曲线的弧微分

1.定 义：

以$y=f(x)$为例，$f(x)\in C[a,b]$，取$A(a,f(a))$，

$M(x,f(x)),x\in C[a,b]$，

称曲线$y=f(x)$在A到M间长度S为$y=f(x)$的**曲线长度函数(弧长)**.

要求：$\lim\limits_{M\to M_0}\dfrac{\overset{\frown}{M_0M}}{|M_0M|}=1.$

$$S'(x)=\lim_{\Delta x\to 0}\frac{S(x+\Delta x)-S(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{S(x+\Delta x)-S(x)}{\sqrt{(\Delta x)^2+(\Delta y)^2}}\cdot\frac{\sqrt{(\Delta x)^2+(\Delta y)^2}}{\Delta x}$$

$$=\sqrt{1+y'^2}$$

曲线的弧微分

$$\mathrm{d}s = \sqrt{1+y'^2}\,\mathrm{d}x$$ ——弧长微分公式1

若曲线为$\begin{cases} x=\phi(t) \\ y=\psi(t) \end{cases}$，则$\mathrm{d}s = \sqrt{1+y'^2}\,\mathrm{d}x = \sqrt{1+\left(\dfrac{\psi'(t)}{\phi'(t)}\right)^2}\,\mathrm{d}\phi(t)$

$$\Rightarrow \mathrm{d}s = \sqrt{(\phi'(t))^2+(\psi'(t))^2}\,\mathrm{d}t$$ ——弧长微分公式2

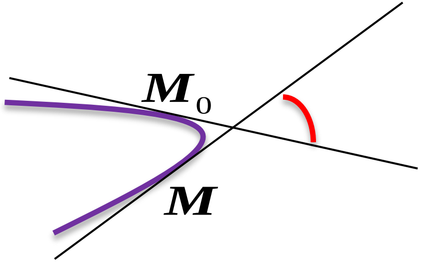
若曲线为$r=r(\theta)$，则$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$，

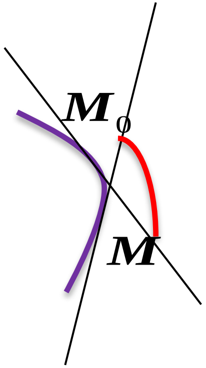
$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = r'\cos\theta - r\sin\theta,\quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = r'\sin\theta + r\cos\theta,$$

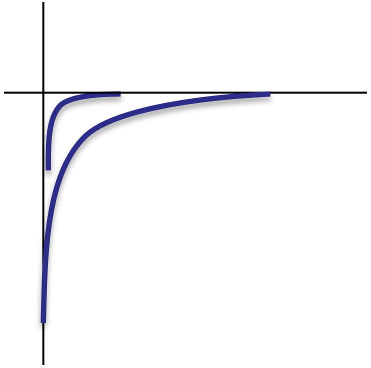
$$\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 = r'(\theta)^2 + r(\theta)^2$$

$$\Rightarrow \mathrm{d}s = \sqrt{(r'(\theta))^2+(r(\theta))^2}\,\mathrm{d}\theta$$ ——弧长微分公式3

曲线的曲率







1.定 义:

设M_0M为一条连续曲线段，$\Delta\alpha$是M_0点切线变到M点处切线转角，ΔS为M_0M的长度，若极限

$\lim\limits_{M\to M_0}\left|\dfrac{\Delta\alpha}{\Delta S}\right|$存在，则称此极限值为曲线$M_0M$在$M_0$点处的**曲率**，记为$K\big|_{M_0}=\lim\limits_{M\to M_0}\left|\dfrac{\Delta\alpha}{\Delta S}\right|.$

