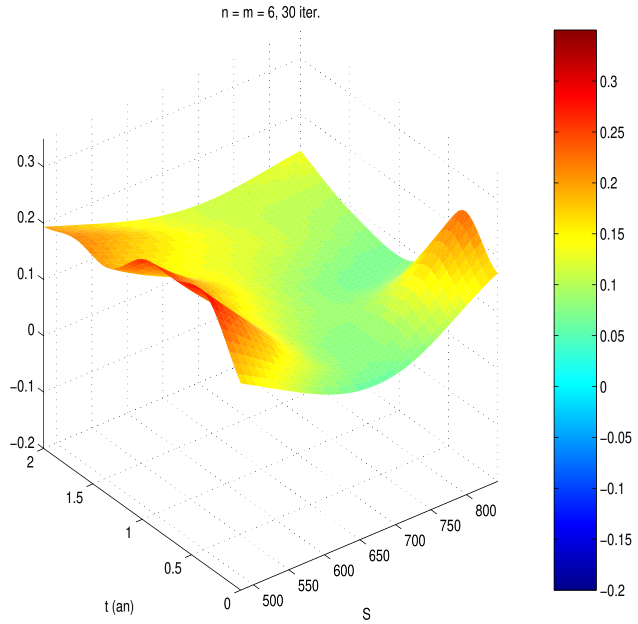
Numerical methods in Finance

University of Ljubljana



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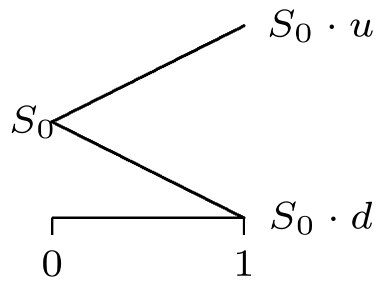
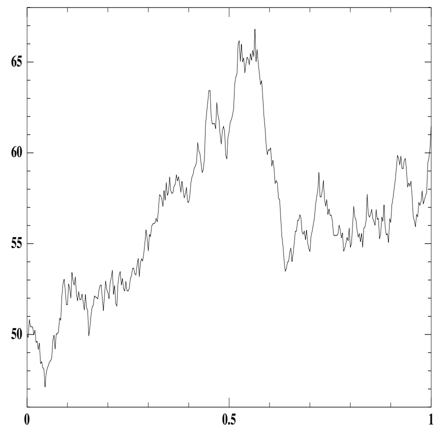
Pricing and hedging methods for derivatives

- *Cox-Ross-Rubinstein discrete model*.



- *Black-Scholes continuous model*



- Numerical methods Tree methods, Monte Carlo methods, Finite Difference methods.

Plan

1. Cox-Ross-Rubinstein model. Pricing and Delta hedging in discrete models. Markov chains. Dynamic programming equations. European and American options in CRR model.
2. Monte Carlo Methods. Simulation methods of classical law. Inverse transform method. Central Limit Theorem. Computation of expectation. Variance reduction techniques (Control Variate, Importance sampling).
3. Geometric brownian motion. Ito's Lemma. Black-Scholes model. Monte Carlo Methods for European options.
4. Greeks. Estimating sensitivities. Dynamic hedging in the Black-Scholes continuous model. Numerical algorithms for portfolio insurance.
5. Tree methods for European and American options. Convergence orders of binomial methods.
6. Monte Carlo methods for Exotic options (Barrier options, Asian options, Lookback options, Rainbow options).
7. Tree methods for exotic options. The Ritchken method. The forward shooting grid methods. The singular points method.
8. Monte Carlo Methods for American options. The Longstaff-Schwartz method.
9. Finite difference methods for the heat equation and the Black-Scholes partial differential equation. Explicit Scheme. Implicit scheme. Cranck-Nicolson scheme. Consistency and stability of the schemes.
10. Matlab sessions with the implementation of the proposed numerical algorithms.

Teaching Material

- Slides of the course.
- J.Hull Options, Futures, and Other Derivatives. Prentice Hall
- N.H. Bingham R. Kiesel. Risk-Neutral Valuation: Pricing and Hedging of Financial Derivatives. Springer Finance
- P.Glasserman. Monte Carlo methods in Financial Engineering. Springer

Examination

- The final assessment will require the solution of exercises on topics examined during lessons.

A derivative product is a financial contract whose value at the expiry date T (maturity) is uniquely determined by the price of a listed security (the underlying financial asset). There are three large groups:

- Options
- Forwards and Futures
- Swaps

Options

They attribute the right (but not the obligation) to carry out a transaction on a certain date at a certain price

Ex: right to buy an *Apple Inc.* stock at $110 on today. Would do you exercise the option?

Options

There are various categories of options:

- Vanilla: Call, Put.
- Exotics: Asian, Lookback, Barrier.
- Optionality: European, American, Bermudian.

Glossary

- **underlying** S_t: the title (financial index) on which the option depends (in our example Apple).
- **payoff** P: the law to compute the amount paid when the option is exercised.
- **strike** K: set sale / purchase price
- **maturity** or **expiration date** T: is the last date the option can be exercised
- **in the money** option: has a positive (intrinsic) value.
- **out the money** option: has an (intrinsic) value of zero.
- **long position** on an asset: you have bought the asset.
- **short position** on an asset: you have sold the asset.
- **writer**: the option seller
- **porfolio**: a collection of financial investments like stocks, bonds, commodities, cash, and cash equivalents

Financial options

European Call options

A **Call option** (a Call) is a financial instrument giving the right (but not to the obligation) to the owner to buy the underlying asset at a given price K (called strike) at prefixed date T (called maturity).

The writer will have an obligation to sell at these conditions.

Because of the asymmetry of the contract :

- the owner of the option has to pay to the writer the premium of the option.
- the writer will provide to the owner $\max(0, S_T - K)$ at maturity.
  The quantity $\max(0, S_T - K)$ is called the payoff of the option.

Financial options

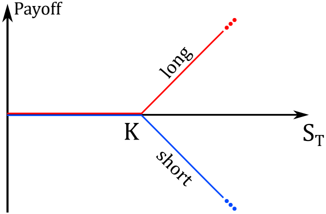
European Put options

A **Put option** (a Put) is a financial instrument giving the right (but not to the obligation) to the owner to sell the underlying asset at a given price K (called strike) at prefixed date T (called maturity).

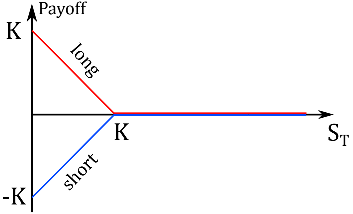
- The payoff of the option is now $\max(0, K - S_T)$.
- As opposed to the call option, the payoff is now a bounded function: writer's exposure to losses is bounded.

Options

Call option (right to buy):



Put option (right to sell):



Forward and Futures

Forward: it is a deferred sales contact of a stock at a previously set price.

- At instant $t = 0$ the selling price S is established
- The money-share exchange will take place at time $T > 0$
- At time $t = 0$ there is no money exchange.

