

# CYCLIC POLYNOMIALS IN DIRICHLET-TYPE SPACES OF THE UNIT BIDISK

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**ABSTRACT.** For  $\alpha \in \mathbb{R}$ , we consider the scale of function spaces, namely the Dirichlet-type space  $\mathcal{D}_\alpha$  consisting of holomorphic functions on the unit bidisk  $\mathbb{D}^2$ ,  $f(z, w) = \sum_{k,l=0}^{\infty} a_{kl} z^k w^l$  such that

$$\sum_{k,l=0}^{\infty} (k+l+1)^\alpha |a_{kl}|^2 < \infty.$$

We present a complete characterization of cyclic polynomials in  $\mathcal{D}_\alpha$ , i.e., given an irreducible polynomial  $p$ , the following holds:

- (i) If  $\alpha \leq 1$ , then  $p$  is cyclic in  $\mathcal{D}_\alpha$ .
- (ii) If  $1 < \alpha \leq 2$ , then  $p$  is cyclic in  $\mathcal{D}_\alpha$  if and only if  $\mathcal{Z}(p) \cap \mathbb{T}^2$  is empty or finite.
- (iii) If  $\alpha > 2$ , then  $p$  is cyclic in  $\mathcal{D}_\alpha$  if and only if  $\mathcal{Z}(p) \cap \mathbb{T}^2$  is empty.

## 1. INTRODUCTION

Let  $\mathbb{C}$  denote the complex plane,  $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$  the open unit disk and  $\mathbb{T} = \{z \in \mathbb{C}, |z| = 1\}$  the unit circle in the complex plane. For  $k \in \mathbb{N}$ , we denote by  $\mathbb{C}^k[z_1, z_2]$  the space of polynomials in two variables with coefficients in the column space  $\mathbb{C}^k$ . Given  $p \in \mathbb{C}[z_1, z_2]$ , its bidegree is the pair  $(m, n)$ , where  $m$  is the highest degree of  $p$  in the variable  $z_1$ , and  $n$  is the highest degree of  $p$  in the variable  $z_2$ . The notation  $\mathcal{Z}(p) = \{(z_1, z_2) \in \mathbb{C}^2 : p(z_1, z_2) = 0\}$  denotes the zero set of  $p$ . A nonzero polynomial  $p$  is said to be *irreducible* if  $p = qr$  with  $q, r \in \mathbb{C}[z_1, z_2]$  implies that  $q \in \mathbb{C}$  or  $r \in \mathbb{C}$ . The notation  $I_n$  stands for the  $n \times n$  identity matrix.

In the classical Hardy space  $H^2(\mathbb{D})$ , a function  $f$  is called *cyclic* if the smallest closed, shift-invariant subspace generated by its polynomial multiples coincides with the whole space. By Beurling's theorem, cyclic functions in this setting are precisely the outer functions, i.e.,  $f(0) \neq 0$  and

$$\log |f(0)| = \int_0^{2\pi} \log |f(e^{i\theta})| \frac{d\theta}{2\pi},$$

making the theory both elegant and complete. In contrast, for the Hardy space on the bidisk  $H^2(\mathbb{D}^2)$ , cyclic functions are outer but there exists an outer function which is not cyclic [17]. With the present understanding, a characterization of cyclic function seems to be a harder problem in several variables. However, cyclic polynomials are characterized in  $H^2(\mathbb{D}^n)$ ; they

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are precisely those polynomials which do not have zeros on the polydisk  $\mathbb{D}^n$  [16]. This gap between the one- and several-variable settings motivates investigations of cyclicity in other spaces, such as the Dirichlet and Dirichlet-type spaces.

In the Dirichlet space  $D$  of the unit disk, Brown and Shields conjectured [5, Question 12] that a function  $f \in D$  is cyclic if and only if it is outer and its boundary zero set has logarithmic capacity zero. The forward direction was proved by Brown and Shields, and the converse, despite several attempts, remains open till now. Several partial results are known [10, 11]. The Brown–Shields conjecture continues to be a central open problem, motivating further exploration of cyclicity in Dirichlet-type spaces [1, 2, 3, 15].

**1.1. Dirichlet type spaces.** We now introduce the following Dirichlet-type space of the unit bidisk where we will investigate the cyclicity of polynomials. For  $\alpha \in \mathbb{R}$ , the Dirichlet-type space denoted as  $\mathcal{D}_\alpha$  on  $\mathbb{D}^2$ , consists of all holomorphic functions  $f(z, w) = \sum_{k,l=0}^{\infty} a_{kl} z^k w^l$  such that

$$\|f\|_\alpha^2 := \sum_{k,l=0}^{\infty} (k+l+1)^\alpha |a_{kl}|^2 < \infty.$$

Note that for  $\alpha = 0$ , we recover the Hardy space  $H^2(\mathbb{D}^2)$  of the unit bidisk and for  $\alpha = 1$ , the space  $\mathcal{D}_1$  was introduced in [4] in connection with toral 2-isometries.

Investigations in this paper are motivated by the cyclicity results obtained for the following Dirichlet-type space on  $\mathbb{D}^2$  in [2, 3]. For  $\alpha \in \mathbb{R}$ , the Dirichlet-type space on the unit bidisk, denoted as  $\mathfrak{D}_\alpha$ , consists of holomorphic functions  $f(z, w) = \sum_{k,l=0}^{\infty} a_{kl} z^k w^l$  such that

$$\|f\|_{\mathfrak{D}_\alpha}^2 := \sum_{k,l=0}^{\infty} (k+1)^\alpha (l+1)^\alpha |a_{kl}|^2 < \infty.$$

From the norm definitions, it is straightforward to see that for  $\alpha \geq 0$  we have

$$\|\cdot\|_\alpha \leq \|\cdot\|_{\mathfrak{D}_\alpha} \leq \|\cdot\|_{2\alpha} \quad (\mathcal{D}_{2\alpha} \subseteq \mathfrak{D}_\alpha \subseteq \mathcal{D}_\alpha), \quad (1.1)$$

and for  $\alpha \leq 0$ ,

$$\|\cdot\|_{2\alpha} \leq \|\cdot\|_{\mathfrak{D}_\alpha} \leq \|\cdot\|_\alpha \quad (\mathcal{D}_\alpha \subseteq \mathfrak{D}_\alpha \subseteq \mathcal{D}_{2\alpha}). \quad (1.2)$$

Inequalities (1.1), (1.2) allow us to transfer properties between both kinds of Dirichlet-type spaces. In particular, we focus here on the notion of cyclicity.

