THE TRUNCATED MOMENT PROBLEM ON CURVES $y = q(x)$ AND $yx^\ell = 1$

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ABSTRACT. In this paper we study the bivariate truncated moment problem (TMP) on curves of the form $y = q(x)$, $q(x) \in \mathbb{R}[x]$, $\deg q \geq 3$, and $yx^\ell = 1$, $\ell \in \mathbb{N} \setminus \{1\}$. For even degree sequences the solution based on the number of moment matrix extensions was first given by Fialkow [Fia11] using the truncated Riesz-Haviland theorem [CF08] and a sum-of-squares representations for polynomials, strictly positive on such curves [Fia11, Sto01]. Namely, the upper bound on this number is quadratic in the degrees of the sequence and the polynomial determining a curve. We use a reduction to the univariate setting technique, introduced in [Zal21, Zal22a, Zal22b], and improve Fialkow's bound to $\deg q - 1$ (resp. $\ell + 1$) for curves $y = q(x)$ (resp. $yx^\ell = 1$). This in turn gives analogous improvements of the degrees in the sum-of-squares representations referred to above. Moreover, we get the upper bounds on the number of atoms in the minimal representing measure, which are $k \deg q$ (resp. $k(\ell + 1)$) for curves $y = q(x)$ (resp. $yx^\ell = 1$) for even degree sequences, while for odd ones they are $k \deg q - \lceil \frac{\deg q}{2} \rceil$ (resp. $k(\ell + 1) - \lfloor \frac{\ell}{2} \rfloor + 1$) for curves $y = q(x)$ (resp. $yx^\ell = 1$). In the even case these are counterparts to the result by Riener and Schweighofer [RS18, Corollary 7.8], which gives the same bound for odd degree sequences on all plane curves, while in the odd case it is a slight improvement of their bound in these special cases. Further on, we give another solution to the TMP on the curves studied based on the feasibility of a linear matrix inequality, corresponding to the univariate sequence obtained, and finally we solve concretely odd degree cases of the TMP on curves $y = x^\ell$, $\ell = 2, 3$, and add a new solvability condition to the even degree case on the curve $y = x^2$.

1. INTRODUCTION

Given a real 2–dimensional sequence $\beta^{(d)} = \{\beta_{0,0}, \beta_{1,0}, \beta_{0,1}, \ldots, \beta_{d,0}, \beta_{d-1,1}, \ldots, \beta_{1,d-1}, \beta_{0,d}\}$ of degree d and a closed subset K of $\mathbb{R}^2$, the **truncated moment problem (K–TMP)** supported on K for $\beta^{(d)}$ asks to characterize the existence of a positive Borel measure μ on $\mathbb{R}^2$ with support in K, such that

$$\beta_{i,j} = \int_K x^i y^j d\mu \quad \text{for} \quad i, j \in \mathbb{Z}_+,\ 0 \leq i + j \leq d. \tag{1.1}$$

If such a measure exists, we say that $\beta^{(d)}$ has a representing measure supported on K and μ is its K–**representing measure**.

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Let $k=\lceil \frac{d}{2}\rceil$. In the degree-lexicographic order $1, X, Y, X^2, XY, Y^2, \ldots, X^k, X^{k-1}Y, \ldots, Y^k$ of rows and columns, the corresponding moment matrix to β is equal to

$$M_k = M_k(\beta) := \begin{pmatrix} M[0,0](\beta) & M[0,1](\beta) & \cdots & M[0,k](\beta) \\ M[1,0](\beta) & M[1,1](\beta) & \cdots & M[1,k](\beta) \\ \vdots & \vdots & \ddots & \vdots \\ M[k,0](\beta) & M[k,1](\beta) & \cdots & M[k,k](\beta) \end{pmatrix}, \tag{1.2}$$

where

$$M[i,j](\beta) := \begin{pmatrix} \beta_{i+j,0} & \beta_{i+j-1,1} & \beta_{i+j-2,2} & \cdots & \beta_{i,j} \\ \beta_{i+j-1,1} & \beta_{i+j-2,2} & \beta_{i+j-3,3} & \cdots & \beta_{i-1,j+1} \\ \beta_{i+j-2,2} & \beta_{i+j-3,3} & \beta_{i+j-4,4} & \cdots & \beta_{i-2,j+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{j,i} & \beta_{j-1,i+1} & \beta_{j-2,i+2} & \cdots & \beta_{0,i+j} \end{pmatrix}$$

and for odd d, the lower right corner $M[k,k]$ of $M_k(\beta)$ is undefined. Until the end of this section we assume that M_k is fully determined, i.e., it corresponds to the even degree sequence $\beta^{(2k)}$. Let $\mathbb{R}[x,y]_k := \{p \in \mathbb{R}[x,y] \colon \deg p \leq k\}$ stand for the set of real polynomials in variables x, y of degree at most k, where for $p \not\equiv 0$ the degree $\deg p$ stands for the maximal sum $i+j$ over all monomials $x^i y^j$ appearing in p with a nonzero coefficient a_{ij}, while for $p \equiv 0$, $\deg p = 0$. For every $p(x,y) = \sum_{i,j} a_{ij} x^i y^j \in \mathbb{R}[x,y]_k$ we denote by $p(X,Y) = \sum_{i,j} a_{ij} X^i Y^j$ the corresponding vector from the column space $\mathcal{C}(M_k)$ of the matrix M_k. If $p(X,Y) = \mathbf{0}$, where $\mathbf{0}$ stands for the zero vector, then we say p is a **column relation** of M_k. Recall from [CF96], that β has a representing measure μ with the support $\mathrm{supp}(\mu)$ being a subset of $\mathcal{Z}(p) := \{(x,y) \in \mathbb{R}^2 \colon p(x,y) = 0\}$ if and only if p is a column relation of M_k. We say that the matrix M_k is **recursively generated (rg)** if for $p, q, pq \in \mathbb{R}[x,y]_k$ such that p is a column relation of M_k, it follows that pq is also a column relation of M_k.

A **concrete solution** to the TMP is a set of necessary and sufficient conditions for the existence of a K–representing measure, that can be tested in numerical examples. Among necessary conditions, M_k must be positive semidefinite (psd) and rg [CF96]. A crucial tool to tackle the TMP, discovered by Curto an Fialkow in 1996, was a **flat extension theorem (FET)** [CF96, Theorem 7.10] (see also [CF05b, Theorem 2.19] and [Lau05] for an alternative proof), which states that $\beta^{(2k)}$ admits a $(\mathrm{rank}\, M_k)$–atomic representing measure if and only if M_k is psd and admits a rank-preserving extension to a moment matrix M_{k+1}. Using the FET as the main tool the bivariate TMP has been concretely solved in the following cases: K is the variety defined by a polynomial $p(x,y) = 0$ with $\deg p \leq 2$ [CF02, CF04, CF05a, Fia14]; $K = \mathbb{R}^2$, $k = 2$ and M_2 is invertible [CY16] (first solved nonconstructively using convex analysis in [FN10]); K is the variety $y = x^3$ [Fia11]; M_k has a special feature such as *recursive determinateness* [CF13] or *extremality* [CFM08]. Some special cases have also been solved in [CY15, Yoo17a, Yoo17b] based on the FET and in [Ble15, BF20, DS18, Fia17, Kim14] using different approaches.

References to some classical work on the TMP are monographs [Akh65, AhK62, KN77], while for a recent development in the area we refer a reader to [Sch17]. We also mention some variants of the TMPs, which attracted a recent research interest, such as versions of the infinite dimensional TMPs [AJK15, GKM16, IKLS17], the TMP for commutative $\mathbb{R}$–algebras [CGIK+], matrix and operator TMPs [And70, BK12, BZ18, BZ21, BW11, DFMT09, DS02, KT22, KW13], etc.

In our previous work we introduced a new approach to tackle the singular bivariate TMP, namely *a reduction to the univariate setting technique*. The idea is to use one of the column relations to transform the problem into the equivalent univariate TMP, where also negative moments of the measure could be present or not all moments between the lowest and the highest degree ones are known. In the case all moments from degree 0 to the highest degree one are known, the situation is well understood in terms of the existence and uniqueness of the representing measure and has been solved in full generality [CF91] for measures with support $\mathbb{R}$, $[a, \infty)$ or $[a, b] \subset \mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$, as well as for even and odd degree sequences. In the presence of negative moments we gave a solution along the lines of the classical case in [Zal22b], where we note that the existence of the solution even in the matrix case was already established by Simonov [Sim06] (but the measure is not constructively obtained and the number of atoms in a minimal measure does not directly follow from this more general approach). Using these results we presented [BZ21, Zal21, Zal22a, Zal22b] alternative solutions with shorter proofs compared to the original ones to the TMPs on the curves $xy = 0$, $y = x^3$, $y^2 = y$, $xy = 1$, but also obtained solutions for new cases, namely on the curve $y^2 = x^3$, on the union of three parallel lines and on $xy^2 = 1$.

The motivation for this paper was to use a reduction technique to the TMP on curves of the form $y = q(x)$ and $yq(x) = 1$, where $q \in \mathbb{R}[x]$. In [Fia11, Section 6] Fialkow gave a solution to the TMP on these curves for even degree sequences in terms of the bound on the degree m for which the existence of a positive extension M_m of M_k is equivalent to the existence of a representing measure. Namely, his bound is quadratic in k and $\deg q$. Using a reduction technique we are able to decrease his bound in the even degree case for all curves of the form $y = q(x)$, $\deg q \geq 3$, to $\deg q - 1$ and for curves of the form $yx^\ell = 1$, $\ell \in \mathbb{N} \setminus \{1\}$, to $\ell + 1$, which is our first main result. Moreover, the reduction technique also works in the odd degree case. A corollary to this improved bounds are also improvements of the sum-of-squares representations for polynomials, strictly positive on such curves, by decreasing the degrees of the polynomials in the representation. Our second main result are the upper bounds on the number of atoms in the minimal representing measure, which are for curves $y = q(x)$, $\deg q \geq 3$, equal to $k \deg q$ in the even and $k \deg q - \lceil \frac{\deg q}{2} \rceil$ in the odd case and for curves $yx^\ell = 1$, $\ell \geq 2$, equal to $k(\ell+1)$ in the even and $k(\ell+1) - \lfloor \frac{\ell}{2} \rfloor + 1$ in the odd case. In the even case these results are counterparts to the result of Riener and Schweighofer [RS18, Corollary 7.8], who proved that for *all* plane curves, odd degree sequence have at most $k \deg q$ atoms in the minimal measure. For curves of the above form we improve their bound slightly in the odd degree case. The third main result of the paper is another solution to the TMPs studied, which is based on the feasibility of a linear matrix inequality corresponding to the univariate sequence obtained. Moreover, we give concrete solutions to the odd degree TMPs on the curves $y = x^2$ and $y = x^3$ and an alternative solution to the even degree case on $y = x^2$ with a new solvability condition, which will be crucially needed in the solution of the TMP on the reducible curve $y(y - x^2) = 0$ in our forthcoming work.

1.1. **Reader's guide.** The paper is organized as follows. In Section 2 we fix some further notation and known results on the TMP, which will be used in the proofs of our results. In Section 3 we give two solutions of the K–TMP for $K = \{(x, y) \in \mathbb{R}^2 \colon y = q(x)\}$, $q \in \mathbb{R}[x]$, $\deg q \geq 3$, one based on the number of psd extensions of the moment matrix needed (see Theorems 3.1 and 3.2 for the even and the odd degree cases, respectively) and the other one based on the feasibility question of a certain linear matrix inequality (see Theorem 3.12). Theorems 3.1 and 3.2 also give bounds on the number of atoms in the minimal K–representing measure. Moreover, Theorem 3.1 gives a Positivstellensatz on K as a corollary (see Corollary 3.4). Further on, we solve concretely the TMPs on the curve $y = x^2$ (see Theorems 3.6 and 3.10 for the even and the odd degree cases, respectively) and on $y = x^3$ for the odd case (see Theorem 3.18). In Section 4 we give the corresponding results to the ones from Section 3 for curves $yx^\ell = 1$, $\ell \in \mathbb{N} \setminus \{1\}$. Theorems 4.1 and

4.2 are the counterparts of Theorems 3.1 and 3.2, respectively, Corollary 4.4 of Corollary 3.4 and Theorem 4.6 of Theorem 3.12.

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2. Preliminaries

In this section we fix some terminology, notation and present some tools needed in the proofs of our main results.

We write $\mathbb{R}^{n\times m}$ for the set of $n\times m$ real matrices. For a matrix M we denote by $\mathcal{C}(M)$ its column space. The set of real symmetric matrices of size n will be denoted by S_n. For a matrix $A\in S_n$ the notation $A\succ 0$ (resp. $A\succeq 0$) means A is positive definite (pd) (resp. positive semidefinite (psd)).

In the rest of this section let $d\in\mathbb{N}$ and $\beta=\beta^{(d)}=\{\beta_{i,j}\}_{i,j\in\mathbb{Z}_+,\ 0\le i+j\le d}$ be a bivariate sequence of degree d.

2.1. **Moment matrix.** Let $k=\lceil \frac{d}{2}\rceil$ and $M_k=M_k(\beta)$ be the moment matrix of β (see (1.2)). Let Q_1,Q_2 be subsets of the set $\{X^iY^j\colon i,j\in\mathbb{Z}_+,\ 0\le i+j\le k\}$. We denote by $(M_k)|_{Q_1,Q_2}$ the submatrix of M_k consisting of the rows indexed by the elements of Q_1 and the columns indexed by the elements of Q_2. In case $Q:=Q_1=Q_2$, we write $(M_k)|_Q=(M_k)|_{Q,Q}$ for short.

Remark 2.1. Whenever Q_1,Q_2 will be subsets of $\{x^iy^j\colon i,j\in\mathbb{Z}_+,\ 0\le i+j\le k\}$ in the rest of the paper, in the notation $(M_k)|_{Q_1,Q_2}$ all monomials from Q_1, Q_2 are meant capitalized, i.e., $x^iy^j\mapsto X^iY^j$.

2.2. **Atomic measures.** For $x\in\mathbb{R}^m$, we use δ_x to denote the probability measure on $\mathbb{R}^m$ such that $\delta_x(\{x\})=1$. By a **finitely atomic positive measure** on $\mathbb{R}^m$ we mean a measure of the form $\mu=\sum_{j=0}^{\ell}\rho_j\delta_{x_j}$, where $\ell\in\mathbb{N}$, each $\rho_j>0$ and each $x_j\in\mathbb{R}^m$. The points x_j are called **atoms** of the measure μ and the constants ρ_j the corresponding **densities**.

2.3. **Riesz functional.** The functional $L_\beta:\mathbb{R}[x,y]_{\le d}\to\mathbb{R}$, defined by

$$L_\beta(p):=\sum_{\substack{i,j\in\mathbb{Z}_+,\\ 0\le i+j\le d}} a_{i,j}\beta_{i,j},\qquad \text{where}\quad p=\sum_{\substack{i,j\in\mathbb{Z}_+,\\ 0\le i+j\le d}} a_{i,j}x^iy^j,$$

is called the **Riesz functional of the sequence** β.

2.4. **Affine linear transformations.** Let $K\subseteq\mathbb{R}^2$. The existence of a K–representing measure for β is invariant under invertible affine linear transformations of the form

$$\phi(x,y)=(\phi(x,y),\phi(x,y)):=(a+bx+cy,d+ex+fy),\ (x,y)\in\mathbb{R}^2, \tag{2.1}$$

$a,b,c,d,e,f\in\mathbb{R}$ with $bf-ce\neq 0$ in the sense which we now explain. We denote by $\widetilde{\beta}$ the 2–dimensional sequence defined by

$$\widetilde{\beta}_{i,j}=L_\beta\big(\phi(x,y)^i\cdot\phi(x,y)^j\big),$$

where L_β is the Riesz functional of β.

Proposition 2.2 ([CF05a, Proposition 1.9])**.** *Assume the notation above and let $d=2k$.*

(1) $M_k(\beta)$ is psd if and only if $M_k(\widetilde{\beta})$ is psd.
(2) $\operatorname{rank} M_k(\beta)=\operatorname{rank} M_k(\widetilde{\beta})$.
(3) $M_k(\beta)$ is rg if and only if $M_k(\widetilde{\beta})$ is rg.

(4) β admits a r–atomic K–representing measure if and only if $\widetilde{\beta}$ admits a r–atomic $\phi(K)$–representing measure.

In case $d = 2k - 1$ is odd, the block $M[k,k]$ of $M_k(\beta)$ is undefined. We say that $M_k(\beta)$ is **psd completable** if there exists an extension $\beta^{(2k)}$ of β such that $M_k(\beta^{(2k)})$ is psd.

Proposition 2.3. *Assume the notation above and let $d = 2k-1$, $k \in \mathbb{N}$.*

(1) $M_k(\beta)$ is psd completable if and only if $M_k(\widetilde{\beta})$ is psd completable.
(2) Let $r \in \mathbb{N}$. There exists an extension $\beta^{(2k)}$ of β such that $\operatorname{rank} M_k(\beta^{(2k)}) = r$ if and only if there exists an extension $\widetilde{\beta}^{(2k)}$ of $\widetilde{\beta}$ such that $\operatorname{rank} M_k(\widetilde{\beta}^{(2k)}) = r$.
(3) Let $r \in \mathbb{N}$. There exists an extension $\beta^{(2k)}$ of β such that $M_k(\beta^{(2k)})$ is rg if and only if there exists an extension $\widetilde{\beta}^{(2k)}$ of $\widetilde{\beta}$ such that $M_k(\widetilde{\beta}^{(2k)})$ is rg.
(4) β admits a r–atomic K–representing measure if and only if $\widetilde{\beta}$ admits a r–atomic $\phi(K)$–representing measure.

Proof. Proposition 2.3 follows easily from Proposition 2.2 by defining the extension $\widetilde{\beta}^{(2k)}$ of $\widetilde{\beta}$ from the extension $\beta^{(2k)}$ of β using the same transformation ϕ together with the Riesz functional $L_{\beta^{(2k)}}$ of the extension. Similarly, for the other direction one uses ϕ^{-1} together with the Riesz functional $L_{\widetilde{\beta}^{(2k)}}$ of the extension. For (4) we notice that any r–atomic K–representing measure of the sequence β generates the extension $\beta^{(2k)}$ and then use (4) of Proposition 2.2. $\square$

2.5. **Hankel matrices and univariate sequences.** Let $k \in \mathbb{N}$. For $v = (v_0, \ldots, v_{2k}) \in \mathbb{R}^{2k+1}$ we define the corresponding Hankel matrix as

$$A_v := (v_{i+j})_{i,j=0}^{k} = \begin{pmatrix} v_0 & v_1 & v_2 & \cdots & v_k \\ v_1 & v_2 & \iddots & \iddots & v_{k+1} \\ v_2 & \iddots & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & v_{2k-1} \\ v_k & v_{k+1} & \cdots & v_{2k-1} & v_{2k} \end{pmatrix} \in S_{k+1}. \tag{2.2}$$

Let $\mathbf{v_j} := (v_{j+\ell})_{\ell=0}^{k}$ be the $(j+1)$–th column of A_v, $0 \le j \le k$. In this notation, we have that

$$A_v = \begin{pmatrix} \mathbf{v_0} & \cdots & \mathbf{v_k} \end{pmatrix}.$$

As in [CF91], the **rank** of v, denoted by $\operatorname{rank} v$, is defined by

$$\operatorname{rank} v = \begin{cases} k+1, & \text{if } A_v \text{ is nonsingular,} \\ \min\{i \colon \mathbf{v_i} \in \operatorname{span}\{\mathbf{v_0}, \ldots, \mathbf{v_{i-1}}\}\}, & \text{if } A_v \text{ is singular.} \end{cases}$$

We denote

- the upper left-hand corner $(v_{i+j})_{i,j=0}^{m} \in S_{m+1}$ of A_v of size $m+1$ by $A_v(m)$.
- the lower right-hand corner $(v_{i+j})_{i,j=k-m}^{k} \in S_{m+1}$ of A_v of size $m+1$ by $A_v[m]$.

We write

$$v^{(\mathrm{rev})} := (v_{2k}, v_{2k-1}, \ldots, v_0)$$

for the **reversed sequence** of v.

