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The Black Scholes model

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Continuous Stochastic Processes

The origin of stochastic processes can be traced back to the field of statistical physics. A physical process is a physical phenomenon whose evolution is studied as a function of time.

In a financial framework, the idea is to give a model of stock price fluctuations in continuous time.

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A continuous-time stochastic process is a family $(X_t)_{t\geq 0}$ of $\mathbb{R}$-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.

- the index t stands for the time.
- for each time t fixed:

$$X_t : \Omega \longrightarrow \mathbb{R}$$

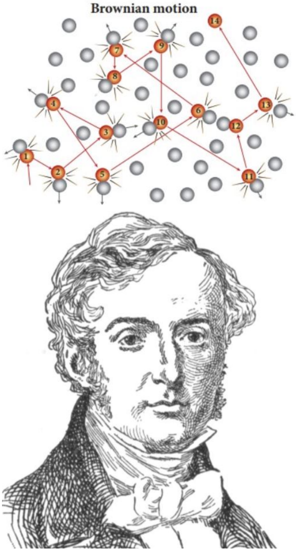
- for each $\omega \in \Omega$ the map $t \longrightarrow X_t(\omega)$ is called the path of the process.

The Brownian motion

In finance, the most common models are constructed on the Brownian motion.

This motion is named after the botanist Robert Brown, who first described the 3-D phenomenon in 1827, while looking through a microscope at some pollen immersed in water: the pollen collides with a large set of smaller particles (molecules of a gas) which move with different velocities in different random directions.

We are going to consider the 1-D version of such a process.



Brownian motion

In finance, the most common models are constructed on the Brownian motion.

Definition

A Brownian motion is a real-valued, continuous stochastic process $(X_t)_{t\geq 0}$ with indipendent, normally distributed and stationary increments. In other words :

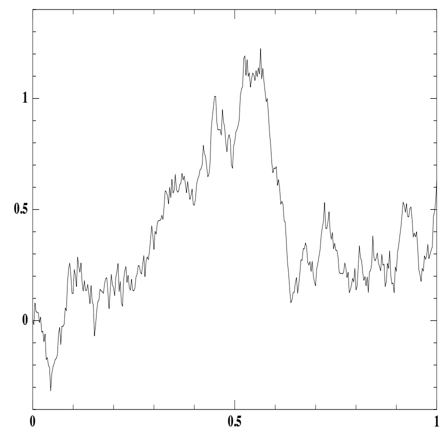
- P1 $B_0 = 0$.
- P1 the function $s \mapsto B_s(\omega)$ is a continuous function.
- P2 indipendent increments : for each $k, 0 \leq t_0 < t_1 < \ldots < t_k$, the increments $B_{t_0}, B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, .., B_{t_k} - B_{t_{k-1}}$ are indipendent.
- P3 for each $t > s \geq 0$, $B_t - B_s \sim N(0, t-s) \Rightarrow$
  $\mathbb{E}_{\mathbb{P}}(B_t - B_s) = 0$ and $\mathbb{E}_{\mathbb{P}}\left[(B_t - B_s)^2\right] = t - s$.
  In particular for $s = 0$ it follows that $\mathbb{E}_{\mathbb{P}}(B_t) = 0$ e $Var(B_t) = t$.

A particle that is undergoing a Brownian motion B_t has the following property.

- P2 Absence of memory.
- P3 The mean of $B_t - B_s$ is 0, so there is not a privilegiate direction.
  The variance of the particle movement is proportional to the observed time.

Brownian motion

The path of the Brownian motion are continuous, but not differentiable.



Financial example 2

$$P = e^{-rT}\mathbb{E}_{\mathbb{P}}\left[(K - e^{\sigma B_T})_+\right].$$

with B_T B.M a time T.
We consider a put option with underlying asset

$$S_t = e^{\sigma B_t}$$

Under $\mathbb{P}$, $B_T \sim N(0,T)$. So $B_T = g\sqrt{T}$ con $g \sim N(0,1)$.
We can approximate the put price with

$$P \approx e^{-rT}\frac{f(X_1) + \cdots + f(X_n)}{n}$$

$X_1, .., X_n \sim N(0,T)$.

Monte Carlo algorithm

```
main()
{
  double mean_price,mean2_price,brownian,price,price_sample,error_price,inf_price,sup_price;
  mean_price= 0.0;
  mean2_price= 0.0;
  for(i=1;i<=N;i++)
    {
      /*Brownian motion simulation*/
      brownian=gaussian()*sqrt(T);

      price_sample=MAX(0.0,K-exp(sigma*brownian));

      mean_price= mean_price+price_sample;
      mean2_price= mean2_price+SQR(price_sample);
    }
  /* Price */
  price=exp(-r*T)*(mean_price/N);
  error_price= sqrt(exp(-2.*r*T)*mean2_price/N - SQR(price))/sqrt(N-1);
  inf_price= price - 1.96*(error_price);
  sup_price= price + 1.96*(error_price);
}
```

Simulation of Brownian motion path

Let $[0, T]$ be divided using N time intervals of lenght $\Delta T = \frac{T}{N}$.

$$B_T = B_{N\Delta T} = \sum_{k=1}^{N} (B_{k\Delta T} - B_{(k-1)\Delta T}) = \sum_{k=1}^{N} \Delta B_k$$

The increments $\Delta B_k \sim N(0, \Delta T)$ are indipendent and normally distribuited :

$$\Delta B_k = g\sqrt{\Delta T}$$

with $\quad g \sim N(0, 1)$.

