Purpose

Detection and quantification of fissile material from statistical properties of the measured radiation (neutrons and/or gamma photons), emitted spontaneously from the sample.

Principle

- Source event: a spontaneous fission with a number distribution (=multiplicity) of neutrons and photons. The factorial moments can be determined in coincidence measurements.
  The multiplicities are characteristic to the fissile nuclide.
- Reaction before escape leads to internal multiplication (*short fission chain*).
  Internal multiplication is assumed instantaneous with the source event.
  It changes the original (source) multiplicities, hence the factorial moments → possibility of determining sample mass.

Objective of work

To calculate the various individual and mixed (joint) moments of the neutron and gamma numbers, as functions of the non-escape (=reaction) probability p through derivation of a **master equation** for the **generating function** of the **joint distribution**. Of particular interest is the covariance between the number of neutrons and gamma photons.

In an individual fission event, the neutrons and gamma photons are generated independently. The internal multiplication will introduce correlations between the neutron and photon numbers.

Notations, definitions

Distributions:

- $\boldsymbol{\nu}_s$, $p_s(n)$ = number and distribution of source fission neutrons (known)
- $\boldsymbol{\nu}_r$, $p_r(n)$ = number and distribution of induced fission neutrons (known)
- $\boldsymbol{\mu}_s$, $f_s(n)$ = number and distribution of source fission gamma photons (known)
- $\boldsymbol{\mu}_r$, $f_r(n)$ = number and distribution of induced fission photons (known)
- $\boldsymbol{\nu}$, $p(n)$ = n.+ distr. of all neutrons from sample for one *initial neutron*
- $\tilde{\boldsymbol{\nu}}$, $P(n)$ = n.+ distr. of all neutrons from sample for one *source event*
- $\boldsymbol{\mu}$, $f(n)$ = n.+ distr. of all gammas from sample for one *initial neutron*
- $\tilde{\boldsymbol{\mu}}$, $F(n)$ = n.+ distr. of all gammas from sample for one *source event*

The generation of neutrons and gamma photons is assumed to be independent in a fission event (both spontaneous and induced).

Generating functions

The following generating functions are defined for the above quantities, in the same order: $q_s(z)$, $q_r(z)$, $r_s(z)$, $q_r(z)$, $h(z)$, $H(z)$, $g(z)$, $G(z)$. Here e.g.

$$q_s(z) = \mathbf{E}\{z^{\boldsymbol{\nu}_s}\} = \sum_{n=0}^{\infty} p_s(n)\, z^n$$

and so on.
The factorial moments of the various random variables will be denoted as e.g.

$$\left[\frac{d^k q_s(z)}{dz^k}\right]_{z=1} = \mathbf{E}\{\boldsymbol{\nu}_s(\boldsymbol{\nu}_s-1)\cdots(\boldsymbol{\nu}_s-k+1)\} = \nu_{s,k}$$

Here the moments $\nu_{s,k}$, $\nu_{r,k}$, $\mu_{s,k}$ and $\mu_{r,k}$ are **known** from nuclear physics; ν_k, $\widetilde{\nu_k}$, μ_k and $\widetilde{\mu_k}$ are to be derived.

Basic equations

Let $w(n_1, n_2|1)$ be the probability that n_1 neutrons and n_2 gamma photons will be generated in a cascade by one initial neutron. Then one can write the backward master equation as

$$w(n_1,n_2|1) = (1-\mathrm{p})\,\delta_{n_1,1}\,\delta_{n_2,0} + \mathrm{p}\sum_{k=0}^{\infty} p_r(k) \sum_{\ell=0}^{\infty} f_r(\ell) \sum_{\substack{n_{11}+\cdots+n_{1k}=n_1 \\ n_{21}+\cdots+n_{2k}=n_2-\ell}} \prod_{i=1}^{k} w(n_{1i}, n_{2i}|1). \tag{1}$$

