高阶导数定义

定义. 如果$y=f(x)$可导，且其导函数$f'(x)$仍可导，则称$f'(x)$的导数为$f(x)$的二阶导数，记为$y''=f''(x)=\dfrac{d^2y}{dx^2}$.如果$f''(x)$还可导，称其导数为$y=f(x)$的三阶导数，记为$y'''=f'''(x)=\dfrac{d^3y}{dx^3}$，……—直这样下去可以类似的定义$y=f(x)$的$n$阶导数，记为

$$y^{(n)}=f^{(n)}(x)=\frac{d^ny}{dx^n}=\lim_{\Delta x\to 0}\frac{f^{(n-1)}(x+\Delta x)-f^{(n-1)}(x)}{\Delta x},$$

这里$n=0,1,2,\cdots$

记$f^{(0)}(x)=f(x)$.

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例. 设$f(x), g(x)$互为反函数，且$f''(x)$存在，又$f'(x) \neq 0$，求$g''(x)$.

解. 设$y = g(x)$，则$x = f(y)$.

由反函数求导法则，有$g'(x) = \dfrac{1}{f'(y)}$

$$\text{则} g''(x) = \left(\frac{1}{f'(y)}\right)_x' = -\frac{f''(y)}{[f'(y)]^2} \cdot y'$$

$$= -\frac{f''(y)}{[f'(y)]^3}$$

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例1. 设 $\begin{cases} x = \arctan t \\ y = e^t \end{cases}$，求 $\dfrac{d^2 y}{dx^2}$.

解. $\dfrac{dy}{dx} = \dfrac{\left(e^t\right)'}{(\arctan t)'} = \dfrac{e^t}{1/(1+t^2)} = e^t(1+t^2)$

得一新的参数式函数 $\begin{cases} Y = e^t\left(1+t^2\right) \\ x = \arctan t \end{cases}$

$$\frac{d^2 y}{dx^2} = \frac{\left(e^t\left(1+t^2\right)\right)'}{(\arctan t)'} = \frac{e^t(t+1)^2}{1/\left(1+t^2\right)}$$

$$= e^t(t+1)^2\left(1+t^2\right)$$

例2. 设$xe^{y}=y+1$,求y''.

解. 方程两边关于x求导

$$\left(xe^{y}\right)'_{x}=(y+1)'_{x}\Rightarrow e^{y}+xe^{y}y'=y'$$

得$y'=\dfrac{e^{y}}{1-xe^{y}}=-\dfrac{e^{y}}{y}$

$$y''=\left(-\frac{e^{y}}{y}\right)'=-\frac{ye^{y}y'-e^{y}y'}{y^{2}}=\frac{e^{2y}(y-1)}{y^{3}}$$

高阶导数公式

$$1.[f(x)\pm g(x)]^{(n)}=f^{(n)}(x)\pm g^{(n)}(x)$$

$$2.[Cf(x)]^{(n)}=Cf^{(n)}(x)$$

$$3.[f(x)g(x)]^{(n)}\triangleq(f+g)^{n}$$ --------莱布尼兹公式

$$=f^{(n)}g^{(0)}+C_n^1f^{(n-1)}g^{(1)}+C_n^2f^{(n-2)}g^{(2)}+\cdots C_n^nf^{(0)}g^{(n)}$$

分析.$[f(x)g(x)]^{(1)}=f^{(1)}(x)g^{(0)}(x)+f^{(0)}(x)g^{(1)}(x)$

$$[f(x)g(x)]^{(2)}=\left[f^{(1)}(x)g^{(0)}(x)+f^{(0)}(x)g^{(1)}(x)\right]'$$

$$=f^{(2)}(x)g^{(0)}(x)+2f^{(1)}(x)g^{(1)}(x)+f^{(0)}(x)g^{(2)}(x)$$

$$\triangleq(f+g)^{2}$$

$$4.\ [f(ax+b)]^{(n)} = a^n f^{(n)}(ax+b)$$

$$5.\ [C]^{(n)} = 0$$

$$6.\ \left[x^{\alpha}\right]^{(n)} = \alpha(\alpha-1)\cdots(\alpha-n+1)\,x^{\alpha-n}$$

$$7.\ \left(\frac{1}{x}\right)^{(n)} = \frac{(-1)^n n!}{x^{n+1}}$$

$$8.\ (a^x)^{(n)} = (\ln a)^n \cdot a^x$$

$$9.\ (e^x)^{(n)} = e^x$$

$$10.\ (\ln x)^{(n)} = \left(\frac{1}{x}\right)^{(n-1)} = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$11.\ [\sin x]^{(n)} = \sin(x + \frac{n\pi}{2})$$

分析. $[\sin x]^{(1)} = \cos x \quad = \sin\left(x + \frac{\pi}{2}\right)$

$$[\sin x]^{(2)} = -\sin x \quad = \sin(x + \pi)$$

$$[\sin x]^{(3)} = -\cos x \quad = \sin\left(x + \frac{3\pi}{2}\right)$$

$$[\sin x]^{(4)} = \sin x \quad = \sin\left(x + \frac{4\pi}{2}\right)$$

$$12.\ [\cos x]^{(n)} = \cos(x + \frac{n\pi}{2})$$

高阶导数计算举例

例1. 设$y=\dfrac{x^3}{x^2+3x+2}$，求$y^{(n)}$.

解. $$\frac{x^3}{x^2+3x+2}=\frac{x\left(x^2+3x+2\right)-3\left(x^2+3x+2\right)+7x+6}{x^2+3x+2}$$

$$=(x-3)+\frac{7x+6}{(x+1)(x+2)}=(x-3)+\frac{8}{x+2}-\frac{1}{x+1}$$

$$y'=1-\frac{8}{(x+2)^2}+\frac{1}{(x+1)^2}$$

$$y^{(n)}=\frac{8(-1)^n n!}{(x+2)^{n+1}}-\frac{(-1)^n n!}{(x+1)^{n+1}},\quad n\geq 2$$

例2. 设$y=\left(x^2+1\right)a^x$，求$y^{(n)}$.

解. 由莱布尼兹公式，

$$y^{(n)}=\left(x^2+1\right)\left(a^x\right)^{(n)}+C_n^1\left(x^2+1\right)^{(1)}\left(a^x\right)^{(n-1)}$$

$$+C_n^2\left(x^2+1\right)^{(2)}\left(a^x\right)^{(n-2)}$$

$$=\left(x^2+1\right)(\ln a)^n a^x+2xn(\ln a)^{n-1}a^x$$

$$+n(n-1)(\ln a)^{n-2}a^x$$