Matrix Fejďż˝r-Riesz theorem with gaps

Aljaďż˝ Zalar, University of Ljubljana, Slovenia

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Notation

R - the ring of complex polynomials $\mathbb{C}[x]$ $(x^* = \overline{x} = x)$ or complex Laurent polynomials $\mathbb{C}[z, \frac{1}{z}]$ $(z^* = \overline{z} = \frac{1}{z})$

$M_n(R)$ - matrix polynomials $(F^* = \overline{F}^T)$

$H_n(R)$ - hermitian matrix polynomials

$\sum M_n(R)^2$ - SOHS matrix polynomials, i.e., finite sums of the form $\sum A_i^* A_i$, where $A_i \in M_n(R)$

Matrix Fejďż˝r-Riesz theorem

Theorem (Fejér-Riesz theorem on $\mathbb{T}$)

Let

$$A(z) = \sum_{m=-N}^{N} A_m z^m \in M_n\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$$

be a $n \times n$ matrix Laurent polynomial, such that $A(z)$ is positive semidefinite for every $z \in \mathbb{T} := \{z \in \mathbb{C} \colon \ |z| = 1\}$. Then there exists a matrix polynomial $B(z) = \sum_{m=0}^{N} B_m z^m \in M_n(\mathbb{C}[z])$, such that

$$A(z) = B(z)^* B(z).$$

Matrix Fejďż˝r-Riesz theorem

Theorem (Fejér-Riesz theorem on $\mathbb{R}$)

Let

$$F(x) = \sum_{m=0}^{2N} F_m x^m \in M_n(\mathbb{C}[x])$$

be a $n \times n$ matrix polynomial, such that $F(x)$ is positive semidefinite for every $x \in \mathbb{R}$. Then there exists a matrix polynomial $G(x) = \sum_{m=0}^{N} G_m x^m \in M_n(\mathbb{C}[x])$, such that

$$F(x) = G(x)^* G(x).$$

Main problem

Problem

1. *Characterize univariate matrix Laurent polynomials, which are positive semidefinite on a union of points and arcs in* $\mathbb{T}$.
2. *Characterize univariate matrix polynomials, which are positive semidefinite on a union of points and intervals (not necessarily bounded) in* $\mathbb{R}$.

Notation

A *basic closed semialgebraic set* $K_S \subseteq \mathbb{R}$ associated to a finite subset $S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[x]$ is given by

$$K := K_S = \{x \in \mathbb{R} \colon g_j(x) \geq 0,\ j = 1, \ldots, s\}.$$

We define the *n-th matrix quadratic module* M_S^n by

$$M_S^n := \{\sigma_0 + \sum_{j=1}^{s} \sigma_j g_j \colon \sigma_j \in \sum M_n(\mathbb{C}[x])^2 \text{ for } j = 0, \ldots, s\}.$$

Let $\prod S := \{g_1^{e_1} \cdots g_s^{e_s} \colon e_j \in \{0,1\},\ j = 1, \ldots, s\}$. The *n-th matrix preordering* T_S^n is $M_{\prod S}^n$.

Notation

Let $\text{Pos}^n_{\succeq 0}(K_S)$ be the set of all $n \times n$ hermitian matrix polynomials, which are positive semidefinite on K_S.

A matrix quadratic module M^n_S is *saturated* if $M^n_S = \text{Pos}^n_{\succeq 0}(K_S)$.

A saturated matrix quadratic module M^n_S is *boundedly saturated*, if every $F \in \text{Pos}^n_{\succeq 0}(K_S)$ is of the form $\sigma_0 + \sum_{j=1}^{s} \sigma_j g_j$, where

$$\deg(\sigma_0), \deg(\sigma_j g_j) \leq \deg(F) \quad \text{for} \quad j = 1, \dots s.$$

Notation

Let $K \subseteq \mathbb{R}$ be a basic closed semialgebraic set.

A set $S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[x]$ is the *natural description* of K, if it satisfies the following conditions:

(a) If K has the least element a, then $x - a \in S$.

(b) If K has the greatest element a, then $a - x \in S$.

(c) For every $a \neq b \in K$, if $(a, b) \cap K = \emptyset$, then $(x - a)(x - b) \in S$.

(d) These are the only elements of S.

Notation

Let $K = \cup_{j=1}^{m} [x_j, y_j] \subseteq \mathbb{R}$ be a basic compact semialgebraic set.

A set $S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[x]$ with $K = K_S$ is the *saturated description* of K, if it satisfies the following conditions:

(a) For every left endpoint x_j there exists $k \in \{1, \ldots, s\}$, such that $g_k(x_j) = 0$ and $g_k'(x_j) > 0$.

