第一换元积分法

设$F'(u)=f(u), u=g(x)$可导，有

$$\left(F(g(x))\right)'=f(g(x))g'(x).$$

$$\Rightarrow \int f(g(x))g'(x)\mathrm{d}x=F(g(x))+C.$$

$$\int f(u)\mathrm{d}u=F(u)+C.$$

$$\Rightarrow \underline{\int f(g(x))g'(x)\mathrm{d}x}=\int f(u)\mathrm{d}u=\underline{\int f(g(x))\mathrm{d}g(x)}$$

——第一换元积分公式(凑微分法)

当$g(x)=\ln x, \arcsin x, ar\cos x, \arctan x, arc\cot x.$

$g'(x)$	$\dfrac{1}{x}$	$\dfrac{1}{\sqrt{1-x^2}}$	$\dfrac{1}{1+x^2}$

例1. $\displaystyle\int \frac{\ln^2 x}{x}\mathrm{d}x = \int \ln^2 x\,\mathrm{d}(\ln x) = \frac{1}{3}\ln^3 x + C$

例2. $\displaystyle\int \frac{\sin\arctan x}{1+x^2}\mathrm{d}x = \int \sin\arctan x\,\mathrm{d}\arctan x$

$$= -\cos\arctan x + C$$

第一换元积分法：基本公式补充

$$\int f(g(x))g'(x)\mathrm{d}x=\int f(g(x))\mathrm{d}g(x)$$

设$g(x)=ax+b$,有

$$\int f(ax+b)\mathrm{d}x=\frac{1}{a}\int f(ax+b)\,\mathrm{d}(ax+b)$$

例1.$\displaystyle\int\frac{1}{a^2+x^2}\mathrm{d}x\,(a>0)=\frac{1}{a^2}\int\frac{1}{1+\left(\frac{x}{a}\right)^2}\mathrm{d}x$

$$=\frac{1}{a}\int\frac{1}{1+\left(\frac{x}{a}\right)^2}\mathrm{d}\left(\frac{x}{a}\right)=\frac{1}{a}\arctan\frac{x}{a}+C$$

公式12:$\displaystyle\int\frac{1}{1+u^2}\mathrm{d}x=\arctan u+C$

基本公式补充

$$\int \frac{1}{a^2+x^2}\mathrm{d}x = \frac{1}{a}\arctan\frac{x}{a}+C\ (a>0)$$ ——公式12′

$$\int \frac{1}{\sqrt{a^2-x^2}}\mathrm{d}x = \arcsin\frac{x}{a}+C\ (a>0)$$ ——公式11′

例2. $\int \tan x\mathrm{d}x = \int \frac{\sin x}{\cos x}\mathrm{d}x = -\int \frac{1}{\cos x}\mathrm{d}\cos x$

$$= -\ln|\cos x| + C$$

$$\int \tan x\mathrm{d}x = -\ln|\cos x| + C$$ ——公式13

$$\int \cot x\mathrm{d}x = \ln|\sin x| + C$$ ——公式14

例3. $$\int \frac{1}{\cos x}\mathrm{d}x = \int \frac{\cos x}{1-\sin^2 x}\mathrm{d}x = \int \frac{1}{1-\sin^2 x}\mathrm{d}\sin x$$

$$= \int \frac{1}{(1+\sin x)(1-\sin x)}\mathrm{d}\sin x$$

$$= \frac{1}{2}\int\left(\frac{1}{1-\sin x}+\frac{1}{1+\sin x}\right)\mathrm{d}\sin x$$

$$= -\frac{1}{2}\int \frac{1}{1-\sin x}\mathrm{d}(1-\sin x)+\frac{1}{2}\int \frac{1}{1+\sin x}\mathrm{d}(1+\sin x)$$

$$= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|+C = \ln\sqrt{\frac{(1+\sin x)^2}{1-\sin^2 x}}+C$$

$$= \ln\left|\frac{1+\sin x}{\cos x}\right|+C$$

公式15 $$\int \frac{1}{\cos x}\mathrm{d}x = \ln\left|\frac{1}{\cos x}+\tan x\right|+C$$

公式16

$$\int \frac{1}{\sin x}\mathrm{d}x = \ln\left|\frac{1}{\sin x} - \cot x\right| + C$$

第一换元积分法：三角函数（i）

1. $\int \sin^m x \cos^n x \,dx$

当m,n至少有一个是奇数时，不妨设$n=2k+1$,有

$$\int \sin^m x\cos^n x\,dx = \int \sin^m x\cos^{2k+1} x\,dx$$
$$= \int \sin^m x\cos^{2k} x\,d\sin x$$

例1. $\int \sin^4 x\cos^3 x dx = \int \sin^4 x\cos^2 x\,d\sin x$

$$= \int \sin^4 x\left(1-\sin^2 x\right)d\sin x$$
$$= \int \sin^4 x d\sin x - \int \sin^6 x d\sin x$$
$$= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C$$

$1.\int \sin^m x \cos^n x \, dx$

当m,n全为偶数时，利用倍角公式

$$\sin^2 x = \frac{1-\cos 2x}{2}, \cos^2 x = \frac{1+\cos 2x}{2}$$ 来降次.

