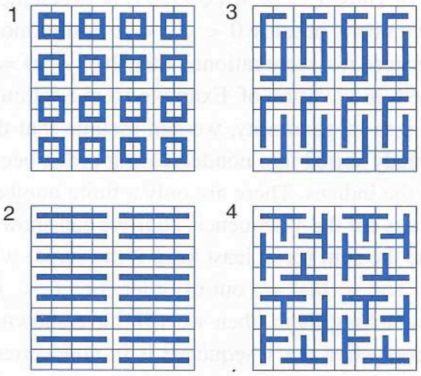
b) The picture shows tilings for the first four patterns.



To show that pattern 5 cannot tile the checkerboard, label the squares from 1 to 64, one row at a time from the top, from left to right in each row. Thus, square 1 is the upper left corner, and square 64 is the lower right. Suppose we did have a tiling. By symmetry and without loss of generality, we may suppose that the tile is positioned in the upper left corner, covering squares 1, 2, 10, and 11. This forces a tile to be adjacent to it on the right, covering squares 3, 4, 12, and 13. Continue in this manner and we are forced to have a tile covering squares 6, 7, 15, and 16. This makes it impossible to cover square 8. Thus, no tiling is possible.

Supplementary Exercises

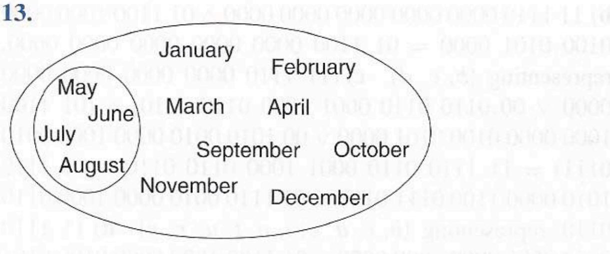
1. a) $q \to p$ **b)** $q \wedge p$ **c)** $\neg q \vee \neg p$ **d)** $q \leftrightarrow p$ **3. a)** The proposition cannot be false unless $\neg p$ is false, so p is true. If p is true and q is true, then $\neg q \wedge (p \to q)$ is false, so the conditional statement is true. If p is true and q is false, then $p \to q$ is false, so $\neg q \wedge (p \to q)$ is false and the conditional statement is true. **b)** The proposition cannot be false unless q is false. If q is false and p is true, then $(p \vee q) \wedge \neg p$ is false, and the conditional statement is true. If q is false and p is false, then $(p \vee q) \wedge \neg p$ is false, and the conditional statement is true. **5.** $\neg q \to \neg p$; $p \to q$; $\neg p \to \neg q$ **7.** $(p \wedge q \wedge r \wedge \neg s) \vee (p \wedge q \wedge \neg r \wedge s) \vee (p \wedge \neg q \wedge r \wedge s) \vee (\neg p \wedge q \wedge r \wedge s)$ **9.** Translating these statements into symbols, using the obvious letters, we have $\neg t \to \neg g$, $\neg g \to \neg q$, $r \to q$, and $\neg t \wedge r$. Assume the statements are consistent. The fourth statement tells us that $\neg t$ must be true. Therefore by modus ponens with the first statement, we know that $\neg g$ is true, hence (from the second statement), that $\neg q$ is true. Also, the fourth statement tells us that r must be true, and so again modus ponens (third statement) makes q true. This is a contradiction: $q \wedge \neg q$. Thus the statements are inconsistent. **11.** Reject-accept-reject-accept, accept-accept-accept-accept, accept-accept-reject-accept, reject-reject-reject-reject, reject-reject-accept-reject, and reject-accept-accept-accept **13.** Aaron is a knave and Crystal is a knight; it cannot be determined what Bohan is. **15.** Brenda **17.** The premises cannot both be true, because they are contradictory. Therefore it is (vacuously) true that whenever all the premises are true, the conclusion is also true, which by definition makes this a valid argument. Because the premises are not both true, we cannot conclude that the conclusion is true. **19.** Use the same propositions as were given in Section 1.3 for a 9×9 Sudoku puzzle, with the variables indexed from 1 to 16, instead of from 1 to 9, and with a similar change for the propositions for the 4×4 blocks: $\bigwedge_{r=0}^{3} \bigwedge_{s=0}^{3} \bigwedge_{n=1}^{16} \bigvee_{i=1}^{4} \bigvee_{j=1}^{4} p(4r+i, 4s+j, n)$. **21. a)** F **b)** T **c)** F **d)** T **e)** F **f)** T **23.** Many answers are possible. One example is United States senators. **25.** $\forall x \exists y \exists z\, (y \neq z \wedge \forall w (P(w, x) \leftrightarrow (w = y \vee w = z)))$ **27. a)** $\neg \exists x P(x)$ **b)** $\exists x (P(x) \wedge \forall y (P(y) \to y = x))$ **c)** $\exists x_1 \exists x_2 (P(x_1) \wedge P(x_2) \wedge x_1 \neq x_2 \wedge \forall y (P(y) \to (y = x_1 \vee y = x_2)))$ **d)** $\exists x_1 \exists x_2 \exists x_3 (P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3 \wedge \forall y (P(y) \to (y = x_1 \vee y = x_2 \vee y = x_3)))$ **29.** Suppose that $\exists x (P(x) \to Q(x))$ is true. Then either $Q(x_0)$ is true for some x_0, in which case $\forall x P(x) \to \exists x Q(x)$ is true; or $P(x_0)$ is false for some x_0, in which case $\forall x P(x) \to \exists x Q(x)$ is true. Conversely, suppose that $\exists x (P(x) \to Q(x))$ is false. That means that $\forall x (P(x) \wedge \neg Q(x))$ is true, which implies $\forall x P(x)$ and $\forall x (\neg Q(x))$. This latter proposition is equivalent to $\neg \exists x Q(x)$. Thus, $\forall x P(x) \to \exists x Q(x)$ is false. **31.** No **33.** $\forall x \forall z \exists y\, T(x, y, z)$, where $T(x, y, z)$ is the statement that student x has taken class y in department z, where the domains are the set of students in the class, the set of courses at this university, and the set of departments in the school of mathematical sciences **35.** $\exists! x \exists! y\, T(x, y)$ and $\exists x \forall z ((\exists y \forall w (T(z, w) \leftrightarrow w = y)) \leftrightarrow z = x)$, where $T(x, y)$ means that student x has taken class y and the domain is all students in this class **37.** $P(a) \to Q(a)$ and $Q(a) \to R(a)$ by universal instantiation; then $\neg Q(a)$ by modus tollens and $\neg P(a)$ by modus tollens **39.** We give a proof by contraposition and show that if $\sqrt{x}$ is rational, then x is rational, assuming throughout that $x \geq 0$. Suppose that $\sqrt{x} = p/q$ is rational, $q \neq 0$. Then $x = (\sqrt{x})^2 = p^2/q^2$ is also rational (q^2 is again nonzero). **41.** We can give a constructive proof by letting $m = 10^{500} + 1$. Then $m^2 = (10^{500} + 1)^2 > (10^{500})^2 = 10^{1000}$. **43.** 23 cannot be written as the sum of eight cubes. **45.** 223 cannot be written as the sum of 36 fifth powers.

CHAPTER 2

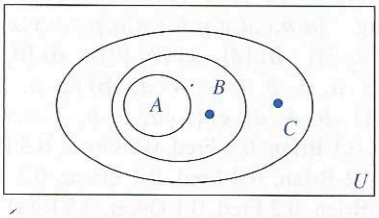
Section 2.1

1. a) $\{-1, 1\}$ **b)** $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ **c)** $\{0, 1, 4, 9, 16, 25, 36, 49, 64, 81\}$ **d)** $\emptyset$ **3. a)** The second is a subset of the first, but the first is not a subset of the second. **b)** Neither is a subset of the other. **c)** The first is a subset of the second, but the second is not a subset of the first. **5. a)** Yes **b)** No **c)** No **7. a)** Yes **b)** No **c)** Yes **d)** No **e)** No **f)** No **9. a)** False **b)** False **c)** False **d)** True **e)** False **f)** False **g)** True **11. a)** True **b)** True **c)** False **d)** True **e)** True **f)** False

13.