(b) For every right endpoint y_j there exists $k \in \{1, \ldots, s\}$, such that $g_k(y_j) = 0$ and $g_k'(y_j) < 0$.

Known results - scalar case

1. (Kuhlmann, Marshall, 2002) If S is the natural description of K, then the preordering $T_S^1 = M^1_{\prod S}$ is boundedly saturated.

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1 (Kuhlmann, Marshall, 2002) If S is the natural description of K, then the preordering $T_S^1 = M^1_{\prod S}$ is boundedly saturated.

- K not compact: T_S^1 is saturated if and only if S contains each of the polynomials in the natural description of K up to scaling by positive constants.
- K compact (Scheiderer, 2003): T_S^1 is saturated if and only if S is a saturated description of K. Moreover, $T_S^1 = M_S^1$.

Known results - scalar case

2 (Scheiderer, version for curves, 2003) Let I be a prime ideal of $\mathbb{R}[\underline{x}] := \mathbb{R}[x_1, \ldots, x_n]$ such that $\dim(\frac{\mathbb{R}[\underline{x}]}{I}) = 1$. Suppose $K_S \cap \mathcal{Z}(I)$ is compact and each $p \in K_S \cap \mathcal{Z}(I)$ is a non-singular zero of I. Then $M_S^1 + I$ is saturated if and only if the following conditions hold:

1. For each boundary point $p \in K_S \cap \mathcal{Z}$, which is not an isolated point of $K_S \cap \mathcal{Z}$, there exists $k \in \{1, \ldots, s\}$, such that $v_p(g_k) = 1$.
2. For each isolated point $p \in K_S \cap \mathcal{Z}$, there exist $k, l \in \{1, \ldots, s\}$, such that $v_p(g_k) = v_p(g_l) = 1$ and $g_k g_l \leq 0$ in some neighbourhood of p in $\mathcal{Z}$.

Known results - matrix case

1. (Gohberg, Krein, 1958) For $K = \mathbb{R}$, $M^n_\emptyset$ is boundedly saturated for every $n \in \mathbb{N}$.
2. (Dette, Studden, 2002) For $K = K_{\{x,1-x\}} = [0,1]$, $T^n_{\{x,1-x\}}$ is boundedly saturated for every $n \in \mathbb{N}$.
3. (Hol, Scherer, 2006) For a finite set $S \subseteq \mathbb{R}[x]$ with a compact set $K = K_S$, M^n_S contains every $F \in M_n(\mathbb{R}[x])$ such that $F|_K \succ 0$.
4. (Schmďż˝dgen, Savchuk, 2012) For $K = K_{\{x\}} = [0,\infty)$, $M^n_{\{x\}}$ is boundedly saturated for every $n \in \mathbb{N}$.

New results

Theorem (Compact Nichtnegativstellensatz for $\mathbb{R}$)

Let $K \subset \mathbb{R}$ be compact. The n-th matrix quadratic module M_S^n is saturated for every $n \in \mathbb{N}$ if and only if S is a saturated description of K.

Sketch of the proof of compact Nsatz

Proposition

Suppose K is a non-empty basic closed semialgebraic set in $\mathbb{R}$ and S a saturated description of K. Then for every $F \in Pos^n_{\succeq 0}(K)$ and every $w \in \mathbb{C}$ there exists $h \in \mathbb{R}[x]$, such that $h(w) \neq 0$ and $h^2 F \in M^n_S$.

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Sketch of the proof of compact Nsatz

To conclude the proof we need the following:

Proposition (Scheiderer, 2006)

Suppose R is a commutative ring with 1 and $\mathbb{Q} \subseteq R$. Let $\Phi : R \to C(K, \mathbb{R})$ be a ring homomorphism, where K is a topological space which is compact and Hausdorff. Suppose $\Phi(R)$ separates points in K. Suppose $f_1, \ldots, f_k \in R$ are such that $\Phi(f_j) \geq 0$, $j = 1, \ldots, k$ and $(f_1, \ldots, f_k) = (1)$. Then there exist $s_1, \ldots, s_k \in R$ such that $s_1 f_1 + \ldots + s_k f_k = 1$ and such that each $\Phi(s_j)$ is strictly positive.