Note that both definitions can be viewed as a generalization of the univariate Dirichlet space  $D_\alpha, \alpha \in \mathbb{R}$ , which consists of holomorphic function  $f : \mathbb{D} \rightarrow \mathbb{C}$ ,  $f(z) = \sum_{k=0}^{\infty} a_k z^k$ , such that

$$\|f\|_{D_\alpha}^2 := \sum_{k=0}^{\infty} (k+1)^\alpha |a_k|^2 < \infty. \quad (1.3)$$

**1.2. Cyclic functions.** In this work, our focus lies on the natural pair of shift operators  $(M_{z_1}, M_{z_2})$  acting on Dirichlet-type spaces  $\mathcal{D}_\alpha$ . These operators are defined by

$$(M_{z_1} f)(z_1, z_2) = z_1 f(z_1, z_2), \quad (M_{z_2} f)(z_1, z_2) = z_2 f(z_1, z_2), \quad f \in \mathcal{D}_\alpha.$$

It is straightforward to verify that both  $M_{z_1}$  and  $M_{z_2}$  are bounded linear operators on  $\mathcal{D}_\alpha$ . From the operator-theoretic point of view, an important problem is to describe the closed subspaces of  $\mathcal{D}_\alpha$  that are invariant under these shifts, namely those  $\mathcal{M} \subseteq \mathcal{D}_\alpha$  for which

$$M_{z_1}\mathcal{M} \subseteq \mathcal{M} \quad \text{and} \quad M_{z_2}\mathcal{M} \subseteq \mathcal{M}.$$

A key step towards this description is to understand when a function  $f \in \mathcal{D}_\alpha$  is cyclic, i.e., when the closed linear span

$$[f] := \overline{\text{span}}\{z_1^k z_2^\ell f : k, \ell \geq 0\}$$

coincides with the entire space  $\mathcal{D}_\alpha$ . It is clear from the definition that  $[f]$  is the smallest closed subspace that contains  $f$  and is invariant under the shift operators  $M_{z_1}$  and  $M_{z_2}$ . Clearly, at least one cyclic vector always exists, e.g., the constant function  $f(z_1, z_2) \equiv 1$  is cyclic, since polynomials in two variables are dense in  $\mathcal{D}_\alpha$ . In the next section, we will see that a necessary condition for cyclicity is that  $f$  must have no zeros in  $\mathbb{D}^2$ .

Note that if  $g \in [f]$ , then  $[g] \subseteq [f]$ . Thus to check  $f$  is cyclic in  $\mathcal{D}_\alpha$ , it suffices to show that there exists a sequence of polynomials  $p_n \in \mathbb{C}[z_1, z_2]$  such that

$$\|p_n f - 1\|_\alpha \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

**1.3. Cyclic polynomials in  $\mathfrak{D}_\alpha$ .** Recent work of Bénéteau et al. [3, Théorème] provides a complete characterization of cyclic polynomials in  $\mathfrak{D}_\alpha$ ,  $\alpha \in \mathbb{R}$ , on the bidisk (see [15] for anisotropic setting and [9] for the unit ball in  $\mathbb{C}^2$ ). Their main result shows that the cyclicity of an irreducible polynomial depends intricately on the structure of its zero set on the distinguished boundary  $\mathbb{T}^2$ . In particular, while non-vanishing in the bidisk is necessary for cyclicity, additional restrictions on the boundary zero set become decisive when the parameter  $\alpha$  of the Dirichlet-type space  $\mathfrak{D}_\alpha$  lies in the range  $(\frac{1}{2}, \infty)$ . We recall the result for the reader's convenience.

**Theorem 1.1** ([3, Théorème]). *Let  $p \in \mathbb{C}[z_1, z_2]$  be an irreducible polynomial with no zeros in the bidisk. We have the following:*

- (i) *If  $\alpha \leq \frac{1}{2}$ , then  $p$  is cyclic in  $\mathfrak{D}_\alpha$ .*
- (ii) *If  $\frac{1}{2} < \alpha \leq 1$ , then  $p$  is cyclic in  $\mathfrak{D}_\alpha$  if and only if  $\mathbb{Z}(p) \cap \mathbb{T}^2$  is an empty or finite set or  $p$  is a constant multiple of  $\zeta - z_1$  or of  $\zeta - z_2$  for some  $\zeta \in \mathbb{T}$ .*
- (iii) *If  $\alpha > 1$ , then  $p$  is cyclic in  $\mathfrak{D}_\alpha$  if and only if  $\mathbb{Z}(p) \cap \mathbb{T}^2$  is empty.*

**1.4. Cyclic polynomials in  $\mathcal{D}_\alpha$ .** In this paper, we solve the problem of characterizing cyclic polynomials in  $\mathcal{D}_\alpha$ . Our main result is an analog of Theorem 1.1 for the Dirichlet-type space  $\mathcal{D}_\alpha$ :

**Theorem 1.2.** *Let  $p \in \mathbb{C}[z_1, z_2]$  be an irreducible polynomial with no zeros in the bidisk. We have the following.*

- (i) *If  $\alpha \leq 1$ , then  $p$  is cyclic in  $\mathcal{D}_\alpha$ .*
- (ii) *If  $1 < \alpha \leq 2$ , then  $p$  is cyclic in  $\mathcal{D}_\alpha$  if and only if  $\mathbb{Z}(p) \cap \mathbb{T}^2$  is empty or finite.*
- (iii) *If  $\alpha > 2$ , then  $p$  is cyclic in  $\mathcal{D}_\alpha$  if and only if  $\mathbb{Z}(p) \cap \mathbb{T}^2$  is empty.*

- Remark 1.3.* (i) Note that the Dirichlet-type spaces studied in [3] do not distinguish between the cyclicity of a polynomial having finitely many zeros on  $\mathbb{T}^2$  and that of  $1 - z_i$ ,  $i = 1, 2$ , which has infinitely many zeros on  $\mathbb{T}^2$  (Theorem 1.1(ii)). In fact, the latter case was used there to establish cyclicity for polynomials having finitely many zeros on  $\mathbb{T}^2$ . In contrast, in our setting, these two cases exhibit different behaviour in  $\mathcal{D}_\alpha$ .
- (ii) The statement (iii) of Theorem 1.2 follows directly from the statement (iii) of Theorem 1.1 using the first inclusion of (1.1). Establishing (i), (ii) of Theorem 1.1 appears to be less obvious and we borrow ideas developed in [3, 15] to prove them. Specifically, the proof of (i) is achieved by carefully examining the proof of Theorem 1.1(ii).

**1.5. Organization of the paper.** In Section 2, we recall some definitions and necessary results to prove our main result. We also present a few necessary conditions for the cyclicity of a function in  $\mathcal{D}_\alpha$ ,  $\alpha \in \mathbb{R}$ . In Section 3, we present a proof of Theorem 1.2. This is divided into three subsections. In Subsection 3.1, we consider the spaces  $\mathcal{D}_\alpha$  for  $\alpha \leq 1$ , where we show that the necessary condition for cyclicity – that the polynomial does not vanish on the bidisk – is also sufficient. In Subsection 3.2, we address the case  $\alpha \leq 2$ . In this case, we show that the known necessary condition mentioned above, with the additional condition that the polynomial has at most finitely many zeros on  $\mathbb{T}^2$  is required for the cyclicity of an irreducible polynomial in  $\mathcal{D}_\alpha$ . In Subsection 3.3, we present the proof of Theorem 1.2 based on the ideas developed in the preceding sections. In Section 4, we conclude the paper with a discussion of the cyclicity of  $1 - z_1 z_2$  and a result based on capacity.

