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Finite Difference schemes for Option Pricing

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Feynamc Kac Formula for Brownian motions

Suppose that the function f is continuous and bounded. Set

$$u(t,x) = \mathbb{E}\Big[f(x+B_t)\Big]$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion.
Then u is the unique smooth solution of the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} \text{ for } t>0 \text{ and } x\in\mathbb{R}, \\ u(0,x) = f(x), \quad \text{x}\in\mathbb{R}. \end{cases}$$

By definition $u(t,x)$:

$$u(t,x) = \int f(y) e^{-\frac{(y-x)^2}{2t}} \frac{dy}{\sqrt{2\pi t}}.$$

Because f is bounded for $t > 0$ it holds :

$$\begin{aligned}
\frac{\partial u}{\partial t}(t,x) &= \int f(y) e^{-\frac{(y-x)^2}{2t}} \left(\frac{(y-x)^2}{2t^2} - \frac{1}{2t} \right) \frac{dy}{\sqrt{2\pi t}}, \\
\frac{\partial u}{\partial x}(t,x) &= \int f(y) e^{-\frac{(y-x)^2}{2t}} \left(\frac{(y-x)}{t} \right) \frac{dy}{\sqrt{2\pi t}}, \\
\frac{\partial^2 u}{\partial x^2}(t,x) &= \int f(y) e^{-\frac{(y-x)^2}{2t}} \left(\frac{(y-x)^2}{t^2} - \frac{1}{t} \right) \frac{dy}{\sqrt{2\pi t}}.
\end{aligned}$$

Therefore for $t > 0$:

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(t,x).$$

Moreover $u(0,x) = f(x)$.

Heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} \text{ for } t > 0 \text{ and } x \in \mathbb{R}, \\ u(0,x) = f(x), \quad \text{x} \in \mathbb{R}. \end{cases}$$

Such problem is called a evolution problem in time, as the solution at time $t \geq 0$ is determined from the values at time $t = 0$, which is called initial condition.

We are interested in the numerical computation of the price function u.

We will consider a deterministic numerical method: the finite difference method. The numerical procedure consists in two steps:

- Discretize the problem by using a consistent and stable approximation method.
- Implement a computational method to solve the discrete equation.

Finite Difference methods for the Heat equation

- We start by limiting the integration domain in space. The parabolic problem is localized to a bounded domain in space $\Omega_l =]-l, l[$;
- an approximate solution is sought by means of finite difference methods involving discrete functions. This leads to a problem in finite dimension.

The basic idea of the finite difference scheme consists in approximating the derivation operator by a discrete operator. For example

$$u''(x) \cong \frac{u(x+\Delta x) - 2u(x) + u(x-\Delta x)}{\Delta x^2}$$

The parameter Δx chosen arbitrarily small has a fixed non zero value.

Approximation

Consider a function $u(x) : [-l, l] \to \mathbb{R}$, $u \in C^4(-l, l)$.
Consider a uniform grid

$$x_i = -l + i\Delta x \quad \text{for } 0 \leq i \leq N + 1.$$

with $\Delta x = \frac{2l}{N+1}$.
By Taylor expansion

$$u(x_i + \Delta x) = u(x_i) + \Delta x u'(x_i) + \frac{1}{2}\Delta x^2 u''(x_i + \nu\Delta x), \quad 0 \leq \nu \leq 1$$

and

$$u(x_i + \Delta x) = u(x_i) + \Delta x u'(x_i) + \frac{1}{2}\Delta x^2 u''(x_i) + \frac{1}{6}\Delta x^3 u^{(3)}(x_i) + \frac{1}{24}\Delta x^4 u^{(4)}(x_i + \nu_x^+ \Delta x)$$

$$u(x_i - \Delta x) = u(x_i) - \Delta x u'(x_i) + \frac{1}{2}\Delta x^2 u''(x_i) - \frac{1}{6}\Delta x^3 u^{(3)}(x_i) + \frac{1}{24}\Delta x^4 u^{(4)}(x_i + \nu_x^- \Delta x)$$

with $-1 \leq \nu_x^- \leq 0$, $0 \leq \nu_x^+ \leq 1$.