Futures: similar to forward contracts but they are standardized and listed contracts

- the difference between the market price and the established price is paid according to a predetermined plan (marked to market)
- the clearing house (guarantee mechanism) acts as guarantor

Swap contract

A **swap contract** is a derivative contract stipulated between two counterparties that exchanges flows of payments (which we improperly call ”installments”), within a predetermined period, usually calculated on the trend of interest (or exchange) rates.

- literally ”exchange”
- Examples: cross currency swap (Dollars for Euros) and interest rate swap (exchange of fixed interest against a variable one)

Stocks

A stock S represents a share of the ownership of a company.

- provide partial ownership of the company, pro rata with investment
- they have a price that varies on the market based on their supply and demand
- they reflect the value of the company but can be subject to market speculation
- allow the owner to receive dividends (share of the net profit for the year)

Bonds

A bond B (also obligation), is a debt instrument which works like a bank account: the issuer owes the holders a debt and has to pay the interest (the coupon) plus the initial debt (the nominal, principal, facial value) at the maturity T.

- it is a riskless instrument: $B_t = B_0 \cdot F^t =$ with F the capitalization factor
- $F = 1 + R$ with R the (annual) interest rate
- $F = e^r$ with r the short rate
- $R = e^r - 1$ and $r = \ln(1 + r)$
- Usually* bonds value increases with time, so $R \geq 0$ and $r \geq 0$.

Financial markets

Financial products are traded through two types of stock exchange circuits:

- organized exchanges
  - subject to regulatory rules
  - require a certain degree of standardization of the traded instruments
  - physical location at which trade takes place
  - Example NYSE New York Stock exchange.
- OTC markets (over the counter)
  - there are no official price lists
  - trading takes place exclusively electronically

Traders

Three main types of traders.

- **Hedgers**: use the market to hedge a risk (eg a flour producer who buys a forward on wheat, or a farmer who sells a forward on wheat). Observation: the producer fears the rise in wheat prices, while the farmer fears the decline.
- **Speculators**: they use the market to make profits, taking on themselves the risks feared by hedgers (e.g. an investor selling a forward on orange juice). They have no particular investment preferences: the important thing is making a profit.
- **Arbitrageurs**: take advantage of some mismatches in the market to make (small) profits without risk.

Figure: The protagonists and the antagonists.

Whether you are a hedger, a speculator or an arbitrageur, the crucial question to answer before selling or buying is:

what is the ”fair” price for a specific financial product?

Valuation of options or option pricing deals with this question.

Market Assumptions

General assumption:

1. Friction-free market: no selling or buying costs, no taxes, no bid-ask spread, no limitations on short selling.
2. No risk of default: no outstanding debts, no late payments, single interest rate for all
3. Rational agents: all market players aim to maximize their wealth
4. Absence of arbitrage opportunities: no free lunch.

Unrealistic but necessary hypotheses for the development of the theory. Moreover:

- Nobody can influence prices;
- No restrictions on the quantities to be exchanged;
- Infinitely divisible shares.

Pricing of financial options

What is the fair price of these financial derivatives products?
The problem of the evaluation of this contingent claim is the problem of the evaluation of a random variable received at maturity (the payoff).

The main message of Black-Scholes-Merton(1973) is that the fair price of a financial derivative is the price obtained using a hedging procedure under absence of arbitrage opportunities (AOA).

We will study numerical methods for two models :

- Discrete model of *Cox-Ross-Rubinstein*, based on Markov chains.
- Continous model of *Black-Scholes*, based on continuous stochastic process.

Additional Market hypothesis

- The interest rate is known and is constant through time.
- The stock pays no dividends or other distributions.
- There are no penalties to short-selling.
- It is possible to borrow any fraction of the price of a security to buy it or to hold it, at the short-time interest rate.

Proposition (Equivalent portfolios)

If AOA is assumed, if two portfolios X e Y have the same value in T, then they must have the same value an any $t \in [0, T]$.

$$X_T = Y_T \rightarrow \forall t \in [0, T]\,,\ X_t = Y_t$$

Proof

Exercise (proof by contradiction).

Theorem (Put-Call Theorem Parity)

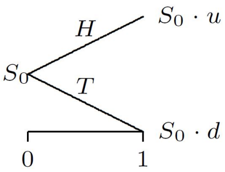
Let us consider a Call and a Put option on the same underlying, with the same strike K and the same maturity T. Let C_t and P_t denote their price at time $0 < t < T$ respectively. Moreover let R denote the interest rate. Then, the following relation, between the prices of the underlying asset S_t and European call and put options on stocks that pay no dividends, holds:

$$C_t - P_t = S_t - K\,(1 + R)^{-(T-t)}$$

The One-period binomial model

Risky asset *H1*: $0 < d < 1 < u$.
Let us imagine that we are tossing a coin.
When we get “Head”, the stock price moves up.
When we get a “Tail”, the price moves down. Consider the sample space $\Omega = \{H, T\}$, $\omega \in \Omega$.

$$S_1(\omega) = \begin{cases} S_1(H) = S_0 u \\ S_1(T) = S_0 d \end{cases}$$



Remark

The probability of getting a Head or a Tail is called natural probability. While it may seem obvious, we will demonstrate that this probability has no influence on the price of an option.

Risk-free asset *H2*: $d < 1 + R < u$, R annual interest rate.

$$\begin{array}{ccc} 1 & & 1+R \\ 0 & & 1 \end{array}$$

Exercise

Assume $1 + R < d < u$ *and construct an arbitrage.*

Let us consider an European call option with strike K and maturity $T = 1$.

$$V_1(\omega) = \begin{cases} \max\{0, S_0 u - K\} = (S_0 u - K)_+ & \text{if} \quad \omega = H \\ \max\{0, S_0 d - K\} = (S_0 d - K)_+ & \text{if} \quad \omega = T \end{cases}$$

Example

$S_0 = 50$, $u = 1.1$, $d = 0.9$, $K = 50$

$$V_1(\omega) = \begin{cases} (55 - 50)_+ = 5 & \textit{if} \quad \omega = H \\ (45 - 50)_+ = 0 & \textit{if} \quad \omega = T \end{cases}$$

Replicating portfolio

The seller of the option at time 1 has to pay

$$V_1(\omega) = \begin{cases} (S_0 u - K)_+ & \text{if} \quad \omega = H \\ (S_0 d - K)_+ & \text{if} \quad \omega = T \end{cases}$$

How to compute V_0, the arbitrage price of this options at time zero?

Solution: by using the *equivalent portfolios* proposition! We create a portfolio $\mathcal{P}$ including α stocks and β units of cash. We want $\mathcal{P}$ to be equivalent to the option (a portfolio which includes the option only) and its value at time $t = 0$ to be known. Such a portfolio $\mathcal{P}$ is termed replicating portfolio.

Idea : Hedging the option by using a portfolio $\mathcal{P} = (\alpha, \beta) \in \mathbb{R}^2$ where

- α the quantity invested in the risky asset at time zero.
- β the quantity invested in the bank account at time zero.

