Integrality Gap Instances and Combinatorial Structure of Gilmore-Gomory LP for Bin Packing

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Masters Thesis Seminar Talk – 21st January, 2019

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Bin Packing: Definition

Input: Items with sizes $s_1, s_2, \ldots, s_n \in (0, 1]$ with their multiplicities $b_1, b_2, \ldots, b_n \in \mathbb{Z}_{\geq 0}$.

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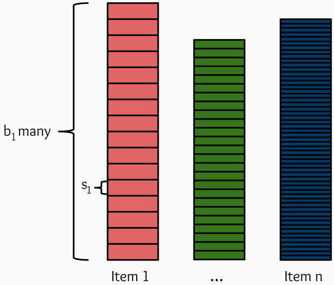

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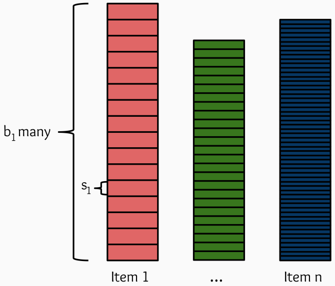

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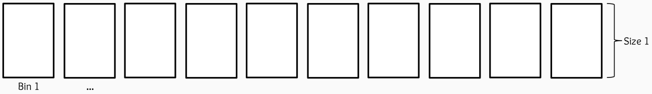

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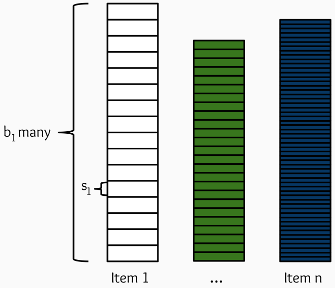

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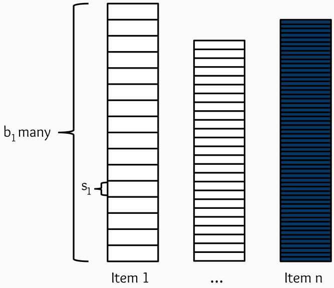

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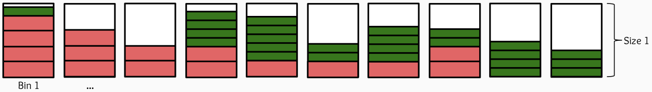

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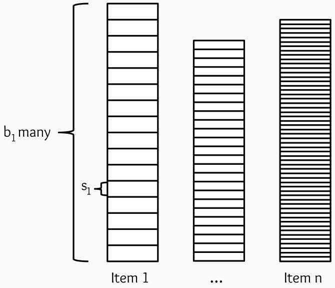

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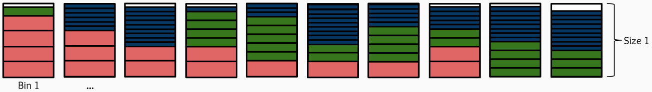

Bin Packing: Polynomial Time Algorithms

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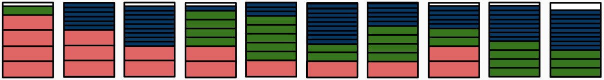
- Strongly NP-hard even if $s_i \in (\frac{1}{4}, \frac{1}{2})$ (3-Partition Case)

The Gilmore-Gomory LP Relaxation

- Recall that $b_i = \#$ of items of size s_i.
- **Feasible Patterns:**

$$\mathcal{P} = \{p \in \mathbb{Z}^n_{\geq 0} \mid \sum_{i=1}^n s_i\, p_i \leq 1\}$$



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- **Gilmore-Gomory LP Relaxation:**

$$\begin{aligned} \min \quad & \sum_{p \in \mathcal{P}} x_p && \leftarrow \text{pattern multiplicity} \\ \text{s.t.} \quad & \sum_{p \in \mathcal{P}} x_p\, p_i = b_i && \forall\, i \in [n] \\ & x_p \in \mathbb{R}_{\geq 0} && \forall\, p \in \mathcal{P} \end{aligned}$$

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The Gilmore-Gomory LP Relaxation: Integrality Gap

Modified Integer Round Up Conjecture (MIRUP)

$\mathrm{OPT} - \lceil \mathrm{OPT_f} \rceil \leq 1$

- True, if # of different item types $n \leq 6$[1]
- Best known upper bound: $\mathrm{OPT} - \mathrm{OPT_f} \leq O(\log n)$

[1] [Sebö and Shmonin, 2009] claimed to prove for $n \leq 7$, but no manuscript!

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- **Additive Integrality Gap** $= \mathrm{OPT} - \mathrm{OPT_f}$

Question [Williamson and Shmoys, 2011]

Is *additive integrality gap* constant for GGLP?