Difference operators

Let us denote $u_i = u(x_i), i = 1, .., N$. Then

$$u'(x_i) = \frac{u_{i+1} - u_i}{\Delta x} + O(\Delta x)$$

and

$$| - u'(x_i) + \frac{u_{i+1} - u_i}{\Delta x}| \leq \frac{\Delta x}{2} \max_{x \in [-l,l]} |u''(x)|$$

Moreover

$$u''(x_i) = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + O(\Delta x^2)$$

and

$$| - u''(x_i) + \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}| \leq \frac{\Delta x^2}{12} \max_{x \in [-l,l]} |u^{(4)}(x)|$$

Localized problem

The problem is solved in a finite interval $\Omega_l =]-l, l[$.

$$(1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2} \dfrac{\partial^2 u}{\partial x^2} \text{ for } t > 0 \text{ and } x \in]-l, l[\\ u(t, \pm l) = 0 \text{ for each } t > 0. \\ u(0, x) = f(x). \end{cases}$$

We impose Dirichlet boundary condition $u(t, \pm l) = 0$ for each $t > 0$.

Finite difference

For the numerical solution of the problem by finite difference method, we introduce a grid of mesh points

$$(t_n, x_i) = (n\Delta t, -l + i\Delta x), \quad n = 0, .., M \quad and \quad i = 0, ..., N+1$$

where

$$\Delta t = \frac{T}{M}, \quad \Delta x = \frac{2l}{N+1}$$

are mesh parameters which are thought of as tending to zero.

Δt is the time step and Δx is the space step.

We approximate $\frac{\partial}{\partial t}$ and $\frac{\partial^2}{\partial x^2}$.

$$\frac{\partial}{\partial t} u(t_n, x_i) \cong \frac{u(t_{n+1}, x_i) - u(t_n, x_i)}{\Delta t}$$

and

$$\frac{\partial^2}{\partial x^2} u(t_n, x_i) \cong \frac{u(t_n, x_{i+1}) - 2u(t_n, x_i) + u(t_n, x_{i-1})}{\Delta x^2}$$

Let be u_i^n an approximation of exact solution u at the node x_i and at time $t_n = n\Delta t$.

$$u_i^n \cong u(t_n, x_i)$$

We obtain the following explicit scheme.

Explicit scheme

We have to solve directly at each time step

$$\begin{cases} \frac{u_i^{n+1}-u_i^n}{\Delta t} = \frac{1}{2}\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{\Delta x^2}, \quad i=1,..,N \qquad n=0,..,M-1 \\ u_i^0 = f(x_i), \quad i=0,..,N+1 \\ u_0^n = u_{N+1}^n = 0, \quad \forall n>0 \end{cases}$$

We can write

$$u_i^{n+1} = \frac{\lambda}{2}u_{i-1}^n + (1-\lambda)u_i^n + \frac{\lambda}{2}u_{i+1}^n$$

with

$$\lambda = \frac{\Delta t}{\Delta x^2}$$

Convergence of the Explicit scheme

We need that the solution of the scheme approximate the solution of the corresponding PDE.
Moreover the approximation need improves as the grid spacings, Δx and Δt, tend to zero.

A scheme that has such behaviour is called a convergent scheme.

- Probabilistic intepretation. The space step Δx and the time step Δt have cannot be chosen indipendently one from the other. $0 < \lambda \leq 1$. Limit central theorem
- The numerical analysis studies the convergence checking different properties. Consistency, Stability

- Consistency The notion of consistency enables us to measure the error produced by approximating the continuous operator by a discrete operator.
  It can be computed on the exact solution of the continuous problem, thanks to a Taylor expansion.
- Stability The approximation is bounded.

Consistency

Definition We say that the scheme

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{1}{2}\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

is consistent with the operator

$$\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2}$$

if for any smooth function $v = v(t,x)$ the difference

$$\Big[\frac{v(t+\Delta t,x) - v(t,x)}{\Delta t} - \frac{1}{2}\frac{v(t,x+\Delta x) - 2v(t,x) + v(t,x-\Delta x)}{\Delta x^2}\Big] - \Big(\frac{\partial v}{\partial t} - \frac{1}{2}\frac{\partial^2 v}{\partial x^2}\Big)(t,x)$$

goes to zero when $\Delta x, \Delta t \to 0$.

