or in A and C, or in both.

- if $x \in (A \cap B)$, Then $x \in (A \cap B)$
- if $x \in (A \cap C)$, Then $x \in (A \cap C)$

$\therefore \quad x \in (A \cap B) \cup (A \cap C)$

Let $y \in (A \cap B) \cup (A \cap C)$

$\Rightarrow \quad y \in (A \cap B) \vee y \in (A \cap C)$.

if $y \in (A \cap B)$ Then $y \in A \cap (B \cup C)$.

if $y \in (A \cap C)$ Then $y \in A \cap (B \cup C)$

$\therefore \quad y \in A \cap (B \cup C)$

The above Two results Shows that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

D) Idempotent Laws

D.1) $A \cup A = A$

Proof

Let $x \in (A \cup A)$

$\Rightarrow \quad x \in A \vee x \in A$

Since x is in A, it is also in A

This Shows that Every Element in $A \cup A$ is also in A

Let $y \in A$

$\Rightarrow \quad y \in (A \cup A)$

This Shows that Every Element in A is Also in $A \cup A$.

$\therefore \quad A \cup A = A$

D.2) $(A \cap A) = A$

Let $x \in (A \cap A)$

$\Rightarrow \quad x \in A \wedge x \in A$

Since x is in $(A \cap A)$ it is also in A

This Shows that Every Element of $(A \cap A)$ is in A.

Let $y \in A$

$\Rightarrow \quad y \in (A \cap A)$

This Shows that Every Element of A is in $(A \cap A)$

$\therefore \quad (A \cap A) = A$

E) De Morgan's Law

E.1) $(A \cup B)^c = A^c \cap B^c$

Proof:

Let $x \in (A \cup B)^c$

$\Rightarrow \quad x \notin (A \cup B)$

$\Rightarrow \quad x \notin A \wedge x \notin B$

$\Rightarrow \quad x \in A^c \wedge x \in B^c$

$\therefore \quad x \in (A^c \cap B^c)$

Let $y \in (A^c \cap B^c)$

$\Rightarrow \quad y \in A^c \wedge y \in B^c$

$y \notin A \wedge y \notin B$

$y \notin (A \cup B)$

$y \in (A \cup B)^c$

$\therefore \quad (A \cup B)^c = A^c \cap B^c$

E.2). $(A \cap B)^c = (A^c \cup B^c)$

Proof:

Let $x \in (A \cap B)^c$

$\Rightarrow \quad x \notin (A \cap B)$

$\Rightarrow$ If $x \notin (A \cap B)$ Then $x \notin A$ and $x \notin B$

$x \in (A^c \cup B^c)$.

$\therefore \quad x \in (A^c \cup B^c)$

Let $y \in (A^c \cup B^c)$

$\Rightarrow \quad y \in A^c \vee y \in B^c$

if $y \notin (A \cap B)$ Then $y \in (A \cap B)^c$

$\therefore \quad y \in (A \cap B)^c$

Hence, $(A \cap B)^c = (A^c \cup B^c)$

A.3) More advanced Theorms

A.1) if $A \subseteq B$, $A \subseteq C \Rightarrow A \subseteq (B \cap C)$

Proof: We need To Show that $\forall x \in A$, $x \in (B \cap C)$

Let $x \in A$

if $x \in A$ Then $x \in B$ {Cooz $A \subseteq B$}

if $x \in A$ Then $x \in C$ {Cooz $A \subseteq C$}

Now, Since $x \in B \wedge x \in C$, x most also be in $(B \cap C)$

$\therefore$ Every Element in A is also in $B \cap C$ which proves that $A \subseteq (B \cap C)$

A.2) $A-(B\cup C) = (A-B)\cap(A-C)$

Proof

Let $x \in A-(B\cup C)$

$\Rightarrow$ $x \in A \wedge x \notin (B\cup C)$

$x \in A \wedge x \notin B \wedge x \notin C$

Hence, $x \in (A-B)$ and $x \in (A-C)$

$\therefore$ $x \in (A-B)\cap(A-C)$

Let $y \in (A-B)\cap(A-C)$

$\Rightarrow$ $y \in (A-B) \wedge y \in (A-C)$

$(y \in A \wedge y \notin B) \wedge (y \in A \wedge y \notin C)$

$\therefore$ $y \in A \wedge y \notin (B\cup C)$

So, $y \in A-(B\cup C)$

This Shows that Every Element of $A-(B\cup C)$ is in $(A-B)\cap(A-C)$ and Viceversa