例2. $\int \sin^4 x dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \cdots\cdots$

$2.\int \frac{1}{\sin^m x \cos^n x} dx$（利用$1=\sin^2 x + \cos^2 x$来降次.）

$$\int \frac{\sin^2 x + \cos^2 x}{\sin^m x \cos^n x} dx \longrightarrow \begin{array}{l} \int \frac{1}{\sin^{m-2} x \cos^n x} dx \\ \int \frac{1}{\sin^m x \cos^{n-2} x} dx \end{array}$$

最终得到$\int \frac{\cos x}{\sin^k x} dx, \int \frac{\sin x}{\cos^k x} dx, \int \frac{1}{\cos^k x} dx, \int \frac{1}{\sin^k x} dx.$

（凑微分）　　（分部积分）

第一换元积分法：三角函数（ii）

$3.\int \tan^n x\,dx, \int \cot^n x\,dx\,(n>2)$

$n=1$时，$\int \tan x\,dx = -\ln|\cos x| + C$

$n=2$时，$\int \tan^2 x\,dx = \int \left(\dfrac{1}{\cos^2 x} - 1\right)dx$

$n>2$时，$\int \tan^n x dx = I_n$

$$= \int \tan^{n-2} x \tan^2 x\,dx$$

$$= \int \tan^{n-2} x \left(\frac{1}{\cos^2 x} - 1\right)dx$$

$$= \int \tan^{n-2} x d\tan x - \int \tan^{n-2} x dx$$

$$= \frac{1}{n-1}\tan^{n-1} x - I_{n-2}$$

4. $\int \frac{1}{a+b\sin^2 x}\mathrm{d}x$和$\int \frac{1}{a+b\cos^2 x}\mathrm{d}x$

$$\int \frac{1}{a+b\sin^2 x}\mathrm{d}x=\int \frac{1}{a\cos^2 x+a\sin^2 x+b\sin^2 x}\mathrm{d}x$$

$$=\int \frac{1}{a+(a+b)\tan^2 x}\cdot\frac{1}{\cos^2 x}\mathrm{d}x=\int \frac{1}{a+(a+b)\tan^2 x}\mathrm{d}\tan x$$

例. $\int \frac{1}{1+\sin^2 x}\mathrm{d}x=\int \frac{1}{\cos^2 x+2\sin^2 x}\mathrm{d}x$

$$=\int \frac{1}{1+2\tan^2 x}\mathrm{d}\tan x=\frac{1}{\sqrt{2}}\int \frac{1}{1+\left(\sqrt{2}\tan x\right)^2}\mathrm{d}\left(\sqrt{2}\tan x\right)$$

$$=\frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}\tan x\right)+C$$

第一换元积分法：其他函数类型

例1. $\int \frac{1}{1+e^x}\mathrm{d}x$

$$=\int \frac{1+e^x-e^x}{1+e^x}\mathrm{d}x$$

$$=\int \left(1-\frac{e^x}{1+e^x}\right)\mathrm{d}x$$

$$=\int \mathrm{d}x-\int \frac{1}{1+e^x}\mathrm{d}e^x$$

$$=x-\int \frac{1}{1+e^x}\mathrm{d}\left(e^x+1\right)$$

$$=x-\ln\left(e^x+1\right)+C$$

例2. $\int \frac{x}{x^2+2x+2}\mathrm{d}x$ 　　（局部求导寻找$g(x)$）

$$\left(x^2+2x+2\right)'=2x+2$$

$$=\frac{1}{2}\int\frac{2x+2-2}{x^2+2x+2}\mathrm{d}x$$

$$=\frac{1}{2}\int\frac{2x+2}{x^2+2x+2}\mathrm{d}x-\int\frac{1}{x^2+2x+2}\mathrm{d}x$$

$$=\frac{1}{2}\int\frac{\left(x^2+2x+2\right)'}{x^2+2x+2}\mathrm{d}x-\int\frac{1}{(x+1)^2+1}\mathrm{d}x$$

$$=\frac{1}{2}\ln\left(x^2+2x+2\right)-\arctan(1+x)+C$$