Introducing the generating function

$$u(z_1,z_2|1) \equiv u(z_1,z_2) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} w(n_1,n_2|1)\, z_1^{n_1}\, z_2^{n_2}, \tag{2}$$

one obtains

$$u(z_1,z_2) = (1-\mathrm{p})\,z_1 + \mathrm{p}\, r_r(z_2)\, q_r\left[u(z_1,z_2)\right]. \tag{3}$$

Likewise, let

$$W(n_1,n_2) \tag{4}$$

be the probability that n_1 neutrons and n_2 gamma quanta are generated by one *source* event. With similar considerations as above, one obtains the master equation

$$W(n_1,n_2) = \sum_{\ell=0}^{\infty} f_s(\ell) \sum_{k=0}^{\infty} p_s(k) \sum_{\substack{n_{11}+\cdots+n_{1k}=n_1 \\ n_{21}+\cdots+n_{2k}=n_2-\ell}} \prod_{i=1}^{k} w\left(n_{1i}, n_{2i}|1\right), \tag{5}$$

Introducing the generating function

$$U(z_1,z_2) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} W(n_1,n_2)\, z_1^{n_1}\, z_2^{n_2} \tag{6}$$

it can be easily shown that it satisfies the equation

$$U(z_1,z_2) = r_s(z_2)\, q_s\left[u(z_1,z_2)\right] \tag{7}$$

with $u(z_1,z_2)$ obeying Eq. (3).

From Eqs (3) and (7) all moments, and the whole probability distribution, can be determined by taking the derivatives. These can be calculated to arbitrarily high order by symbolic computation (MATHEMATICA).

Equations for the individual distributions of neutrons and gamma photons are recovered as special cases by taking $z_2 = 1$ and $z_1 = 1$, respectively.

Calculation of the covariance

The covariance $\mathbf{Cov}\,\{\tilde{\boldsymbol{\nu}}, \tilde{\boldsymbol{\mu}}\}$ is defined as

$$\mathbf{Cov}\,\{\tilde{\boldsymbol{\nu}}, \tilde{\boldsymbol{\mu}}\} = \mathbf{E}\{\tilde{\boldsymbol{\nu}}\,\tilde{\boldsymbol{\mu}}\} - \mathbf{E}\{\tilde{\boldsymbol{\nu}}\}\,\mathbf{E}\{\tilde{\boldsymbol{\mu}}\} \equiv \left[\frac{\partial^2 U(z_1,z_2)}{\partial z_1 \partial z_2}\right]_{z_1=z_2=1} - \left[\frac{\partial U(z_1,z_2)}{\partial z_1}\right]_{z_1=z_2=1} \left[\frac{\partial U(z_1,z_2)}{\partial z_2}\right]_{z_1=z_2=1}. \tag{8}$$

Using the generating function (7) one obtains

$$\left[\frac{\partial U(z_1,z_2)}{\partial z_1}\right]_{z_1=z_2=1} = \mathbf{E}\{\tilde{\boldsymbol{\nu}}\} = \frac{1-p}{1-p\nu_{r,1}}\,\nu_{s,1} = \mathbf{M}\nu_{s,1}, \tag{9}$$

$$\left[\frac{\partial U(z_1,z_2)}{\partial z_2}\right]_{z_1=z_2=1} = \mathbf{E}\{\tilde{\boldsymbol{\mu}}\} = \mu_{s,1} + p\,\frac{\mu_{r,1}}{1-p\nu_{r,1}}\nu_{s,1} = \mu_{s,1} + \nu_{s,1}\,\mathbf{M}_\gamma, \tag{10}$$

and

$$\left[\frac{\partial^2 U(z_1,z_2)}{\partial z_1 \partial z_2}\right]_{z_1=z_2=1} = \mu_{s,1}\,\nu_{s,1}\,h_1 + \nu_{s,2}\,h_1 g_1 + \nu_{s,1}\,c_{1,1}, \tag{11}$$