A sequence $v = (v_0, \ldots, v_{2k})$ is called **positively recursively generated (prg)** if, denoting $r = \operatorname{rank} v$, it holds that $A_v(r-1) \succ 0$ and in case $r < k+1$, also

$$v_j = \sum_{i=0}^{r-1} \varphi_i v_{j-r+i} \quad \text{for} \quad j = r, \ldots, 2k,$$

where

$$\begin{pmatrix} \varphi_0 & \cdots & \varphi_{r-1} \end{pmatrix} := A_v(r-1)^{-1} \begin{pmatrix} v_r & \cdots & v_{2r-1} \end{pmatrix}^T. \tag{2.3}$$

A sequence $v = (v_0, \ldots, v_{2k})$ is called **negatively recursively generated (nrg)** if, denoting $r = \operatorname{rank} v^{(\mathrm{rev})}$, it holds that $A_v[r-1] \succ 0$ and in case $r < k+1$, also

$$v_{2k-r-j} = \sum_{i=0}^{r-1} \psi_i v_{2k-r+1-j+i} \quad \text{for} \quad j = 0, \ldots, 2k-r,$$

where

$$\begin{pmatrix} \psi_0 & \cdots & \psi_{r-1} \end{pmatrix} := A_v[r-1]^{-1} \begin{pmatrix} v_{2k-2r+1} & \cdots & v_{2k-r} \end{pmatrix}^T.$$

2.6. **Univariate truncated moment problem.** Given a real sequence

$$\gamma^{(k_1,k_2)} = (\gamma_{k_1}, \gamma_{k_1+1}, \ldots, \gamma_{k_2-1}, \gamma_{k_2})$$

of degree (k_1, k_2), $k_1, k_2 \in \mathbb{Z}$, $k_1 \leq k_2$, a subset K of $\mathbb{R}$, the **truncated moment problem supported on K for $\gamma^{(k_1,k_2)}$ ((K, k_1, k_2)–TMP)** asks to characterize the existence of a positive Borel measure μ on $\mathbb{R}$ with support in K, such that

$$\gamma_i = \int_K x^i d\mu \quad \text{for} \quad i \in \mathbb{Z},\ k_1 \leq i \leq k_2. \tag{2.4}$$

If such a measure exists, we say that $\gamma^{(k_1,k_2)}$ has a representing measure supported on K and μ is its K**–representing measure**. Note that:

- The $(\mathbb{R}, 0, k)$–TMP with $k \in \mathbb{Z}_+$ is the usual **truncated Hamburger moment problem (THMP) of degree** k, which was solved in full generality in [CF91].
- The $(\mathbb{R} \setminus \{0\}, k_1, k_2)$–TMP with $k_1, k_2 \in \mathbb{Z}$, $k_1 < 0 < k_2$ is the **strong truncated Hamburger moment problem (STHMP) of degree** (k_1, k_2). For even k_1 and k_2 the solution is [Zal22b, Theorem 3.1], but the technique in the proof can be extended to establish also the cases, where k_1, k_2 are not both even.

Let $\mathbb{R}[x^{-1}, x] = \left\{\sum_{i=r_1}^{r_2} a_i x^i \colon a_i \in \mathbb{R}, r_1, r_2 \in \mathbb{Z}, r_1 \leq r_2\right\}$ be the set of Laurent polynomials. For $k_1, k_2 \in \mathbb{Z}$, $k_1 \leq k_2$, we denote by $V_{(k_1,k_2)}$ a vector subspace in $\mathbb{R}[x^{-1}, x]$ generated by the set $\{x^{k_1}, x^{k_1+1}, \ldots, x^{k_2}\}$. For a sequence $\gamma := \gamma^{(k_1,k_2)}$ the functional $L_\gamma : V_{(k_1,k_2)} \to \mathbb{R}$, defined by

$$L_\gamma(p) := \sum_{k_1 \leq i \leq k_2} a_i \gamma_i, \qquad \text{where} \quad p = \sum_{k_1 \leq i \leq k_2} a_i x^i,$$

is called the **Riesz functional of the sequence** γ.

3. The TMP on the curves $y = q(x)$

In this section we study the K–TMP for K being a curve of the form $y = q(x)$, $q \in \mathbb{R}[x]$. In Subsection 3.1 we first give a solution of the K–TMP, $\deg q \geq 3$, based on the number of positive semidefinite extensions of the moment matrix needed and also bound the number of atoms in the K–representing measure with the smallest number of atoms (see Theorem 3.1 for the even degree and Theorem 3.2 for the odd degree sequences). As a result we obtain a sum-of-squares representation for polynomials, which are strictly positive on K (see Corollary 3.4). This improves bounds on the degrees in the previously known result [Fia11, Proposition 6.3]. In Subsection 3.2 we apply the technique from the proofs of the results from Subsection 3.1 to give a concrete solution of the TMP on the curve $y = x^2$, which is an alternative solution to the one from [CF04] in the even case (see Theorem 3.6) and is new in the odd case (see Theorem 3.10). In Subsection 3.3 we give a solution to the K–TMP based on a feasibility of the corresponding linear matrix inequality (see Theorem 3.12). Finally, in Subsection 3.4 we concretely solve the TMP on the curve $y = x^3$ in the odd degree case (see Theorem 3.18).

3.1. Solution to the TMP in terms of psd extensions of M_k, bounds on the number of atoms in the minimal measure and a Positivstellensatz.

Theorem 3.1. *Let $K := \{(x, y) \in \mathbb{R}^2 \colon y = q(x)\}$, where $q \in \mathbb{R}[x]$ with $\deg q \geq 3$, and $\beta := \beta^{(2k)} = (\beta_{i,j})_{i,j \in \mathbb{Z}_+, i+j \leq 2k}$ with $k \geq \deg q$. The following statements are equivalent:*

(1) β has a K–representing measure.

(2) β has a s–atomic K–representing measure for some s satisfying

$$\operatorname{rank} M_k \leq s \leq k \deg q.$$

(3) M_k satisfies $Y = q(X)$ and admits a positive semidefinite, recursively generated extension $M_{k+\deg q-2}$.

(4) M_k satisfies $Y = q(X)$ and admits a positive semidefinite extension $M_{k+\deg q-1}$.

Theorem 3.2. *Let $K := \{(x, y) \in \mathbb{R}^2 \colon y = q(x)\}$, where $q \in \mathbb{R}[x]$ with $\deg q \geq 3$, and $\beta^{(2k-1)} = (\beta_{i,j})_{i,j \in \mathbb{Z}_+, i+j \leq 2k-1}$ with $k \geq \deg q$. Then the following statements are equivalent:*

(1) β has a K–representing measure.

(2) β has a s–atomic K–representing measure for some s satisfying

$$\operatorname{rank} M_{k-1} \leq s \leq k \deg q - \left\lceil \frac{\deg g}{2} \right\rceil.$$

(3) $\beta^{(2k-1)}$ can be extended to a sequence $\beta^{(2k)}$ such that M_k is satisfies $Y = q(X)$ and admits a positive semidefinite, recursively generated extension $M_{k+\deg q-2}$.

(4) $\beta^{(2k-1)}$ can be extended to a sequence $\beta^{(2k)}$ such that M_k is satisfies $Y = q(X)$ and admits a positive semidefinite extension $M_{k+\deg q-1}$.

Remark 3.3. (1) In [CF08], Curto and Fialkow studied polynomials $p \in \mathbb{R}[x, y]$ for which the existence of the $\mathcal{Z}(p)$–representing measure is equivalent to the psd moment matrix extension of some size. In [Fia11, Section 6] the author considered polynomials of the form $p(x, y) = y - q(x)$, where $q \in \mathbb{R}[x]$, and proved that the number of psd extensions of the moment matrix needed is $(2k + 1) \deg q - k$ [Fia11, Propositions 6.1, 6.3], where $2k$ is the degree of the sequence. The proof of this result relies on the truncated Riesz-Haviland theorem [CF08, Theorem 1.2] and a sum-of-squares representations for polynomials, strictly positive on $\mathcal{Z}(p)$ ([Fia11, Proposition 6.3] and [Sto01, Proposition 5.1]). Part (4) of Theorem 3.1 improves Fialkow's result by decreasing the number of extensions to $\deg q - 1$. We mention that this was known for the case of the curve $y = x^3$ [Fia11, Corollary 5.3].

(2) In [RS18], the authors also studied odd degree sequences β, which are moments of a positive Borel measure supported on a plane curve $\mathcal{Z}(p)$, $p \in \mathbb{R}[x,y]$, and proved that every such sequence admits a $(k \deg p)$–atomic $\mathcal{Z}(p)$–representing measure [RS18, Corollary 7.6]. In the proof they use their variant of Bézout's theorem on the number of intersection points of two plane algebraic curves [RS18, Theorem 7.3]. Part (2) of Theorem 3.1 gives an analogue of [RS18, Corollary 7.6] for even degree sequences on curves $\mathcal{Z}(y-q(x))$, $\deg q \geq 3$, while part (2) of Theorem 3.2 improves [RS18, Corollary 7.6] for curves $\mathcal{Z}(y-q(x))$, $\deg q \geq 3$, by decreasing the upper bound on the number of atoms needed by $\lceil \frac{\deg q}{2} \rceil$.

(3) If $\deg q = 2$ in Theorem 3.1, then $y = q_2x^2 + q_1x + q_0 \in \mathbb{R}[x]$ with $q_2 \neq 0$ or equivalently $\frac{1}{q_2}y - q_1x - q_0 = x^2$. By applying an affine linear transformation $\phi(x,y) = (x, \frac{1}{q_2}y - q_1x - q_0)$ to the sequence β we get a sequence $\widetilde{\beta}$ with the moment matrix $M_k(\widetilde{\beta})$ satisfying $Y = X^2$. So it is enough to observe the case of a parabola, which was concretely solved in [CF04] by the use of the FET. The technique used in the proof of Theorem 3.1 can be used to give an alternative proof of the solution from [CF04] and also obtain a new solvability condition (see Theorem 3.6 below). This condition will be essentially used in the solution of TMP on the cubic reducible curve $y(y - x^2) = 0$ in our forthcoming work, similarly as for the TMP on the union of three parallel lines [Zal22a, Theorem 4.2], where we needed such version of the solution of the TMP on the union of two parallel lines [Zal22a, Theorem 3.1]. The upper bound on the number of atoms in the minimal representing measure is $2k + 1$ and this is sharp (e.g., if M_k has only column relations coming from $Y = X^2$ by rg, then it is of rank $2k + 1$ and so every representing measure must have at least $2k + 1$ atoms). So the equivalence (1) $\Leftrightarrow$ (2) of Theorem 3.1 does not extend to $\deg q = 2$. (The moment $\gamma_{k \deg q - 1}$ is not independent from β for $\deg q = 2$ as opposed to $\deg q > 2$ and hence in the last step of the proof below decreasing the number of atoms in the representing measure from $k \deg q + 1$ to $k \deg q$ cannot be done.) Also the equivalence (1) $\Leftrightarrow$ (3) of Theorem 3.1 is not true for $\deg q = 2$, but we need to replace $k + \deg q - 2$ by $k + \deg q - 1$ in (3), because we do not get the information about $\gamma_{2k \deg q + 1} = \gamma_{4k+1}$ and $\gamma_{2k \deg q + 2} = \gamma_{4k+2}$ from $M_{k+\deg q - 2} = M_k$ for $\deg q = 2$ as opposed to $\deg q > 2$. However, the equivalence (1) $\Leftrightarrow$ (4) still holds for $\deg q = 2$ with the argument given in Theorem 3.6 below.

(4) If $\deg q \leq 1$ in Theorem 3.1, then $q(x) = ax + by + c$, $a, b, c \in \mathbb{R}$. If $(a,b) \neq (0,1)$, then the following statements are equivalent:
   (a) β has a K–representing measure.
   (b) β has a s–atomic K–representing measure for some s satisfying
$$\operatorname{rank} M_k \leq s \leq k+1.$$
   (c) M_k satisfies $Y = aX + bY + c$, is positive semidefinite and recursively generated.

The equivalence (4a) $\Leftrightarrow$ (4c) follows from [CF08, Proposition 3.11], while the equivalence (4a) $\Leftrightarrow$ (4b) follows from the solution [CF91, Theorem 3.9] of the univariate $\mathbb{R}$–TMP, which corresponds to $(M_k)|_{\{1,X,\ldots,X^k\}}$. Namely, if the atoms $x_1, \ldots, x_m$ represent $\beta_{i,0}$, $i = 0, \ldots, 2k$, then the atoms (x_i, y_i), where $y_i = \frac{1}{1-b}(ax_i + c)$ will represent β if $b \neq 1$. If $b = 1$ and $a \neq 0$, then we change the roles of x and y in the argument above. If $b = 1$ and $a = 0$, then $y = q(x)$ only makes sense if $c = 0$, but in this case there are no relations in the moment matrix and for $k > 2$ the solution to the TMP is not known (for $k = 2$ the solution is known [FN10, CY16]).

The proof of Theorem 3.1 for a polynomial $q(x)$ of the form x^ℓ, $\ell \geq 3$, compared to a general $q(x)$, is technically much less involved, so before proving a general case we first demonstrate the basic idea on this special case.

Proof of Theorem 3.1 for $q(x) = x^\ell$, $\ell \geq 3$. The implications (1) $\Rightarrow$ (4) and (2) $\Rightarrow$ (1) are trivial. The implication (4) $\Rightarrow$ (3) is [CF96, Theorem 3.14]. It remains to prove the implication (3) $\Rightarrow$ (2). Assume that $Y = X^\ell$ is a column relation and M_k admits a psd, rg extension $M_{k+\ell-2}$. Let

$$\mathcal{B} = \left\{1, x, \ldots, x^{\ell-1}, y, yx, \ldots, yx^{\ell-1}, \ldots, y^{k-1}, \ldots, y^{k-1}x^{\ell-1}, y^k, y^k x\right\} \tag{3.1}$$

be a set of monomials and V a vector subspace in $\mathbb{R}[x, y]_{k+\ell-2}$ generated by the set $\mathcal{B}$. Since $M_{k+\ell-2}$ satisfies $X^iY^j = X^{i+j\ell}$ for every $i, j \in \mathbb{Z}_+$ such that $i + j\ell \leq k + \ell - 2$, it follows that the columns (capitalized) from $\mathcal{B}$ span $\mathcal{C}(M_{k+\ell-2})$. Let $p(x, y) = \sum_{i,j} p_{ij}x^iy^j \in V$ be a polynomial and $\widehat{p}$ a vector of its coefficients ordered in the basis $\mathcal{B}$. We define a univariate polynomial $g_p(x)$ corresponding to $p(x, y)$ by

$$g_p(x) := p(x, x^\ell) = \sum_{i,j} p_{ij}x^{i+\ell j} = \sum_{s=0}^{k\ell+1} g_{p,s}x^s \in \mathbb{R}[x]_{k\ell+1}. \tag{3.2}$$

Let $\widehat{g_p}$ be its vector of coefficients in the basis

$$\mathcal{B}_1 = \{1, x, \ldots, x^{k\ell+1}\}. \tag{3.3}$$

The monomials $x^{i_1}y^{j_1}$, $x^{i_2}y^{j_2}$ from $\mathcal{B}$ correspond to the same monomial x^s by the correspondence (3.2) iff $i_1 + \ell j_1 = i_2 + \ell j_2$, which is further equivalent to $i_1 = i_2$ and $j_1 = j_2$ (since i_1 and i_2 are at most $\ell - 1$ in $\mathcal{B}$). Therefore

$$\widehat{g_p} = \widehat{p}. \tag{3.4}$$

We define two univariate sequences

$$\gamma := \gamma^{(0,2k\ell)} = (\gamma_0, \gamma_1, \ldots, \gamma_{2k\ell}) \in \mathbb{R}^{2k\ell+1}, \quad \widetilde{\gamma} := \gamma^{(2k\ell+2)} = (\gamma, \gamma_{2k\ell+1}, \gamma_{2k\ell+2}) \in \mathbb{R}^{2k\ell+3}$$

by the formula

$$\gamma_t = \beta_{t \bmod \ell, \lfloor \frac{t}{\ell} \rfloor}. \tag{3.5}$$

Note that $t \bmod \ell + \lfloor \frac{t}{\ell} \rfloor \leq \ell - 1 + 2k$ (here we used that $\ell \geq 3$, $t \leq 2k + 2$ and thus $\lfloor \frac{t}{\ell} \rfloor \leq 2k$) and so $\beta_{t \bmod \ell, \lfloor \frac{t}{\ell} \rfloor}$ is well-defined being an element of the matrix $M_{k+\ell-2}$ (since $2(k+\ell-2) \geq \ell-1+2k$ for $\ell \geq 3$).

By the following claim solving the K–TMP for β is equivalent to solving the $\mathbb{R}$–TMP for γ.

Claim 1. Let $u \in \mathbb{N}$. A sequence γ admits a u–atomic $\mathbb{R}$–representing measure if and only if β admits a u–atomic K–representing measure.

Proof of Claim 1. First we prove the implication ($\Rightarrow$). Let $x_1, \ldots, x_u$, be the atoms in the $\mathbb{R}$–representing measure for γ with the corresponding densities $\rho_1, \ldots, \rho_u$. We will prove that the atoms $(x_1,(x_1)^\ell),\ldots,(x_u,(x_u)^\ell)$ with densities $\rho_1,\ldots,\rho_p$ are the K–representing measure for β. We have that

$$\beta_{i,j}=\beta_{i \bmod \ell, j+\lfloor\frac{i}{\ell}\rfloor}=\gamma_{i \bmod \ell+(j+\lfloor\frac{i}{\ell}\rfloor)\ell}=\gamma_{i+j\ell}=\sum_{p=0}^{u}\rho_p(x_p)^{i+j\ell}=\sum_{p=0}^{u}\rho_p(x_p)^i((x_p)^\ell)^j,$$

where we used that β is rg in the first equality, (3.5) with $t=i \bmod \ell+(j+\lfloor\frac{i}{\ell}\rfloor)\ell=i+j\ell$ in the second and third equalities noticing that $i+j\ell=i+j+j(\ell-1)\leq 2k+2k(\ell-1)\leq 2k\ell$, the definitions of ρ_p, x_p in the fourth equality, and split $(x_p)^{i+j\ell}$ into two parts in the last equality. So the atoms $(x_1,(x_1)^\ell),\ldots,(x_u,(x_u)^\ell)$ with densities $\rho_1,\ldots,\rho_p$ indeed represent $\beta_{i,j}$ for $i,j\in\mathbb{Z}_+$ such that $i+j\leq 2k$. This proves the implication ($\Rightarrow$).

It remains to prove the implication ($\Leftarrow$). Let $(x_1,(x_1)^\ell),\ldots,(x_u,(x_u)^\ell)$ be the atoms in the K–representing measure for β with the corresponding densities $\rho_1,\ldots,\rho_u$. We will prove that the atoms $(x_1,\ldots,x_u)$ with densities $\rho_1,\ldots,\rho_p$ are the $\mathbb{R}$–representing measure for γ. We have that

$$\gamma_t=\beta_{t \bmod \ell,\lfloor\frac{t}{\ell}\rfloor}=\sum_{p=0}^{u}\rho_p(x_p)^{t \bmod \ell}((x_p)^\ell)^{\lfloor\frac{t}{\ell}\rfloor}=\sum_{p=0}^{u}\rho_p(x_p)^t,$$

where we used the definition (3.5) of γ_t in the first equality, the definitions of ρ_p, x_p in the second equality (note that the measure for β generates an rg extension $\beta^{(2(k+\ell-2))}$ and $\beta_{t \bmod \ell,\lfloor\frac{t}{\ell}\rfloor}$ is an element of $M_{k+\ell-2}$ by the same argument as in the lines after (3.5)), and used the fact that $t \bmod \ell+\ell\lfloor\frac{t}{\ell}\rfloor=t$ in the last equality. This proves the implication ($\Leftarrow$). ■

Let $(M_{k+\ell-2})|_{\mathcal{B}}$ be the restriction of $M_{k+\ell-2}$ to the rows and columns indexed by monomials from $\mathcal{B}$ (see Remark 2.1). The following claim gives an explicit connection between $(M_{k+\ell-2})|_{\mathcal{B}}$ and the Hankel matrix $A_{\widetilde{\gamma}}$ of the sequence $\widetilde{\gamma}$.