The value of the portfolio at time 0 is given by:

$$\widehat{V_0} = \alpha S_0 + \beta \Rightarrow \beta = \widehat{V_0} - \alpha S_0$$

For hedging purposes we need

$$\widehat{V_1}(\omega) = V_1(\omega)$$

No-Arbitrage conditions requires that

$$\widehat{V_0} = V_0$$

The value of the portfolio $\mathcal{P}$ at time 1 is given by:

$$\begin{cases} (1) & \alpha S_1(H) + \beta(1+R) = V_1(H) \\ (2) & \alpha S_1(T) + \beta(1+R) = V_1(T) \end{cases}$$

Solving the system in the unknown variables V_0, α, β :

$$\widehat{\alpha} = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} \quad \widehat{\beta} = V_0 - \widehat{\alpha} S_0$$

Now we can compute V_0.
From (1) and $\beta = \widehat{V_0} - \alpha S_0$, we have:

$$\widehat{\alpha} S_1(H) + (V_0 - \widehat{\alpha} S_0)(1+R) = \widehat{\alpha} S_1(H) + V_0(1+R) - \widehat{\alpha} S_0(1+R) = V_1(H)$$

$$\begin{aligned} V_0 &= \frac{1}{(1+R)}\Big[V_1(H) - \widehat{\alpha} S_1(H) + \widehat{\alpha} S_0(1+R)\Big] \\ &= \frac{1}{(1+R)}\Big[V_1(H) + \widehat{\alpha}(S_0(1+R) - S_0 u)\Big] \end{aligned}$$

Then

$$
\begin{aligned}
V_0 &= \frac{1}{(1+R)}\Big[V_1(H) + \widehat{\alpha}(S_0(1+R) - S_0 u)\Big] \\
&= \frac{1}{(1+R)}\Big[V_1(H) + \frac{V_1(H) - V_1(T)}{S_0 u - S_0 d}(S_0(1+R) - S_0 u)\Big] \\
&= \frac{1}{(1+R)}\Big[\frac{(u-d)V_1(H) + (1+R)V_1(H) - uV_1(H) - (1+R)V_1(T) + uV_1(T)}{(u-d)}\Big] \\
&= \frac{1}{(1+R)}\Big[\frac{((1+R)-d)}{(u-d)}V_1(H) + \frac{(u-(1+R))}{(u-d)}V_1(T)\Big]
\end{aligned}
$$

Consider

$$q = \frac{(1+R)-d}{u-d}$$

and

$$\widehat{q} = \frac{u-(1+R)}{u-d}$$

Risk-neutral pricing formula

$$V_0 = \frac{1}{(1+R)}\left[qV_1(H) + \widehat{q}V_1(T)\right] = \mathbb{E}_q\left[\frac{1}{(1+R)}V_1\right] \tag{3.1}$$

Remark

Recall the hypothesis (H2 : $d < 1 + R < u$).
Therefore

$$q = \frac{(1+R)-d}{u-d} > 0 \qquad \widehat{q} = \frac{u-(1+R)}{u-d} > 0$$

$$q + \widehat{q} = 1$$

q is called the risk-neutral probability.

Remark

The pricing formula (3.1) holds for each derivatives.

Risk-neutral valuation formula

$$V_0 = \mathbb{E}_q\left[\frac{1}{(1+R)}V_1\right] \tag{3.2}$$

$$\mathbb{E}_q\left[\frac{V_1}{V_0}\right] = (1+R) \tag{3.3}$$

Remark

The price of a contingent claim is the expected value of the discounted payoff with respect to an equivalent martingale measure[a].

The expected return of each contingent claim is equal to the return of the risk-free asset.

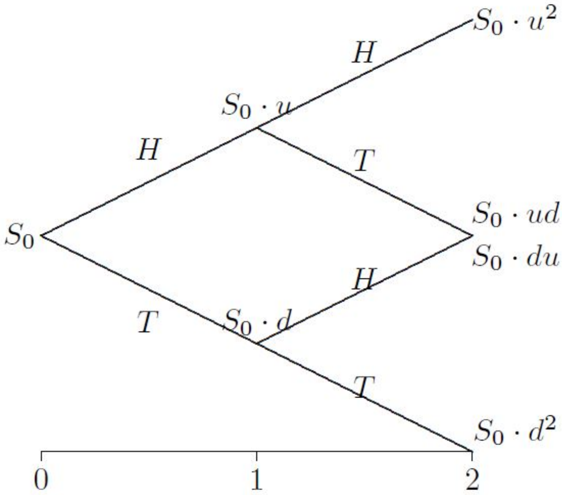
[a] a probability measure, equivalent to the model's probability measure, under which discounted market prices are martingales

The two-period binomial model

- Risky asset *H1*: $0 < d < 1 < u$.
  Consider $\Omega = \{HH, HT, TH, TT\}$, $\omega \in \Omega$ $\omega = (\omega_1, \omega_2)$.
  The asset price at time 2 is given by

$$S_2 = \begin{cases} S_2(HH) = S_0 u^2 \\ S_2(HT) = S_2(TH) = S_0 ud \\ S_2(TT) = S_0 d^2 \end{cases}$$

- Risk-free asset *H2*: $d < 1 + R < u$.


$S_0 \cdot u^2$
H
$S_0 \cdot u$
H
T
$S_0 \cdot ud$
S_0
$S_0 \cdot du$
H
T
$S_0 \cdot d$
T
$S_0 \cdot d^2$
0
1
2

Let us consider an European Call option with strike K e maturity 2.
The value of the option at time 2 is given by:

$$V_2(\omega) = \begin{cases} (S_0u^2 - K)_+ & \text{if} \quad \omega = HH \\ (S_0ud - K)_+ & \text{if} \quad \omega = HT \quad or \quad \omega = TH \\ (S_0d^2 - K)_+ & \text{if} \quad \omega = TT \end{cases}$$

Example

$S_0 = 45.454545$, $u = 1.1$, $d = 0.9$, $K = 40$

$$V_2(\omega) = \begin{cases} (55 - 40)_+ = 15 & if \quad \omega = HH \\ (45 - 40)_+ = 5 & if \quad \omega = HT \quad or \quad \omega = TH \\ (36.81 - 40)_+ = 0 & if \quad \omega = TT \end{cases}$$

Dynamic hedging

In order to compute the option value at time $t = 0$ let's try again to construct a replicating portfolio.
The value of the portfolio at time 0 is given by:

$$\widehat{V_0} = \alpha_0 S_0 + \beta_0 \Rightarrow \beta_0 = \widehat{V_0} - \alpha S_0$$

For hedging purposes we need

$$\widehat{V_2}(\omega) = V_2(\omega)$$

No-Arbitrage conditions requires that

$$\widehat{V_0} = V_0 \qquad \widehat{V_1} = V_1$$

The value of the portfolio at time $t = 1$ is given by:

$$\begin{cases} \widehat{V_1}(H) & = \alpha_0 S_1(H) + (V_0 - \alpha S_0)(1+R) \\ \widehat{V_1}(T) & = \alpha_0 S_1(T) + (V_0 - \alpha S_0)(1+R) \end{cases}$$

$$\begin{cases} \widehat{V_1}(H) = \alpha_0 S_0 u + (V_0 - \alpha_0 S_0)(1+R) =_{AOA} V_1(H) & \text{if } \omega_1 = H \quad (3.4a) \\ \widehat{V_1}(T) = \alpha_0 S_0 d + (V_0 - \alpha_0 S_0)(1+R) =_{AOA} V_1(T) & \text{if } \omega_1 = T \quad (3.4b) \end{cases}$$

Then $\widehat{V_1}$ depends on ω_1 the outcome of first coin toss.