The difference is called the truncation error for the function v.

Accuracy

Definition The previous scheme is accurate of order q in time and p in space for the operator

$$\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2}$$

if for any smooth function v the truncation error goes to zero as

$$O(\Delta x^p + \Delta t^q)$$

Consistency of the explicit scheme

Lemma The explicit scheme is consistent, accurate of order one in time and two in space for the the heat equation operator $u_t - \frac{1}{2}u_{xx}$.

Proof From Taylor expansion we have for each $v \in C^{2,4}(\mathbb{R}_+, \Omega_l)$

$$\frac{v(t+\Delta t, x) - v(t,x)}{\Delta t} - \frac{1}{2}\frac{v(t, x+\Delta x) - 2v(t,x) + v(t, x-\Delta x)}{\Delta x^2} =$$

$$v_t(t,x) - \frac{1}{2}v_{xx}(t,x) + \frac{\Delta t}{2}v_{tt}(t+\nu_t\Delta t, x) - \frac{\Delta x^2}{48}(v_{xxxx}(t, x+\nu_x^-\Delta x) + v_{xxxx}(t, x+\nu_x^+\Delta x)),$$

with $0 \le \nu_t \le 1$, $-1 \le \nu_x^- \le 0$, $0 \le \nu_x^+ \le 1$.

Stability

But the consistency is not enough to let us prove the convergence of the scheme. Another notion is required, that of stability.

Definition The scheme is said to be stable in the L^∞ norm iff there is a constant $K > 0$ indipendent of Δt and Δx such that for each n

$$\|u^n\|_\infty \leq K \|f\|_\infty,$$

where $\|u^n\|_\infty = \sup_{0 \leq i \leq N+1} |u_i^n|$.

The idea is that there can be no growth over time.

Stability of the explicit scheme

Lemma The explicit scheme is stable in the L^∞ norm iff $\lambda = \frac{\Delta t}{\Delta x^2} \leq 1$.

Proof

$$u_i^{n+1} = \frac{\lambda}{2} u_{i-1}^n + (1-\lambda) u_i^n + \frac{\lambda}{2} u_{i+1}^n.$$

Under the hypothesis $\lambda \leq 1$, the coefficients λ and $1-\lambda$ of the linear combinations are positive or vanishes.

So that

$$|u_i^{n+1}| \leq \frac{\lambda}{2} |u_{i-1}^n| + (1-\lambda) |u_i^n| + \frac{\lambda}{2} |u_{i+1}^n|,$$

and

$$\|u^{n+1}\|_\infty \leq \|u^n\|_\infty.$$

The stability result is given by recurrence.

Using apagoge, suppose now $\lambda > 1$. Let us consider $f_i = \alpha(-1)^i$.

$u_i^1 = \dfrac{\lambda}{2}\alpha(-1)^{i-1} + (1-\lambda)\alpha(-1)^i + \dfrac{\lambda}{2}\alpha(-1)^{i+1} = (-1)^i\alpha(1-2\lambda) = f_i(1-2\lambda)$.

Because $|1-2\lambda| > 1$, we have not stability in the L^∞ norm

$$\|u^1\|_\infty > \|f\|_\infty.$$

Convergence theorem : Explicit Scheme

Theorem Let u the exact solution of the heat problem.
Suppose that $u \in C^{2,4}(\mathbb{R}_+, \Omega_l)$.
Let u_i^n the numerical solution obtained with the explicit scheme. Suppose that

$$\lambda = \frac{\Delta t}{\Delta x^2} \leq 1$$

(i.e. the scheme is stable).
Then, for each $T > 0$, it exists a constant $C_T > 0$ (depending only on u and T) such that

$$\sup_{\substack{0 \leq i \leq N+1 \\ 0 \leq n \leq M}} |u_i^n - u(t_n, x_i)| \leq C_T(\Delta t + \Delta x^2).$$

