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Monte Carlo methods for American Options

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American options in a unidimensional model

The stock price process satisfies the following SDE:

$$\frac{dS_t}{S_t} = r dt + \sigma dB_t$$

The value at time $t = 0$ of an American Put option on the risky underlying, with maturity T and payoff function $\psi(x) = (K - x)_+$, is, in the connection with Optimal Stopping Theory, given by:

$$v(0, s_0) = \sup_{\tau \in \mathcal{T}_{0,T}} E_Q \left[e^{-r\tau} \psi(S_\tau) \right]$$

where $\mathcal{T}_{0,T}$ is the set of all stopping times with values in $[0, T]$.
Since it is American, plain Monte Carlo simulation is not feasible.

Longstaff–Schwartz Method

Idea : Approximation of Conditional Continuation values with regression

- Discrete time steps. Bermudan option.
- Monte Carlo simulation of the underlying asset during the lifetime of the option.
- Early exercise backwards in time: at each time steps comparison between the exercise value and the continuation value computed using a regression.
- discounting cashflows and averaging the paths.

Bermudan Options

- Exercise times $t_0 = 0 < t_1 < \ldots < t_N = T$

Price in $t = 0$

$$P_0 = \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}\Big[e^{-r\tau}\psi(S_\tau)\Big]$$

Backward Dynamic Programming for P

$$\begin{cases} P_N := \psi\left(S_{t_N}\right) \\ P_j := \max\left(\psi\left(S_{t_j}\right), \mathbb{E}\left[e^{-r(t_{j+1}-t_j)} P_{j+1} | \mathcal{F}_{t_j}\right]\right) \quad 0 \le j \le N-1 \end{cases}$$

where

$$P_j = P(t_j, S_{t_j}) = \sup_{\tau \in \mathcal{T}_{t_j,T}} \mathbb{E}\Big[e^{-r(\tau - t_j)}\psi(S_\tau) | \mathcal{F}_{t_j}\Big]$$

Optimal Stopping Time

$$\tau_0^* := \min \left\{ t_k \geq 0; \psi \left(S_{t_k} \right) = P_k \right\}$$

$$P_0 = \mathbb{E} \left[e^{-r\tau_0^*} \psi \left(S_{\tau_0^*} \right) \right]$$

Moreover

$$\tau_j^* := \min \left\{ t_k \geq t_j; \psi \left(S_{t_k} \right) = P_k \right\}$$

$$P_j = \mathbb{E} \left[e^{-r(\tau_j^* - t_j)} \psi \left(S_{\tau_j^*} \right) | \mathcal{F}_{t_j} \right]$$

Backward Dynamic Programming for τ^*

$$\begin{cases} \tau_N^* := T \\ \tau_j^* := t_j \mathbf{1}_{A_j} + \tau_{j+1}^* \mathbf{1}_{A_j^c} \quad 0 \leq j \leq N-1 \end{cases}$$

where

$$A_j := \left\{ \psi\left(S_{t_j}\right) \geq P_j \right\}$$

We can eliminate the dependency on P thanks to

$$\psi\left(S_{t_j}\right) \geq P_j \Longleftrightarrow \psi\left(S_{t_j}\right) \geq \mathbb{E}\left[e^{-r(t_{j+1}-t_j)} P_{j+1} \,|\mathcal{F}_{t_j}\right] \Longleftrightarrow$$

$$\psi\left(S_{t_j}\right) \geq \mathbb{E}\Big[e^{-r(t_{j+1}-t_j)} \mathbb{E}\big[e^{-r(\tau_{j+1}^*-t_{j+1})} \psi\left(S_{\tau_{j+1}^*}\right) |\mathcal{F}_{t_{j+1}}\big] |\mathcal{F}_{t_j}\Big]$$

Therefore

$$A_j := \left\{ \psi\left(S_{t_j}\right) \geq \mathbb{E}\Big[e^{-r(\tau_{j+1}^*-t_j)} \psi\left(S_{\tau_{j+1}^*}\right) |\mathcal{F}_{t_j}\Big] = \right.$$

$$\left. \psi\left(S_{t_j}\right) \geq \mathbb{E}\Big[e^{-r(\tau_{j+1}^*-t_j)} \psi\left(S_{\tau_{j+1}^*}\right) |S_{t_j}\Big] \right\}$$

So we use the backward procedure

$$\begin{cases} \tau_N^* := T \\ \tau_j^* := t_j \mathbf{1}_{A_j} + \tau_{j+1}^* \mathbf{1}_{A_j^c} \quad 0 \leq j \leq N-1 \end{cases}$$

where

$$A_j := \left\{ \psi\left(S_{t_j}\right) \geq \mathbb{E}\left[e^{-r\left(\tau_{j+1}^* - t_j\right)} \psi\left(S_{\tau_{j+1}^*}\right) | S_{t_j} \right] \right\}$$

We can consider only in-the-money paths in the estimation

It is useless to compute

$$\mathbb{E}\left[e^{-r\left(\tau_{j+1}^* - t_j\right)} \psi\left(S_{\tau_{j+1}^*}\right) | S_{t_j} \right]$$

when $\psi\left(S_{t_j}\right) = 0$.

Longstaff–Schwartz Method

Compute τ_0^*

$$P_0 = \mathbb{E}\left[e^{-r\tau_0^*} \psi\left(S_{\tau_0^*}\right) \right]$$

using the backward induction on the optimal stopping times.

