Computation of the polynomial Q(θ) for a (y^2-x^3)-moment sequence with positive semidefinite M(3), without a measure.

Out[ ]=

We do reverse engineering and generate a univariate sequence γ= (γ_i)_{i<=18} of degree 18, such that the restriction of the Hankel matrix A_γ to columns and rows indexed by 1,X^2,X^3,...,X^9 is positive definite, but the restriction to columns and rows X,X^2,...,X^9 is not recursively generate sequence, i.e., column X^8 is in the span of the previous ones, but X^9 is not. Then we generate a sequence β={ β_{i,j}}_{i+j<=6} by β_{i,j}=γ_{2i+3j}. By the solution to the TMP on Y^2–X^3 (Corollary 4.3 in A. Zalar, The truncated Hamburger moment problems with gaps in the index set, Integ. Equ. Oper. Theory 93 (2021)), β will not have a (y^2–x^3)–supported representing measure.

First we generate a univariate sequence γ with 9 atoms 0,1/2,1/3,...,1/8.

In[1]:=
```
H[γ0_, γ1_, γ2_, γ3_, γ4_, γ5_, γ6_, γ7_, γ8_, γ9_, γ10_, γ11_, γ12_, γ13_, γ14_, γ15_,
   γ16_, γ17_, γ18_] = HankelMatrix[{γ0, γ1, γ2, γ3, γ4, γ5, γ6, γ7, γ8, γ9},
   {γ9, γ10, γ11, γ12, γ13, γ14, γ15, γ16, γ17, γ18}];
```

In[2]:=
```
f[x_] = H[1, x, x^(2), x^(3), x^(4), x^(5), x^(6), x^(7), x^(8), x^(9),
   x^(10), x^(11), x^(12), x^(13), x^(14), x^(15), x^(16), x^(17), x^(18)];
```

In[3]:=
```
F = f[0] + f[1/2] + f[1/3] + f[1/4] + f[1/5] + f[1/6] + f[1/7] + f[1/8];
```

In[4]:=
```
MatrixForm[F]
```

Out[4]//MatrixForm=

8	481 / 280	372 149 / 705 600	12 8… / 65 8…
481 / 280	372 149 / 705 600	12 852 473 / 65 856 000	40 71… / 497 87…
372 149 / 705 600	12 852 473 / 65 856 000	40 717 994 801 / 497 871 360 000	5 141 2… / 139 403…
12 852 473 / 65 856 000	40 717 994 801 / 497 871 360 000	5 141 275 445 731 / 139 403 980 800 000	6 091 02… / 351 298 0…
40 717 994 801 / 497 871 360 000	5 141 275 445 731 / 139 403 980 800 000	6 091 026 858 664 049 / 351 298 031 616 000 000	821 221 2… / 98 363 448…
5 141 275 445 731 / 139 403 980 800 000	6 091 026 858 664 049 / 351 298 031 616 000 000	821 221 274 202 733 619 / 98 363 448 852 480 000 000	1 010 667 59… / 247 875 891 1…
6 091 026 858 664 049 / 351 298 031 616 000 000	821 221 274 202 733 619 / 98 363 448 852 480 000 000	1 010 667 599 322 175 009 601 / 247 875 891 108 249 600 000 000	15 488 077 72… / 7 711 694 390…
821 221 274 202 733 619 / 98 363 448 852 480 000 000	1 010 667 599 322 175 009 601 / 247 875 891 108 249 600 000 000	15 488 077 721 039 289 426 059 / 7 711 694 390 034 432 000 000 000	173 952 332 188… / 174 901 228 765 9…
1 010 667 599 322 175 009 601 / 247 875 891 108 249 600 000 000	15 488 077 721 039 289 426 059 / 7 711 694 390 034 432 000 000 000	173 952 332 188 994 804 190 119 249 / 174 901 228 765 980 917 760 000 000 000	24 201 572 009 80… / 48 972 344 054 474…
15 488 077 721 039 289 426 059 / 7 711 694 390 034 432 000 000 000	173 952 332 188 994 804 190 119 249 / 174 901 228 765 980 917 760 000 000 000	24 201 572 009 804 864 462 019 724 819 / 48 972 344 054 474 656 972 800 000 000 000	30 369 616 473 726… / 123 410 307 017 276 1…

We increase the moment γ_18 so that the restriction of A_γ to columns/rows X,...,X^9 is not RG.

In[5]:=
```
F⟦10, 10⟧ = F⟦10, 10⟧ + 1/10;
```

In[6]:=
```
N[NullSpace[F⟦{2, 3, 4, 5, 6, 7, 8, 9, 10}, {2, 3, 4, 5, 6, 7, 8, 9, 10}⟧]]
```

Out[6]=
```
{{-0.0000248016, 0.000868056, -0.0126736,
  0.0998264, -0.456944, 1.21181, -1.71786, 1., 0.}}
```

We check that the restriction of A_γ to columns/rows 1,X^2, ..., X^9 is positive definite .

In[7]:=
```
PositiveDefiniteMatrixQ[F⟦{1, 3, 4, 5, 6, 7, 8, 9, 10}, {1, 3, 4, 5, 6, 7, 8, 9, 10}⟧]
```

Out[7]=
```
True
```

Now we define the sequence β={ β_{i,j}}_{i+j<=6} by β_{i,j}=γ_{2i+3j}.

