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Monte Carlo methods for Exotic Options

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Exotic options in the Black Scholes model

- Barrier Options
- Asian options
- Lookback options
- Rainbow Options

Simulation of diffusion in Black and Scholes model

In the Black and Scholes model, the underlying asset price S_t follows the diffusion:

$$dS_t = rS_t dt + \sigma S_t dB_t$$

and then the price is a geometric Brownian process:

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) t + \sigma B_t\right)$$

- **Forward simulation:**

With scheme $\mathcal{T}$ for the discretization, we have:

$$S_{t_{k+1}} = S_{t_k} \exp\left((r - \frac{\sigma^2}{2})(t_{k+1} - t_k) + \sigma B_{t_{k+1}-t_k}\right)$$

and for a discretization with evenly spaced intervals of size h, we simply have:

$$S_{t_{k+1}} = S_{t_k} \exp\left((r - \frac{\sigma^2}{2})h + \sigma\sqrt{h}g_k\right)$$

Barrier options

The payoff of a knock-out single or double barrier is given by $f(S_T)$ provided that the underlying asset price S does not hit on the barrier(s) during the time interval $[0, T]$; if it does, a pre-specified cash rebate R is paid out.

For example, let us consider a call-down-and-out options.
Let

$$\tau_L = \inf\{u > 0 \; ; \; S_u \leq L\}$$

be the hitting time on the barrier L.

Payoff function

$$\begin{cases} e^{-rT}(S_T(x) - K)_+ & \text{if} \quad \tau_{\mathrm{L}} \geq \mathrm{T} \\ e^{-r\tau_L} R & \text{if} \quad \tau_{\mathrm{L}} < \mathrm{T} \end{cases}$$

Asian options

The price of an European Asian option is given by

$$P(0, s, s) = E\left[e^{-rT} f(S_T, A_T) | S_0 = s, A_0 = s\right].$$

where A_T is the integral mean

$$A_T = \frac{1}{T}\int_0^T S_t$$

Payoff examples

- Fixed Asian Call: the payoff is $(A_T - K)_+$.
- Fixed Asian Put: the payoff is $(K - A_T)_+$.
- Floating Asian Call: the payoff is $(S_T - A_T)_+$.
- Floating Asian Put: the payoff is $(A_T - S_T)_+$.

Mean Approximations

Payoff

$$(S_{mean} - K)_+ = (\frac{1}{T}\int_0^T S_t dt - K)_+$$

Let $h = \frac{T}{N}$ be the time discretization step.

- Riemann

$$\int_0^T S_t dt \approx \sum_{i=0}^{N-1} S_{ih} h = h\Big(S_0 + S_h + \cdots + S_{(N-2)h} + S_{(N-1)h}\Big)$$

- Trapezoidal

$$\int_0^T S_t dt \approx \frac{1}{2} h\Big(S_0 + 2S_h + \cdots + 2S_{(N-2)h} + S_{(N-1)h}\Big)$$

Lookback options

Lookback options are options whose payoff depend on the maximum or minimum of the underlying asset price reached during the life of the option.
The price of an European lookback option is given by

$$P(0, s, s) = E\left[e^{-rT} f(S_T, M_T) | S_0 = s, M_0 = s\right].$$

where M_T

$$M_T = \max_{0 \le t \le T} S_t$$

$$m_T = \min_{0 \le t \le T} S_t$$

Payoff example:

- Fixed Lookback Call: the payoff is $(M_T - K)_+$.
- Fixed Lookback Put: the payoff is $(K - m_T)_+$.
- Floating Lookback Call: the payoff is $(S_T - m_T)_+$.
- Floating Lookback Put: the payoff is $(M_T - S_T)_+$.

Multivariate normal random variables (Gaussian vector)

Multidimensional models will generally involve Gaussian processes with values in $\mathbb{R}^n$.

Let $X = (X_1, \ldots, X_n)$ be a random vector with values in $\mathbb{R}^n$. Its distribution is characterized by

- the vector of its expectations

$$m = (m_1, \cdots, m_n)^t = (\mathbb{E}(X_1), \cdots, \mathbb{E}(X_n))^t$$

- its variance-covariance matrix

$$\Sigma = (\Sigma_{ij})_{1 \leq i \leq n, 1 \leq j \leq n}$$

  where

$$\Sigma_{ij} = Cov(X_i, X_j) = \mathbb{E}(X_i X_j) - \mathbb{E}(X_i)\mathbb{E}(X_j).$$

A random vector $X = (X_1, \ldots, X_n)$ is a Gaussian vector, if for each $a_1, \ldots, a_n$, the real valued random variable $\sum_{i=1}^n a_i X_i$ is normal.

