

CYCLIC POLYNOMIALS IN DIRICHLET-TYPE SPACES OF THE UNIT BIDISK

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ABSTRACT. For $\alpha \in \mathbb{R}$, we consider the scale of function spaces, namely the Dirichlet-type space \mathcal{D}_α consisting of holomorphic functions on the unit bidisk \mathbb{D}^2 , $f(z, w) = \sum_{k,l=0}^{\infty} a_{kl} z^k w^l$ such that

$$\sum_{k,l=0}^{\infty} (k+l+1)^\alpha |a_{kl}|^2 < \infty.$$

In this paper, we solve an open problem posed in [18, Open problem 1], which asks whether the polynomial $2 - z_1 - z_2$ is cyclic in \mathcal{D}_α for $\frac{3}{2} < \alpha \leq 2$. We provide an affirmative answer and, as a consequence, complete the characterization of cyclic polynomials in \mathcal{D}_α . In addition, we establish several properties of cyclic functions and present alternative proofs of some cases previously obtained in [18].

1. INTRODUCTION

Let \mathbb{C} denote the complex plane, $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$ the open unit disk and $\mathbb{T} = \{z \in \mathbb{C}, |z| = 1\}$ the unit circle in the complex plane. Given a polynomial $p \in \mathbb{C}[z_1, z_2]$, its bidegree is the pair (m, n) , where m is the highest degree of p in the variable z_1 , and n is the highest degree of p in the variable z_2 . We write

$$\mathcal{Z}(p) := \{(z_1, z_2) \in \mathbb{C}^2 : p(z_1, z_2) = 0\}$$

for the vanishing set of p . A nonzero polynomial p is said to be *irreducible* if $p = qr$ with $q, r \in \mathbb{C}[z_1, z_2]$ implies that $q \in \mathbb{C}$ or $r \in \mathbb{C}$.

In the classical Hardy space $H^2(\mathbb{D})$, a function f is called *cyclic* if the smallest closed, shift-invariant subspace generated by its polynomial multiples coincides with the whole space. By Beurling's theorem, cyclic functions in this setting are precisely the outer functions, i.e., $f(0) \neq 0$ and

$$\log |f(0)| = \int_0^{2\pi} \log |f(e^{i\theta})| \frac{d\theta}{2\pi},$$

making the theory both elegant and complete. In contrast, for the Hardy space on the bidisk $H^2(\mathbb{D}^2)$, cyclic functions are outer but there exists an outer function which is not cyclic [17]. With the present understanding, a characterization of cyclic function seems to be a harder problem in several

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variables. However, cyclic polynomials are characterized in $H^2(\mathbb{D}^n)$; they are precisely those polynomials which do not have zeros on the polydisk \mathbb{D}^n [16]. This gap between the univariate and multivariate settings motivates investigations of cyclicity in other spaces, such as the Dirichlet space and the Dirichlet-type spaces.

In the Dirichlet space D of the unit disk, Brown and Shields conjectured [5, Question 12] that a function $f \in D$ is cyclic if and only if it is outer and its boundary zero set has logarithmic capacity zero. They were able to establish the forward direction, while the converse, despite several attempts, remains open till now. Several partial results are known [10, 11]. The Brown–Shields conjecture continues to be a central open problem, motivating further exploration of cyclicity in Dirichlet-type spaces [1, 2, 3, 15].

1.1. Dirichlet type spaces. We now introduce the following Dirichlet-type space of the unit bidisk where we will investigate the cyclicity of polynomials. For $\alpha \in \mathbb{R}$, the Dirichlet-type space denoted by \mathcal{D}_α on \mathbb{D}^2 , consists of all holomorphic functions $f(z, w) = \sum_{k,l=0}^{\infty} a_{kl} z^k w^l$ such that

$$\|f\|_\alpha^2 := \sum_{k,l=0}^{\infty} (k+l+1)^\alpha |a_{kl}|^2 < \infty.$$

Note that for $\alpha = 0$, we recover the Hardy space $H^2(\mathbb{D}^2)$ of the unit bidisk and for $\alpha = 1$, the space \mathcal{D}_1 was introduced in [4] in connection with toral 2-isometries. This space has also been considered by Torkinejad Ziarati in [18], where the weight $(k+l+1)^\alpha$ is replaced by $(k+l+2)^\alpha$. However, it can be seen that these two norms are equivalent. Indeed, for $\alpha \geq 0$ we have

$$(k+l+1)^\alpha \leq (k+l+2)^\alpha \leq 2^\alpha (k+l+1)^\alpha,$$

while for $\alpha \leq 0$ it holds that

$$2^\alpha (k+l+1)^\alpha \leq (k+l+2)^\alpha \leq (k+l+1)^\alpha.$$

Investigations in this paper are motivated by the cyclicity results [2, 3] obtained for the Dirichlet-type space \mathfrak{D}_α , $\alpha \in \mathbb{R}$, on \mathbb{D}^2 , which consists of holomorphic functions $f(z, w) = \sum_{k,l=0}^{\infty} a_{kl} z^k w^l$ such that

$$\|f\|_{\mathfrak{D}_\alpha}^2 := \sum_{k,l=0}^{\infty} (k+1)^\alpha (l+1)^\alpha |a_{kl}|^2 < \infty.$$

From the norm definitions, it is straightforward to see that for $\alpha \geq 0$ we have

$$\|\cdot\|_\alpha \leq \|\cdot\|_{\mathfrak{D}_\alpha} \leq \|\cdot\|_{2\alpha} \quad (\mathcal{D}_{2\alpha} \subseteq \mathfrak{D}_\alpha \subseteq \mathcal{D}_\alpha), \quad (1.1)$$

and for $\alpha \leq 0$,

$$\|\cdot\|_{2\alpha} \leq \|\cdot\|_{\mathfrak{D}_\alpha} \leq \|\cdot\|_\alpha \quad (\mathcal{D}_\alpha \subseteq \mathfrak{D}_\alpha \subseteq \mathcal{D}_{2\alpha}). \quad (1.2)$$

Inequalities (1.1), (1.2) allow us to transfer properties between both kinds of Dirichlet-type spaces. In particular, we focus here on the notion of cyclicity.

