设函数 $\lim \dfrac{f(x)}{g(x)}$ 为 “$\dfrac{0}{0}$” 或 “$\dfrac{\infty}{\infty}$” 型未定式，而

$\lim \dfrac{f'(x)}{g'(x)} = A$(或$\infty$)，则有 $\lim \dfrac{f(x)}{g(x)} = \lim \dfrac{f'(x)}{g'(x)}$.

证明.(只能证明$\lim f(x)=0$和$\lim g(x)=0$时情形.)

(1) $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)}$ 情形,定义 $f(x_0) = g(x_0) = 0$,

则$f(x)$、$g(x)$在x_0处连续，从而有

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(x_0)}{g(x)-g(x_0)} \overset{\text{柯西}}{=} \frac{f'(\xi)}{g'(\xi)} \quad (\xi\text{介于}x_0, x\text{之间.})$$

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(\xi)}{g'(\xi)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)} = A(\infty)$$

(2) $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}$ 情形，令 $x=\dfrac{1}{t}$， 当 $x\to\infty$ 时， $t\to 0.$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{t\to 0}\frac{f\left(\frac{1}{t}\right)}{g\left(\frac{1}{t}\right)}=\lim_{t\to 0}\frac{\left[f\left(\frac{1}{t}\right)\right]'}{\left[g\left(\frac{1}{t}\right)\right]'}$$

$$=\lim_{t\to 0}\frac{-\frac{1}{t^2}f'\left(\frac{1}{t}\right)}{-\frac{1}{t^2}g'\left(\frac{1}{t}\right)}=\lim_{t\to 0}\frac{f'\left(\frac{1}{t}\right)}{g'\left(\frac{1}{t}\right)}$$

$$=\lim_{x\to\infty}\frac{f'(x)}{g'(x)}=A(\infty)$$

例1. $\lim\limits_{x\to\infty}\dfrac{x-\sin x}{x+\sin x} = \lim\limits_{x\to\infty}\dfrac{1-\cos x}{1+\cos x}$ (不存在)

$$= \lim_{x\to\infty}\frac{1-\dfrac{\sin x}{x}}{1+\dfrac{\sin x}{x}} = 1$$

例2. $\lim\limits_{x\to 0}\dfrac{e^{-\frac{1}{x^2}}}{x} = \lim\limits_{x\to 0}\dfrac{\left(e^{-\frac{1}{x^2}}\right)'}{(x)'} = \lim\limits_{x\to 0}\dfrac{2e^{-\frac{1}{x^2}}}{x^3} \quad \left("\dfrac{0}{0}"\right)$

$$\overset{t=\frac{1}{x}}{=} \lim_{t\to\infty}\frac{t}{e^{t^2}} = \lim_{t\to\infty}\frac{1}{2te^{t^2}} = 0.$$

罗比达法则应用举例

例3. $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2}-\cot^2 x\right)$

$$=\lim_{x\to 0}\frac{\sin^2 x-x^2\cos^2 x}{x^2\sin^2 x}=\lim_{x\to 0}\frac{\sin^2 x-x^2\cos^2 x}{x^4}$$

$$=\lim_{x\to 0}\frac{(\sin x+x\cos x)(\sin x-x\cos x)}{x^4}$$

$$=\lim_{x\to 0}\frac{\sin x+x\cos x}{x}\lim_{x\to 0}\frac{\sin x-x\cos x}{x^3}$$

$$=2\lim_{x\to 0}\frac{(\sin x-x\cos x)'}{(x^3)'}$$

$$=2\lim_{x\to 0}\frac{\cos x-\cos x+x\sin x}{3x^2}$$

$$=\frac{2}{3}$$

罗比达法则应用举例

函数的未定式：

(1) $"\dfrac{0}{0}"$型　　(2) $"\dfrac{\infty}{\infty}"$型

(3) $"0\cdot\infty"$型 $\longrightarrow$ $"\dfrac{\infty}{\frac{1}{0}}"$ 或 $"\dfrac{0}{\frac{1}{\infty}}"$ $\longrightarrow$ $"\dfrac{0}{0}"$型或$"\dfrac{\infty}{\infty}"$型

(4) $"\infty-\infty"$型 $\longrightarrow$ $"0\cdot\infty"$型

(5) $"1^{\infty}"$型　　(6) $"0^{0}"$型　　(7) $"\infty^{0}"$型

$\downarrow$　　$\downarrow$　　$\downarrow$

$$\underbrace{"e^{\infty\cdot\ln 1}" \qquad "e^{0\cdot\ln 0}" \qquad "e^{0\cdot\ln\infty}"}_{"e^{0\cdot\infty}"\text{型}}$$

罗比达法则应用举例

例1. $\lim\limits_{x\to+\infty}\left(x-x^2\ln\left(1+\frac{1}{x}\right)\right)$ ("$\infty-\infty$"型)

$$=\lim_{x\to+\infty}x^2\left(\frac{1}{x}-\ln\left(1+\frac{1}{x}\right)\right)$$

$$=\lim_{x\to+\infty}\frac{\frac{1}{x}-\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x^2}}\qquad\left(令\frac{1}{x}=t\right)$$

$$=\lim_{t\to0^+}\frac{t-\ln(1+t)}{t^2}=\lim_{t\to0^+}\frac{1-\frac{1}{1+t}}{2t}$$

$$=\lim_{t\to0^+}\frac{t}{2t(1+t)}=\frac{1}{2}$$

罗比达法则应用举例

例2. $\lim\limits_{x\to0}\left(\dfrac{3^x+5^x}{2}\right)^{\frac{1}{x}}$ ("1^∞"型)

$$=\lim_{x\to0}e^{\frac{\ln\left(\frac{3^x+5^x}{2}\right)}{x}}=\exp\left(\lim_{x\to0}\frac{\ln\left(\frac{3^x+5^x}{2}\right)}{x}\right)$$

$$=\exp\left(\lim_{x\to0}\frac{3^x\ln3+5^x\ln5}{3^x+5^x}\right)=\sqrt{15}$$

例3. $\lim\limits_{n\to\infty}\left(3^n+5^n\right)^{\frac{1}{n}}$ ("∞^0"型)

$$\lim_{x\to+\infty}\left(3^x+5^x\right)^{\frac{1}{x}}=\lim_{x\to+\infty}e^{\frac{\ln\left(3^x+5^x\right)}{x}}=\exp\left(\lim_{x\to+\infty}\frac{\ln\left(3^x+5^x\right)}{x}\right)$$

$$=\exp\left(\lim_{x\to+\infty}\frac{3^x\ln3+5^x\ln5}{3^x+5^x}\right)=\exp\left(\lim_{x\to+\infty}\frac{\left(\frac{3}{5}\right)^x\ln3+\ln5}{\left(\frac{3}{5}\right)^x+1}\right)$$

$$=5$$