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- *Crucial* for the size of search trees in branch-and-bound.
- Stronger formulations perform better than weaker ones as the instances grow.
- In practice, largest known gap is 1.1.
- Hence, there is a dire need to address this difference between the bounds in theory and practice.

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- Examine structural properties of GGLP in the context of approximation solutions and design an approximate algorithm.

Theoretical investigations on special class of instances

Structural Properties to MIRUP Counterexamples

Theorem [Shmonin, 2007]

For an instance $I = (n, s, b)$ the following properties are true:

(a) Modify I to $I' = (n, s, b')$ (*residual*) by updating $b' = \sum_{p \in \mathcal{P}} \{x_p\}\, p$, then $\mathrm{OPT}(I) - \mathrm{OPT_f}(I) \leq \mathrm{OPT}(I') - \mathrm{OPT_f}(I')$.

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(b) Modify I' to I'' by deleting all items of $s_i \leq \frac{1}{s^\intercal b}$, then $\mathrm{OPT}(I') - \lceil \mathrm{OPT}_\mathrm{f}(I') \rceil \leq \max\{\mathrm{OPT}(I'') - \lceil \mathrm{OPT}_\mathrm{f}(I'') \rceil, 1\}$.

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(c) A counterexample instance I to MIRUP conjecture still holds even if items with $s_i > \frac{1}{n}$ are eliminated.

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IRUP Property of Large Items

Special instances adhering to MIRUP conjecture: Large Item sizes

Theorem

Given an instance $I = (n, s)$ with $s_i \in (\frac{1}{3}, 1]$. Then, integrality gap for I is either 0 or $\frac{1}{2}$.

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Preliminaries:

- $p_{\{i,j\}}$ denotes the pattern that packs item i and j in one bin.
- $p_{i,1}$ and $p_{i,2}$ denote the pattern p packing item i once and twice resp.
- Support graph $G = (V, E)$ where $V = [n]$ and $E = \{\{i, j\} \mid x^*_{p_{\{i,j\}}} > 0\}$.

Large Item sizes: Useful Lemma

- x^* is an optimal basic solution to GGLP such that $\forall p \in \text{supp}(x^*)$ it holds that $\mathbf{1}^T p \leq 2$.

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Black box Lemma

x^* can be modified without changing the objective function value so that the support graph G is either a cycle of odd length or an isolated vertex.

Large Item sizes: Theorem Proof

Proof. Can formulate an integral solution x that uses $\lceil \mathrm{OPT_f}(I) \rceil$ bins.

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- Define integral solution x as:

$$x_p = \begin{cases} 1, & \text{for all } p_{e_i}, \text{ if } i \text{ is even} \\ 1, & \text{for all } p_{v,1}, \text{ if } v = e_1 \cap e_n \\ 0, & \text{otherwise} \end{cases}$$

$$\implies \mathrm{OPT}(I) \leq \frac{n-1}{2} + 1 = \frac{n}{2} + \frac{1}{2} = \lceil \mathrm{OPT_f}(I) \rceil.$$

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- G is an isolated vertex $\implies \mathrm{OPT_f}(I) = \frac{1}{2}$ and $\mathrm{OPT}(I) = 1$.

$\blacksquare$

Theoretical investigations on special class of instances

MIRUP Property of Instances with 7 different items

Special instances adhering to MIRUP conjecture: $n \leq 7$

Theorem [Nitsche et al., 1998]

For all instances $I = (6, s, b)$, $\mathrm{OPT}(I) \leq \lceil \mathrm{OPT_f}(I) \rceil + 1$.

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Recall that,

$$\text{Residual instance} \implies \mathrm{OPT}(I) \leq n$$

$$\text{Small items deleted} \implies s_i > \frac{1}{\mathrm{size}(I)} = \frac{1}{s^T b} \geq \frac{1}{\mathrm{OPT}(I)} \geq \frac{1}{n}$$

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  $\implies$ $\mathrm{OPT_f}(I') \geq \mathrm{OPT}(I)$ (Property of Duality)
- $size(I') = w^\intercal b > m \implies$ MIRUP instance ($size(I') \leq \mathrm{OPT_f}(I')$) ■

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Example for $\mathrm{OPT_f} = 4$**:**

By *residual instances:* need to verify if $\lceil \mathrm{OPT_f}(I) \rceil = 1, 2, 3, 4, 5, 6, 7$

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Optimum solution: $\mathrm{OPT}(I) = p_1 + p_2 + p_3 + p_4 + \mathrm{R}$

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Weights:

- 1 to 1-bins.
- $\frac{1}{2}$ to 2-bins and largest item in R.

