

Matrix Fejér-Riesz theorem with gaps

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Notation

R - the ring of complex polynomials $\mathbb{C}[x]$ ($x^* = \bar{x} = x$) or complex Laurent polynomials $\mathbb{C}[z, \frac{1}{z}]$ ($z^* = \bar{z} = \frac{1}{z}$)

$M_n(R)$ - matrix polynomials ($F^* = \bar{F}^T$)

$H_n(R)$ - hermitian matrix polynomials

$\sum M_n(R)^2$ - SOHS matrix polynomials, i.e., finite sums of the form $\sum A_i^* A_i$, where $A_i \in M_n(R)$

Matrix Fejđž“r-Riesz theorem

Theorem (Fejér-Riesz theorem on \mathbb{T})

Let

$$A(z) = \sum_{m=-N}^N A_m z^m \in M_n \left(\mathbb{C} \left[z, \frac{1}{z} \right] \right)$$

be a $n \times n$ matrix Laurent polynomial, such that $A(z)$ is positive semidefinite for every $z \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Then there exists a matrix polynomial $B(z) = \sum_{m=0}^N B_m z^m \in M_n(\mathbb{C}[z])$, such that

$$A(z) = B(z)^* B(z).$$

Matrix Fejđž"r-Riesz theorem

Theorem (Fejér-Riesz theorem on \mathbb{R})

Let

$$F(x) = \sum_{m=0}^{2N} F_m x^m \in M_n(\mathbb{C}[x])$$

be a $n \times n$ matrix polynomial, such that $F(x)$ is positive semidefinite for every $x \in \mathbb{R}$. Then there exists a matrix polynomial $G(x) = \sum_{m=0}^N G_m x^m \in M_n(\mathbb{C}[x])$, such that

$$F(x) = G(x)^* G(x).$$

Main problem

Problem

- 1 *Characterize univariate matrix Laurent polynomials, which are positive semidefinite on a union of points and arcs in \mathbb{T} .*
- 2 *Characterize univariate matrix polynomials, which are positive semidefinite on a union of points and intervals (not necessarily bounded) in \mathbb{R} .*

Notation

A *basic closed semialgebraic set* $K_S \subseteq \mathbb{R}$ associated to a finite subset $S = \{g_1, \dots, g_s\} \subset \mathbb{R}[x]$ is given by

$$K := K_S = \{x \in \mathbb{R} : g_j(x) \geq 0, j = 1, \dots, s\}.$$

We define the *n-th matrix quadratic module* M_S^n by

$$M_S^n := \left\{ \sigma_0 + \sum_{j=1}^s \sigma_j g_j : \sigma_j \in \sum M_n(\mathbb{C}[x])^2 \text{ for } j = 0, \dots, s \right\}.$$

Let $\prod S := \{g_1^{e_1} \cdots g_s^{e_s} : e_j \in \{0, 1\}, j = 1, \dots, s\}$. The *n-th matrix preordering* T_S^n is $M_{\prod S}^n$.

Notation

Let $\text{Pos}_{\geq 0}^n(K_S)$ be the set of all $n \times n$ hermitian matrix polynomials, which are positive semidefinite on K_S .

A matrix quadratic module M_S^n is *saturated* if $M_S^n = \text{Pos}_{\geq 0}^n(K_S)$.

A saturated matrix quadratic module M_S^n is *boundedly saturated*, if every $F \in \text{Pos}_{\geq 0}^n(K_S)$ is of the form $\sigma_0 + \sum_{j=1}^s \sigma_j g_j$, where

$$\deg(\sigma_0), \deg(\sigma_j g_j) \leq \deg(F) \quad \text{for } j = 1, \dots, s.$$

Notation

Let $K \subseteq \mathbb{R}$ be a basic closed semialgebraic set.

A set $S = \{g_1, \dots, g_s\} \subset \mathbb{R}[x]$ is the *natural description* of K , if it satisfies the following conditions:

- (a) If K has the least element a , then $x - a \in S$.
- (b) If K has the greatest element a , then $a - x \in S$.
- (c) For every $a \neq b \in K$, if $(a, b) \cap K = \emptyset$, then $(x - a)(x - b) \in S$.
- (d) These are the only elements of S .

Notation

Let $K = \cup_{j=1}^m [x_j, y_j] \subseteq \mathbb{R}$ be a basic compact semialgebraic set.