We can conclude that the explicit scheme is convergent when $\lambda \leq 1$

$$\lim_{\Delta t, \Delta x \to 0} \left(\sup_{\substack{0 \leq i \leq N+1 \\ 0 \leq n \leq M}} |u_i^n - u(t_n, x_i)| \right) = 0.$$

Proof By Taylor expansion at point x_i in space and t_n in time (Consistency lemma) we have

$$\frac{u(t_{n+1},x_i)-u(t_n,x_i)}{\Delta t}-\frac{1}{2}\frac{u(t_n,x_{i+1})-2u(t_n,x_i)+u(t_n,x_{i-1})}{\Delta x^2}=\epsilon_i^n$$

with

$$\epsilon_i^n=\frac{\Delta t}{2}u_{tt}(t_n+\nu_t\Delta t,x_i)-\frac{\Delta x^2}{48}(u_{xxxx}(t_n,x_i+\nu_x^-\Delta x)+u_{xxxx}(t_n,x_i+\nu_x^+\Delta x)),$$

with $0\le\nu_t\le1$, $-1\le\nu_x^-\le0$, $0\le\nu_x^+\le1$.
Denote by

$$K=\sup_{-l\le x\le l,0\le t\le T}(|u_{tt}(t,x)|+|u_{xxxx}(t,x)|).$$

When $n\Delta t\le T$ we have

$$|\epsilon_i^n|\le\frac{K\Delta t}{2}+\frac{K\Delta x^2}{24}$$

.

Let us introduce the numerical error $z_i^n = u_i^n - u(t_n, x_i)$. It is easy to see that

$$\frac{z_i^{n+1} - z_i^n}{\Delta t} - \frac{1}{2}\frac{z_{i+1}^n - 2z_i^n + z_{i-1}^n}{\Delta x^2} = -\epsilon_i^n.$$

So that

$$z_i^{n+1} = \frac{\lambda}{2} z_{i-1}^n + (1-\lambda) z_i^n + \frac{\lambda}{2} z_{i+1}^n - \Delta t \epsilon_i^n.$$

Using the hypothesis that

$$\lambda = \frac{\Delta t}{\Delta x^2} \leq 1,$$

we have

$$|z_i^{n+1}| \leq \|z^n\|_\infty + \Delta t (\frac{K \Delta t}{2} + \frac{K \Delta x^2}{24}).$$

So that

$$\|z^{n+1}\|_\infty \le \|z^n\|_\infty + \Delta t \frac{K}{2}(\Delta t + \frac{\Delta x^2}{12}).$$

and

$$\|z^{n+1}\|_\infty \le \|z^n\|_\infty + \Delta t \frac{K}{2}(1 + \frac{1}{12})(\Delta t + \Delta x^2).$$

Because $\|z^0\|_\infty = 0$ ($u_i^0 = f(x_i)$) we conclude that for each $n \le M$

$$\|z^n\|_\infty \le n\Delta t \frac{K}{2}(1 + \frac{1}{12})(\Delta t + \Delta x^2).$$

This gives the result

$$\sup_{\substack{0\le i\le N+1 \\ 0\le n\le M}} |u_i^n - u(t_n, x_i)| \le C_T(\Delta t + \Delta x^2).$$

with $C_T = \frac{KT}{2}(1 + \frac{1}{12})$.

Fully Implicit scheme

The fully implicit scheme is given by:

$$\begin{cases} \frac{u_i^{n+1}-u_i^n}{\Delta t} = \frac{1}{2}\frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{\Delta x^2}, \quad i=1,..,N \qquad n=0,..,M-1 \\ u_i^0 = f(x_i), \quad i=0,..,N+1 \\ u_0^n = u_{N+1}^n = 0, \quad \forall n>0 \end{cases}$$

We have to solve at each time step a linear system.