Remark

Static hedging (α and β do not change in time) does not work! So, α and β need to be time dependent: it's dynamic hedging.

We have to reallocate the portfolio value $\widehat{V_1}$ to hedge against next time step! From

$$\widehat{V_1} = \alpha_0 S_1 + \beta_0(1 + R)$$

to

$$\widehat{V_1} = \alpha_1 S_1 + \beta_1$$

No money is added or withdrawn: the portfolio is self-financing. Now

$$\widehat{V_1} = \alpha_1 S_1 + \beta_1 \Rightarrow \beta_1 = \widehat{V_1} - \alpha_1 S_1$$

where α_1, β_1, S_1 depends on ω_1.

Rebalancing the portfolio

The value of the portfolio $\widehat{V_2}$ at time 2 is given by:

$$
\begin{cases}
\hat{V}_2(HH) = \alpha_1(H)S_2(HH) + (V_1(H) - \alpha_1(H)S_1(H))(1+R) = V_2(HH) & (3.5a) \\
\text{if } \omega_1 = H, \omega_2 = H & \\
\hat{V}_2(HT) = \alpha_1(H)S_2(HT) + (V_1(H) - \alpha_1(H)S_1(H))(1+R) = V_2(HT) & (3.5b) \\
\text{if } \omega_1 = H, \omega_2 = T & \\
\hat{V}_2(TH) = \alpha_1(T)S_2(TH) + (V_1(T) - \alpha_1(T)S_1(T))(1+R) = V_2(TH) & (3.5c) \\
\text{if } \omega_1 = T, \omega_2 = H & \\
\hat{V}_2(TT) = \alpha_1(T)S_2(TT) + (V_1(T) - \alpha_1(T)S_1(T))(1+R) = V_2(TT) & (3.5d) \\
\text{if } \omega_1 = T, \omega_2 = T &
\end{cases}
$$

From (3.5c)-(3.5d) it follows

$$\alpha_1(T) = \frac{V_2(TH) - V_2(TT)}{S_2(TH) - S_2(TT)}$$

substituting this into (3.5c)

$$V_1(T) = \frac{1}{(1+R)}\Big[qV_2(TH) + \hat{q}V_2(TT)\Big]$$

From (3.5b)-(3.5c) it follows

$$\alpha_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)}$$

substituting this into (3.5b)

$$V_1(H) = \frac{1}{(1+R)}\Big[qV_2(HH) + \hat{q}V_2(HT)\Big]$$

From (3.5a)-(3.5b) it follows

$$\alpha_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}, \quad \text{and} \quad V_0 = \frac{1}{(1+R)}\Big[qV_1(H) + \hat{q}V_1(T)\Big]$$

Risk-neutral pricing formula

$$V_0 = \frac{1}{(1+R)^2}\left[q^2 V_2(HH) + q\widehat{q} V_2(TH) + q\widehat{q} V_2(HT) + \widehat{q}^2 V_2(TT)\right] \tag{3.6}$$

$$= \mathbb{E}_q\left[\frac{1}{(1+R)^2} V_2\right] \tag{3.7}$$

Remark

The price of an option is the expected value of the discounted payoff with respect to an equivalent martingale measure.

The n-periods binomial model : the CRR model. The binomial

model was first proposed by William Sharpe in the 1978 and formalized by Cox, Ross and Rubinstein in 1979.

- Risky asset H1: $0 < d < 1 < u$.
  $\omega \in \Omega$ with 2^n, $\omega = (\omega_1, \dots, \omega_n)$.

$$S_k(\omega_1, \dots, \omega_{k-1}, \omega_k) = \begin{cases} S_{k-1}(\omega_1, \dots, \omega_{k-1})u & \text{if} \quad \omega_k = H \\ S_{k-1}(\omega_1, \dots, \omega_{k-1})d & \text{if} \quad \omega_k = T \end{cases}$$

- Risk-free asset H2: $d < 1 + R < u$.

$$S^0_{k+1} = (1 + R) S^0_k$$

Remark

If $n = 66$, $2^{66} = 7 \times 10^{19}$, but because of recombining property there are $n + 1$ final nodes.

Let's consider a Call option. Then:

$$V_n(\omega) = (S_n(\omega) - K)_+$$

More generally V_k is the value of the option at time $t = k$. It holds:

$$V_k(\omega_1, \ldots, \omega_k) = \frac{1}{(1+R)}\Big[qV_{k+1}(\omega_1, \ldots, \omega_k, H) + \widehat{q}V_{k+1}(\omega_1, \ldots, \omega_k, T)\Big]$$

The number of shares to hold in the replicating portfolio is given by

$$\alpha_k(\omega_1, \ldots, \omega_k) = \frac{V_{k+1}(\omega_1, \ldots, \omega_k, H) - V_{k+1}(\omega_1, \ldots, \omega_k, T)}{S_{k+1}(\omega_1, \ldots, \omega_k, H) - S_{k+1}(\omega_1, \ldots, \omega_k, T)}$$

while the cash amount is given by

$$\beta_k(\omega_1, \ldots, \omega_k) = V_k(\omega_1, \ldots, \omega_k) - \alpha_k(\omega_1, \ldots, \omega_k)S_k(\omega_1, \ldots, \omega_k)$$

Risk-neutral pricing formula

The pricing formula becomes:

$$
\begin{aligned}
V_0 &= \frac{1}{(1+R)^n}\mathbb{E}_q\Big[V_n\Big] && (3.8)\\
&= \frac{1}{(1+R)^n}\mathbb{E}_q\Big[(S_n-K)_+\Big] && (3.9)\\
&= \frac{1}{(1+R)^n}\Big[\sum_{h=0}^{n}\binom{n}{h} q^h \hat{q}^{n-h}(S_0u^hd^{n-h}-K)_+\Big] && (3.10)
\end{aligned}
$$

More generally, option value at time $t=k$ can be computed as follows:

$$V_k = \mathbb{E}_q\Big[\frac{1}{(1+R)^{n-k}}V_n|\mathcal{F}_k\Big]$$

The discounted option value is a martingale:

$$\frac{1}{(1+R)^k} V_k = \mathbb{E}_q\Big[\frac{1}{(1+R)^n} V_n | \mathcal{F}_k\Big]$$

