函数的收敛准则

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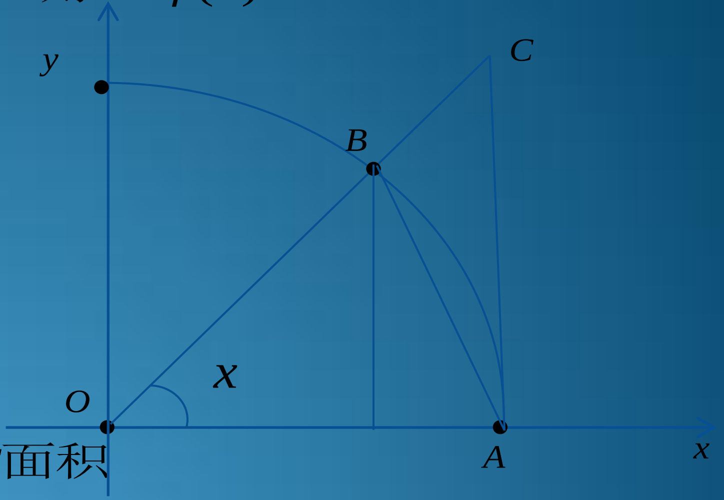
定理. 若$g(x) \le f(x) \le h(x)$, 且$\lim g(x) = \lim h(x) = A$, 则$\lim f(x) = A$.

两个重要极限

1. $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$

易见$\dfrac{\sin x}{x}$是偶函数，只需考虑$x>0$时的情形.



如图，有ΔOAB面积 $\le$ 扇形OAB面积$\le$ ΔOAC面积

$$\Rightarrow \frac{1}{2}\sin x \le \frac{1}{2}x \le \frac{1}{2}\tan x$$

$$\Rightarrow \sin x \le x \le \frac{\sin x}{\cos x} \Rightarrow 1 \le \frac{x}{\sin x} \le \frac{1}{\cos x}$$

$$\Rightarrow \cos x \le \frac{\sin x}{x} \le 1$$

故$\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$

重要极限 $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$，推广至 $\lim\dfrac{\sin\square}{\square}=1(\square\to 0)$.

例1. 求 $\lim\limits_{x\to 0}\dfrac{\arcsin x}{x}$.

设 $u=\arcsin x$， 原式 $=\lim\limits_{u\to 0}\dfrac{u}{\sin u}=1.$

例2. 求 $\lim\limits_{x\to 0}\dfrac{\tan x}{x}$.

原式 $=\lim\limits_{x\to 0}\dfrac{\sin x}{x}\cdot\dfrac{1}{\cos x}=1.$

例3. 求 $\lim\limits_{x\to 0}\dfrac{\sin 3x}{\sin 5x}$.

原式 $=\lim\limits_{x\to 0}\dfrac{\dfrac{\sin x}{3x}}{\dfrac{\sin 5x}{5x}}\cdot\dfrac{3}{5}=\dfrac{3}{5}.$

重要极限

2. $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x$ 或 $\lim\limits_{t\to 0}(1+t)^{\frac{1}{t}}$

1）先证 $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$ 存在. 记 $x_n=\left(1+\dfrac{1}{n}\right)^n$

$$x_n=\left(1+\frac{1}{n}\right)^n=1+1+\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2+\cdots+\frac{n(n-1)\cdots(n-n+1)}{n!}\left(\frac{1}{n}\right)^n$$

$$=2+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

$$x_{n+1}=2+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right)$$

$$+\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right)$$

易见 $x_{n+1}>x_n$, 故 $\{x_n\}$ 单调上升.

又$2 \le x_n = \left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$

$$\le 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}\cdots+\frac{1}{n!}$$

$$=1+1+\frac{1}{2}+\frac{1}{2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}\cdots+\frac{1}{2\cdot 3\cdots n}$$

$$\le 1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}\cdots+\frac{1}{2^{n-1}}$$

$$=1+\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} \le 1+\frac{1}{1-\frac{1}{2}} = 3$$

故$\{x_n\}$有界，$\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n$存在. 记$\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \mathrm{e}$.

