泰勒公式的引入

微分. $f(x) = f(x_0) + f'(x_0)(x - x_0) + o(x - x_0)$

问题: 能否有$P_n(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n$,

使$f(x) = P_n(x) + o(x - x_0)^n$, 如果有，有多少个？

先证唯一性.

若有$P_n(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n$

$$Q_n(x) = b_0 + b_1(x - x_0) + \cdots + b_n(x - x_0)^n,$$

使$f(x) = P_n(x) + o(x - x_0)^n = Q_n(x) + o(x - x_0)^n$

即$a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n = b_0 + b_1(x - x_0) + \cdots + b_n(x - x_0)^n + o(x - x_0)^n$

令$x \to x_0$，取极限得$a_0 = b_0$. 方程两端除以$x - x_0$, 得

$$a_1 + \cdots + a_n(x - x_0)^{n-1} = b_1 + \cdots + b_n(x - x_0)^{n-1} + o(x - x_0)^{n-1}$$

令$x \to x_0$，取极限得$a_1 = b_1$.

同理$a_i = b_i (i = 0, 1, 2, \cdots, n)$, 即表示唯一.

泰勒公式的引入

下考虑存在性.

引理. 设$f(x)$，$P_n(x)$在x_0的邻域内有定义， 在x_0点有n阶导数，

若$f^{(k)}(x_0)=P_n^{(k)}(x_0), k=0,1,2,\cdots,n$， 则$f(x)=P_n(x)+o\left(x-x_0\right)^n$.

证明：$\lim\limits_{x\to x_0}\dfrac{f(x)-P_n(x)}{\left(x-x_0\right)^n}=\lim\limits_{x\to x_0}\dfrac{f'(x)-P_n'(x)}{n\left(x-x_0\right)^{n-1}}=\cdots$

$$=\frac{1}{n!}\lim_{x\to x_0}\frac{\left[f^{(n-1)}(x)-P_n^{(n-1)}(x)\right]-\left[f^{(n-1)}(x_0)-P_n^{(n-1)}(x_0)\right]}{x-x_0}$$

$$=\frac{1}{n!}\left[f^{(n)}(x_0)-P_n^{(n)}(x_0)\right]$$

$$=0$$

即$f(x)=P_n(x)+o\left(x-x_0\right)^n$.

泰勒公式的引入

假定$P_n(x)=a_0+a_1(x-x_0)+\cdots+a_n(x-x_0)^n$

令$x=x_0$，得$P_n(x_0)=a_0=f(x_0)$.

方程两端关于x求导,得

$P_n'(x)=a_1+2a_2(x-x_0)\cdots+na_n(x-x_0)^{n-1}$

令$x=x_0$，得$P_n'(x_0)=a_1=f'(x_0)$.

同理$P_n^{(n)}(x_0)=n!a_n=f^{(n)}(x_0),n=0,1,2\cdots$

即$P_n(x)=f(x_0)+f'(x_0)(x-x_0)+\cdots+\dfrac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

泰勒公式

定理1. 设函数$f(x)$在x_0点处有n阶导数，则在x_0点附近$f(x)$可表示为

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+R_n(x)$$

其中$R_n(x)=o\left(|x-x_0|^n\right)$.

称上式为函数$f(x)$在x_0点处展开成n阶的泰勒公式，称$R_n(x)$为佩亚诺余项.

定理2. 设函数$f(x)$在x_0点邻域内有$n+1$阶导数，则在x_0点$f(x)$可表示为

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+R_n(x)$$

其中$R_n(x)=\dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$($\xi$介于$x,x_0$之间).

称上式为函数$f(x)$在x_0点处展开的具有拉格朗日余项的泰勒公式.

又称该定理为泰勒中值定理.

泰勒公式

证明： 由$R_n^{(k)}(x_0)=f^{(k)}(x_0)-P_n^{(k)}(x_0)=0, k=0,1,2,\cdots,n,$

$$\frac{R_n(x)}{(x-x_0)^{n+1}}=\frac{R_n(x)-R_n(x_0)}{(x-x_0)^{n+1}-(x_0-x_0)^{n+1}}$$

$$=\frac{R_n'(\xi_1)}{(n+1)(\xi_1-x_0)^n}(\xi_1\text{介于}x,x_0\text{之间})$$

$$=\frac{R_n'(\xi_1)-R_n'(x_0)}{(n+1)\left[(\xi_1-x_0)^n-(x_0-x_0)^n\right]}$$

$$=\frac{R_n''(\xi_2)}{(n+1)n(\xi_2-x_0)^{n-1}}(\xi_2\text{介于}x_0,\xi_1\text{之间}).$$

连续应用柯西中值定理$n+1$次，得

$$\frac{R_n(x)}{(x-x_0)^{n+1}}=\frac{R_n^{(n+1)}(\xi)}{(n+1)!}(\xi\text{介于}x_0,x\text{之间}).$$

又$R_n^{(n+1)}(x)=f^{(n+1)}(x)$,

$$R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}(\xi\text{介于}x_0,x\text{之间}).$$

常用的泰勒公式

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

当$x_0=0$时，称此时展开式为具有拉格朗日余项的麦克劳林公式.