Moreover

$$\mathbb{E}_q\left[\frac{V_n}{V_k} | \mathcal{F}_h\right] = (1+R)^{n-k}$$

Replicating portfolio algortihm

Start $t_0 = 0,\ S_0 = x.$
$V_0 = C(0, x)$
Compute $\alpha_0 = \Big(C(1, x*u) - C(1, x*d)\Big)/\Big(x*u - x*d\Big);$
$\beta_0 = V_0 - S_0\alpha_0;$

for $k = 1, \ldots, N-1$
BEGIN;
simulation of S_k;
$V_k = \alpha_{k-1}S_k + \beta_{k-1}(1+R);$
rebalancing the portfolio;
$\alpha_k = \Big(C(k+1, u*S_k) - C(k+1, S_k*d)\Big)/\Big(S_k*u - S_k*d\Big);$
$\beta_k = V_k - S_k\alpha_k;$
END;

simulation of S_N;

$V_N = \alpha_{N-1}S_N + \beta_{N-1}(1+R);$

A portfolio V is self-financing if there is no consumption or investment at any time $t > 0$.

Trading strategies

$$V_k = \alpha_k S_k + \beta_k, \quad k = 0, .., n$$

V is self-financing iff

$$V_k = \alpha_k S_k + \beta_k = \alpha_{k-1} S_k + \beta_{k-1}(1 + R)$$

The variation in its value is only due to the variations in the value of the underlying assets.

An arbitrage is a self-financing portfolio such that

$$\begin{cases} V_0 = 0 \\ V_k \geq 0 \\ P(V_N > 0) > 0 \end{cases}$$

Remark The CRR model with $d < (1 + R) < u$ is a complet market: every contigent claim with payoff $G = f(S_N)$ can be replicated perfectly with a self-financing portfolio composed of risky asset and risk-free asset.

Probability theory

Definition

A σ-algebra is a collection $\mathcal{F}$ of subsets of Ω if:

(i) $\Omega \in \mathcal{F}$,

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$,

(iii) *if for any sequence $A_n \in \mathcal{F}$ we have*

$$\cup_{n=1}^{\infty} A_n \in \mathcal{F}$$

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A real random variable is a function $X : \Omega \longrightarrow \mathbb{R}$ such that

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}$$

for all Borel sets $B \in \mathcal{B}$. We say that X is $\mathcal{F}$-measurable

Conditional expectation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G}$ a sub-σ algebra of $\mathcal{F}$.

Proposition

Let X a real random variable integrable ($\mathbb{E}(|X|) < +\infty$). Then there exists a random variable Y $\mathcal{G}$-measurable integrable such that for each $G \in \mathcal{G}$

$$\mathbb{E}(X\mathbf{1}_G) = \mathbb{E}(Y\mathbf{1}_G).$$

The random variable Y is called the conditional expectation and is denoted by

$$\mathbb{E}\left(X|\mathcal{G}\right)$$

Properties

1. If X is $\mathcal{G}$-measurable, $\mathbb{E}(X|\mathcal{G}) = X$, a.s.
2. $\mathbb{E}\left(\mathbb{E}\left(X|\mathcal{G}\right)\right) = \mathbb{E}\left(X\right)$.
3. Linearity :
$$\mathbb{E}\left(aX + bY|\mathcal{G}\right) = a\mathbb{E}\left(X|\mathcal{G}\right) + b\mathbb{E}\left(Y|\mathcal{G}\right) \text{ a.s.}$$
4. 'Taking out what is known' :
If Z is $\mathcal{G}$-measurable, $\mathbb{E}\left(ZX|\mathcal{G}\right) = Z\mathbb{E}\left(X|\mathcal{G}\right)$ a.s.
5. Tower property : if $\mathcal{A}$ is a sub-σ algebra of $\mathcal{G}$, then:
$$\mathbb{E}\left(\mathbb{E}\left(X|\mathcal{G}\right)|\mathcal{A}\right) = \mathbb{E}\left(X|\mathcal{A}\right) \text{ a.s.}$$
and
$$\mathbb{E}\left(\mathbb{E}\left(X|\mathcal{A}\right)|\mathcal{G}\right) = \mathbb{E}\left(X|\mathcal{A}\right) \text{ a.s.}$$
6. Best approximation : if Z is $\mathcal{G}$-measurable and square integrable,
$$\mathbb{E}\left[\left(X - \mathbb{E}\left(X|\mathcal{G}\right)\right)^2\right] \leq \mathbb{E}\left[\left(X - Z\right)^2\right] \text{ a.s.}$$
$\mathbb{E}\left(X|\mathcal{G}\right)$ is the best approximation in least square sense of X using a $\mathcal{G}$-measurable random variable.

Martingale

We recall that a filtration is a sequence of sub σ-algebra of $\mathcal{A}$ such that $\mathcal{F}_m \subset \mathcal{F}_n$ for $m \leq n$.

We consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, that is to say a space Ω equipped with a σ-algebra $\mathcal{A}$, a probability $\mathbb{P}$ and a filtration $\mathcal{F} := (\mathcal{F}_n, n \in \mathbb{N})$.

Martingale

Definition

A sequence $(M_n, n \geq 0)$ of $\mathbb{R}$-valued random variables is a $\mathcal{F}$-martingale if

(i) *M_n is $\mathcal{F}_n$-measurable for all n,*

(ii) *$\mathbb{E}(|M_n|) < +\infty$ for all n,*

(iii) *$\mathbb{E}\left[M_{n+1}|\mathcal{F}_n\right] = M_n$ for all n.*

Remark

If $(M_n, n \geq 0)$ is a martingale, then

$$\mathbb{E}\left[M_p|\mathcal{F}_n\right] = M_n, \ \forall p \geq n. \tag{4.1}$$

Markov chain

Definition

Let $(S_n, n \geq 0)$ *be a sequence of random variables taking values in a finite or countable set* $\mathcal{E}$*.* S_n *is a Markov chain if:*

$$\mathbb{P}\left(S_{n+1} = y | S_0 = x_0, \ldots, S_n = x_n\right) = \mathbb{P}\left(S_{n+1} = y | S_n = x_n\right).$$

The intuitive meaning of the Markov property is that the future behavior of the process $(S_n)_{n \geq 0}$ after n depends only on the value S_n and is not influenced by the history of the process before n.

The stock prices follow a Markov property, where the current price of the stock contains all the information of the past. The weak form of *Efficient Market Hypothesis* say that the current state contains more information about the future state than all prior states combined.

Markov chain and random walks

The Markov chain is said *time homogenous* if $\mathbb{P}\left(S_{n+1} = y | S_n = x\right)$ does not depend on n.
One then sets:

$$P(x, y) = \mathbb{P}\left(S_{n+1} = y | S_n = x\right).$$

The matrix $(P(x, y))_{x \in \mathcal{E}, y \in \mathcal{E}}$ is called the transition matrix of the Markov chain

Remark

$\forall x, y \in \mathcal{E} \quad P(x, y) \geq 0$ *and,* $\forall x \ \sum_{y \in \mathcal{E}} P(x, y) = 1.$

Random walks

Example

Binomial random walk Let $(X_i, i \geq 1)$ a sequence of i.i.d. (indipendent, identically distributed) random variables with $\mathbb{P}(X_i = \pm 1) = 1/2$. Then $S_n = X_1 + \cdots + X_n$ is a homogenous Markov chain with transition matrix $P(x, x+1) = P(x, x-1) = 1/2$, $P(x, y) = 0$ otherwise.