where

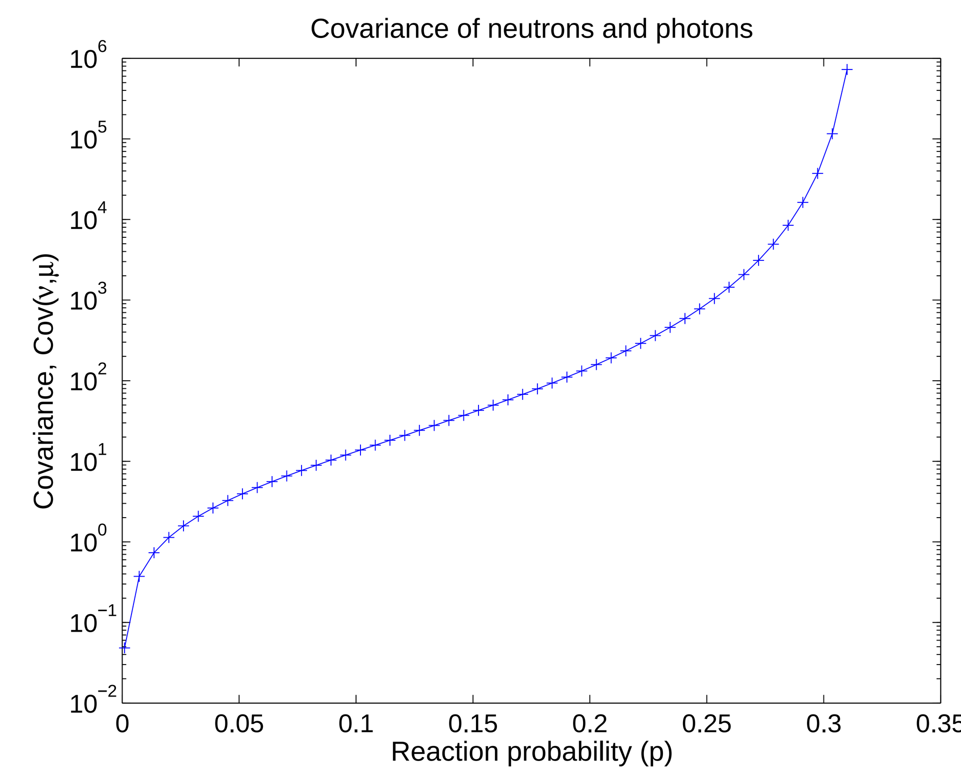
$$h_1 = \left[\frac{\partial u(z_1,z_2)}{\partial z_1}\right]_{z_1=z_2=1} \equiv \mathbf{M}, \quad g_1 = \left[\frac{\partial g(z_1,z_2)}{\partial z_2}\right]_{z_1=z_2=1} \equiv \mathbf{M}_\gamma,$$

where $\mathbf{M}$ is the so-called *leakage multiplication*, $\mathbf{M}_\gamma$ the gamma multiplication, both per one initial neutron. Further, from Eq. (3) one obtain

$$c_{1,1} = \left[\frac{\partial^2 u(z_1,z_2)}{\partial z_1 \partial z_2}\right]_{z_1=z_2=1} = c_{1,1} = p\,\left(\mu_{r,1}\nu_{r,1}\,h_1 + \nu_{r,2}\,h_1 g_1 + \nu_{r,1}\,c_{1,1}\right).$$

Final result:

$$\mathbf{Cov}\,\{\tilde{\boldsymbol{\nu}}, \tilde{\boldsymbol{\mu}}\} = \frac{p}{1-p}\left(\nu_{s,1}\nu_{r,1} + \nu_{s,2} - \nu_{s,1}^2\right)\mu_{r,1}\,\mathbf{M}^2 + \left(\frac{p}{1-p}\right)^2 \left(\nu_{s,1} + \mu_{r,1}\nu_{r,2}\right)\,\mathbf{M}^3. \tag{12}$$



Conclusions

- Without self-multiplication: the covariance is zero.
- With increasing self-multiplication: the covariance is positive and increasing.
- The dependence on the non-leakage probability is strongly non-linear.