Property A linear transformation of a Gaussian vector is a Gaussian vector.

$$X \sim N(m, \Sigma) \Rightarrow AX \sim N(Am, A\Sigma A^t)$$

- Example. Standard gaussian vector

$$G = (g_1, \cdots, g_n)^t$$

with $g_1, \cdots, g_n$ i.i.d $N(0,1)$. Then

$$G \sim N(0, I)$$

- Example. Brownian vector

$$(B_{t_1}, \ldots, B_{t_N})$$

is a Gaussian vector.

Moreover

$$\mathbb{E}(B_{t_i}) = 0.$$

Assume that $0 < s < t$

$$Cov(B_s, B_t) = Cov\Big(B_s, B_s + (B_t - B_s)\Big) = Cov(B_s, B_s) + Cov(B_s, (B_t - B_s)) = s = \min(s, t).$$

$$\Sigma_{ij} = \min(t_i, t_j).$$

Then

$$(B_{t_1}, \ldots, B_{t_N}) \sim N(0, \Sigma).$$

Simulation of Gaussian vectors

- Remember that to simulate a Gaussian random variable $X \sim N(\mu, \sigma^2)$ with mean μ and variance σ^2

$$X = \mu + \sigma g,$$

with $g \sim N(0, 1)$

- Because of the linear transformation property

$$X = m + AG \sim N(m, AIA^t) = N(m, AA^t)$$

- Let Σ be a invertible matrix. In order to simulate a Gaussian vector

$$X \sim N(m, \Sigma)$$

we compute A lower triangular such that

$$AA^t = \Sigma$$

A is called the square root of Σ.

In order to compute A we can use the Cholevski algorithm.

Algorithm

Simulation of a Gaussian vector $X \sim N(m, \Sigma)$

- Compute the square root of the matrix Σ, say A the lower triangular matrix.
- Simulate n indipendent standard random variables $\sim N(0,1)$ $G = (g_1, \cdots, g_n)^t$.
- Return $m + AG$.

Cholevski Algorithm

- **STEP 0,**
  - $a_{11} = \sqrt{\Sigma_{11}}$
  - $a_{i1} = \frac{\Sigma_{i1}}{a_{11}}$ $1 \leq i \leq n$

  STEP 1
  - $a_{ii} = \sqrt{\Sigma_{ii} - \sum_{j=1}^{i-1} a_{ij}^2}$ for $1 < i \leq n$
  - $a_{ij} = \frac{\Sigma_{ij} - \sum_{k=1}^{j-1} a_{ik} a_{jk}}{a_{jj}}$ $1 < j < i \leq n$
  - $a_{ij} = 0$ $1 < i < j \leq n$

```
-->C=[1 -0.7 0.2;-0.7 1 0.2;0.2 0.2 1]
 C  =

!   1.    - 0.7    0.2 !
! - 0.7     1.     0.2 !
!   0.2     0.2    1.  !

-->m=[2 2 2]
 m  =

!   2.    2.    2. !

-->G=rand(1,3,'normal')
 G  =
! - 1.7350313    0.5546874  - 0.2143931 !

-->G=G'
 G  =

! - 1.7350313 !
!   0.5546874 !
! - 0.2143931 !
```

```
-->m=m'
 m  =

!   2. !
!   2. !
!   2. !
-->A=chol(C)
 A  =

!   1.  - 0.7          0.2        !
!   0.    0.7141428    0.4760952 !
!   0.    0.           0.8563488 !
-->A=A'
 A  =

!   1.     0.           0.         !
! - 0.7    0.7141428    0.         !
!   0.2    0.4760952    0.8563488 !
-->X=m+A*G
 X  =
!   0.2649687 !
!   3.610648  !
!   1.7334825 !
```

Example : Brownian vector

Let Σ be the variance-covariance matrix of the vector $(B_{t_1}, \ldots, B_{t_N})$

$$\Sigma_{ij} = \min(t_i, t_j)$$

Then

$$(B_{t_1}, \ldots, B_{t_N}) \sim N(0, \Sigma)$$

We compute the square root matrix of using the Cholevski algorithm

$$AA^t = \Sigma$$

$$(1) \qquad A = \begin{pmatrix} \sqrt{t_1} & 0 & 0 & \cdots & 0 & 0 \\ \sqrt{t_1} & \sqrt{t_2 - t_1} & 0 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \vdots & \ddots & \ddots & \ddots & \vdots \\ \sqrt{t_1} & & & & & \\ \sqrt{t_1} & \sqrt{t_2 - t_1} & \cdots & \cdots & \cdots & \cdots \\ \sqrt{t_1} & \sqrt{t_2 - t_1} & \cdots & \cdots & & \sqrt{t_n - t_{n1}} \end{pmatrix}.$$

Consider $G = (g_1, \cdots, g_n)^t$. Then $AG \sim N(0, \Sigma)$.