15. The dots in certain regions indicate that those regions are not empty.



17. Suppose that $x \in A$. Because $A \subseteq B$, this implies that $x \in B$. Because $B \subseteq C$, we see that $x \in C$. Because $x \in A$ implies that $x \in C$, it follows that $A \subseteq C$. **19. a)** 1 **b)** 1 **c)** 2 **d)** 3 **21. a)** $\{\emptyset, \{a\}\}$ **b)** $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ **c)** $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$ **23. a)** 8 **b)** 16 **c)** 2 **25.** For the "if" part, given $A \subseteq B$, we want to show that that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, i.e., if $C \subseteq A$ then $C \subseteq B$. But this follows directly from Exercise 17. For the "only if" part, given that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, we want to show that $A \subseteq B$. Suppose $a \in A$. Then $\{a\} \subseteq A$, so $\{a\} \in \mathcal{P}(A)$. Since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, it follows that $\{a\} \in \mathcal{P}(B)$, which means that $\{a\} \subseteq B$. But this implies $a \in B$, as desired. **27. a)** $\{(a, y), (b, y), (c, y), (d, y), (a, z), (b, z), (c, z), (d, z)\}$ **b)** $\{(y, a), (y, b), (y, c), (y, d), (z, a), (z, b), (z, c), (z, d)\}$ **29.** The set of triples (a, b, c), where a is an airline and b and c are cities. A useful subset of this set is the set of triples (a, b, c) for which a flies between b and c. **31.** $\emptyset \times A = \{(x, y) \mid x \in \emptyset \text{ and } y \in A\} = \emptyset = \{(x, y) \mid x \in A \text{ and } y \in \emptyset\} = A \times \emptyset$ **33. a)** $\{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (1, 3), (3, 0), (3, 1), (3, 3)\}$ **b)** $\{(1, 1), (1, 2), (1, a), (1, b), (2, 1), (2, 2), (2, a), (2, b), (a, 1), (a, 2), (a, a), (a, b), (b, 1), (b, 2), (b, a), (b, b)\}$ **35.** mn **37.** m^n **39.** The elements of $A \times B \times C$ consist of 3-tuples (a, b, c), where $a \in A$, $b \in B$, and $c \in C$, whereas the elements of $(A \times B) \times C$ look like $((a, b), c)$—ordered pairs, the first coordinate of which is again an ordered pair. **41. a)** The square of a real number is never -1. True **b)** There exists an integer whose square is 2. False **c)** The square of every integer is positive. False **d)** There is a real number equal to its own square. True **43. a)** $\{-1, 0, 1\}$ **b)** $\mathbf{Z} - \{0, 1\}$ **c)** $\emptyset$ **45.** We must show that $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ if and only if $a = c$ and $b = d$. The "if" part is immediate. So assume these two sets are equal. First, consider the case when $a \neq b$. Then $\{\{a\}, \{a, b\}\}$ contains exactly two elements, one of which contains one element. Thus, $\{\{c\}, \{c, d\}\}$ must have the same property, so $c \neq d$ and $\{c\}$ is the element containing exactly one element. Hence, $\{a\} = \{c\}$, which implies that $a = c$. Also, the two-element sets $\{a, b\}$ and $\{c, d\}$ must be equal. Because $a = c$ and $a \neq b$, it follows that $b = d$. Second, suppose that $a = b$. Then $\{\{a\}, \{a, b\}\} = \{\{a\}\}$, a set with one element. Hence, $\{\{c\}, \{c, d\}\}$ has only one element, which can happen only when $c = d$, and the set is $\{\{c\}\}$. It then follows that $a = c$ and $b = d$. **47.** Let $S = \{a_1, a_2, \ldots, a_n\}$. Represent each subset of S with a bit string of length n, where the ith bit is 1 if and only if $a_i \in S$. To generate all subsets of S, list all 2^n bit strings of length n (for instance, in increasing order), and write down the corresponding subsets.

Section 2.2

1. a) The set of students who live within one mile of school and walk to classes **b)** The set of students who live within one mile of school or walk to classes (or do both) **c)** The set of students who live within one mile of school but do not walk to classes **d)** The set of students who walk to classes but live more than one mile away from school **3. a)** $\{0,1,2,3,4,5,6\}$ **b)** $\{3\}$ **c)** $\{1, 2, 4, 5\}$ **d)** $\{0, 6\}$ **5.** $\overline{\overline{A}} = \{x \mid \neg(x \in \overline{A})\} = \{x \mid \neg(\neg x \in A)\} = \{x \mid x \in A\} = A$ **7. a)** $A \cup U = \{x \mid x \in A \vee x \in U\} = \{x \mid x \in A \vee \mathbf{T}\} = \{x \mid \mathbf{T}\} = U$ **b)** $A \cap \emptyset = \{x \mid x \in A \wedge x \in \emptyset\} = \{x \mid x \in A \wedge \mathbf{F}\} = \{x \mid \mathbf{F}\} = \emptyset$ **9. a)** $A \cup \overline{A} = \{x \mid x \in A \vee x \notin A\} = U$ **b)** $A \cap \overline{A} = \{x \mid x \in A \wedge x \notin A\} = \emptyset$ **11. a)** $A \cup B = \{x \mid x \in A \vee x \in B\} = \{x \mid x \in B \vee x \in A\} = B \cup A$ **b)** $A \cap B = \{x \mid x \in A \wedge x \in B\} = \{x \mid x \in B \wedge x \in A\} = B \cap A$ **13.** Suppose $x \in A \cap (A \cup B)$. Then $x \in A$ and $x \in A \cup B$ by the definition of intersection. Because $x \in A$, we have proved that the left-hand side is a subset of the right-hand side. Conversely, let $x \in A$. Then by the definition of union, $x \in A \cup B$ as well. Therefore $x \in A \cap (A \cup B)$ by the definition of intersection, so the right-hand side is a subset of the left-hand side. **15. a)** $x \in \overline{A \cup B} \equiv x \notin A \cup B \equiv \neg(x \in A \vee x \in B) \equiv \neg(x \in A) \wedge \neg(x \in B) \equiv x \notin A \wedge x \notin B \equiv x \in \overline{A} \wedge x \in \overline{B} \equiv x \in \overline{A} \cap \overline{B}$

b)

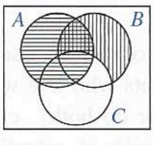
A	B	$A \cup B$	$\overline{A \cup B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$
1	1	1	0	0	0	0
1	0	1	0	0	1	0
0	1	1	0	1	0	0
0	0	0	1	1	1	1

17. a) $x \in \overline{A \cap B \cap C} \equiv x \notin A \cap B \cap C \equiv x \notin A \vee x \notin B \vee x \notin C \equiv x \in \overline{A} \vee x \in \overline{B} \vee x \in \overline{C} \equiv x \in \overline{A} \cup \overline{B} \cup \overline{C}$

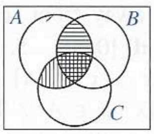
b)

A	B	C	$A \cap B \cap C$	$\overline{A \cap B \cap C}$	$\overline{A}$	$\overline{B}$	$\overline{C}$	$\overline{A} \cup \overline{B} \cup \overline{C}$
1	1	1	1	0	0	0	0	0
1	1	0	0	1	0	0	1	1
1	0	1	0	1	0	1	0	1
1	0	0	0	1	0	1	1	1
0	1	1	0	1	1	0	0	1
0	1	0	0	1	1	0	1	1
0	0	1	0	1	1	1	0	1
0	0	0	0	1	1	1	1	1