Hence, $A-(B\cup C) = (A-B)\cap(A-C)$

A.3) $A-(B\cup C) = (A-B)\cup(A-C)$

Proof: By Using Same analogy (as above), We Can Prove this also.

B) COMBINATORICS

B.1) Permutation and Combination

Conceptually distinguish b/w P and C. When is Permutation needed and When is Combination needed? Give Some Everyday Example

What's the Connection b/w binomial theorm and Combinitorics?

Solution: ~~Conceptually distinguish b/w Permutation and Combination~~

Permutation: A Permutation of a Set of (distinct) Object is an Ordering of the Objects in row.

A Permutation of n things taken r at a time, Written $P(n,r)$ is an arrangment in a row of r things, Taken from a set of n distinct things. In Permutation Order matters.

In Simpler Terms permutation is a product rule without repetation.

$$P(n,r) = \frac{n!}{(n-r)!} \quad (0 \le r \le n) \quad \left\{ \text{Without repetation} \right\}$$

$$P(n,r) = n^r \quad \left\{ \text{With repetation} \right\}$$

Combination: Combinations are the Selection of items from a group of items where the Order of Selection doesn't matter.

Combination is same as Division rule.

$$C(n,r) = \frac{n!}{(n-r)!\,r!} \quad (0 \le r \le n) \quad \left\{ \text{Without repetation} \right\}$$

$$C(n,r) = {}^{n+r-1}C_r \quad \left\{ \text{With repetition} \right\}$$

Permutation is required when we're dealing with problems of arrangments, and Combination is required when we're dealing with problems of Selection

Example: ↳ Find # of Possible Passwords that can be generated of Length 8 By Using (A....Z) and (0,...9) Characters.

↳ Similarly Possible # of Licence plates of Length 4.

↳ Select a team of 11 Players from 14 players.

The Connection b/w binomial Thorm and Combinatorics Lies in the Counting of Combinations or arrangments of objects, which are Encapsulated in the Coefficients of the binomial Expension.

3.2) Giving out Chocolates

Solution: # Kids = 10

Chocolates = 16

Condition 01: Each kid must get atleast one Chocolate

$$x_1 + x_2 + \cdots + x_{10} = 16 \qquad (x_i > 0)$$

Give one Chocolate to Each Student

$$(x_1+1) + (x_2+1) + \cdots + (x_{10}+1) = 16$$

$$x_1 + x_2 + \cdots + x_{10} = 6$$

Now we have to distribute 6 Chocolates among 10 kids.

This problem is same as "STAR BAR" (IODB Template)

Stars = 6

Bars = 9

$$\# \text{ Permutations} = \frac{15!}{6! \times 9!} = 5005$$

OR.

$$\# \text{ Permutations} = {}^{15}C_6 = 5005$$

★ Condition 02: No Child can get more than 3 Chocolates.

After Apply Condition 01, New Equation is

$$x_1 + x_2 + \cdots + x_{10} = 6 \qquad (x_i > 0)$$

Permutation = 5005 - (Atleast ~~Any~~ one got more than or equal. 3 chocolates).

Computing Atleast one got more than or equal 3 Chocolates

Such permutation = Y

Let x_1 get ~~more~~ 3 Chocolates

$$(x_1+3) + x_2 + \cdots + x_{10} = 6$$

$$x_1 + x_2 + \cdots + x_{10} = 3$$

Again it becomes Star Bar Problem.

12.

