定积分第一换元积分法

$$\int_a^b f(g(x))g'(x)\mathrm{d}x = \int_a^b f(g(x))\,\mathrm{d}g(x)$$

例1. 计算 $\int_0^1 x^3\sqrt{1+x^2}\,\mathrm{d}x.$

解： 原式 $=\frac{1}{2}\int_0^1 x^2\sqrt{1+x^2}\,\mathrm{d}\left(x^2+1\right)$

$$=\frac{1}{2}\int_0^1\left(x^2+1-1\right)\sqrt{1+x^2}\,\mathrm{d}\left(x^2+1\right)$$

$$=\frac{1}{2}\int_0^1\left[\left(x^2+1\right)^{\frac{3}{2}}-\sqrt{1+x^2}\right]\mathrm{d}\left(x^2+1\right)$$

$$=\frac{1}{2}\cdot\frac{2}{5}\int_0^1\mathrm{d}\left(x^2+1\right)^{\frac{5}{2}}-\frac{1}{2}\cdot\frac{2}{3}\int_0^1\mathrm{d}\left(x^2+1\right)^{\frac{3}{2}}$$

$$=\frac{1}{5}\left(x^2+1\right)^{\frac{5}{2}}\Big|_0^1-\frac{1}{3}\left(x^2+1\right)^{\frac{3}{2}}\Big|_0^1$$

定积分分部积分法

$$\int_a^b f(x)\mathrm{d}g(x) = f(x)g(x)\Big|_a^b - \int_a^b g(x)\,\mathrm{d}f(x)$$

例2. 计算 $\lim\limits_{n\to\infty}\int_0^1 e^{x^2}\cos nx\,\mathrm{d}x.$

解： 原式 $=\lim\limits_{n\to\infty}\dfrac{1}{n}\int_0^1 e^{x^2}\mathrm{d}\sin nx$

$$=\lim_{n\to\infty}\left[\frac{1}{n}e^{x^2}\sin nx\Big|_0^1 - \frac{1}{n}\int_0^1 \sin nx\,\mathrm{d}e^{x^2}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}e\sin n - \lim_{n\to\infty}\frac{1}{n}\int_0^1 2x\sin nx e^{x^2}\,\mathrm{d}x$$

$$=-\lim_{n\to\infty}\frac{2\xi\sin n\xi e^{\xi^2}}{n}\text{其中}\xi\in[0,1]$$

$$=0$$

分段函数的积分

例2. 设$f(x)=\begin{cases} x^2+1, x\geq 1 \\ x\sin x, x<1 \end{cases}$，求$\int_0^2 f(x)\mathrm{d}x$.

解：原式 $=\int_0^1 x\sin x\mathrm{d}x+\int_1^2\left(x^2+1\right)\mathrm{d}x$

$$=-\int_0^1 x\mathrm{d}\cos x+\frac{1}{3}x^3\Big|_1^2+x\Big|_1^2$$

$$=-x\cos x\Big|_0^1+\int_0^1\cos x\mathrm{d}x+\frac{1}{3}(8-1)+1$$

$$=-\cos 1+\sin 1+\frac{10}{3}$$

定积分第二换元积分法

$$\int f(x)\,\mathrm{d}x=\int f(g(t))g'(t)\mathrm{d}t$$

$$\Rightarrow \int_a^b f(x)\mathrm{d}x \overset{x=g(t)}{=} \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(t))\,g'(t)\mathrm{d}t$$ “换元换限”

注：不要求$g(x)$是一对一函数，只要求$g(x)$可积、连续.

结论1. 设$f(x)$是可积的奇函数，则$\int_{-a}^{a} f(x)\mathrm{d}x=0.$

证明：设$f(-x)=-f(x)$, $\int_{-a}^{a} f(x)\mathrm{d}x=\int_{-a}^{0} f(x)\mathrm{d}x+\int_{0}^{a} f(x)\mathrm{d}x$

设$x=-t$, 对$\int_{-a}^{0} f(x)\mathrm{d}x$, 当$x=-a, t=a; x=0, t=0.$

$$\int_{-a}^{0} f(x)\mathrm{d}x=\int_{a}^{0} f(-t)\mathrm{d}(-t)=\int_{a}^{0} f(t)\mathrm{d}t$$

故$\int_{-a}^{a} f(x)\mathrm{d}x=\int_{a}^{0} f(t)\mathrm{d}t+\int_{0}^{a} f(x)\mathrm{d}x=0$

例. 证明：连续奇函数的原函数全为偶函数.