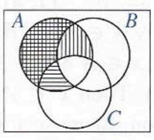
19. a) Both sides equal $\{x \mid x \in A \wedge x \notin B\}$. **b)** $A = A \cap U = A \cap (B \cup \overline{B}) = (A \cap B) \cup (A \cap \overline{B})$ **21.** $x \in A \cup (B \cup C) \equiv (x \in A) \vee (x \in (B \cup C)) \equiv (x \in A) \vee (x \in B \vee x \in C) \equiv (x \in A \vee x \in B) \vee (x \in C) \equiv x \in (A \cup B) \cup C$ **23.** $x \in A \cup (B \cap C) \equiv (x \in A) \vee (x \in (B \cap C)) \equiv (x \in A) \vee (x \in B \wedge x \in C) \equiv (x \in A \vee x \in B) \wedge (x \in A \vee x \in C) \equiv x \in (A \cup B) \cap (A \cup C)$ **25. a)** $\{4,6\}$ **b)** $\{0,1,2,3,4,5,6,7,8,9,10\}$ **c)** $\{4, 5, 6, 8, 10\}$ **d)** $\{0,2,4, 5,6,7,8,9,10\}$ **27. a)** The double-shaded portion is the desired set.



b) The desired set is the entire shaded portion.



c) The desired set is the entire shaded portion.



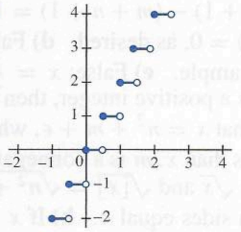
29. a) $B \subseteq A$ **b)** $A \subseteq B$ **c)** $A \cap B = \emptyset$ **d)** Nothing, because this is always true **e)** $A = B$ **31.** $A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B) \equiv \forall x(x \notin B \rightarrow x \notin A) \equiv \forall x(x \in \overline{B} \rightarrow x \in \overline{A}) \equiv \overline{B} \subseteq \overline{A}$ **33.** The set of students who are computer science majors but not mathematics majors or who are mathematics majors but not computer science majors **35.** An element is in $(A \cup B) - (A \cap B)$ if it is in the union of A and B but not in the intersection of A and B, which means that it is in either A or B but not in both A and B. This is exactly what it means for an element to belong to $A \oplus B$. **37. a)** $A \oplus A = (A - A) \cup (A - A) = \emptyset \cup \emptyset = \emptyset$ **b)** $A \oplus \emptyset = (A - \emptyset) \cup (\emptyset - A) = A \cup \emptyset = A$ **c)** $A \oplus U = (A - U) \cup (U - A) = \emptyset \cup \overline{A} = \overline{A}$ **d)** $A \oplus \overline{A} = (A - \overline{A}) \cup (\overline{A} - A) = A \cup \overline{A} = U$ **39.** $B = \emptyset$ **41.** Yes **43.** Yes **45.** If $A \cup B$ were finite, then it would have n elements for some natural number n. But A already has more than n elements, because it is infinite, and $A \cup B$ has all the elements that A has, so $A \cup B$ has more than n elements. This contradiction shows that $A \cup B$ must be infinite. **47. a)** $\{1, 2, 3, \ldots, n\}$ **b)** $\{1\}$ **49. a)** A_n **b)** $\{0, 1\}$ **51. a)** $\mathbf{Z}$, $\{-1, 0, 1\}$ **b)** $\mathbf{Z} - \{0\}$, $\emptyset$ **c)** $\mathbf{R}$, $[-1, 1]$ **d)** $[1, \infty)$, $\emptyset$ **53. a)** $\{1, 2, 3, 4, 7, 8, 9, 10\}$ **b)** $\{2, 4, 5, 6, 7\}$ **c)** $\{1, 10\}$ **55.** The bit in the ith position of the bit string of the difference of two sets is 1 if the ith bit of the first string is 1 and the ith bit of the second string is 0, and is 0 otherwise. **57. a)** 11 1110 0000 0000 0000 0000 $\vee$ 01 1100 1000 0000 0100 0101 0000 = 11 1110 1000 0000 0100 0101 0000, representing $\{a, b, c, d, e, g, p, t, v\}$ **b)** 11 1110 0000 0000 0000 0000 0000 $\wedge$ 01 1100 1000 0000 0100 0101 0000 = 01 1100 0000 0000 0000 0000 0000, representing $\{b, c, d\}$ **c)** (11 1110 0000 0000 0000 0000 0000 $\vee$ 00 0110 0110 0001 1000 0110 0110) $\wedge$ (01 1100 1000 0000 0100 0101 0000 $\vee$ 00 1010 0010 0000 1000 0010 0111) = 11 1110 0110 0001 1000 0110 0110 $\wedge$ 01 1110 1010 0000 1100 0111 0111 = 01 1110 0010 0000 1000 0110 0110, representing $\{b, c, d, e, i, o, t, u, x, y\}$ **d)** 11 1110 0000 0000 0000 0000 0000 $\vee$ 01 1100 1000 0000 0100 0101 0000 $\vee$ 00 1010 0010 0000 1000 0010 0111 $\vee$ 00 0110 0110 0001 1000 0110 0110 = 11 1110 1110 0001 1100 0111 0111, representing $\{a,b,c,d,e,g,h,i,n,o,p,t,u,v,x,y,z\}$ **59. a)** $\{1, 2, 3, \{1, 2, 3\}\}$ **b)** $\{\emptyset\}$ **c)** $\{\emptyset, \{\emptyset\}\}$ **d)** $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ **61. a)** $\{3 \cdot a,\ 3 \cdot b,\ 1 \cdot c,\ 4 \cdot d\}$ **b)** $\{2 \cdot a,\ 2 \cdot b\}$ **c)** $\{1 \cdot a,\ 1 \cdot c\}$ **d)** $\{1 \cdot b,\ 4 \cdot d\}$ **e)** $\{5 \cdot a,\ 5 \cdot b,\ 1 \cdot c, 4 \cdot d\}$ **63.** $\overline{F}$ = {0.4 Alice, 0.1 Brian, 0.6 Fred, 0.9 Oscar, 0.5 Rita}, $\overline{R}$ = {0.6 Alice, 0.2 Brian, 0.8 Fred, 0.1 Oscar, 0.3 Rita} **65.** {0.4 Alice, 0.8 Brian, 0.2 Fred, 0.1 Oscar, 0.5 Rita}