Let us denove

$$U^n = (u_1^n, \dots, u_N^n) \ ,$$

and :

$$B = \begin{pmatrix} 1+\lambda & -\frac{\lambda}{2} & 0 & \dots & 0 \\ -\frac{\lambda}{2} & 1+\lambda & -\frac{\lambda}{2} & \dots & 0 \\ 0 & -\frac{\lambda}{2} & 1+\lambda & -\frac{\lambda}{2} & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -\frac{\lambda}{2} & 1+\lambda \end{pmatrix},$$

Then at each time step we have to solve :

$$BU^{n+1} = U^n.$$

B is invertible.

In fact, B is strictly diagonally dominant, i.e.

$$|b_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |b_{ij}| = \Lambda_i, \quad i = 1, .., n$$

Therefore, the union of the disks $|z - b_{ii}| \leq \Lambda_i$, does not include the origin $z = 0$ of the complex plan, and from the Gerschgorin Theorem, $\lambda = 0$ is not an eigenvalue of B, proving B is nonsingular.

Moreover, given the linear system $Bx = g$, then for each i :

$$-\frac{\lambda}{2}x_{i-1} + (1+\lambda)x_i - \frac{\lambda}{2}x_{i+1} = g_i.$$

If $\lambda > 0$ we have :

$$|x_i| \leq \frac{\|g\|_\infty + \lambda\|x\|_\infty}{1+\lambda},$$

so that $\|x\|_\infty \leq \|g\|_\infty$. Therefore B^{-1} verify

$$\|B^{-1}g\|_\infty \leq \|g\|_\infty.$$

We obtain a result of unconditionally stability.

For each Δx and for each Δt

$$\|U^{n+1}\|_\infty \leq \|U^n\|_\infty.$$

Convergence theorem : Fully Implicit Scheme

Theorem Let u the exact solution of the heat problem.
Suppose that $u \in C^{2,4}(\mathbb{R}_+, \Omega_l)$.
Let u_i^n the numerical solution obtained with the fully implicit scheme. Then, for each $T > 0$, there exists a constant $C_T > 0$ (depending only on u and T) such that

$$\sup_{\substack{0 \leq i \leq N+1 \\ 0 \leq n \leq M}} |u_i^n - u(t_n, x_i)| \leq C_T(\Delta t + \Delta x^2).$$

The fully implicit scheme is unconditionally stable.

Proof By Taylor expansion at point x_i in space and t_{n+1} in time we have

$$\frac{u(t_{n+1},x_i)-u(t_n,x_i)}{\Delta t}-\frac{1}{2}\frac{u(t_{n+1},x_{i+1})-2u(t_{n+1},x_i)+u(t_{n+1},x_{i-1})}{\Delta x^2}=\epsilon_i^{n+1}$$

with

$$\epsilon_i^{n+1}=-\frac{\Delta t}{2}u_{tt}(t_{n+1}+\nu_t^-\Delta t,x_i)$$

$$-\frac{\Delta x^2}{48}(u_{xxxx}(t_{n+1},x_i+\nu_x^-\Delta x)+u_{xxxx}(t_{n+1},x_i+\nu_x^+\Delta x)),$$

with $-1\leq\nu_t^-\leq 0$, $-1\leq\nu_x^-\leq 0$, $0\leq\nu_x^+\leq 1$.

Denote by

$$K=\sup_{-l\leq x\leq l,0\leq t\leq T}(|u_{tt}(t,x)|+|u_{xxxx}(t,x)|).$$

When $n\Delta t\leq T$ we have

$$|\epsilon_i^{n+1}|\leq\frac{K\Delta t}{2}+\frac{K\Delta x^2}{24}$$

.

Let us introduce the numerical error $z_i^n = u_i^n - u(t_n, x_i)$. It is easy to see that

$$\frac{z_i^{n+1} - z_i^n}{\Delta t} - \frac{1}{2}\frac{z_{i+1}^{n+1} - 2z_i^{n+1} + z_{i-1}^{n+1}}{\Delta x^2} = -\epsilon_i^{n+1}.$$

So that

$$Bz^{n+1} = z^n - \Delta t \epsilon^{n+1}.$$

Using the fact that B is invertible and that

$$\|B^{-1}g\|_\infty \leq \|g\|_\infty,$$

we have

$$\|z^{n+1}\|_\infty \leq \|z^n\|_\infty + \Delta t(\frac{K\Delta t}{2} + \frac{K\Delta x^2}{24}).$$