Start $t_0 = 0,\ B_0 = 0,\ \Delta T = \frac{T}{N}$

for $k = 1, \ldots, N$

BEGIN;

$t_k = t_{k-1} + \Delta T$;

simulation of $g \sim N(0, 1)$;

$B_{k\Delta T} = B_{(k-1)\Delta T} + g\sqrt{\Delta T}$;

END;

Algorithm

```
main()
{
  double k,T,brownian,B_T,time;
  int N;

  k=T/N;
  brownian=0.;
  time=0.;
  for(i=1;i<=N;i++)
    {
      /*Time*/
      time+=k;

      /*Brownian path simulation*/
      brownian=brownian+gaussian()*sqrt(k);
    }

  /* B_T */
  B_T=brownian;
}
```

Brownian motion and random walk

One of the standard ways used to approximate a Brownian motion is to use a random walk. Here we use the standard symmetric random walk.

Proposition

Let $(X_i, i \geq 1)$ be a sequence of independent random variables such that $\mathbb{P}(X_i = \pm 1) = 1/2$.
Set $S_n = X_1 + \cdots + X_n$.
Let $\Delta T = T/N$ the time step. Set

$$B_N = \sqrt{\Delta T} S_N.$$

Then, the sequence B_N converges in distribution to B_T.
Consequently, if f is a bounded continuous function then

$$\mathbb{E}_{\mathbb{P}}\Big[f(B_N)\Big] \text{ converges to } \mathbb{E}_{\mathbb{P}}\Big[f(B_T)\Big].$$

In the financial example 2, $f(B_T) = (K - e^{\sigma B_T})_+$. We use $f(B_N) = (K - e^{\sigma B_N})_+$, or $g(S_N) = (K - e^{\sigma \sqrt{\Delta T} S_N})_+$

Besides, $E(g(SN))$ can be computed as follows :

$$\begin{cases} u(N\Delta T, x) = g(x), \\ u(n\Delta T, x) = \dfrac{1}{2}u((n+1)\Delta T, x+1) + \dfrac{1}{2}u((n+1)\Delta T, x-1). \end{cases}$$

Exercise

Compute

$$P = e^{-rT}\mathbb{E}_{\mathbb{P}}\left[(K - e^{\sigma B_T})_+\right].$$

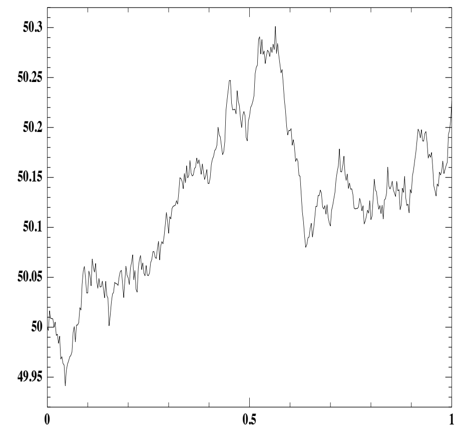
with $K = 100$, $r = 0.03$, $\sigma = 0.2$ using

- Monte Carlo algorithm
- Tree method

Brownian motion with drift

$$X_t = \mu t + \sigma B_t$$

with B_t standard brownian motion, with μ and σ costants.



A french matematician Bachelier introduces it in first years of 1900 for modelling stock prices. But there is the problem of the negative prices:

$$X_t \sim N(\mu t, \sigma^2 t).$$

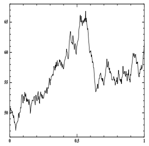
Geometric Brownian motion

Definition

A Geometric Brownian motion S_t is a continuous stochastic process such that:

- P1 $S_0 = x$.
- P2 $S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$

B_t standard Brownian motion, μ and σ costant.



The log-returns $\log \frac{S_t}{S_0}$ have normal distribution (the returns are normal).

$\frac{S_t}{S_0}$ is log-normal of parameters $(\mu - \frac{1}{2}\sigma^2)t$ and $\sigma^2 t$.

Property of the GBM

P1 Consider s<t. Then

$$\log(\frac{S_t}{S_s}) \sim N((\mu - \frac{1}{2}\sigma^2)(t-s), \sigma^2(t-s))$$

Expectation

$$\mathbb{E}(\frac{S_t}{S_s}) = e^{\mu(t-s)}$$

Variance

$$Var(\frac{S_t}{S_s}) = e^{2\mu(t-s)}(e^{\sigma^2(t-s)} - 1)$$

P2 for each $0 \leq t_0 < t_1 < \ldots < t_n$, the relative increments $S_{t_k}/S_{t_{k-1}}$ are indipendent and have common law.

Financial example 3

$$P = e^{-rT}\mathbb{E}_{\mathbb{P}}\left[(K - S_T)_+\right],$$

with S_T value of the GBM at time T T.
We consider a put option with underlying asset

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$$

The payoff can be written

$$h(B_T) = (K - S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma B_T})_+$$

Then

$$P \approx e^{-rT}\frac{h(X_1) + \cdots + h(X_n)}{n}$$

$X_1, .., X_n \sim N(0, T)$.