Section 2.3

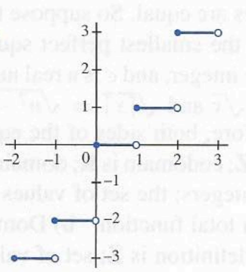
1. a) $f(0)$ is not defined. **b)** $f(x)$ is not defined for $x < 0$. **c)** $f(x)$ is not well-defined because there are two distinct values assigned to each x. **3. a)** Not a function **b)** A function **c)** Not a function **5. a)** Domain the set of bit strings; range the set of integers **b)** Domain the set of bit strings; range the set of even nonnegative integers **c)** Domain the set of bit strings; range the set of nonnegative integers not exceeding 7 **d)** Domain the set of positive integers; range the set of squares of positive integers $= \{1, 4, 9, 16, \ldots\}$ **7. a)** Domain $\mathbf{Z}^+ \times \mathbf{Z}^+$; range $\mathbf{Z}^+$ **b)** Domain $\mathbf{Z}^+$; range $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ **c)** Domain the set of bit strings; range $\mathbf{N}$ **d)** Domain the set of bit strings; range $\mathbf{N}$ **9. a)** 1 **b)** 0 **c)** 0 **d)** -1 **e)** 3 **f)** -1 **g)** 2 **h)** 1 **11.** Only the function in part (a) **13.** Only the functions in parts (a) and (d) **15. a)** Onto **b)** Not onto **c)** Onto **d)** Not onto **e)** Onto **17. a)** Depends on whether teachers share offices **b)** One-to-one assuming only one teacher per bus **c)** Most likely not one-to-one, especially if salary is set by a collective bargaining agreement **d)** One-to-one **19.** Answers will vary. **a)** Set of offices at the school; probably not onto **b)** Set of buses going on the trip; onto, assuming every bus gets a teacher chaperone **c)** Set of real numbers; not onto **d)** Set of strings of nine digits with hyphens after third and fifth digits; not onto **21. a)** The function $f(x)$ with $f(x) = 3x + 1$ when $x \geq 0$ and $f(x) = -3x + 2$ when $x < 0$ **b)** $f(x) = |x| + 1$ **c)** The function $f(x)$ with $f(x) = 2x + 1$ when $x \geq 0$ and $f(x) = -2x$ when $x < 0$ **d)** $f(x) = x^2 + 1$ **23. a)** Yes **b)** No **c)** Yes **d)** No **25.** Suppose that f is strictly decreasing. This means that $f(x) > f(y)$ whenever $x < y$. To show that g is strictly increasing, suppose that $x < y$. Then $g(x) = 1/f(x) < 1/f(y) = g(y)$. Conversely, suppose that g is strictly increasing. This means that $g(x) < g(y)$ whenever $x < y$. To show that f is strictly decreasing, suppose that $x < y$. Then $f(x) = 1/g(x) > 1/g(y) = f(y)$. **27. a)** Let f be a given strictly decreasing function from $\mathbf{R}$ to itself. If

$a < b$, then $f(a) > f(b)$; if $a > b$, then $f(a) < f(b)$. Thus if $a \neq b$, then $f(a) \neq f(b)$. **b)** Answers will vary; for example, $f(x) = 0$ for $x < 0$ and $f(x) = -x$ for $x \geq 0$. **29.** The function is not one-to-one, so it is not invertible. On the restricted domain, the function is the identity function on the nonnegative real numbers, $f(x) = x$, so it is its own inverse. **31. a)** $f(S) = \{0, 1, 3\}$ **b)** $f(S) = \{0, 1, 3, 5, 8\}$ **c)** $f(S) = \{0, 8, 16, 40\}$ **d)** $f(S) = \{1, 12, 33, 65\}$ **33. a)** Let x and y be distinct elements of A. Because g is one-to-one, $g(x)$ and $g(y)$ are distinct elements of B. Because f is one-to-one, $f(g(x)) = (f \circ g)(x)$ and $f(g(y)) = (f \circ g)(y)$ are distinct elements of C. Hence, $f \circ g$ is one-to-one. **b)** Let $y \in C$. Because f is onto, $y = f(b)$ for some $b \in B$. Now because g is onto, $b = g(x)$ for some $x \in A$. Hence, $y = f(b) = f(g(x)) = (f \circ g)(x)$. It follows that $f \circ g$ is onto. **35.** No. For example, suppose that $A = \{a\}$, $B = \{b, c\}$, and $C = \{d\}$. Let $g(a) = b$, $f(b) = d$, and $f(c) = d$. Then f and $f \circ g$ are onto, but g is not. **37.** $(f + g)(x) = x^2 + x + 3$, $(fg)(x) = x^3 + 2x^2 + x + 2$ **39.** f is one-to-one because $f(x_1) = f(x_2) \rightarrow ax_1 + b = ax_2 + b \rightarrow ax_1 = ax_2 \rightarrow x_1 = x_2$. f is onto because $f((y - b)/a) = y$. $f^{-1}(y) = (y - b)/a$. **41. a)** $A = B = \mathbf{R}$, $S = \{x \mid x > 0\}$, $T = \{x \mid x < 0\}$, $f(x) = x^2$ **b)** It suffices to show that $f(S) \cap f(T) \subseteq f(S \cap T)$. Let $y \in B$ be an element of $f(S) \cap f(T)$. Then $y \in f(S)$, so $y = f(x_1)$ for some $x_1 \in S$. Similarly, $y = f(x_2)$ for some $x_2 \in T$. Because f is one-to-one, it follows that $x_1 = x_2$. Therefore $x_1 \in S \cap T$, so $y \in f(S \cap T)$. **43. a)** $\{x \mid 0 \leq x < 1\}$ **b)** $\{x \mid -1 \leq x < 2\}$ **c)** $\emptyset$ **45.** $f^{-1}(\overline{S}) = \{x \in A \mid f(x) \notin S\} = \overline{\{x \in A \mid f(x) \in S\}} = \overline{f^{-1}(S)}$ **47.** Let $x = \lfloor x \rfloor + \epsilon$, where ϵ is a real number with $0 \leq \epsilon < 1$. If $\epsilon < \frac{1}{2}$, then $\lfloor x \rfloor - 1 < x - \frac{1}{2} < \lfloor x \rfloor$, so $\lceil x - \frac{1}{2} \rceil = \lfloor x \rfloor$ and this is the integer closest to x. If $\epsilon > \frac{1}{2}$, then $\lfloor x \rfloor < x - \frac{1}{2} < \lfloor x \rfloor + 1$, so $\lceil x - \frac{1}{2} \rceil = \lfloor x \rfloor + 1$ and this is the integer closest to x. If $\epsilon = \frac{1}{2}$, then $\lceil x - \frac{1}{2} \rceil = \lfloor x \rfloor$, which is the smaller of the two integers that surround x and are the same distance from x. **49.** Write the real number x as $\lfloor x \rfloor + \epsilon$, where ϵ is a real number with $0 \leq \epsilon < 1$. Because $\epsilon = x - \lfloor x \rfloor$, it follows that $0 \leq -\lfloor x \rfloor < 1$. The first two inequalities, $x - 1 < \lfloor x \rfloor$ and $\lfloor x \rfloor \leq x$, follow directly. For the other two inequalities, write $x = \lceil x \rceil - \epsilon'$, where $0 \leq \epsilon' < 1$. Then $0 \leq \lceil x \rceil - x < 1$, and the desired inequality follows. **51. a)** If $x < n$, because $\lfloor x \rfloor \leq x$, it follows that $\lfloor x \rfloor < n$. Suppose that $x \geq n$. By the definition of the floor function, it follows that $\lfloor x \rfloor \geq n$. This means that if $\lfloor x \rfloor < n$, then $x < n$. **b)** If $n < x$, then because $x \leq \lceil x \rceil$, it follows that $n \leq \lceil x \rceil$. Suppose that $n \geq x$. By the definition of the ceiling function, it follows that $\lceil x \rceil \leq n$. This means that if $n < \lceil x \rceil$, then $n < x$. **53.** If n is even, then $n = 2k$ for some integer k. Thus, $\lfloor n/2 \rfloor = \lfloor k \rfloor = k = n/2$. If n is odd, then $n \;\, 2k + 1$ for some integer k. Thus, $\lfloor n/2 \rfloor = \lfloor k + \frac{1}{2} \rfloor = k = (n - 1)/2$. **55.** Assume that $x \geq 0$. The left-hand side is $\lceil -x \rceil$ and the right-hand side is $-\lfloor x \rfloor$. If x is an integer, then both sides equal $-x$. Otherwise, let $x = n + \epsilon$, where n is a natural number and ϵ is a real number with $0 \leq \epsilon < 1$. Then $\lceil -x \rceil = \lceil -n - \epsilon \rceil = -n$ and $-\lfloor x \rfloor = -\lfloor n + \epsilon \rfloor = -n$ also. When $x < 0$, the equation also holds because it can be obtained by substituting $-x$ for x. **57.** $\lceil b \rceil - \lfloor a \rfloor - 1$ **59. a)** 1 **b)** 3 **c)** 126 **d)** 3600 **61. a)** 100 **b)** 256 **c)** 1030 **d)** 30,200

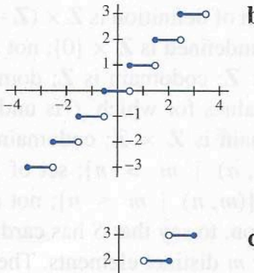
63.