As in the explicit case, this gives the result

$$\sup_{\substack{0\le i\le N+1\\0\le n\le M}} |u_i^n - u(t_n, x_i)| \le C_T(\Delta t + \Delta x^2).$$

with $C_T = \frac{KT}{2}(1 + \frac{1}{12})$.
We can conclude that the fully implicit scheme is convergent

$$\lim_{\Delta t, \Delta x \to 0} \Big(\sup_{\substack{0\le i\le N+1\\0\le n\le M}} |u_i^n - u(t_n, x_i)| \Big) = 0.$$

Gauss factorization

We have to solve linear systems

$$Bx = g,$$

where x and g, are N dimensional vectors et B is tridiagonal matrix :

$$B = \begin{pmatrix} b_1 & c_1 & 0 & \cdots & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{N-1} & b_{N-1} & c_{N-1} \\ 0 & 0 & 0 & \cdots & a_N & b_N \end{pmatrix}.$$

Let be $x = (x_i)_{1 \le i \le N}$ and $g = (g_i)_{1 \le i \le N}$.

We proceed by this way: the matrix B is reduced to a lower triangular matrix with the pivot method.

Up Steps:

$b'_N = b_N$

$g'_N = g_N$

For $\quad i = N-1, .., 1$

$$b'_i = b_i - c_i a_{i+1} / b'_{i+1}$$

$$g'_i = g_i - c_i g'_{i+1} / b'_{i+1}$$

After this transformation we obtain the equivalent system $B'x = g'$, with :

$$B' = \begin{pmatrix} b'_1 & 0 & 0 & \cdots & 0 & 0 \\ a_2 & b'_2 & 0 & 0 & \cdots & 0 \\ 0 & a_3 & b'_3 & 0 & \cdots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{N-1} & b'_{N-1} & 0 \\ 0 & 0 & 0 & \cdots & a_N & b'_N \end{pmatrix}.$$

Finally, x is computed.

Downs Steps :

$$x_1 = g'_1 / b'_1$$

For $\quad i = 2, .., N,$

$$x_i = (g'_i - a_i x_{i-1}) / b'_i$$

The complexity of the algorithm is linear.

Computational complexity

At each time time step

- the implicit scheme costs $6N$ multiplications et $3N$ additions.
- the explicit scheme with $\lambda = 1$ costs $2N$ multiplications et N additions.

But we need consider the stability condition for the explicit scheme.
In a implicit scheme we can choose arbitrarily M.
For example we choose $l = 1$, $T = 1$, $N + 1 = M = 200$ so that $\Delta x = 0.01$ and $\Delta t = 0.005$
In a explicit scheme we cannot choose arbitrarily M because of the stability condition.
For $\lambda = 1$, $\Delta t = \Delta x^2 = 10^{-4}$, therefore $M = int(\frac{T}{\Delta t}) = 10000$!!.

Option Pricing and Partial Differential equation

In the case of complete markets, one can prove by arbitrage technique that the fair price at time 0 of an European option which guarantees the cash flow $\psi(S_T^x)$ at time T is by given

$$u(0,x) = \mathbb{E}_Q\left[e^{-rT}\psi(S_T^x)\right]$$

where S_t^x is the price of the underlying asset at time t with initial price $S_0 = x$. Now we use the Feynman Kac formula that give a relation between second order partial differential equations and stochastic differential equation.

Feynam Kac Formula

Let $(X_t)_{t\geq 0}$ be the solution of the stochastic differential equation

$$X_t^x = x + \int_0^t \mu(s, X_s^x)ds + \int_0^t \sigma(X_s^x)dB_s,$$

Define the second-order operator

$$Af(x) = \frac{\sigma^2(t,x)}{2}\frac{\partial^2 f}{\partial x^2} + \mu(t,x)\frac{\partial f}{\partial x}$$

with f any real-valued function of class $C^2(\mathbb{R})$.