In[8]:=
```
(*Degree 0*)
β00 = F⟦1, 1⟧;
(*Degree 1*)
β10 = F⟦3, 1⟧;
β01 = F⟦4, 1⟧;
(*Degree 2*)
β20 = F⟦5, 1⟧;
β11 = F⟦6, 1⟧;
β02 = F⟦7, 1⟧;
(*Degree 3*)
β30 = F⟦7, 1⟧;
β21 = F⟦8, 1⟧;
β12 = F⟦9, 1⟧;
β03 = F⟦10, 1⟧;
(*Degree 4*)
β40 = F⟦9, 1⟧;
β31 = F⟦10, 1⟧;
β22 = F⟦10, 2⟧;
β13 = F⟦10, 3⟧;
β04 = F⟦10, 4⟧;
(*Degree 5*)
β50 = F⟦10, 2⟧;
β41 = F⟦10, 3⟧;
β32 = F⟦10, 4⟧;
β23 = F⟦10, 5⟧;
β14 = F⟦10, 6⟧;
β05 = F⟦10, 7⟧;
(*Degree 4*)
β60 = F⟦10, 4⟧;
β51 = F⟦10, 5⟧;
β42 = F⟦10, 6⟧;
β33 = F⟦10, 7⟧;
β24 = F⟦10, 8⟧;
β15 = F⟦10, 9⟧;
β06 = F⟦10, 10⟧;
```

We generate the matrix M(3).

In[36]:=
```
M[β00_,
   β10_, β01_,
   β20_, β11_, β02_,
   β30_, β21_, β12_, β03_,
   β40_, β31_, β22_, β13_, β04_,
   β50_, β41_, β32_, β23_, β14_, β05_,
   β60_, β51_, β42_, β33_, β24_, β15_, β06_] =
  {{β00, β10, β01, β20, β11, β02, β30, β21, β12, β03},
   {β10, β20, β11, β30, β21, β12, β40, β31, β22, β13},
   {β01, β11, β02, β21, β12, β03, β31, β22, β13, β04},
   {β20, β30, β21, β40, β31, β22, β50, β41, β32, β23},
   {β11, β21, β12, β31, β22, β13, β41, β32, β23, β14},
   {β02, β12, β03, β22, β13, β04, β32, β23, β14, β05},
   {β30, β40, β31, β50, β41, β32, β60, β51, β42, β33},
   {β21, β31, β22, β41, β32, β23, β51, β42, β33, β24},
   {β12, β22, β13, β32, β23, β14, β42, β33, β24, β15},
   {β03, β13, β04, β23, β14, β05, β33, β24, β15, β06}
  };
```

We generate the block B(4) of the extension M(4).

In[37]:=
```
B[β00_,
   β10_, β01_,
   β20_, β11_, β02_,
   β30_, β21_, β12_, β03_,
   β40_, β31_, β22_, β13_, β04_,
   β50_, β41_, β32_, β23_, β14_, β05_,
   β60_, β51_, β42_, β33_, β24_, β15_, β06_,
   β70_, β61_, β52_, β43_, β34_, β25_, β16_, β07_] =
  {
   {β40, β31, β22, β13, β04},
   {β50, β41, β32, β23, β14},
   {β41, β32, β23, β14, β05},
   {β60, β51, β42, β33, β24},
   {β51, β42, β33, β24, β15},
   {β42, β33, β24, β15, β06},
   {β70, β61, β52, β43, β34},
   {β61, β52, β43, β34, β25},
   {β52, β43, β34, β25, β16},
   {β43, β34, β25, β16, β07}
  };
```

We compute the smallest C (4) in the extension M (4) .

```
In[38]:= Cblock[θ_, ϕ_, ψ_] = Transpose[B[β00,
          β10, β01,
          β20, β11, β02,
          β30, β21, β12, β03,
          β40, β31, β22, β13, β04,
          β50, β41, β32, β23, β14, β05,
          β60, β51, β42, β33, β24, β15, β06,
          β42, β33, β24, β15, β06, θ, ϕ, ψ]⟦{1, 2, 3, 4, 5, 6, 8, 9, 10}, {1, 2, 3, 4, 5}⟧].
       Inverse[(M[β00,
           β10, β01,
           β20, β11, β02,
           β30, β21, β12, β03,
           β40, β31, β22, β13, β04,
           β50, β41, β32, β23, β14, β05,
           β60, β51, β42, β33, β24, β15, β06]⟦
          {1, 2, 3, 4, 5, 6, 8, 9, 10}, {1, 2, 3, 4, 5, 6, 8, 9, 10}⟧)].
       (B[β00,
          β10, β01,
          β20, β11, β02,
          β30, β21, β12, β03,
          β40, β31, β22, β13, β04,
          β50, β41, β32, β23, β14, β05,
          β60, β51, β42, β33, β24, β15, β06,
          β42, β33, β24, β15, β06, θ, ϕ, ψ]⟦{1, 2, 3, 4, 5, 6, 8, 9, 10}, {1, 2, 3, 4, 5}⟧);
```

We compute the quadratic polynomial Q (θ) = Q2 θ^2 + Q1 θ + Q0 determining the existence of a flat extension .

```
In[39]:= hatC = Simplify[Cblock[θ, ϕ, ψ]];
         hankelEQ =
           Simplify[{hatC⟦4, 2⟧ - hatC⟦3, 3⟧, hatC⟦5, 2⟧ - hatC⟦4, 3⟧, hatC⟦5, 3⟧ - hatC⟦4, 4⟧}];
         stheta = Simplify[Solve[{hankelEQ⟦1⟧ == 0, hankelEQ⟦2⟧ == 0}, {ϕ, ψ}, Reals]];
         Length[stheta];
         critpoly = Simplify[hankelEQ⟦3⟧ /. stheta⟦1⟧]

Out[43]=
         - 16 257 024
```

The polynomial Q (θ) is a constant -16257024. Hence, a flat extension does not exist.