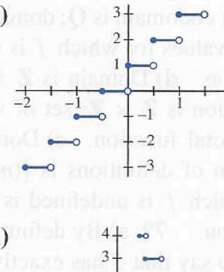
65.



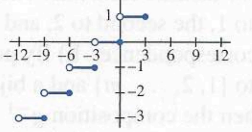
67. a)



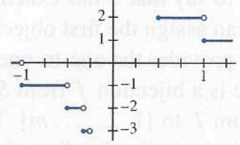
b)



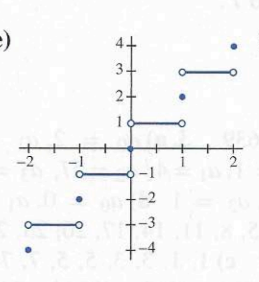
c)



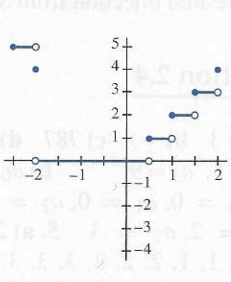
d)



e)



f)



g) See part (a). **69.** $f^{-1}(y) = (y - 1)^{1/3}$ **71. a)** $f_{A \cap B}(x) = 1 \leftrightarrow x \in A \cap B \leftrightarrow x \in A$ and $x \in B \leftrightarrow f_A(x) = 1$ and $f_B(x) = 1 \leftrightarrow f_A(x) f_B(x) = 1$ **b)** $f_{A \cup B}(x) = 1 \leftrightarrow x \in A \cup B \leftrightarrow x \in A$ or $x \in B \leftrightarrow f_A(x) = 1$ or $f_B(x) = 1 \leftrightarrow f_A(x) + f_B(x) - f_A(x) f_B(x) = 1$ **c)** $f_{\overline{A}}(x) = 1 \leftrightarrow x \in \overline{A} \leftrightarrow x \notin A \leftrightarrow f_A(x) = 0 \leftrightarrow 1 - f_A(x) = 1$ **d)** $f_{A \oplus B}(x) = 1 \leftrightarrow x \in A \oplus B \leftrightarrow (x \in A$ and $x \notin B)$ or $(x \notin A$ and $x \in B) \leftrightarrow f_A(x) + f_B(x) - 2 f_A(x) f_B(x) = 1$ **73. a)** True; because $\lfloor x \rfloor$ is already an integer, $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$. **b)** False; $x = \frac{1}{2}$ is a counterexample. **c)** True; if x or y is an integer, then by property 4b in Table 1, the difference is 0. If neither x nor y is an integer, then $x = n + \epsilon$ and $y = m + \delta$, where n and m are integers and ϵ and δ are positive real numbers less than 1. Then $m + n < x + y < m + n + 2$, so $\lceil x + y \rceil$ is either $m+n+1$ or $m+n+2$. Therefore, the given expression is either $(n + 1) + (m + 1) - (m + n + 1) = 1$ or $(n + 1) + (m + 1) - (m + n + 2) = 0$, as desired. **d)** False; $x = \frac{1}{4}$ and $y = 3$ is a counterexample. **e)** False; $x = \frac{1}{2}$ is a counterexample. **75. a)** If x is a positive integer, then the two sides are equal. So suppose that $x = n^2 + m + \epsilon$, where n^2 is the largest perfect square less than x, m is a nonnegative integer, and $0 < \epsilon \le 1$. Then both $\sqrt{x}$ and $\sqrt{\lfloor x \rfloor} = \sqrt{n^2 + m}$ are between n and $n + 1$, so both sides equal n. **b)** If x is a positive integer, then the two sides are equal. So suppose that $x = n^2 - m - \epsilon$, where n^2 is the smallest perfect square greater than x, m is a nonnegative integer, and ϵ is a real number with $0 < \epsilon \le 1$. Then both $\sqrt{x}$ and $\sqrt{\lceil x \rceil} = \sqrt{n^2 - m}$ are between $n - 1$ and n. Therefore, both sides of the equation equal n. **77. a)** Domain is **Z**; codomain is **R**; domain of definition is the set of nonzero integers; the set of values for which f is undefined is $\{0\}$; not a total function. **b)** Domain is **Z**; codomain is **Z**; domain of definition is **Z**; set of values for which f is undefined is $\emptyset$; total function. **c)** Domain is $\mathbf{Z} \times \mathbf{Z}$; codomain is **Q**; domain of definition is $\mathbf{Z} \times (\mathbf{Z} - \{0\})$; set of values for which f is undefined is $\mathbf{Z} \times \{0\}$; not a total function. **d)** Domain is $\mathbf{Z} \times \mathbf{Z}$; codomain is **Z**; domain of definition is $\mathbf{Z} \times \mathbf{Z}$; set of values for which f is undefined is $\emptyset$; total function. **e)** Domain is $\mathbf{Z} \times \mathbf{Z}$; codomain is **Z**; domain of definitions is $\{(m, n) \mid m > n\}$; set of values for which f is undefined is $\{(m, n) \mid m \le n\}$; not a total function. **79. a)** By definition, to say that S has cardinality m is to say that S has exactly m distinct elements. Therefore we can assign the first object to 1, the second to 2, and so on. This provides the one-to-one correspondence. **b)** By part (a), there is a bijection f from S to $\{1, 2, \dots, m\}$ and a bijection g from T to $\{1, 2, \dots, m\}$. Then the composition $g^{-1} \circ f$ is the desired bijection from S to T.