Monte Carlo algorithm

```
main()
{
  double mean_price,mean2_price,brownian,price,price_sample,error_price,inf_price,sup_price;
  mean_price= 0.0;
  mean2_price= 0.0;
  for(i=1;i<=N;i++)
    {

      brownian=gaussian()*sqrt(T);

      price_sample=MAX(0.0,K-x*exp((mu-0.5*sigma*sigma)*T+sigma*brownian));

      mean_price=mean_price+price_sample;
      mean2_price=mean2_price+SQR(price_sample);
    }

  price=exp(-r*T)*(mean_price/N);
  error_price= sqrt(exp(-2.*r*T)*mean2_price/N - SQR(price))/sqrt(N-1);
  inf_price= price - 1.96*(error_price);
  sup_price= price + 1.96*(error_price);
}
```

Simulation of Geometric Brownian motion path

Let $[0, T]$ be divided using N time intervals of lenght $\Delta T = \frac{T}{N}$.

$$S_T = S_{N\Delta T} = S_0 \prod_{k=1}^{N} \frac{S_{k\Delta T}}{S_{(k-1)\Delta T}} = S_0 \prod_{k=1}^{N} e^{(\mu - \frac{1}{2}\sigma^2)\Delta T + \sigma \Delta B_k},$$

with

$$(\mu - \frac{1}{2}\sigma^2)\Delta T + \sigma \Delta B_k = (\mu - \frac{1}{2}\sigma^2)\Delta T + \sigma g \sqrt{\Delta T} \quad con \quad g \sim N(0,1)$$

Simulation of the GBM path $(S_t)_{0 \le t \le T}$:

Start $t_0 = 0$, $S_0 = x$, $\Delta T = \frac{T}{N}$

for $k = 1, \dots, N$

BEGIN;

$t_k = t_{k-1} + \Delta T$;

simulation of $g \sim N(0,1)$;

$S_{k\Delta T} = S_{(k-1)\Delta T} e^{(\mu - \frac{1}{2}\sigma^2)\Delta T + \sigma g \sqrt{\Delta T}}$;

END.

Algorithm

```
main()
{
  double k,T,w_derive,s,S_N,mu=0.1,sigma=0.2,time;
  int N;

  k=T/N;
  s=50.;
  time=0.;
  for(i=1;i<=N;i++)
    {
      /*Timew*/
      time=time+k;
      /*Geometric Brownian simulation*/
      s=s*exp((mu-0.5*sigma*sigma)*k+sigma*gaussian()*sqrt(k));
    }
  /* S_T */
   S_T=s;
}
```

Differential property of the Brownian motion

$$\mathbb{E}\Big[(B_t - B_s)^2\Big] = t - s$$

Let us consider the random variable.

$$X = \Big(B_{t+\Delta t} - B_t\Big)^2$$

Then

$$\mathbb{E}\Big[X\Big] = (t + \Delta t) - t = \Delta t$$

and

$$V\Big[X\Big] = 2(\Delta t)^2$$

When Δt is close to zero the r.v. X is "not to much random" and is very close to his mean Δt:

$$X = \Big(B_{t+\Delta t} - B_t\Big)^2 \approx \Delta t$$

We write

$$\Big(dB_t\Big)^2 = dB_t dB_t = dt$$

and

$$dB_t = \sqrt{dt}$$

The quadratic variation of the Brownian motion in $[0,T]$ is equal to his variance.

Let $(B_t, t \geq 0)$ be a standard Brownian motion. For each $T > 0$ and partition $0 = t_0^n < t_1^n < \cdots < t_n^n = T$ so that $\pi = \sup_{i \leq n}(t_i^n - t_{i-1}^n)$ goes to zero when $n \to \infty$:

$$\sum_{i=1}^{n} \left(B(t_i^n) - B(t_{i-1}^n)\right)^2 \to T,$$

in the sense if the quadratic mean, for $n \to \infty$

Proof

$$\mathbb{E}\left[\sum_{i=1}^{n} \left(B(t_i^n) - B(t_{i-1}^n)\right)^2\right] = T$$

The random variables $\left(B(t_i^n) - B(t_{i-1}^n)\right)^2$, $i = 1, 2, \ldots, n$ are indipendent.

$$\begin{aligned} \mathrm{Var}\left[\sum_{i=1}^{n} \left(B(t_i^n) - B(t_{i-1}^n)\right)^2\right] &= \sum_{i=1}^{n} \mathrm{Var}\left[\left(B((t_i^n) - B(t_{i-1}^n)\right)^2\right] \\ &= 2\sum_{i=1}^{n} (t_i^n - t_{i-1}^n)^2 \leq 2\pi T. \end{aligned}$$

This variance goes to 0 when $n \to \infty$.

Stochastic integral

Consider the stochastic integral

$$\int_0^T f(t, B_t) dB_t.$$

We can define $X_t = \int_0^T f(t, B_t) dB_t$ as the limit of discrete sums of the type

$$X_n = \sum_{j=0}^{n-1} f(t_j^n, B_{t_j^n})(B_{t_{j+1}^n} - B_{t_j^n}),$$

as n goes to infinity.

When can think X_n as a "Riemann sum" in which the representative point inside each subinterval is the left-most point.

This definition of the stochastic integral is called the Ito integral.

Of course, conditions on f are necessary to ensure that X_n converge in a reasonable sense and that the limit does not depende on the sequence on meshes t_i^n.