Note that both spaces \mathcal{D}_α and \mathfrak{D}_α can be viewed as a generalization of the univariate Dirichlet space D_α , $\alpha \in \mathbb{R}$, which consists of holomorphic function

$f : \mathbb{D} \rightarrow \mathbb{C}$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$, such that

$$\|f\|_{D_\alpha}^2 := \sum_{k=0}^{\infty} (k+1)^\alpha |a_k|^2 < \infty. \quad (1.3)$$

1.2. Cyclic functions. In this subsection we explain the motivation to study cyclic function in the Dirichlet-type space \mathcal{D}_α . Let (M_{z_1}, M_{z_2}) be a pair of shift operators acting on \mathcal{D}_α , i.e.,

$$(M_{z_1} f)(z_1, z_2) := z_1 f(z_1, z_2), \quad (M_{z_2} f)(z_1, z_2) := z_2 f(z_1, z_2), \quad f \in \mathcal{D}_\alpha.$$

It is straightforward to verify that both M_{z_1} and M_{z_2} are bounded linear operators on \mathcal{D}_α . From the operator-theoretic point of view, an important problem is to describe the closed subspaces of \mathcal{D}_α that are invariant under these shifts, namely those $\mathcal{M} \subseteq \mathcal{D}_\alpha$ for which

$$M_{z_1} \mathcal{M} \subseteq \mathcal{M} \quad \text{and} \quad M_{z_2} \mathcal{M} \subseteq \mathcal{M}.$$

A key step towards this description is to understand when a function $f \in \mathcal{D}_\alpha$ is cyclic, i.e., when the closed linear span

$$[f] := \overline{\text{span}}\{z_1^k z_2^\ell f : k, \ell \geq 0\}$$

coincides with the entire space \mathcal{D}_α . It is clear from the definition that $[f]$ is the smallest closed subspace that contains f and is invariant under the shift operators M_{z_1} and M_{z_2} . Clearly, at least one cyclic vector always exists, e.g., the constant function $f(z_1, z_2) \equiv 1$ is cyclic, because polynomials in two variables are dense in \mathcal{D}_α . In the next section, we will see that a necessary condition for cyclicity is that f has no zeros in \mathbb{D}^2 .

Note that if $g \in [f]$, then $[g] \subseteq [f]$. Thus to check f is cyclic in \mathcal{D}_α , it suffices to show that there exists a sequence of polynomials $p_n \in \mathbb{C}[z_1, z_2]$ such that

$$\|p_n f - 1\|_\alpha \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

1.3. Cyclic polynomials in \mathfrak{D}_α . Recent work of Bénéteau et al. [3, Theorem] provides a complete characterization of cyclic polynomials in \mathfrak{D}_α , $\alpha \in \mathbb{R}$, on the bidisk (see [15] for anisotropic setting and [9] for the unit ball in \mathbb{C}^2). Their main result shows that the cyclicity of an irreducible polynomial depends intricately on the structure of its zero set on the distinguished boundary \mathbb{T}^2 . In particular, while non-vanishing in the bidisk is necessary for cyclicity, additional restrictions on the boundary zero set become decisive when the parameter α of the Dirichlet-type space \mathfrak{D}_α lies in the range $(\frac{1}{2}, \infty)$. We recall the result for the reader's convenience.

Theorem 1.1 ([3, Theorem]). *Let $p \in \mathbb{C}[z_1, z_2]$ be an irreducible polynomial with no zeros in the bidisk. We have the following:*

- (i) *If $\alpha \leq \frac{1}{2}$, then p is cyclic in \mathfrak{D}_α .*
- (ii) *If $\frac{1}{2} < \alpha \leq 1$, then p is cyclic in \mathfrak{D}_α if and only if $\mathbb{Z}(p) \cap \mathbb{T}^2$ is an empty or finite set or p is a constant multiple of $\zeta - z_1$ or of $\zeta - z_2$ for some $\zeta \in \mathbb{T}$.*
- (iii) *If $\alpha > 1$, then p is cyclic in \mathfrak{D}_α if and only if $\mathbb{Z}(p) \cap \mathbb{T}^2$ is empty.*

1.4. Cyclic polynomials in \mathcal{D}_α . The question of characterization of cyclic polynomials in \mathcal{D}_α was very recently studied by Torkinejad Ziarati [18], leaving one open case open which depends on the following questions posed by the author (see [18, Open problem 1]).

Question 1.2. For $\frac{3}{2} \leq \alpha \leq 2$, determine whether the polynomial $2 - z_1 - z_2$ is cyclic in \mathcal{D}_α .