解：设$f(-x)=-f(x)$,由$\left(\int_0^x f(t)\mathrm{d}t\right)'=f(x)$,

$F(x)=\int_0^x f(t)\mathrm{d}t+C$是$f(x)$的所有原函数.

$F(-x)=\int_0^{-x} f(t)\mathrm{d}t+C$，设$t=-u$,

$$F(-x)=-\int_0^x f(-u)\mathrm{d}u+C$$

$$=\int_0^x f(u)\mathrm{d}u+C=F(x)$$

故结论成立.

注：连续偶函数的原函数不一定是奇函数. (留做习题)

结论2. 可积的周期函数$f(x)$，其周期为$T(\neq 0)$，

则对任意实数α，有$\int_{\alpha}^{\alpha+T} f(x)\mathrm{d}x=\int_{0}^{T} f(x)\mathrm{d}x$.

证明: $\int_{\alpha}^{\alpha+T} f(x)\mathrm{d}x=\int_{\alpha}^{0} f(x)\mathrm{d}x+\int_{0}^{T} f(x)\mathrm{d}x+\int_{T}^{\alpha+T} f(x)\mathrm{d}x$

设$x=t+T$，

$$\int_{T}^{\alpha+T} f(x)\mathrm{d}x=\int_{0}^{\alpha} f(t+T)\mathrm{d}(t+T)$$

$$=\int_{0}^{\alpha} f(t)\mathrm{d}t=\int_{0}^{\alpha} f(x)\mathrm{d}x$$

故$\int_{\alpha}^{\alpha+T} f(x)\mathrm{d}x=\int_{\alpha}^{0} f(x)\mathrm{d}x+\int_{0}^{T} f(x)\mathrm{d}x+\int_{0}^{\alpha} f(x)\mathrm{d}x$

$$=\int_{0}^{T} f(x)\mathrm{d}x$$

例1. 计算 $\int_{-\pi/4}^{\pi/4}\frac{\cos x}{1+e^{-x}}\,dx.$ “换元换限”

解： 原式 $=\int_{-\pi/4}^{0}\frac{\cos x}{1+e^{-x}}\,dx+\int_{0}^{\pi/4}\frac{\cos x}{1+e^{-x}}\,dx$

对 $\int_{-\pi/4}^{0}\frac{\cos x}{1+e^{-x}}\,dx$，令 $x=-t$，

$$\int_{-\pi/4}^{0}\frac{\cos x}{1+e^{-x}}\,dx=\int_{\pi/4}^{0}\frac{\cos(-t)}{1+e^{t}}\,d(-t)=\int_{0}^{\pi/4}\frac{\cos t}{1+e^{t}}\,dt$$

上式 $=\int_{0}^{\pi/4}\frac{\cos x}{1+e^{x}}\,dx+\int_{0}^{\pi/4}\frac{\cos x}{1+e^{-x}}\,dx$

$$=\int_{0}^{\pi/4}\frac{\cos x}{1+e^{x}}\,dx+\int_{0}^{\pi/4}\frac{e^{x}\cos x}{1+e^{x}}\,dx=\int_{0}^{\pi/4}\cos x\,dx=\frac{\sqrt{2}}{2}$$

例2. 证明：$\int_0^{\pi/2} f(\sin x)\mathrm{d}x = \int_0^{\pi/2} f(\cos x)\mathrm{d}x$;

解： 设$x = \dfrac{\pi}{2} - t$,

当$x = \dfrac{\pi}{2}$时， $t=0$； $x = 0$时， $t=\dfrac{\pi}{2}$.