Claim 2. We have that

$$(M_{k+\ell-2})|_{\mathcal{B}}=A_{\widetilde{\gamma}}. \tag{3.6}$$

Proof of Claim 2. Let $p(x,y)=\sum_{i,j}p_{ij}x^iy^j\in V$ and $r(x,y)=\sum_{i,j}r_{ij}x^iy^j\in V$ be polynomials from the vector subspace V and $\widehat{p}$, $\widehat{r}$ vectors of their coefficients ordered in the basis $\mathcal{B}$. Let

$\widetilde{\beta} := \beta^{(2(k+\ell-2))}$. Then we have

$$\begin{aligned}
(\widehat{r})^T\left((M_{k+\ell-2})|_{\mathcal{B}}\right)\widehat{p} &=^1 L_{\widetilde{\beta}}(pr) = L_{\widetilde{\beta}}\big(\sum_{i_1,i_2,j_1,j_2} p_{i_1j_1}r_{i_2j_2}x^{i_1+i_2}y^{j_1+j_2}\big)\\
&=^2 \sum_{i_1,i_2,j_1,j_2} p_{i_1j_1}r_{i_2j_2}\beta_{i_1+i_2,j_1+j_2}\\
&=^3 \sum_{i_1,i_2,j_1,j_2} p_{i_1j_1}r_{i_2j_2}\gamma_{i_1+i_2+(j_1+j_2)\ell}\\
&=^4 L_{\widetilde{\gamma}}\left(\sum_{i_1,i_2,j_1,j_2} p_{i_1j_1}r_{i_2j_2}x^{i_1+i_2+(j_1+j_2)\ell}\right)\\
&=^5 L_{\widetilde{\gamma}}\left(\sum_{i_1,i_2,j_1,j_2} p_{i_1j_1}x^{i_1+j_1\ell}\cdot r_{i_2j_2}x^{i_2+j_2\ell}\right)\\
&=^6 L_{\widetilde{\gamma}}\Big(\underbrace{\Big(\sum_{i_1,j_1} p_{i_1j_1}x^{i_1+j_1\ell}\Big)}_{g_p(x)}\underbrace{\Big(\sum_{i_2,j_2} r_{i_2j_2}x^{i_2+j_2\ell}\Big)}_{g_r(x)}\Big)\\
&=^7 (\widehat{g_r})^T A_{\widetilde{\gamma}}\widehat{g_p} = (\widehat{r})^T A_{\widetilde{\gamma}}\widehat{p},
\end{aligned}$$

where in the first line we used the correspondence between the moment matrix of $\widetilde{\beta}$ and the Riesz functional $L_{\widetilde{\beta}}$, the definition of $L_{\widetilde{\beta}}$ in the second, (3.5) and the fact that β is rg in the third (rg is needed if $i_1 + i_2 \geq \ell$), the definition of $L_{\widetilde{\gamma}}$ in the fourth, in the fifth line we decomposed the exponent of x into two parts, in the sixth we decomposed a sum into the product of two sums, in the seventh we used the correspondence between the moment matrix of $\widetilde{\gamma}$ and $L_{\widetilde{\gamma}}$, where $\widehat{g_p}$, $\widehat{g_r}$ are the vectors of coefficents of g_p and g_r in the basis $\mathcal{B}_1$ (see (3.3)) and (3.4). Since p and q were arbitrary from V, this proves Claim 2. ■

Since $(M_{k+\ell-2})|_{\mathcal{B}}$ is psd, it follows from (3.6) that $A_{\widetilde{\gamma}}$ is also psd. We separate two cases. Either $A_{\widetilde{\gamma}}$ is pd or $A_{\widetilde{\gamma}}$ is singular. In the first case in particular $A_{\widetilde{\gamma}}(k\ell) = A_\gamma$ is pd, while in the second case $A_{\widetilde{\gamma}}(k\ell)$ is psd and prg by [CF91, Theorem 2.6]. By [CF91, Theorem 3.9], γ admits a $(\operatorname{rank} A_\gamma)$–atomic $\mathbb{R}$–representing measure. Since $\operatorname{rank} M_k \leq \operatorname{rank} A_\gamma \leq k\ell + 1$, using Claim 1 the following holds:

(2') β has a s–atomic K–representing measure for some s satisfying

$$\operatorname{rank} M_k \leq s \leq \operatorname{rank} A_\gamma \leq k\ell + 1. \tag{3.7}$$

To obtain (2) of Theorem 3.1 we need to decrease the upper bound in (3.7) by 1. Note that the bound $k\ell+1$ occurs only in the case A_γ is pd, which we assume in the rest of the proof. We denote by $\gamma(\mathrm{z})$ a sequence obtained from the sequence γ by replacing $\gamma_{2k\ell-1}$ with a variable z. The matrix $A_{\gamma(\mathrm{z})}$ is a partially positive definite matrix and by [Zal21, Lemma 2.11] there exist two choices of z, which we denote by $z^\pm$, such that $A_{\gamma(z^\pm)}$ is psd and has rank $k\ell$. Since $\operatorname{rank} A_{\gamma(z^\pm)}(k\ell - 1) = \operatorname{rank} A_{\gamma(z^\pm)} = k\ell$, the sequence $\gamma(z^\pm)$ is prg and by [CF91, Theorem 3.9] it admits a $k\ell$–atomic $\mathbb{R}$–representing measure. If none of the moments $\beta_{i,j}$ of the sequence β depends on $\gamma_{2k\ell-1}$, the $\mathbb{R}$–representing measure for $\gamma(z^\pm)$ will generate a K–representing measure for β as in the proof of Claim 1. But by definition (3.5), there is indeed no moment of $\beta = \beta^{(2k)}$, which depends on

$\gamma_{2k\ell-1}$ (since we need to represent only moments of degree at most $2k$, while $\gamma_{2k\ell-1}$ corresponds to $\beta_{\ell-1,2k-1}$ in some extension of β), and this concludes the proof of Theorem 3.1 for $q(x) = x^\ell$. $\square$

To prove Theorem 3.2 for $q(x) = x^\ell$, $\ell \geq 3$, only a little adaptation of the last part of the proof of Theorem 3.1 is needed, which we now explain.

Proof of Theorem 3.2 for $q(x) = x^\ell$, $\ell \geq 3$. The implications (1) $\Rightarrow$ (4) and (2) $\Rightarrow$ (1) are trivial. The implication (4) $\Rightarrow$ (3) follows from [CF96, Theorem 3.14]. It remains to prove the implication (3) $\Rightarrow$ (2). Following the proof of Theorem 3.1 everything remains the same until (2'). It remains to justify that the upper bound in (3.7) can be decreased to $m := k\ell - \lceil \frac{\ell}{2} \rceil$. If $\operatorname{rank} A_\gamma \leq m$, then we are already done. From now on we assume that $r := \operatorname{rank} A_\gamma > m$. Since γ admits a $\mathbb{R}$–representing measure, which we denote by μ, γ is prg and $\operatorname{rank} \gamma = \operatorname{rank} A_\gamma = \operatorname{rank} A_\gamma(r-1)$ by [CF91, Theorem 3.9]. Hence, $A_\gamma(r-1)$ is pd and in particular also its submatrix $A_\gamma(m)$ is pd. We denote by $\gamma(\mathrm{z}_1, \ldots, \mathrm{z}_\ell)$ a sequence obtained from the sequence γ by replacing $\gamma_{(2k-1)\ell+1}, \gamma_{(2k-1)\ell+2}, \ldots, \gamma_{2k\ell}$ with variables $\mathrm{z}_1, \ldots, \mathrm{z}_\ell$. The sequence $\gamma^{(0,(2k-1)\ell)} := (\gamma_0, \ldots, \gamma_{(2k-1)\ell})$ is represented by μ, being a subsequence of γ. If ℓ is even, then $(2k-1)\ell$ is also even and by [CF91, Theorem 3.9], $\gamma^{(0,(2k-1)\ell)}$ has a $(\operatorname{rank} \gamma^{(0,(2k-1)\ell)})$–atomic $\mathbb{R}$–representing measure. Otherwise ℓ is odd, $(2k-1)\ell$ is also odd and by [CF91, Theorem 3.1], $\gamma^{(0,(2k-1)\ell)}$ has a $(\operatorname{rank} \gamma^{(0,(2k-1)\ell-1)})$–atomic $\mathbb{R}$–representing measure, where $\gamma^{(0,(2k-1)\ell-1)} := (\gamma_0, \ldots, \gamma_{(2k-1)\ell-1})$. We denote the measure obtained in this way by μ_1 and generate its moment sequence $\gamma(z_1, \ldots, z_\ell)$, where $z_1, \ldots, z_\ell$ are the moments of degrees $(2k-1)\ell+1, \ldots, 2k\ell$. Hence, $\operatorname{rank} A_{\gamma(z_1,\ldots,z_\ell)}$ is equal to $\operatorname{rank} A_{\gamma^{(0,(2k-1)\ell)}}$ for even ℓ and $\operatorname{rank} A_{\gamma^{(0,(2k-1)\ell-1)}}$ for odd ℓ. Since $(2k-1)\ell = 2m$ for even ℓ and $(2k-1)\ell - 1 = 2m$ for odd ℓ, μ_1 is m–atomic (since $A_\gamma(m) \succ 0$ by assumption). If none of the moments $\beta_{i,j}$ of the sequence $\beta^{(2k-1)}$ depends on $\gamma_{(2k-1)\ell+1}, \gamma_{(2k-1)\ell+2}, \ldots, \gamma_{2k\ell}$, then μ_1 will generate a K–representing measure for $\beta^{(2k-1)}$ as in the proof of Claim 1 of Theorem 3.1. But by definition (3.5), none of the moments of $\beta^{(2k-1)}$ depends on $\gamma_{(2k-1)\ell+1}, \gamma_{(2k-1)\ell+3}, \ldots, \gamma_{2k\ell}$, which concludes the proof of Theorem 3.2 for $q(x) = x^\ell$. $\square$

Now we prove Theorem 3.1 in the general case.

Proof of Theorem 3.1 for general $q(x)$. Before starting a proof we do an affine linear transformation ϕ which will be used in the proof of the implication (3) $\Rightarrow$ (2) to justify in an easier way that the upper bound in (2) is $k \deg q$ instead of $k \deg q + 1$. We write $\ell := \deg q$ and let $q(x) = \sum_{i=0}^{\ell} q_i x^i$, where $q_\ell \neq 0$ and each $q_i \in \mathbb{R}$.

Claim 1. We may assume that $q_{\ell-1} = 0$.

Proof of Claim 1. Defining $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ by $\varphi(x,y) = \Big(x + \dfrac{q_{\ell-1}}{\ell q_\ell}, y\Big) =: (\tilde{x}, y)$, note that the relation $y = q(x)$ becomes

$$y = q\Big(\tilde{x} - \frac{q_{\ell-1}}{\ell q_\ell}\Big) = \sum_{i=0}^{\ell} q_i \Big(\tilde{x} - \frac{q_{\ell-1}}{\ell q_\ell}\Big)^i = q_\ell \tilde{x}^\ell + \underbrace{\Big(-q_\ell \cdot \ell \frac{q_{\ell-1}}{\ell q_\ell} + q_{\ell-1}\Big)}_{=0} \tilde{x}^{\ell-1} + \sum_{i=0}^{\ell-2} \tilde{q}_i \tilde{x}^i,$$

for some $\tilde{q}_0, \ldots, \tilde{q}_{\ell-2} \in \mathbb{R}$. Since the solution of the K–TMP is invariant under applying ϕ by Proposition 2.2, the conclusion of Claim 1 follows. $\blacksquare$

Now we start the proof of the theorem. The implications (1) $\Rightarrow$ (4) and (2) $\Rightarrow$ (1) are trivial. The implication (4) $\Rightarrow$ (3) is [CF96, Theorem 3.14]. It remains to prove the implication (3) $\Rightarrow$ (2). Assume that M_k admits a psd, rg extension $M_{k+\ell-2}$. Let

$$\mathcal{B} = \{1, x, \ldots, x^{\ell-1}, y, yx, \ldots, yx^{\ell-1}, \ldots, y^{k-1}, \ldots, y^{k-1}x^{\ell-1}, y^k, y^k x\} \tag{3.8}$$

be a set of monomials and V a vector subspace in $\mathbb{R}[x,y]_{k+\ell-2}$ generated by the set $\mathcal{B}$. Since $M_{k+\ell-2}$ satisfies $X^iY^j = X^i q(X)^j$ for every $i, j \in \mathbb{Z}_+$ such that $i + j\ell \leq k + \ell - 2$, it follows that the columns from $\mathcal{B}$ span $\mathcal{C}(M_{k+\ell-2})$. Let $p(x,y) = \sum_{i,j} p_{ij} x^i y^j \in V$ be a polynomial and $\widehat{p}$ a vector of its coefficients ordered in the basis $\mathcal{B}$. Before we define a univariate polynomial $g_p(x)$ corresponding to $p(x,y)$ we prepare some computations. We have that

$$\begin{aligned}
x^i(q(x))^j &= x^i \Big(\sum_{0 \leq i_1, \ldots, i_j \leq \ell} q_{i_1} q_{i_2} \cdots q_{i_j} x^{i_1 + \ldots + i_j} \Big) \\
&= \sum_{p=0}^{j\ell} \Big(\sum_{\substack{0 \leq i_1, \ldots, i_j \leq \ell, \\ i_1 + \ldots + i_j = p}} q_{i_1} q_{i_2} \cdots q_{i_j} \Big) x^{i+p} \\
&= \sum_{s=i}^{i+j\ell} q_{i,j,s} x^s,
\end{aligned} \tag{3.9}$$

for all $i, j \in \mathbb{Z}_+$, where

$$q_{i,j,s} = \begin{cases} \displaystyle\sum_{\substack{0 \leq i_1, \ldots, i_j \leq \ell, \\ i_1 + \ldots + i_j = s-i}} q_{i_1} q_{i_2} \ldots q_{i_j}, & \text{if } i \leq s \leq i + j\ell, \\ 0, & \text{otherwise.} \end{cases} \tag{3.10}$$

Later on we will need the following observation about the numbers $q_{i,j,s}$.

Claim 2. Let $i_1, i_2, j_1, j_2, s \in \mathbb{Z}_+$. Then

$$q_{i_1+i_2, j_1+j_2, s} = \sum_{t=i_1}^{s} q_{i_1, j_1, t} q_{i_2, j_2, s-t}. \tag{3.11}$$

Proof of Claim 2. We write $m_1 := i_1 + i_2$ and $m_2 := i_1 + i_2 + (j_1 + j_2)\ell$. We separate two cases: $s \in \{m_1, m_1 + 1, \ldots, m_2\}$ and $s \notin \{m_1, m_1 + 1, \ldots, m_2\}$.

Case 1: $s \in \{m_1, m_1+1, \ldots, m_2\}$. We have that

$$\begin{aligned}
q_{i_1+i_2,j_1+j_2,s} &=^1 \sum_{\substack{0\le k_1,\ldots,k_{j_1+j_2}\le \ell,\\ k_1+\ldots+k_{j_1+j_2}=s-i_1-i_2}} q_{k_1}q_{k_2}\cdots q_{k_{j_1}}q_{k_{j_1+1}}\cdots q_{k_{j_1+j_2}}\\
&=^2 \sum_{t=i_1}^{s} \sum_{\substack{0\le k_1,\ldots,k_{j_1}\le \ell,\\ k_1+\ldots+k_{j_1}=t-i_1}} \sum_{\substack{0\le k_{j_1+1},\ldots,k_{j_2}\le \ell,\\ k_{j_1+1}+\ldots+k_{j_2}=s-t-i_2}} q_{k_1}q_{k_2}\cdots q_{k_{j_1}}q_{k_{j_1+1}}\cdots q_{k_{j_1+j_2}}\\
&=^3 \sum_{t=i_1}^{s}\Big(\sum_{\substack{0\le k_1,\ldots,k_{j_1}\le \ell,\\ k_1+\ldots+k_{j_1}=t-i_1}} q_{k_1}q_{k_2}\cdots q_{k_{j_1}}\Big)\Big(\sum_{\substack{0\le k_{j_1+1},\ldots,k_{j_2}\le \ell,\\ k_{j_1+1}+\ldots+k_{j_2}=s-t-i_2}} q_{k_{j_1+1}}\cdots q_{k_{j_1+j_2}}\Big)\\
&=^4 \sum_{t=i_1}^{s} q_{i_1,j_1,t}q_{i_2,j_2,s-t},
\end{aligned}$$

where the first equality follows by definition (3.10) of $q_{i_1+i_2,j_1+j_2,s}$, in the second we decomposed the sum into three sums, in the third we used independence of the inner two sums, while the last equality follows by definitions (3.10) of $q_{i_1,j_1,t}$ and $q_{i_2,j_2,s-t}$.

Case 2: $s \notin \{m_1, m_1+1, \ldots, m_2\}$. For $s > m_2$ we have $q_{i_1+i_2,j_1+j_2,s} = 0$ and

$$\sum_{t=i_1}^{s} q_{i_1,j_1,t}q_{i_2,j_2,s-t} = \sum_{t=i_1}^{i_1+j_1\ell} q_{i_1,j_1,t}\underbrace{q_{i_2,j_2,s-t}}_{\substack{=0,\text{ since}\\ s-t>i_2+j_2\ell}} + \sum_{t=i_1+j_1\ell+1}^{s} \underbrace{q_{i_1,j_1,t}}_{\substack{=0,\text{ since}\\ t>i_1+j_1\ell}} q_{i_2,j_2,s-t} = 0,$$

which implies that (3.10) holds. Similarly, for $s < m_1$ we again have $q_{i_1+i_2,j_1+j_2,s} = 0$ and $\sum_{t=i_1}^{s} q_{i_1,j_1,t}q_{i_2,j_2,s-t} = 0$, since $q_{i_2,j_2,s-t} = 0$ for every t due to $s-t < i_2$. Also in this case (3.10) holds. ■

Now we define a univariate polynomial $g_p(x)$ corresponding to $p(x,y)$ by

$$g_p(x) := p(x, q(x)) = \sum_{i,j} p_{ij} \sum_{s=i}^{i+j\ell} q_{i,j,s}x^s =: \sum_{s=0}^{k\ell+1} g_{p,s}x^s \in \mathbb{R}[x]_{k\ell+1},$$

where we used (3.9) in the second equality. Let $\widehat{g_p}$ be its vector of coefficients in the basis

$$\mathcal{B}_1 = \{1, x, \ldots, x^{k\ell+1}\}. \tag{3.12}$$

The following claim expresses $\widehat{g_p}$ by $\widehat{p}$.

Claim 3. It holds that

$$\widehat{g_p} = P^T\widehat{p}, \tag{3.13}$$

where

$$P = \begin{pmatrix} I_\ell & 0 & \cdots & \cdots & 0 & 0 \\ P[1,0] & P[1,1] & 0 & & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ P[k-1,0] & P[k-1,1] & \cdots & \cdots & P[k-1,k-1] & 0 \\ Q[0] & Q[1] & \cdots & \cdots & Q[k-1] & Q[k] \end{pmatrix} \in \mathbb{R}^{(k\ell+2)\times(k\ell+2)}$$

and

$$P[c,d] = \begin{pmatrix} q_{0,c,d\ell} & q_{0,c,d\ell+1} & \cdots & q_{0,c,d\ell+\ell-1} \\ q_{1,c,d\ell} & q_{1,c,d\ell+1} & \cdots & q_{1,c,d\ell+\ell-1} \\ \vdots & \cdots & \cdots & \vdots \\ q_{\ell-1,c,d\ell} & q_{\ell-1,c,d\ell+1} & \cdots & q_{\ell-1,c,d\ell+\ell-1} \end{pmatrix} \in \mathbb{R}^{\ell\times\ell} \quad \text{for each } c, d,$$

$$Q[d] = \begin{pmatrix} q_{0,k,d\ell} & q_{0,k,d\ell+1} & \cdots & q_{0,k,d\ell+\ell-1} \\ q_{1,k,d\ell} & q_{1,k,d\ell+1} & \cdots & q_{1,k,d\ell+\ell-1} \end{pmatrix} \in \mathbb{R}^{2\times\ell} \quad \text{for } d = 0, \ldots, k-1,$$

$$Q[k] = \begin{pmatrix} (q_\ell)^k & 0 \\ q_{1,k,k\ell} & (q_\ell)^k \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

Proof of Claim 3. We write $\vec{v}_{x,y}$ for the vector of monomials x^iy^j from the basis $\mathcal{B}$ (see (3.8)) and $\vec{v}_x$ for the vector of monomials from the basis $\mathcal{B}_1$ (see (3.12)). We have that $p(x,y) = \big(\vec{v}_{x,y}\big)^T\widehat{p}$ and $g_p(x) = \big(\vec{v}_x\big)^T\widehat{g}_p$. By (3.9) it follows that $\vec{v}_{x,y} = P\vec{v}_x$. Hence, the definition of g_p implies that $g_p(x) = \big(P\vec{v}_x\big)^T\widehat{p} = \big(\vec{v}_x\big)^T P^T\widehat{p}$. Thus, $\widehat{g}_p = P^T\widehat{p}$, which proves Claim 3. ∎

Note that

$$q_{i,j,i+j\ell} = (q_\ell)^j \neq 0 \tag{3.14}$$

and hence we can express $x^{i+j\ell}$ from (3.9) by the formula

$$x^{i+j\ell} = \frac{1}{(q_\ell)^j}\Big(x^i(q(x))^j - \sum_{s=0}^{i+j\ell-1} q_{i,j,s}x^s\Big). \tag{3.15}$$

We define two univariate sequences

$$\gamma := \gamma^{(0,2k\ell)} = (\gamma_0, \gamma_1, \ldots, \gamma_{2k\ell}) \in \mathbb{R}^{2k\ell+1}, \quad \widetilde{\gamma} := \gamma^{(2k\ell+2)} = (\gamma, \gamma_{2k\ell+1}, \gamma_{2k\ell+2}) \in \mathbb{R}^{2k+3},$$

recursively for $t = 0, 1, \ldots, 2k\ell+2$ by the formula

$$\gamma_t = \frac{1}{(q_\ell)^{\lfloor \frac{t}{\ell} \rfloor}}\Big(\beta_{t \bmod \ell, \lfloor \frac{t}{\ell} \rfloor} - \sum_{s=0}^{t-1} q_{t \bmod \ell, \lfloor \frac{t}{\ell} \rfloor, s} \cdot \gamma_s\Big). \tag{3.16}$$

Note that $t \bmod \ell + \lfloor \frac{t}{\ell} \rfloor \leq \ell - 1 + 2k$ (here we used that $\ell \geq 3$ and thus $\lfloor \frac{t}{\ell} \rfloor \leq 2k$) and so $\beta_{t \bmod \ell, \lfloor \frac{t}{\ell} \rfloor}$ is well-defined being an element of the matrix $M_{k+\ell-2}$ (since $2(k+\ell-2) \geq \ell - 1 + 2k$ for $\ell \geq 3$).

By the following claim solving the K–TMP for β is equivalent to solving the $\mathbb{R}$–TMP for γ.

Claim 4. Let $u \in \mathbb{N}$. A sequence γ admits a u–atomic $\mathbb{R}$–representing measure if and only if β admits a u–atomic K–representing measure.