## 2. PRELIMINARIES

In this section, we list some properties of cyclic function which are needed to give a self-contained treatment of the proof of Theorem 1.2 (see Subsection 2.1), two simple criteria to determine cyclicity of a function (see Subsections 2.2 and 2.3) and an inequality on the value of a real analytic function (see Subsection 2.4). For further details, we refer the reader to [3, Section 2].

**2.1. Some properties of cyclic functions in  $\mathcal{D}_\alpha$ .** Note that  $\mathcal{D}_\alpha$ ,  $\alpha \in \mathbb{R}$ , is a reproducing kernel Hilbert space, i.e., the evaluation map  $e_w$ , for  $w \in \mathbb{D}$ ,  $e_w(f) := f(w)$  is a continuous linear functional on  $\mathcal{D}_\alpha$  and its multiplier space is defined as

$$M(\mathcal{D}_\alpha) = \{\phi : \mathbb{D}^2 \rightarrow \mathbb{C} : \phi f \in \mathcal{D}_\alpha \text{ for all } f \in \mathcal{D}_\alpha\}.$$

Elements of  $M(\mathcal{D}_\alpha)$  are called *multipliers* of  $\mathcal{D}_\alpha$ . It is easy to verify that all polynomials are multipliers of  $\mathcal{D}_\alpha$ ,  $\alpha \in \mathbb{R}$ .

Some properties of cyclic functions are the following:

- (i) A cyclic function in  $\mathcal{D}_\alpha$  can not vanish on the bidisk. To see this, take  $p_n$  to be a sequence of polynomials such that  $\|p_n f - 1\|_\alpha \rightarrow 0$ . Since evaluations are continuous, the conclusion follows from the following expression

$$p_n(z_1, z_2)f(z_1, z_2) - 1 = e_{(z_1, z_2)}(p_n f - 1).$$

- (ii) Given a cyclic function  $f \in \mathcal{D}_\alpha$ ,  $\alpha \in \mathbb{R}$ , the function defined by  $g(z_1, z_2) := f(\zeta z_1, \eta z_2)$ , where  $\zeta, \eta \in \mathbb{T}$ , is clearly also cyclic in  $\mathcal{D}_\alpha$ . Indeed, if there

exists a sequence of polynomials  $\{p_n\}_{n \in \mathbb{Z}_+}$  such that

$$\|p_n f - 1\|_\alpha \rightarrow 0, \quad (2.1)$$

then the sequence defined by  $q_n(z_1, z_2) := p_n(\zeta z_1, \eta z_2)$  also satisfies

$$\|q_n g - 1\|_\alpha = \|p_n f - 1\|_\alpha \rightarrow 0,$$

proving cyclicity of  $g$ .

- (iii) Assume  $f$  is a reducible polynomial with  $f = gh$  for some nonconstant polynomials  $g$  and  $h$ . Then  $f = gh$  is cyclic in  $\mathcal{D}_\alpha$  if and only if  $g$  and  $h$  are cyclic in  $\mathcal{D}_\alpha$ . Let us verify:

If  $f$  is cyclic, then there is a sequence of polynomials  $\{p_n\}_{n \in \mathbb{Z}_+}$  satisfying (2.1). But then the sequence  $\{r_n\}_{n \in \mathbb{Z}_+}$  and  $\{s_n\}_{n \in \mathbb{Z}_+}$ , where  $r_n := p_n g$  and  $s_n := p_n h$  satisfy  $\|r_n h - 1\|_\alpha \rightarrow 0$  and  $\|s_n g - 1\|_\alpha \rightarrow 0$ , proving cyclicity of  $g$  and  $h$ .

Conversely, assume that  $g$  and  $h$  are cyclic. Then there exists a sequence of polynomials  $\{r_n\}_{n \in \mathbb{Z}_+}$  such that  $\|r_n g - 1\|_\alpha \rightarrow 0$ . Note that

$$\|r_n g h - h\|_\alpha \leq \|M_h\| \|r_n g - 1\|_\alpha$$

where  $\|M_h\|$  is operator norm of the multiplication operator  $M_h$ . This shows that  $h \in [f]$ . Hence  $[h] \subseteq [f]$ . Since  $h$  is cyclic,  $f$  is cyclic.

Therefore, it suffices to characterize cyclicity of irreducible polynomials in  $\mathcal{D}_\alpha$ .

**2.2. Slices of a function.** In this subsection we establish a result on cyclicity of one-variable slices of a function, which serves as quick tools for determining whether a function is a suitable candidate for being cyclic.

Let  $f = f(z_1, z_2)$  be a holomorphic function in the bidisk. By fixing one variable, say  $z_1$ , the slice

$$f_{z_1} : \mathbb{D} \rightarrow \mathbb{C}, \quad f_{z_1}(z_2) = f(z_1, z_2),$$

is a holomorphic function in the unit disk. The slice  $f_{z_2}$  is defined analogously.

**Proposition 2.1.** *If  $f$  is cyclic in  $\mathcal{D}_\alpha$ , then  $f_{z_1}$  and  $f_{z_2}$  are cyclic in  $D_\alpha$ .*

*Proof.* Let  $\alpha \geq 0$  and  $z_1 \in \mathbb{D}$ . Consider

$$\begin{aligned} \|f_{z_1}\|_{D_\alpha}^2 &\stackrel{(1.3)}{=} \sum_{j \geq 0} (j+1)^\alpha \left| \sum_{i \geq 0} a_{ij} z_1^i \right|^2 \\ &\leq \sum_{j \geq 0} \left| \sum_{i \geq 0} (i+j+1)^{\alpha/2} a_{ij} z_1^i \right|^2 \\ &\leq \sum_{j \geq 0} \left( \sum_{i \geq 0} (i+j+1)^\alpha |a_{ij}|^2 \right) \left( \sum_{i \geq 0} |z_1|^{2i} \right) \\ &\leq \frac{1}{1 - |z_1|^2} \|f\|_\alpha^2. \end{aligned}$$

Let  $\alpha \leq 0$ . Note that

$$\|f_{z_1}\|_{D_\alpha} \leq \|k_{z_1}\|_{D_\alpha} \|f\|_{\mathcal{D}_\alpha} \underbrace{\leq}_{(1.2)} \|k_{z_1}\|_{D_\alpha} \|f\|_{\mathcal{D}_\alpha}$$

where  $k_{z_1}$  is the reproducing kernel of  $D_\alpha$  at  $z_1$  (see [2, Proposition 2.1]).  $\square$

A natural question is whether the converse of the above result holds. To examine this, consider  $p(z_1, z_2) = 2 - z_1 - z_2$ . Note that the slices of  $p$  are cyclic in  $D_\alpha$  for all  $\alpha$ , but  $p$  itself is not cyclic in  $\mathcal{D}_\alpha$  for  $\alpha > 2$  (by Theorem 1.2).