In this paper, we present a solution to this problem and provide alternative proofs of several results from [18, Section 5], thus completing the characterization of cyclic polynomials in \mathcal{D}_α (see also [18, Theorem 31]):

Theorem 1.3. *Let $p \in \mathbb{C}[z_1, z_2]$ be an irreducible polynomial with no zeros in the bidisk. We have the following:*

- (i) *If $\alpha \leq 1$, then p is cyclic in \mathcal{D}_α .*
- (ii) *If $1 < \alpha \leq 2$, then p is cyclic in \mathcal{D}_α if and only if $\mathcal{Z}(p) \cap \mathbb{T}^2$ is empty or finite.*
- (iii) *If $\alpha > 2$, then p is cyclic in \mathcal{D}_α if and only if $\mathcal{Z}(p) \cap \mathbb{T}^2$ is empty.*

Remark 1.4. (i) Note that the Dirichlet-type spaces studied in [3] do not distinguish between the cyclicity of a polynomial having finitely many zeros on \mathbb{T}^2 and that of $1 - z_i$, $i = 1, 2$, which has infinitely many zeros on \mathbb{T}^2 (see Theorem 1.1.(ii)). In fact, the latter case was used there to establish cyclicity for polynomials having finitely many zeros on \mathbb{T}^2 . In contrast, in our setting, these two cases exhibit different behaviour in \mathcal{D}_α .

(ii) Theorem 1.3.(iii) follows directly from Theorem 1.1.(iii) together with the first inclusion in (1.1). In contrast, establishing parts (i) and (ii) of Theorem 1.1 is less straightforward. For the proof of part (i) we refer to [18]. The proof of part (ii) relies on the techniques used in the proof of [3, Theorem 3.1], together with arguments from [15, Appendix A].

1.5. Organization of the paper. In Section 2, we recall some definitions and necessary results to prove our main result. In Section 3, we present a solution to Question 1.2. In Section 4, we present a proof of Theorem 1.3. This is done by first proving the case of cyclicity of the polynomial having at most finitely many zeros on \mathbb{T}^2 in \mathcal{D}_α , $\alpha \leq 2$. We then complete the proof of Theorem 1.3 using the results of the preceding sections. In Section 5, we present a few necessary conditions for the cyclicity of a function in \mathcal{D}_α , $\alpha \in \mathbb{R}$. In Section 6, we conclude the paper with a discussion on the cyclicity of $z_i - a$, with $|a| \geq 1$, $1 - z_1 z_2$ and a result based on capacity.

2. PRELIMINARIES

In this section, we list some properties of cyclic function which are needed to give a self-contained treatment of the proof of Theorem 1.3 (see Subsection 2.1) and recall an inequality on the value of a real analytic function (see Subsection 2.2).

2.1. Some properties of cyclic functions in \mathcal{D}_α . Note that \mathcal{D}_α , $\alpha \in \mathbb{R}$, is a reproducing kernel Hilbert space, i.e., the evaluation map e_w , for $w \in \mathbb{D}$,

$e_w(f) := f(w)$ is a continuous linear functional on \mathcal{D}_α and its multiplier space is defined as

$$M(\mathcal{D}_\alpha) = \{\phi : \mathbb{D}^2 \rightarrow \mathbb{C} : \phi f \in \mathcal{D}_\alpha \text{ for all } f \in \mathcal{D}_\alpha\}.$$

Elements of $M(\mathcal{D}_\alpha)$ are called *multipliers* of \mathcal{D}_α . It is easy to verify that all polynomials are multipliers of \mathcal{D}_α , $\alpha \in \mathbb{R}$.

Some properties of cyclic functions are the following:

- (i) A cyclic function in \mathcal{D}_α can not vanish on the bidisk. To see this, take p_n to be a sequence of polynomials such that $\|p_n f - 1\|_\alpha \rightarrow 0$. Since evaluations are continuous, the conclusion follows from the following expression

$$p_n(z_1, z_2)f(z_1, z_2) - 1 = e_{(z_1, z_2)}(p_n f - 1).$$

- (ii) Given a cyclic function $f \in \mathcal{D}_\alpha$, $\alpha \in \mathbb{R}$, the function defined by $g(z_1, z_2) := f(\zeta z_1, \eta z_2)$, where $\zeta, \eta \in \mathbb{T}$, is clearly also cyclic in \mathcal{D}_α . Indeed, if there exists a sequence of polynomials $\{p_n\}_{n \in \mathbb{Z}_+}$ such that

$$\|p_n f - 1\|_\alpha \rightarrow 0, \tag{2.1}$$

then the sequence defined by $q_n(z_1, z_2) := p_n(\zeta z_1, \eta z_2)$ satisfies

$$\|q_n g - 1\|_\alpha = \|p_n f - 1\|_\alpha \rightarrow 0,$$

proving cyclicity of g .

- (iii) Assume that f is a reducible polynomial with $f = gh$ for some nonconstant polynomials g and h . Then $f = gh$ is cyclic in \mathcal{D}_α if and only if g and h are cyclic in \mathcal{D}_α . Let us verify:

If f is cyclic, then there is a sequence of polynomials $\{p_n\}_{n \in \mathbb{Z}_+}$ satisfying (2.1). But then the sequence $\{r_n\}_{n \in \mathbb{Z}_+}$ and $\{s_n\}_{n \in \mathbb{Z}_+}$, where $r_n := p_n g$ and $s_n := p_n h$, satisfy $\|r_n h - 1\|_\alpha \rightarrow 0$ and $\|s_n g - 1\|_\alpha \rightarrow 0$, proving cyclicity of g and h .