Proof of Claim 4. First we prove the implication ($\Rightarrow$). Let $x_1, \ldots, x_u$, be the atoms in the $\mathbb{R}$–representing measure for γ with the corresponding densities $\rho_1, \ldots, \rho_u$. We will prove that the atoms $(x_1, q(x_1)), \ldots, (x_u, q(x_u))$ with densities $\rho_1, \ldots, \rho_p$ are the K–representing measure for β. We use induction on the index i in $\beta_{i,j}$, where $i + j \leq 2k$. For $i < \ell$ and any j such that $i + j \leq 2k$ we have that

$$\begin{aligned}
\beta_{i,j} &=^1 (q_\ell)^j \gamma_{i+j\ell} + \sum_{s=0}^{i+j\ell-1} q_{i,j,s}\gamma_s \\
&=^2 (q_\ell)^j \Big(\sum_{p=0}^{u} \rho_p (x_p)^{i+j\ell} \Big) + \sum_{s=0}^{i+j\ell-1} q_{i,j,s} \Big(\sum_{p=0}^{u} \rho_p (x_p)^s \Big) \\
&=^3 \sum_{p=0}^{u} \rho_p (q_\ell)^j (x_p)^{i+j\ell} + \sum_{p=0}^{u} \Big(\rho_p \sum_{s=0}^{i+j\ell-1} q_{i,j,s} (x_p)^s \Big) \\
&=^4 \sum_{p=0}^{u} \Big(\rho_p \Big((q_\ell)^j (x_p)^{i+j\ell} + \sum_{s=0}^{i+j\ell-1} q_{i,j,s} (x_p)^s \Big) \Big) \\
&=^5 \sum_{p=0}^{u} \Big(\rho_p (x_p)^i (q(x_p))^j \Big),
\end{aligned}$$

where we used (3.16) with $t = i + j\ell$ in the first equality noticing that

$$i + j\ell = i + j + j(\ell - 1) \leq 2k + 2k(\ell - 1) \leq 2k\ell,$$

implying well-definedness of γ_s by s being bounded above by $2k\ell$, the definitions of ρ_p, x_p in the second equality, we interchanged the order of summation in the third and fourth equalities and in the last we used (3.15) for $x = x_p$. So the atoms $(x_1, q(x_1)), \ldots, (x_u, q(x_u))$ with densities $\rho_1, \ldots, \rho_p$ indeed represent $\beta_{i,j}$ for $i < \ell$ and any j such that $i + j \leq 2k$. We now assume that this holds for all $i = 0, \ldots, m$ and j such that $i + j \leq 2k$, where $m \geq \ell - 1$ and prove it for $i = m + 1$

and any $j \leq 2k - i$. We have that

$$\begin{aligned}
\beta_{m+1,j} &=^1 \frac{1}{q_\ell}\big(\beta_{m+1-\ell,j+1} - \sum_{s=0}^{\ell-1} q_s \beta_{m+1-\ell+s,j}\big)\\
&=^2 \frac{1}{q_\ell}\Big(\Big(\sum_{p=0}^{u} \rho_p (x_p)^{m+1-\ell}(q(x_p))^{j+1}\Big) - \sum_{s=0}^{\ell-1} q_s\Big(\sum_{p=0}^{u}\rho_p (x_p)^{m+1-\ell+s}(q(x_p))^j\Big)\Big)\\
&=^3 \frac{1}{q_\ell}\sum_{p=0}^{u}\Big(\rho_p\Big((x_p)^{m+1-\ell}(q(x_p))^{j+1} - \sum_{s=0}^{\ell-1} q_s (x_p)^{m+1-\ell+s}(q(x_p))^j\Big)\Big)\\
&=^4 \frac{1}{q_\ell}\sum_{p=0}^{u}\Big(\rho_p (x_p)^{m+1-\ell}(q(x_p))^j\Big(q(x_p) - \sum_{s=0}^{\ell-1} q_s (x_p)^s\Big)\Big)\\
&=^5 \sum_{p=0}^{u}\Big(\rho_p (x_p)^{m+1}(q(x_p))^j\Big),
\end{aligned}$$

where we used that β is rg in the first equality, the induction hypothesis in the second, in the third we interchanged the order of summation, factored out $(x_p)^{m+1-\ell}(q(x_p))^j$ in the fourth and in the last we used that $q(x) - \sum_{s=0}^{\ell-1} q_s x^s = q_\ell x^\ell$ by definition of q. This proves the implication $(\Rightarrow)$.

It remains to prove the implication $(\Leftarrow)$. Let $(x_1, q(x_1)), \ldots, (x_u, q(x_u))$ be the atoms in the K–representing measure for β with the corresponding densities $\rho_1, \ldots, \rho_u$. We will prove that the atoms $(x_1, \ldots, x_u)$ with densities $\rho_1, \ldots, \rho_p$ are the $\mathbb{R}$–representing measure for γ. We use induction on the index t in γ_t. For $t = 0$ the claim is trivial, since $\gamma_0 = \beta_{0,0} = \sum_{p=0}^{u} \rho_u$. We now assume that the claim holds for all $t-1$ with $0 \leq t-1 \leq 2k\ell - 1$ and prove it for t. We have that

$$\begin{aligned}
\gamma_t &=^1 \frac{1}{(q_\ell)^{\lfloor \frac{t}{\ell}\rfloor}}\Big(\beta_{t \bmod \ell, \lfloor \frac{t}{\ell}\rfloor} - \sum_{s=0}^{t-1} q_{t \bmod \ell, \lfloor \frac{t}{\ell}\rfloor, s}\cdot \gamma_s\Big)\\
&=^2 \frac{1}{(q_\ell)^{\lfloor \frac{t}{\ell}\rfloor}}\Big(\sum_{p=0}^{u}\rho_p (x_p)^{t \bmod \ell}(q(x_p))^{\lfloor \frac{t}{\ell}\rfloor} - \sum_{s=0}^{t-1} q_{t \bmod \ell, \lfloor \frac{t}{\ell}\rfloor, s}\cdot\Big(\sum_{p=0}^{u}\rho_p (x_p)^s\Big)\Big)\\
&=^3 \frac{1}{(q_\ell)^{\lfloor \frac{t}{\ell}\rfloor}}\sum_{p=0}^{u}\Big(\rho_p\Big((x_p)^{t \bmod \ell}(q(x_p))^{\lfloor \frac{t}{\ell}\rfloor} - \sum_{s=0}^{t-1} q_{t \bmod \ell, \lfloor \frac{t}{\ell}\rfloor, s}\cdot (x_p)^s\Big)\Big)\\
&=^4 \frac{1}{(q_\ell)^{\lfloor \frac{t}{\ell}\rfloor}}\sum_{p=0}^{u}\Big(\rho_p q_{t \bmod \ell, \lfloor \frac{t}{\ell}\rfloor, t \bmod \ell + \lfloor \frac{t}{\ell}\rfloor \ell}\cdot (x_p)^{t \bmod \ell + \lfloor \frac{t}{\ell}\rfloor \ell}\Big)\\
&=^5 \frac{1}{(q_\ell)^{\lfloor \frac{t}{\ell}\rfloor}}\sum_{p=0}^{u}\Big(\rho_p (q_\ell)^{\lfloor \frac{t}{\ell}\rfloor}(x_p)^t\Big)\\
&=^6 \sum_{p=0}^{u}\rho_p (x_p)^t,
\end{aligned}$$

where we used the definition (3.16) of γ_t in the first equality, the definitions of ρ_p, x_p and the induction hypothesis in the second equality, we interchanged the order of summation in the third equality, used (3.9) for $(i,j) = (t \bmod \ell, \lfloor \frac{t}{\ell}\rfloor)$ in the fourth equality and the observation (3.14) for

$(i,j) = (t \bmod \ell, \lfloor \frac{t}{\ell} \rfloor)$ in the fifth equality. This proves the implication $(\Leftarrow)$. ■

Let $(M_{k+\ell-2})|_{\mathcal{B}}$ be the restriction of $M_{k+\ell-2}$ to the rows and columns indexed by monomials (capitalized) from $\mathcal{B}$. The following claim gives an explicit connection between $(M_{k+\ell-2})|_{\mathcal{B}}$ and the Hankel matrix $A_{\widetilde{\gamma}}$ of the sequence $\widetilde{\gamma}$.

Claim 5. We have that

$$(M_{k+\ell-2})|_{\mathcal{B}} = P A_{\widetilde{\gamma}} P^T. \tag{3.17}$$

Proof of Claim 5. Let $p(x,y) = \sum_{i,j} p_{ij} x^i y^j \in V$ and $r(x,y) = \sum_{i,j} r_{ij} x^i y^j \in V$ be polynomials from the vector subspace V and $\widehat{p}$, $\widehat{r}$ vectors of their coefficients ordered in the basis $\mathcal{B}$ (see (3.8)). Let $\widetilde{\beta} := \beta^{(2(k+\ell-2))}$. Then we have

$$
\begin{aligned}
(\widehat{r})^T\left((M_{k+\ell-2})|_{\mathcal{B}}\right)\widehat{p} &=^1 L_{\widetilde{\beta}}(pr) = L_{\widetilde{\beta}}\Big(\sum_{i_1,i_2,j_1,j_2} p_{i_1j_1} r_{i_2j_2} x^{i_1+i_2} y^{j_1+j_2}\Big) \\
&=^2 \sum_{i_1,i_2,j_1,j_2} p_{i_1j_1} r_{i_2j_2} \beta_{i_1+i_2,j_1+j_2} \\
&=^3 \sum_{i_1,i_2,j_1,j_2} p_{i_1j_1} r_{i_2j_2} \left(\sum_{s=i_1+i_2}^{i_1+i_2+(j_1+j_2)\ell} q_{i_1+i_2,j_1+j_2,s}\gamma_s\right) \\
&=^4 \sum_{i_1,i_2,j_1,j_2} p_{i_1j_1} r_{i_2j_2} \left(\sum_{s=i_1+i_2}^{i_1+i_2+(j_1+j_2)\ell} \left(\sum_{t=i_1}^{s} q_{i_1,j_1,t} q_{i_2,j_2,s-t}\right)\gamma_s\right) \\
&=^5 \sum_{i_1,i_2,j_1,j_2} \left(\sum_{s=i_1+i_2}^{i_1+i_2+(j_1+j_2)\ell} \left(\sum_{t=i_1}^{s} p_{i_1j_1} q_{i_1,j_1,t} r_{i_2j_2} q_{i_2,j_2,s-t}\right)\gamma_s\right) \\
&=^6 L_{\widetilde{\gamma}}\left(\sum_{i_1,i_2,j_1,j_2} \left(\sum_{s=i_1+i_2}^{i_1+i_2+(j_1+j_2)\ell} \left(\sum_{t=i_1}^{s} p_{i_1j_1} q_{i_1,j_1,t} r_{i_2j_2} q_{i_2,j_2,s-t}\right) x^s\right)\right) \\
&=^7 L_{\widetilde{\gamma}}\left(\sum_{i_1,i_2,j_1,j_2} \left(\sum_{s=i_1+i_2}^{i_1+i_2+(j_1+j_2)\ell} \sum_{t=i_1}^{s} p_{i_1j_1} q_{i_1,j_1,t} x^t \cdot r_{i_2j_2} q_{i_2,j_2,s-t} x^{s-t}\right)\right) \\
&=^8 L_{\widetilde{\gamma}}\Big(\sum_{i_1,i_2,j_1,j_2} \Big(\sum_{t=i_1}^{i_1+j_1\ell} p_{i_1j_1} q_{i_1,j_1,t} x^t\Big)\Big(\sum_{u=i_2}^{i_2+j_2\ell} r_{i_2j_2} q_{i_2,j_2,u} x^u\Big) \\
&=^9 L_{\widetilde{\gamma}}\Big(\underbrace{\Big(\sum_{i_1,j_1}\sum_{t=i_1}^{i_1+j_1\ell} p_{i_1j_1} q_{i_1,j_1,t} x^t\Big)}_{g_p(x)} \underbrace{\Big(\sum_{i_2,j_2}\sum_{s=i_1+i_2}^{i_2+j_2\ell} r_{i_2j_2} q_{i_2,j_2,s} x^s\Big)}_{g_r(x)}\Big) \\
&=^{10} \widehat{g_r}^T A_{\widetilde{\gamma}} \widehat{g_p} = (P^T\widehat{r})^T A_{\widetilde{\gamma}} (P^T\widehat{p}) = \widehat{r}^T (P A_{\widetilde{\gamma}} P^T)\widehat{p},
\end{aligned}
$$

where in the first line we used the correspondence between the moment matrix and the Riesz functional $L_{\widetilde{\beta}}$, the definition $L_{\widetilde{\beta}}$ in the second, (3.16) and the fact that β is rg in the third (rg is needed if $i_1+i_2 \geq \ell$), Claim 2 in the fourth, we moved the factor $p_{i_1j_1} r_{i_2j_2}$ into the inner sum in the fifth, used the definition of $L_{\widetilde{\gamma}}$ in the sixth, split x^s into two parts and moved it into the inner sum in the seventh, decomposed a double sum into the product of two sums in the eight using that $q_{i_1,j_1,t}$ is nonzero only for $t \leq i_1 + j_1\ell$ and $q_{i_2,j_1,u}$ is nonzero only for $u \leq i_2 + j_2\ell$, decomposed a sum into the product of two sums using independence of the factors in the ninth line, in the tenth we used the correspondence between $A_{\widetilde{\gamma}}$ and the Riesz functional $L_{\widetilde{\gamma}}$, where $\widehat{g_p}$, $\widehat{g_r}$ are the vectors of coefficents of g_p and g_r in the basis $\mathcal{B}_1$ (see (3.12)) and also Claim 3. Since p and q were arbitrary from V, this proves Claim 5. ■

Note that

$$P[c,c] = \begin{pmatrix} (q_\ell)^c & 0 & \cdots & \cdots & 0 \\ q_{1,c,c\ell} & (q_\ell)^c & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ q_{\ell-1,c,c\ell} & \cdots & \cdots & q_{\ell-1,c,c\ell+\ell-1} & q_\ell^c \end{pmatrix} \in \mathbb{R}^{\ell\times\ell} \quad \text{for } c = 1,\ldots,k-1.$$

Since P is a lower triangular matrix with all nonzero diagonal entries, it is invertible. Claim 5 implies that

$$A_{\widetilde{\gamma}} = P^{-1}\left((M_{k+\ell-2})|_{\mathcal{B}}\right)(P^{-1})^T. \tag{3.18}$$

Since $(M_{k+\ell-2})|_{\mathcal{B}}$ is psd, it follows from (3.18) that $A_{\widetilde{\gamma}}$ is also psd. We separate two cases. Either $A_{\widetilde{\gamma}}$ is pd or $A_{\widetilde{\gamma}}$ is singular. In the first case in particular $A_{\widetilde{\gamma}}(k\ell) = A_\gamma$ is pd, while in the second case $A_{\widetilde{\gamma}}(k\ell)$ is psd and prg by [CF91, Theorem 2.6]. By [CF91, Theorem 3.9], γ admits a $(\operatorname{rank} A_\gamma)$–atomic $\mathbb{R}$–representing measure. Since $\operatorname{rank} M_k \leq \operatorname{rank} A_\gamma \leq k\ell + 1$, using Claim 4 the following holds:

(2') β has a s–atomic K–representing measure for some s satisfying

$$\operatorname{rank} M_k \leq s \leq A_\gamma \leq k\ell + 1. \tag{3.19}$$

To obtain (2) of Theorem 3.1 we need to decrease the upper bound in (3.19) by 1. Note that the bound $k\ell+1$ occurs only in the case A_γ is pd, which we assume in the rest of the proof. We denote by $\gamma(\mathrm{z})$ a sequence obtained from the sequence γ by replacing $\gamma_{2k\ell-1}$ with a variable z. The matrix $A_{\gamma(\mathrm{z})}$ is a partially pd matrix and by [Zal21, Lemma 2.11] there exist two choices of z, which we denote by $z^\pm$, such that $A_{\gamma(z^\pm)}$ is psd and has rank $k\ell$. Since $\operatorname{rank} A_{\gamma(z^\pm)}(k\ell-1) = \operatorname{rank} A_{\gamma(z^\pm)} = k\ell$, the sequence $\gamma(z^\pm)$ is prg and by [CF91, Theorem 3.9] it admits a $k\ell$–atomic $\mathbb{R}$–representing measure. If none of the moments $\beta_{i,j}$ of the sequence β depends on $\gamma_{2k\ell-1}$, the $\mathbb{R}$–representing measure for $\gamma(z^\pm)$ will generate a K–representing measure for β as in the proof of Claim 4. By (3.9), the only moment from β, which could depend on $\gamma_{2k\ell-1}$, is $\beta_{0,2k}$. Note that if $q_{0,2k,2k\ell-1} = 0$, then also $\beta_{0,2k}$ is independent from the value of $\gamma_{2k\ell-1}$. We have that

$$q_{0,2k,2k\ell-1} = \sum_{\substack{0\leq i_1,\ldots,i_{2k}\leq \ell,\\ i_1+\ldots+i_{2k}=2k\ell-1}} q_{i_1}q_{i_2}\cdots q_{i_{2k}} = 2k(q_\ell)^{2k-1}q_{\ell-1},$$

where in the first equality we used the definition (3.10) of $q_{0,2k,2k\ell-1}$, while in the second we used the fact that $i_1+\ldots+i_{2k} = 2k\ell-1$ could be fullfilled only if $2k-1$ indices i_j are ℓ and one is $\ell-1$. So $q_{0,2k,2k\ell-1} = 0$ iff $q_{\ell-1} = 0$. But this is true by Claim 1 and concludes the proof of Theorem 3.1. □

To prove Theorem 3.2 in the general case a little adaptation of the last part of the proof of Theorem 3.1 is needed, which we now explain.

Proof of Theorem 3.2 for any $q(x)$. The justification is the same as in the proof of the special case $q(x) = x^\ell$, $\ell \geq 3$, above, only that in last sentence one uses the definition of γ_t in the general case, i.e., (3.16) instead of (3.5). □

A corollary to Theorem 3.1 is an improvement of the bounds on the degrees of sums of squares in the Positivstellensatz [Fia11, Corollary 6.3] for the curves of the form $y = q(x)$, $q \in \mathbb{R}[x]$, $\deg q \geq 3$.

Corollary 3.4. *Let $K := \{(x,y) \in \mathbb{R}^2 \colon y = q(x)\}$, where $q \in \mathbb{R}[x]$ satisfies $\deg q \geq 3$. Let $k \geq \deg q$. If $r(x,y) \in \mathbb{R}[x,y]_{2k}$, is strictly positive on K, then r admits a decomposition*

$$r(x,y) = \sum_{i=1}^{\ell_1} f_i(x,y)^2 + (y - q(x)) \sum_{i=1}^{\ell_2} g_i(x,y)^2 - (y - q(x)) \sum_{i=1}^{\ell_3} h_i(x,y)^2,$$

where $\ell_1, \ell_2, \ell_3 \in \mathbb{Z}_+$, $f_i, g_i \in \mathbb{R}[x,y]$ and

$$\deg f_i^2 \leq 2m, \ \deg((y - q(x))g_i^2) \leq 2m, \ \deg((y - q(x))h_i^2) \leq 2m$$

with $m = k + \deg q - 1$.

Proof. By the equivalence (1) $\Leftrightarrow$ (3) of Theorem 3.1, the set K has the property $(R_{k,\deg q-2})$ in the notation of [CF08, p. 2713]. Now the result follows by [CF08, Theorem 1.5]. □

Remark 3.5. The bound on m in Theorem 3.4 from [Fia11, Corollary 5.4] is quadratic in k and $\deg q$, namely $(2k+1)\deg q$.

3.2. **Solution of the parabolic TMP.** The following is a concrete solution of the parabolic TMP, first solved in [CF04]. We give an alternative proof together with a new solvability condition, i.e., (6) below, where the variety condition is removed. See also Remark 3.3.(3) above.

Theorem 3.6 (Solution to the parabolic TMP, even case)**.** *Let $K := \{(x,y) \in \mathbb{R}^2 \colon y = x^2\}$ be the parabola and $\beta := \beta^{(2k)} = (\beta_{i,j})_{i,j\in\mathbb{Z}_+, i+j\leq 2k}$, where $k \geq 2$. The following statements are equivalent:*

(1) β has a K–representing measure.
(2) β has a $(\operatorname{rank} M_k)$–atomic K–representing measure.
(3) M_k is positive semidefinite, recursively generated, satisfies the column relation $Y = X^2$ and $\operatorname{rank} M_k \leq \operatorname{card} \mathcal{V}(\beta)$, where

$$\mathcal{V}(\beta) := \bigcap_{\substack{g\in\mathbb{R}[x,y]_{\leq k},\\ g(X,Y)=\mathbf{0}}} \mathcal{Z}(g).$$

(4) M_k satisfies $Y = X^2$ and admits a positive semidefinite, recursively generated extension M_{k+1}.
(5) M_k satisfies $Y = X^2$ and admits a positive semidefinite extension M_{k+1}.
(6) M_k is positive semidefinite, the relations $\beta_{i,j+1} = \beta_{i+2,j}$ hold for every $i, j \in \mathbb{Z}_+$ with $i + j \leq 2k - 2$ and, defining

$$\mathcal{B} = \{1, x, y, yx, \ldots, y^{k-1}, y^{k-1}x, y^k\}, \tag{3.20}$$

one of the following statements holds:

(a) $(M_k)|_{\mathcal{B}\setminus\{y^k\}}$ is positive definite.

(b) $\operatorname{rank}(M_k)|_{\mathcal{B}\setminus\{y^k\}} = \operatorname{rank} M_k$.

Proof. Let us start by proving the equivalences (1) ⇔ (2) ⇔ (6). By [Ric57], (1) is equivalent to:

(1') β has a s–atomic K–representing measure for some $s \in \mathbb{N}$.

Let

$$\gamma := \gamma^{(0,4k)} = (\gamma_0, \gamma_1, \ldots, \gamma_{4k}) \in \mathbb{R}^{4k+1} \tag{3.21}$$

be defined by (3.5) with $\ell = 2$ as in the proof of the case $q(x) = x^\ell$ of Theorem 3.1. Claim 1 in the proof of the case $q(x) = x^\ell$ of Theorem 3.1 holds with the same proof also for $\ell = 2$. Using Claim 1 and [CF91, Theorem 3.9] for γ, the equivalences $(1')$ ⇔ (2) ⇔ (6) follow by noticing that $A_\gamma = (M_k)|_{\mathcal{B}}$ and $A_\gamma(2k-1) = (M_k)|_{\mathcal{B}\setminus\{Y^k\}}$.