Two dimensional Black-Scholes model

$$\frac{dS_t^i}{S_t^i} = r dt + \sigma_i dW_t^i, \quad S_0^i = x_i, \quad i = 1, 2$$

The two Brownian motion are correlated

$$\mathbb{E}[W_t^1 W_t^2] = \rho t$$

$$\mathbb{E}\Big[(W_{t+\Delta t}^1 - W_t^1)(W_{t+\Delta t}^2 - W_t^2)\Big] = \rho \Delta t$$

$$dW_t^1 dW_t^2 = \rho dt$$

Using Cholevski algorithm we can write

$$\frac{dS_t^i}{S_t^i} = r dt + \sum_{j=1}^{2} \sigma_{ij} dB_t^j, \quad S_0^i = x_i, \quad i = 1, 2$$

where B_t^1 and B_t^2 are indipendent Brownanian motions.

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix}$$

Monte Carlo Algorithm

The Exchange option is a particular case of Rainbow option

$$(S_T^1 - \lambda S_T^2)_+$$

Payoff function

$$\psi(x_1, x_2) = (x_1 - \lambda x_2)_+$$

With the Box-Muller algorithm we can draw (g_1, g_2) with $g_i \sim N(0,1)$ indipendent.

Tree methods

Let us consider random walk

$$B_N = \sqrt{\Delta T} S_N = (\sqrt{\Delta T} S_N^1, \sqrt{\Delta} T S_N^2)$$

$S_N = (S_N^1, S_N^2)$ where S_N^1, S_N^2 are the two-dimensional Wiener random walk with transition probabilities

$$\mathbb{Q}\Big((x, y), (x+1, y+1)\Big) = 1/4$$

$$\mathbb{Q}\Big((x, y), (x+1, y-1)\Big) = 1/4$$

$$\mathbb{Q}\Big((x, y), (x-1, y+1)\Big) = 1/4$$

$$\mathbb{Q}\Big((x, y), (x-1, y-1)\Big) = 1/4$$

Then, B_N converges in law to $B_T = (B_T^1, B_T^2)$ with B_T^1, B_T^2 indipendent Brownian motions.

$$\mathbb{E}_{\mathbb{Q}}\Big[f(B_N)\Big] \textit{ converges to } \mathbb{E}_{\mathbb{Q}}\Big[f(B_T^1, B_T^2)\Big]$$

.

In order to compute $e^{-rT}\mathbb{E}_{\mathbb{Q}}\Big[f(S_N)\Big]$ we use the dynamic backward algorithm :

$$\begin{cases} u\Big(N\Delta T, x, y\Big) = f(x,y), \\ u\Big(n\Delta T, x, y\Big) = e^{-r\Delta T}\Big[\dfrac{1}{4}u\Big((n+1)\Delta T, x+1, y+1\Big) + \dfrac{1}{4}u\Big((n+1)\Delta T, x+1, y-1\Big) + \\ \frac{1}{4}u\Big((n+1)\Delta T, x-1, y+1\Big) + \frac{1}{4}u\Big((n+1)\Delta T, x-1, y-1\Big)\Big]. \end{cases}$$

Multidimensional Black-Scholes model

$$\begin{cases} dS_t^1 & = & S_t^1 \left(rdt + \sum_{j=1}^d \sigma_{1j} dB_t^j \right), S_0^1 = x_1 \\ \dots & & \dots \\ dS_t^d & = & S_t^d \left(rdt + \sum_{j=1}^d \sigma_{dj} dB_t^j \right), S_0^d = x_d \end{cases}$$

where$\left(B_t = (B_t^1, \cdot, B_t^d, t \geq 0) \right)$ is a d-dimensional Brownian motion with indipendent components (Cholevski).

Basket options

$$(\frac{1}{d} \sum_{i=1}^{d} S_T^i - K)_+$$

Simulation of diffusions

We consider the general diffusion process:

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t), X(0) = x,$$

If we don't have any explicit solution for X_t (like for Black and Scholes model), we have to use approximation schemes with a discretization of the process.