Approximation of the conditional expectations

$$\mathbb{E}\left[e^{-r(\tau_{j+1}^* - t_j)} \psi\left(S_{\tau_{j+1}^*}\right) | S_{t_j} \right]$$

using regressions

Let

$$Y_j = e^{-r(\tau_{j+1}^* - t_j)} \psi\left(S_{\tau_{j+1}^*}\right)$$

We need compute

$$\mathbb{E}\left[Y_j | S_{t_j} \right]$$

The regression method

$$\mathbb{E}\Big[Y_j | S_{t_j}\Big]$$

can be expressed as $\phi_j(S_{t_j})$, where ϕ_j minimizes

$$\mathbb{E}\left[\left(Y_j - f(S_{t_j})\right)^2\right]$$

among all functions f such that $\mathbb{E}\left[(f(S_{t_j}))^2\right] < +\infty$.

Since L^2 is a Hilbert space the conditional expectation can be represented as a linear function of a total basis of L^2

$$\phi_j = \sum_{l \geq 1} \alpha_l g_l$$

Algorithm in finite dimensional space

1. Initialize $\tau_N^* := T$

2. Define $\alpha^j = (\alpha_l^j, 1 \leq l \leq k)$ as the vector wich minimizes

$$\mathbb{E}\left[\left(e^{-r(\tau_{j+1}^*-t_j)}\psi\left(S_{\tau_{j+1}^*}\right)-(\alpha^j,g)(S_{t_j})\right)^2\right]$$

$(\alpha^j, g) = \sum_{1 \leq l \leq k} \alpha_l g_l$

3. Define

$$\tau_j^* := t_j \mathbf{1}_{A_j} + \tau_{j+1}^* \mathbf{1}_{A_j^c} \quad 0 \leq j \leq N-1$$

$$\mathrm{A}_j := \left\{\psi\left(S_{t_j}\right) \geq (\alpha^j, g)(S_{t_j})\right\}$$

Empirical version

1. Initialize $\tau_N^m := T$

2. Define $\alpha_M^j = (\alpha_l^j, 1 \leq l \leq k)$ as the vector wich minimizes

$$\frac{1}{M} \sum_{1 \leq m \leq M} \left(e^{-r(\tau_{j+1}^m - t_j)} \psi \left(S_{\tau_{j+1}^m}^m \right) - (\alpha^j, g)(S_{t_j}^m) \right)^2$$

3. Define for each trajectory m

$$\tau_j^m := t_j \mathbf{1}_{A_j} + \tau_{j+1}^m \mathbf{1}_{A_j^c} \quad 0 \leq j \leq N-1.$$

$$\mathrm{A}_j := \left\{ \psi \left(S_{t_j}^m \right) \geq (\alpha_M^j, g)(S_{t_j}^m) \right\}$$

Estimator of the price is given by

$$P_0 = \max \left(\psi(x_0), \frac{1}{M} \sum_{1 \leq m \leq M} e^{-r\tau_1^m} \psi \left(S_{\tau_1^m}^m \right) \right).$$

Remarks

- The minimization problem is standard least-squares approximation problem.

- Choiche of Basis Functions (Canonical, Legendre, Laguerre).

  – Canonical basis functions: $g_1(x) = 1$, $g_2(x) = x$, $g_3(x) = x^2$, $g_n(x) = x^n$

  – Laguerre basis functions: $g_1(x) = e^{-x^2}$, $g_2(x) = e^{-x^2}(1-x)$, $g_3(x) = e^{-x^2}(1 - 2x + \frac{x^2}{2})$, $g_{n+1}(x) = e^{-\frac{x}{2}} \frac{e^x}{n!} \frac{d^n}{dx^n}(x^n e^{-x})$

- We can restrict outself in the regression to trajectories such that $\left\{\psi\left(S_{t_j}\right) > 0\right\}$

Numerical example

- American Put option on one asset
- The maturity is $T = 3$ years and the strike $K = 1.1$
- $r = 0.06$. We have the discount factors $e^{-r} = 0.94176$ and $e^{-2r} = 0.88692$
- We need to compute at each time step $t_j = 1, 2$ the conditional expectations

$$\mathbb{E}\Big[Y_j | S_{t_j}\Big],$$

where $Y_j = e^{-r(\tau^*_{j+1} - t_j)}\psi\left(S_{\tau^*_{j+1}}\right)$ is the discounted payoff.

- We regress

$$Y_j$$

on the canonical basis functions $1, S_j, S_j^2$

$$\min_{\alpha_1^j, \alpha_2^j, \alpha_3^j} \quad E\left\{\left[Y_j - (\alpha_1^j + \alpha_2^j S_j + \alpha_3^j S_j^2)\right]^2\right\}$$

Final Remarks

- We can use backward approach which uses Brownian bridge law $B_0 = 0$ and $B_{t_{j+1}} = b$. Then

$$B_{t_j} \sim \mathcal{N}\left(\frac{t_j}{t_{j+1}} b, \frac{t_j}{t_{j+1}} (t_{j+1} - t_j)\right).$$

- The Longstaff-Schwartz method is very useful with several underlying assets.

- A rigorous proof of the convergence of the algorithm is given by Clement, Lamberton, Protter (Finance and Stochastics 2002).