Suppose that u is solution of the following backward partial differential equation

$$\begin{cases} \frac{\partial u}{\partial t} + Au - ru = 0 \ \text{ in } [0, T[\times\mathbb{R} \\ u(T,x) = \psi(x) \quad \text{in } \mathbb{R} \end{cases}$$

Then

$$u(0,x) = \mathbb{E}\Big[e^{-rT}\psi(X_T^x)\Big]$$

Black-Scholes equation

We recall that the price of an European option in the Black and Scholes model

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t$$

can be formulated in terms of the solution to a Partial Differential Equation.
After logarithmic transformation $X_t = \log(S_t)$ the price at time t of the option is $V_t = u(t, X_t)$ where u solves the parabolic equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}(t,x) + (r - \frac{\sigma^2}{2})\frac{\partial u}{\partial x}(t,x) - ru(t,x) = 0 \text{ in } [0,T) \times \mathbb{R}, \\ u(T,x) = \psi(x), \forall x \in \mathbb{R}, \end{cases}$$

Localization

Let be $x = \log(S_0)$. We start by limiting the integration domain in space: the problem will be solved in a finite interval $[x - l, x + l]$.
For the numerical solution of the problem by finite difference method, we introduce a grid of mesh points

$$(t_n, x_i) = (n\Delta t, x - l + i\Delta x), \quad n = 0, .., M \qquad i = 0, ..., N + 1$$

where

$$\Delta t = \frac{T}{M}, \quad \Delta x = \frac{2l}{N + 1}$$

are mesh parameters which are thought of as tending to zero.
Δt is the time step and Δx is the space step.

θ-scheme

Using a θ-scheme in time, $0 \le \theta \le 1$,we have

$$\begin{cases} \frac{u_i^{n+1}-u_i^n}{\Delta t} + \theta \frac{\sigma^2}{2} \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} + (1-\theta)\frac{\sigma^2}{2} \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} + \\ \quad \theta (r - \frac{\sigma^2}{2}) \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} + (1-\theta)(r - \frac{\sigma^2}{2}) \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} - \\ \quad \theta r u_i^n - (1-\theta) r u_i^{n+1} = 0, \quad i = 1, .., N \qquad n = M-1, .., 0 \\ u_i^N = \psi(x_i), \quad i = 0, .., N+1. \end{cases}$$

with Dirichlet boundary conditions

$$\forall n = N-1, ..., 0 \quad u_0^n = u_{N+1}^n = 0$$

Explicit scheme

First, let us discuss the case $\theta = 0$.
We obtain

$$u_i^{n+1} = p_1 u_{i-1}^n + p_2 u_i^n + p_3 u_{i+1}^n$$

where

$$p_1 = \Delta T(\frac{\sigma^2}{2\Delta x^2} - \frac{\mu}{2\Delta x}) \quad p_2 = 1 - \Delta T(r + \frac{\sigma^2}{\Delta x^2}) \quad p_3 = \Delta T(\frac{\sigma^2}{2\Delta x^2} + \frac{\mu}{2\Delta x})$$

and $\mu = r - \frac{1}{2}\sigma^2$.
This scheme is stable if $\Delta T \leq \frac{\Delta x^2}{\sigma^2 + r\Delta x^2}$.

Implicit scheme

In the case $\frac{1}{2} \leq \theta \leq 1$ we have to solve at each time step, a linear system of the type

$$\mathbf{T}U^n = \mathbf{S}U^{n+1}$$

where $\mathbf{T}$ and $\mathbf{S}$ are tridiagonal matrix of the type

$$\begin{pmatrix} b_1 & c_1 & 0 & \cdots & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{M-1} & b_{M-1} & c_{M-1} \\ 0 & 0 & 0 & \cdots & a_M & b_M \end{pmatrix}.$$

The coefficient of $\mathbf{T}$ are given by

$$a_i = \theta \Delta T(\frac{\mu}{2\Delta x} - \frac{\sigma^2}{2\Delta x^2}), \quad b_i = 1 + \theta \Delta T(r + \frac{\sigma^2}{\Delta x^2}), \quad c_i = -\theta \Delta T(\frac{\mu}{2\Delta x} + \frac{\sigma^2}{2\Delta x^2}).$$