Example

$$\int_0^T dB_s = B_T$$

Example

$$\int_0^T B_s dB_s = -\frac{1}{2}T + \frac{1}{2}B_T^2$$

$$
\begin{aligned}
\int_0^T B_s \, dB_s &= \lim_{n\to\infty} \sum_{i=0}^{n-1} B_{t_j} \cdot \left(B_{t_{j+1}} - B_{t_j} \right) \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \left(B_{t_j} B_{t_{j+1}} - B_{t_j}^2 \right) \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \left(-\frac{1}{2} \left(B_{t_{j+1}} - B_{t_j} \right)^2 - \frac{1}{2} B_{t_j}^2 + \frac{1}{2} B_{t_{j+1}}^2 \right) \\
&= \lim_{n\to\infty} \frac{1}{2} \left[-\sum_{i=0}^{n-1} \left(B_{t_{j+1}} - B_{t_j} \right)^2 + \sum_{i=0}^{n-1} \left(B_{t_{j+1}}^2 - B_{t_j}^2 \right) \right] \\
&= -\frac{1}{2} T + \frac{1}{2} B_T^2 .
\end{aligned}
$$

Ito integral property

$$\int_0^T f(t, B_t) dB_t.$$

- Linearity
- Expectation

$$\mathbb{E}\Big[\int_0^T f(t, B_t) dB_t\Big] = 0.$$

- Quadratic mean

$$\mathbb{E}\Big[\Big(\int_0^T f(t, B_t) dB_t\Big)^2\Big] = \mathbb{E}\Big[\int_0^T f^2(t, B_t) dt\Big].$$

Stochastic differential equations

Definition A process $(X_t)_{t\geq 0}$ which satisfies

$$X_t = x + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s, \tag{1}$$

is called a solution of the stochastic differential equation with coefficient μ and σ, intial condition x and Brownian motion $(B_t)_{t\geq 0}$.

$(X_t)_{t\geq 0}$ is called the diffusion process corresponding to the coefficients μ and σ. We can write the differential simbolic notation

$$\begin{cases} dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_0 = x. \end{cases}$$

Example

The standard Brownian motion, the Brownian motion with drift and the geometric Brownian motion are solution of particular s.d.e.

Example : Brownian motion with drift

The Brownian motion with drift is solution of the following s.d.e.

$$\begin{cases} dX_t = \mu dt + \sigma dB_t \\ X_0 = x. \end{cases}$$

It is the diffusion process corresponding to the coefficients $\mu(t, X_t) = \mu$ and $\sigma(t, X_t) = 1$.

Stochastic Integral

$$X_t = x + \int_0^t \mu ds + \int_0^t \sigma dB_s$$

The solution is

$$X_t = x + \mu t + \sigma B_t$$

Example : Geometric Brownian motion

The Geometric Brownian motion with drift is solution of the following s.d.e.

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dB_t \\ S_0 = x. \end{cases}$$

Stochastic Integral

$$S_t = S_0 + \int_0^t \mu S_u du + \int_0^t \sigma S_u dB_u \tag{2}$$

The solution is

$$S_t = x e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$$

The results is obtained using the Ito's Lemma.

Theorem (Existence and Uniqueness)

$$X_t = x + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s, \tag{3}$$

If μ and σ are continuous functions, and if there exists a constant $K < +\infty$, such that :

1. $|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K|x-y|$
2. $|\mu(t,x)| + |\sigma(t,x)| \leq K(1+|x|)$

then, for any $T \geq 0$, (3) admist a unique solution in the interval $[0,T]$.
Moreover, this solution $(X_s)_{0\leq s\leq T}$ satisfies :

$$\mathbb{E}\left(\sup_{0\leq s\leq T} |X_s|^2\right) < +\infty$$

Ito's Lemma

Lemma

Let $(X_t)_{t\geq 0}$ *the solution of*

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t$$
$$X_0 = x_0$$

and let $f(t, X_t)$ *be a real-valued function of class* $C^{1,2}$.
Then

$$df(t, X_t) = \left(\frac{\partial f(t,X_t)}{\partial t} + \mu(t, X_t)\frac{\partial f(t,X_t)}{\partial X_t} + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2 f(t,X_t)}{\partial X_t^2}\right)dt + \sigma(t, X_t)\frac{\partial f(t,X_t)}{\partial X_t}dB_t$$

We can write

$$df(t, X_t) = \alpha(t, X_t)dt + \frac{\partial f}{\partial X_t}dX_t$$

with

$$\alpha(t, X_t) = \frac{\partial f}{\partial t} + \frac{\sigma^2(t, X_t)}{2}\frac{\partial^2 f}{\partial X_t^2}$$

Example

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$
$$S_0 = x$$

Using Ito's lemma

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$$

Let us consider $X_t = B_t$ and

$$S_t = f(t, B_t) = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$$

Ito's lemma implies that

$$dS_t = df(t, B_t) = \Big((\mu - \frac{1}{2}\sigma^2)S_t + \frac{1}{2}\sigma^2 S_t\Big)dt + \sigma S_t dB_t = \mu S_t dt + \sigma S_t dB_t$$

On the contrary, let is consider $X_t = S_t$ and

$$f(t, S_t) = \ln(S_t)$$

Ito's lemma implies that

$$dln(S_t) = df(t, S_t) \quad = \quad (\mu - \frac{1}{2}\sigma^2)dt + \sigma dB_t$$

or in the integral form

$$\begin{aligned}
\int_0^T 1\; dln(S_t) &= \int_0^T (\mu - \frac{1}{2}\sigma^2)\; dt + \int_0^T \sigma\; dB_t \\
[ln(S_t)]_0^T &= (\mu - \frac{1}{2}\sigma^2)\; [t]_0^T + \sigma\; [B_t]_0^T \\
ln(\frac{S_T}{S_0}) &= (\mu - \frac{1}{2}\sigma^2)T + \sigma(B_T - B_0) \\
\frac{S_T}{S_0} &= \exp\left[(\mu - \frac{1}{2}\sigma^2)T + \sigma B_T\right]
\end{aligned}$$