Conversely, assume that g and h are cyclic. Then there exists a sequence of polynomials $\{r_n\}_{n \in \mathbb{Z}_+}$ such that $\|r_n g - 1\|_\alpha \rightarrow 0$. Note that

$$\|r_n g h - h\|_\alpha \leq \|M_h\| \|r_n g - 1\|_\alpha$$

where $\|M_h\|$ is an operator norm of the multiplication operator M_h . This shows that $h \in [f]$. Hence $[h] \subseteq [f]$. Since h is cyclic, it follows that f is cyclic.

Therefore, it suffices to characterize cyclicity of irreducible polynomials in \mathcal{D}_α .

2.2. Łojasiewicz's inequality. In the proof of Theorem 4.1 below, the following inequality will be used essentially.

Theorem 2.1 ([14, Łojasiewicz's inequality]). *Let f be a nonzero real analytic function on an open set $U \subseteq \mathbb{R}^n$. Assume the zero set $\mathcal{Z}(f)$ of f in U is nonempty. Let E be a compact subset of U . Then there are constants $C > 0$ and $q \in \mathbb{N}$, depending on E , such that*

$$|f(x)| \geq C \cdot \text{dist}(x, \mathcal{Z}(f))^q$$

for every $x \in E$.

3. CYCLICITY OF THE POLYNOMIAL $2 - z_1 - z_2$

In the section, we answer the Question 1.2 using the techniques from [15, Appendix A].

Theorem 3.1. *Let $\alpha \leq 2$. Then $p(z_1, z_2) = 2 - z_1 - z_2$ is cyclic in \mathcal{D}_α .*

Proof. It suffices to show that p is cyclic in \mathcal{D}_2 . Let $f \in \mathcal{D}_2$ be such that $f \perp [p]$. Consider the following series of f ,

$$f(z_1, z_2) = \sum_{i,j=0}^{\infty} \frac{b_{i,j}}{(i+j+1)^2} z_1^i z_2^j.$$

We will show that $f = 0$, or equivalently $b_{i,j} = 0$ for $i, j \in \mathbb{Z}_+$. By $f \perp [p]$, it follows that

$$2b_{k,l} = b_{k+1,l} + b_{k,l+1}, \quad k, l \in \mathbb{Z}_+. \quad (3.1)$$

Since $f \in \mathcal{D}_2$, a new function

$$g(z_1, z_2) := \sum_{i,j \geq 0} b_{i,j} z_1^i z_2^j$$

belongs to \mathcal{D}_{-2} , and by (3.1),

$$(z_1 + z_2 - 2z_1 z_2)g(z_1, z_2) = z_1 g(z_1, 0) + z_2 g(0, z_2), \quad (z_1, z_2) \in \mathbb{D}^2. \quad (3.2)$$

We next introduce the substitutions

$$z_1 = \frac{\zeta}{\zeta - 1}, \quad z_2 = \frac{\zeta}{\zeta + 1}. \quad (3.3)$$

Observe that $z_1 \in \mathbb{D}$ precisely when $\Re \zeta < \frac{1}{2}$, and $z_2 \in \mathbb{D}$ precisely when $\Re \zeta > -\frac{1}{2}$, where $\Re \zeta$ denotes the real part of the complex number ζ . By substituting the above expressions for z_1 and z_2 into (3.2), we get

$$0 = \frac{\zeta}{\zeta - 1} g\left(\frac{\zeta}{\zeta - 1}, 0\right) + \frac{\zeta}{\zeta + 1} g\left(0, \frac{\zeta}{\zeta + 1}\right), \quad \text{for } -\frac{1}{2} < \Re \zeta < \frac{1}{2}.$$

Thus, defining $h : \mathbb{C} \rightarrow \mathbb{C}$ by

$$h(\zeta) = \begin{cases} \frac{1}{\zeta - 1} g\left(\frac{\zeta}{\zeta - 1}, 0\right), & \text{if } \Re \zeta < \frac{1}{2}, \\ -\frac{1}{\zeta + 1} g\left(0, \frac{\zeta}{\zeta + 1}\right), & \text{if } \Re \zeta > -\frac{1}{2}, \end{cases}$$

we obtain that h is a well-defined entire function. Note that

$$\sum_{k \geq 0} \frac{|b_{k0}|^2}{(k+1)^2} \asymp \int_{\mathbb{D}} |g(z_1, 0)|^2 (1 - |z_1|^2) dA(z_1).$$

This can be verified by integrating the right-hand side using polar coordinates. Thus we have

$$\begin{aligned} \sum_{k \geq 0} \frac{|b_{k0}|^2}{(k+1)^2} &\asymp \int_{\Re \zeta < 1/2} |(\zeta - 1)h(\zeta)|^2 \left(1 - \left|\frac{\zeta}{\zeta - 1}\right|^2\right) \frac{dA(\zeta)}{|\zeta - 1|^4} \\ &= \int_{\Re \zeta < 1/2} |h(\zeta)|^2 \frac{1 - 2\Re \zeta}{|\zeta - 1|^4} dA(\zeta), \end{aligned} \quad (3.4)$$

and similarly,

$$\begin{aligned} \sum_{l \geq 0} \frac{|b_{0l}|^2}{(l+1)^2} &\asymp \int_{\mathbb{D}} |g(0, z_2)|^2 (1 - |z_2|^2) dA(z_2) \\ &= \int_{\Re \zeta > -1/2} |(\zeta + 1)h(\zeta)|^2 \left(1 - \left|\frac{\zeta}{\zeta + 1}\right|^2\right) \frac{dA(\zeta)}{|\zeta + 1|^4} \\ &= \int_{\Re \zeta > -1/2} |h(\zeta)|^2 \frac{1 + 2\Re \zeta}{|\zeta + 1|^4} dA(\zeta). \end{aligned}$$

