可降阶的高阶方程

1. 方程$y^{(n)} = f(x)$

$$\Rightarrow y^{(n-1)} = \int f(x)\mathrm{d}x + C_1$$

$$\Rightarrow y^{(n-2)} = \int\int f(x)\mathrm{d}x + C_1 x + C_2$$

$$\Rightarrow y = \int\cdots\int f(x)\mathrm{d}x\cdots\mathrm{d}x + C_{n-1}x^{n-1} + C_{n-2}x^{n-2} + \cdots + C_1 x + C_0$$

2. 方程$F\left(x, y^{(n)}, y^{(n+1)}\right) = 0$

设$y^{(n)} = u$，方程化为一阶方程$F(x, u, u') = 0$

例. 求方程$xy''' - y'' = x^2e^x$的通解.

解： 令$y'' = u$，方程化为 $u' = \dfrac{1}{x}u + xe^x$

$$\Rightarrow u = e^{\int p(x)\mathrm{d}x}\left[C_1 + \int q(x)e^{-\int p(x)\mathrm{d}x}\mathrm{d}x\right]$$

$$= e^{\int \frac{1}{x}\mathrm{d}x}\left[C_1 + \int xe^x e^{-\int \frac{1}{x}\mathrm{d}x}\mathrm{d}x\right] = x\left[C_1 + e^x\right]$$

$$\Rightarrow y'' = C_1x + xe^x$$

$$\Rightarrow y' = C_1\int x\mathrm{d}x + \int xe^x\mathrm{d}x$$

$$= \frac{1}{2}C_1x^2 + xe^x - e^x + C_2$$

故通解为$y = \dfrac{1}{6}C_1x^3 + xe^x - 2e^x + C_2x + C_3$

3. 方程$F\left(y, y', y''\right)=0$

设$y'=u=\dfrac{\mathrm{d}y}{\mathrm{d}x}$,

$$y''=\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{\mathrm{d}u}{\mathrm{d}y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}=u\frac{\mathrm{d}u}{\mathrm{d}y}$$

方程化为一阶方程$F\left(y, u, u\dfrac{\mathrm{d}u}{\mathrm{d}y}\right)=0$

伯努利方程

例. 求方程 $\begin{cases} yy'' = 1 + y'^2 \\ y\big|_{x=1} = 1,\ y'\big|_{x=1} = 0 \end{cases}$ 的特解.

解： 令 $y' = u,\ y'' = u\dfrac{\mathrm{d}u}{\mathrm{d}y}$，方程化为 $yu\dfrac{\mathrm{d}u}{\mathrm{d}y} = 1 + u^2$

$$\Rightarrow \int \frac{u\mathrm{d}u}{1+u^2} = \int \frac{\mathrm{d}y}{y} \Rightarrow \frac{1}{2}\ln\left(1+u^2\right) = \ln|y| + C_0$$

$\Rightarrow 1 + u^2 = C_1 y$, 由 $x = 1$ 时， $y = 1, u = 0, \Rightarrow C_1 = 1$

$$\Rightarrow u = \pm\sqrt{y^2 - 1} = y' \Rightarrow \frac{\mathrm{d}y}{\sqrt{y^2-1}} = \pm \mathrm{d}x$$

$$\Rightarrow \int \frac{\mathrm{d}y}{\sqrt{y^2-1}} = \pm\int \mathrm{d}x \Rightarrow \ln\left|y + \sqrt{y^2-1}\right| = \pm x + C_2$$

$\Rightarrow y + \sqrt{y^2 - 1} = C_3 e^{\pm x}$, 由 $x = 1$ 时， $y = 1, \Rightarrow C_3 = e^{\mp 1}$

$$\Rightarrow y + \sqrt{y^2-1} = e^{\pm(x-1)}, \quad \Rightarrow y - \sqrt{y^2-1} = e^{\mp(x-1)},$$

故方程的解 $y = \dfrac{1}{2}\left(e^{x-1} + e^{1-x}\right)$.