**2.3. Diagonal restriction of a function.** In this subsection we establish a result on cyclicity of diagonal restriction of a function, which is another criterion for determining whether a function is a suitable candidate for being cyclic.

Given a holomorphic function  $f$  on  $\mathbb{D}^2$ , define the diagonal restriction of  $f$  by  $(\mathcal{O}f)(z) := f(z, z)$ , being a holomorphic function on  $\mathbb{D}$ . The following proposition gives a necessary condition for the cyclicity of a function in  $\mathcal{D}_\alpha$ . The proof follows closely the proof of [2, Proposition 2.2].

**Proposition 2.2.** *If  $f$  is cyclic in  $\mathcal{D}_\alpha$ , then  $\mathcal{O}f$  is cyclic in  $D_{\alpha-1}$ .*

*Proof.* It suffices to prove that for  $f \in \mathcal{D}_\alpha$ ,  $\alpha \in \mathbb{R}$ , it follows that

$$\|\mathcal{O}f\|_{D_{\alpha-1}} \leq \|f\|_\alpha. \quad (2.2)$$

Let  $f(z_1, z_2) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{k,l} z_1^k z_2^l$ . Then

$$\mathcal{O}f(z) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{k,l} z^{k+l}$$

converges absolutely for every  $|z| < 1$ , hence  $\mathcal{O}f$  can be rewritten as

$$\mathcal{O}f(z) = \sum_{n=0}^{\infty} b_n z^n, \quad b_n = \sum_{k+l=n} a_{k,l} = \sum_{k=0}^n a_{k,n-k}.$$

Thus,

$$\|\mathcal{O}f\|_{D_\alpha}^2 = \sum_{n=0}^{\infty} |b_n|^2 (n+1)^\alpha = \sum_{n=0}^{\infty} \left| \sum_{k=0}^n a_{k,n-k} \right|^2 (n+1)^\alpha. \quad (2.3)$$

By the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} \left| \sum_{k=0}^n a_{k,n-k} \right|^2 &\leq \left( \sum_{k=0}^n |a_{k,n-k}|^2 (n+1)^\alpha \right) \left( \sum_{k=0}^n (n+1)^{-\alpha} \right) \\ &\leq \sum_{k=0}^n |a_{k,n-k}|^2 (n+1)^\alpha (n+1)^{-\alpha+1}. \end{aligned}$$

Using this in (2.3), it follows that

$$\|\mathcal{O}f\|_{D_{\alpha-1}}^2 \leq \sum_{n=0}^{\infty} \sum_{k=0}^n |a_{k,n-k}|^2 (n+1)^\alpha = \|f\|_\alpha^2,$$

proving (2.2).  $\square$

*Remark 2.3.* The diagonal restriction in  $\mathfrak{D}_\alpha$  satisfies (2.2) only for  $\alpha \geq 0$ , while for  $\alpha < 0$ ,  $\alpha - 1$  needs to be replaced by  $2\alpha - 1$  (see [2, Proposition 2.2]).

**2.4. Łojasiewicz's inequality.** In the proof of Theorem 3.3 below, the following inequality will be used essentially.

**Theorem 2.4** ([14, Łojasiewicz's inequality]). *Let  $f$  be a nonzero real analytic function on an open set  $U \subseteq \mathbb{R}^n$ . Assume the zero set  $\mathcal{Z}(f)$  of  $f$  in  $U$  is nonempty. Let  $E$  be a compact subset of  $U$ . Then there are constants  $C > 0$  and  $q \in \mathbb{N}$ , depending on  $E$ , such that*

$$|f(x)| \geq C \cdot \text{dist}(x, \mathcal{Z}(f))^q$$

for every  $x \in E$ .

### 3. PROOF OF THEOREM 1.2

In this section, we provide a proof of Theorem 1.2. In Subsection 3.1, we treat separately the case  $\alpha \leq 1$ . In Subsection 3.2, we study polynomials having finitely many zeroes on  $\mathbb{T}^2$ . Finally, in Subsection 3.3, we complete the proof of Theorem 1.2.

**3.1. Polynomial having no zeros inside the bidisk.** The following theorem establishes the cyclicity of polynomials having no zeros in the bidisk. The proof closely parallels the argument of [3, Theorem 4.1] which deals with the cyclicity in  $\mathfrak{D}_\alpha$ , with only a few modifications needed in the framework of  $\mathcal{D}_\alpha$ .

**Theorem 3.1.** *Assume that  $\alpha \leq 1$ . Any polynomial  $f \in \mathbb{C}[z_1, z_2]$  that does not vanish in the bidisk is cyclic in  $\mathcal{D}_\alpha$ .*

Note that it suffices to prove the statement of Theorem 3.1 for  $\alpha = 1$ . Let us outline the details of the modification in the proof of [3, Theorem 4.1] to establish Theorem 3.1:

- (i) A common property of the spaces  $\mathcal{D}_\alpha$  and  $\mathfrak{D}_\alpha$  is orthogonality of monomials. Because of this property, most of the calculation part in the proof of Theorem 3.1 follows as in proof of [3, Theorem 4.1], but  $(k+1)d/2 + 1$  needs to be replaced with  $(k+1)d$  when bounding  $\|\vec{v}A^{kd}\vec{B}\|_1$  from above in (3.7).
- (ii) Proving Theorem 3.1 separately for univariate polynomials of the form  $z_i - a$ ,  $|a| \geq 1$ , needs to be done separately, since it is later used to prove the theorem in case of general  $f$ . This is Proposition 3.2 below.

For completeness, the full proof of Theorem 3.1 is presented below.

In [5, Lemma 8], Brown and Shields showed that  $z - a$  is cyclic in  $D_1$  if and only if  $|a| \geq 1$ . Since the only candidate for cyclic irreducible polynomial, which depends on only one variable, is  $z_i - a$ ,  $|a| \geq 1$ , we study cyclicity of this polynomial in  $\mathcal{D}_\alpha$ ,  $\alpha \in \mathbb{R}$ , in the next result.

**Proposition 3.2.** *Let  $p(z) = z - a$  with  $|a| \geq 1$ . Then  $P(z_1, z_2) := p(z_1)$  is cyclic in  $\mathcal{D}_\alpha$  if and only if  $\alpha \leq 1$ .*

*Proof.* For  $f \in D_\alpha$ , define  $F(z_1, z_2) = f(z_1)$ . Note that  $\|f\|_{D_\alpha} = \|F\|_{\mathcal{D}_\alpha}$ . Since it is well-known [5] that  $p$  is cyclic in  $D_\alpha$  if and only if  $\alpha \leq 1$ , the implication ( $\Leftarrow$ ) is clear. For the implication ( $\Rightarrow$ ), assume that  $\alpha > 1$ . Let  $q_n \in \mathbb{C}[z_1, z_2]$  be a sequence such that  $\|q_n F - 1\|_\alpha \rightarrow 0$ . By the orthogonality of monomials in  $\mathcal{D}_\alpha$ , one can choose  $q_n(z_1, z_2) =: g_n(z_1)$  where  $g_n \in \mathbb{C}[z]$ . But since  $\|q_n F - 1\|_\alpha = \|g_n p - 1\|_{D_\alpha}$ , this is a contradiction with  $\alpha > 1$ .  $\square$

Now we give a proof of Theorem 3.1.