Both series are finite due to $g \in \mathcal{D}_{-2}$; hence, the sum of the two integrals is finite, and consequently,

$$\int_{|\zeta| > 1} \frac{|h(\zeta)|^2}{|\zeta|^4} dA(\zeta) < \infty.$$

This forces h to be a polynomial of degree at most 1, i.e.,

$$h(\zeta) = a(\zeta - 1) + b, \quad a, b \in \mathbb{C}. \quad (3.5)$$

Assume that $a \neq 0$. Using (3.5) in (3.4), we get

$$\int_{\Re \zeta < 1/2} \frac{1 - 2\Re \zeta}{|\zeta - 1|^2} dA(\zeta) < \infty,$$

which is a contradiction. Hence, $a = 0$. Note that

$$\begin{aligned} g(z_1, 0) &\underset{(3.3)}{=} \frac{1}{z_1 - 1} h\left(\frac{z_1}{z_1 - 1}\right) = \frac{b}{z_1 - 1}, \\ g(0, z_2) &\underset{(3.3)}{=} -\frac{1}{z_2 + 1} h\left(\frac{z_2}{z_2 + 1}\right) = -\frac{b}{z_2 + 1}. \end{aligned} \quad (3.6)$$

Using (3.6) in (3.2), we get

$$\begin{aligned} (z_1 + z_2 - 2z_1 z_2)g(z_1, z_2) &= \frac{bz_1}{z_1 - 1} - \frac{bz_2}{z_2 + 1}, \quad (z_1, z_2) \in \mathbb{D}^2, \\ &= \frac{b(z_1 + z_2)}{(z_1 - 1)(z_2 - 1)}, \quad (z_1, z_2) \in \mathbb{D}^2. \end{aligned} \quad (3.7)$$

In particular, restricting (3.7) to the curve $z_2 = -z_1 + cz_1^2$, $c \in \mathbb{R}$, we get

$$g(z_1, -z_1 + cz_1^2) = \frac{bc}{(c + 2 - cz_1)(z_1 - 1)(-z_1 + cz_1^2 + 1)}.$$

Therefore

$$g(0, 0) = \lim_{z_1 \rightarrow 0} g(z_1, -z_1 + cz_1^2) = \frac{-bc}{c + 2}$$

Since c was arbitrary, this implies $b = 0$. Therefore $h = 0$ and consequently $g = 0$ and $f = 0$. \square

Remark 3.2. The proof shows that if a sequence of complex numbers $\{b_{k,l}\}_{k,l \in \mathbb{Z}_+}$ satisfies (3.1) and

$$\sum_{k,l \in \mathbb{Z}_+} \frac{|b_{k,l}|^2}{(k + l + 1)^2} < \infty,$$

then $b_{k,l} = 0$ for all $k, l \in \mathbb{Z}_+$.

The following result is an immediate and noteworthy consequence.

Corollary 3.3. *For $\zeta_1, \zeta_2 \in \mathbb{T}$, $p(z_1, z_2) = 2 - \zeta_1 z_1 - \zeta_2 z_2$ is cyclic in \mathcal{D}_α for $\alpha \leq 2$.*

Proof. This follows from Lemma 3.1 together with the fact that cyclicity is preserved under rotation. \square

4. A PROOF OF THEOREM 1.3

First, we investigate the cyclicity of polynomials in \mathcal{D}_α , $\alpha \leq 2$, that possess only finitely many zeros on \mathbb{T}^2 . In the previously studied settings (see [3, Section 3]), the cyclicity of the polynomial $1 - z_i$ was essentially used in the proof of the main result [3, Theorem 3.1]. However, in our framework, $1 - z_i$ needs not be replaced by another polynomial cyclic in \mathcal{D}_α , $\alpha \leq 2$, since $1 - z_i$ is cyclic in \mathcal{D}_α only for $\alpha \leq 1$. It turns out that $2 - z_1 - z_2$ serves as a suitable substitute.

Theorem 4.1. *Consider a polynomial $p \in \mathbb{C}[z_1, z_2]$ having no zeros in \mathbb{D}^2 and finitely many on \mathbb{T}^2 . Then p is cyclic in \mathcal{D}_α for $\alpha \leq 2$.*

The proof of Theorem 4.1 will parallel the arguments in the proof of [3, Theorem 3.2], which deals with the cyclicity in \mathfrak{D}_α , using the modifications described in the paragraph before Theorem 4.1.

The following result, which is an analog of [3, Lemma 3.3], will allow us to compare polynomials having finitely many zeros on \mathbb{T}^2 with the polynomials of the type $2 - \zeta z_1 - \eta z_2$, $\zeta, \eta \in \mathbb{T}$, whose cyclicity has already been established in Corollary 3.3.

Lemma 4.2. *Suppose $f \in \mathbb{C}[z_1, z_2]$ has no zeros in \mathbb{D}^2 and finitely many on \mathbb{T}^2 , i.e., $\{(\zeta_j, \eta_j) \in \mathbb{T}^2, j = 1, \dots, k\}$. Then for any integer k there exists sufficiently large N such that the function*

$$Q(z_1, z_2) = \frac{\prod_{i=1}^k (2 - \zeta_i^{-1} z_1 - \eta_i^{-1} z_2)^N}{f(z_1, z_2)}$$

is k -times differentiable on \mathbb{T}^2 .