The implications (2) ⇒ (4) and (4) ⇒ (5) are trivial. The implication (1) ⇒ (3) follows from the neccessary conditions for the existence of a K–representing measure (the variety condition follows from [CF96, Proposition 3.1 and Corollary 3.7]).

Now we prove the implication (5) ⇒ (6). By [CF96, Theorem 3.14], it follows that M_k is rg. Defining the sequence $\widetilde{\gamma} := \gamma^{(0,4k+2)} = (\gamma_0, \gamma_1, \ldots, \gamma_{4k+1}, \gamma_{4k+2}) \in \mathbb{R}^{4k+3}$, where γ_i is defined by (3.5) with $\ell = 2$ as in the proof of the case $q(x) = x^\ell$ of Theorem 3.1, it follows by M_{k+1} being psd that in particular $(M_{k+1})|_{\mathcal{B}\cup\{y^{k+1}\}} = A_{\widetilde{\gamma}}$ is also psd. If $A_{\gamma^{(0,4k)}}$ is pd, then $(M_k)|_{\mathcal{B}\setminus\{y^k\}}$ is pd, which is (6a) of Theorem 3.6. Otherwise $A_{\gamma^{(0,4k)}}$ is singular and prg by [CF91, Theorem 2.6]. In particular, $\operatorname{rank} A_\gamma = \operatorname{rank} A_\gamma(2k-1)$, which, by noticing that $(M_k)|_{\mathcal{B}\setminus\{y^k\}} = A_\gamma(2k-1)$, implies (6b) of Theorem 3.6. This proves (5) ⇒ (6).

It remains to prove the implication (3) ⇒ (6). If $(M_k)|_{\mathcal{B}\setminus\{y^k\}}$ is pd, we are done. Otherwise $(M_k)|_{\mathcal{B}\setminus\{y^k\}}$ is not pd. We have to prove that in this case $\operatorname{rank}(M_k)|_{\mathcal{B}\setminus\{y^k\}} = \operatorname{rank} M_k$. We asssume by contradiction that $\operatorname{rank}(M_k)|_{\mathcal{B}\setminus\{y^k\}} < \operatorname{rank} M_k$. Let γ be as in (3.21). The inequality $\operatorname{rank}(M_k)|_{\mathcal{B}\setminus\{y^k\}} < \operatorname{rank} M_k$ implies that $\operatorname{rank} A_\gamma(2k-1) < \operatorname{rank} A_\gamma$. Let $r = \operatorname{rank}\gamma$. Then, by [CF91, Theorem 2.6], $\operatorname{rank} A_\gamma(2k-1) = r$ and hence, [CF91, Theorems 3.9, 3.10] imply that $\gamma^{(0,4k-2)} = (\gamma_0, \ldots, \gamma_{4k-2})$ has a unique r–atomic $\mathbb{R}$–representing measure with atoms $x_1, \ldots, x_r$. Hence, $\operatorname{rank} A_\gamma = r+1$. Note that for every $g(x,y) \in \mathbb{R}[x,y]$, which is a column relation of M_k, it follows that $g(x,x^2) \in \mathbb{R}[x]$ is a column relation of A_γ (where columns of A_γ are $1, X, \ldots, X^{2k}$). Since $(x,y) \in \mathcal{V}(\beta)$ and $(x,y') \in \mathcal{V}(\beta)$, implies that $y = y'$ (due to $y = x^2$ and $y' = x^2$), it follows that $\mathcal{V}(\beta) \subseteq \{(x_1, x_1^2), \ldots, (x_r, x_r^2)\}$. (This is true, since the atoms of a finitely atomic measure always satisfy all column relations of the moment matrix. Moreover, the sets are equal, but we do not need this in the rest of the proof.) Hence, $|\mathcal{V}(\beta)| \le r$. Since $\operatorname{rank} A_\gamma = \operatorname{rank} M_k$, this leads to a contradiction with the assumption $\operatorname{rank} M_k \le |\mathcal{V}(\beta)|$. ☐

Remark 3.7. (1) The main technique in the proof of the implication (3) ⇒ (2) of Theorem 3.6 used in [CF04] is by considering 5 different cases according to the form of the relations between the columns of $\mathcal{B}$ defined by (3.20). The most demanding cases, which both use the FET as the main tool in the construction of a flat extension M_{k+1} of M_k, are cases where there is only one relation and the column Y^k occurs nontrivially in it or if there is no relation present.

(2) Observe that (6) of Theorem 3.6 does not assume all rg relations and the variety condition $\operatorname{rank} M_k \le \operatorname{card} \mathcal{V}(\beta)$, but only M_k being psd, relations coming from $Y = X^2$ by rg and certain rank conditions.

The following example shows that the variety condition $\operatorname{rank} M_k \le \operatorname{card} \mathcal{V}(\beta)$ from (3) of Theorem 3.6 cannot be removed in contrast to the case of K being a circle [CF02, Theorem 2.1] or a union of two parallel lines [Zal22a, Theorem 3.1]. The *Mathematica* file with numerical computations can be found on the link `https://github.com/ZalarA/TMP_quadratic_curves`.

Example 3.8. Let $\beta = (\beta_{i,j})_{i,j\in\mathbb{Z}_+, i+j\leq 4}$ be a bivariate sequence of degree 4 with the moment matrix M_2 equal to

$$M_2 = \begin{array}{c} \\ 1 \\ X \\ Y \\ X^2 \\ XY \\ Y^2 \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} 1 & X & Y & X^2 & XY & Y^2 \end{array} \\ \begin{pmatrix} 3 & 0 & 2 & 2 & 0 & 2 \\ 0 & 2 & 0 & 0 & 2 & 0 \\ 2 & 0 & 2 & 2 & 0 & 2 \\ 2 & 0 & 2 & 2 & 0 & 2 \\ 0 & 2 & 0 & 0 & 2 & 0 \\ 2 & 0 & 2 & 2 & 0 & 3 \end{pmatrix} \end{array}.$$

M_2 is psd with the eigenvalues $\frac{1}{2}(9+\sqrt{65}) \approx 8.53, 4, 1, \frac{1}{2}(9-\sqrt{65}) \approx 0.47, 0, 0$, and the column relations $Y = X^2$, $XY = X$. Hence, M_2 is psd, rg and satisfies $Y = X^2$. The variety $\mathcal{V}(\beta)$ is equal to $\{(0,0), (-1,1), (1,1)\}$. So $4 = \operatorname{rank} M_2 > \operatorname{card} \mathcal{V}(\beta) = 3$ and the variety condition is not satisfied. Thus, β does not admit a representing measue supported on the parabola $y = x^2$. So M_k being psd, satisfying $Y = X^2$ and rg does not imply the variety condition and the existence of a representing measure. $\triangle$

The following example demonstrates the solution of [CF04, Example 1.6] in the univariate setting. The *Mathematica* file with numerical computations can be found on the link `https://github.com/ZalarA/TMP_quadratic_curves`.

Example 3.9. Let $\beta = (\beta_{i,j})_{i,j\in\mathbb{Z}_+, i+j\leq 6}$ be a bivariate sequence of degree 6 with the moment matrix M_3 equal to

$$M_3 = \begin{array}{c} \\ 1 \\ X \\ Y \\ X^2 \\ XY \\ Y^2 \\ X^3 \\ X^2Y \\ XY^2 \\ Y^3 \end{array}\!\!\begin{array}{c} \begin{array}{cccccccccc} 1 & X & Y & X^2 & XY & Y^2 & X^3 & X^2Y & XY^2 & Y^3 \end{array} \\ \begin{pmatrix} 1 & 0 & a & a & 0 & b & 0 & b & 0 & c \\ 0 & a & 0 & 0 & b & 0 & b & 0 & c & 0 \\ a & 0 & b & b & 0 & c & 0 & c & 0 & d \\ a & 0 & b & b & 0 & c & 0 & c & 0 & d \\ 0 & b & 0 & 0 & c & 0 & c & 0 & d & 0 \\ b & 0 & c & c & 0 & d & 0 & d & 0 & e \\ 0 & b & 0 & 0 & c & 0 & c & 0 & d & 0 \\ b & 0 & c & c & 0 & d & 0 & d & 0 & e \\ 0 & c & 0 & 0 & d & 0 & d & 0 & e & 0 \\ c & 0 & d & d & 0 & e & 0 & e & 0 & f \end{pmatrix} \end{array},$$

with the inequalities $a > 0, b > a^2, c > \frac{b^2}{a}, d > \frac{b^3-2abc+c^2}{b-a^2}$, which ensure that $(M_2)|_{\{1,X,Y,XY,Y^2,X^2Y\}}$ is psd and $(M_2)|_{\{1,X,Y,XY,Y^2\}}$ is pd. Note that M_3 satisfies the column relations $Y = X^2$, $XY = X^3$ and $Y^2 = X^2Y$. We introduce the univariate sequence

$$\gamma \in (1, 0, a, 0, b, 0, c, 0, d, 0, e, 0, f) \in \mathbb{R}^{13}$$

as in the proof of Theorem 3.6. We denote the rows and columns of A_γ by $1, X, \ldots, X^6$. Since $(M_2)|_{\{1,X,Y,XY,Y^2\}}$ is pd, it follows that $A_\gamma(4)$ is pd. For

$$e = \big(\mathbf{v}_\gamma(5,4)\big)^T (A_{(1,0,a,0,b,0,c,0,d)})^{-1} \mathbf{v}_\gamma(5,4) = \frac{-c^3+2bcd-ad^2}{b^2-ac},$$

we have that $A_\gamma(5) \succeq 0$ (e.g., using [Alb69, Theorem 1] for $A_\gamma(5)$) and $X^5 \in \operatorname{span}\{1, X, \ldots, X^4\}$ in $A_\gamma(5)$, where the vector $\mathbf{v}_\gamma(5,4) = \begin{pmatrix} 0 & c & 0 & d & 0 \end{pmatrix}^T$ is the restriction of the column X^5 to the rows indexed by $1, X, X^2, X^3, X^4$. Hence, for γ to admit a $\mathbb{R}$–representing measure, $A_\gamma \succeq 0$ and $X^i \in \operatorname{span}\{1, X, \ldots, X^{i-1}\}$ for $i = 5, 6$ [CF91, Theorem 3.9]. Since $A_\gamma(5) \succeq 0$ and the last column of $A_\gamma(5)$ is a linear combination of the others, it only needs to hold by [Alb69, Theorem 1], that

$$\mathbf{v}_\gamma(6,5) \in \mathcal{C}(A_\gamma(5)) \quad \text{and} \quad \begin{aligned} f &= \big(\mathbf{v}_\gamma(6,5)\big)^T (A_\gamma(5))^\dagger \mathbf{v}_\gamma(6,5) \\ &= \frac{-bc^4 - b^2c^2d - 2ac^3d - b^3d - b^3d^2 + 4abcd^2 - a^2d^2}{(b^2-ac)^2}, \end{aligned}$$

where $\mathbf{v}_\gamma(6,5)$ denotes the restriction of X^6 to the rows indexed by $1, \ldots, X^5$ in A_γ and $(A_\gamma(5))^\dagger$ denotes the Moore-Penrose inverse of $A_\gamma(5)$. Using *Mathematica* we check that the equality $A_\gamma(5)(A_\gamma(5))^\dagger \mathbf{v}_\gamma(6,5) = \mathbf{v}_\gamma(6,5)$ holds, which implies that $\mathbf{v}_\gamma(6,5) \in \mathcal{C}(A_\gamma(5))$ is true. By [CF91, Theorem 3.10], in this case the $\mathbb{R}$–representing measure is unique, 5–atomic and consists of the roots of the polynomial

$$\begin{aligned} p(x) &= \begin{pmatrix} 1 & x & x^2 & x^3 & x^4 & x^5 \end{pmatrix} (A_{(1,0,a,0,b,0,c,0,d)})^{-1} \mathbf{v}_\gamma(5,4) \\ &= x\Big(x^4 + \frac{ad-bc}{b^2-ac}x^2 + \frac{c^2-bd}{b^2-ac}\Big). \end{aligned}$$

So $p(x)$ has roots $0, x_1, -x_1, x_2, -x_2$ and the atoms for the K–representing measure for β are $(0,0)$, (x_1, x_1^2), $(-x_1, x_1^2)$, (x_2, x_2^2), $(-x_2, x_2^2)$. $\triangle$

The following theorem is a concrete solution of the parabolic TMP of odd degree, which can be solved using the same technique as odd cases of the TMP on $y = x^\ell$, $\ell \geq 3$, but for $\ell = 2$ we get explicit conditions for the existence of the solution, similarly as in the even degree case.

Theorem 3.10 (Solution to the parabolic TMP, odd case). *Let $K := \{(x,y) \in \mathbb{R}^2 \colon y = x^2\}$ be the parabola and $\beta := \beta^{(2k-1)} = (\beta_{i,j})_{i,j \in \mathbb{Z}_+, i+j \leq 2k-1}$, where $k \geq 2$. Let $\gamma := (\gamma_0, \gamma_1, \ldots, \gamma_{4k-2})$ be a sequence, defined by $\gamma_t := \beta_{t \bmod 2, \lfloor \frac{t}{2} \rfloor}$ for $t = 0, 1, \ldots, 4k-2$. The following statements are equivalent:*

(1) β has a K–representing measure.
(2) β has a $(\operatorname{rank}\gamma)$–atomic K–representing measure.
(3) β can be extended to a sequence $\beta^{(2k)}$ such that M_k is psd, rg, has a column relation $Y = X^2$ and satisfies $\operatorname{rank} M_k \leq \operatorname{card} \mathcal{V}(\beta^{(2k)})$, where

$$\mathcal{V}(\beta^{(2k)}) := \bigcap_{\substack{g \in \mathbb{R}[x,y]_{\leq k}, \\ g(X,Y) = \mathbf{0} \text{ in } M_k}} \mathcal{Z}(g).$$

(4) β can be extended to a sequence $\beta^{(2k+2)}$ such that M_{k+1} is psd and has a column relation $Y = X^2$.
(5) The relations $\beta_{i,j+1} = \beta_{i+2,j}$ hold for every $i, j \in \mathbb{Z}_+$ with $i + j \leq 2k-1$, $A_\gamma \succeq 0$ and the sequence γ is positively recursively generated.
(6) The relations $\beta_{i,j+1} = \beta_{i+2,j}$ hold for every $i, j \in \mathbb{Z}_+$ with $i + j \leq 2k-1$ and defining $\beta_{i,2k-i} = \beta_{i \bmod 2, 2k-i+\lfloor \frac{i}{2} \rfloor}$ for $2 \leq i \leq 2k$, the moment matrix $(M_k)|_{\mathcal{B} \setminus \{y^k\}}$, where $\mathcal{B} = \{1, x, y, yx, \ldots, y^{k-1}, y^{k-1}x, y^k\}$, is positive semidefinite and

$$\big(\beta_{0,k} \quad \beta_{1,k} \quad \beta_{0,k+1} \quad \beta_{1,k+1} \quad \cdots \quad \beta_{0,2k-1}\big)^T \in \mathcal{C}((M_k)|_{\mathcal{B} \setminus \{xy^{k-1}, y^k\}, \mathcal{B} \setminus \{y^k\}}). \tag{3.22}$$

Proof. The equivalences (1) ⇔ (3) ⇔ (4) follow by Theorem 3.6. By [Ric57], (1) is equivalent to:

(1') β has a s–atomic K–representing measure for some $s \in \mathbb{N}$.

Claim 1 of the case $q(x) = x^\ell$ of Theorem 3.1 holds with the same proof also for $\ell = 2$ and odd degree (i.e., $i+j \leq 2k-1$). Together with [CF91, Theorem 3.9], the equivalences $(1')$ ⇔ (2) ⇔ (5) follow. Note that $(M_k)|_{\mathcal{B}\setminus\{y^k\}} = A_\gamma$. By [BW11, Theorem 2.7.5], γ is prg if and only if (3.22) holds. This establishes the equivalence (5) ⇔ (6). □

Remark 3.11. Note that $\operatorname{rank}\gamma$ in Theorem 3.10 is at most $2k$ and it is $2k$ iff A_γ is positive definite.

3.3. **A solution to the TMP based on a feasibility of a linear matrix inequality.** In this subsection we give another alternative solution to the TMP on curves $y = q(x)$, where $q(x) \in \mathbb{R}[x]$ and $\deg q \geq 3$, which is based on a feasibility of a linear matrix inequality associated to the univariate sequence γ, obtained from the original sequence β as in the proofs of the results of previous subsections. The feasibility question appears as a result of the fact that γ is not fully determined by β, but β admits a K–representing measure if and only if γ can be completed to a sequence admitting a $\mathbb{R}$–representing measure.

Theorem 3.12. *Let* $K := \{(x, y) \in \mathbb{R}^2 \colon y = q(x)\}$*, where* $q(x) = \sum_{i=0}^{\ell} q_i x^i \in \mathbb{R}[x]$*,* $\ell \geq 3$*,* $q_\ell \neq 0$*, and*

$$\beta := \beta^{(d)} = (\beta_{i,j})_{i,j\in\mathbb{Z}_+, i+j\leq d},$$

where $\lceil \frac{d}{2} \rceil \geq \deg q$*. Define*

$$N := \Big\{t \in \mathbb{Z}_+ \colon t \bmod \ell + \Big\lfloor \frac{t}{\ell} \Big\rfloor \leq d\Big\}, \quad V := \{0, 1, \ldots, d\ell\} \setminus N = \{i_1, \ldots, i_{|V|}\},$$

and for $i, j, s \in \mathbb{Z}_+$*, such that* $i + j \leq d$*, numbers*

$$q_{i,j,s} := \begin{cases} \displaystyle\sum_{\substack{0\leq i_1,\ldots,i_j\leq \ell,\\ i_1+\ldots+i_j=s-i}} q_{i_1} q_{i_2} \cdots q_{i_j}, & \text{if } i \leq s \leq i + j\ell, \\ 0, & \text{otherwise}. \end{cases}$$

Let $\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_{|V|}}, \gamma_{d\ell+1}, \gamma_{d\ell+2}$ *be variables,*

$$\gamma_t = \frac{1}{(q_\ell)^{\lfloor \frac{t}{\ell} \rfloor}} \Big(\beta_{t \bmod \ell, \lfloor \frac{t}{\ell} \rfloor} - \sum_{s=0}^{t-1} q_{t \bmod \ell, \lfloor \frac{t}{\ell} \rfloor, s} \cdot \gamma_s\Big) \quad \text{for every } t \in N, \tag{3.23}$$

and

$$F : \begin{cases} \mathbb{R}^{|V|+2} \to \mathbb{R}^{d\ell+3}, & \text{if } d\ell \text{ is even}, \\ \mathbb{R}^{|V|+1} \to \mathbb{R}^{d\ell+2}, & \text{if } d\ell \text{ is odd}, \end{cases}$$

a function, defined by

$$\begin{cases} F(\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_{|V|}}, \gamma_{d\ell+1}, \gamma_{d\ell+2}) = (\gamma_0, \gamma_1, \ldots, \gamma_{d\ell+1}, \gamma_{d\ell+2}), & \text{if } d\ell \text{ is even}, \\ F(\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_{|V|}}, \gamma_{d\ell+1}) = (\gamma_0, \gamma_1, \ldots, \gamma_{d\ell+1}), & \text{if } d\ell \text{ is odd}. \end{cases}$$

Then the following statements are equivalent:

(1) β *has a* K*–representing measure.*

(2) $\beta_{i,j} = \sum_{p=0}^{\ell-1} q_p \beta_{i+p,j-1}$ *for every* $i, j \in \mathbb{Z}_+$*, such that* $i + j \leq d - \ell - 2$ *and there exist* $\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_{|V|}}, \gamma_{d\ell+1}, \gamma_{d\ell+2} \in \mathbb{R}$ *such that*

$$\begin{cases} A_{F(\gamma_{i_1},\gamma_{i_2},\ldots,\gamma_{i_{|V|}},\gamma_{d\ell+1},\gamma_{d\ell+2})} \succeq 0, & \text{if } d\ell \text{ is even}, \\ A_{F(\gamma_{i_1},\gamma_{i_2},\ldots,\gamma_{i_{|V|}},\gamma_{d\ell+1})} \succeq 0, & \text{if } d\ell \text{ is odd}. \end{cases}$$

Proof. Observing the proof of Theorem 3.1 for a general $q(x)$ one can notice that

$$F(\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_{|V|}}, \gamma_{d\ell+1}, \gamma_{d\ell+2}) \tag{3.24}$$

corresponds to the sequence $\widetilde{\gamma}$. The original sequence β determines only γ_t for $t \in N$ by (3.23), while for $t \in V$, γ_t are variables. By the proof of Theorem 3.1, β will have a K–representing measure iff it satisfies the rg relations coming from the column relation $Y = q(X)$ and there exists $\widetilde{\gamma}$ such that $A_{\widetilde{\gamma}} \succeq 0$. This proves Theorem 3.12 for even d.