Discrepancy Theory Inspired Instances

Beck's 3-permutation Conjecture

Given any *three permutations* on m elements, one can always find a color assignment such that in every interval of each of those permutations, the number of red and blue colored elements differ by a constant.

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permutation 2:	7	8	2	5	3	4	1	6
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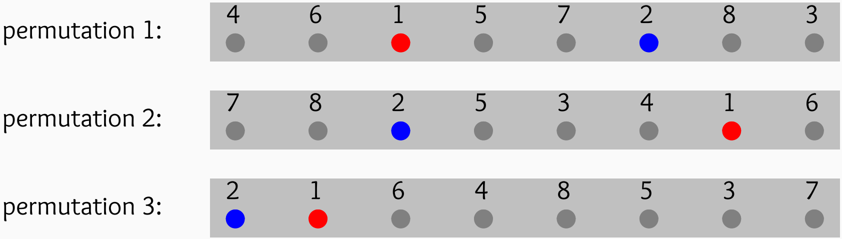

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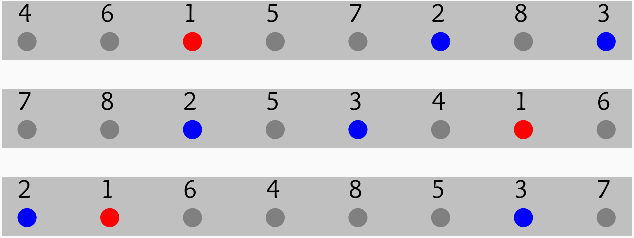

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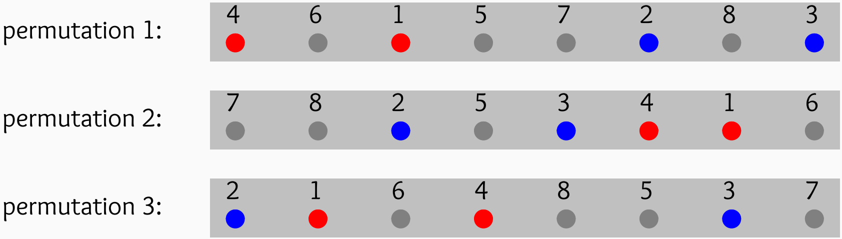

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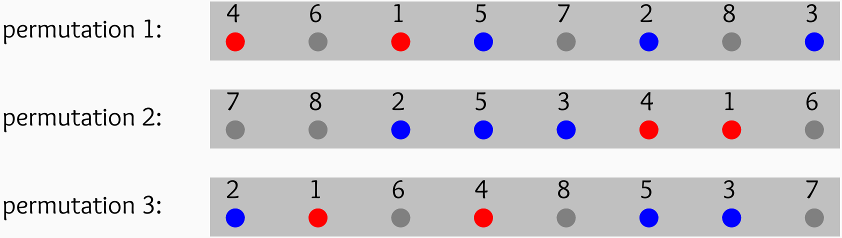

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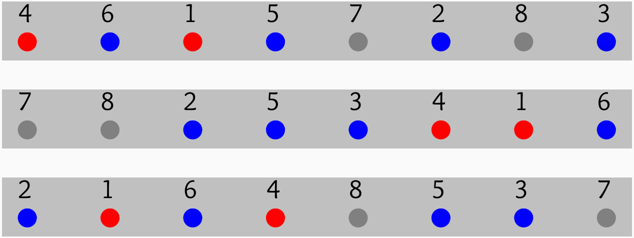

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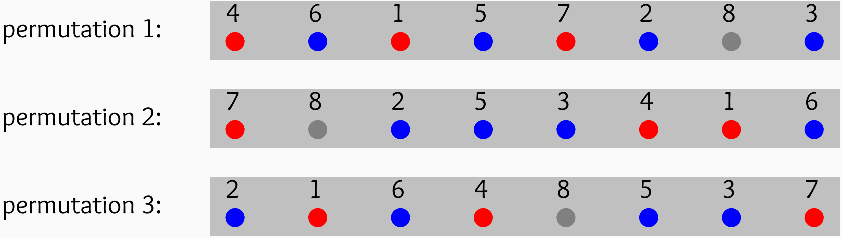

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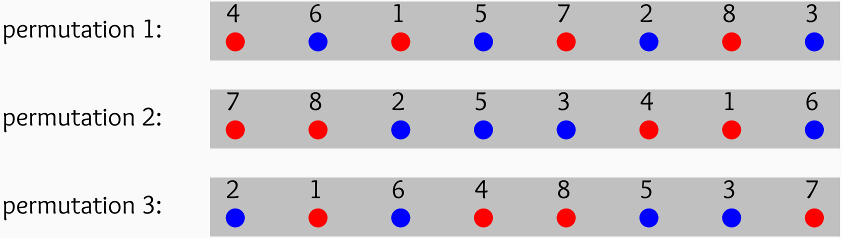