Spirit

$\#$ Permutations $= \dfrac{12!}{3! \times 9!} = 220$ $\qquad$ $\#$ STAR $= 3$

$\#$ Bars $= 9$

But we can give 3 Chocolates to any of 10 kids

$\therefore$ Total $\#$ Permutations $= 220 \times 10$

$= 2200$

$\therefore$ Total $\#$ ways to distribute 16 Chocolates among 10 kids such that they'll receive Atleast 1 and Atmost 3 Chocolates

$= 5005 - 2200 = 2805$

B.3) Money Banking

Solution: $\#$ 1 Rupee Coins $= 5$ $\qquad$ $\#$ 2 Rupee Coins $= 4$

$\#$ 5 Rupee Coins $= 2$ $\qquad$ $\#$ 10 Rupee Coins $= 1$

Let x_1 denotes $\#$ 1 rupee Coins

x_2 $\qquad$ 2

x_5 $\qquad$ 5

x_{10} $\qquad$ 10

$\cancel{x_1 + 4x_2 + 2x_5 + x_{10} =}$

$\cancel{x_1 + 2 \cdot 4 x_2 + 2 x_5 + x_{10} = 13}$

$x_1 + 2x_2 + 5x_5 + 10x_{10} = 13$

$$\left\{ \begin{array}{l} 0 \le x_1 \le 5 \\ 0 \le x_2 \le 4 \\ 0 \le x_5 \le 2 \\ 0 \le x_{10} \le 1 \end{array} \right\}$$

Now the Problem is to find all possible Solutions to above Equation

All Possible Solutions to Above Equation are.

0 4 1 0	3 0 0 1
1 1 0 1	3 0 2 0
1 1 2 0	4 2 1 0
2 3 1 0	5 4 0 0

Date ..../..../.....

```python

import numpy as np
from itertools import product


range_1 = np.arange(0, 6)
range_2 = np.arange(0, 5)
range_3 = np.arange(0, 3)
range_4 = np.arange(0, 2)

count = 0
print("All Possible Solutions are: ")
for x, y, z, t in product(range_1, range_2, range_3, range_4):
    if x + 2*y + 5*z + 10*t == 13:
        print(x, y, z, t)
        count += 1

print(f"Number of all possible solutions is: {count}")
```

$\therefore$ Only 8 Such Solutions are Possible

Hence There are only 8 ways To give ₹ 13 from the Bag

C) SIMPLE PROBABILITY

C.1) Red and Blue

Solution: # Red Balls = 5 , # Blue Balls = 10

Third drawn will be blue only in below Cases

RRB, BRB, BBB, RBB

$P(R_1 R_2 B_3) = P(R_1)\, P(R_2/R_1)\, P(B_3/R_1 \cap R_2)$

$P(B_1 R_2 B_3) = P(B_1)\, P(R_2/B_1)\, P(B_3/B_1 \cap R_2)$

$P(B_1 B_2 B_3) = P(B_1)\, P(B_2/B_1)\, P(B_3/B_1 \cap B_2)$

$P(R_1 B_2 B_3) = P(R_1)\, P(B_2/R_1)\, P(B_3/R_1 \cap B_2)$

1 → first drawn
2 → Second drawn
3 → 3rd drawn

$P(R_1 R_2 B_3) = 5/15 * 5/15 * 10/15 = 2/3^3$

$P(B_1 R_2 B_3) = 10/15 * 5/14 * 9/14 = 30/14^2$

$P(B_1 B_2 B_3) = 10/15 * 9/14 * 8/13 = 24/91$

$P(R_1 B_2 B_3) = 5/15 * 10/15 * 9/14 = 1/7$

P(Third drawn will be a blue ball) $= \left(2/3^3 + 15/98 + 24/91 + 1/7\right)$

$= 0.63$

C.2) Rain Measurement

Solution: if it rains today, Then there is 70% Chance it'll rain tomarrow

Whenever it rains it rains 50mm/day.

14.

