Electromagnetics Summary

r.g.vankrieken

November 2020

Contents

Chapter 1

Introduction

In this book the differential forms of the Maxwell equations are used in addition to the integral forms. It follows the axiomatic introduction method of teaching electrodynamics, starting from electrostatics. The 2 starting equations are Faraday's law of induction and Amperes Flux theorem.

$$\frac{d}{dt}\int_F \mathbf{B}\cdot d\mathbf{F} = -\oint_{C(F)} \mathbf{E}\cdot d\mathbf{l} \tag{1.1}$$

$$\frac{d}{dt}\int_F \mathbf{D}\cdot d\mathbf{F} + \int_F \mathbf{j}\cdot d\mathbf{F} = \oint_{C(F)} \mathbf{H}\cdot d\mathbf{l} \tag{1.2}$$

Where the new unit is $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ with $\mathbf{D}$ is the Dielectric displacement and $\mathbf{P}$ the polarisation vector or dipole moment per unit volume of the medium. The first differential term is the displacement current as $\frac{\partial \mathbf{D}}{\partial t}$ represents a current density. The other new unit is $\mathbf{H}$, the magnetic field strength.

Additionally when enclosing the surface F ,in the integral of $\mathbf{B}$ in equation 1.1, to a volume it will always yield 0. As no magnetic monopoles exist.

$$\int_{F_{enclosed}} \mathbf{B}\cdot d\mathbf{F} = 0 \tag{1.3}$$

As earlier said in this book the differential forms are also used. These are obtained from the divergence theorem. This entails that flux on a surface can be split up by dividing a volume in smaller volumes. The flux on the surface of the internal volumes cancle on the inside. When the volume approaches 0 it approaches the differential dV.

$$\nabla\cdot B = div\ \mathbf{B}(\mathbf{r}) = 0 \tag{1.4}$$

This way equation 1.2 can also be rewritten (and simplified) to

$$\begin{aligned} \int_{F_{closed}} \mathbf{D} \cdot d\mathbf{F} &= Q \\ \nabla \cdot \mathbf{D} &= \frac{q}{V} \end{aligned} \tag{1.5}$$

This says that the divergence of dielectric displacement is equal to the charge density. Where $\mathbf{H} = 0$ as the surface is a closed surface.

The Units are as follows :

- $\mathbf{E}$, electric field intensity (V/m^2)
- $\mathbf{D}$, electric flux density (C/m^2) or (FV/m^2)
- $\mathbf{B}$, magnetic flux density (Wb/m^2)
- $\mathbf{H}$, magnetic field intensity (A/m)

Chapter 2

Electrostatics: Basic Aspects

2.1 Coulombs law and Electric fields

The electrical field is derived from coulombs law, in this case between charge 1 and 2.

$$\mathbf{F}_{12} = k\frac{q_1 q_2}{r^3}\mathbf{r} \tag{2.1}$$

Where $k = \frac{1}{4\pi\epsilon_0}$ in (m/F). The electric field strength is dependent on location and a more generalised representation of the force between the charges. The equation of electric field strength $\mathbf{E(r)}$ is as follows.

$$\mathbf{E(r)} = k\frac{q_1(\mathbf{r} - \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|^3} \tag{2.2}$$

Or in the case of a charge volume instead of a point charge the following holds:

$$\mathbf{E(r)} = k\int_V \rho(\mathbf{r}')\frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}d\mathbf{r}' \tag{2.3}$$

Where **r**' is the location where an observation takes place. (The cubed factor comes from the fact that an unit vector can be expressed in a fraction, multiplied with the integrand $\mathbf{r} = \dfrac{r_1 - r_2}{|r_1 - r_2|}$. Electric field density **D** can be described as $\mathbf{D} = \epsilon\mathbf{E}$ or $\mathbf{D} = \frac{E}{k}$

2.2 Electrostatic Potential

The electric field intensity can also be derived from the electrostatic potential as demonstrated below.

$$\begin{gathered} \nabla_r \frac{1}{|\mathbf{r} - \mathbf{r'}|} = -\frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \text{ as } |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} \\ \mathbf{E(r)} = -k\int_V \rho(\mathbf{r}')\nabla_r \frac{1}{|\mathbf{r} - \mathbf{r}'|^3}d\mathbf{r}' = -\nabla\phi \end{gathered} \tag{2.4}$$

This implies that the electric field is the gradient of the electrostatic potential $\phi(\mathbf{r})$. And since $\nabla \times \nabla = 0$;

$$\nabla \times \mathbf{E} = 0 \tag{2.5}$$

This tells us that the rotation of the electric field is 0; the electric field is curl-free/rotation free. Next the flux is considered. The flux is $\mathbf{E} \cdot d\mathbf{F}$ and represents the amount of vector field coming through a surface. So the equation for flux is as follows.

$$\Phi(\mathbf{r}) = k \int_{V(F)} \rho(\mathbf{r}')d\mathbf{r}' \int_F \frac{(\mathbf{r} - \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|^3} d\mathbf{F}(\mathbf{r}) \tag{2.6}$$

Where V is the volume enclosed by surface F. Assuming that **r**' is outside of V means that the following holds. (Divergence theorem)

$$\nabla \cdot \mathbf{E} = 4k\pi\rho \tag{2.7}$$

For a conductive shell with no enclosed charge no flux is apparent. This is Gauss Law, which says charge is the source of electric fields.

