STABILNOST LU RAZCEPA

Izrek 0.1. Naj bo $A \in \mathbb{R}^{n\times n}$ obrnljiva matrika, pri kateri se izvede LU razcep brez pivotiranja. Za izračunani matriki $\widehat{L}, \widehat{U}$ velja

$$A = \widehat{L}\widehat{U} + E,$$

kjer je

$$|E| \leq 3(n-1)u\left(|A| + |\widehat{L}||\widehat{U}|\right) + \mathcal{O}(u^2). \tag{0.1}$$

Dokaz. Dokazujemo z indukcijo na n. Za $n = 1$ je $A = a \in \mathbb{R}$, $\widehat{L} = 1$ in $\widehat{U} = a$. Zato je $E = 0$ in (0.1) velja. Privzemimo, da ocena (0.1) velja za $(n-1)\times(n-1)$ matrike in dokažimo, da velja z $n \times n$ matrike. Pišimo

$$A = \begin{pmatrix} \alpha & w^T \\ v & B \end{pmatrix},$$

kjer je $\alpha \in \mathbb{R}$, $v, w \in \mathbb{R}^{n-1}$ in $B \in \mathbb{R}^{(n-1)\times(n-1)}$. Naredimo prvi korak Gaussove eliminacije. Dobimo

$$A^{(1)} = \begin{pmatrix} 1 & 0 \\ \widehat{z} & \widehat{A}_1 \end{pmatrix},$$

kjer je

$$\widehat{z} = \mathrm{fl}\left(\frac{v}{\alpha}\right) = \frac{v}{\alpha} + f, \qquad |f| \leq \frac{|v|}{|\alpha|}u$$

in

$$\widehat{A}_1 = \mathrm{fl}(B - \widehat{z}w^T) = B - \widehat{z}w^T + F, \qquad |F| \leq 2u(|B| + |\widehat{z}||w|^T). \tag{0.2}$$

Sedaj izračunamo LU razcep matrike $\widehat{A}_1$. Po indukcijski predpostavki velja

$$\widehat{A}_1 = \widehat{L}_1\widehat{U}_1 + E_1, \qquad |E_1| \leq 3(n-2)u(|\widehat{A}_1| + |\widehat{L}_1||\widehat{U}_1|) + \mathcal{O}(u^2). \tag{0.3}$$

Zato velja

$$\widehat{L}\widehat{U} = \begin{pmatrix} 1 & 0 \\ \widehat{z} & \widehat{L}_1 \end{pmatrix}\begin{pmatrix} \alpha & w^T \\ 0 & \widehat{U}_1 \end{pmatrix} = \begin{pmatrix} \alpha & w^T \\ \alpha\widehat{z} & \widehat{z}w^T + \widehat{A}_1 - E_1 \end{pmatrix} = A + \underbrace{\begin{pmatrix} 0 & 0 \\ \alpha f & F - E_1 \end{pmatrix}}_{=:H}$$

Ocenimo še $|H|$. Velja

$$\begin{aligned} |F - E_1| &\leq |F| + |E_1| \\ &\leq 2u(|B| + |\widehat{z}||w|^T) + 3(n-2)u(|\widehat{A}_1| + |\widehat{L}_1||\widehat{U}_1|) + \mathcal{O}(u^2) \\ &= 3(n-1)u\left(|B| + |\widehat{z}||w|^T + |\widehat{L}_1||\widehat{U}_1|\right) + \mathcal{O}(u^2) \end{aligned}$$

kjer smo v drugi neenakosti uporabili (0.2) in (0.3), v tretji pa

$$|\widehat{A}_1| \leq (1 + 2u)(|B| + |\widehat{z}||w|^T) + \mathcal{O}(u^2).$$

Sledi

$$|H| \leq 3(n-1)u \cdot \Big(\underbrace{\begin{pmatrix} |\alpha| & |w|^T \\ |v| & |B| \end{pmatrix}}_{|A|} + \underbrace{\begin{pmatrix} 1 & 0 \\ |\widehat{z}| & |\widehat{L}_1| \end{pmatrix}}_{|\widehat{L}|} \underbrace{\begin{pmatrix} |\alpha| & |w|^T \\ 0 & |\widehat{U}_1| \end{pmatrix}}_{|\widehat{U}|} \Big).$$

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