2) 再证 $\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{x}\right)^x=e$

当$x>1$时，存在n使得$n\le x\le n+1$

$$\Rightarrow \frac{1}{n}\ge\frac{1}{x}\ge\frac{1}{n+1}\Rightarrow 1+\frac{1}{n}\ge 1+\frac{1}{x}\ge 1+\frac{1}{n+1}$$

$$\Rightarrow \left(1+\frac{1}{x}\right)^x\ge\left(1+\frac{1}{n+1}\right)^x\ge\left(1+\frac{1}{n+1}\right)^n$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^n=\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1}\Big/\left(1+\frac{1}{n+1}\right)=e$$

又$\left(1+\dfrac{1}{x}\right)^x\le\left(1+\dfrac{1}{n}\right)^x\le\left(1+\dfrac{1}{n}\right)^{n+1}$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)=e.\ 故\ \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x=e$$

3) 最后证 $\lim\limits_{x\to-\infty}\left(1+\dfrac{1}{x}\right)^{x}=e$

令 $x=-u$，当 $x\to-\infty$ 时 $u\to+\infty$

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x}=\lim_{u\to+\infty}\left(1-\frac{1}{u}\right)^{-u}=\lim_{u\to+\infty}\left(\frac{u}{u-1}\right)^{u}$$

$$=\lim_{u\to+\infty}\left(1+\frac{1}{u-1}\right)^{u}$$

$$=\lim_{u\to+\infty}\left(1+\frac{1}{u-1}\right)^{u-1}\left(1+\frac{1}{u-1}\right)$$

$$=e$$

综上，$\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x}=e$

重要极限$\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e$，推广至$\lim(1+\square)^{\frac{1}{\square}}=e(\square\to 0)$.

例1. 求$\lim\limits_{x\to 0}\dfrac{\log_a(1+x)}{x}(a>0)$.

$$原式=\lim_{x\to 0}\log_a(1+x)^{\frac{1}{x}}=\log_a e.$$

特别地，$\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}=1$

例2. 求$\lim\limits_{x\to 0}\dfrac{a^x-1}{x}$.

令$a^x-1=t \Rightarrow a^x=t+1 \Rightarrow x=\log_a(t+1)$. 当$x\to 0$时，$t\to 0$

$$原式=\lim_{t\to 0}\frac{t}{\log_a(t+1)}=\frac{1}{\log_a e}=\ln a$$

特别地，$\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1$

重要极限$\lim(1+\square)^{\frac{1}{\square}}=e(\square\to 0)$

例1. 求$\lim\limits_{x\to\infty}\left(\dfrac{x+1}{x+2}\right)^{x}$

$$原式=\lim_{x\to\infty}\left(1+\frac{-1}{x+2}\right)^{x}=\lim_{x\to\infty}\left[\left(1+\frac{-1}{x+2}\right)^{\frac{x+2}{-1}}\right]^{\frac{-1}{x+2}\cdot x}=e^{-1}$$

例2. 求$\lim\limits_{x\to 0}(\cos x+\sin^2 x)^{\frac{1}{x^2}}$.

$$原式=\lim_{x\to 0}(1+\cos x+\sin^2 x-1)^{x^2}$$

$$=\lim_{x\to 0}\left[(1+\cos x+\sin^2 x-1)^{\frac{1}{\cos x+\sin^2 x-1}}\right]^{\frac{\cos x+\sin^2 x-1}{x^2}}$$

$$又\lim_{x\to 0}\frac{\cos x+\sin^2 x-1}{x^2}=\lim_{x\to 0}\left(\frac{\sin^2 x}{x^2}-\frac{2\sin^2\frac{x}{2}}{x^2}\right)=1-\lim_{x\to 0}2\cdot\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\cdot\frac{1}{4}=\frac{1}{2}$$

$$故原式=e^{\frac{1}{2}}$$