Observing the proof of Theorem 3.2 in case d is odd one can notice that only $\gamma^{(0,d\ell)} = (\gamma_0, \ldots, \gamma_{d\ell})$ needs to have a $\mathbb{R}$–representing measure to obtain a K–representing measure for β. In case $d\ell$ is even, this is by [CF91, Theorem 3.9] equivalent to $A_{\gamma^{(0,d\ell+2)}} \succeq 0$, where

$$\gamma^{(0,d\ell+2)} = (\gamma_0, \ldots, \gamma_{d\ell}, \gamma_{d\ell+1}, \gamma_{d\ell+2})$$

for some $\gamma_{d\ell+1}, \gamma_{d\ell+2}$. Since $\gamma^{(0,d\ell+2)}$ corresponds to the sequence (3.24), this proves Theorem 3.12 for even $d\ell$ with d being odd. If $d\ell$ is odd, then by [CF91, Theorem 3.1] it suffices that there is $\gamma_{d\ell+1}$ such that $A_{\gamma^{(0,d\ell+1)}} \succeq 0$, where $\gamma^{(0,d\ell+1)} = (\gamma_0, \ldots, \gamma_{d\ell}, \gamma_{d\ell+1})$, and this proves Theorem 3.12 for odd $d\ell$. $\square$

We will present the statement of Theorem 3.12 on a few examples. The following example is for the case $\deg q = 3$ and a sequence β of even degree.

Example 3.13. Let $\beta = (\beta_{i,j})_{i,j \in \mathbb{Z}_+, i+j \leq 2k}$ be a bivariate sequence of degree $2k$, $k \geq 3$, and $K := \{(x, y) \in \mathbb{R}^2 \colon y = x^3\}$. For the existence of a K–representing measure β must satisfy the relations $\beta_{i,j} = \beta_{i+3,j-1}$ for every $i, j \in \mathbb{Z}_+$ such that $i + j + 2 \leq 2k$. In the notation of Theorem 3.12, we have

$$N := \left\{t \in \mathbb{Z}_+ \colon t \bmod 3 + \left\lfloor \frac{t}{3} \right\rfloor \leq 2k \right\} = \{t \in \mathbb{Z}_+ \colon t \leq 6k,\ t \neq 6k-1\},$$

$$V := \{6k-1\} \quad \text{and} \quad q_{i,j,s} := \left\{ \begin{array}{ll} 1, & \text{if } s = i+3j, \\ 0, & \text{otherwise,} \end{array} \right. \quad \text{for } i,j,s \in \mathbb{Z}_+, \text{ such that } i+j \leq 2k.$$

The formula (3.23) is equal to

$$\gamma_t = \beta_{t \bmod t, \lfloor \frac{t}{3} \rfloor} \quad \text{for every } t \in \mathbb{N},$$

the function $F : \mathbb{R}^3 \to \mathbb{R}^{6k+3}$ is defined by

$$F(\gamma_{\mathbf{6k-1}}, \gamma_{\mathbf{6k+1}}, \gamma_{\mathbf{6k+2}}) := (\gamma_0, \gamma_1, \ldots, \gamma_{6k-2}, \gamma_{\mathbf{6k-1}}, \gamma_{6k}, \gamma_{\mathbf{6k+1}}, \gamma_{\mathbf{6k+2}}).$$

The matrix $A_{F(\gamma_{\mathbf{6k-1}}, \gamma_{\mathbf{6k+1}}, \gamma_{\mathbf{6k+2}})}$ is equal to

$$\left(\begin{array}{cccccccc|c}
\gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \cdots & & \gamma_{3k} & \gamma_{3k+1} \\
\gamma_1 & \gamma_2 & \gamma_3 & \iddots & & & & & \vdots \\
\gamma_2 & \gamma_3 & \iddots & & & & & & \gamma_{6k-2} \\
\gamma_3 & \iddots & & & & & & \gamma_{6k-2} & \gamma_{\mathbf{6k-1}} \\
\vdots & & & & & & \gamma_{6k-2} & \gamma_{\mathbf{6k-1}} & \gamma_{6k} \\
\gamma_{3k} & \cdots & & \cdots & \cdots & \gamma_{6k-2} & \gamma_{\mathbf{6k-1}} & \gamma_{6k} & \gamma_{\mathbf{6k+1}} \\
\hline
\gamma_{3k+1} & \cdots & \cdots & \cdots & \gamma_{6k-2} & \gamma_{\mathbf{6k-1}} & \gamma_{6k} & \gamma_{\mathbf{6k+1}} & \gamma_{\mathbf{6k+2}}
\end{array}\right),$$

The question of feasibility of $A_{F(\gamma_{\mathbf{6k-1}}, \gamma_{\mathbf{6k+1}}, \gamma_{\mathbf{6k+2}})} \succeq 0$ can be answered analytically, since the structure of the missing entries is simple enough. Actually it is even easier to work with $A_{(\gamma_0, \ldots, \gamma_{\mathbf{6k-1}}, \gamma_{6k})}$

and answer the feasibility question together with the condition from the solution of [CF91, Theorem 3.9] (see [Zal21, Theorem 3.1]). $\triangle$

Remark 3.14. If $\deg q = 3$ in Theorem 3.1, then a polynomial is of the form $y = q_3x^3 + q_2x^2 + q_1x + q_0 \in \mathbb{R}[x]$, where $q_3 \neq 0$, and using affine linear transformations (ALTs) it can be transformed to $y = x^3$. Indeed, by first applying an ALT as at the beginning of the proof of Claim 1 of Theorem 3.1, we can assume that $q_2 = 0$, i.e., the polynomial becomes $y = q_3x^3 + q_1x + q_0$. Now we apply an ALT $(x, y) \mapsto \big(x, y - q_1x - q_0\big)$, followed by $(x, y) \mapsto \big(\sqrt[3]{q_3}x, y\big)$ and get a polynomial $y = x^3$.

The following example demonstrates the statement of Theorem 3.12 for the case $\deg q = 3$ and a sequence β of odd degree.

Example 3.15. Let $\beta = (\beta_{i,j})_{i,j\in\mathbb{Z}_+, i+j\le 2k-1}$ be a bivariate sequence of degree $2k-1$, $k \ge 3$, and $K := \{(x, y) \in \mathbb{R}^2 \colon y = x^3\}$. For the existence of a K–representing measure β must satisfy the relations $\beta_{i,j} = \beta_{i+3,j-1}$ for every $i, j \in \mathbb{Z}_+$ such that $i+j+2 \le 2k-1$. In the notation of Theorem 3.12, we have

$$N := \left\{t \in \mathbb{Z}_+ \colon t \bmod 3 + \left\lfloor \frac{t}{3} \right\rfloor \le 2k-1\right\} = \{t \in \mathbb{Z}_+ \colon t \le 6k-3 \text{ and } t \neq 6k-4\},$$

$$V := \{6k-4\}, \quad q_{i,j,s} := \begin{cases} 1, & \text{if } s = i+3j, \\ 0, & \text{otherwise.} \end{cases} \quad \text{for } \ i, j, s \in \mathbb{Z}_+, \text{ such that } i+j \le 2k-1.$$

The formula (3.23) is equal to

$$\gamma_t = \beta_{t \bmod t, \lfloor \frac{t}{3} \rfloor} \quad \text{for every } t \in \mathbb{N},$$

the function $F : \mathbb{R}^2 \to \mathbb{R}^{6k-1}$ is defined by

$$F(\gamma_{\mathbf{6k-4}}, \gamma_{\mathbf{6k-2}}) := (\gamma_0, \gamma_1, \ldots, \gamma_{6k-5}, \gamma_{\mathbf{6k-4}}, \gamma_{6k-3}, \gamma_{\mathbf{6k-2}}).$$

Since $6k-3 = (2k-1)\cdot 3$ is odd, only feasibility of the inequality $A_{F(\gamma_{\mathbf{6k-4}}, \gamma_{\mathbf{6k-2}})} \succeq 0$ is important for the existence of the representing measure for β, where $A_{F(\gamma_{\mathbf{6k-4}}, \gamma_{\mathbf{6k-2}})}$ is equal to

$$\begin{pmatrix}
\gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \cdots & \gamma_{3k-1} \\
\gamma_1 & \gamma_2 & \gamma_3 & \iddots & & & \vdots \\
\gamma_2 & \gamma_3 & \iddots & & & & \gamma_{6k-5} \\
\gamma_3 & \iddots & & & & \gamma_{6k-5} & \gamma_{\mathbf{6k-4}} \\
\vdots & & & & \gamma_{6k-5} & \gamma_{\mathbf{6k-4}} & \gamma_{6k-3} \\
\gamma_{3k-1} & \cdots & \cdots & \gamma_{6k-5} & \gamma_{\mathbf{6k-4}} & \gamma_{6k-3} & \gamma_{\mathbf{6k-2}}
\end{pmatrix}.$$

This feasibility question can be answered analytically, since the structure of missing entries is simple enough. See Theorem 3.18 below. $\triangle$

The following example demonstrates the statement of Theorem 3.12 for the case $y = x^4$ and a sequence β of even degree.

Example 3.16. Let $\beta = (\beta_{i,j})_{i,j\in\mathbb{Z}_+, i+j\le 2k}$ be a bivariate sequence of degree $2k$, $k \ge 4$, and $K := \{(x, y) \in \mathbb{R}^2 \colon y = x^4\}$. For the existence of a K–representing measure β must satisfy the relations $\beta_{i,j} = \beta_{i+4,j-1}$ for every $i, j \in \mathbb{Z}_+$ such that $i+j+3 \le 2k$. In the notation of Theorem

3.12, we have

$$\begin{aligned}
N &:= \left\{ t \in \mathbb{Z}_+ \colon t \bmod 4 + \left\lfloor \frac{t}{4} \right\rfloor \leq 2k \right\} \\
&= \left\{ t \in \mathbb{Z}_+ \colon t \leq 8k,\ t \notin \{8k-5, 8k-2, 8k-1\} \right\}, \\
V &:= \{8k-5, 8k-2, 8k-1\}, \\
q_{i,j,s} &:= \left\{ \begin{array}{ll} 1, & \text{if } s = i + 4j, \\ 0, & \text{otherwise.} \end{array} \right. \quad \text{for } \ i,j,s \in \mathbb{Z}_+, \text{ such that } i + j \leq 2k.
\end{aligned}$$

The formula (3.23) is equal to

$$\gamma_t = \beta_{t \bmod t, \left\lfloor \frac{t}{4} \right\rfloor} \quad \text{for every } t \in \mathbb{N},$$

the function $F : \mathbb{R}^5 \to \mathbb{R}^{8k+3}$ is defined by

$$\begin{aligned}
F(\gamma_{\mathbf{8k-5}}, \gamma_{\mathbf{8k-2}}, \gamma_{\mathbf{8k-1}}, \gamma_{\mathbf{8k+1}}, \gamma_{\mathbf{8k+2}}) := (&\gamma_0, \gamma_1, \ldots, \gamma_{8k-6}, \gamma_{\mathbf{8k-5}}, \gamma_{8k-4}, \gamma_{8k-3}, \\
&\gamma_{\mathbf{8k-2}}, \gamma_{\mathbf{8k-1}}, \gamma_{8k}, \gamma_{\mathbf{8k+1}}, \gamma_{\mathbf{8k+2}}).
\end{aligned}$$

The matrix $A_{F(\gamma_{\mathbf{8k-5}}, \gamma_{\mathbf{8k-2}}, \gamma_{\mathbf{8k-1}}, \gamma_{\mathbf{8k+1}}, \gamma_{\mathbf{8k+2}})}$ is equal to

$$\left(\begin{array}{ccccccccc|c}
\gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 & \cdots & & & \cdots & \gamma_{4k} & \gamma_{4k+1} \\
\gamma_1 & \gamma_2 & \gamma_3 & \iddots & & & & & & \vdots \\
\gamma_2 & \gamma_3 & \iddots & & & & & \gamma_{8k-6} & \gamma_{\mathbf{8k-5}} & \gamma_{8k-4} \\
\gamma_3 & \iddots & & & & & \gamma_{8k-6} & \gamma_{\mathbf{8k-5}} & \gamma_{8k-4} & \gamma_{8k-3} \\
\vdots & & & & & \gamma_{8k-6} & \gamma_{\mathbf{8k-5}} & \gamma_{8k-4} & \gamma_{8k-3} & \gamma_{\mathbf{8k-2}} \\
 & & & & \gamma_{8k-6} & \gamma_{\mathbf{8k-5}} & \gamma_{8k-4} & \gamma_{8k-3} & \gamma_{\mathbf{8k-2}} & \gamma_{\mathbf{8k-1}} \\
\vdots & & & \gamma_{8k-6} & \gamma_{\mathbf{8k-5}} & \gamma_{8k-4} & \gamma_{8k-3} & \gamma_{\mathbf{8k-2}} & \gamma_{\mathbf{8k-1}} & \gamma_{8k} \\
\gamma_{4k} & & \gamma_{8k-6} & \gamma_{\mathbf{8k-5}} & \gamma_{8k-4} & \gamma_{8k-3} & \gamma_{\mathbf{8k-2}} & \gamma_{\mathbf{8k-1}} & \gamma_{8k} & \gamma_{\mathbf{8k+1}} \\
\hline
\gamma_{4k+1} & \cdots & \gamma_{\mathbf{8k-5}} & \gamma_{8k-4} & \gamma_{8k-3} & \gamma_{\mathbf{8k-2}} & \gamma_{\mathbf{8k-1}} & \gamma_{8k} & \gamma_{\mathbf{8k+1}} & \gamma_{\mathbf{8k+2}}
\end{array}\right)$$

In contrast to the situation $y = x^3$ from Example 3.13, the structure of missing entries here is too complicated for the analytic approach and we believe the feasibility question can only be answered numerically using linear matrix inequality solvers. $\triangle$

The following example demonstrates the statement of Theorem 3.12 for the case $y = x^4$ and a sequence β of odd degree.

Example 3.17. Let $\beta = (\beta_{i,j})_{i,j \in \mathbb{Z}_+, i+j \leq 2k-1}$ be a bivariate sequence of degree $2k-1$, $k \geq 4$, and $K := \{(x,y) \in \mathbb{R}^2 \colon y = x^4\}$. For the existence of a K–representing measure β must satisfy the relations $\beta_{i,j} = \beta_{i+4,j-1}$ for every $i,j \in \mathbb{Z}_+$ such that $i + j + 3 \leq 2k-1$. In this case

$$\begin{aligned}
N &:= \left\{ t \in \mathbb{Z}_+ \colon t \bmod 4 + \left\lfloor \frac{t}{4} \right\rfloor \leq 2k-1 \right\} \\
&= \left\{ t \in \mathbb{Z}_+ \colon t \leq 8k \text{ and } t \notin \{8k-9, 8k-6, 8k-5\} \right\}, \\
V &:= \{8k-9, 8k-6, 8k-5\}, \\
q_{i,j,s} &:= \left\{ \begin{array}{ll} 1, & \text{if } s = i + 4j, \\ 0, & \text{otherwise.} \end{array} \right. \quad \text{for } \ i,j,s \in \mathbb{Z}_+, \text{ such that } i + j \leq 2k-1.
\end{aligned}$$

The formula (3.23) is equal to

$$\gamma_t = \beta_{t \bmod t, \lfloor \frac{t}{4} \rfloor} \quad \text{for every } t \in \mathbb{N},$$

the function $F : \mathbb{R}^5 \to \mathbb{R}^{8k-1}$ is defined by

$$\begin{aligned} F(\gamma_{\mathbf{8k-9}}, \gamma_{\mathbf{8k-6}}, \gamma_{\mathbf{8k-5}}, \gamma_{\mathbf{8k-3}}, \gamma_{\mathbf{8k-2}}) := \big(&\gamma_0, \gamma_1, \ldots, \gamma_{8k-10}, \gamma_{\mathbf{8k-9}}, \gamma_{8k-8}, \gamma_{8k-7}, \\ &\gamma_{\mathbf{8k-6}}, \gamma_{\mathbf{8k-5}}, \gamma_{8k-4}, \gamma_{\mathbf{8k-3}}, \gamma_{\mathbf{8k-2}}\big), \end{aligned}$$

and the matrix $A_{F(\gamma_{\mathbf{8k-9}}, \gamma_{\mathbf{8k-6}}, \gamma_{\mathbf{8k-5}}, \gamma_{\mathbf{8k-3}}, \gamma_{\mathbf{8k-2}})}$ is equal to

$$\left(\begin{array}{ccccccccc|c}
\gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 & \cdots & & & \cdots & \gamma_{4k-2} & \gamma_{4k-1} \\
\gamma_1 & \gamma_2 & \gamma_3 & \iddots & & & & & & \vdots \\
\gamma_2 & \gamma_3 & \iddots & & & & & \gamma_{8k-10} & \gamma_{\mathbf{8k-9}} & \gamma_{8k-8} \\
\gamma_3 & \iddots & & & & & \gamma_{8k-10} & \gamma_{\mathbf{8k-9}} & \gamma_{8k-8} & \gamma_{8k-7} \\
\vdots & & & & & \gamma_{8k-10} & \gamma_{\mathbf{8k-9}} & \gamma_{8k-8} & \gamma_{8k-7} & \gamma_{\mathbf{8k-6}} \\
 & & & & \gamma_{8k-10} & \gamma_{\mathbf{8k-9}} & \gamma_{8k-8} & \gamma_{8k-7} & \gamma_{\mathbf{8k-6}} & \gamma_{\mathbf{8k-5}} \\
\vdots & & & \gamma_{8k-10} & \gamma_{\mathbf{8k-9}} & \gamma_{8k-8} & \gamma_{8k-7} & \gamma_{\mathbf{8k-6}} & \gamma_{\mathbf{8k-5}} & \gamma_{8k-4} \\
\gamma_{4k-2} & & \gamma_{8k-10} & \gamma_{\mathbf{8k-9}} & \gamma_{8k-8} & \gamma_{8k-7} & \gamma_{\mathbf{8k-6}} & \gamma_{\mathbf{8k-5}} & \gamma_{8k-4} & \gamma_{\mathbf{8k-3}} \\
\hline
\gamma_{4k-1} & \cdots & \gamma_{\mathbf{8k-9}} & \gamma_{8k-8} & \gamma_{8k-7} & \gamma_{\mathbf{8k-6}} & \gamma_{\mathbf{8k-5}} & \gamma_{8k-4} & \gamma_{\mathbf{8k-3}} & \gamma_{\mathbf{8k-2}}
\end{array}\right)$$

Since $8k-4 = (2k-1) \cdot 4$ is even, the problem has the same structure as in the even degree case (see Example 3.16). $\triangle$

3.4. **A solution to the odd degree TMP on $y = x^3$.** The following theorem is a concrete solution to the TMP of odd degree on the curve $y = x^3$, which can be solved using the same technique as odd cases of the TMP on $y = x^\ell$, $\ell \geq 3$. However, for $\ell = 3$ we get explicit conditions for the existence of the solution, similarly as in the even degree case [Zal21, Theorem 3.1].

Theorem 3.18 (Solution to the TMP on $y = x^3$, odd case)**.** *Let $K := \{(x, y) \in \mathbb{R}^2 \colon y = x^3\}$ and $\beta := \beta^{(2k-1)} = (\beta_{i,j})_{i,j \in \mathbb{Z}_+, i+j \leq 2k-1}$, where $k \geq 3$. Let $\gamma(z) := (\gamma_0, \gamma_1, \ldots, \gamma_{6k-5}, z, \gamma_{6k-3})$ be a sequence, defined by $\gamma_t := \beta_{t \bmod 3, \lfloor \frac{t}{3} \rfloor}$ for $t = 0, 1, \ldots, 6k-5, 6k-3$, and z is a variable. The following statements are equivalent:*

(1) β has a K–representing measure.
(2) β has a $(\operatorname{rank} M_{k-1})$–atomic or $(\operatorname{rank} M_{k-1} + 1)$–atomic K–representing measure.
(3) The relations $\beta_{i,j+1} = \beta_{i+3,j}$ hold for every $i, j \in \mathbb{Z}_+$ with $i + j \leq 2k-4$ and denoting $\mathcal{B} = \{1, x, x^2, y, yx, yx^2, \ldots, y^{k-1}\}$, one of the following holds:
   (a) $(M_{k-1})|_{\mathcal{B}} \succ 0$.
   (b) $(M_{k-1})|_{\mathcal{B}} \not\succ 0$, $(M_{k-1})|_{\mathcal{B}} \succeq 0$, denoting $\gamma := (\gamma_0, \gamma_1, \ldots, \gamma_{6k-6})$, $r := \operatorname{rank} \gamma$ and

$$\begin{pmatrix} \varphi_0 & \cdots & \varphi_{r-1} \end{pmatrix} := A_\gamma(r-1)^{-1} \begin{pmatrix} \gamma_r & \cdots & \gamma_{2r-1} \end{pmatrix}^T, \tag{3.25}$$

it holds that

$$\gamma_{6k-u} = \sum_{i=0}^{r-1} \varphi_i \gamma_{6k-u-r+i} \quad \textit{for } u = 3, 5, \tag{3.26}$$

where γ_{6k-4} is defined by (3.26) for $u = 4$.