Proof. Let $\{(\zeta_j, \eta_j) \in \mathbb{T}^2, j = 1, \dots, k\}$ be as in the statement of the lemma and define the polynomials

$$p_j(z_1, z_2) = 2 - \zeta_j^{-1} z_1 - \eta_j^{-1} z_2.$$

Clearly, p_j has only one zero (ζ_j, η_j) on \mathbb{T}^2 . Note that

$$|p_j(z_1, z_2)|^2 \leq |z_1 - \zeta_j|^2 + |z_2 - \eta_j|^2 + 2|(z_1 - \zeta_j)(z_2 - \eta_j)|. \quad (4.1)$$

Writing $z_1, z_2, \zeta_j, \eta_j \in \mathbb{T}$ as

$$z_1 = e^{ix_1}, z_2 = e^{ix_2}, \zeta_j = e^{iy_{1,j}}, \eta_j = e^{iy_{2,j}},$$

where $x_1, x_2, y_{1,j}, y_{2,j} \in [0, 2\pi)$, respectively, (4.1) becomes

$$\begin{aligned} |p_j(e^{ix_1}, e^{ix_2})|^2 &\leq |e^{ix_1} - e^{iy_{1,j}}|^2 + |e^{ix_2} - e^{iy_{2,j}}|^2 \\ &\quad + 2|(e^{ix_1} - e^{iy_{1,j}})(e^{ix_2} - e^{iy_{2,j}})|. \end{aligned} \quad (4.2)$$

Define

$$r(x_1, x_2) = |f(e^{ix_1}, e^{ix_2})|^2, \quad (x_1, x_2) \in \mathbb{R}^2,$$

and let $\mathcal{Z}(r)$ denote its zero set. Consider the compact set $E = [0, 2\pi]^2$. By Theorem 2.1, there exist constant $C > 0$ and $q \in \mathbb{N}$ such that

$$r(x) \geq C \operatorname{dist}(x, \mathcal{Z}(r))^q, \quad x \in E.$$

Since the set $\mathcal{Z}(r) \cap E$ consists of finitely many points, there is a constant $c > 0$ satisfying

$$\operatorname{dist}(x, \mathcal{Z}(r))^2 \geq c \prod_{y \in \mathcal{Z}(r) \cap E} |x - y|^2, \quad x \in E. \quad (4.3)$$

Also, we have

$$|x - y|^2 = \sum_{j=1,2} |x_j - y_j|^2 \geq \sum_{j=1,2} |e^{ix_j} - e^{iy_j}|^2 \geq 2 \prod_{j=1,2} |e^{ix_j} - e^{iy_j}|. \quad (4.4)$$

Using (4.2), (4.3) and (4.4), we have a constant $C_1 > 0$ such that

$$\operatorname{dist}(x, \mathcal{Z}(r))^2 \geq C_1 \prod_j |p_j|^2.$$

This yields that the function

$$\frac{\prod_j |p_j|^{q/2}}{|f(z_1, z_2)|^2},$$

is bounded on \mathbb{T}^2 . We now apply the standard trick, which is to increase the exponent in the numerator and assign the value zero at the zeros of f , to obtain a function that is k -times continuously differentiable on \mathbb{T}^2 . This completes the proof of Lemma 4.2. \square

Finally, we can prove Theorem 4.1

Proof of Theorem 4.1. By Lemma 4.2, we obtain a function

$$Q(z_1, z_2) = \frac{\left(\prod_{i=1}^k (2 - \zeta_i^{-1} z_1 - \eta_i^{-1} z_2) \right)^N}{p(z_1, z_2)} =: \frac{g(z_1, z_2)}{p(z_1, z_2)},$$

where $(\zeta_i, \eta_i) \in \mathbb{T}^2$, which is twice continuously differentiable on \mathbb{T}^2 . Thus its Fourier coefficients $\widehat{Q}(k, l)$ satisfy

$$\sum_{k,l} |\widehat{Q}(k, l)|^2 (k+1)^2 (l+1)^2 < \infty.$$

But since

$$\sum_{k,l} |\widehat{Q}(k, l)|^2 (k+l+1)^2 \leq \sum_{k,l} |\widehat{Q}(k, l)|^2 (k+1)^2 (l+1)^2,$$

we obtain $Q \in \mathcal{D}_\alpha$, $\alpha \leq 2$. Hence $g(z_1, z_2) \in p\mathcal{D}_\alpha$, $\alpha \leq 2$. Since g is cyclic in \mathcal{D}_2 and p is a multiplier, we obtain that p is also cyclic in \mathcal{D}_2 . \square

4.1. Proof of Theorem 1.3. For completeness we provide a proof of Theorem 1.3 in this section.

We recall the following result from [18, p. 17]. This establishes the cyclicity of polynomials having no zeros in the bidisk. The proof closely parallels the argument of [3, Theorem 4.1] which deals with the cyclicity in \mathfrak{D}_α , with a few modifications needed in the framework of \mathcal{D}_α .

Theorem 4.3. *Assume that $\alpha \leq 1$. Any polynomial $f \in \mathbb{C}[z_1, z_2]$ that does not vanish in the bidisk is cyclic in \mathcal{D}_α .*

The proof of Theorem 1.3 is now straightforward using the results above.

Proof of Theorem 1.3. We divide the argument into three cases according to the value of α : $\alpha > 2$, $\alpha \in (1, 2]$ and $\alpha \leq 1$.

Case 1: $\alpha > 2$. Using the norm inequalities $\|f\|_{\mathfrak{D}_{\alpha/2}} \leq \|f\|_{\mathcal{D}_\alpha} \leq \|f\|_{\mathfrak{D}_\alpha}$, Theorem 1.3.(iii) follows from Theorem 1.1.(iii).