New results

Theorem (Nichtnegativstellensatz for $\mathbb{T}$)

Let $\mathscr{K} \subseteq \mathbb{T}$ be a basic closed semialgebraic set and $\mathscr{S} = \{b_1, \ldots, b_s\} \subset H_1(\mathbb{C}[z, \frac{1}{z}])$ a finite set, such that $\mathscr{K} = \mathscr{K}_{\mathscr{S}}$. The n-th matrix quadratic module $\mathcal{M}^n_{\mathscr{S}} := \left\{\sum_{i=0}^s \tau_e b_i : \tau_e \in \sum M_n(\mathbb{C}[z])^2 \text{ for } i = 0, \ldots, s\right\}$ is saturated for every integer $n \in \mathbb{N}$ if and only if $\mathscr{S}$ satisfies the following conditions:

(a) *For every boundary point $a \in \mathscr{K}$, which is not isolated, there exists $k \in \{1, \ldots, s\}$, such that $b_k(a) = 0$ and $\frac{db_k}{dz}(a) \neq 0$.*

(b) *For every isolated point $a \in \mathscr{K}$, there exist $k, l \in \{1, \ldots, s\}$, such that $b_k(a) = b_l(a) = 0$, $\frac{db_k}{dz}(a) \neq 0$, $\frac{db_l}{dz}(a) \neq 0$ and $b_k b_l \leq 0$ on some neighborhood of a.*

New results

Theorem (Compact Nichtnegativstellensatz for curves)

Under the hypothesis of and if the coordinate ring $\frac{\mathbb{R}[x]}{I}$ is regular, then the n-th matrix quadratic module $M_S^n + M_n(I)$ is saturated.

Counterexample for non-compact case

Example

The matrix polynomial $F(x) := \begin{bmatrix} x+2 & \sqrt{6} \\ \sqrt{6} & x^2-2x+3 \end{bmatrix}$ is positive semidefinite on $K := [-1,0] \cup [1,\infty)$, but $F \notin T_S^2 = M^2_{\prod S}$, where S is the natural description of K. Moreover, for $\epsilon > 0$ small enough, even $F + \epsilon I \notin T_S^2$.

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Example

The matrix polynomial

$G(x) := x^2 F(\frac{1}{x} - 2) = \begin{bmatrix} x & \sqrt{6}x^2 \\ \sqrt{6}x^2 & 1-6x+11x^2 \end{bmatrix}$ is positive semidefinite on $K := [0, \frac{1}{3}] \cup [\frac{1}{2}, 1]$, but $G \notin T^2_{S,b} = M^2_{\prod S, b}$, where S is the natural description of K. However, $G \in T^2_{S,4}$.

Classification of non-compact sets K

Let K be a non-compact closed semialgebraic set with a natural description S. The classification of sets K according to T_S^n being saturated is the following:

Classification of non-compact sets K

K	T_S^n sat.
an unbounded interval	Yes
a union of an unbounded interval and an isolated point	?
a union of an unbounded interval and m isolated points with $m \geq 2$	No
a union of two unbounded intervals	Yes
a union of two unbounded intervals and an isolated point	?
a union of two unbounded intervals and m isolated points with $m \geq 2$	No
includes a bounded and an unbounded interval	No

Classification of compact sets K

Let K be a compact closed semialgebraic set with a natural description S. The classification of sets K according to T_S^n being boundedly saturated is the following:

Classification of compact sets K

K	T_S^n sat.	T_S^n bsat.
a union of at most three points	Yes	Yes
a union of m points with $m \geq 4$	Yes	No / stable
a bounded interval	Yes	Yes
a union of a bounded interval and an isolated point	Yes	?
a union of a bounded interval and m isolated points with $m \geq 2$	Yes	No
a compact set containing at least two intervals	Yes	No

Non-compact Nichtnegativstellensatz

Theorem (Non-compact Nichtnegativstellensatz)

Suppose K is an unbounded basic closed semialgebraic set in $\mathbb{R}$ and S a natural description of K. Then, for a hermitian $F \in M_n(\mathbb{C}[x])$, the following are equivalent:

1. $F \in Pos^n_{\succeq 0}(K)$.
2. $(1+x^2)^k F \in T^n_S$ *for some* $k \in \mathbb{N} \cup \{0\}$.

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1. $F \in Pos^n_{\succeq 0}(K)$.
2. $(1+x^2)^k F \in T^n_S$ *for some* $k \in \mathbb{N} \cup \{0\}$.
3. *There exists* $h \in \mathbb{R}[x]$, *such that* $h^2 > 0$ *on* $\mathbb{R}$, $\deg(h) \leq \deg(F)(3^n - 1)$ *and* $h^2 F \in T^n_{S,b}$.

Thank you for your attention!