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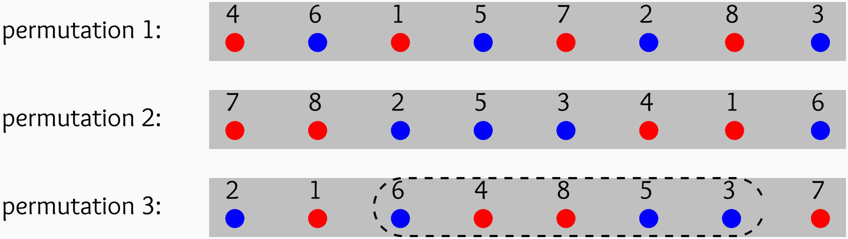
Given any *three permutations* on m elements, one can always find a color assignment such that in every interval of each of those permutations, the number of red and blue colored elements differ by a constant.



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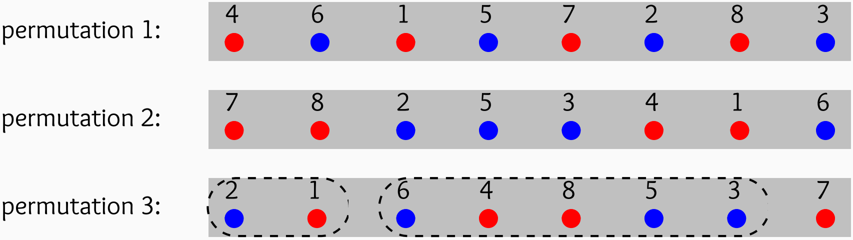

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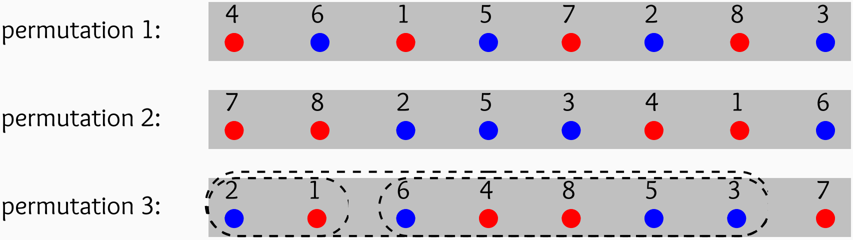

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Beck's Conjecture: Implications

Recall: Prominent Open Problem

Is *additive integrality gap* constant for GGLP?

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[Eisenbrand et al., 2011] (SODA '11)

Beck's Conjecture $\implies$ Integrality gap of 3-partition instances is $O(1)$.

Why Discrepancy Theory Inspired 3-Partition Instances?

- For 3-Partition case, all three, [Karmarkar and Karp, 1982, Rothvoß, 2013, Hoberg and Rothvoß, 2017] give an upper bound of $O(\log n)$ on integrality gap.

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Let $\mathcal{S}_k$ be the family of prefixes of three permutations π_1, π_2, π_3 on $m = 3^k$ elements. Then, $disc(\mathcal{S}_k)$ is $\Omega(\log m)$.

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[Newman et al., 2012] (Consequence to bin packing)

There exists a bin packing instance $I = (n, s)$ with an optimal basic feasible solution x to GGLP, such that any integral solution y to GGLP satisfying $\text{supp}(y) \subseteq \text{supp}(x)$ has value at least $\text{OPT}(I) + \Omega(\log n)$.

Badly Three Colorable Permutations: Illustration

π_1	A	B	C
π_2	C	A	B
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BP Instance from Badly Colorable Three Permutations: Construction

$\pi_1^{k=2}$	$\underline{1\quad 2}$	$\underline{\mathbf{3}\quad \mathbf{4}}$	$\underline{5\quad 6}$	$\underline{7\quad 8}$	$\underline{9\quad 0}$
$\pi_2^{k=2}$	$\underline{9\quad 7}$	$\underline{8\quad 3}$	$\underline{1\quad 2}$	$\underline{\mathbf{6}\quad \mathbf{4}}$	$\underline{5\quad 0}$
$\pi_3^{k=2}$	$\underline{5\quad 6}$	$\underline{\mathbf{4}\quad \mathbf{8}}$	$\underline{9\quad 7}$	$\underline{2\quad 3}$	$\underline{1\quad 0}$