Spinal points

Let's assume it rains continously for 5 days (Excluding day 1)

$\therefore$ Total Rain on 5 days = $5 \times 50mm = 250mm$

$= 25cm$

Clearly even if it rains continously for 5 days then total rain won't be $> 25cm$

$\therefore P(\underbrace{\text{Total rain} > 150cm \text{ in 6 days}}_{X} \mid \underbrace{\text{it didn't rain on day 1}}_{Y})$

$$P(X|Y) = \frac{P(Y|X)\,P(X)}{P(Y)}$$

$$P(X|Y) = \frac{P(Y|X) \times 0}{P(Y)} = 0$$

Hence $P(X|Y) = 0$

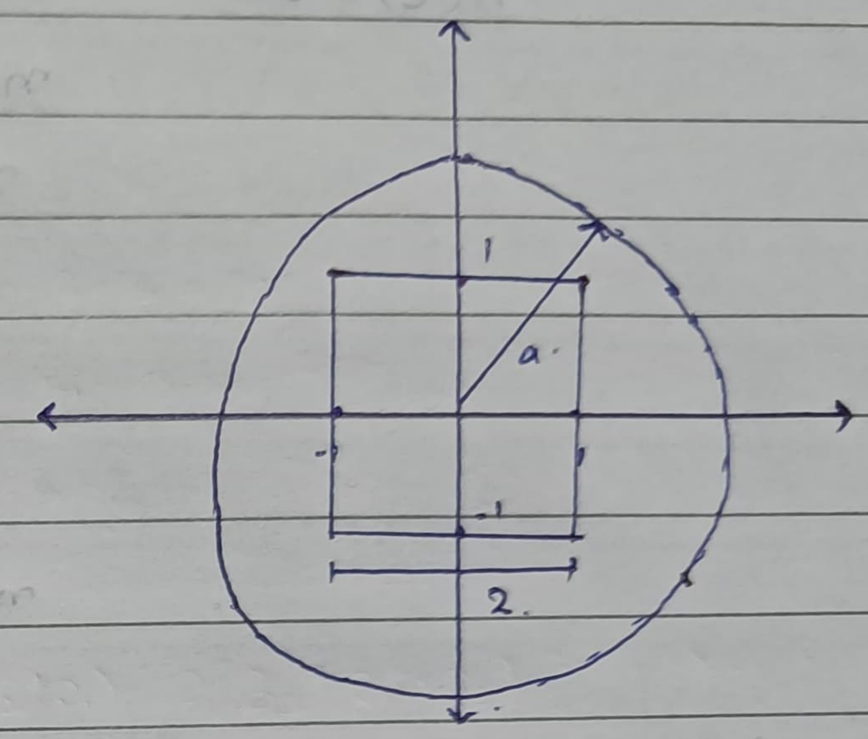
C.3) Geomertic Probability

Solution: $x^2+y^2 \le a^2$ $\quad x,y \in [-1,1]$ and $a \in R$

$$P(x^2+y^2 \le a^2 \mid x,y \in [-1,1] \wedge a \in R) = ?$$

Case 01: $a > 1$

$$P(x^2+y^2 \le a^2 \mid x,y \in [-1,1] \wedge a > 1) = \frac{\text{Area of } \cancel{\text{Circle}}\ \text{Square}}{\text{Area of } \cancel{\text{Square}}\ \text{Circle}}$$

