An early spin-off of the text editor EMACS was named EINE (EINE Is Not EMACS), and one of its successors was called ZWEI (ZWEI Was EINE Initially).

Certain well-known camp songs have recursive definitions, for example, "99 bottles of beer on the wall ...." This song is a good example of top-down thinking.

Exercises 3.3

Rosen Page 158

1. Write the recurrence relation for the Fibonacci numbers (~~Definition 3.1~~) in the form of a recursive definition, with two base cases and one recursive case.
   lecture notes
2. See ~~Example 3.16.~~ Why is the first part of the definition necessary? In other words, why must λ be defined as a palindrome? (Hint: Try forming the palindrome `otto`.)
3. Give a recursive definition for the set X of all binary strings with an even number of 0's.
4. ~~See Example 3.17. Give a recursive definition for the set Y of all binary strings with *more* 0's than 1's. (Hint: Use the set X of Example 3.17 in your definition of Y.)~~
5. Define a set X of strings in the symbols 0 and 1 as follows.

   B. 0 and 1 are in X.

   $\mathbf{R_1}$. If x and y are in X, so is $xxyy$.

   $\mathbf{R_2}$. If x and y are in X, so is xyx.

   Explain why the string $01001011 \in X$ using the definition. Build up the string step by step, and justify each step by referring to the appropriate part of the definition.

   lecture notes
6. Use the definition of the reverse of a string in ~~Example 3.18.~~ to compute $(\texttt{cubs})^R$. Justify each step using the definition.
7. ~~Refer to Example 3.15. Suppose that the symbols can be compared, so for any i and j with $i \neq j$, either $a_i < a_j$ or $a_j < a_i$. Modify the definition so that it defines the set of all strings whose symbols are in increasing order.~~
8. Let K be the set of all cities that you can get to from Toronto by taking flights (or sequences of flights) on commercial airlines. Give a recursive definition of K.

9. Create your own example of an object or situation whose recursive definition is the same as the Kevin Bacon movie club in Example 3.14.

10. Define a set X of numbers as follows.

    B. $2 \in X$.

    R$_1$. If $x \in X$, so is $10x$.

    R$_2$. If $x \in X$, so is $x + 4$.

    (a) List all the elements of X that are less than 30.

    (b) Explain why there are no odd numbers in X.

11. Define a set X of integers recursively as follows.

    B. 10 is in X.

    R$_1$. If x is in X and $x > 0$, then $x - 3$ is in X.

    R$_2$. If x is in X and $x < 0$, then $x + 4$ is in X.

    List all elements of X.

12. Give a recursive definition for the set Y of all positive multiples of 5. That is,
$$Y = \{5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, \ldots\}.$$
Your definition should have a base case and a recursive part.

13. The following recursive definition defines a set Z of ordered pairs.

    B. $(2, 4)$ is in Z.

    R$_1$. If (x, y) is in Z with $x < 10$ and $y < 10$, then $(x + 1, y + 1)$ is in Z.

    R$_2$. If (x, y) is in Z with $x > 1$ and $y < 10$, then $(x - 1, y + 1)$ is in Z.

    Plot these ordered pairs in the xy-plane.

14. Give a recursive definition for the set of even integers (including both positive and negative even integers).

15. Give a recursive definition for the set of all powers of 2.

16. Define a set X recursively as follows.

    B. 3 and 7 are in X.

    R. If x and y are in X, so is $x + y$. (Here it is possible that $x = y$.)

Decide which of the following numbers are in X. Explain each decision.

(a) 24

(b) 1,000,000

(c) 11

17. Define a set X recursively as follows.

B. $12 \in X$.

R$_1$. If $x \in X$ and x is even, then $x/2 \in X$.

R$_2$. If $x \in X$ and x is odd, then $x + 1 \in X$.

List all the elements of X.

18. Give a recursive definition for the set X of all natural numbers that are one or two more than a multiple of 10. In other words, give a recursive definition for the set $\{1, 2, 11, 12, 21, 22, 31, 32, \ldots\}$.

19. Let S be a set of sets with the following recursive definition.

B. $\emptyset \in S$.

R. If $X \subseteq S$, then $X \in S$.

(a) List three different elements of S.

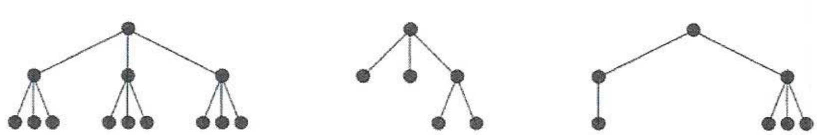
(b) Explain why S has infinitely many elements.

20. In ~~Example 3.22~~ lecture notes, we gave a recursive definition of a binary tree. Suppose we modify this definition by deleting part $\mathbf{B_1}$, so that an empty tree is not a binary tree. A tree satisfying this revised definition is called a *full binary tree.*

(a) Give an example of a full binary tree with five nodes.

(b) Give an example of a binary tree with five nodes that is not a full binary tree.

21. A *ternary* tree is a tree where every parent node has (at most) three child nodes. For example, the following are ternary trees.



Give a recursive definition for a ternary tree.

19. Find a formula for the area of (the black part of) $S(n)$, the nth term in the sequence of shapes whose limit is the Sierpinski gasket fractal in Figure 3.11 on page 185. Assume that $S(1)$ is a black equilateral triangle with area 1. Prove that your formula is correct.

20. Let X be the set defined in Example 3.21. p 177

    (a) Prove, by induction on n, that $2n+1 \in X$ for all $n \geq 0$. (This shows that X contains all the odd natural numbers.)

    (b) Prove by induction that every element in X is odd. (This shows that the set of all odd natural numbers contains X.)

    (c) Together, what do (a) and (b) show?

21. Define a set X recursively as follows.

    B. $2 \in X$.

    R. If $x \in X$, so is $x + 10$.

    Use induction to prove that every element of X is even.

22. Define a set X recursively as follows.

    B. 3 and 7 are in X.

    R. If x and y are in X, so is $x + y$. (Here it is possible that $x = y$.)

    Prove that, for every natural number $n \geq 12$, $n \in X$. (Hint: For the base case, show that 12, 13, and 14 are in X.)

23. Define a *Q-sequence* recursively as follows.

    B. $x,\ 4 - x$ is a Q-sequence for any real number x.

    R. If $x_1, x_2, \ldots, x_j$ and $y_1, y_2, \ldots, y_k$ are Q-sequences, so is

    $$x_1 - 1,\ x_2, \ldots, x_j, y_1, y_2, \ldots,\ y_k - 3.$$

    Use structural induction (i.e., induction on the recursive definition) to prove that the sum of the numbers in any Q-sequence is 4.

24. In the game of chess, a knight moves by jumping to a square that is two units away in one direction and one unit away in another. For example, in Figure 3.18, the knight at K can move to any of the squares marked with an asterisk $*$. Prove by induction that a knight can move from any square