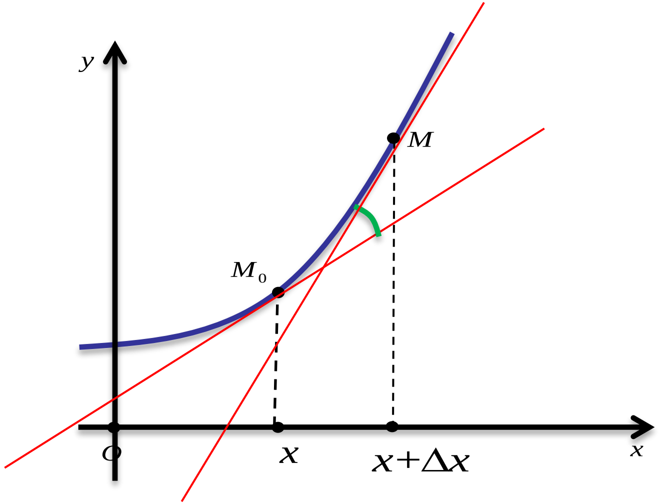
曲线的曲率

$$K\big|_{M_0}=\lim_{M\to M_0}\left|\frac{\Delta\alpha}{\Delta S}\right|.$$

$$=\lim_{M\to M_0}\left|\frac{\Delta\alpha/\Delta x}{\Delta S/\Delta x}\right|$$

注意到$\tan\alpha=y'$

$$\Rightarrow\alpha=\arctan y'$$



故$K\big|_{M_0}=\left|\dfrac{(\arctan y')'}{\sqrt{1+y'^2}}\right|=\left|\dfrac{y''}{(1+y'^2)^{\frac{3}{2}}}\right|$

曲率公式：$K=\left|\dfrac{y''}{(1+y'^2)^{\frac{3}{2}}}\right|$

例. 求半径为R的圆的曲率.

解：$K=\left|\dfrac{y''}{\left(1+y'^2\right)^{\frac{3}{2}}}\right|$，设圆的方程为$x^2+y^2=R^2$.

两端关于x求导，$2x+2yy'=0$，$\Rightarrow y'=-\dfrac{x}{y}$

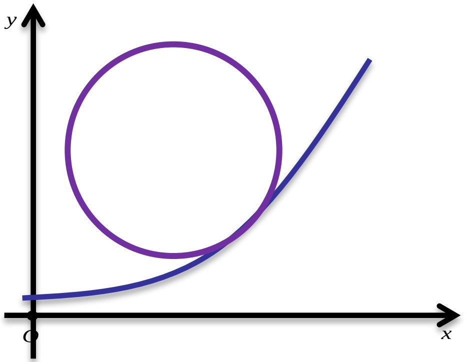
$$y''=-\frac{y-xy'}{y^2}=-\frac{y+\dfrac{x^2}{y}}{y^2}=-\frac{R^2}{y^3}$$

$$1+y'^2=\frac{x^2+y^2}{y^2}=\frac{R^2}{y^2},\quad \left(1+y'^2\right)^{\frac{3}{2}}=\frac{R^3}{y^3}$$

$$\Rightarrow K=\frac{1}{R}$$

定义：

设Γ为一曲线，Ω为一个圆，
若Γ在M点处满足：



1）曲线Γ与Ω在M点处相切；

2）曲线Γ与Ω在M点处凹向相同；

3）Γ在M点处曲率等于Ω半径的倒数；

则称Ω为曲线Γ在M点处的**曲率圆**.

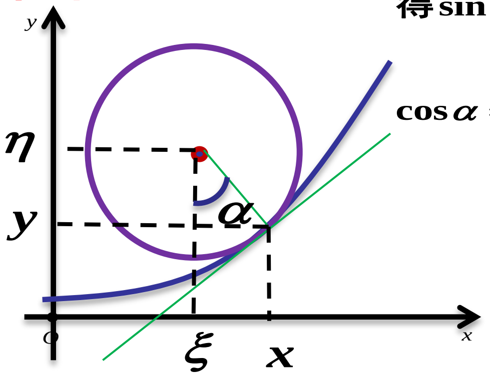
此圆的半径为Γ在M点处的**曲率半径**. $R=\left|\dfrac{\left(1+y'^2\right)^{\frac{3}{2}}}{y''}\right|$

曲率圆

此圆的圆心为Γ在M点处的曲率中心.

记为(ξ,η).

下以凹的曲线为例加以推导.



由$y'=\tan\alpha$,

得$\sin\alpha=\dfrac{y'}{\sqrt{1+y'^2}}$

$\cos\alpha=\dfrac{1}{\sqrt{1+y'^2}}$

$$x-\xi=R\sin\alpha$$

$$=\frac{(1+y'^2)^{\frac{3}{2}}}{y''}\cdot\frac{y'}{\sqrt{1+y'^2}}$$

$$=\frac{1+y'^2}{y''}\cdot y'$$

$$\eta-y=R\cos\alpha=\frac{(1+y'^2)^{\frac{3}{2}}}{y''}\cdot\frac{1}{\sqrt{1+y'^2}}=\frac{1+y'^2}{y''}$$

$$\begin{cases}\xi=x-\dfrac{1+y'^2}{y''}\cdot y'\\[2ex]\eta=y+\dfrac{1+y'^2}{y''}\end{cases}$$

——曲率中心公式

定义. 若$M(x,y)$为曲线Γ的动点，M沿曲线移动时得到其曲率中心运动轨迹，称此运动轨迹为Γ的渐屈线.