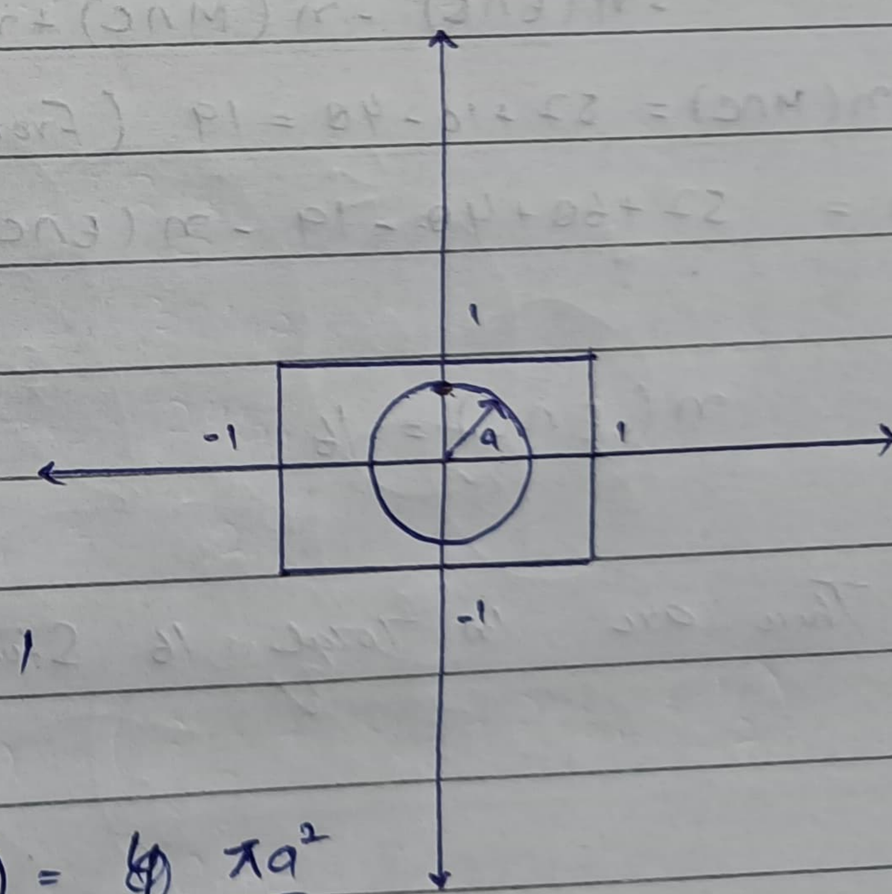


$$= \frac{\cancel{\pi a^2}\ 4}{\cancel{4}\ \pi a^2} \quad ; (a > 1)$$

$$\therefore P(x^2+y^2 \le a^2 \mid x,y \in [-1,1] \wedge a > 1) = \frac{\cancel{\pi a^2}\ 4}{\cancel{4}\ \pi a^2}$$

Case 02: $a \le 1$

$$P(x^2+y^2 \le a^2 \mid x,y \in [-1,1] \wedge a \le 1) = \frac{\text{Area of } \cancel{\text{Square}}\ \text{Circle}}{\text{Area of } \cancel{\text{Circle}}\ \text{Square}}$$



$$= \frac{\cancel{4}\ \pi a^2}{\cancel{\pi a^2}\ 4} \quad ; a \le 1$$

$$\therefore P(x^2+y^2 \le a^2 \mid x,y \in [-1,1] \wedge a \le 1) = \frac{\cancel{4}\ \pi a^2}{\cancel{\pi a^2}\ 4}$$

D) MODELLING AND GAMES

D.2) Class Test

Solution:

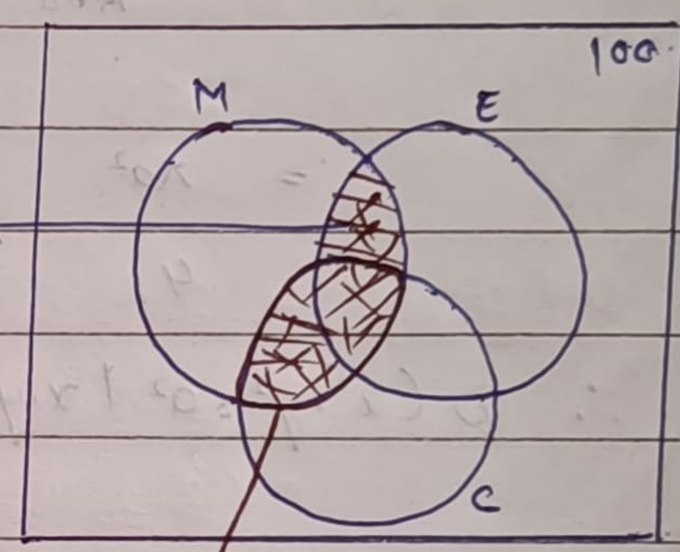
$n(T) = 100$ $\quad$ $n(C) = 48$

$n(M) = 57$ $\quad$ $n(M) = n(M \cap E) + n(M \cap C) - n(M \cap E \cap C) = 57$ (1)