$$\int_0^{\pi/2} f(\sin x)\mathrm{d}x$$

$$= \int_{\pi/2}^{0} f(\cos t)\mathrm{d}\left(\frac{\pi}{2} - t\right)$$

$$= \int_0^{\pi/2} f(\cos t)\mathrm{d}t$$

例3. 证明：$\int_0^{\pi} xf(\sin x)\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\mathrm{d}x = \pi\int_0^{\pi/2} f(\sin x)\mathrm{d}x.$

证明： 设$t = \pi - x$，

$$\int_0^{\pi} xf(\sin x)\mathrm{d}x = \int_{\pi}^{0}(\pi - t)\, f(\sin(\pi - t))\mathrm{d}(\pi - t)$$

$$= \pi\int_0^{\pi} f(\sin t)\mathrm{d}t - \int_0^{\pi} tf(\sin t)\mathrm{d}t$$

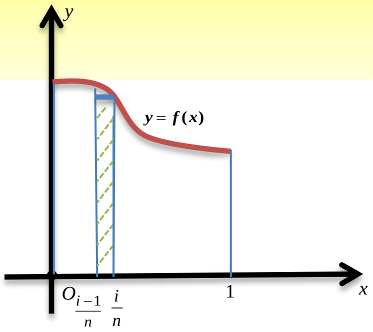
$$\Rightarrow \int_0^{\pi} tf(\sin t)\mathrm{d}t = \frac{\pi}{2}\int_0^{\pi} f(\sin t)\mathrm{d}t.$$

$$\int_0^{\pi} f(\sin x)\mathrm{d}x = \int_0^{\pi/2} f(\sin x)\mathrm{d}x + \int_{\pi/2}^{\pi} f(\sin x)\mathrm{d}x$$

$$= \int_0^{\pi/2} f(\sin x)\mathrm{d}x + \int_{\pi/2}^{0} f(\sin(\pi - t))\mathrm{d}(\pi - t)$$

$$= 2\int_0^{\pi/2} f(\sin x)\mathrm{d}x$$

定积分定义求极限



$$\int_a^b f(x)\mathrm{d}x = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

若$f(x)$在$[0,1]$可积，则

$$\int_0^1 f(x)\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n} = \lim_{n\to\infty}\frac{1}{n}\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\cdots+f\left(\frac{n}{n}\right)\right]$$

例1. 求$\lim\limits_{n\to\infty}\left(\dfrac{1}{n+1}+\dfrac{1}{n+2}+\cdots+\dfrac{1}{n+n}\right)$.

解：$\lim\limits_{n\to\infty}\dfrac{1}{n}\left(\dfrac{1}{1+\frac{1}{n}}+\dfrac{1}{1+\frac{2}{n}}+\cdots+\dfrac{1}{1+\frac{n}{n}}\right)=\displaystyle\int_0^1\frac{1}{1+x}\mathrm{d}x=\ln 2$

例2. 求 $\lim\limits_{n\to\infty}\frac{1}{n}\sqrt[n]{(n+1)(n+2)\cdots(n+n)}$.

解: $\lim\limits_{n\to\infty} e^{\ln\frac{1}{n}((n+1)(n+2)\cdots(n+n))^{1/n}} = \lim\limits_{n\to\infty} e^{\frac{1}{n}\ln\frac{(n+1)(n+2)\cdots(n+n)}{n^n}}$

$$= \lim_{n\to\infty} e^{\frac{1}{n}\left[\ln(1+\frac{1}{n})+\ln(1+\frac{2}{n})+\cdots+\ln(1+\frac{n}{n})\right]}$$

$$= e^{\lim\limits_{n\to\infty}\frac{1}{n}\left[\ln(1+\frac{1}{n})+\ln(1+\frac{2}{n})+\cdots+\ln(1+\frac{n}{n})\right]}$$

$$= e^{\int_0^1 \ln(1+x)\mathrm{d}x} = e^{\int_0^1 \ln(1+x)\mathrm{d}(x+1)}$$

$$= e^{\ln(1+x)(x+1)\big|_0^1 - \int_0^1 (x+1)\mathrm{d}\ln(1+x)} = e^{2\ln 2-1} = \frac{4}{e}$$