*Proof of Theorem 3.1.* We start with a few reductions:

- If  $f$  has finitely many or no zeros on  $\mathbb{T}^2$ , then it is cyclic in  $\mathcal{D}_2$  by Theorem 1.2.(ii) and by (1.1), also in  $\mathcal{D}_\alpha$  for  $\alpha \in (0, 2]$ . Thus, to complete the proof of Theorem 3.1, we can assume that  $f$  has infinitely many zeros on  $\mathbb{T}^2$ .
- By (1.1), it suffices to prove Theorem 3.1 for  $\alpha = 1$ , which we assume from now on and write  $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_1$ .
- We can assume that  $f$  is irreducible (see Subsection 2.1).
- Since  $f$  has infinitely many zeros on  $\mathbb{T}^2$ , it is of the form  $\underline{f} = \lambda \tilde{f}$  for some  $\lambda \in \mathbb{T}$ , where  $(n, m)$  is bidegree of  $f$  and  $\tilde{f}(z, w) = z_1^n z_2^m f\left(\frac{1}{z_1}, \frac{1}{z_2}\right)$ . This is due to Bézout theorem, since  $f$  and  $\tilde{f}$  have a common non-constant factor and are irreducible.

By the reductions above and [3, Proposition 4.2], there exist a unitary matrix  $U$  of size  $n+m$ , a column vector polynomial  $\vec{B}(z_1, z_2) \in \mathbb{C}^{n+m}[z_1, z_2]$  and a row vector polynomial  $\vec{v} \in \mathbb{C}^{n+m}[z_2]$ , such that

$$(I_{n+m} - \underbrace{U(z_1 I_n \oplus z_2 I_m)}_{=: A(z_1, z_2)}) \vec{B}(z_1, z_2) \in f(z_1, z_2) \cdot \mathbb{C}^{n+m}[z_1, z_2], \quad (3.1)$$

$$p(z_2) := \vec{v}(z_2) \vec{B}(z_1, z_2) \in \mathbb{C}[z_2] \quad \text{satisfies} \quad \mathcal{Z}(p) \cap \mathbb{D} = \emptyset. \quad (3.2)$$

Suppose  $g \in \mathcal{D}_1$  is orthogonal to  $[f]$ . Our goal is to show  $g = 0$ . For this aim we first prove the following claim:

**Claim:** For every row vector polynomial  $\vec{u} \in \mathbb{C}^{n+m}[z_1, z_2]$ , it holds that

$$\langle \vec{u} \vec{B}, g \rangle = 0.$$

*Proof of Claim.* For every  $k \in \mathbb{N}$  we have

$$\begin{aligned} (I - A(z_1, z_2)^k) \vec{B}(z_1, z_2) &= \sum_{j=0}^{k-1} A(z_1, z_2)^j (I - A(z_1, z_2)) \vec{B}(z_1, z_2) \\ &\stackrel{(3.1)}{\in} f(z_1, z_2) \cdot \mathbb{C}^{n+m}[z_1, z_2]. \end{aligned} \quad (3.3)$$

For every row vector polynomial  $\vec{u} \in \mathbb{C}^{n+m}[z_1, z_2]$  and every  $k \in \mathbb{N}$ , we have

$$s_k(z_1, z_2) := \vec{u}(z_1, z_2) (I - A(z_1, z_2)^k) \vec{B}(z_1, z_2) \stackrel{(3.3)}{\in} [f]. \quad (3.4)$$

Since  $g \perp [f]$ , (3.4) in particular implies that  $s_k \perp g$ , or equivalently

$$\langle \vec{u} \vec{B}, g \rangle = \langle \vec{u} A^k \vec{B}, g \rangle. \quad (3.5)$$

If  $\langle \vec{u}A^k\vec{B}, g \rangle = 0$  for some  $k \in \mathbb{N}$ , then (3.5) implies the statement of the Claim. Assume that  $\langle \vec{u}A^k\vec{B}, g \rangle \neq 0$  for every  $k \in \mathbb{N}$ . Let

$$d := \deg(\vec{u}(z_1, z_2)) + \deg(\vec{B}(z_1, z_2)) + 1,$$

where  $\deg(\cdot)$  denotes the maximum of total degrees of all entries. Since  $\vec{u}A^{kd}\vec{B}$  is a linear combination of monomials of degrees between  $kd$  and  $kd + d - 1$ , it follows that  $\{\vec{u}A^{kd}\vec{B}\}_{k \geq 0}$  are pairwise orthogonal in  $\mathcal{D}_\alpha$ .

By Bessel's inequality and (3.5),

$$\|g\|_1^2 \geq \sum_{k \geq 0} \frac{|\langle \vec{u}\vec{B}, g \rangle|^2}{\|\vec{u}A^{kd}\vec{B}\|_1^2}. \quad (3.6)$$

Further on,

$$\|\vec{u}A^{kd}\vec{B}\|_1^2 \leq (k+1)d \|\vec{u}A^{kd}\vec{B}\|_{H^2}^2 \leq (k+1) \underbrace{d\|u\|_{H^\infty}\|\vec{B}\|_{H^2}^2}_{=:C} \quad (3.7)$$

where the first inequality follows from the fact that  $\vec{u}A^{kd}\vec{B}$  has degree at most  $(k+1)d-1$ , while in the second we used that  $\|A^{kd}(z_1, z_2)\|_{H^\infty} = 1$  for  $z_1, z_2 \in \mathbb{T}^2$ . Using (3.7) in (3.6), it follows that

$$\sum_{k \geq 0} \frac{|\langle \vec{u}\vec{B}, g \rangle|^2}{C(k+1)} < \infty,$$

which is only possible if  $\langle \vec{u}\vec{B}, g \rangle = 0$ , proving the Claim.  $\blacksquare$

Choosing  $\vec{u}(z_1, z_2) = z_1^j z_2^k \vec{v}(z_2)$ , where  $\vec{v}$  is as in (3.2), the Claim implies that

$$0 = \langle z_1^j z_2^k \vec{v}(z_2) \vec{B}(z), g \rangle = \langle z_1^j z_2^k p(z_2), g \rangle,$$

whence  $g$  is orthogonal to  $[p]$ . Since  $p$  has no zero in  $\mathbb{D}$ , it is cyclic in  $\mathcal{D}_1$  by Proposition 3.2 and hence  $[p] = \mathcal{D}_1$ . Therefore  $g = 0$  and  $f$  is cyclic in  $\mathcal{D}_\alpha$  for all  $\alpha \leq 1$ .  $\square$

**3.2. Polynomials having finitely many zeroes on  $\mathbb{T}^2$ .** In this subsection, we investigate the cyclicity of polynomials in  $\mathcal{D}_\alpha$  for  $\alpha \leq 2$  that possess only finitely many zeros on  $\mathbb{T}^2$ . In the previously studied settings (see [3, Section 3]), the cyclicity of the polynomial  $1 - z_i$  was essentially used in the proof of the main result [3, Theorem 3.1]. However, in our framework,  $1 - z_i$  needs not be replaced by another polynomial cyclic in  $\mathcal{D}_\alpha$  for  $\alpha \leq 2$ , since  $1 - z_i$  is cyclic in  $\mathcal{D}_\alpha$  only for  $\alpha \leq 1$ . It turns out, that  $2 - z_1 - z_2$  serves as a suitable substitute.