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_0
s_1	1	1	0	**0**	0	0	0	0	0	0
s_2	0	0	1	**1**	0	0	0	0	0	0
s_3	0	0	0	**0**	1	1	0	0	0	0
s_4	0	0	0	**0**	0	0	1	1	0	0
s_5	0	0	0	**0**	0	0	0	0	1	1
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s_8	1	1	0	**0**	0	0	0	0	0	0
s_9	0	0	0	**1**	0	1	0	0	0	0
s_{10}	0	0	0	**0**	1	0	0	0	0	1
s_{11}	0	0	0	**0**	1	1	0	0	0	0
s_{12}	0	0	0	**1**	0	0	0	1	0	0
s_{13}	0	0	0	**0**	0	0	1	0	1	0
s_{14}	0	1	1	**0**	0	0	0	0	0	0
s_{15}	1	0	0	**0**	0	0	0	0	0	1

Implementation and Results

Generating Instances: Discrepancy Inspired 3-Partition Instances

- Recall that, for 3-Partition instance $s_i \in (\frac{1}{4}, \frac{1}{2})$. Let $\mathcal{P}_k$ be the set of $3^k + 1$ patterns with $n = 3 \cdot \frac{3^k+1}{2}$ items.

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- SLP:

$$
\begin{aligned}
\min \quad & 0 & \\
\text{s.t.} \quad & s^T p = 1 & \forall p \in \mathcal{P}_k \\
& s_i \leq \frac{1}{2} - \frac{1}{8} & \forall i \in [n] \\
& s_i \geq \frac{1}{4} + \frac{1}{16} & \forall i \in [n]
\end{aligned}
$$

Generating Instances: Hindrance and its Remedy

Hindrance:

- Infeasible SLP:
  - Due to the constraint $s^T p = 1$ as the size of the third item s_o is determined by $s_o = 1 - s_l - s_m$ (infeasible if $1 - s_l - s_m < \frac{1}{4}$).

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  - Them mark all items s_l, s_m, s_o contained in the current pattern together with $s_{l+1}, s_{m+1}, s_{o+1}$ if they exist.

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- GGLP and GGLP* are equivalent formulations.

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- Performed integrality tests for $70,000$ instances.

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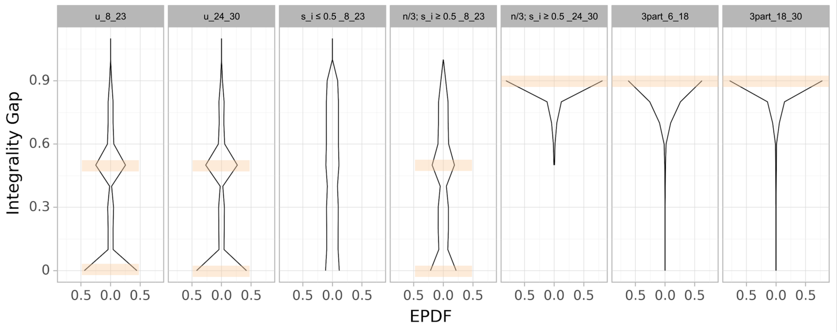

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Instances	Integrality Gap			
		0	[0.9,1)	1
	k=2	10000	-	-
	k=3	6202	986	2811
	k=4	0	1568	3182

Problem II: Bin Packing for $O(1)$ different item types

Bin Packing in Fixed Dimension: Polynomial Time Algorithms

For constant n:

- OPT $+ 1$ in time $2^{2^{O(n)}} \cdot (\log \Delta)^{O(1)}$ [Jansen and Solis-Oba, 2010]

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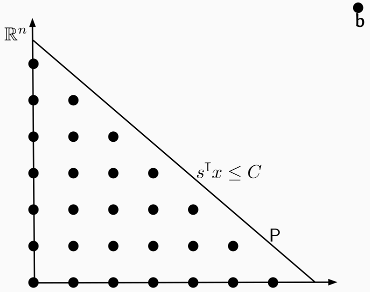
  In time $|\mathcal{V}_I|^{2^{O(n)}} O(\log \Delta)^{O(1)}$ parameterized by $\mathcal{V}_I$

Geometric View: From the Bird's Eye

- Define $P = \{x \in \mathbb{R}^n \mid s^\top x \leq C\}$



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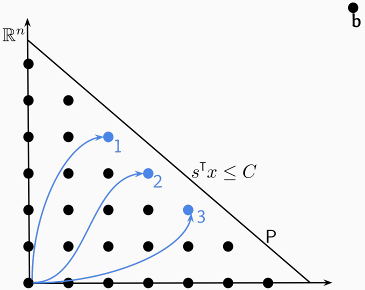