Case 2: $\alpha \in (1, 2]$. The norm inequality $\|f\|_\alpha \geq \|f\|_{\mathfrak{D}_{\alpha/2}}$ implies that if a polynomial p is cyclic in \mathcal{D}_α , it must also be cyclic in $\mathfrak{D}_{\alpha/2}$. By Theorem 1.1, the zero set $\mathcal{Z}(p) \cap \mathbb{T}^2$ is empty or a finite set.

Subcase 2.1: $\mathcal{Z}(p) \cap \mathbb{T}^2 = \emptyset$. Using the case $\alpha > 2$, we obtain that p is cyclic in \mathcal{D}_3 . Since $\|\cdot\|_\alpha \leq \|\cdot\|_3$ for $\alpha \leq 2$, cyclicity in \mathcal{D}_α follows.

Subcase 2.2: $\mathcal{Z}(p) \cap \mathbb{T}^2$ is finite. This subcase follows by Theorem 4.1.

Case 3: $\alpha \leq 1$. This subcase follows by Theorem 4.3. \square

Remark 4.4. (i) This case $\alpha > 2$, has been solved in [18, Theorem 31(c)] by proving that \mathcal{D}_α is an algebra. We have provided alternative proof.
(ii) Note that if p has no zeros in the bidisk and only finitely many zeros on \mathbb{T}^2 , then its cyclicity in \mathcal{D}_α , $\alpha \leq 1$, also follows from Theorem 1.1 by a direct comparison of norms. However, this result excludes the cyclicity of a polynomial having infinitely many zeros on \mathbb{T}^2 as in the case of $1 - z_1 z_2$.

5. GENERAL PROPERTIES OF CYCLIC FUNCTION

In this section, we discuss some general properties of cyclic functions.

5.1. Slices of a function. In this subsection, we establish a result on the cyclicity of univariate slices of a function, which serve as quick tools for determining whether a function is a suitable candidate for being cyclic.

Let $f = f(z_1, z_2)$ be a holomorphic function on the bidisk. By fixing one variable, say z_1 , the slice

$$f_{z_1} : \mathbb{D} \rightarrow \mathbb{C}, \quad f_{z_1}(z_2) = f(z_1, z_2),$$

is a holomorphic function on the unit disk. The slice f_{z_2} is defined analogously.

Proposition 5.1. *If f is cyclic in \mathcal{D}_α , then f_{z_1} and f_{z_2} are cyclic in \mathcal{D}_α .*

Proof. Let $\alpha \geq 0$ and $z_1 \in \mathbb{D}$. Consider

$$\begin{aligned} \|f_{z_1}\|_{D_\alpha}^2 &\stackrel{(1.3)}{=} \sum_{j \geq 0} (j+1)^\alpha \left| \sum_{i \geq 0} a_{ij} z_1^i \right|^2 \\ &\leq \sum_{j \geq 0} \left| \sum_{i \geq 0} (i+j+1)^{\alpha/2} a_{ij} z_1^i \right|^2 \\ &\leq \sum_{j \geq 0} \left(\sum_{i \geq 0} (i+j+1)^\alpha |a_{ij}|^2 \right) \left(\sum_{i \geq 0} |z_1|^{2i} \right) \\ &\leq \frac{1}{1 - |z_1|^2} \|f\|_\alpha^2. \end{aligned}$$

Let $\alpha \leq 0$. Note that

$$\|f_{z_1}\|_{D_\alpha} \leq \|k_{z_1}\|_{D_\alpha} \|f\|_{\mathcal{D}_\alpha} \stackrel{(1.2)}{\leq} \|k_{z_1}\|_{D_\alpha} \|f\|_{\mathcal{D}_\alpha}$$

where k_{z_1} is the reproducing kernel of D_α at z_1 (see [2, Proposition 2.1]). \square

A natural question is whether the converse of the above result holds. To examine this, consider $p(z_1, z_2) = 2 - z_1 - z_2$. Note that the slices of p are cyclic in D_α for all α , but p itself is not cyclic in \mathcal{D}_α for $\alpha > 2$ (by Theorem 1.3).

5.2. Diagonal restriction of a function. In this subsection, we establish a result on cyclicity of diagonal restriction of a function, which is another criterion for determining whether a function is a suitable candidate for being cyclic.

Given a holomorphic function f on \mathbb{D}^2 , define the diagonal restriction of f by $(\mathcal{O}f)(z) := f(z, z)$, being a holomorphic function on \mathbb{D} . The following proposition gives a necessary condition for the cyclicity of a function in \mathcal{D}_α . The proof follows closely the proof of [2, Proposition 2.2].