**Theorem 3.3.** *Consider a polynomial  $p \in \mathbb{C}[z_1, z_2]$  having no zeros in  $\mathbb{D}^2$  and finitely many on  $\mathbb{T}^2$ . Then  $p$  is cyclic in  $\mathcal{D}_\alpha$  for  $\alpha \leq 2$ .*

The proof of Theorem 3.3 will parallel the arguments in the proof of [3, Theorem 3.2], which deals with the cyclicity in  $\mathfrak{D}_\alpha$ , using the modifications described in the paragraph before Theorem 3.3. For this reason we first establish Theorem 3.3 in the special case when  $p(z_1, z_2) = 2 - z_1 - z_2$ . The proof presented here for this special case is motivated by the argument given in [15, Appendix A].

**Lemma 3.4.** *Let  $\alpha \leq 2$ . Then  $p(z_1, z_2) = 2 - z_1 - z_2$  is cyclic in  $\mathcal{D}_\alpha$ .*

*Proof.* It suffices to show that  $p$  is cyclic in  $\mathcal{D}_2$ . Let  $f \in \mathcal{D}_2$  such that  $f \perp [p]$ . Consider the following series of  $f$ ,

$$f(z_1, z_2) = \sum_{i,j=0}^{\infty} \frac{b_{i,j}}{(i+j+1)^2} z_1^i z_2^j.$$

We will show that  $f = 0$ , or equivalently  $b_{i,j} = 0$  for  $i, j \in \mathbb{Z}_+$ . By  $f \perp [p]$ , it follows that

$$2b_{k,l} = b_{k+1,l} + b_{k,l+1}, \quad k, l \in \mathbb{Z}_+. \quad (3.8)$$

Since  $f \in \mathcal{D}_2$ , a new function

$$g(z_1, z_2) := \sum_{i,j \geq 0} b_{i,j} z_1^i z_2^j$$

belongs to  $\mathcal{D}_{-2}$ , and by (3.8),

$$(z_1 + z_2 - 2z_1 z_2)g(z_1, z_2) = z_1 g(z_1, 0) + z_2 g(0, z_2), \quad (z_1, z_2) \in \mathbb{D}^2. \quad (3.9)$$

We next introduce the substitutions

$$z_1 = \frac{\zeta}{\zeta - 1}, \quad z_2 = \frac{\zeta}{\zeta + 1}. \quad (3.10)$$

Observe that  $z_1 \in \mathbb{D}$  precisely when  $\Re \zeta < \frac{1}{2}$ , and  $z_2 \in \mathbb{D}$  precisely when  $\Re \zeta > -\frac{1}{2}$ , where  $\Re \zeta$  denotes the real part of the complex number  $\zeta$ . By substituting the above expressions for  $z_1$  and  $z_2$  into (3.9), we get

$$0 = \frac{\zeta}{\zeta - 1} g\left(\frac{\zeta}{\zeta - 1}, 0\right) + \frac{\zeta}{\zeta + 1} g\left(0, \frac{\zeta}{\zeta + 1}\right), \quad \text{for } -\frac{1}{2} < \Re \zeta < \frac{1}{2}.$$

Thus, defining  $h : \mathbb{C} \rightarrow \mathbb{C}$  by

$$h(\zeta) = \begin{cases} \frac{1}{\zeta - 1} g\left(\frac{\zeta}{\zeta - 1}, 0\right), & \text{if } \Re \zeta < \frac{1}{2}, \\ -\frac{1}{\zeta + 1} g\left(0, \frac{\zeta}{\zeta + 1}\right), & \text{if } \Re \zeta > -\frac{1}{2}, \end{cases}$$

we obtain that  $h$  is a well-defined entire function. Note that

$$\sum_{k \geq 0} \frac{|b_{k0}|^2}{(k+1)^2} \asymp \int_{\mathbb{D}} |g(z_1, 0)|^2 (1 - |z_1|^2) dA(z_1).$$

This can be verified by integrating right hand side using polar coordinates. Thus we have

$$\begin{aligned} \sum_{k \geq 0} \frac{|b_{k0}|^2}{(k+1)^2} &\asymp \int_{\Re \zeta < 1/2} |(\zeta - 1)h(\zeta)|^2 \left(1 - \left|\frac{\zeta}{\zeta - 1}\right|^2\right) \frac{dA(\zeta)}{|\zeta - 1|^4} \\ &= \int_{\Re \zeta < 1/2} |h(\zeta)|^2 \frac{1 - 2\Re \zeta}{|\zeta - 1|^4} dA(\zeta), \end{aligned} \quad (3.11)$$

and similarly,

$$\begin{aligned} \sum_{l \geq 0} \frac{|b_{0l}|^2}{(l+1)^2} &\asymp \int_{\mathbb{D}} |g(0, z_2)|^2 (1 - |z_2|^2) dA(z_2) \\ &= \int_{\Re \zeta > -1/2} |(\zeta + 1)h(\zeta)|^2 \left(1 - \left|\frac{\zeta}{\zeta + 1}\right|^2\right) \frac{dA(\zeta)}{|\zeta + 1|^4} \\ &= \int_{\Re \zeta > -1/2} |h(\zeta)|^2 \frac{1 + 2\Re \zeta}{|\zeta + 1|^4} dA(\zeta). \end{aligned}$$

Both series are finite since  $g \in \mathcal{D}_{-2}$ ; hence, the sum of the two integrals is finite, and consequently,

$$\int_{|\zeta| > 1} \frac{|h(\zeta)|^2}{|\zeta|^4} dA(\zeta) < \infty.$$

This forces  $h$  to be a polynomial of degree at most 1, i.e., .

