设函数
$$\lim \frac{f(x)}{g(x)}$$
为 " $\frac{0}{0}$ " 或 " $\frac{\infty}{\infty}$ " 型未定式,而

$$\lim \frac{f'(x)}{g'(x)} = A(或\infty), \quad 贝有\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}.$$

证明.(只能证明 $\lim f(x)=0$ 和 $\lim g(x)=0$ 时情形.)

$$(1)\lim_{x\to x_0} \frac{f(x)}{g(x)}$$
情形,定义 $f(x_0) = g(x_0) = 0$,

则f(x)、g(x)在 x_0 处连续,从而有

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} \stackrel{\text{柯西}}{=} \frac{f'(\xi)}{g'(\xi)} \quad (\xi \Upsilon + x_0, x \ge i).$$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(\xi)}{g'(\xi)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = A(\infty)$$

罗比达法则



$$(2)\lim_{x\to\infty}\frac{f(x)}{g(x)}$$
情形,令 $x=\frac{1}{t}$,当 $x\to\infty$ 时, $t\to0$.

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{t \to 0} \frac{f\left(\frac{1}{t}\right)}{g\left(\frac{1}{t}\right)} = \lim_{t \to 0} \frac{\left[f\left(\frac{1}{t}\right)\right]'}{\left[g\left(\frac{1}{t}\right)\right]'}$$

$$= \lim_{t \to 0} \frac{-\frac{1}{t^2} f'\left(\frac{1}{t}\right)}{-\frac{1}{t^2} g'\left(\frac{1}{t}\right)} = \lim_{t \to 0} \frac{f'\left(\frac{1}{t}\right)}{g'\left(\frac{1}{t}\right)}$$

$$= \lim_{x \to \infty} \frac{f'(x)}{g'(x)} = A(\infty)$$



例1.
$$\lim_{x \to \infty} \frac{x - \sin x}{x + \sin x} = \lim_{x \to \infty} \frac{1 - \cos x}{1 + \cos x}$$
 (不存在)

$$= \lim_{x \to \infty} \frac{1 - \frac{\sin x}{x}}{1 + \frac{\sin x}{x}} = 1$$

例2.
$$\lim_{x \to 0} \frac{e^{-\frac{1}{x^2}}}{x} = \lim_{x \to 0} \frac{\left(e^{-\frac{1}{x^2}}\right)}{\left(x\right)'} = \lim_{x \to 0} \frac{2e^{-\frac{1}{x^2}}}{x^3} \quad \left("\frac{0}{0}"\right)$$

$$= \lim_{t \to \infty} \frac{t}{e^{t^2}} = \lim_{t \to \infty} \frac{1}{2te^{t^2}} = 0.$$



例3.
$$\lim_{x\to 0} \left(\frac{1}{x^2} - \cot^2 x \right)$$

$$= \lim_{x \to 0} \frac{\sin^2 x - x^2 \cos^2 x}{x^2 \sin^2 x} = \lim_{x \to 0} \frac{\sin^2 x - x^2 \cos^2 x}{x^4}$$

$$= \lim_{x \to 0} \frac{\left(\sin x + x \cos x\right)\left(\sin x - x \cos x\right)}{x^4}$$

$$= \lim_{x \to 0} \frac{\sin x + x \cos x}{x} \lim_{x \to 0} \frac{\sin x - x \cos x}{x^3}$$

$$=2\lim_{x\to 0}\frac{\left(\sin x - x\cos x\right)'}{\left(x^3\right)'}$$

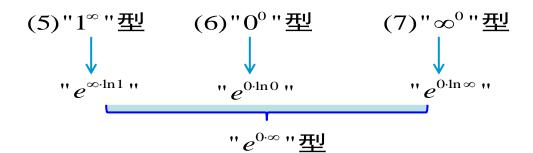
$$=2\lim_{x\to 0}\frac{\cos x - \cos x + x\sin x}{3x^2}$$

$$=\frac{2}{3}$$



函数的未定式:

$$(4)"\infty-\infty"型\longrightarrow "0\cdot\infty"型$$







例2.
$$\lim_{x \to 0} \left(\frac{3^{x} + 5^{x}}{2} \right)^{\frac{1}{x}} \qquad \qquad ("1^{\infty}" 型)$$

$$= \lim_{x \to 0} e^{\frac{\ln\left(\frac{3^{x} + 5^{x}}{2}\right)}{x}} = \exp\left(\lim_{x \to 0} \frac{\ln\left(\frac{3^{x} + 5^{x}}{2}\right)}{x}\right)$$

$$= \exp\left(\lim_{x \to 0} \frac{3^{x} \ln 3 + 5^{x} \ln 5}{3^{x} + 5^{x}}\right) = \sqrt{15}$$