Example

Trinomial random walk. Let $(X_i, i \geq 1)$ a sequence of i.i.d. random variables $\mathbb{P}(X_i = \pm 1) = \lambda/2$ and $\mathbb{P}(X_i = 0) = 1 - \lambda$, with $0 < \lambda \leq 1$. The transition matrix is given by
$P(x, x+1) = P(x, x-1) = \lambda/2$, $P(x, x) = 1 - \lambda$, $P(x, y) = 0$ otherwise.

Example

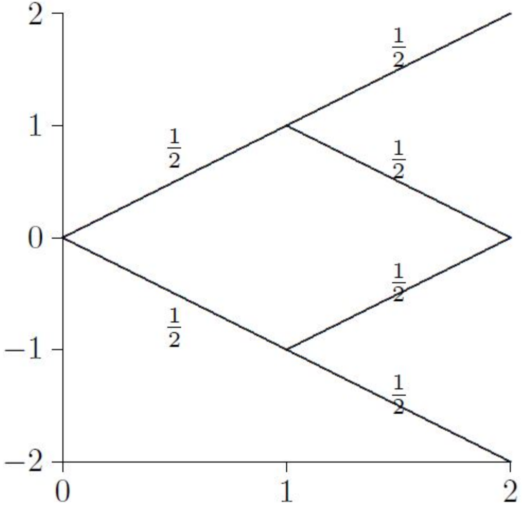
Random walk of Cox Ross-Rubinstein. Let $(U_n, n \geq 0)$ a sequence of i.i.d. random variables with $\mathbb{P}(U_n = u) = p$, $\mathbb{P}(U_n = d) = 1 - p$ and $0 < p < 1$, u and d real numbers.Let $S_0 = x$ and :

$$S_{n+1} = S_n U_{n+1}.$$

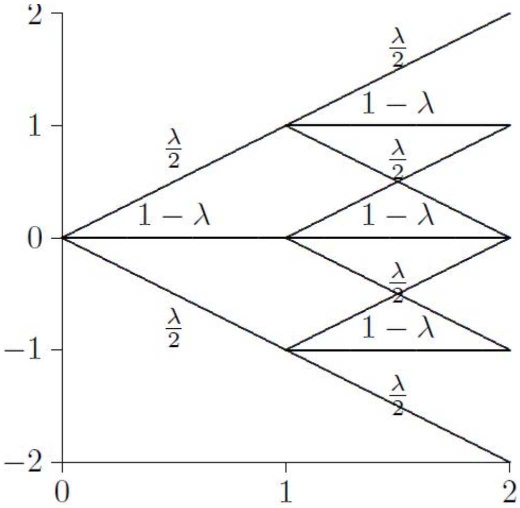
Let $S_n = x \prod_{i=1}^{n} U_i$. S_n is a homogenous Markov chain with transition matrix:

$$\begin{array}{lclcl} P(x, xu) & = & \mathbb{P}\left(S_{n+1} = xu | S_n = x\right) & = & p \\ P(x, xd) & = & \mathbb{P}\left(S_{n+1} = xd | S_n = x\right) & = & 1 - p \\ P(x, y) & = & 0 & \text{otherwise} & \end{array}$$

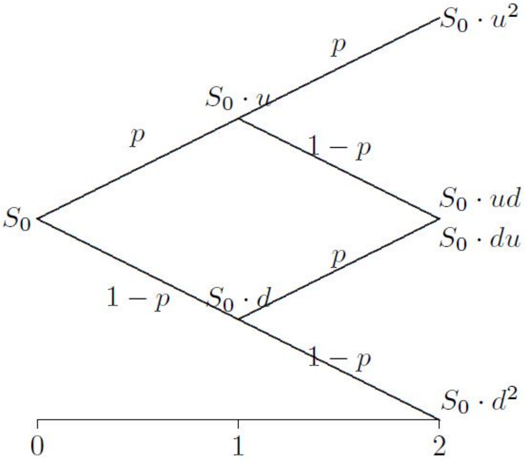
Binomial random walk



Trinomial random walk



Cox-Ross-Rubinstein random walk



More general definition of Markov chain

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $(\mathcal{F}_n, n \geq 1)$ a filtration. A process $(S_n, n \geq 1)$ taking values in a finite or countable set $\mathcal{E}$ is a Markov chain *with the family of transition matrices (P_n) if :*

- *For all n, S_n is $\mathcal{F}_n$-measurable.*
- *For any bounded function ϕ*

$$\mathbb{E}\left(\phi(S_{n+1})|\mathcal{F}_n\right) = \mathbb{E}\left(\phi(S_{n+1})|S_n\right)$$

Dynamic programming algorithm

Option pricing can be solved through a dynamic programming algorithm. Problem: compute

$$\mathbb{E}\left(\phi(S_N)\right).$$

Proposition

Let $\phi(x)$ a bounded function. Let $(S_n, n \geq 0)$ a Markov chain with transition matrix P. Let u be the unique solution of:

$$\begin{cases} u(N,x) = \phi(x), \\ u(n,x) = \sum_{y \in \mathcal{E}} P(x,y) u(n+1,y). \end{cases} \tag{4.5}$$

Then:

$$\mathbb{E}\left(\phi(S_N) | \mathcal{F}_n\right) = u(n, S_n),$$

In particular:

$$u(0,x) = \mathbb{E}\left(\phi(S_N) | \, S_0 = x\right).$$

Proof

The process (M_n) defined by $M_n := u(n, S_n)$ is a $\mathcal{F}$-martingale.
In fact because u is the solution to (4.5) and by the Markov property

$$u(n, S_n) = \mathbb{E}\left[u(n+1, S_{n+1})|S_n\right] = E\left[u(n+1, S_{n+1})|\mathcal{F}_n\right]$$

Any martingale satisfies

$$\mathbb{E}\left[M_N|\mathcal{F}_n\right] = M_n, \ \forall n \leq N.$$

Thus,

$$\mathbb{E}\left[u(N, S_N)|\mathcal{F}_n\right] = u(n, S_n).$$

As $u(N, x) = \phi(x)$,

$$u(n, S_n) = \mathbb{E}\left[\phi(S_N)|\mathcal{F}_n\right].$$

For $n = 0$ one gets

$$u(0, S_0) = \mathbb{E}\left[\phi(S_N)|\mathcal{F}_0\right].$$

As a result, if $S_0 = x$ then

$$u(0, x) = \mathbb{E}\left[\phi(S_N)\right].$$

Examples

Example

Binomial random walk. S_n is a Markov chain with transition matrix $P(x, x+1) = P(x, x-1) = 1/2$. We have $u(0, x)) = \mathbb{E}(\phi(S_N)|S_0 = x)$, where u satisfies :

$$\begin{cases} u(N, x) = \phi(x), \\ u(n, x) = \frac{1}{2}u(n+1, x+1) + \frac{1}{2}u(n+1, x-1). \end{cases}$$