$$h(\zeta) = a(\zeta - 1) + b, \quad a, b \in \mathbb{C}. \quad (3.12)$$

Assume that  $a \neq 0$ . Using (3.12) in (3.11), we get

$$\int_{\Re \zeta < 1/2} \frac{1 - 2\Re \zeta}{|\zeta - 1|^2} dA(\zeta) < \infty,$$

which is a contradiction. Hence,  $a = 0$ . Note that

$$\begin{aligned} g(z_1, 0) &\underset{(3.10)}{=} \frac{1}{z_1 - 1} h\left(\frac{z_1}{z_1 - 1}\right) = \frac{b}{z_1 - 1}, \\ g(0, z_2) &\underset{(3.10)}{=} -\frac{1}{z_2 + 1} h\left(\frac{z_2}{z_2 + 1}\right) = -\frac{b}{z_2 + 1}. \end{aligned} \quad (3.13)$$

Using (3.13) in (3.9), we get

$$\begin{aligned} (z_1 + z_2 - 2z_1 z_2)g(z_1, z_2) &= \frac{bz_1}{z_1 - 1} - \frac{bz_2}{z_2 + 1}, \quad (z_1, z_2) \in \mathbb{D}^2, \\ &= \frac{b(z_1 + z_2)}{(z_1 - 1)(z_2 + 1)}, \quad (z_1, z_2) \in \mathbb{D}^2. \end{aligned} \quad (3.14)$$

In particular, restricting (3.14) to the curve  $z_2 = -z_1 + cz_1^2$ ,  $c \in \mathbb{R}$ , we get

$$g(z_1, -z_1 + cz_1^2) = \frac{bc}{(c + 2 - cz_1)(z_1 - 1)(-z_1 + cz_1^2 + 1)}.$$

Therefore

$$g(0, 0) = \lim_{z_1 \rightarrow 0} g(z_1, -z_1 + cz_1^2) = \frac{-bc}{c + 2}$$

Since  $c$  was arbitrary, this implies  $b = 0$ . Therefore  $h = 0$  and consequently  $g = 0$  and  $f = 0$ .

□

*Remark 3.5.* The proof shows that if a sequence of complex numbers  $\{b_{k,l}\}_{k,l \in \mathbb{Z}_+}$  satisfies (3.8) and

$$\sum_{k,l \in \mathbb{Z}_+} \frac{|b_{k,l}|^2}{(k + l + 1)^2} < \infty,$$

then  $b_{k,l} = 0$  for all  $k, l \in \mathbb{Z}_+$ .

The following result is an immediate and noteworthy consequence.

**Proposition 3.6.** *For  $\zeta_1, \zeta_2 \in \mathbb{T}$ ,  $p(z_1, z_2) = 2 - \zeta_1 z_1 - \zeta_2 z_2$  is cyclic in  $\mathcal{D}_\alpha$  for  $\alpha \leq 2$ .*

*Proof.* This follows from Lemma 3.4 together with the fact that cyclicity is preserved under rotation.  $\square$

The following result, which is an analog of [3, Lemma 3.3], will allow us to compare polynomials having finitely many zeros on  $\mathbb{T}^2$  to the polynomials of the type  $2 - \zeta z_1 - \eta z_2$ ,  $\zeta, \eta \in \mathbb{T}$ , whose cyclicity has already been established in Proposition 3.6.

**Lemma 3.7.** *Suppose  $f \in \mathbb{C}[z_1, z_2]$  has no zeros in  $\mathbb{D}^2$  and finitely many on  $\mathbb{T}^2$ , i.e.,  $\{(\zeta_j, \eta_j) \in \mathbb{T}^2, j = 1, \dots, k\}$ . Then for any integer  $k$  there exists sufficiently large  $N$  such that the function  $Q$  defined by*

$$Q(z_1, z_2) = \frac{\prod_{i=1}^k (2 - \zeta_i^{-1} z_1 - \eta_i^{-1} z_2)^N}{f(z_1, z_2)}$$

is  $k$ -times differentiable on  $\mathbb{T}^2$ .

*Proof.* Let  $\{(\zeta_j, \eta_j) \in \mathbb{T}^2, j = 1, \dots, k\}$  be as in the statement of the lemma and define the polynomials

$$p_j(z_1, z_2) = 2 - \zeta_j^{-1} z_1 - \eta_j^{-1} z_2.$$

Clearly  $p_j$  has only one zero  $(\zeta_j, \eta_j)$  on  $\mathbb{T}^2$ . Note that

$$|p_j(z_1, z_2)|^2 \leq |z_1 - \zeta_j|^2 + |z_2 - \eta_j|^2 + 2|(z_1 - \zeta_j)(z_2 - \eta_j)|. \quad (3.15)$$

Writing  $z_1, z_2, \zeta_j, \eta_j \in \mathbb{T}$  as

$$z_1 = e^{ix_1}, z_2 = e^{ix_2}, \zeta_j = e^{iy_{1,j}}, \eta_j = e^{iy_{2,j}},$$

where  $x_1, x_2, y_{1,j}, y_{2,j} \in [0, 2\pi)$ , respectively, (3.15) becomes

$$\begin{aligned} |p_j(e^{ix_1}, e^{ix_2})|^2 &\leq |e^{ix_1} - e^{iy_{1,j}}|^2 + |e^{ix_2} - e^{iy_{2,j}}|^2 \\ &\quad + 2|(e^{ix_1} - e^{iy_{1,j}})(e^{ix_2} - e^{iy_{2,j}})|. \end{aligned} \quad (3.16)$$

Define

$$r(x_1, x_2) = |f(e^{ix_1}, e^{ix_2})|^2, \quad (x_1, x_2) \in \mathbb{R}^2,$$

and let  $\mathcal{Z}(r)$  denote its zero set. Consider the compact set  $E = [0, 2\pi]^2$ . By Theorem 2.4, there exist constants  $C > 0$  and  $q \in \mathbb{N}$  such that

$$r(x) \geq C \operatorname{dist}(x, \mathcal{Z}(r))^q, \quad x \in E.$$

Since the set  $\mathcal{Z}(r) \cap E$  consists of finitely many points, there is a constant  $c > 0$  satisfying

$$\operatorname{dist}(x, \mathcal{Z}(r))^2 \geq c \prod_{y \in \mathcal{Z}(r) \cap E} |x - y|^2, \quad x \in E. \quad (3.17)$$

Also, we have

$$|x - y|^2 = \sum_{j=1,2} |x_j - y_j|^2 \geq \sum_{j=1,2} |e^{ix_j} - e^{iy_j}|^2 \geq 2 \prod_{j=1,2} |e^{ix_j} - e^{iy_j}|. \quad (3.18)$$

Using (3.16), (3.17) and (3.18), we have a constant  $C_1 > 0$  such that

$$\text{dist}(x, \mathcal{Z}(r))^2 \geq C_1 \prod_j |p_j|^2.$$

This yields that the function defined by

$$\frac{\prod_j |p_j|^{q/2}}{|f(z_1, z_2)|^2},$$

is bounded on  $\mathbb{T}^2$ . We now apply the standard trick, which is to increase the exponent in the numerator and assign the value zero at the zeros of  $f$ , to obtain a function that is  $k$ -times continuously differentiable on  $\mathbb{T}^2$ . This completes the proof of Lemma 3.7.  $\square$