Section 2.4

1. a) 3 **b)** −1 **c)** 787 **d)** 2639 **3. a)** $a_0 = 2, a_1 = 3, a_2 = 5, a_3 = 9$ **b)** $a_0 = 1, a_1 = 4, a_2 = 27, a_3 = 256$ **c)** $a_0 = 0, a_1 = 0, a_2 = 1, a_3 = 1$ **d)** $a_0 = 0, a_1 = 1, a_2 = 2, a_3 = 3$ **5. a)** 2, 5, 8, 11, 14, 17, 20, 23, 26, 29 **b)** 1, 1, 1, 2, 2, 2, 3, 3, 3, 4 **c)** 1, 1, 3, 3, 5, 5, 7, 7, 9, 9 **d)** −1, −2, −2, 8, 88, 656, 4912, 40064, 362368, 3627776 **e)** 3, 6, 12, 24, 48, 96, 192, 384, 768, 1536 **f)** 2, 4, 6, 10, 16, 26, 42, 68, 110, 178 **g)** 1, 2, 2, 3, 3, 3, 3, 4, 4, 4 **h)** 3, 3, 5, 4, 4, 3, 5, 5, 4, 3 **7.** Each term could be twice the previous term; the nth term could be obtained from the previous term by adding $n - 1$; the terms could be the positive integers that are not multiples of 3; there are infinitely many other possibilities. **9. a)** 2, 12, 72, 432, 2592 **b)** 2, 4, 16, 256, 65,536 **c)** 1, 2, 5, 11, 26 **d)** 1, 1, 6, 27, 204 **e)** 1, 2, 0, 1, 3 **11. a)** 6, 17, 49, 143, 421 **b)** $49 = 5 \cdot 17 - 6 \cdot 6$, $143 = 5 \cdot 49 - 6 \cdot 17$, $421 = 5 \cdot 143 - 6 \cdot 49$ **c)** $5a_{n-1} - 6a_{n-2} = 5(2^{n-1} + 5 \cdot 3^{n-1}) - 6(2^{n-2} + 5 \cdot 3^{n-2}) = 2^{n-2}(10 - 6) + 3^{n-2}(75 - 30) = 2^{n-2} \cdot 4 + 3^{n-2} \cdot 9 \cdot 5 = 2^n + 3^n \cdot 5 = a_n$ **13. a)** Yes **b)** No **c)** No **d)** Yes **e)** Yes **f)** Yes **g)** No **h)** No **15. a)** $a_{n-1} + 2a_{n-2} + 2n - 9 = -(n - 1) + 2 + 2[-(n - 2) + 2] + 2n - 9 = -n + 2 = a_n$ **b)** $a_{n-1} + 2a_{n-2} + 2n - 9 = 5(-1)^{n-1} - (n - 1) + 2 + 2[5(-1)^{n-2} - (n - 2) + 2] + 2n - 9 = 5(-1)^{n-2}(-1 + 2) - n + 2 = a_n$ **c)** $a_{n-1} + 2a_{n-2} + 2n - 9 = 3(-1)^{n-1} + 2^{n-1} - (n - 1) + 2 + 2[3(-1)^{n-2} + 2^{n-2} - (n - 2) + 2] + 2n - 9 = 3(-1)^{n-2}(-1 + 2) + 2^{n-2}(2 + 2) - n + 2 = a_n$ **d)** $a_{n-1} + 2a_{n-2} + 2n - 9 = 7 \cdot 2^{n-1} - (n - 1) + 2 + 2[7 \cdot 2^{n-2} - (n - 2) + 2] + 2n - 9 = 2^{n-2}(7 \cdot 2 + 2 \cdot 7) - n + 2 = a_n$ **17. a)** $a_n = 2 \cdot 3^n$ **b)** $a_n = 2n + 3$ **c)** $a_n = 1 + n(n + 1)/2$ **d)** $a_n = n^2 + 4n + 4$ **e)** $a_n = 1$ **f)** $a_n = (3^{n+1} - 1)/2$ **g)** $a_n = 5n!$ **h)** $a_n = 2^n n!$ **19. a)** $a_n = 3a_{n-1}$ **b)** 5,904,900 **21. a)** $a_n = n + a_{n-1}$, $a_0 = 0$ **b)** $a_{12} = 78$ **c)** $a_n = n(n + 1)/2$ **23.** $B(k) = [1 + (0.07/12)]B(k - 1) - 100$, with $B(0) = 5000$ **25. a)** One 1 and one 0, followed by two 1s and two 0s, followed by three 1s and three 0s, and so on; 1, 1, 1 **b)** The positive integers are listed in increasing order with each even positive integer listed twice; 9, 10, 10. **c)** The terms in odd-numbered locations are the successive powers of 2; the terms in even-numbered locations are all 0; 32, 0, 64. **d)** $a_n = 3 \cdot 2^{n-1}$; 384, 768, 1536 **e)** $a_n = 15 - 7(n - 1) = 22 - 7n$; −34, −41, −48 **f)** $a_n = (n^2 + n + 4)/2$; 57, 68, 80 **g)** $a_n = 2n^3$; 1024, 1458, 2000 **h)** $a_n = n! + 1$; 362881, 3628801, 39916801 **27.** Among the integers $1, 2, \dots, a_n$, where a_n is the nth positive integer not a perfect square, the nonsquares are $a_1, a_2, \dots, a_n$ and the squares are $1^2, 2^2, \dots, k^2$, where k is the integer with $k^2 < n + k < (k + 1)^2$. Consequently, $a_n = n + k$, where $k^2 < a_n < (k + 1)^2$. To find k, first note that $k^2 < n + k < (k + 1)^2$, so $k^2 + 1 \le n + k \le (k + 1)^2 - 1$. Hence, $(k - \frac{1}{2})^2 + \frac{3}{4} = k^2 - k + 1 \le n \le k^2 + k = (k + \frac{1}{2})^2 - \frac{1}{4}$. It follows that $k - \frac{1}{2} < \sqrt{n} < k + \frac{1}{2}$, so $k = \{\sqrt{n}\}$ and $a_n = n + k = n + \{\sqrt{n}\}$. **29. a)** 20 **b)** 11 **c)** 30 **d)** 511 **31. a)** 1533 **b)** 510 **c)** 4923 **d)** 9842 **33. a)** 21 **b)** 78 **c)** 18 **d)** 18 **35.** $\sum_{j=1}^{n}(a_j - a_{j-1}) = a_n - a_0$ **37. a)** n^2 **b)** $n(n + 1)/2$ **39.** 15150 **41.** $\frac{n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2} + (n + 1)(m - (n + 1)^2 + 1)$, where $n = \lfloor \sqrt{m} \rfloor - 1$ **43. a)** 0 **b)** 1680 **c)** 1 **d)** 1024 **45.** 34

Section 2.5

1. a) Countably infinite, −1, −2, −3, −4, ... **b)** Countably infinite, 0, 2, −2, 4, −4, ... **c)** Countably infinite, 99, 98, 97, ... **d)** Uncountable **e)** Finite **f)** Countably infinite, 0, 7, −7, 14, −14, ... **3. a)** Countable: match n with the string of n 1s. **b)** Countable. To find a correspondence, follow the path in Example 4, but omit fractions in the top three rows (as well as continuing to omit fractions not in lowest terms). **c)** Uncountable **d)** Uncountable **5.** Suppose m new guests arrive at the fully occupied hotel. Move the guest in Room n to Room $m + n$ for $n = 1, 2, 3, \dots$; then the new guests can occupy rooms 1 to m. **7.** For $n = 1, 2, 3, \dots$, put the guest currently in Room $2n$ into Room n, and the guest currently in Room $2n-1$ into Room n of the new building. **9.** Move the guess currently Room i to Room $2i+1$ for $i = 1, 2, 3, \ldots$. Put the jth guest from the kth bus into Room $2^k(2j+1)$. **11. a)** $A = [1, 2]$ (closed interval of real numbers from 1 to 2), $B = [3, 4]$ **b)** $A = [1, 2] \cup \mathbf{Z}^+$, $B = [3, 4] \cup \mathbf{Z}^+$ **c)** $A = [1, 3]$, $B = [2, 4]$ **13.** Suppose that A is countable. Then either A has cardinality n for some nonnegative integer n, in which case there is a one-to-one function from A to a subset of $\mathbf{Z}^+$ (the range is the first n positive integers), or there exists a one-to-one correspondence f from A to $\mathbf{Z}^+$; in either case we have satisfied Definition 2. Conversely, suppose that $|A| \leq |\mathbf{Z}^+|$. By definition, this means that there is a one-to-one function from A to $\mathbf{Z}^+$, so A has the same cardinality as a subset of $\mathbf{Z}^+$ (namely the range of that function). By Exercise 16 we conclude that A is countable. **15.** Assume that B is countable. Then the elements of B can be listed as $b_1, b_2, b_3, \ldots$. Because A is a subset of B, taking the subsequence of $\{b_n\}$ that contains the terms that are in A gives a listing of the elements of A. Because A is uncountable, this is impossible. **17.** Assume that $A - B$ is countable. Then, because $A = (A - B) \cup (A \cap B)$, the elements of A can be listed in a sequence by alternating elements of $A - B$ and elements of $A \cap B$. This contradicts the uncountability of A. **19.** We are given bijections f from A to B and g from C to D. Then the function from $A \times C$ to $B \times D$ that sends (a, c) to $(f(a), g(c))$ is a bijection. **21.** By the definition of $|A| \leq |B|$, there is a one-to-one function $f : A \to B$. Similarly, there is a one-to-one function $g : B \to C$. By Exercise 33 in Section 2.3, the composition $g \circ f : A \to C$ is one-to-one. Therefore by definition $|A| \leq |C|$. **23.** Using the Axiom of Choice from set theory, choose distinct elements $a_1, a_2, a_3, \ldots$ of A one at a time (this is possible because A is infinite). The resulting set $\{a_1, a_2, a_3, \ldots\}$ is the desired infinite subset of A. **25.** The set of finite strings of characters over a finite alphabet is countably infinite, because we can list these strings in alphabetical order by length. Therefore the infinite set S can be identified with an infinite subset of this countable set, which by Exercise 16 is also countably infinite. **27.** Suppose that $A_1, A_2, A_3, \ldots$ are countable sets. Because A_i is countable, we can list its elements in a sequence as $a_{i1}, a_{i2}, a_{i3}, \ldots$. The elements of the set $\bigcup_{i=1}^n A_i$ can be listed by listing all terms a_{ij} with $i + j = 2$, then all terms a_{ij} with $i + j = 3$, then all terms a_{ij} with $i + j = 4$, and so on. **29.** There are a finite number of bit strings of length m, namely, 2^m. The set of all bit, strings is the union of the sets of bit strings of length m for $m = 0, 1, 2, \ldots$. Because the union of a countable number of countable sets is countable (see Exercise 27), there are a countable number of bit strings. **31.** It is clear from the formula that the range of values the function takes on for a fixed value of $m + n$, say $m + n = x$, is $(x-2)(x-1)/2 + 1$ through $(x-2)(x-1)/2 + (x-1)$, because m can assume the values $1, 2, 3, \ldots, (x-1)$ under these conditions, and the first term in the formula is a fixed positive integer when $m + n$ is fixed. To show that this function is one-to-one and onto, we merely need to show that the range of values for $x + 1$ picks up precisely where the range of values for x left off, i.e., that $f(x-1, 1) + 1 = f(1, x)$. We have $f(x-1, 1) + 1 = \frac{(x-2)(x-1)}{2} + (x-1) + 1 = \frac{x^2 - x + 2}{2} = \frac{(x-1)x}{2} + 1 = f(1, x)$. **33.** By the Schröder-Bernstein theorem, it suffices to find one-to-one functions $f : (0, 1) \to [0, 1]$ and $g : [0, 1] \to (0, 1)$. Let $f(x) = x$ and $g(x) = (x + 1)/3$. **35.** Each element A of the power set of the set of positive integers (i.e., $A \subseteq \mathbf{Z}^+$) can be represented uniquely by the bit string $a_1a_2a_3\ldots$, where $a_i = 1$ if $i \in A$ and $a_i = 0$ if $i \notin A$. Assume there were a one-to-one correspondence $f : \mathbf{Z}^+ \to \mathcal{P}(\mathbf{Z}^+)$. Form a new bit string $s = s_1s_2s_3\ldots$ by setting s_i to be 1 minus the ith bit of $f(i)$. Then because s differs in the i bit from $f(i)$, s is not in the range of f, a contradiction. **37.** For any finite alphabet there are a finite number of strings of length n, whenever n is a positive integer. It follows by the result of Exercise 27 that there are only a countable number of strings from any given finite alphabet. Because the set of all computer programs in a particular language is a subset of the set of all strings of a finite alphabet, which is a countable set by the result from Exercise 16, it is itself a countable set. **39.** Exercise 37 shows that there are only a countable number of computer programs. Consequently, there are only a countable number of computable functions. Because, as Exercise 38 shows, there are an uncountable number of functions, not all functions are computable.