A set $S = \{g_1, \dots, g_s\} \subset \mathbb{R}[x]$ with $K = K_S$ is the *saturated description* of K , if it satisfies the following conditions:

- (a) For every left endpoint x_j there exists $k \in \{1, \dots, s\}$, such that $g_k(x_j) = 0$ and $g'_k(x_j) > 0$.
- (b) For every right endpoint y_j there exists $k \in \{1, \dots, s\}$, such that $g_k(y_j) = 0$ and $g'_k(y_j) < 0$.

Known results - scalar case

- 1 (Kuhlmann, Marshall, 2002) If S is the natural description of K , then the preordering $T_S^1 = M_{\prod S}^1$ is boundedly saturated.

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- 1 (Kuhlmann, Marshall, 2002) If S is the natural description of K , then the preordering $T_S^1 = M_{\prod S}^1$ is boundedly saturated.
- K not compact: T_S^1 is saturated if and only if S contains each of the polynomials in the natural description of K up to scaling by positive constants.
 - K compact (Scheiderer, 2003): T_S^1 is saturated if and only if S is a saturated description of K . Moreover, $T_S^1 = M_S^1$.

Known results - scalar case

- 2 (Scheiderer, version for curves, 2003) ▶ Let I be a prime ideal of $\mathbb{R}[\underline{x}] := \mathbb{R}[x_1, \dots, x_n]$ such that $\dim(\frac{\mathbb{R}[\underline{x}]}{I}) = 1$. Suppose $K_S \cap \mathcal{Z}(I)$ is compact and each $p \in K_S \cap \mathcal{Z}(I)$ is a non-singular zero of I . Then $M_S^1 + I$ is saturated if and only if the following conditions hold:
- ① For each boundary point $p \in K_S \cap \mathcal{Z}$, which is not an isolated point of $K_S \cap \mathcal{Z}$, there exists $k \in \{1, \dots, s\}$, such that $v_p(g_k) = 1$.
 - ② For each isolated point $p \in K_S \cap \mathcal{Z}$, there exist $k, l \in \{1, \dots, s\}$, such that $v_p(g_k) = v_p(g_l) = 1$ and $g_k g_l \leq 0$ in some neighbourhood of p in \mathcal{Z} .

Known results - matrix case

- 1 (Gohberg, Krein, 1958) For $K = \mathbb{R}$, M_\emptyset^n is boundedly saturated for every $n \in \mathbb{N}$.
- 2 (Dette, Studden, 2002) For $K = K_{\{x, 1-x\}} = [0, 1]$, $T_{\{x, 1-x\}}^n$ is boundedly saturated for every $n \in \mathbb{N}$.
- 3 (Hol, Scherer, 2006) For a finite set $S \subseteq \mathbb{R}[x]$ with a compact set $K = K_S$, M_S^n contains every $F \in M_n(\mathbb{R}[x])$ such that $F|_K \succ 0$.
- 4 (Schmüdgen, Savchuk, 2012) For $K = K_{\{x\}} = [0, \infty)$, $M_{\{x\}}^n$ is boundedly saturated for every $n \in \mathbb{N}$.

New results

Theorem (Compact Nichtnegativstellensatz for \mathbb{R})

Let $K \subset \mathbb{R}$ be compact. The n -th matrix quadratic module M_S^n is saturated for every $n \in \mathbb{N}$ if and only if S is a saturated description of K .

Sketch of the proof of compact Nsatz

Proposition

Suppose K is a non-empty basic closed semialgebraic set in \mathbb{R} and S a saturated description of K . Then for every $F \in \text{Pos}_{\geq 0}^n(K)$ and every $w \in \mathbb{C}$ there exists $h \in \mathbb{R}[x]$, such that $h(w) \neq 0$ and $h^2 F \in M_S^n$.

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Sketch of the proof of compact Nsatz

To conclude the proof we need the following:

Proposition (Scheiderer, 2006)

Suppose R is a commutative ring with 1 and $\mathbb{Q} \subseteq R$. Let $\Phi : R \rightarrow C(K, \mathbb{R})$ be a ring homomorphism, where K is a topological space which is compact and Hausdorff. Suppose $\Phi(R)$ separates points in K . Suppose $f_1, \dots, f_k \in R$ are such that $\Phi(f_j) \geq 0$, $j = 1, \dots, k$ and $(f_1, \dots, f_k) = (1)$. Then there exist $s_1, \dots, s_k \in R$ such that $s_1 f_1 + \dots + s_k f_k = 1$ and such that each $\Phi(s_j)$ is strictly positive.

New results

Theorem (Nichtnegativstellensatz for \mathbb{T})

Let $\mathcal{K} \subseteq \mathbb{T}$ be a basic closed semialgebraic set and $\mathcal{S} = \{b_1, \dots, b_s\} \subset H_1(\mathbb{C}[z, \frac{1}{z}])$ a finite set, such that $\mathcal{K} = \mathcal{K}_{\mathcal{S}}$.