Then

$$S_T = S_0 \exp\left[(\mu - \frac{1}{2}\sigma^2)T + \sigma B_T.\right] \tag{4}$$

Example

Consider $f(t,x) = x^2$ and $X_t = B_t$. Then

$$f(t, B_t) = B_t^2$$

Ito's lemma implies that

$$dB_t^2 = df(t, B_t) = \left(\frac{1}{2}2\right)dt + 2B_t dB_t = dt + 2B_t dB_t$$

In the integral form

$$B_t^2 = B_0^2 + \int_0^t \frac{1}{2}2du + \int_0^t 2B_u dB_u$$

so that

$$\int_0^t B_u dB_u = \frac{1}{2}(B_t^2 - t).$$

Simulation diffusions paths

Euler Discretization Scheme

$$\Delta X_t = X_{t+\Delta T} - X_t = \mu(t, X_t)\Delta t + \sigma(t, X_t)\Delta B_t$$
$$X_0 = x_0$$

Start $t_0 = 0,\ x_0,\ \Delta T = \frac{T}{N}$

for $k = 1, \dots, N$

BEGIN;

$t_k = t_{k-1} + \Delta T$;

simulation of $g \sim N(0, 1)$;

$x_{k\Delta T} = x_{(k-1)\Delta T} + \mu(x_{(k-1)\Delta T}, t_{k-1})\Delta T + \sigma(x_{(k-1)\Delta T}, t_{k-1}) g\sqrt{\Delta T}$

END;

Brownian motion

Definition

A Brownian motion is a real-valued, continuous stochastic process $(X_t)_{t\geq 0}$ with indipendent, normally distribuited and stationary increments. In other words :

- $B_0 = 0$.
- continuity.
- *indipendent increments* : if $s \leq t$, $B_t - B_s$ is indipendent of $\mathcal{F}_s = \sigma(B_u, u \leq s)$.
- *stationary increments* : if $s \leq t$, $B_t - B_s$ and B_{t-s} have the same law.

Continuous-time martingale

Let us consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a filtration $\mathcal{F} := (\mathcal{F}_t, t \geq 0)$ on this space.

Definition An adapted family $(M_t, t \geq 0)$ of integrable random variables is a $(\mathcal{F}_t)$-martingale if for each $s \leq t$,

$$\mathbb{E}\left(M_t | \mathcal{F}_s\right) = M_s.$$

It follows from the definition that if $(M_t)_{t \geq 0}$ is a martingale, then $\mathbb{E}(M_t) = \mathbb{E}(M_0)$, for each t.

Example

B_t is an $\mathcal{F}_t$-martingale.

Markov property

The intuitive meaning of the Markov property is that the future behaviour of the process $(X_t)_{t\geq 0}$ after t depends only on the value X_t and is not influenced by the history of the process before t.

Mathematically speaking, $(X_t)_{t\geq 0}$ satisfies the Markov property if, for any function f bounded and measurable and for any s and t, such that $s \leq t$, we have :

$$\mathbb{E}\left(f\left(X_t\right)|\mathcal{F}_s\right) = \mathbb{E}\left(f\left(X_t\right)|X_s\right).$$

This property is satisfied for a solution of the equation (3).

This is a crucial property of the Markovian model and it will have great conseguences in the pricing of options.

Black-Scholes model

- Risk-free asset

$$\begin{cases} dS_t^0 = rS_t^0 dt \\ S_0^0 = 1. \end{cases}$$

- Risk asset

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dB_t \\ S_0 = x. \end{cases}$$

  with B_t under the historical probability $\mathbb{P}$.

- The short-term interest rate is known and is costant through time.
- The stock pays no dividends or other distributions.
- There are no penalties to short-selling.
- It is possible to borrow any fraction of the price of a security to buy it or to hold it, at the short-time interest rate.
- Absence of arbitrage opportunities.

Financial interpretation of the parameters

- r istantaneous interest rate : $[0\%, 12\%]$

- μ expected return of the risky asset.

$$\mathbb{E}(\frac{S_t}{S_0}) = e^{\mu t}$$

- σ is the volatility σ.
  This is vey important parameters : $[30\%, 70\%]$ in the equity market.

- risk premium λ

$$\lambda = \frac{\mu - r}{\sigma}$$

Then

$$\mu = r + \lambda\sigma$$

The expected return μ of the risky asset is the sum of the return of the no-risky asset plus something proportional to σ.

We can write

$$dS_t = rS_t dt + \sigma S_t (dB_t + \lambda dt)$$

The Girsanov theorem gives

$$d\widehat{B}_t = dB_t + \lambda dt$$

with $\widehat{B}_t$ standard Brownian motion under the risk neutral probability Q.

Dinamics under the risk neutral probability measure

$$
(5) \qquad \begin{cases} dS_t = rS_t dt + \sigma S_t d\widehat{B}_t \\ S_0 = x. \end{cases}
$$

with $\widehat{B}_t$ standard Brownian motion under Q.
The solution(5) is

$$
S_t = xe^{(r-\frac{1}{2}\sigma^2)t+\sigma\widehat{B}_t}
$$

Then

$$
\mathbb{E}_Q\left(\frac{S_T}{S_t}|\mathcal{F}_t\right) = e^{r(T-t)}
$$

Radon-Nikodyn Theorem

Let $\mathbb{P}$ and Q be two probabilty measure on $(\Omega, \mathcal{F})$
If Q is absolutely continuous with respect to $\mathbb{P}$, $(A \in \mathcal{F}, \mathbb{P}(A) = 0 \to Q(A) = 0)$, then there existe a unique r.v. $X \geq 0$, $\mathcal{F}$-misurable such that

$$Q(A) = \int_A X \, d\mathbb{P}$$

The random variable X is commonly written as

$$\frac{dQ}{d\mathbb{P}} = X$$

X called the Radon-Nikodyn derivative.