Moreover, if a K–representing measure exists, then there does not exist a $(\operatorname{rank} M_{k-1})$*–atomic one if and only if* $(M_{k-1})|_{\mathcal{B}} \succ 0$ *and* γ_{6k-3} *does not satisfy* (3.26) *for* $u = 3$*, where* γ_{6k-4} *is obtained by* (3.26) *for* $u = 4$ *and one uses* (3.25) *with* $r = 3k - 2$*.*

Proof. By Theorem 3.12, (1) is equivalent to the validity of the relations $\beta_{i,j+1} = \beta_{i+3,j}$ for every $i, j \in \mathbb{Z}_+$ with $i + j \leq 2k - 4$ and feasibility of $A_{F(\boldsymbol{\gamma_{6k-4}},\boldsymbol{\gamma_{6k-2}})} \succeq 0$, where the linear matrix function $A_{F(\boldsymbol{\gamma_{6k-4}},\boldsymbol{\gamma_{6k-2}})}$ is as in Example 3.15. The latter is further equivalent to the existence of γ_{6k-4} and γ_{6k-2} such that $F(\gamma_{6k-4}, \gamma_{6k-2})$ has a $\mathbb{R}$–representing measure. Here we note that if $A_{F(\gamma_{6k-4},\gamma_{6k-2})} \succeq 0$ is such that $\operatorname{rank} F(\gamma_{6k-4}, \gamma_{6k-2}) < A_{F(\gamma_{6k-4},\gamma_{6k-2})}$, then $\operatorname{rank} F(\gamma_{6k-4}, \gamma_{6k-2}) = A_{F(\gamma_{6k-4},\gamma_{6k-2})} - 1$ by [CF91, Corollary 2.5]. Since γ_{6k-2} occurs only in the bottom right corner of $A_{F(\gamma_{6k-4},\gamma_{6k-2})}$, we can replace it with $\widetilde{\gamma}_{6k-2}$ such that $A_{F(\gamma_{6k-4},\widetilde{\gamma}_{6k-2})} \succeq 0$ and $\operatorname{rank} F(\gamma_{6k-4}, \widetilde{\gamma}_{6k-2}) = A_{F(\gamma_{6k-4},\widetilde{\gamma}_{6k-2})}$, which by [CF91, Theorem 3.9] indeed implies the existence of a $\mathbb{R}$–representing measure. We have that $(M_{k-1})|_{\mathcal{B}} = A_\gamma$. If $(M_{k-1})|_{\mathcal{B}} \succ 0$, there exists γ_{6k-4} such that $A_{(\gamma_0,\gamma_1,\ldots,\gamma_{6k-4})} \succ 0$ and by [CF91, Theorem 3.1], the sequence $(\gamma_0, \gamma_1, \ldots, \gamma_{6k-3})$ has a $(3k-1)$–atomic $\mathbb{R}$–representing measure. Hence, one also gets γ_{6k-2} such that $F(\gamma_{6k-4}, \gamma_{6k-2})$ has a $\mathbb{R}$–representing measure. If $(M_{k-1})|_{\mathcal{B}} \succeq 0$ and $(M_{k-1})|_{\mathcal{B}} \not\succ 0$, then by [CF91, Theorem 3.8], $(\gamma_0, \gamma_1, \ldots, \gamma_{6k-5})$ has a unique $\mathbb{R}$–representing measure. This measure also represents γ_{6k-3} iff (3.26) for $u = 4$ and $u = 3$ holds. This establishes the equivalence (1) $\Leftrightarrow$ (3). The equivalence of both with (2) follows by observing that $F(\gamma_{6k-4}, \gamma_{6k-2})$ admits a $(\operatorname{rank}\gamma)$–atomic or $(\operatorname{rank}\gamma + 1)$–atomic $\mathbb{R}$–representing measure. The first case happens iff (3b) holds or (3a) holds and γ_{6k-4} is obtained by (3.26) for $u = 4$, where one uses (3.25) with $r = 3k - 2$, and γ_{6k-3} is obtained by (3.26) for $u = 3$. Since $\operatorname{rank} A_\gamma = \operatorname{rank} M_{k-1}$, the equivalence follows. □

4. The TMP on the curves $yx^\ell = 1$

In this section we study the K–TMP for K being a curve of the form $yx^\ell = 1$, $\ell \in \mathbb{N}$, $\ell \geq 2$. In Subsection 4.1 we first give a solution of the K–TMP, based on the number of positive semidefinite extensions of the moment matrix needed and also bound the number of atoms in the K–representing measure with the smallest number of atoms (see Theorem 4.1 for the even degree and Theorem 4.2 for the odd degree sequences). As a result we obtain a sum-of-squares representation for polynomials, which are strictly positive on K (see Corollary 4.4). This improves bounds in the previously known result [Fia11, Proposition 6.4]. In Subsection 4.2 we give a solution to the K–TMP, based on a feasibility of the corresponding linear matrix inequality (see Theorem 4.6).

4.1. **Solution to the TMP in terms of psd extensions of M_k, bounds on the number of atoms in the minimal measure and a Positivstellensatz.**

Theorem 4.1. *Let* $K := \{(x, y) \in \mathbb{R}^2 \colon yx^\ell = 1\}$*, where* $\ell \in \mathbb{N} \setminus \{1\}$*, and* $\beta := \beta^{(2k)} = (\beta_{i,j})_{i,j\in\mathbb{Z}_+,i+j\leq 2k}$*, where* $k \geq \ell + 1$*. The following statements are equivalent:*

1. *β has a K–representing measure.*
2. *β has a s–atomic K–representing measure for some s satisfying*
$$\operatorname{rank} M_k \leq s \leq k(\ell + 1).$$
3. *M_k satisfies $Y = X^\ell$ and admits a positive semidefinite, recursively generated extension $M_{k+\ell}$.*
4. *M_k satisfies $Y = X^\ell$ and admits a positive semidefinite extension $M_{k+\ell+1}$.*

Theorem 4.2. *Let* $K := \{(x, y) \in \mathbb{R}^2 \colon yx^\ell = 1\}$*, where* $\ell \in \mathbb{N} \setminus \{1\}$*, and* $\beta := \beta^{(2k-1)} = (\beta_{i,j})_{i,j\in\mathbb{Z}_+,i+j\leq 2k-1}$*, where* $k \geq \ell + 1$*. The following statements are equivalent:*

(1) β has a K–representing measure.
(2) β has a s–atomic K–representing measure for some s satisfying

$$\operatorname{rank} M_k \leq s \leq k(\ell+1) - \left\lfloor \frac{\ell}{2} \right\rfloor + 1.$$

(3) $\beta^{(2k-1)}$ can be extended to a sequence $\beta^{(2k)}$ such that M_k satisfies $YX^\ell = 1$ and admits a positive semidefinite, recursively generated extension $M_{k+\ell}$.
(4) $\beta^{(2k-1)}$ can be extended to a sequence $\beta^{(2k)}$ such that M_k satisfies $YX^\ell = 1$ and admits a positive semidefinite extension $M_{k+\ell+1}$.

Remark 4.3. (1) In [Fia11, Section 6] the author considered TMPs on $\mathcal{Z}(p)$ in terms of the number of psd extension of the moment matrix also for polynomials of the form $p(x,y) = yq(x)$, where $q \in \mathbb{R}[x]$, and proved that the number of psd extensions of the moment matrix ensuring the existence of the measure is $(2k+2)(2+\deg q) - (1+\deg q + k)$ [Fia11, Propositions 6.1, 6.4], where $2k$ is the degree of the sequence. The proof of this result relies on the truncated Riesz-Haviland theorem [CF08, Theorem 1.2] and a sum-of-squares representations for polynomials, strictly positive on $\mathcal{Z}(p)$ ([Fia11, Proposition 6.4] and [Sto01, Proposition 5.1]). Part (4) of Theorem 4.1 improves Fialow's result in case $q(x) = x^\ell$, $\ell \geq 2$, by decreasing the number of extensions to $\ell + 1$.
(2) Similarly as in Remark 3.3.(2), part (2) of Theorem 4.1 is a counterpart of [RS18, Corollary 7.6] for even degree sequences on curves $\mathcal{Z}(yx^\ell - 1)$, while part (2) of Theorem 4.2 improves [RS18, Corollary 7.6] for curves $\mathcal{Z}(yx^\ell - 1)$ by decreasing it for $\lfloor \frac{\ell}{2} \rfloor - 1$.

Proof of Theorem 4.1. The implications (1) $\Rightarrow$ (4) and (2) $\Rightarrow$ (1) are trivial. The implication (4) $\Rightarrow$ (3) follows by [CF96, Theorem 3.14]. It remains to prove the implication (3) $\Rightarrow$ (2). Assume that YX^ℓ is a column relation and M_k admits a psd, rg extension $M_{k+\ell}$. Let

$$\mathcal{B} = \left\{ y^{k+1}x^{\ell-1}, y^k, y^k x, \ldots, y^k x^{\ell-1}, \ldots, y, yx, \ldots, yx^{\ell-1}, 1, x, \ldots, x^{k+1} \right\} \tag{4.1}$$

be the set of monomials and V the vector subspace in $\mathbb{R}[x,y]_{k+\ell}$ generated by the set $\mathcal{B}$. Since $M_{k+\ell}$ satisfies

$$X^i Y^j = \begin{cases} X^{i \bmod \ell} Y^{j - \lfloor \frac{i}{\ell} \rfloor}, & \text{if } i,j \in \mathbb{Z}_+,\ i+j \leq k,\ j \geq \lfloor \frac{i}{\ell} \rfloor, \\ X^{i-j\ell}, & \text{if } i,j \in \mathbb{Z}_+,\ i+j \leq k,\ j < \lfloor \frac{i}{\ell} \rfloor, \end{cases}$$

it follows that the columns from $\mathcal{B}$ span $\mathcal{C}(M_{k+\ell})$. Let $p(x,y) = \sum_{i,j} p_{ij} x^i y^j \in V$ be a polynomial and $\widehat{p}$ a vector of its coefficients ordered in the basis $\mathcal{B}$. We define a univariate polynomial $q_p(x)$ corresponding to $p(x,y)$, by

$$g_p(x) := p(x, x^{-\ell}) = \sum_{i,j} p_{ij} x^{i-\ell j} =: \sum_{s=-k\ell-1}^{k+1} g_{p,s} x^s \in \mathbb{R}[x]_{k\ell+1}. \tag{4.2}$$

Let $\widehat{g_p}$ be its vector of coefficients in the basis

$$\mathcal{B}_1 = \{x^{-k\ell-1}, x^{-k\ell}, \ldots, x^{k+1}\}. \tag{4.3}$$

The monomials $x^{i_1}y^{j_1}$, $x^{i_2}y^{j_2}$ from $\mathcal{B}$ correspond to the same monomial x^s by the correspondence (4.2) iff $i_1 - \ell j_1 = i_2 - \ell j_2$, which is further equivalent to $i_1 = i_2$ and $j_1 = j_2$ (since i_1 and i_2 are at most $\ell - 1$ in $\mathcal{B}$). Therefore

$$\widehat{g_p} = \widehat{p}. \tag{4.4}$$

We define two univariate sequences

$$\gamma := \gamma^{(-2k\ell,2k)} = \big(\gamma_{-2k\ell}, \gamma_{-2k\ell+1}, \ldots, \gamma_0, \ldots, \gamma_{2k}\big) \in \mathbb{R}^{2k(1+\ell)+1},$$
$$\widetilde{\gamma} := \gamma^{(-2k\ell-2,2k+2)} = \big(\gamma_{-2k\ell-2}, \gamma_{-2k\ell-1}, \gamma, \gamma_{2k+1}, \gamma_{2k+2}\big) \in \mathbb{R}^{2k(1+\ell)+5},$$

by the formula

$$\gamma_t = \begin{cases} \beta_{t,0}, & \text{if } t \geq 0, \\ \beta_{t+\ell\lceil\frac{|t|}{\ell}\rceil, \lceil\frac{|t|}{\ell}\rceil}, & \text{if } t < 0. \end{cases} \tag{4.5}$$

Note that for $t < 0$ we have that $t + \ell\lceil\frac{|t|}{\ell}\rceil \leq \ell - 1$, $\lceil\frac{|t|}{\ell}\rceil \leq 2k + 1$ (since $\ell \geq 3$) and hence

$$t + \ell\left\lceil\frac{|t|}{\ell}\right\rceil + \left\lceil\frac{|t|}{\ell}\right\rceil \leq \ell - 1 + 2k + 1 = 2k + \ell.$$

Therefore $\beta_{t+\ell\lceil\frac{|t|}{\ell}\rceil, \lceil\frac{|t|}{\ell}\rceil}$ is well-defined being an element of the matrix $M_{k+\ell}$ (since $2k + \ell \geq 2k + 2\ell$).

By the following claim solving the K–TMP for β is equivalent to solving the $(\mathbb{R} \setminus \{0\})$–TMP for γ.

Claim 1. Let $u \in \mathbb{N}$. A sequence γ admits a u–atomic $(\mathbb{R} \setminus \{0\})$–representing measure if and only if β admits a u–atomic K–representing measure .

Proof of Claim 1. First we prove the implication $(\Rightarrow)$. Let $x_1, \ldots, x_u$, be the atoms in the $(\mathbb{R}\setminus\{0\})$–representing measure for γ with the corresponding densities $\rho_1, \ldots, \rho_u$. We will prove that the atoms $(x_1, (x_1)^{-\ell}), \ldots, (x_u, (x_u)^{-\ell})$ with densities $\rho_1, \ldots, \rho_p$ are the K–representing measure for β. We separate two cases:

(1) $\lfloor\frac{i}{\ell}\rfloor \geq j$:

$$\beta_{i,j} = \beta_{i-\ell j,0} = \gamma_{i-\ell j} = \sum_{p=0}^{u} \rho_p (x_p)^{i-\ell j} = \sum_{p=0}^{u} \rho_p (x_p)^i ((x_p)^{-\ell})^j,$$

were we used the fact that β is rg in the first equality, (4.5) in the second equality, the definitions of ρ_p, x_p in the third equality and split $(x_p)^{i-\ell j}$ into two parts in the last equality.

(2) $\lfloor\frac{i}{\ell}\rfloor < j$:

$$\begin{aligned} \beta_{i,j} = \beta_{i \bmod \ell, j-\lfloor\frac{i}{\ell}\rfloor} = \gamma_{-(j-\lfloor\frac{i}{\ell}\rfloor)\ell + i \bmod \ell} &= \sum_{p=0}^{u} \rho_p (x_p)^{-(j-\lfloor\frac{i}{\ell}\rfloor)\ell + i \bmod \ell} \\ &= \sum_{p=0}^{u} \rho_p (x_p)^{\lfloor\frac{i}{\ell}\rfloor\ell + i \bmod \ell} ((x_p)^{-\ell})^j \\ &= \sum_{p=0}^{u} \rho_p (x_p)^i ((x_p)^{-\ell})^j, \end{aligned}$$

were we used the fact that β is rg in the first equality, (4.5) in the second equality, the definitions of ρ_p, x_p in the third equality, split the exponent at x_p into two parts in the fourth equality and used that $\lfloor\frac{i}{\ell}\rfloor\ell + i \bmod \ell = i$ in the last equality.

This proves the implication $(\Rightarrow)$.

It remains to prove the implication $(\Leftarrow)$. Let $(x_1,(x_1)^{-\ell}),\ldots,(x_u,(x_u)^{-\ell})$ be the atoms in the K–representing measure for β with the corresponding densities $\rho_1,\ldots,\rho_u$. We will prove that the atoms $(x_1,\ldots,x_u)$ with densities $\rho_1,\ldots,\rho_p$ are the $(\mathbb{R}\setminus\{0\})$–representing measure for γ:

- For $t\geq 0$ we have that
$$\gamma_t=\beta_{t,0}=\sum_{p=0}^{u}\rho_p(x_p)^t,$$
where we use the definition (4.5) in the first equality and the definitions of ρ_p,x_p in the second.
- For $t<0$ we have that
$$\gamma_t=\beta_{t+\ell\left\lceil\frac{|t|}{\ell}\right\rceil,\left\lceil\frac{|t|}{\ell}\right\rceil}=\sum_{p=0}^{u}\rho_p(x_p)^{t+\ell\left\lceil\frac{|t|}{\ell}\right\rceil}((x_p)^{-\ell})^{\left\lceil\frac{|t|}{\ell}\right\rceil}=\sum_{p=0}^{u}\rho_p(x_p)^t,$$
where we use the definition (4.5) in the first equality and the definitions of ρ_p,x_p in the second.

This proves the implication $(\Leftarrow)$. ■

Let $(M_{k+\ell})|_{\mathcal{B}}$ be the restriction of $M_{k+\ell}$ to the rows and columns indexed by monomials (capitalized) from $\mathcal{B}$. The following claim gives an explicit connection between $(M_{k+\ell})|_{\mathcal{B}}$ and the Hankel matrix $A_{\widetilde{\gamma}}$ of the sequence $\widetilde{\gamma}$.

Claim 2. We have that

$$(M_{k+\ell})|_{\mathcal{B}}=A_{\widetilde{\gamma}}. \tag{4.6}$$

Proof of Claim 2. Let $p(x,y)=\sum_{i,j}p_{ij}x^iy^j\in V$ and $r(x,y)=\sum_{i,j}r_{ij}x^iy^j\in V$ be polynomials from the vector subspace V and $\widehat{p}$, $\widehat{r}$ vectors of their coefficients ordered in the basis $\mathcal{B}$. Let $\widetilde{\beta}:=\beta^{(2(k+1)\ell+2(\ell-1))}$. Then we have

$$\begin{aligned}
(\widehat{r})^T\left((M_{k+\ell})|_{\mathcal{B}}\right)\widehat{p} &=^1 L_{\widetilde{\beta}}(pr)=L_{\widetilde{\beta}}\Big(\sum_{i_1,i_2,j_1,j_2}p_{i_1j_1}r_{i_2j_2}x^{i_1+i_2}y^{j_1+j_2}\Big)\\
&=^2 \sum_{i_1,i_2,j_1,j_2}p_{i_1j_1}r_{i_2j_2}\beta_{i_1+i_2,j_1+j_2}\\
&=^3 \sum_{i_1,i_2,j_1,j_2}p_{i_1j_1}r_{i_2j_2}\gamma_{i_1+i_2-(j_1+j_2)\ell}\\
&=^4 L_{\widetilde{\gamma}}\left(\sum_{i_1,i_2,j_1,j_2}p_{i_1j_1}r_{i_2j_2}x^{i_1+i_2-(j_1+j_2)\ell}\right)\\
&=^5 L_{\widetilde{\gamma}}\left(\sum_{i_1,i_2,j_1,j_2}p_{i_1j_1}x^{i_1-j_1\ell}\cdot r_{i_2j_2}x^{i_2-j_2\ell}\right)\\
&=^6 L_{\widetilde{\gamma}}\Big(\underbrace{\Big(\sum_{i_1,j_1}p_{i_1j_1}x^{i_1-j_1\ell}\Big)}_{g_p(x)}\underbrace{\Big(\sum_{i_2,j_2}r_{i_2j_2}x^{i_2-j_2\ell}\Big)}_{g_r(x)}\Big)\\
&=^7 \widehat{g_r}^{\,T}A_{\widetilde{\gamma}}\widehat{g_p}=\widehat{r}^TA_{\widetilde{\gamma}}\widehat{p},
\end{aligned}$$

where in the first line we used the correspondence between the moment matrix and the Riesz functional $L_{\widetilde{\beta}}$, the definition $L_{\widetilde{\beta}}$ in the second, (4.5) and the fact that β is rg in the third (rg is needed if $i_1 + i_2 \geq \ell$), the definition of $L_{\widetilde{\gamma}}$ in the fourth, decomposed the exponent of x into two parts in the fifth, decomposed a sum into the product of two sums in the sixth, in the seventh we used the correspondence between $A_{\widetilde{\gamma}}$ and the Riesz functional $L_{\widetilde{\gamma}}$, where $\widehat{g_p}$, $\widehat{g_r}$ are the vectors of coefficents of g_p and g_r in the basis $\mathcal{B}_1$ (see (4.3)) and (4.4). Since p and q were arbitrary from V, this proves Claim 2. ■

Since $(M_{k+\ell})|_{\mathcal{B}}$ is psd, it follows from (4.6) that $A_{\widetilde{\gamma}}$ is psd. We separate two cases. Either A_γ is pd or A_γ is psd, singular, prg by [CF91, Theorem 2.6] and nrg by [Zal22b, Proposition 2.1.(5)]. By [Zal22b, Theorem 3.1], γ admits a $(\operatorname{rank} A_\gamma)$–atomic $(\mathbb{R} \setminus \{0\})$–representing measure. Since $\operatorname{rank} M_k \leq \operatorname{rank} A_\gamma \leq k(\ell + 1) + 1$, using Claim 2 the following holds:

(2') β has a s–atomic K–representing measure for some s satisfying

$$\operatorname{rank} M_k \leq s \leq \operatorname{rank} A_\gamma \leq k(\ell + 1) + 1. \tag{4.7}$$

To obtain (2) of Theorem 4.1 we need to decrease the upper bound in (4.7) by 1. Note that the bound $k(\ell + 1) + 1$ occurs only in the case A_γ is pd. We denote by $\gamma(\mathrm{z})$ a sequence obtained from the sequence γ by replacing $\gamma_{-2k\ell+1}$ with a variable z. The matrix $A_{\gamma(\mathrm{z})}$ is a partially positive definite matrix and by [Zal21, Lemma 2.11] there exist two choices of z, which we denote by $z^\pm$, such that $A_{\gamma(z^\pm)}$ is psd and has rank $k(\ell + 1)$. Using [Zal22b, Proposition 2.5] for the reversed sequence $(\gamma(z^\pm))^{(\mathrm{rev})}$ of $\gamma(z^\pm)$, we see that at least one of $(\gamma(z^\pm))^{(\mathrm{rev})}$ admits a $(\mathbb{R} \setminus \{0\})$–representing measure. Hence, at least one of $\gamma(z^\pm)$ is prg and nrg, and admits a $k(\ell + 1)$–atomic $(\mathbb{R} \setminus \{0\})$–representing measure. If none of the moments $\beta_{i,j}$ of the sequence β depends on $\gamma_{-2k\ell+1}$, the $(\mathbb{R} \setminus \{0\})$–representing measure for $(\gamma(z^\pm))^{(\mathrm{rev})}$ will generate the K–representing measure for β as in the proof of Claim 2. But by definition (4.5), there is indeed no moment from β, which depends on $\gamma_{-2k\ell+1}$ (since we need to represent moments of degree at least $-2k\ell$ and at most $2k$, while $\gamma_{-2k\ell+1}$ corresponds to $\beta_{\ell-1,2k}$ in some extension of β), which concludes the proof of Theorem 4.1. □

To prove Theorem 4.2 only a little adaptation of the last part of the proof of Theorem 4.1 is needed, which we now explain.