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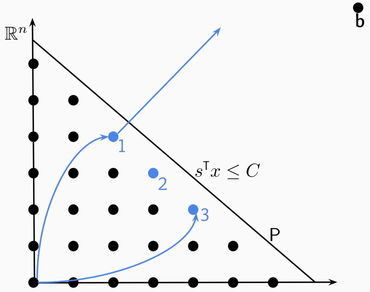

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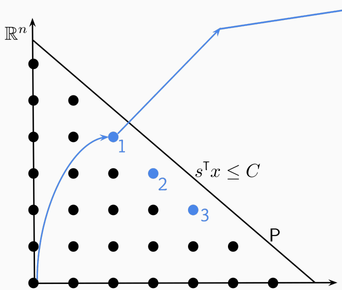

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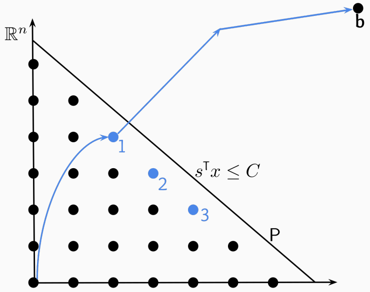

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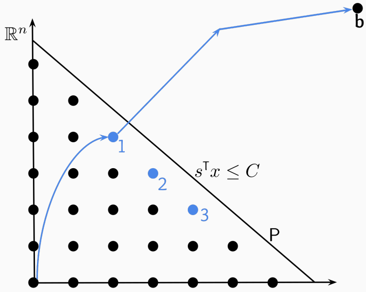

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- Integral points in P are exponentially many.
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Bin Packing decision variant: Given an instance $I = (n, s, b, C, m)$, the decision variant consists in determining if there exists an assignment of items to m bins of size C.

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Trivial UB:

$UB = \max\{s^{\mathsf{T}}x \mid x \in \{0, \ldots, b_1\} \times \ldots \times \{0, \ldots, b_n\}, s^{\mathsf{T}}x \leq C\}$

Feasibility Certificates

- **Bottleneck Instance:** An instance $I = (n, s, b, C, m)$ is *bottleneck* if C adheres to the upper bound UB
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As a partial converse, if $\overline{b} \notin$ P, then I is infeasible.

Bottleneck Instances and Feasibility

Consider an instance, $I = (3, (6, 10, 15)^{\mathsf{T}}, (4, 2, 1)^{\mathsf{T}}, 30, 2)$

Bottleneck Instances and Feasibility

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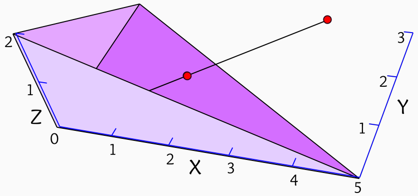
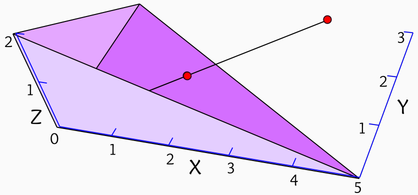
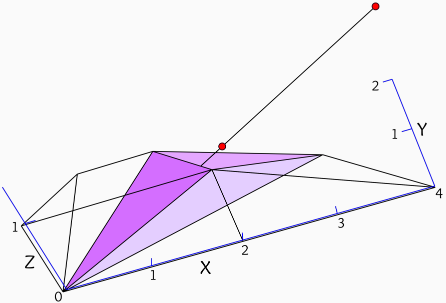


Bottleneck Instances and Feasibility

Consider an instance, $I = (3, (6, 10, 15)^\mathsf{T}, (4, 2, 1)^\mathsf{T}, \color{red}{28}, 2)$





Modified Feasibility Certificate

Corresponding Polytope: For a bottleneck instance $I = (n, s, b, C, m)$ and the polytope $\mathrm{P} = \{x \in \mathbb{R}^n \mid s^\mathsf{T} x \leq C, 0 \leq x \leq b_i, \ \forall i \in [n]\}$, then $\mathrm{P}^* = \mathrm{conv}(\mathrm{P} \cap \mathbb{Z}^n)$ is the corresponding polytope.

Modified Theorem

For an instance $I = (n, s, b, C, m)$ and its corresponding polytope P^*, I is infeasible *if* $\overline{b} = \frac{b}{m} \notin \mathrm{P}^*$. Otherwise, I might be feasible.