Proposition 5.2. *If f is cyclic in \mathcal{D}_α , then $\mathcal{O}f$ is cyclic in $D_{\alpha-1}$.*

Proof. It suffices to prove that for $f \in \mathcal{D}_\alpha$, $\alpha \in \mathbb{R}$, it follows that

$$\|\mathcal{O}f\|_{D_{\alpha-1}} \leq \|f\|_\alpha. \quad (5.1)$$

Let $f(z_1, z_2) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{k,l} z_1^k z_2^l$. Then

$$\mathcal{O}f(z) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{k,l} z^{k+l}$$

converges absolutely for every $|z| < 1$, hence $\mathcal{O}f$ can be rewritten as

$$\mathcal{O}f(z) = \sum_{n=0}^{\infty} b_n z^n, \quad b_n = \sum_{k+l=n} a_{k,l} = \sum_{k=0}^n a_{k,n-k}.$$

Thus,

$$\|\mathcal{O}f\|_{D_\alpha}^2 = \sum_{n=0}^{\infty} |b_n|^2 (n+1)^\alpha = \sum_{n=0}^{\infty} \left| \sum_{k=0}^n a_{k,n-k} \right|^2 (n+1)^\alpha. \quad (5.2)$$

By the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} \left| \sum_{k=0}^n a_{k,n-k} \right|^2 &\leq \left(\sum_{k=0}^n |a_{k,n-k}|^2 (n+1)^\alpha \right) \left(\sum_{k=0}^n (n+1)^{-\alpha} \right) \\ &\leq \sum_{k=0}^n |a_{k,n-k}|^2 (n+1)^\alpha (n+1)^{-\alpha+1}. \end{aligned}$$

Using this in (5.2), it follows that

$$\|\mathcal{O}f\|_{D_{\alpha-1}}^2 \leq \sum_{n=0}^{\infty} \sum_{k=0}^n |a_{k,n-k}|^2 (n+1)^\alpha = \|f\|_\alpha^2,$$

proving (5.1). \square

Remark 5.3. The diagonal restriction in \mathfrak{D}_α satisfies (5.1) only for $\alpha \geq 0$, while for $\alpha < 0$, $\alpha-1$ needs to be replaced by $2\alpha-1$ (see [2, Proposition 2.2]).

6. CONCLUDING REMARKS

We provided a self-contained proof of the cyclicity of the polynomial $2 - z_1 - z_2$; we now give a simple proofs of the cyclicity of $1 - z_1 z_2$ and $z_1 - a$, $|a| \geq 1$ using standard techniques. These are model polynomials and often play a crucial role in establishing cyclicity results for broader classes of functions (i.e., see the proof of [15, Theorem 5]), and therefore merit particular attention to ensure independent proof.

The following has been proved in [18, Proposition 38] using a dilation argument. We provide an alternative proof.

Proposition 6.1. *Let $p(z) = z - a$ with $|a| \geq 1$. Then $P(z_1, z_2) := p(z_1)$ is cyclic in \mathcal{D}_α if and only if $\alpha \leq 1$.*

Proof. For $f \in D_\alpha$, define $F(z_1, z_2) = f(z_1)$. Note that $\|f\|_{D_\alpha} = \|F\|_{\mathcal{D}_\alpha}$. Since it is well-known [5] that p is cyclic in D_α if and only if $\alpha \leq 1$, the implication (\Leftarrow) is clear. For the implication (\Rightarrow), assume that $\alpha > 1$. Let $q_n \in \mathbb{C}[z_1, z_2]$ be a sequence such that $\|q_n F - 1\|_\alpha \rightarrow 0$. By the orthogonality of monomials in \mathcal{D}_α , one can choose $q_n(z_1, z_2) =: g_n(z_1)$ where $g_n \in \mathbb{C}[z]$. But since $\|q_n F - 1\|_\alpha = \|g_n p - 1\|_{D_\alpha}$, this is a contradiction with $\alpha > 1$. \square

Proposition 6.2. *$1 - z_1 z_2$ is cyclic in \mathcal{D}_α if and only if $\alpha \leq 1$.*

Proof. Note that if f is a function in one variable say z and F is defined by $F(z_1, z_2) := f(z_1 z_2)$, then we have the following: for $\alpha > 0$

$$\|f\|_{D_\alpha} \leq \|F\|_\alpha \leq 2^\alpha \|f\|_{D_\alpha}. \quad (6.1)$$

Let $\alpha \in \mathbb{R}$. Assume that $p(z_1, z_2) = 1 - z_1 z_2$ is cyclic in \mathcal{D}_α . Then there exists sequence of polynomials $P_n(z_1, z_2) := p_n(z_1 z_2)$ such that $\|P_n p - 1\|_\alpha \rightarrow 0$, where p_n is a univariate polynomial. By (6.1), we have that $1 - z$ is cyclic in D_α and hence $\alpha \leq 1$. The converse also follows using similar reasoning. \square

We conclude the paper with a few comments on capacity. Finite logarithmic capacity, or Riesz α -capacity plays an important role in identifying non-cyclic functions in \mathcal{D}_α [2, Proposition 4.2]. This approach originates in the work of Brown and Shields [5] and was later extended to several variables by several authors. We record the following straightforward result, since it provides a necessary condition for the cyclicity of a general function in \mathcal{D}_α , $\alpha \in \mathbb{R}$.

Proposition 6.3. *Let $\alpha \in \mathbb{R}$, and $f \in \mathcal{D}_\alpha$. Let f^* denote the radial limit of f , i.e., $f^*(e^{i\theta_1}, e^{i\theta_2}) = \lim_{r \rightarrow 1^-} f(re^{i\theta_1}, re^{i\theta_2})$. Then the following holds:*

- (a) *If $\mathcal{Z}(f^*)$ has positive logarithmic capacity, then f is not cyclic in \mathcal{D}_α , $\alpha \geq 2$.*
- (b) *For $0 < \alpha < 1$, if $\mathcal{Z}(f^*)$ has positive Riesz α -capacity, then f is not cyclic in $\mathcal{D}_{2\alpha}$.*

Proof. This follows from the fact that f is not cyclic in respective \mathcal{D}_α and by doing norm comparison with $\mathcal{D}_{2\alpha}$. \square

Statements and Declarations

Data Availability Statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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