Rombergova metoda za integracijo

Fakulteta za računalništvo in informatiko
Univerza v Ljubljani

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Naj bo $T(h)$ aproksimacija integrala $I = \int_a^b f(x)\,dx$ z uporabo sestavljenega trapeznega pravila s korakom h.

Če je naša funkcija $(2k+2)$-krat odvedljiva, lahko napako trapezne formule (z uporabo Euler-Maclaurinove vrste) razvijemo v konvergentno vrsto po sodih potencah h:

$$\boxed{E(h) := T(h) - I = C_1 h^2 + C_2 h^4 + \ldots + C_k h^{2k} + \mathcal{O}(h^{2k+2})}, \tag{1}$$

pri čemer so konstante $C_1, \ldots, C_k$ neodvisne od h.

V (1) vnesemo h, $h/2$, $h/4$, ..., $h/2^k$ in dobimo:

$$I = T(h) + C_1 h^2 + C_2 h^4 + \ldots + C_\ell h^{2\ell} + \mathcal{O}(h^{2\ell+2}), \tag{2}$$

$$I = T(h/2) + C_1 \left(\frac{h}{2}\right)^2 + C_2 \left(\frac{h}{2}\right)^4 + \ldots + C_\ell \left(\frac{h}{2}\right)^{2\ell} + \mathcal{O}\left(\left(\frac{h}{2}\right)^{2\ell+2}\right), \tag{3}$$

$$I = T(h/4) + C_1 \left(\frac{h}{4}\right)^2 + C_2 \left(\frac{h}{4}\right)^4 + \ldots + C_\ell \left(\frac{h}{4}\right)^{2\ell} + \mathcal{O}\left(\left(\frac{h}{4}\right)^{2\ell+2}\right), \tag{4}$$

$$\vdots$$

$$I = T\left(\frac{h}{2^k}\right) + C_1 \left(\frac{h}{2^k}\right)^2 + C_2 \left(\frac{h}{2^k}\right)^4 + \ldots + C_\ell \left(\frac{h}{2^k}\right)^{2\ell} + \mathcal{O}\left(\left(\frac{h}{2^k}\right)^{2\ell+2}\right).$$

Z uporabo (2) in (3) bi se radi znebili člena pri h^2. Pomnožimo (3) s 4, da dobimo

$$4I = 4T(h/2) + C_1 h^2 + 4C_2 \left(\frac{h}{2}\right)^4 + \ldots + 4C_\ell \left(\frac{h}{2}\right)^{2\ell} + \mathcal{O}\left(\left(\frac{h}{2}\right)^{2\ell+2}\right), \qquad (5)$$

Sedaj od (5) odštejemo (2) in dobimo

$$3I = 4T(h/2) - T(h) + C_2 \left(\frac{1}{2^2} - 1\right) h^4 + \ldots + C_\ell \left(\frac{1}{2^{2\ell-2}} - 1\right) h^{2\ell} + \mathcal{O}(h^{2\ell+2}),$$

oziroma

$$I = \underbrace{\frac{4T(h/2) - T(h)}{3}}_{T_1(h/2)} + \frac{C_2}{3} \left(\frac{1}{2^2} - 1\right) h^4 + \ldots + \frac{C_\ell}{3} \left(\frac{1}{2^{2\ell-2}} - 1\right) h^{2\ell} + \mathcal{O}(h^{2\ell+2}).$$

Podobno z uporabo (3) in (4) pridemo do enakosti

$$I = \underbrace{\frac{4T(h/4) - T(h/2)}{3}}_{T_1(h/4)} + \frac{C_2}{3} \left(\frac{1}{2^2} - 1\right) \left(\frac{h}{2}\right)^4 + \ldots + \frac{C_\ell}{3} \left(\frac{1}{2^{2\ell-2}} - 1\right) \left(\frac{h}{2}\right)^{2\ell} + \ldots .$$

V splošnem dobimo:

$$I = \underbrace{\frac{4T\left(\frac{h}{2^k}\right) - T\left(\frac{h}{2^{k-1}}\right)}{3}}_{T_1(h/2^k)} + \frac{C_2}{3}\left(\frac{1}{2^2} - 1\right)\left(\frac{h}{2^{k-1}}\right)^4 + \ldots + \frac{C_\ell}{3}\left(\frac{1}{2^{2\ell-2}} - 1\right)\left(\frac{h}{2^{k-1}}\right)^{2\ell}$$
$$+ \mathcal{O}\left(\left(\frac{h}{2^{k-1}}\right)^{2\ell+2}\right).$$

Torej so

$$\boxed{T_1(h/2^\ell) = \frac{4T(h/2^\ell) - T(h/2^{\ell-1})}{3}}, \quad \ell = 1, 2, 3, \ldots, k$$

boljši približki za I, saj je

$$I = T_1(h/2^\ell) + \mathcal{O}\left(\left(\frac{h}{2^{\ell-1}}\right)^4\right).$$

Lahko bi nadaljevali zgornji postopek in dobili;

$$\boxed{T_2(h/2^\ell) = \frac{4^2 T_1(h/2^\ell) - T_1(h/2^{\ell-1})}{4^2 - 1}}, \quad \ell = 2, 3, 4, \ldots, k$$

ki so še boljši približki za I, saj je

$$I = T_2(h/2^\ell) + \mathcal{O}\left(\left(\frac{h}{2^{\ell-1}}\right)^6\right).$$

Na m-tem koraku, kjer je $m = 1, \ldots, k$, dobimo

$$\boxed{T_m(h/2^\ell) = \frac{4^m T_{m-1}(h/2^\ell) - T_{m-1}(h/2^{\ell-1})}{4^m - 1}}, \quad \ell = m, m+1, m+2, \ldots, k$$

in

$$I = T_m(h/2^\ell) + \mathcal{O}\left(\left(\frac{h}{2^{\ell-1}}\right)^{2(m+1)}\right).$$

Vrednosti $T_m(h/2^\ell)$ v kompaktni obliki računamo s pomočjo trikotne tabele:

$$\begin{array}{ccccc}
T(h) & & & & \\
T(h/2) & T_1(h/2) & & & \\
T(h/2^2) & T_1(h/2^2) & T_2(h/2^2) & & \\
T(h/2^3) & T_1(h/2^3) & T_2(h/2^3) & T_3(h/2^3) & \\
T(h/2^4) & T_1(h/2^4) & T_2(h/2^4) & T_3(h/2^4) & T_4(h/2^4),
\end{array}$$

pri čemer $T_m(h/2^\ell)$ izračunamo tako, da:

- pomnožimo element levo od njega z utežjo $\dfrac{4^m}{4^m - 1}$
- odštejemo element za eno levo in eno navzgor pomnožen z utežjo $\dfrac{1}{4^m - 1}$.

Algoritem:

https://zalara.github.io/Rombergova.m