Section 2.6

1. a) 3×4 **b)** $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$ **c)** $\begin{bmatrix} 2 & 0 & 4 & 6 \end{bmatrix}$ **d)** 1 **e)** $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 4 & 3 \\ 3 & 6 & 7 \end{bmatrix}$ **3. a)** $\begin{bmatrix} 1 & 11 \\ 2 & 18 \end{bmatrix}$ **b)** $\begin{bmatrix} 2 & -2 & -3 \\ 1 & 0 & 2 \\ 9 & -4 & 4 \end{bmatrix}$ **c)** $\begin{bmatrix} -4 & 15 & -4 & 1 \\ -3 & 10 & 2 & -3 \\ 0 & 2 & -8 & 6 \\ 1 & -8 & 18 & -13 \end{bmatrix}$ **5.** $\begin{bmatrix} 9/5 & -6/5 \\ -1/5 & 4/5 \end{bmatrix}$ **7.** $\mathbf{0} + \mathbf{A} = [0 + a_{ij}] = [a_{ij} + 0] = \mathbf{0} + \mathbf{A}$ **9.** $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = [a_{ij} + (b_{ij} + c_{ij})] = [(a_{ij} + b_{ij}) + c_{ij}] = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$ **11.** The number of rows of $\mathbf{A}$ equals the number of columns of $\mathbf{B}$, and the number of columns of $\mathbf{A}$ equals the number of rows of $\mathbf{B}$. **13.** $\mathbf{A}(\mathbf{BC}) = \left[\sum_q a_{iq}\left(\sum_r b_{qr}c_{rl}\right)\right] = \left[\sum_q \sum_r a_{iq}b_{qr}c_{rl}\right] = \left[\sum_r \sum_q a_{iq}b_{qr}c_{rl}\right] = \left[\sum_r \left(\sum_q a_{iq}b_{qr}\right)c_{rl}\right] = (\mathbf{AB})\mathbf{C}$ **15.** $\mathbf{A}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ **17. a)** Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$. Then $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$. We have $(\mathbf{A} + \mathbf{B})^t = [a_{ji} + b_{ji}] = [a_{ji}] + [b_{ji}] = \mathbf{A}^t + \mathbf{B}^t$. **b)** Using the same notation as in part (a), we have $\mathbf{B}^t\mathbf{A}^t =$

$\left[\sum_q b_{qi}a_{jq}\right] = \left[\sum_q a_{jq}b_{qi}\right] = (\mathbf{AB})^t$, because the (i, j)th entry is the (j, i)th entry of $\mathbf{AB}$. **19.** The result follows because $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = (ad-bc)\mathbf{I}_2 = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. **21.** $\mathbf{A}^n(\mathbf{A}^{-1})^n = \mathbf{A}(\mathbf{A}\cdots(\mathbf{A}(\mathbf{A}\mathbf{A}^{-1})\mathbf{A}^{-1})\cdots\mathbf{A}^{-1})\mathbf{A}^{-1}$ by the associative law. Because $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$, working from the inside shows that $\mathbf{A}^n(\mathbf{A}^{-1})^n = \mathbf{I}$. Similarly $(\mathbf{A}^{-1})^n\mathbf{A}^n = \mathbf{I}$. Therefore $(\mathbf{A}^n)^{-1} = (\mathbf{A}^{-1})^n$. **23.** The (i, j)th entry of $\mathbf{A} + \mathbf{A}^t$ is $a_{ij} + a_{ji}$, which equals $a_{ji} + a_{ij}$, the (j, i)th entry of $\mathbf{A} + \mathbf{A}^t$, so by definition $\mathbf{A} + \mathbf{A}^t$ is symmetric. **25.** $x_1 = 1$, $x_2 = -1$, $x_3 = -2$

