微分的定义

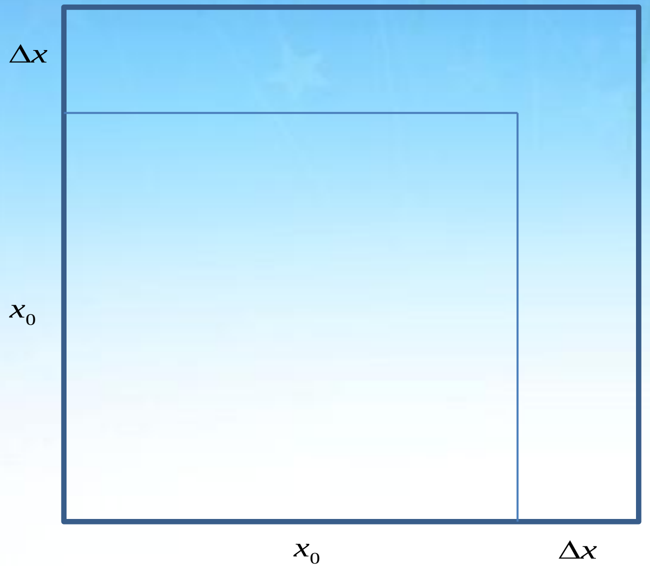
薄片热胀冷缩问题

$$\Delta y=(x_0+\Delta x)^2-x_0{}^2$$

$$=x_0{}^2+(\Delta x)^2+2x_0\Delta x-x_0{}^2$$

$$=2x_0\Delta x+(\Delta x)^2$$

$$=A\Delta x+o(\Delta x)$$



定义1. 设$y=f(x)$在x_0的某邻域内有定义，给x_0一个增量Δx，如果相应的函数增量$\Delta y=f(x_0+\Delta x)-f(x_0)=A\Delta x+o(\Delta x)$此时则说$y=f(x)$在$x_0$点可微. 其中$A\Delta x$为$y=f(x)$在$x_0$点的微分（$A$为常数），记$A\Delta x=dy\big|_{x=x_0}$.

定义2. 设$y = f(x)$在D上每一点都可微，则其微分又是D上一个新的函数，称此函数为$y = f(x)$在D上的微分函数，即$dy=df(x) = f(x)\Delta x.$

哈尔滨工业大学数学系
Department of Mathematics,HIT

微分与导数的关系

定理. 函数$y=f(x)$在点x_0可导$\Leftrightarrow$函数$y=f(x)$在点x_0可微，

且$\mathrm{d}y\big|_{x=x_0}=f'(x_0)\Delta x$.

分析. "$\Rightarrow$" 若可导，$f'(x_0)=\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}$存在

$$\frac{\Delta y}{\Delta x}=f'(x_0)+\alpha\Rightarrow\Delta y=f'(x_0)\Delta x+\alpha\Delta x$$

$\Rightarrow\Delta y=f'(x_0)\Delta x+o(\Delta x)$，即可微.

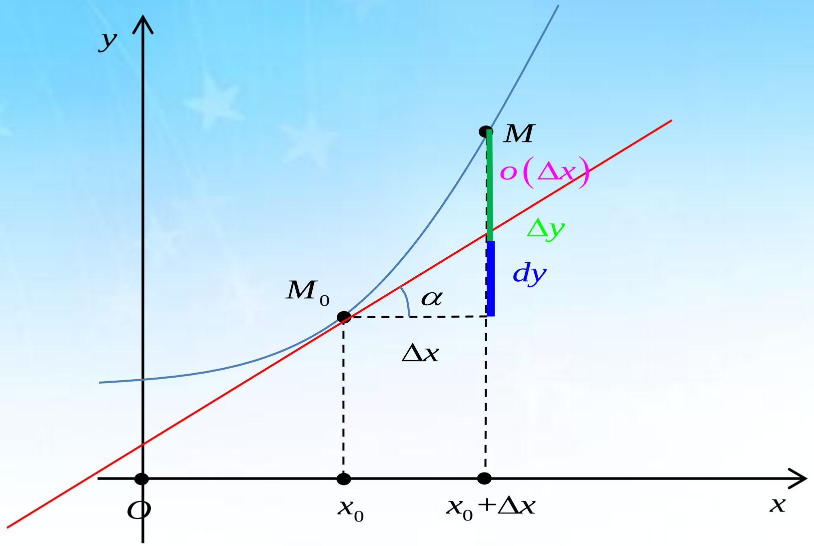
"$\Leftarrow$" 若可微，$\Delta y=A\Delta x+o(\Delta x)$

$\Rightarrow\dfrac{\Delta y}{\Delta x}=A+\dfrac{o(\Delta x)}{\Delta x}$，两边取极限

$\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}=A$存在，即可导.

哈尔滨工业大学数学系
Department of Mathematics,HIT

微分的几何意义



$$\Delta y = f(x_0 + \Delta x) - f(x_0)$$

$$dy = f'(x_0)\Delta x = \tan\alpha\Delta x$$

微分的计算

1.微分的基本公式

$$y = x \Rightarrow dy = dx = \Delta x$$

故$dy = df(x) = f'(x)\Delta x = f'(x)dx$

2.微分的四则运算

1)$d(u \pm v) = du \pm dv$;

2)$d(uv) = vdu + udv$; $\left(d(uv) = (uv)' dx = v'udx + u'vdx\right)$

3)$d\left(\dfrac{u}{v}\right) = \dfrac{vdu - udv}{v^2}(v \neq 0)$;

3.复合函数的微分

设$y=f(u)$可微，$u=g(x)$可微，且$y=f[g(x)]$在x的邻域内有定义

u看成是自变量时，$dy=f'(u)du$

u看成是中间变量时，

$$dy=d\{f[g(x)]\}=\{f[g(x)]\}'dx$$

$$=f'[g(x)]g'(x)dx$$

$$=f'(u)du$$

称之为一阶微分形式不变性.

微分的应用：近似计算

由微分定义$\Delta y = f(x_0 + \Delta x) - f(x_0) = f'(x_0)\Delta x + o(\Delta x)$

如果$f'(x_0)$存在，有$f(x_0 + \square) \approx f(x_0) + f'(x_0)\cdot\square, \quad (\square \to 0)$

例. 近似计算$\sqrt[5]{1.01}$的值.

解. 设$f(x) = \sqrt[5]{x}$，取$x_0 = 1$，取$\Delta x = 0.01$，$f'(x) = \dfrac{1}{5}x^{-\frac{4}{5}}$

$$f(x_0 + \Delta x) = \sqrt[5]{1.01} \approx f(x_0) + f'(x_0)\Delta x$$
$$= \sqrt[5]{1} + \frac{1}{5}(1)^{-\frac{4}{5}} \cdot 0.01 = 1.002$$