Let there be n vectors of pressure difference

$$dP = \sum_{i=1}^{n} dp_i$$

Each vector consists of the recorded pressure differences dp in one microphone and n is the number of the microphones.

The target of the experiment was to position the microphones in such way that we could make the exact phase delay between each couple of microphones in the same cluster. For this porpuse we have positioned them at a distance of less then 1 meter apart. The drone used was quadcopter and generated anywhere from 0 to 4000 rpm (I'm approximating here). Thus, the natural modes of frequency are

$$f = \left[0,..,\frac{4000}{60}\right] \cdot 4 \leq 270 \rightarrow \lambda = \frac{C_0}{f} = \frac{\approx 343}{270} \geq 1.3$$

Which means we were

Now that we have the data, let us make some black magic to produce the location of the target.

Some definition:

Autocorrelation:

$$R_X(\tau) = \langle X(t) X(t+\tau) \rangle$$

The power spectral density (PSD) of a stationary stochastic process can be computed by taking the Fourier transform of its autocorrelation function.

The Fourier transform of $R_X(\tau)$ is defined as:

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-2\pi j f \tau} d\tau$$

To make a connection between two variables we can use the Cross-Correlation function:

$$C_{X_1,X_2}(\tau) = \langle X_1(t) X_2(t+\tau) \rangle$$

and the cross power spectral density (CPSD):

$$S_{X_1,X_2}(f) = \int_{-\infty}^{\infty} C_{X_1,X_2}(\tau) e^{-2\pi j f \tau} d\tau$$

Now, let us talk business:

Let's say we take two vector signals and they overlap quite nicely What do we want? We want to find the phase. How can we find it if the frequency is changing in time?

If the frequency is changing in time, we need to consider the time-varying phase between the two signals. One way to do this is to use the time-frequency analysis, which allows us to analyze the spectral content of a signal as it changes over time.

The short-time Fourier transform (STFT), which decomposes a signal into its constituent frequencies at different time points using the Fourier transform.

Once we have obtained the time-frequency representation of the two signals, we can then compute the cross-spectrum between them to obtain the cross power spectral density (CPSD).

the cross-power spectral density (CPSD) between two signals $x(t)$ and $y(t)$ is defined as:

$$CPSD_{xy}(f) = E[X(f) Y^*(f)]$$

where $X(f)$ and $Y(f)$ are the Fourier transforms of $x(t)$ and $y(t)$, respectively. The complex conjugate of $Y(f)$, denoted by $Y^*(f)$, is the complex conjugate of the Fourier transform of $y(t)$ at frequency f.

The phase difference between $x(t)$ and $y(t)$ at frequency f can be defined as the phase angle of the complex number $CPSD_{xy}(f)$:

$$d\phi(f) = \text{angle}(CPSD_{xy}(f))$$

where angle(.) is the complex argument function that returns the angle of a complex number.

The CPSD can be written in terms of its magnitude and phase as:

$$CPSD_{xy}(f) = \text{abs}(CPSD_{xy}(f)) \cdot e^{j \cdot d\phi(f)}$$

Using the above expressions, we can derive the relationship between the phase difference and the CPSD:

$$d\phi(f) = \text{angle}(CPSD_{xy}(f)) \quad = \text{angle}(\text{abs}(CPSD_{xy}(f)) \cdot e^{j \cdot d\phi(f)}) \ = \text{angle}(e^{j \cdot d\phi(f)})$$

$$\underset{\text{angle}(\text{abs}(CPSD_{xy}(f)))=0}{=} d\phi(f)$$

Solving for $d\phi(f)$, we get:

$$d\phi(f) = -\frac{\text{imag}(CPSD_{xy}(f))}{\text{abs}(CPSD_{xy}(f))}$$

$$d\phi = -\frac{Im\{S_{X,Y}(f)\}}{|S_{X,Y}(f)|}$$

The phase difference is expressed in radians and represents the delay between the two signals at each frequency.

Note: This derivation is for two signals of one system. In order to make a sound mathematical rule there should be some alterations. Let's take $x(t)$ and $y(t)$ to be input and output of a fictional system $H(s)$