- The **Euler approximation scheme** for this diffusion is expressed as:

$$X_{t_{k+1}} = X_{t_k} + b(X_{t_k})h + \sigma(X_{t_k})(B_{t_{k+1}} - B_{t_k})$$

Simulation is obtained with a forward algorithm by:

$$X_{t_{k+1}} = X_{t_k} + b(X_{t_k})h + \sigma(X_{t_k})\sqrt{h}g_k$$

for $k = 0, \ldots, M - 1$.

Convergence theorem

Let $(\bar{X}(kh), k \geq 0)$ the sequence of r.v. definded by the Euler scheme with $h = \frac{T}{N}$.
Suppose that b and σ are functions of class C^4 with bounded derivatives up the order 4.
Suppose that f is a function of class C^4 and of polynomial growth.
Then there exist a costant C_T indipendent of h such that the Euler scheme satisfy :

$$\left| \mathbb{E}\left(f(X(T))\right) - \mathbb{E}\left(f(\bar{X}(T))\right) \right| \leq \frac{C_T}{N}.$$

Vasicek model

$$\begin{cases} dr_t = a(b - r_t)dt + \sigma dB_t \\ r_0 = x. \end{cases}$$

with a, b, σ positive costants.
Mean-reversion model.

$$r_t = r_0 e^{-at} + b\left(1 - e^{-at}\right) + \sigma e^{-at} \int_0^t e^{as} dB_s$$

Then

$$r_t \sim N(\mu, \widehat{\sigma}^2)$$

$$\mu = \mathbb{E}[r_t] = r_0 e^{-at} + b\left(1 - e^{-at}\right)$$

$$\widehat{\sigma}^2 = \mathrm{Var}[r_t] = \sigma^2 \left(\frac{1 - e^{-2at}}{2a}\right)$$

Remark $\mathbb{Q}(r_t < 0) > 0$
Remark

$$\mathbb{E}[r_\infty] = b$$

b is the long term mean value

a is the speed of the mean-reversion.

Ornstein-Uhlenbeck process

The Ornstein-Uhlenbeck process is solution of the following s.d.e.

$$\begin{cases} dX_t &= -cX_t dt + \sigma dB_t \\ X_0 &= x \end{cases}$$

If $Y_t = X_t e^{ct}$ we have

$$dY_t = dX_t e^{ct} + X_t d(e^{ct}).$$

Then

$$dY_t = \sigma e^{ct} dB_t$$

$$X_t = xe^{-ct} + \sigma e^{-ct} \int_0^t e^{cs} dB_s.$$

Mean and variance of an Ornstein-Uhlenbeck process

$$X_t = xe^{-ct} + \sigma e^{-ct} \int_0^t e^{cs} dB_s.$$

We can compute the mean

$$\mathbb{E}(X_t) = xe^{-ct} + \sigma e^{-ct} \mathbb{E}\left(\int_0^t e^{cs} dB_s \right) = xe^{-ct}$$

and the variance

$$\begin{aligned}
\mathrm{Var}(X_t) &= \mathbb{E}\left(X_t^2\right) - \left[\mathbb{E}(X_t)\right]^2 \\
&= \sigma^2 \mathbb{E}\left(e^{-2ct} \left(\int_0^t e^{cs} dB_s \right)^2 \right) \\
&= \sigma^2 e^{-2ct} \mathbb{E}\left(\int_0^t e^{2cs} ds \right) \\
&= \sigma^2 \frac{1 - e^{-2ct}}{2c}
\end{aligned}$$

Explicit solution in the Vasicek model

$$X_t = r_t - b,$$

(X_t) is solutions of s.d.e. :

$$dX_t = -aX_t + \sigma dB_t,$$

so that (X_t) is a Ornstein-Uhlenbeck process.

$$r_t = r_0 e^{-at} + b\left(1 - e^{-at}\right) + \sigma e^{-at} \int_0^t e^{as} dB_t$$

Monte Carlo method for ZCB price

$$P(0,T) = \mathbb{E}_Q\left[\exp\left(-\int_0^T r_s ds\right)\right]$$

Integral approximation

$$\int_0^T r_s ds \approx \sum_{i=0}^{N-1} r_{ih} h = h\Big(r_0 + r_h + \cdots + r_{(N-2)h} + r_{(N-1)h}\Big)$$

Consider $g \sim N(0,1)$. Discretization schemes:

- Explicit

$$r_{t+\Delta t} = r_t e^{-a\Delta t} + b\left(1 - e^{-a\Delta t}\right) + g\sigma \frac{\sqrt{1 - e^{a\Delta t}}}{2a}$$

- Euler

$$r_{t+\Delta t} = r_t + a(b - r_t)\Delta t + \sigma g \sqrt{\Delta t}$$