Example

Trinomial random walk. S_n is a Markov chain with transition matrix $P(x, x+1) = P(x, x-1) = \lambda/2$, $P(x, x) = 1 - \lambda$. We have $u(0, x)) = \mathbb{E}(\phi(S_N)|S_0 = x)$, where u satisfies :

$$\begin{cases} u(N, x) = \phi(x), \\ u(n, x) = \frac{\lambda}{2}u(n+1, x+1) + (1-\lambda)u(n+1, x) + \frac{\lambda}{2}u(n+1, x-1). \end{cases}$$

Corollary

Let $\phi(x)$ a bounded function from $\mathcal{E}$ to $\mathbb{R}$ and R a bounded function from $\mathcal{E}$ to $\mathbb{R}_+$. Let $(S_n, n \geq 0)$ be a Markov chain with transition matrix P.

Problem: compute

$$\mathbb{E}\left(\prod_{i=0}^{N-1} \frac{1}{1+R(S_i)} \phi(S_N)\right).$$

Let u be the unique solution of:

$$\text{(10)} \qquad \begin{cases} v(N,x) = \phi(x), \\ v(n,x) = \sum_{y \in \mathcal{E}} \frac{P(x,y)}{1+R(x)} u(n+1,y). \end{cases}$$

Then :

$$\mathbb{E}\left(\prod_{i=n}^{N-1} \frac{1}{1+R(S_i)} \phi(S_N) | \mathcal{F}_n\right) = v(n, S_n),$$

In particular:

$$v(0,x) = \mathbb{E}\left(\prod_{i=0}^{N-1} \frac{1}{1+R(S_i)} \phi(S_N) | S_0 = x\right).$$

Examples

Example

Cox-Ross-Rubinstein random walk

$$V_0 = \mathbb{E}\left(\frac{1}{(1+R)^N}\phi(S_N)\right),$$

we have $V_0 = v(0, S_0)$*, where* v *satisfies:*

$$\begin{cases} v(N,x) = \phi(x), \\ v(n,x) = \dfrac{q}{1+R}v(n+1,xu) + \dfrac{1-q}{1+R}v(n+1,xd). \end{cases}$$

with q *Risk neutral probability*

Tree algorithm
European Put options in discrete model

```
/*Risk neutral probability*/
pu=((1+R)-d)/(u-d);
pd=1-pu;

/* Conditions at maturity*/
for (j=0;j<=N;j++)
    P[j]=MAX(0.,K-S_0*pow(u,N-j)*pow(d,j));

/* Backward induction */
  for (i=1;i<=N;i++)
    for (j=0;j<=N-i;j++)
 P[j]=pow(1.+R,-1.)*(pu*P[j]+pd*P[j+1]);

/*  E(\phi(S_N)|S_0=x) is given in P[0] */
```

Optimal stopping problem

One of the simplest optimal control problems is the optimal stopping problem, where at any time the only two possible control actions are:

- to stop the process (i.e. exercise the option);
- to let it continue (i.e. keeping option alive).

This problem will illustrate the basic ideas of dynamic programming for Markov chains and introduce the fundamental principle of optimality in a simple way.

Stopping times

Definition

A random time τ is a random variable with values in $\mathbb{N} \cup \{+\infty\}$. A random time τ is a stopping time *w.r.t. a filtration $\mathcal{F} := (\mathcal{F}_n, n \in \mathbb{N})$ if $\{\tau \leq n\} \in \mathcal{F}_n$ for all n.*

Proposition

Let $(M_n, n \geq 0)$ be a $\mathcal{F}$-martingale and τ a stopping time w.r.t to $\mathcal{F}$. Then the stopped process

$$M_{\min(n,\tau)}$$

is a martingale.

Theorem (Optional Stopping Theorem)

Let N be a strictly positive integer. Let $(M_n, n \geq 0)$ be a $\mathcal{F}$-martingale.
For any bounded stopping time τ such that $n \leq \tau \leq N$, a.s., there holds

$$\mathbb{E}\left[M_\tau | \mathcal{F}_n\right] = M_n.$$

American option pricing

The American options can be exercised at any time between 0 and T.
The price at time 0 of an American option guaranteeing the cash–flow $\phi(S_p)$ if it is exercised at time $0 \le p \le N$ is given by

$$V_0 = \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}\left[\frac{1}{(1+R)^{\tau}}\phi(S_{\tau})\right]. \tag{5.1}$$

where $\tau \in \mathcal{T}_{0,T}$ is the set of $\mathcal{F}-$ stopping times taking values in $\{0, \cdots, T\}$.

Remark

One of the simplest optimal control problems is the optimal stopping problem, where the only two possible control actions at any time are to stop the process or to let it continue (if it has not yet been stopped). This problem will illustrate the basic ideas of dynamic programming for Markov chains and introduce the fundamental principle of optimality *in a simple way.*

At each stage, the management maker observes the current state of the system and decides either to continue the process or to stop it. If there is only one choice other than stopping, then each policy is characterized at each period by the stopping set, *that is, the set of states where the policy stops the system.*

Dynamic programming algorithm

Proposition

Let $(S_n, n \geq 0)$ be a Markov chain with transition matrix $P(x,y)$. Let u be the unique solution to

$$\begin{cases} u(N,x) & = \phi(x), \\ u(n,x) & = \max\left(\sum_{y\in\mathcal{E}} P(x,y)u(n+1,y), \phi(x)\right) \end{cases} \tag{5.2}$$

Then, for all $0 \leq n \leq N$,

$$\sup_{\tau\in\mathcal{T}_{n,N}} \mathbb{E}\left[\phi(S_\tau)|\mathcal{F}_n\right] = u(n, S_n).$$

In particular, if $S_0 = x$ is deterministic, then

$$\sup_{\tau\in\mathcal{T}_{0,N}} \mathbb{E}\left[\phi(S_\tau)\right] = u(0, x).$$

Proof

- Set
$$Y_n := u(n, S_n) - \mathbb{E}\left[u(n, S_n) | \mathcal{F}_{n-1}\right]$$
and
$$M_n := Y_1 + \cdots + Y_n .$$
Then (M_n) is a $\mathcal{F}$–martingale. In fact
$$\begin{aligned} \mathbb{E}(M_{n+1} - M_n | \mathcal{F}_n) &= \mathbb{E}(Y_{n+1} | \mathcal{F}_n) \\ &= \mathbb{E}\left[u(n+1, S_{n+1}) - \mathbb{E}\left[u(n+1, S_{n+1}) | \mathcal{F}_n\right] | \mathcal{F}_n\right] \\ &= 0. \end{aligned}$$

- Owing to the definition of u, it holds that
$$u(n, S_n) \geq \mathbb{E}\left[u(n+1, S_{n+1}) | \mathcal{F}_n\right] .$$
So,
$$u(n+1, S_{n+1}) - u(n, S_n) \leq u(n+1, S_{n+1}) - \mathbb{E}\left[u(n+1, S_{n+1}) | \mathcal{F}_n\right] = Y_{n+1} .$$

- As $Y_{n+1} \geq u(n+1, S_{n+1}) - u(n, S_n)$, a straightforward computation leads to

$$M_p - M_n \geq u(p, S_p) - u(n, S_n),$$

for all n and all $p \geq n$.