2.3 The Equations of Electrostatics

The three important equations of electrostatics are:

$$\begin{aligned}
&\mathbf{E} = -\nabla\phi \\
&\nabla \cdot \mathbf{E} = 4\pi k\rho(\mathbf{r}) \text{ or } \int \mathbf{E} \cdot d\mathbf{F} = 4\pi k \Sigma_i q_i \\
&\nabla \times \mathbf{E} = 0
\end{aligned} \tag{2.8}$$

From the first two equations the Poisson equation can be determined.

$$\Delta\phi = -4\pi k\rho(\mathbf{r}) = \nabla \cdot \nabla\phi \tag{2.9}$$

Or in Cartesian (x,y,z) and cylindrical coordinates(r,θ,z) respectively:

$$\begin{aligned}
\Delta\phi &= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \\
\Delta\phi &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}
\end{aligned} \tag{2.10}$$

With these it can be concluded that the potential ϕ outside of a uniformly charged sphere where $\rho(\mathbf{r}) = 0$ is $\phi = -\dfrac{constant}{r}$

2.4 Dirac's Delta Distribution

The delta function is an approximate impulse. It is called a function but in reality it is a distribution, defined as a functional by the following property

$$f(a) = \int_{-\infty}^{\infty} f(x)\delta(x-a)dx \tag{2.11}$$

The quantity $\delta(x)$ can be represented as

$$\delta(x) = \lim_{x \to 0} \frac{1}{2\sqrt{\pi a}} e^{\frac{x^2}{4a}} \tag{2.12}$$

Which entails that $\delta(x)$ is infinity for x = 0 and 0 otherwise. The following are key properties of the delta function.

- $\delta(x) = \delta(-x)$
- $\delta'(x) = -\delta'(-x)$
- $x\delta(x) = 0$
- $x\delta'(x) = -\delta(x)$
- $\delta(ax) = \frac{1}{a}\delta(x)$

When it holds that $\delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$ a solution to $\Delta G(\mathbf{r}) = \delta(\mathbf{r})$ is called Green's Function. When $G(\mathbf{r})$ is known, the inhomogenious solution of the Poisson equation $\phi(\mathbf{r})$ of $\Delta\phi = -4k\pi\rho$ follows from the relation as shown here:

$$\phi(r) = -k \int G(\mathbf{r} - \mathbf{r}') 4\pi\rho d\mathbf{r}' \tag{2.13}$$

This holds as the Laplacian ($\Delta = \nabla^2$) of this relation yields:

$$\Delta\phi(\mathbf{r}) = -k \int \delta(\mathbf{r} - \mathbf{r}') 4\pi\rho(\mathbf{r}') d\mathbf{r}' = -4k\pi\rho(\mathbf{r}) \tag{2.14}$$

For $\phi(\mathbf{r}) = k \int \frac{\rho(\mathbf{r}')d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$ to satisfy equation 2.13 the following must hold:

$$\begin{aligned} &G(\mathbf{r} - \mathbf{r}') = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} \\ &\text{Or in other words} \\ &\Delta_{\mathbf{r}} = -4\pi\delta(\mathbf{r}) \end{aligned} \tag{2.15}$$

Which makes sense for $\mathbf{r} \neq 0$ which after the following:

$$\begin{gathered} r = \sqrt{x^2 + y^2 + z^2}, \quad \triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \\ \frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\frac{x}{r^3}, \quad \frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3x^2}{r^5} \\ \triangle\frac{1}{r} = -\frac{3}{r^3} + \frac{3r^2}{r^5} = 0 \end{gathered} \tag{2.16}$$

Can be integrated in the following relation:

$$\begin{aligned} &\int_V \Delta_{\mathbf{r}} \frac{1}{\mathbf{r}} d\mathbf{r} = \int_V \nabla \cdot (\nabla \frac{1}{r}) d\mathbf{r} = \int_F (\nabla \frac{1}{r}) \cdot d\mathbf{F} \\ &= -\int_F \frac{\mathbf{r}}{r^3} \cdot d\mathbf{F} = -\int \frac{dF}{r^2} = -\int d\Omega = -4\pi \end{aligned} \tag{2.17}$$

This then verifies 2.15 holds and then shows that the Green's function is the potential of a negative unit point charge multiplied with ϵ_0.

$$\rho(\mathbf{r}') = -\epsilon_0 \delta(\mathbf{r}') \tag{2.18}$$

When this is kept in mind the equation for the electrostatic potential is:

$$\phi(\mathbf{r}) = -k \int G(\mathbf{r} - \mathbf{r}') 4\pi \rho(\mathbf{r}') d\mathbf{r}' = G(\mathbf{r}) \tag{2.19}$$

2.5 Potential Energy of Charges

When a charge q is moved from A to B against the field **E** the work done can be described as follows:

$$\begin{aligned} W &= -\int_A^B q\mathbf{E}(\mathbf{r}) \cdot d\mathbf{r} = q \int_A^B \nabla\phi \cdot d\mathbf{r} = q \int_A^B d\phi \\ &= q(*\phi(B) - \phi(A)) \end{aligned} \tag{2.20}$$

The minus sign infront of the integral represents the opposing force against the electricfield **E**

Bibliography