Recursive Definitions

Def a recursive definition of a given object has the following parts:

B. a base case, which usually defines the simplest possible such object and

R. a recursive case, which defines a more complicated object in terms of a simpler 1.

Note any recurrence relation is a recursive definition of a function.

Example 1 recursive definition of a^n where a is a nonzero real number and n is a non-negative integer.

B. (base case) $a^0 = 1$

R. (recursive case) $a^{n+1} = a \cdot a^n$

Definition The set Σ^* of strings over the alphabet Σ is defined recursively by

B. $\lambda \in \Sigma^*$ (where λ is the empty string)

R. If $w \in \Sigma^*$ and $x \in \Sigma$ then $wx \in \Sigma^*$

Note We don't write λ after it has been concatenated with another string

bobλ = bob = λbob

Example If $\Sigma = \{0, 1\}$ then Σ^* consists of all bitstrings and λ

Definition Let Σ be a set a symbols and Σ^* the set of strings formed from the symbols in Σ. Then concatenation of two strings is defined recursively as

B. If $\omega \in \Sigma^*$ then $\omega\lambda = \omega$ (λ = empty string)

R. If $\omega_1 \in \Sigma^*$ and $\omega_2 \in \Sigma^*$ and $x \in \Sigma$ then $\omega_1(\omega_2 x) = (\omega_1\omega_2)x$

Definition $s \in \Sigma^*$, a palindrome, is defined recursively as

B_1 λ is a palendrome

B_2 any symbol $a \in \Sigma$ is a palindrome

R. If x and y are palindromes then yxy is a palindrome

Example let $\Sigma = \{0, 1\}$

1 is a palendrome
0 is a palendrome } by B_2
101 is a palendrome by R
1010101 is a palendrome by R

Example

let $\Sigma = \{a, b, c, \ldots, x, y, z\}$

hannah is a palendrome

n is a palindrome B_2

λ is a palendrome B_1

$n\lambda n = nn$ is a palendrome R

a is a palendrome B_2

anna is a palendrome R

h is a palendrome B_1

hannah is a palendrome R

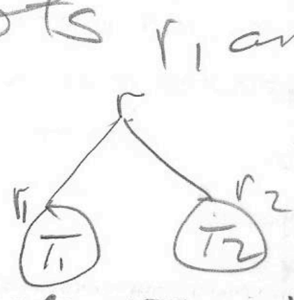
Example [binary tree a tree data structure in which each <u>node</u> has has atmost 2 child nodes]

<u>Recursive definition</u> of the set of all binary trees

<u>B_1</u> The empty tree is a binary tree

<u>B_2</u> A single vertex is a binary tree (called the root of the tree)

<u>R</u> If T_1 and T_2 are binary trees with roots r_1 and r_2, respectively, then the tree



is a binary tree

with root r. If further T_i $(i = 1, 2)$, the empty tree then there is no edge from r to T_i

Example Reverse string

Let $s \in \Sigma^*$ be a string then its reverse s^R is defined as follows

B. $\lambda^R = \lambda$

R. If s has one or more symbols, write $s = ra$ where a is a symbol and r is a string (possibly empty)

Then $s^R = (ra)^R = a r^R$

Theorem 1 If a is symbol, then $a^R = a$

insertion of empty string

Proof: $a^R = (\lambda a)^R = a \lambda^R = a \lambda$

def by R

$= a$ [removal of empty string]

Consider the string cat find its reverse

$(cat)^R = t(ca)^R$ by R

$= ta(c)^R$ by R

$=$ tac by theorem 1

Example: Give a recursive definition for all even positive integers (positive integers)

B. $2 \in X$

R. If $x \in X$, so is $x+2$

Structural Induction

A proof by Structural Induction consists of 2 steps.

1. Base case: show that the result holds for all elements specified in the basis step of the recursive definition.

2. Inductive step
Inductive hypothesis: assume that the claim holds for some object.

Show that if the statement is true for each element used to construct new elements in the recursive step of the definition, the results holds for these new elements

Example 1: Define a set $X \subseteq \mathbb{Z}$ recursively

B: $4 \in X$

R_1: If $x \in X$ then $x-12 \in X$

R_2: If $x \in X$ then $x^2 \in X$

Prove: Every element of X is devisible by 4

Proof: by structural induction

Base Case: $4 = 1 \cdot 4$ so $4 | 4$, true for base case

Inductive Step

Inductive hypothesis: Suppose that some $x \in X$ is divisible by 4 then $x = 4a$ for some integer a

Then $x^2 = (4a)^2 = 4(4a^2)$ true for x^2

$x - 12 = 4a - 12 = 4(a-3)$ true for $x-12$

Cases R_1 and R_2 always produce integers divisible by 4 if $4 | x$ and the base case gives an integer divisible by 4. So by induction all elements of X are divisible by 4.

QED

Example 2. $S \subseteq \mathbb{Z} \times \mathbb{Z}$
Recursive definition for S
B. $(0,0) \in S$
R. If $(m,n) \in S$ then $(m+2, n+3) \in S$

List 4 members of S
$(0,0)$, $(2,3)$, $(4,6)$, $(6,9)$
Using Structural Induction prove that
If $(m,n) \in S$ then $5 | (m+n)$
Proof by Structural Induction
Base Case $(0,0) \in S$ and $0 = 5 \cdot 0$
so $5 | (0+0)$, true for base case
Inductive Step
Inductive hypothesis; Suppose that
$(i,k) \in S$ and $(i+k)$ is divisible by
5 then $(i+k) = 5a$ for some integer a

$(i+2, k+3) \in S$ and $(i+2) + (k+3)$
$= (i+k) + (2+3) = 5a + 5 = 5(a+1)$
and is divisible by 5
So the B. and R. cases always
produce integers divisible by 5.
By induction for all elements $(m,n) \in S$
$5 | (m+n)$