Suppose $X \subseteq \mathbb{R}^n$ is a set of points. Then,

$$C(X) = \text{cone}(X) = \Big\{ \sum_{x \in X} \lambda_x \cdot x \mid \lambda_x \in \mathbb{R}_{\geq 0}, \forall x \in X \Big\}$$

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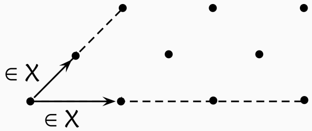

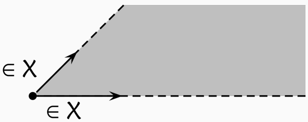

Some Definitions

Suppose $X \subseteq \mathbb{R}^n$ is a set of points. Then,

$$C(X) = \text{cone}(X) = \{ \sum_{x \in X} \lambda_x \cdot x \mid \lambda_x \in \mathbb{R}_{\geq 0}, \forall x \in X \}$$

Assumption: All cones are pointed!

Caratheodary Thm: If $b \in \mathit{cone}(X)$, then there is a conic combination $\lambda \in \mathbb{R}^X_{\geq 0}$ with only $|\text{supp}(\lambda)| \leq n$ many non-zero entries and $b = \sum_{x \in X} \lambda_x \cdot x$.

Concentric Cones

Concentric Convex Cones: Given l n-dimensional convex cones C_i, we call them concentric if they satisfy that $C_1 \subset C_2 \subset \ldots \subset C_l$, where $C_i = \mathrm{conv}(\mathbf{0}, F_i),\ \forall i \in [l]$ and $F_i = \mathrm{conv}(i \cdot f_i, \ldots, i \cdot f_n)$. In particular, all C_i are closed.

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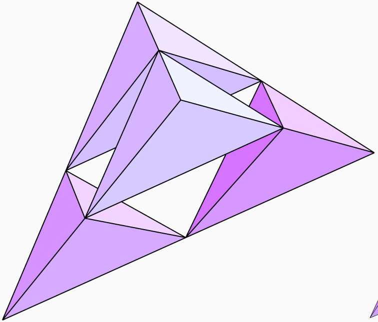

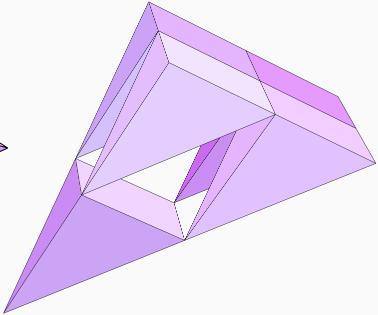

Structural Theorem: Construction of Solution

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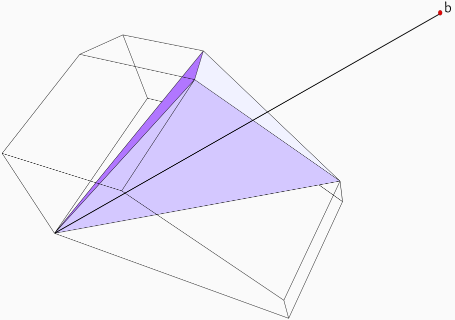

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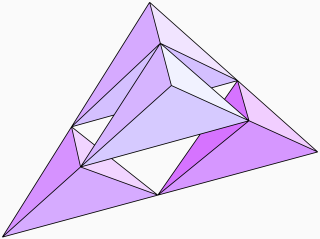
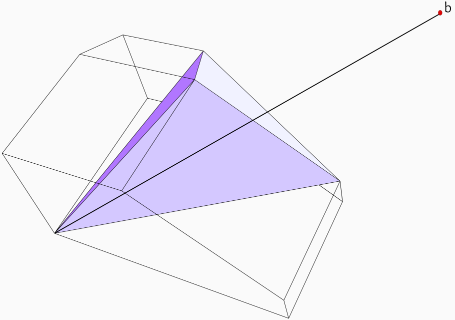


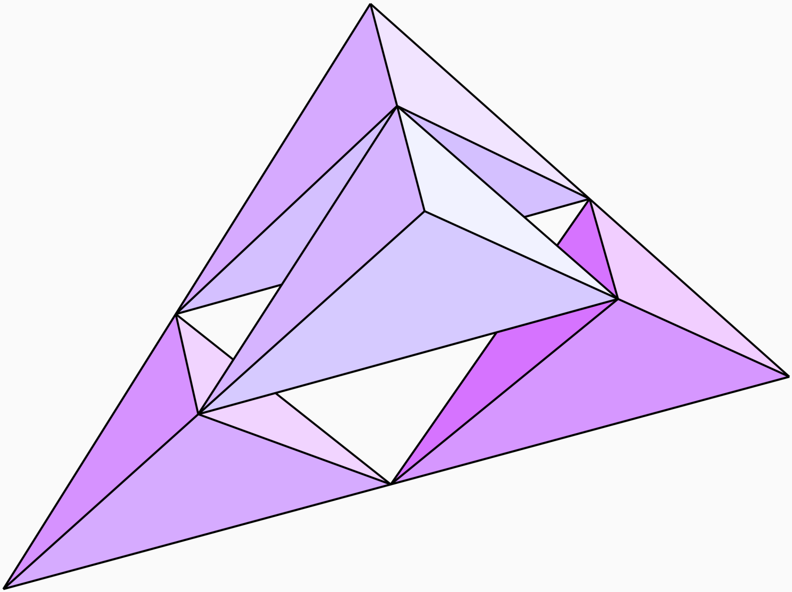
Structural Theorem: Construction of Solution

