# 求面积

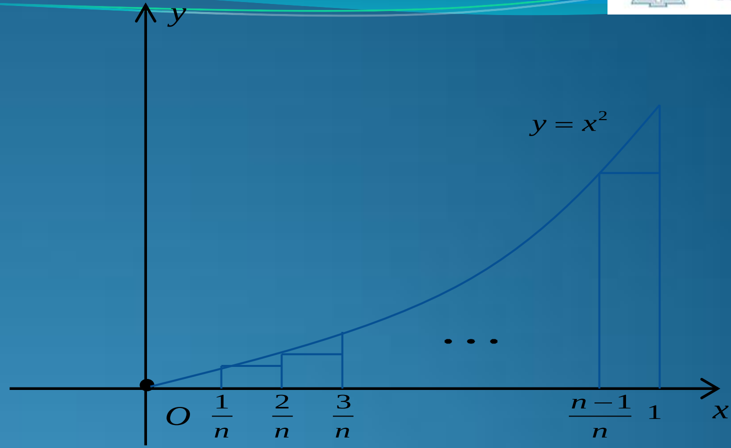
$$S_n \approx \frac{1}{n}\left[\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\cdots+\left(\frac{n-1}{n}\right)^2\right]$$

$$=\frac{1^2+2^2+\cdots+(n-1)^2}{6n^3}$$

$$=\frac{(n-1)n(2n-1)}{6n^3}$$

$$=\frac{2n^2-3n+1}{6n^2}$$

$$=\frac{1}{3}-\frac{1}{2n}+\frac{1}{6n^2}\to\frac{1}{3}$$



# 数列极限的概念

设$\{x_n\}$为一数列，$A$为一实数，对$\forall\varepsilon>0,\exists N,$使得当$n>N$时，有$|x_n-A|<\varepsilon$或$A-\varepsilon<x_n<A+\varepsilon$. 则称数列$\{x_n\}$在$n$趋向于无穷大时有极限或收敛，$A$为其极限值（或说$x_n$收敛于$A$），记为$\lim\limits_{n\to\infty}x_n=A$或$x_n\to A(n\to\infty)$.


1) $\varepsilon$ : 度量数列与极限值的距离.

2) $N$ : 与$\varepsilon$相关，标示这变化的进程.

# 利用极限定义的证明题

$\lim\limits_{n\to\infty} x_n = A \Leftrightarrow \forall\varepsilon > 0, \exists N, s.t.$ 当$n > N$时，有$|x_n - A| < \varepsilon.$

注：只能判断是否是极限，不能用来求极限

例1. 证明$\lim\limits_{n\to\infty}\dfrac{1}{n} = 0.$

分析：找$N$, 当$n > N$时, $\left|\dfrac{1}{n} - 0\right| < \varepsilon \Rightarrow \dfrac{1}{n} < \varepsilon \Rightarrow n > \dfrac{1}{\varepsilon}.$

证明：$\forall\varepsilon>0$, 取$N = \left[\dfrac{1}{\varepsilon}\right] + 1$, 当$n > N$时，

$$n > \left[\frac{1}{\varepsilon}\right] + 1 > \frac{1}{\varepsilon}$$

即$\left|\dfrac{1}{n} - 0\right| < \varepsilon.$

例2.　证明$\lim\limits_{n\to\infty}\sqrt[n]{a}=1.(a>1)$

分析：找$N$,当$n>N$时,$\left|\sqrt[n]{a}-1\right|<\varepsilon \Rightarrow \sqrt[n]{a}<\varepsilon+1 \Rightarrow \ln a^{\frac{1}{n}}<\ln(\varepsilon+1)$

$$\Rightarrow \frac{1}{n}\ln a<\ln(\varepsilon+1) \quad \Rightarrow n>\frac{\ln a}{\ln(\varepsilon+1)}.$$

证明：$\forall\varepsilon>0$, 取$N=\left[\dfrac{\ln a}{\ln(\varepsilon+1)}\right]+1$,当$n>N$时，

$$n>\left[\frac{\ln a}{\ln(\varepsilon+1)}\right]+1>\frac{\ln a}{\ln(\varepsilon+1)} \quad \Rightarrow \frac{1}{n}\ln a<\ln(\varepsilon+1)$$

$$\Rightarrow \ln a^{\frac{1}{n}}<\ln(\varepsilon+1) \quad \Rightarrow \sqrt[n]{a}<\varepsilon+1$$

即有$\left|\sqrt[n]{a}-1\right|<\varepsilon.$

例3. 证明$\lim\limits_{n\to\infty}\sqrt[n]{n}=1.$

分析：找$N$,当$n>N$时,$\left|\sqrt[n]{n}-1\right|<\varepsilon \Rightarrow \sqrt[n]{n}<\varepsilon+1$

$$\Rightarrow \sqrt[n]{n}=\sqrt[n]{n\cdot 1\cdots 1}\le \frac{n+(n-1)}{n}=2-\frac{1}{n} \quad <\frac{1}{n}+2 \quad <\varepsilon+1$$

$$\Rightarrow \frac{1}{n}<\varepsilon-1$$

$$\Rightarrow \sqrt[n]{n}=\sqrt[n]{\sqrt{n}\cdot\sqrt{n}\cdot 1\cdots 1}\le \frac{2\sqrt{n}+n-2}{n}=\frac{2}{\sqrt{n}}+1-\frac{2}{n}<\frac{2}{\sqrt{n}}+1<\varepsilon+1$$

$$\Rightarrow n>\frac{4}{\varepsilon^2}$$

证明：$\forall\varepsilon>0$, 取$N=\left[\frac{4}{\varepsilon^2}\right]+1$,当$n>N$时，

$$n>\left[\frac{4}{\varepsilon^2}\right]+1>\frac{4}{\varepsilon^2} \quad \Rightarrow \frac{2}{\sqrt{n}}+1<\varepsilon+1 \Rightarrow \frac{2\sqrt{n}+n-2}{n}=\frac{2}{\sqrt{n}}+1-\frac{2}{n}<\varepsilon+1$$

$$\Rightarrow \sqrt[n]{n}=\sqrt[n]{\sqrt{n}\cdot\sqrt{n}\cdot 1\cdots 1} \le \frac{2\sqrt{n}+n-2}{n}<\varepsilon+1$$

即有$\left|\sqrt[n]{n}-1\right|<\varepsilon.$