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$$

(ξ介于0，x之间.)

常用的麦克劳林公式：

$$1.\ e^x=1+x+\frac{1}{2!}x^2+\cdots+\frac{1}{n!}x^n+\frac{e^{\xi}}{(n+1)!}x^{n+1}$$

$$2.\ \sin x=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\cdots+(-1)^m\frac{x^{2m+1}}{(2m+1)!}+(-1)^{m+1}\frac{\cos\xi}{(2m+3)!}x^{2m+3}$$

$$3.\ \cos x=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\cdots+(-1)^m\frac{x^{2m}}{(2m)!}+(-1)^{m+1}\frac{\cos\xi}{(2m+2)!}x^{2m+2}$$

$$4.\ \ln(x+1)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+(-1)^{n-1}\frac{x^n}{n}+(-1)^n\frac{x^{n+1}}{(n+1)(1+\xi)^{n+1}}$$

$$5.\ (1+x)^{\alpha}=1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n$$

$$+\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{(n+1)!}(1+\xi)^{\alpha-n-1}x^{n+1}$$

常用的泰勒公式

证明$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + (-1)^{m+1}\frac{\cos\xi}{(2m+3)!}x^{2m+3}$

$$\left(\text{分析. } f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}\right)$$

证明：设$f(x) = \sin x$，则$f^{(n)}(x) = \sin(x + \frac{n}{2}\pi)(n = 0, 1, 2, \cdots)$

所以$f(0) = 0$，$f'(0) = 1, f''(0) = 0, f'''(0) = -1,$

$$f^{(k)}(0) = \sin\frac{k\pi}{2} = \begin{cases} 0, & k = 2m \\ (-1)^m, & k = 2m+1 \end{cases}$$

$$f^{(n+1)}(x) = \sin\left(x + \frac{(n+1)\pi}{2}\right)$$

当$n+1 = 2m+3$时，$f^{(n+1)}(\xi) = \sin\left(\xi + \frac{(2m+3)\pi}{2}\right)$

$$= (-1)^{m+1}\cos\xi$$

1.近似计算

例1. 用$e \approx 1+1+\frac{1}{2!}+\cdots+\frac{1}{n!}$来计算$e$的值，使其误差不超过0.001，求$n$的最小值.

解.$e^x = 1+x+\frac{1}{2!}x^2+\cdots+\frac{1}{n!}x^n+\frac{e^{\xi}}{(n+1)!}x^{n+1}$ (ξ介于0，x之间)

取$x=1$，得$e=1+1+\frac{1}{2!}+\cdots+\frac{1}{n!}+\frac{e^{\xi}}{(n+1)!}$ (ξ介于0,1之间)

$$\frac{e^{\xi}}{(n+1)!} \le \frac{e}{(n+1)!} < \frac{3}{(n+1)!} \le 0.001$$

$$\Rightarrow (n+1)! \ge 3000$$

$$\Rightarrow n = 6$$

2.极限计算

例2. 当$x \to 0$时，问$1-x+\dfrac{x^2}{2}-e^x+2\sin x$是$x$的几阶无穷小？

解.$e^x=1+x+\dfrac{1}{2!}x^2+\cdots+\dfrac{1}{n!}x^n+o\left(x^n\right)$

$$\sin x = x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\cdots+(-1)^m\frac{x^{2m+1}}{(2m+1)!}+o(x^{2m+1})$$

代入，得

$$1-x+\frac{x^2}{2}-e^x+2\sin x$$

$$=1-x+\frac{x^2}{2}-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+o(x^3)\right)+2\left(x-\frac{x^3}{3!}+o(x^3)\right)$$

$$=-\frac{x^3}{2}+o(x^3)$$

故为x的3阶无穷小.

3.证明题

例3. 设$f(x)$在$[-1,1]$上有连续3阶导数， 且$f(-1)=0,\ f(1)=1,$

$f'(0)=0,$证明： 存在$\xi\in(-1,1)$， 使$f'''(\xi)=3$成立.

证明： 将$f(x)$在$x=0$点处展成麦克劳林公式,

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(\eta)}{3!}x^3\ (\eta\text{介于}0,x\text{之间})$$

$$f(-1)=0=f(0)+\frac{f''(0)}{2}-\frac{f'''(\xi_1)}{6}(\xi_1\in(-1,0))\quad\cdots\cdots\quad ①$$

$$f(1)=1=f(0)+\frac{f''(0)}{2}+\frac{f'''(\xi_2)}{6}(\xi_2\in(0,1))\quad\cdots\cdots\quad ②$$

② - ①, 得$1=\frac{1}{6}[f'''(\xi_1)+f'''(\xi_2)]$

$$\min[f'''(\xi_1),f'''(\xi_2)]\le 3=\frac{1}{2}[f'''(\xi_1)+f'''(\xi_2)]\le\max[f'''(\xi_1),f'''(\xi_2)]$$

由介值定理，存在$\xi\in(-1,1)$， 使$f'''(\xi)=3.$