- Besides, if τ is a stopping time such that $n \leq \tau \leq N$,

$$M_\tau - M_n \geq u(\tau, S_\tau) - u(n, S_n).$$

The Optional Stopping Theorem imply that

$$0 = \mathbb{E}\left[M_\tau - M_n | \mathcal{F}_n\right] \geq \mathbb{E}\left[u(\tau, S_\tau) | \mathcal{F}_n\right] - u(n, S_n)$$

Thus, we have just checked that

$$u(n, S_n) \geq \mathbb{E}\left[u(\tau, S_\tau) | \mathcal{F}_n\right].$$

For all stopping time taking values in $[n, N]$, there holds

$$u(n, S_n) \geq \mathbb{E}\left[\phi(S_\tau) | \mathcal{F}_n\right],$$

because from the definition u, $u(n, x) \geq \phi(x)$.

- Consequently,

$$u(n, S_n) \geq \sup_{\tau \in \mathcal{T}_{n,N}} \mathbb{E}\left[\phi(S_\tau)|\mathcal{F}_n\right]$$

- It remains to find a stopping time τ_n^* taking values in $[n, N]$ and such that

$$u(n, S_n) = \mathbb{E}\left[\phi(S_{\tau_n^*})|\mathcal{F}_n\right].$$

  To this end, set

$$\tau_n^* := \inf\left\{p > n, u(p, S_p) = \phi(S_p)\right\}.$$

  One can easily check that τ_n^* is a stopping time.
  Besides, $\tau_n^* \leq N$ since $u(N, S_N) = \phi(S_N)$.

- On the set $\{\omega; p < \tau_n^*(\omega)\}$ we have

$$u(p, S_p) = \mathbb{E}\left[u(p+1, S_{p+1})|\mathcal{F}_p\right],$$

  so that

$$Y_{p+1} = u(p+1, S_{p+1}) - u(p, S_p).$$

- Consequently,

$$\begin{array}{rcl} Y_{n+1} & = & u(n+1, S_{n+1}) - u(n, S_n) \\ Y_{n+2} & = & u(n+2, S_{n+2}) - u(n+1, S_{n+1}) \\ \vdots & \vdots & \vdots \\ Y_{\tau_n^*} & = & u(\tau_n^*, S_{\tau_n^*}) - u(\tau_n^* - 1, S_{\tau_n^*-1}). \end{array}$$

- Therefore,

$$M_{\tau_n^*} - M_n = u(\tau_n^*, S_{\tau_n^*}) - u(n, S_n).$$

Using the Optional Sampling Theorem, one gets

$$0 = \mathbb{E}\left[M_{\tau_n^*} - M_n | \mathcal{F}_n\right] = \mathbb{E}\left[u(\tau_n^*, S_{\tau_n^*}) | \mathcal{F}_n\right] - u(n, S_n)$$

- So we have
$$u(n, S_n) = \mathbb{E}\left[\phi(S_{\tau_n^*})|\mathcal{F}_n\right].$$
because by definition of τ_n^*
$$u(\tau_n^*, S_{\tau_n^*}) = \phi(S_{\tau_n^*}).$$
Remember that
$$u(n, S_n) \geq \sup_{\tau \in \mathcal{T}_{n,N}} \mathbb{E}\left[\phi(S_\tau)|\mathcal{F}_n\right].$$
This implies that
$$u(n, S_n) = \sup_{\tau \in \mathcal{T}_{n,N}} \mathbb{E}\left[\phi(S_\tau)|\mathcal{F}_n\right].$$

Optimal stopping time

The stopping time

$$\tau_0^* := \inf \{p > 0,\ u(p, S_p) = \phi(S_p)\}$$

satisfies

$$\sup_{\tau \in \mathcal{T}_{0,N}} \mathbb{E}\left[\phi(S_\tau)\right] = \mathbb{E}\left[\phi(S_{\tau_0^*})\right].$$

The stopping time τ_0^* is an optimal stopping time.

Dynamic programming algorithm

Let $(S_n, n \geq 0)$ be a Markov chain with transition matrix $P(x, y)$. Let u be the unique solution to

$$\text{(13)} \qquad \begin{cases} u(N, x) &= \phi(x), \\ u(n, x) &= \max\left(\sum_{y \in \mathcal{E}} \frac{1}{1+R} P(x, y) u(n+1, y), \phi(x)\right). \end{cases}$$

Then, for all $0 \leq n \leq N$,

$$\sup_{\tau \in \mathcal{T}_{n,N}} \mathbb{E}\left[(1+R)^{-(\tau-n)} \phi(S_\tau) | \mathcal{F}_n\right] = u(n, S_n).$$

In particular, if $S_0 = x$ is deterministic, then

$$\sup_{\tau \in \mathcal{T}_{0,N}} \mathbb{E}\left[(1+R)^{-\tau} \phi(S_\tau)\right] = u(0, x).$$

American Call Options

If the underlying pays no dividends, one can prove that exercising at maturity is the optimal strategy for a Call option.

The price of an American Call option equates the European price for the same option.

In fact by the Call-Put Parity theorem the European Call option price is decomposed in the sum of

$$C_t^{EU} = S_t - K + K - K(1+R)^{-(T-t)} + P_t^{EU}$$

intrinsic value (value obtained by exercising at time t), time premium (interest premium, purely financial component), Put option price.

It holds

$$C_t^{AM} \geq C_t^{EU} \geq S_t - K,$$

then early exercise is not convenient and in particular

$$C_0^{AM} = C_0^{EU}.$$

American Put Options

On the contrary, for the Put options it is different.

$$P_t^{EU} = K - S_t + K(1+R)^{-(T-t)} - K + C_t^{EU}.$$

Early exercise may be convenient.

Binomial algorithm
American Put option in discrete model

```
/*Risk neutral probability*/
pu=((1+R)-d)/(u-d);
pd=1-pu;

/*Intrinsic values*/
for (j=0;j<=2*N;j++)
    InV[j]=max(0.,K-x*pow(u,N-j));
/*Terminal condition*/
for (j=0;j<=N;j++)
    P[j]=InV[2*j];

/*Dynamic programming*/
  for (i=1;i<=N;i++)
    for (j=0;j<=N-i;j++)
 P[j]=MAX(pow(1.+R,-1.)*(pu*P[j]+pd*P[j+1]),InV[i+2*j]);

  /* Price in P[0] */
```