Theorem

Given $I = (n, s, b, C, m)$ and $\frac{b}{m} \in \mathsf{P}^*$. Let $\mathsf{F}_1 = \{f_1, \ldots, f_l\} \subseteq \mathsf{P}^*$ induced by the intersection of straight line between $\mathbf{0}$ and b. Then $\exists$ concentric cones $\mathsf{C}_1 \subset \mathsf{C}_2 \subset \ldots \subset \mathsf{C}_m$ with $b \in \mathsf{C}_m$ s.t. b can be represented by $O(n)$ integral vectors chosen from the vertices of F_1 or dominated by these, and an integral residue $r \in \mathsf{P}^*$.



Construction of Solution: Proof Case 1,2,3



Computation of Relevant Patterns: Conditions

Want to find explicit representation of $F \subset P^*$ satisfying following conditions:

- $\frac{b}{m} \in \text{conv}(0, f_1, \ldots, f_n)$.
- $\frac{s^\intercal b}{m} \leq s^\intercal f_i \leq \widetilde{C}$, for all $i \in [n]$, where $\widetilde{C}$ is defined according to the upper bound UB.
- all f_i are linearly independent.

Hindrance:

- No explicit representation of the corresponding polytope P^*.

Explicit Representation of P^*

Integral points in P^* [Cook et al., 1992, Hartmann, 1988]

Let $\mathcal{V}^*$ be the vertices of P^*. Then $|\mathcal{V}^*| \leq m^n \cdot (O(\log(\Delta)))^n$ and can be computed in $|\mathcal{V}^*| \cdot n^{O(n)} \cdot (m\log(\Delta))^{O(1)}$.

- As n is fixed, the explicit representation of P^* can be computed in polynomial time.
- Let $\mathcal{V}^* = \{p_1, \ldots, p_k\}$ be the vertex set that define P^*.

Computation of Relevant Patterns: Primal Dual Algorithm

- $\frac{b}{m} \in \text{conv}(0, f_1, \ldots, f_n)$.
- $\frac{s^\intercal b}{m} \leq s^\intercal f_i \leq \widetilde{C}$, for all $i \in [n]$, where $\widetilde{C}$ is defined according to the upper bound UB.
- all f_i are linearly independent.

- Can compute the vertices $\{f_1, \ldots, f_n\}$ using Primal-Dual method.
- To check the feasibility of the solution to the dual LP, we solve the separation problem which is knapsack problem in fixed dimension.

From Patterns to Solution

- **Diophantine equation:**
  $\sigma_1 f_1 + \ldots + \sigma_n f_n = b - r$, where $\sigma_i \in \mathbb{Z}_{\geq 0},\ \forall i \in [n]$

From Patterns to Solution

- **Diophantine equation:**
  $\sigma_1 f_1 + \ldots + \sigma_n f_n = b - r$, where $\sigma_i \in \mathbb{Z}_{\geq 0},\ \forall i \in [n]$
- **Problem:** The residual r is not known *a priori*, and hence without r the solution to the Diophantine equation is not necessary integral.

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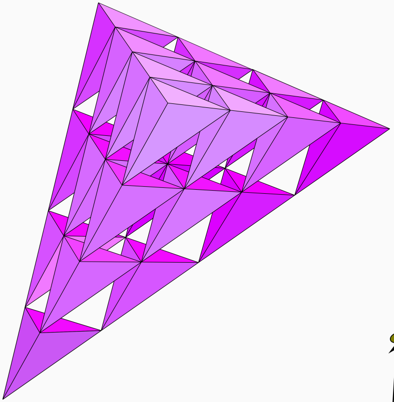

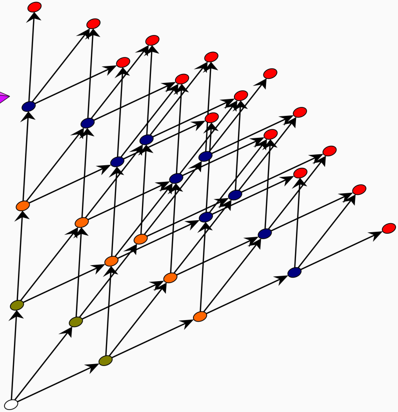

Thank You

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