$n(E) = 68$ $\quad$ $n(C) = n(C \cap M) = 48$

$n(M \cap E \cap C) = 10$

$n(E \cap C) = ?$



$n(M \cap E) + n(M \cap C)$

$- n(M \cap E \cap C) = 57$

$n(C \cap M) = 48$

$n(M \cup E \cup C) = n(M) + n(E) + n(C) - n(M \cap E)$

$- n(E \cap C) - n(M \cap C) + n(M \cap E \cap C)$

$n(M \cap C) = 57 + 10 - 48 = 19$ (From (1))

$100 = 57 + 68 + 48 - 19 - n(E \cap C) - 48 + 10$

$n(E \cap C) = 16$

$\therefore$ There are in Total 16 Student who have passed both in English & civics.

D.2)

A) With replacement

Solution:

$$F(x) = \begin{cases} x^2 ; & \text{if } x\%2 == 0 \\ -x ; & \text{if } x\%2 != 0 \end{cases}$$

To Select any 3 No.s From 6 Number There are ${}^6C_3 = 20$ Such. Selections Possible.

$P(\text{To Win}) = n(\text{Event})/n(S) = ?$

$n(S) = 20$ $\quad S \rightarrow$ Sample Space.

Event = Getting Profit in Each 3 Selections

$n(\text{Event}) = ?$

Event = $\{(1,2,4), (1,2,6), (1,4,6), (2,3,4), (2,3,6), (2,4,6), (3,4,6)\}$

$n(\text{Event}) = 7$

$P(\text{To Win}) = 7/20 = 0.35 = 35\%$

$\therefore$ There's a 35% Chance of Winning the game With replacement.

B) Without Replacement

$P(\text{To Win}) = n(\text{Event})/n(S)$

Event = Getting profit in Each 3 Selections

Event = $\{(1,2,4), (1,2,6), (1,4,6), (2,3,4), (2,3,6), (2,4,6), (3,4,6), (1,2,4), (2,3,4), (1,2,4), (2,3,4)\}$

$n(\text{Event}) = 11$

$n(S) = {}^6C_3 + {}^5C_3 + {}^4C_3 + {}^3C_3$

$P(\text{To Win}) = 11/35 = 0.3142 = 31.42\%$

Conclusion if we don't replace the Cards Then Probability of Winning is less.

```python
import numpy as np
from itertools import combinations

range_1 = np.arange(1, 7)
all_combinations = np.array(list(combinations(range_1, 3)))

#Function that'll compute associated value for each number
def compute_value(num):
    if num % 2 == 0:
        return np.sqrt(num)
    else:
        return -num

#Function that'll compute net profit or loss for any three selection
def profit_loss(x, y, z):
    return compute_value(x) + compute_value(y) + compute_value(z)

#Function that'll calculte probability
def cal_prob():
    count = 0
    card_nums = 6
    sample_space = 0

    while card_nums:
        range_1 = np.arange(1, card_nums + 1)
        all_combinations = np.array(list(combinations(range_1, 3)))
        sample_space += all_combinations.shape[0]
        for picked_num in all_combinations:
            if profit_loss(*picked_num) > 0:
                count += 1
        card_nums -= 1

    print(f"Probability of getting profit 'Without Replacement' is: {round(count/sample_space, 2)}")

cal_prob()
```

```python
import numpy as np
from itertools import combinations

range_1 = np.arange(1, 7)
all_combinations = np.array(list(combinations(range_1, 3)))

#Function that'll compute associated value for each number
def compute_value(num):
    if num % 2 == 0:
        return np.sqrt(num)
    else:
        return -num

#Function that'll compute net profit or loss for any three selection
def profit_loss(x, y, z):
    return compute_value(x) + compute_value(y) + compute_value(z)

#Function that'll calculte probability
def cal_prob():
    count = 0
    for selection in all_combinations:
        if profit_loss(*selection) > 0:
            count += 1

    print(f"Probability of getting profit 'With Replacement' is: {round(count/all_combinations.shape[0], 2)}")

cal_prob()
```