The n -th matrix quadratic module

$\mathcal{M}_{\mathcal{S}}^n := \left\{ \sum_{i=0}^s \tau_i b_i : \tau_i \in \sum M_n(\mathbb{C}[z])^2 \text{ for } i = 0, \dots, s \right\}$ is saturated for every integer $n \in \mathbb{N}$ if and only if \mathcal{S} satisfies the following conditions:

- (a) For every boundary point $a \in \mathcal{K}$, which is not isolated, there exists $k \in \{1, \dots, s\}$, such that $b_k(a) = 0$ and $\frac{db_k}{dz}(a) \neq 0$.
- (b) For every isolated point $a \in \mathcal{K}$, there exist $k, l \in \{1, \dots, s\}$, such that $b_k(a) = b_l(a) = 0$, $\frac{db_k}{dz}(a) \neq 0$, $\frac{db_l}{dz}(a) \neq 0$ and $b_k b_l \leq 0$ on some neighborhood of a .

New results

Theorem (Compact Nichtnegativstellensatz for curves)

Under the hypothesis of  and if the coordinate ring $\frac{\mathbb{R}[x]}{I}$ is regular, then the n -th matrix quadratic module $M_S^n + M_n(I)$ is saturated.

Counterexample for non-compact case

Example

The matrix polynomial $F(x) := \begin{bmatrix} x+2 & \sqrt{6} \\ \sqrt{6} & x^2 - 2x + 3 \end{bmatrix}$ is positive semidefinite on $K := [-1, 0] \cup [1, \infty)$, but $F \notin T_S^2 = M_{\prod S}^2$, where S is the natural description of K . Moreover, for $\epsilon > 0$ small enough, even $F + \epsilon I \notin T_S^2$.

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Example

The matrix polynomial

$G(x) := x^2 F\left(\frac{1}{x} - 2\right) = \begin{bmatrix} x & \sqrt{6}x^2 \\ \sqrt{6}x^2 & 1 - 6x + 11x^2 \end{bmatrix}$ is positive semidefinite on $K := [0, \frac{1}{3}] \cup [\frac{1}{2}, 1]$, but $G \notin T_{S,b}^2 = M_{\prod S,b}^2$, where S is the natural description of K . However, $G \in T_{S,4}^2$.

Classification of non-compact sets K

Let K be a non-compact closed semialgebraic set with a natural description S . The classification of sets K according to T_S^n being saturated is the following:

Classification of non-compact sets K

K	T_S^n sat.
an unbounded interval	Yes
a union of an unbounded interval and an isolated point	?
a union of an unbounded interval and m isolated points with $m \geq 2$	No
a union of two unbounded intervals	Yes
a union of two unbounded intervals and an isolated point	?
a union of two unbounded intervals and m isolated points with $m \geq 2$	No
includes a bounded and an unbounded interval	No

Classification of compact sets K

Let K be a compact closed semialgebraic set with a natural description S . The classification of sets K according to T_S^n being boundedly saturated is the following:

Classification of compact sets K

K	T_S^n sat.	T_S^n bsat.
a union of at most three points	Yes	Yes
a union of m points with $m \geq 4$	Yes	<u>No</u> stable
a bounded interval	Yes	Yes
a union of a bounded interval and an isolated point	Yes	?
a union of a bounded interval and m isolated points with $m \geq 2$	Yes	No
a compact set containing at least two intervals	Yes	No

Non-compact Nichtnegativstellensatz

Theorem (Non-compact Nichtnegativstellensatz)

Suppose K is an unbounded basic closed semialgebraic set in \mathbb{R} and S a natural description of K . Then, for a hermitian $F \in M_n(\mathbb{C}[x])$, the following are equivalent:

- ❶ $F \in \text{Pos}_{\sum_0}^n(K)$.
- ❷ $(1 + x^2)^k F \in T_S^n$ for some $k \in \mathbb{N} \cup \{0\}$.

Non-compact Nichtnegativstellensatz

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- 1 $F \in \text{Pos}_{\sum_0}^n(K)$.
- 2 $(1 + x^2)^k F \in T_S^n$ for some $k \in \mathbb{N} \cup \{0\}$.
- 3 There exists $h \in \mathbb{R}[x]$, such that $h^2 > 0$ on \mathbb{R} , $\deg(h) \leq \deg(F)(3^n - 1)$ and $h^2 F \in T_{S,b}^n$.

Thank you for your attention!