Finally, we can prove Theorem 3.3

*Proof of Theorem 3.3.* By Lemma 3.7, we obtain a function

$$Q(z_1, z_2) = \frac{\left(\prod_{i=1}^k (2 - \zeta_i^{-1} z_1 - \eta_i^{-1} z_2)\right)^N}{p(z_1, z_2)} =: \frac{g(z_1, z_2)}{p(z_1, z_2)},$$

where  $(\zeta_i, \eta_i) \in \mathbb{T}^2$ , which is twice continuously differentiable on  $\mathbb{T}^2$ . Thus its Fourier coefficients  $\widehat{Q}(k, l)$  satisfy

$$\sum_{k,l} |\widehat{Q}(k, l)|^2 (k+1)^2 (l+1)^2 < \infty.$$

But since

$$\sum_{k,l} |\widehat{Q}(k, l)|^2 (k+l+1)^2 \leq \sum_{k,l} |\widehat{Q}(k, l)|^2 (k+1)^2 (l+1)^2,$$

we obtain  $Q \in \mathcal{D}_\alpha$ ,  $\alpha \leq 2$ . Hence  $g(z_1, z_2) \in p\mathcal{D}_\alpha$ ,  $\alpha \leq 2$ . Since  $g$  is cyclic in  $\mathcal{D}_2$  and  $p$  is a multiplier, we obtain that  $p$  is also cyclic in  $\mathcal{D}_2$ .  $\square$

**3.3. Proof of Theorem 1.2.** The proof of Theorem 1.2 is now straightforward using the results above.

We divide the argument into three cases according to the value of  $\alpha$ :  $\alpha > 2$ ,  $\alpha \in (1, 2]$  and  $\alpha \leq 1$ .

**Case 1:**  $\alpha > 2$ . Using the norm inequalities  $\|f\|_{\mathcal{D}_{\alpha/2}} \leq \|f\|_{\mathcal{D}_\alpha} \leq \|f\|_{\mathcal{D}_\alpha}$ , Theorem 1.2.(iii) follows from Theorem 1.1.(iii).

**Case 2:**  $\alpha \in (1, 2]$ . The norm inequality  $\|f\|_\alpha \geq \|f\|_{\mathcal{D}_{\alpha/2}}$  implies that if a polynomial  $p$  is cyclic in  $\mathcal{D}_\alpha$ , it must also be cyclic in  $\mathcal{D}_{\alpha/2}$ . By Theorem 1.1, the zero set  $Z(p) \cap \mathbb{T}^2$  is empty or a finite set.

**Subcase 2.1:**  $Z(p) \cap \mathbb{T}^2 = \emptyset$ . Using the case  $\alpha > 2$ , we obtain that  $p$  is cyclic in  $\mathcal{D}_3$ . Since  $\|\cdot\|_\alpha \leq \|\cdot\|_3$  for  $\alpha \leq 2$ , cyclicity in  $\mathcal{D}_\alpha$  follows.

**Subcase 2.2:**  $Z(p) \cap \mathbb{T}^2$  is finite. This subcase follows from Theorem 3.3.

**Case 3:**  $\alpha \leq 1$ . The case  $\alpha \in (0, 1]$  is covered by Theorem 3.1. By comparing the norm, the case  $\alpha \leq 0$  is also achieved.

*Remark 3.8.* Note that if  $p$  has no zeros in the bidisk and only finitely many zeros on  $\mathbb{T}^2$ , then its cyclicity in  $\mathcal{D}_\alpha$ ,  $\alpha \leq 1$ , also follows from Theorem 1.1 by a direct comparison of norms. However, this result excludes the cyclicity of a polynomial having infinitely many zeros on  $\mathbb{T}^2$  as in the case of  $1 - z_1 z_2$ .

#### 4. CONCLUDING REMARKS

We provided a self-contained proof of the cyclicity of the polynomial  $2 - z_1 - z_2$ ; we now give a simple proof of the cyclicity of  $1 - z_1 z_2$  using standard technique. These are model polynomials and often play a crucial role in establishing cyclicity results for broader classes of functions (i.e., see the proof of [15, Theorem 5]), and therefore merit particular attention to ensure independent proof.

**Lemma 4.1.**  $1 - z_1 z_2$  is cyclic in  $\mathcal{D}_\alpha$  if and only if  $\alpha \leq 1$ .

*Proof.* Note that if  $f$  is a function in one variable say  $z$  and  $F$  is defined by  $F(z_1, z_2) := f(z_1 z_2)$ , then we have the following: for  $\alpha > 0$

$$\|f\|_{D_\alpha} \leq \|F\|_\alpha \leq 2^\alpha \|f\|_{D_\alpha}. \quad (4.1)$$

Let  $\alpha \in \mathbb{R}$ . Assume that  $p(z_1, z_2) = 1 - z_1 z_2$  is cyclic in  $\mathcal{D}_\alpha$ . Then there exists sequence of polynomials  $P_n(z_1, z_2) := p_n(z_1 z_2)$  such that  $\|P_n p - 1\|_\alpha \rightarrow 0$ , where  $p_n$  is a univariate polynomial. By (4.1), we have that  $1 - z$  is cyclic in  $D_\alpha$  and hence  $\alpha \leq 1$ . The converse also follows using similar reasoning.  $\square$

We conclude the paper with a few comments on capacity. Finite logarithmic capacity, or Riesz  $\alpha$ -capacity plays an important role in identifying non-cyclic functions in  $\mathfrak{D}_\alpha$  [2, Proposition 4.2]. This approach originates in the work of Brown and Shields [5] and was later extended to several variables by several authors. We record the following straightforward result, since it provides a necessary condition for the cyclicity of a general function in  $\mathcal{D}_\alpha$ ,  $\alpha \in \mathbb{R}$ .

**Proposition 4.2.** Let  $\alpha \in \mathbb{R}$ , and  $f \in \mathcal{D}_\alpha$ . Let  $f^*$  denote the radial limit of  $f$  i.e.  $f^*(e^{i\theta_1}, e^{i\theta_2}) = \lim_{r \rightarrow 1^-} f(re^{i\theta_1}, re^{i\theta_2})$ . Then the following holds:

- (a) If  $\mathcal{Z}(f^*)$  has positive logarithmic capacity, then  $f$  is not cyclic in  $\mathcal{D}_\alpha$ ,  $\alpha \geq 2$ .
- (b) For  $0 < \alpha < 1$ , if  $\mathcal{Z}(f^*)$  has positive Riesz  $\alpha$ -capacity, then  $f$  is not cyclic in  $\mathcal{D}_{2\alpha}$ .

*Proof.* This follows from the fact that  $f$  is not cyclic in respective  $\mathfrak{D}_\alpha$  and by doing norm comparison with  $\mathcal{D}_{2\alpha}$ .  $\square$

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