Proof of Theorem 4.2. The implications (1) $\Rightarrow$ (4) and (2) $\Rightarrow$ (1) are trivial. The implication (4) $\Rightarrow$ (3) follows from [CF96, Theorem 3.14]. It remains to prove the implication (3) $\Rightarrow$ (2). Following the proof of Theorem 4.1 everything remains the same until (2'). It remains to justify that the upper bound in (4.7) can be decreased to $m := k(\ell + 1) - \lfloor \frac{\ell}{2} \rfloor + 1$. If $\operatorname{rank} A_\gamma \leq m$, then we are already done. From now on we assume that $r := \operatorname{rank} A_\gamma > m$. Since γ admits a $(\mathbb{R} \setminus \{0\})$–representing measure, which we denote by μ, it is nrg and $\operatorname{rank} \gamma = \operatorname{rank} A_\gamma = \operatorname{rank} A_\gamma[r-1]$. Hence, $A_{\gamma^{(2k-2(r-1),2k)}}$ is pd and in particular also its submatrix $A_{\gamma^{(2k-2(m-1),2k)}}$ is pd. We denote by $\gamma(\mathrm{z}_1, \ldots, \mathrm{z}_\ell)$ a sequence obtained from the sequence γ by replacing the moments $\gamma_{-2k\ell}, \gamma_{-2k\ell+1}, \ldots, \gamma_{-2k\ell+\ell-1}$ with variables $\mathrm{z}_1, \ldots, \mathrm{z}_\ell$. By [Zal22b, Theorem 3.1], the sequence $\gamma^{(2k-2(m-1),2k)}$ has a m–atomic $(\mathbb{R} \setminus \{0\})$–representing measure (to apply [Zal22b, Theorem 3.1] we used that $2k - 2(m - 1) = -2k\ell + 2\lfloor \frac{\ell}{2} \rfloor < 0$). We denote the measure obtained in this way by μ_1 and generate its moment sequence $\gamma(z_1, \ldots, z_\ell)$, where $z_1, \ldots, z_\ell$ are the moments of degree $-2k\ell, -2k\ell+1, \ldots, -2k\ell+\ell-1$, respectively. If none of the moments $\beta_{i,j}$ of the sequence $\beta^{(2k-1)}$ depends on $\gamma_{-2k\ell}, \gamma_{-2k\ell+1}, \ldots, \gamma_{-2k\ell+\ell-1}$, then μ_1 will generate the K–representing measure for $\beta^{(2k-1)}$ as in the proof of Claim 1 of Theorem 4.1. But by definition (4.5), there is indeed no moment from $\beta^{(2k-1)}$ depending on $\gamma_{-2k\ell}, \gamma_{-2k\ell+1}, \ldots, \gamma_{-2k\ell+\ell-1}$, which concludes the proof of Theorem 4.2. □

A corollary to Theorem 4.1 is an improvement of the bounds on the degrees of sums of squares in the Positivstellensatz [Fia11, Corollary 6.4] for the curves of the form $yx^\ell = 1$, $\ell \in \mathbb{N} \setminus \{1\}$.

Corollary 4.4. *Let $K := \{(x, y) \in \mathbb{R}^2 \colon yx^\ell = 1\}$, where $\ell \in \mathbb{N} \setminus \{1\}$ and $k \geq \ell + 1$. If $r(x, y) \in \mathbb{R}[x, y]_{2k}$ is strictly positive on K, then r admits a decomposition*

$$r(x, y) = \sum_{i=1}^{\ell_1} f_i(x, y)^2 + (yx^\ell - 1) \sum_{i=1}^{\ell_2} g_i(x, y)^2 - (yx^\ell - 1) \sum_{i=1}^{\ell_2} h_i(x, y)^2,$$

where $\ell_1, \ell_2, \ell_3 \in \mathbb{Z}_+$, $f_i, g_i, h_i \in \mathbb{R}[x, y]$ and

$$\deg f_i^2 \leq 2m,\ \deg((yx^\ell - 1)g_i^2) \leq 2m,\ \deg((yx^\ell - 1)h_i^2) \leq 2m$$

with $m = k + \ell + 1$. where $\ell_1, \ell_2, \ell_3 \in \mathbb{Z}_+$.

Proof. By the equivalence (1) $\Leftrightarrow$ (3) of Theorem 4.1, the set K has the property $(R_{k,\ell})$ in the notation of [CF08, p. 2713]. Now the result follows by [CF08, Theorem 1.5]. □

Remark 4.5. The bound on m in Theorem 4.4 in [Fia11, Corollary 6.4] is quadratic in k and ℓ, namely $(2k + 2)(2 + \ell) - (1 + \ell)$.

4.2. **A solution to the TMP based on the feasibility of a linear matrix inequality.** In this subsection we give another alternative solution to the TMP on curves yx^ℓ, where $\ell \in \mathbb{N} \setminus \{1\}$, which is based on the feasibility of a linear matrix inequality associated to the univariate sequence γ, obtained from the original sequence β as in the proof of the results in the previous subsection. The feasibility question appears as a result of the fact that γ is not fully determined by β, but β admits a K–representing measure if and only if γ can be completed to a sequence admitting a $(\mathbb{R} \setminus \{0\})$–representing measure.

Theorem 4.6. *Let $K := \{(x, y) \in \mathbb{R}^2 \colon yx^\ell = 1\}$, $\ell \in \mathbb{N} \setminus \{1\}$, and*

$$\beta := \beta^{(d)} = (\beta_{i,j})_{i,j \in \mathbb{Z}_+, i+j \leq d},$$

where $\lceil \frac{d}{2} \rceil \geq \ell + 1$. Let

$$N := \{t \in \mathbb{Z}_- \colon t = -i\ell + j \text{ for some } 0 \leq j < \ell, i \in \mathbb{Z}_+ \text{ and } i + j \leq d\},$$
$$V := \{-d\ell, -d\ell + 1, \ldots, d - 1, d\} \setminus N = \{i_1, \ldots, i_{|V|}\},$$

$\{\gamma_t\}_{t \in V} = \{\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_{|V|}}\}$ be a sequence of variables and

$$\gamma_t = \begin{cases} \beta_{t,0}, & \text{if } t \geq 0, \\ \beta_{t+\ell\lceil \frac{|t|}{\ell} \rceil, \lceil \frac{|t|}{\ell} \rceil}, & \text{if } t < 0. \end{cases} \tag{4.8}$$

Let

$$F : \begin{cases} \mathbb{R}^{|V|+4} \to \mathbb{R}^{d(\ell+1)+5}, & \text{if } d \text{ is even}, \\ \mathbb{R}^{|V|+3} \to \mathbb{R}^{d(\ell+1)+4}, & \text{if only } \ell \text{ is even}, \\ \mathbb{R}^{|V|+2} \to \mathbb{R}^{d(\ell+1)+3}, & \text{if } d, \ell \text{ are odd}, \end{cases}$$

be a function, defined by

$$\begin{cases} F(\gamma_{-d\ell-2}, \gamma_{-d\ell-1}, \{\gamma_t\}_{t \in V}, \gamma_{d+1}, \gamma_{d+2}) = (\gamma_{-d\ell-2}, \gamma_{-d\ell-1}, \ldots, \gamma_{d+1}, \gamma_{d+2}), & \text{if } d \text{ is even}, \\ F(\gamma_{-d\ell-2}, \gamma_{-d\ell-1}, \{\gamma_t\}_{t \in V}, \gamma_{d+1}) = (\gamma_{-d\ell-2}, \gamma_{-d\ell-1}, \ldots, \gamma_{d+1}), & \text{if only } \ell \text{ is even}, \\ F(\gamma_{-d\ell-1}, \{\gamma_t\}_{t \in V}, \gamma_{d+1}) = (\gamma_{-d\ell-1}, \ldots, \gamma_{d+1}), & \text{if } d, \ell \text{ are odd}. \end{cases}$$

Then the following statements are equivalent:

(1) *β has a K–representing measure.*

(2) *$\beta_{i+\ell,j+1} = \beta_{i,j}$ for every $i, j \in \mathbb{Z}_+$ such that $i + j \le d - \ell - 1$ and there exist*

$$\gamma_{-d\ell-2}, \gamma_{-d\ell-1}, \{\gamma_t\}_{t\in V}, \gamma_{d+1}, \gamma_{d+2}$$

such that

$$\begin{cases} A_{F(\gamma_{-d\ell-2},\gamma_{-d\ell-1},\{\gamma_t\}_{t\in V},\gamma_{d+1},\gamma_{d+2})} \succeq 0, & \text{if } d \text{ is even},\\ A_{F(\gamma_{-d\ell-2},\gamma_{-d\ell-1},\{\gamma_t\}_{t\in V},\gamma_{d+1})} \succeq 0, & \text{if only } \ell \text{ is even},\\ A_{F(\gamma_{-d\ell-1},\{\gamma_t\}_{t\in V},\gamma_{d+1})} \succeq 0, & \text{if } d, \ell \text{ are odd}. \end{cases}$$

Proof. Observing the proof of Theorem 4.1 one can notice that

$$F(\gamma_{-d\ell-2}, \gamma_{-2d\ell-1}, \{\gamma_t\}_{t\in V}, \gamma_{d+1}, \gamma_{d+2})$$

corresponds to the sequence $\widetilde{\gamma}$. The original sequence β determines only γ_t for $t \in N$ by (4.8), while for $t \in V$, γ_t are variables. By the proof of Theorem 4.1, β will have a K–representing measure iff it satisfies the rg relations coming from the column relation $YX^\ell = 1$ and there exists $\widetilde{\gamma}$ such that $A_{\widetilde{\gamma}} \succeq 0$. This proves Theorem 4.6 for even d.

Observing the proof of Theorem 4.2 in case d is odd one can notice that only

$$\gamma^{(-d\ell,d)} = (\gamma_{-d\ell}, \gamma_{-d\ell+1}, \ldots, \gamma_{d-1}, \gamma_d)$$

needs to have a $(\mathbb{R} \setminus \{0\})$–representing measure to obtain a K–representing measure for β. In case $d\ell$ is even, this is equivalent to $A_{\gamma^{(-d\ell-2,d+1)}} \succeq 0$, where

$$\gamma^{(-d\ell-2,d+1)} = (\gamma_{-d\ell-2}, \gamma_{-d\ell-1}, \ldots, \gamma_d, \gamma_{d+1})$$

for some $\gamma_{-d\ell-2}$, $\gamma_{-d\ell-1}$ and γ_{d+1}. Since $\gamma^{(-d\ell-2,d+1)}$ corresponds to the sequence

$$F(\gamma_{-d\ell-2}, \gamma_{-d\ell-1}, \{\gamma_t\}_{t\in V}, \gamma_{d+1}),$$

this proves Theorem 4.2 for even ℓ with d being odd. If $d\ell$ is odd, then it suffices that there are $\gamma_{-d\ell-1}$ and γ_{d+1} such that $A_{\gamma^{(-d\ell-1,d+1)}} \succeq 0$, where

$$\gamma^{(-d\ell-1,d+1)} = (\gamma_{-d\ell-1}, \gamma_{-d\ell}, \ldots, \gamma_d, \gamma_{d+1}).$$

Since $\gamma^{(-d\ell-1,d+1)}$ corresponds to the sequence

$$F(\gamma_{-d\ell-1}, \gamma_{-d\ell}, \{\gamma_t\}_{t\in V}, \gamma_{d+1}),$$

this proves Theorem 4.2 for odd $d\ell$. $\square$

We will present the statement of Theorem 4.6 on a few examples. The following example is for $\ell = 2$ and a sequence β of even degree.

Example 4.7. Let $\beta = (\beta_{i,j})_{i,j\in\mathbb{Z}_+, i+j\le 2k}$ be a bivariate sequence of degree $2k$, $k \ge 3$, and $K := \{(x, y) \in \mathbb{R}^2 \colon yx^2 = 1\}$. For the existence of a K–representing measure β must satisfy the relations $\beta_{i,j} = \beta_{i+2,j+1}$ for every $i, j \in \mathbb{Z}_+$ such that $i + j \le 2k - 3$. In the notation of Theorem 4.6, $V = \{-4k + 1\}$, the formula (4.5) is equal to

$$\gamma_t = \begin{cases} \beta_{t,0}, & \text{if } t \ge 0,\\ \beta_{t+2\lceil \frac{|t|}{2} \rceil, \lceil \frac{|t|}{2} \rceil}, & \text{if } t < 0, \end{cases}$$

and the function $F:\mathbb{R}^5\to\mathbb{R}^{6k+4}$ is defined by

$$F(\gamma_{\mathbf{-4k-2}},\gamma_{\mathbf{-4k-1}},\gamma_{\mathbf{-4k+1}},\gamma_{\mathbf{2k+1}},\gamma_{\mathbf{2k+2}}) := (\gamma_{\mathbf{-4k-2}},\gamma_{\mathbf{-4k-1}},\gamma_{-4k},\gamma_{\mathbf{-4k+1}},\\ \gamma_{-4k+2},\ldots,\gamma_{2k},\gamma_{\mathbf{2k+1}},\gamma_{\mathbf{2k+2}}).$$

The matrix $A_{F(\gamma_{\mathbf{-4k-2}},\gamma_{\mathbf{-4k-1}},\gamma_{\mathbf{-4k+1}},\gamma_{\mathbf{2k+1}},\gamma_{\mathbf{2k+2}})}$ is equal to

$$\left(\begin{array}{c|ccccccc|c}
\gamma_{\mathbf{-4k-2}} & \gamma_{\mathbf{-4k-1}} & \gamma_{-4k} & \gamma_{\mathbf{-4k+1}} & \gamma_{-4k+2} & \cdots & & \gamma_k & \gamma_{k+1}\\ \hline
\gamma_{\mathbf{-4k-1}} & \gamma_{-4k} & \gamma_{\mathbf{-4k+1}} & \gamma_{-4k+2} & \iddots & & & \gamma_{k+1} & \gamma_{k+2}\\
\gamma_{-4k} & \gamma_{\mathbf{-4k+1}} & \gamma_{-4k+2} & \iddots & & & & \gamma_{k+2} & \vdots\\
\gamma_{\mathbf{-4k+1}} & \beta_{0,2k-1} & \iddots & & & & & \vdots & \vdots\\
\gamma_{-4k+2} & \iddots & & & & & & \vdots & \gamma_{2k}\\
\vdots & & & & & & & \gamma_{2k} & \gamma_{\mathbf{2k+1}}\\ \hline
\gamma_{k+1} & \cdots & \cdots & \cdots & \gamma_{2k-1} & \gamma_{2k} & & \gamma_{\mathbf{2k+1}} & \gamma_{\mathbf{2k+2}}
\end{array}\right)$$

The question of feasibility of $A_{F(\gamma_{\mathbf{-4k-2}},\gamma_{\mathbf{-4k-1}},\gamma_{\mathbf{-4k+1}},\gamma_{\mathbf{2k+1}},\gamma_{\mathbf{2k+2}})}$ can be answered analytically, since the structure of the missing entries is simple enough. Actually it is even easier to work with $A_{(\gamma_{-4k},\gamma_{\mathbf{-4k+1}},\gamma_{-4k+2},\ldots,\gamma_{2k})}$ and answer the feasibility question together with the conditions from the solution of [Zal22b, Theorem 2.1] (see [Zal22b, Theorem 4.1]). $\triangle$

The following example demonstrates the statement of Theorem 4.6 for the case $yx^3 = 1$ and a sequence β of even degree.

Example 4.8. Let $\beta = (\beta_{i,j})_{i,j\in\mathbb{Z}_+,i+j\le 2k}$ be a bivariate sequence of degree $2k$, $k \ge 4$, and $K := \{(x,y)\in\mathbb{R}^2 : yx^3 = 1\}$. For the existence of a K–representing measure β must satisfy the relations $\beta_{i,j} = \beta_{i+3,j+1}$ for every $i,j\in\mathbb{Z}_+$ such that $i+j\le 2k-4$. In the notation of Theorem 4.6, we have $V = \{-6k+1,-6k+2,-6k+5\}$, the formula (4.5) is equal to

$$\gamma_t = \begin{cases} \beta_{t,0}, & \text{if } t\ge 0,\\ \beta_{t+3\lceil\frac{|t|}{3}\rceil,\lceil\frac{|t|}{3}\rceil}, & \text{if } t<0.\end{cases}$$

and the function $F:\mathbb{R}^7\to\mathbb{R}^{8k+4}$ is defined by

$$F(\gamma_{\mathbf{-6k-2}},\gamma_{\mathbf{-6k-1}},\gamma_{\mathbf{-6k+1}},\gamma_{\mathbf{-6k+2}},\gamma_{\mathbf{-6k+5}},\gamma_{\mathbf{2k+1}},\gamma_{\mathbf{2k+2}}) := (\gamma_{\mathbf{-6k-2}},\gamma_{\mathbf{-6k-1}},\gamma_{-6k},\gamma_{\mathbf{-6k+1}},\\ \gamma_{\mathbf{-6k+2}},\gamma_{-6k+3},\gamma_{-6k+4},\gamma_{\mathbf{-6k+5}},\gamma_{-6k+6},\ldots,\gamma_{2k},\gamma_{\mathbf{2k+1}},\gamma_{\mathbf{2k+2}}).$$

The matrix $A_{F(\boldsymbol{\gamma_{-6k-2}},\boldsymbol{\gamma_{-6k-1}},\boldsymbol{\gamma_{-6k+1}},\boldsymbol{\gamma_{-6k+2}},\boldsymbol{\gamma_{-6k+5}},\boldsymbol{\gamma_{2k+1}},\boldsymbol{\gamma_{2k+2}})}$ is equal to

$$
\left(\begin{array}{c|ccccccccc|c}
\boldsymbol{\gamma_{-6k-2}} & \boldsymbol{\gamma_{-6k-1}} & \gamma_{-6k} & \boldsymbol{\gamma_{-6k+1}} & \boldsymbol{\gamma_{-6k+2}} & \gamma_{-6k+3} & \gamma_{-6k+4} & \boldsymbol{\gamma_{-6k+5}} & \cdots & \gamma_{k} & \gamma_{k+1} \\
\hline
\boldsymbol{\gamma_{-6k-1}} & \gamma_{-6k} & \boldsymbol{\gamma_{-6k+1}} & \boldsymbol{\gamma_{-6k+2}} & \gamma_{-6k+3} & \gamma_{-6k+4} & \boldsymbol{\gamma_{-6k+5}} & \cdots & \cdots & \gamma_{k+1} & \gamma_{k+2} \\
\gamma_{-6k} & \boldsymbol{\gamma_{-6k+1}} & \boldsymbol{\gamma_{-6k+2}} & \gamma_{-6k+3} & \gamma_{-6k+4} & \boldsymbol{\gamma_{-6k+5}} & \iddots & & & \gamma_{k+2} & \vdots \\
\boldsymbol{\gamma_{-6k+1}} & \boldsymbol{\gamma_{-6k+2}} & \gamma_{-6k+3} & \gamma_{-6k+4} & \boldsymbol{\gamma_{-6k+5}} & \iddots & & & & \vdots & \vdots \\
\boldsymbol{\gamma_{-6k+2}} & \gamma_{-6k+3} & \gamma_{-6k+4} & \boldsymbol{\gamma_{-6k+5}} & \iddots & & & & & \vdots & \vdots \\
\vdots & \gamma_{-6k+4} & \boldsymbol{\gamma_{-6k+5}} & \iddots & & & & & & \vdots & \vdots \\
\vdots & \iddots & \iddots & & & & & & & \vdots & \vdots \\
\vdots & \iddots & & & & & & & & \vdots & \gamma_{2k} \\
\vdots & & & & & & & & & \gamma_{2k} & \boldsymbol{\gamma_{2k+1}} \\
\hline
\gamma_{k+1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \gamma_{2k-1} & \gamma_{2k} & \boldsymbol{\gamma_{2k+1}} & \boldsymbol{\gamma_{2k+2}}
\end{array}\right)
$$

In contrast to the situation $yx^2 = 1$ from Example 4.7, the structure of the missing entries here is too complicated for the analytic approach and we believe the feasibility question can only be answered numerically using linear matrix inequality solvers. $\triangle$

Remark 4.9. It would be interesting to know, to what extent does the result [RS18, Corollary 7.6] (see Remark 3.3.(2)) extend to even degree sequences on plane curves. As explained in Remark 3.3.(3), one needs one more atom in the upper bound for curves of the form $y = q(x)$ with $\deg q = 2$ and the same is true if $\deg q \leq 1$ by Remark 3.3.(4). On the other hand the results of the present paper suggest that for curves $\mathcal{Z}(p)$, $\deg p \geq 3$, the upper bound could be $k \deg p$. Also from the concrete solution of the TMP on the curve $\mathcal{Z}(y^2 - x^3)$ [Zal21, Corollary 4.3] it follows that the same bound works. However, the forthcoming result of Bhardwaj [Bha+] shows that also for degree 3 curves the upper bound has to be loosened, by constructing a truncated moment sequence of degree $2k = 6$ on a curve $\mathcal{Z}(p)$, where $p(x,y) = y^2 - x^3 + ax - 1$, $a = \frac{524287}{262144}$, with a minimal measure consisting of 10 atoms, which is $k \deg p + 1$.

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