Change of probability measure in the Gaussian case

Let use consider $Z \sim N(\mu, 1)$ under $\mathbb{P}$.
Then there exists Q so that $Z(0,1)$ under Q where

$$dQ = e^{-\mu Z + \frac{1}{2}\mu^2} d\mathbb{P}.$$

In fact

$$\mathbb{P}(Z \leq z) = \int_{\{\omega : Z(\omega) \leq z\}} d\mathbb{P}(\omega) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2}} dx,$$

and

$$Q(Z \leq z) = \int_{\{\omega : Z(\omega) \leq z\}} dQ(\omega) = \int_{\{\omega : Z(\omega) \leq z\}} e^{-\mu Z(\omega) + \frac{1}{2}\mu^2} d\mathbb{P}(\omega) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx.$$

Moreover it holds

$$\mathbb{E}_{\mathbb{P}}\Big[f(Z)\Big] = \mathbb{E}_Q\Big[f(Z)e^{\mu Z - \frac{1}{2}\mu^2}\Big] \quad \textit{and} \quad \mathbb{E}_Q\Big[f(Z)\Big] = \mathbb{E}_P\Big[f(Z)e^{-\mu Z + \frac{1}{2}\mu^2}\Big].$$

Girsanov's Theorem

Let B_t be a Brownian motion under $(\Omega, \mathcal{F}, \mathbb{P})$ adapted to the filtration $\mathcal{F}_t$.
Let $(Z_t)_{0\leq t\leq T}$ be the process defined by :

$$Z_t = \exp\Big(-\lambda B_t - \frac{1}{2}\lambda^2 t\Big).$$

Then, under the probability measure Q with density Z_T with respect to $\mathbb{P}$

$$dQ = Z_T d\mathbb{P}$$

the process $(\widehat{B}_t)_{0\leq t\leq T}$ given by $\widehat{B}_t = B_t + \lambda t$, is a standard Brownian motion under Q.

Risk neutral pricing formula

$$\mathbb{E}_Q\left(\frac{S_T}{S_t}|\mathcal{F}_t\right) = e^{r(T-t)}$$

This holds for each asset:

$$\mathbb{E}_Q\left[\frac{V_T}{V_t}|\mathcal{F}_t\right] = e^{r(T-t)}$$

Equivalently

$$V_t = \mathbb{E}_Q\left(e^{-r(T-t)}V_T|\mathcal{F}_t\right).$$

The price of a contingent claim is the expected value of the discounted payoff.

$$e^{-rt}V_t = \mathbb{E}_Q\left(e^{-rT}V_T|\mathcal{F}_t\right).$$

Discounted prices are martingales.

Black-Scholes formula for European Call options

The price at time t of an European Call option in the Black-Scholes model

$$C_t = \mathbb{E}_Q\left(e^{-r(T-t)}C_T|\mathcal{F}_t\right) = \mathbb{E}_Q\left(e^{-r(T-t)}(S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(B_T-B_t)} - K)_+\right)$$

is given by

$$C_t = S_t N(d_1) - K e^{-r(T-t)} N(d_2)$$

with

$$d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

and $N(x)$ the distribution function of the standard Gaussian variable

$$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-x^2/2}dx.$$

Black-Scholes formula for European Put options

The price at time t of an European put option in the Black-Scholes model

$$P_t = \mathbb{E}_Q\left(e^{-r(T-t)}P_T|\mathcal{F}_t\right) = \mathbb{E}_Q\left(e^{-r(T-t)}(K - S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(B_T - B_t)})_+\right)$$

is given by

$$P_t = Ke^{-r(T-T)}N(-d_2) - S_t N(-d_1)$$

Implementation of the formula

The price of the call option depends on six parameters.