27. a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

29. a) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

31. a) $\mathbf{A} \vee \mathbf{B} = [a_{ij} \vee b_{ij}] = [b_{ij} \vee a_{ij}] = \mathbf{B} \vee \mathbf{A}$ **b)** $\mathbf{A} \wedge \mathbf{B} = [a_{ij} \wedge b_{ij}] = [b_{ij} \wedge a_{ij}] = \mathbf{B} \wedge \mathbf{A}$ **33. a)** $\mathbf{A} \vee (\mathbf{B} \wedge \mathbf{C}) = [a_{ij}] \vee [b_{ij} \wedge c_{ij}] = [a_{ij} \vee (b_{ij} \wedge c_{ij})] = [(a_{ij} \vee b_{ij}) \wedge (a_{ij} \vee c_{ij})] = [a_{ij} \vee b_{ij}] \wedge [a_{ij} \vee c_{ij}] = (\mathbf{A} \vee \mathbf{B}) \wedge (\mathbf{A} \vee \mathbf{C})$ **b)** $\mathbf{A} \wedge (\mathbf{B} \vee \mathbf{C}) = [a_{ij}] \wedge [b_{ij} \vee c_{ij}] = [a_{ij} \wedge (b_{ij} \vee c_{ij})] = [(a_{ij} \wedge b_{ij}) \vee (a_{ij} \wedge c_{ij})] = [a_{ij} \wedge b_{ij}] \vee [a_{ij} \wedge c_{ij}] = (\mathbf{A} \wedge \mathbf{B}) \vee (\mathbf{A} \wedge \mathbf{C})$ **35.** $\mathbf{A} \odot (\mathbf{B} \odot \mathbf{C}) = \left[\bigvee_q a_{iq} \wedge \left(\bigvee_r (b_{qr} \wedge c_{rl})\right)\right] = \left[\bigvee_q \bigvee_r (a_{iq} \wedge b_{qr} \wedge c_{rl})\right] = \left[\bigvee_r \bigvee_q (a_{iq} \wedge b_{qr} \wedge c_{rl})\right] = \left[\bigvee_r \left(\bigvee_q (a_{iq} \wedge b_{qr})\right) \wedge c_{rl}\right] = (\mathbf{A} \odot \mathbf{B}) \odot \mathbf{C}$

Supplementary Exercises

1. a) $\overline{A}$ **b)** $A \cap B$ **c)** $A - B$ **d)** $\overline{A} \cap \overline{B}$ **e)** $A \oplus B$ **3.** Yes **5.** $A - (A - B) = A - (A \cap \overline{B}) = A \cap (\overline{A \cap \overline{B}}) = A \cap (\overline{A} \cup B) = (A \cap \overline{A}) \cup (A \cap B) = \emptyset \cup (A \cap B) = A \cap B$ **7.** Let $A = \{1\}$, $B = \emptyset$, $C = \{1\}$. Then $(A - B) - C = \emptyset$, but $A - (B - C) = \{1\}$. **9.** No. For example, let $A = B = \{a, b\}$, $C = \emptyset$, and $D = \{a\}$. Then $(A - B) - (C - D) = \emptyset - \emptyset = \emptyset$, but $(A - C) - (B - D) = \{a, b\} - \{b\} = \{a\}$. **11. a)** $|\emptyset| \le |A \cap B| \le |A| \le |A \cup B| \le |U|$ **b)** $|\emptyset| \le |A - B| \le |A \oplus B| \le |A \cup B| \le |A| + |B|$ **13. a)** Yes, no **b)** Yes, no **c)** f has inverse with $f^{-1}(a) = 3$, $f^{-1}(b) = 4$, $f^{-1}(c) = 2$, $f^{-1}(d) = 1$; g has no inverse. **15.** If f is one-to-one, then f provides a bijection between S and $f(S)$, so they have the same cardinality. If f is not one-to-one, then there exist elements x and y in S such that $f(x) = f(y)$. Let $S = \{x, y\}$. Then $|S| = 2$ but $|f(S)| = 1$. **17.** Let $x \in A$. Then $S_f(\{x\}) = \{f(y) \mid y \in \{x\}\} = \{f(x)\}$. By the same reasoning, $S_g(\{x\}) = \{g(x)\}$. Because $S_f = S_g$, we can conclude that $\{f(x)\} = \{g(x)\}$, and so necessarily $f(x) = g(x)$. **19.** The equation is true if and only if the sum of the fractional parts of x and y is less than 1. **21.** The equation is true if and only if either both x and y are integers, or x is not an integer but the sum of the fractional parts of x and y is less than or equal to 1. **23.** If x is an integer, then $\lfloor x \rfloor + \lfloor m - x \rfloor = x + m - x = m$. Otherwise, write x in terms of its integer and fractional parts: $x = n + \epsilon$, where $n = \lfloor x \rfloor$ and $0 < \epsilon < 1$. In this case $\lfloor x \rfloor + \lfloor m - x \rfloor = \lfloor n + \epsilon \rfloor + \lfloor m - n - \epsilon \rfloor = n + m - n - 1 = m - 1$. **25.** Write $n = 2k + 1$ for some integer k. Then $n^2 = 4k^2 + 4k + 1$, so $n^2/4 = k^2 + k + \frac{1}{4}$. Therefore, $\lceil n^2/4 \rceil = k^2 + k + 1$. But $(n^2 + 3)/4 = (4k^2 + 4k + 1 + 3)/4 = k^2 + k + 1$. **27.** Let $x = n + (r/m) + \epsilon$, where n is an integer, r is a nonnegative integer less than m, and ϵ is a real number with $0 \le \epsilon < 1/m$. The left-hand side is $\lfloor nm + r + m\epsilon \rfloor = nm + r$. On the right-hand side, the terms $\lfloor x \rfloor$ through $\lfloor x + (m + r - 1)/m \rfloor$ are all just n and the terms from $\lfloor x + (m - r)/m \rfloor$ on are all $n + 1$. Therefore, the right-hand side is $(m - r)n + r(n + 1) = nm + r$, as well. **29.** 101 **31.** $a_1 = 1$; $a_{2n+1} = n \cdot a_{2n}$ for all $n > 0$; and $a_{2n} = n + a_{2n-1}$ for all $n > 0$. The next four terms are 5346, 5353, 37471, and 37479. **33.** If each $f^{-1}(j)$ is countable, then $S = f^{-1}(1) \cup f^{-1}(2) \cup \cdots$ is the countable union of countable sets and is therefore countable by Exercise 27 in Section 2.5. **35.** Because there is a one-to-one correspondence between $\mathbf{R}$ and the open interval (0, 1) (given by $f(x) = 2\arctan(x)/\pi$), it suffices to shows that $|(0, 1) \times (0, 1)| = |(0, 1)|$. By the Schröder-Bernstein theorem it suffices to find injective functions $f : (0, 1) \to (0, 1) \times (0, 1)$ and $g : (0, 1) \times (0, 1) \to (0, 1)$. Let $f(x) = (x, \frac{1}{2})$. For g we follow the hint. Suppose $(x, y) \in (0, 1) \times (0, 1)$, and represent x and y with their decimal expansions $x = 0.x_1x_2x_3\ldots$ and $y = 0.y_1y_2y_3\ldots$, never choosing the expansion of any number that ends in an infinite string of 9s. Let $g(x, y)$ be the decimal expansion obtained by interweaving these two strings, namely $0.x_1y_1x_2y_2x_3y_3\ldots$. **37.** $\mathbf{A}^{4n} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{A}^{4n+1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\mathbf{A}^{4n+2} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\mathbf{A}^{4n+3} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, for $n \ge 0$ **39.** Suppose that $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Let $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Because $\mathbf{AB} = \mathbf{BA}$, it follows that $c = 0$ and $a = d$. Let $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Because $\mathbf{AB} = \mathbf{BA}$, it follows that $b = 0$. Hence, $\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = a\mathbf{I}$.

41. a) Let $\mathbf{A} \odot \mathbf{0} = [b_{ij}]$. Then $b_{ij} = (a_{i1} \wedge 0) \vee \cdots \vee (a_{ip} \wedge 0) = 0$. Hence, $\mathbf{A} \odot \mathbf{0} = \mathbf{0}$. Similarly $\mathbf{0} \odot \mathbf{A} = \mathbf{0}$. **b)** $\mathbf{A} \vee \mathbf{0} = [a_{ij} \vee 0] = [a_{ij}] = \mathbf{A}$. Hence $\mathbf{A} \vee \mathbf{0} = \mathbf{A}$. Similarly $\mathbf{0} \vee \mathbf{A} = \mathbf{A}$. **c)** $\mathbf{A} \wedge \mathbf{0} = [a_{ij} \wedge 0] = [0] = \mathbf{0}$. Hence $\mathbf{A} \wedge \mathbf{0} = \mathbf{0}$. Similarly $\mathbf{0} \wedge \mathbf{A} = \mathbf{0}$.