The coefficient of $\mathbf{S}$ are given by

$$a_i = (1-\theta)\Delta T(\frac{\sigma^2}{2\Delta x^2} - \frac{\mu}{2\Delta x}), \quad b_i = 1 - (1-\theta)\Delta T(r + \frac{\sigma^2}{\Delta x^2}), \quad c_i = (1-\theta)\Delta T(\frac{\mu}{2\Delta x} + \frac{\sigma^2}{2\Delta x^2}).$$

American option

The price of an American option u solve the following variational inequality

$$\begin{cases} \max(\frac{\partial u}{\partial t} + Au, \psi - u) = 0, \quad (t,x) \in [0,T[\times\mathbb{R} \\ u(T,x) = \psi(x), \quad x \in \mathbb{R}. \end{cases}$$

American options

Consider the solution of this system of inequalities

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au \leq 0, \quad u \geq \psi, \quad (t,x) \in [0,T) \times \mathbb{R} \\ \left(\dfrac{\partial u}{\partial t} + Au\right)(\psi - u) = 0, \quad (t,x) \in [0,T) \times \mathbb{R} \\ u(T,x) = \psi(x), \quad x \in \mathbb{R} \end{cases}$$

Then $u(0,x) = \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}\left(e^{-r}\psi\left(S_\tau\right)\right)$.

Barrier options

We consider the PDE in the logarithmic variable

$$\begin{cases} \dfrac{\partial u}{\partial t} + (r - \dfrac{\sigma^2}{2})\dfrac{\partial u}{\partial x} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 u}{\partial x^2} - ru = 0 \\ u(0,x) = (K - e^x)_+, \end{cases}$$

together with the Dirichlet boundary conditions steming from the rebate on one (resp. both) side(s). For example in the down-and-out case

$$u(t, log(D)) = R$$

Rainbow options

We will consider American options written on two dividend-paying stocks. Let S_t^i $(i = 1, 2)$ be the stock-price at time t of the stock i which satisfies the following stochastic differential equation:

$$\frac{dS_t^i}{S_t^i} = (r - \delta_i)dt + \sum_{j=1}^{2} \sigma_{ij} dW_t^j \quad i = 1, 2$$

Let $\alpha_i = r - \delta_i - \frac{1}{2}\sigma_i^2$ and $x_i = \log s_i$, $i = 1, 2$.
After a standard logarithmic transformation $(X_t^1, X_t^2) = (\log(S_t^1), \log(S_t^2))$, the price at time 0 of an European option can be formulated in terms of the solution $u(t, x_1, x_2)$ to the following partial differential equation

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\sigma_1^2}{2}\frac{\partial^2 u}{\partial x_1^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 u}{\partial x_2^2} + \alpha_1 \frac{\partial u}{\partial x_1} + \alpha_2 \frac{\partial u}{\partial x_2} + \rho\sigma_1\sigma_2 \frac{\partial^2 u}{\partial x_1 \partial x_2} - ru = 0 \ \text{ in } [0, T[\times \mathbb{R}^2 \\ u(T, x_1, x_2) = \psi(e^{x_1}, e^{x_2}) \end{cases}$$

Asian options

The price at time t of an Asian option is given by:

$$V(t, S_t, A_t) = e^{-r(T-t)}\mathbb{E}(g(S_T, A_T)|\mathcal{F}_t).$$

The price of the Asian option V is solution of the following PDE:

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \frac{1}{t}(S-A)\frac{\partial V}{\partial A} - rV = 0, \\ V(T, S, A) = g(S, A). \end{cases}$$

The PDE is difficult to solve since the parabolic operator is degenerated in the A-variable.

Lookback options

The price of an European Lookback option can be formulated, after a logarithm change of variable, in terms of the solution u to the following PDE

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + (r-q)\frac{\partial u}{\partial x} - ru = 0 \;\; \text{in } [0,T[\times\{x<y\} \\ \frac{\partial u}{\partial y} = 0 \quad in \quad \{x=y\} \\ u(T,x,y) = \psi(e^x, e^y) \end{cases}$$