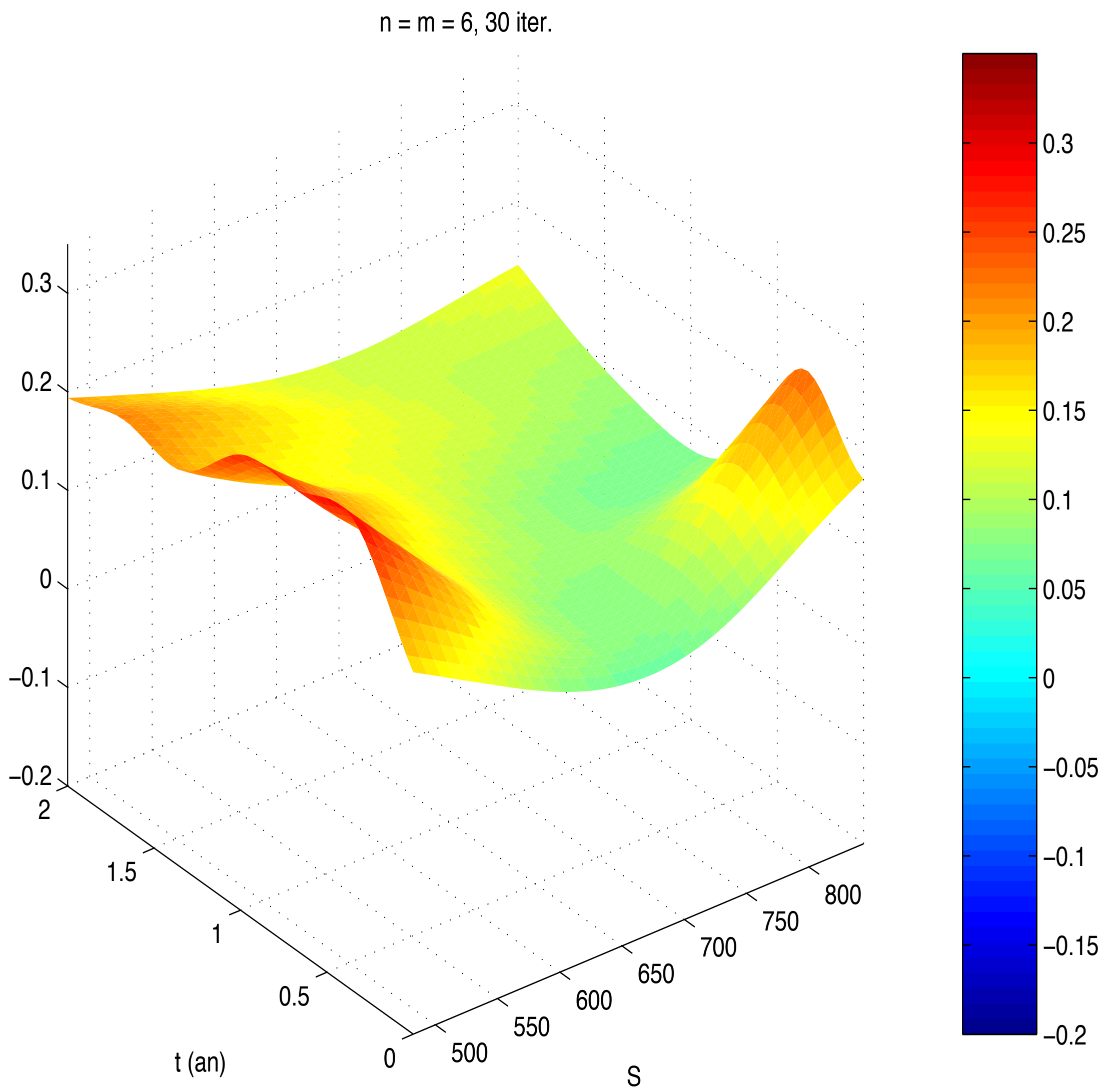
$$C = C(S_t = x, t, T, K, \sigma, r)$$

- The strike K and the maturity T are specified in the contract.
- r is constant. But in general this is not true (Vasicek or CIR model).

- The volatility cannot be observed directly. In practice, two methods are used to evaluate σ
  – The historical method: in the BS model, $\sigma^2 T$ is the variance of $\log(S_T)$ and the variables $\log(S_{\Delta T}/S_0)$, $\log(S_2/S_{\Delta T})$, . . . , $\log(S_{N\Delta T}/S_{(N-1)\Delta T})$ are i.i.d random variables.
    Therefore, σ can be estimated by statistical means using past observations of the asset price.
  – the "implied volatility" method: some options are quoted on organized markets; the price of options being an increasing function of σ, we can associate an "implied" volatility to each quoted option, by inversion of the Black-Scholes formula.

$$C^{Obs}(S_0, 0, T, K) = C(S_0, 0, T, K, \Sigma(K, T), r)$$

  Σ is called implied volatility. Due to the market imperfections Σ has a typical dependence on K called SMILE EFFECT.


n = m = 6, 30 iter.
0.3
0.2
0.1
0
−0.1
−0.2
2
1.5
1
0.5
0
t (an)
500
550
600
650
700
750
800
S
0.3
0.25
0.2
0.15
0.1
0.05
0
−0.05
−0.1
−0.15
−0.2

e

Approximating the distribution function of $g \sim N(0,1)$

Set $t = \frac{1}{1+px}$, then:

$$N(x) = \begin{cases} 1 - \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})(b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5) & \text{if} \quad x \geq 0 \\ \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})(b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5) & \text{if} \quad x < 0 \end{cases}$$

with the following constants :

$$p = 0.2316419;$$

$$b_1 = 0.319381530;$$

$$b_2 = -0.356563782;$$

$$b_3 = 1.781477937;$$

$$b_4 = -1.821255978;$$

$$b_5 = 1.330274429;$$

Approximating the distribution function of $g \sim N(0,1)$

```
double N(double x)
{ const double p= 0.2316419;
  const double b1= 0.319381530;
  const double b2= -0.356563782;
  const double b3= 1.781477937;
  const double b4= -1.821255978;
  const double b5= 1.330274429;
  const double one_over_twopi= 0.39894228;
  double t;
  if(x >= 0.0)
    {
      t = 1.0 / ( 1.0 + p * x );
      return (1.0 - one_over_twopi * exp( -x * x / 2.0 )
* t * ( t *( t * ( t * ( t * b5 + b4 ) + b3 ) + b2 ) + b1 ));
    }
  else
    {/* x < 0 */
      t = 1.0 / ( 1.0 - p * x );
      return ( one_over_twopi * exp( -x * x / 2.0 ) *
       t * ( t *( t * ( t * ( t * b5 + b4 ) + b3 ) + b2 ) + b1 ));
    }
}
```

Scilab

```
function [y]=Norm(x)
  [y,Q]=cdfnor("PQ",x,0,1);
endfunction
```

Price of a Call option

```
main()
{
    double sigmasqrt,d1,d2,delta,price;

    sigmasqrt=sigma*sqrt(t);
    d1=(log(s/k)+r*t)/sigmasqrt+sigmasqrt/2.;
    d2=d1-sigmasqrt;
    delta=N(d1);
    /*Price*/
    price=s*delta-exp(-r*t)*k*N(d2);
}
```

Price of a put option

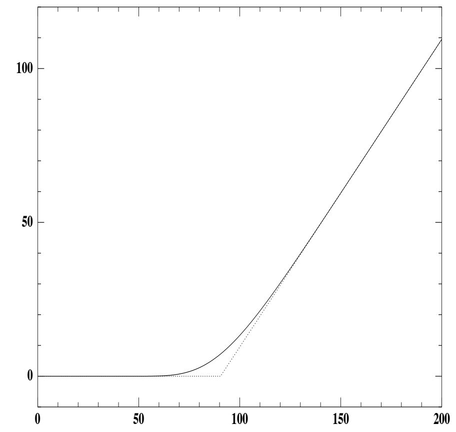
```
main()
{
    double sigmasqrt,d1,d2,delta,price;

    sigmasqrt=sigma*sqrt(t);
    d1=(log(s/k)+r*t)/sigmasqrt+sigmasqrt/2.;
    d2=d1-sigmasqrt;
    delta=N(-d1);

    /*Price*/
    price=exp(-r*t)*k*N(-d2)-delta*s;
}
```

Put-Call Theorem Parity

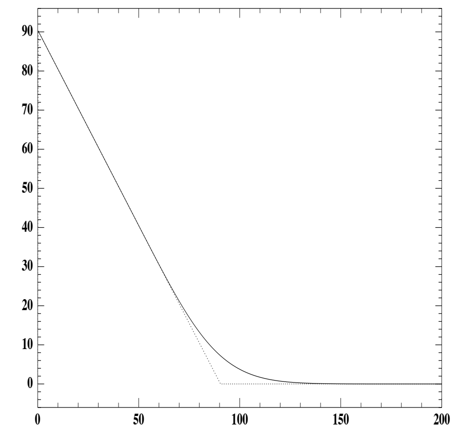
We have the following put-call parity between the prices of the underlying asset S_t and European call and put options on stocks that pay no dividends:

$$C_t = P_t + S_t - Ke^{-r(T-t)}.$$

Payoff Call



Payof Put



Proof of the Black-Scholes formula

$$C(t,x) = E_Q\left[e^{-r(T-t)}(xe^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(B_T-B_t)} - K)_+\right]$$

Then

$$C(t,x) = E_Q\left[\left(xe^{-\frac{1}{2}\sigma^2(T-t)+\sigma\sqrt{T-t}g} - Ke^{-r(T-t)}\right)_+\right]$$

with $g \sim N(0,1)$.
Set

$$d_1 = \frac{\log\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

$$\begin{aligned}
C(t,x) &= \mathbb{E}\left[\left(xe^{\sigma\sqrt{(T-t)}g-\frac{1}{2}\sigma^2(T-t)} - Ke^{-r(T-t)}\right) g \geq -d_2\right] \\
&= \int_{-d_2}^{+\infty} \left(xe^{\sigma\sqrt{(T-t)}y-\frac{1}{2}\sigma^2(T-t)} - Ke^{-r(T-t)}\right) \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy \\
&= \int_{-\infty}^{d_2} \left(xe^{-\sigma\sqrt{(T-t)}y-\frac{1}{2}\sigma^2(T-t)} - Ke^{-r(T-t)}\right) \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy.
\end{aligned}$$

$$C(t,x) = \int_{-\infty}^{d_2} \left(x e^{-\sigma\sqrt{(T-t)}y - \sigma^2 (T-t)/2} - K e^{-r(T-t)} \right) \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy.$$

The change of variable $z = y + \sigma\sqrt{(T-t)}$, gives :

$$C(t,x) = xN(d_1) - Ke^{-r(T-t)}N(d_2),$$

with :

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-x^2/2} dx.$$

Monte Carlo method in the Black-Scholes model

We want compute

$$P = e^{-rT}\mathbb{E}_Q\left[(K - S_T)_+\right].$$

in the Black-Scholes model

$$S_T = S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma B_T}$$

The payoff function can be written in the following way

$$h(B_T) = (K - S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma B_T})_+$$

We can approximate the price with

$$P \approx e^{-rT}\frac{h(X_1)+\cdots+h(X_n)}{n}$$

$X_1, .., X_n \sim N(0, T)$.

Monte Carlo algorithm
European Put in the Black-Scholes model

```
main()
{
  double mean_price,mean2_price,brownian,price,price_sample,error_price,inf_price,sup_price;
  mean_price= 0.0;
  mean2_price= 0.0;
  for(i=1;i<=N;i++)
    {

      brownian=gaussian()*sqrt(T);

      price_sample=MAX(0.0,K-x*exp((r-0.5*sigma*sigma)*T+sigma*brownian));

      mean_price= mean_price+price_sample;
      mean2_price= mean2_price+SQR(price_sample);
    }
  /* Price */
  price=exp(-r*T)*(mean_price/N);
  error_price= sqrt(exp(-2.*r*T)*mean2_price/N - SQR(price))/sqrt(N-1);
  inf_price= price - 1.96*(error_price);
  sup_price= price + 1.96*(error_price);
}
```