数列极限的收敛准则

夹挤定理. 设从某项以后有$y_n \le x_n \le z_n$, 且$\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} z_n = A$, 则$\lim\limits_{n\to\infty} x_n = A$.

(分析). $A-\varepsilon < y_n < A+\varepsilon$, 当$n > N_1$

$A-\varepsilon < z_n < A+\varepsilon$, 当$n > N_2$

$A-\varepsilon < y_n \le x_n \le z_n < A+\varepsilon$ 当$n > \max\{N_1, N_2\}$

数列极限的计算

例1．计算$\lim\limits_{n\to\infty}\left(\dfrac{1}{n^2+n+1}+\dfrac{2}{n^2+n+2}+\cdots+\dfrac{n}{n^2+n+n}\right).$

$$\cdots<\frac{1}{n^2+n+1}+\frac{2}{n^2+n+1}+\cdots+\frac{n}{n^2+n+1}=\frac{n(n+1)}{2\left(n^2+n+1\right)}\to\frac{1}{2}$$

$$\cdots>\frac{1}{n^2+n+n}+\frac{2}{n^2+n+n}+\cdots+\frac{n}{n^2+n+n}=\frac{n(n+1)}{2\left(n^2+n+n\right)}\to\frac{1}{2}$$

由夹挤定理，$\lim\limits_{n\to\infty}\left(\dfrac{1}{n^2+n+1}+\dfrac{2}{n^2+n+2}+\cdots+\dfrac{n}{n^2+n+n}\right)=\dfrac{1}{2}$

例2. 计算$\lim\limits_{n\to\infty}\left(\dfrac{1}{a_1^n}+\dfrac{1}{a_2^n}+\cdots+\dfrac{1}{a_k^n}\right)^{\frac{1}{n}}$，其中$0<a_1<a_2<\cdots<a_k$.

$$\frac{1}{a_1}=\left(\frac{1}{a_1^n}\right)^{\frac{1}{n}}<\cdots<\left(\frac{1}{a_1^n}+\frac{1}{a_1^n}+\cdots+\frac{1}{a_1^n}\right)^{\frac{1}{n}}=\left(\frac{k}{a_1^n}\right)^{\frac{1}{n}}=\frac{\sqrt[n]{k}}{a_1}\to\frac{1}{a_1}$$

由夹挤定理，$\lim\limits_{n\to\infty}\left(\dfrac{1}{a_1^n}+\dfrac{1}{a_2^n}+\cdots+\dfrac{1}{a_k^n}\right)^{\frac{1}{n}}=\dfrac{1}{a_1}$

数列极限的单调有界原理

单调有界原理. 单调有界数列必收敛.

（分析）. 以数列$\{x_n\}$单调上升且有界为例.

由确界公理有$A=\sup\{x_n\}$.

即对$\forall\varepsilon>0,\exists N,$使$x_N>A-\varepsilon$.

当$n>N$时，$A-\varepsilon<x_N\le x_n<A+\varepsilon$

由数列极限的定义，$\lim\limits_{n\to\infty}x_n=A$.

数列极限的单调有界原理

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例1. 设$x_1>0,\ x_{n+1}=\dfrac{1}{2}\left(x_n+\dfrac{4}{x_n}\right)$，求$\lim\limits_{n\to\infty}x_n$.

分析. $\dfrac{x_{n+1}}{x_n}=\dfrac{1}{2}\left(1+\dfrac{4}{x_n^2}\right)\geq 1(\text{或}\leq 1)$. $\Rightarrow 1+\dfrac{4}{x_n^2}\geq 2 \quad \Rightarrow \dfrac{4}{x_n^2}\geq 1 \quad \Rightarrow x_n^2\leq 4$

证明. 易见$x_n>0$, $\quad x_{n+1}=\dfrac{1}{2}\left(x_n+\dfrac{4}{x_n}\right)\geq 2\sqrt{\dfrac{1}{2}x_n\,\dfrac{1}{2}\,\dfrac{4}{x_n}}=2$

$$\Rightarrow x_{n+1}^2\geq 4 \quad \Rightarrow 1+\frac{4}{x_{n+1}^2}\leq 2 \Rightarrow \frac{x_{n+2}}{x_{n+1}}=\frac{1}{2}\left(1+\frac{4}{x_{n+1}^2}\right)\leq 1$$

即数列$\{x_n\}$从第二项开始单调递减. 又$0<x_n\leq x_1$，故数列有界.

由单调有界原理，极限存在，记$\lim\limits_{n\to\infty}x_n=A$.

对$x_{n+1}=\dfrac{1}{2}\left(x_n+\dfrac{4}{x_n}\right)$两端取极限，得$A=\dfrac{1}{2}\left(A+\dfrac{4}{A}\right)$

得$A=\pm 2$，由极限保序性，$\lim\limits_{n\to\infty}x_n=2$.

数列极限的单调有界原理

例2. 设$x_1 = 7, x_{n+1} = \sqrt{2 + x_n}$，求$\lim\limits_{n\to\infty} x_n$.

证明. 易见$x_n > 0$, $x_{n+1} - x_n = \sqrt{2 + x_n} - \sqrt{2 + x_{n-1}} = \dfrac{x_n - x_{n-1}}{\sqrt{2 + x_n} + \sqrt{2 + x_{n-1}}}$

$x_{n+1} - x_n$与$x_n - x_{n-1}$同号 $\Rightarrow x_{n+1} - x_n$与$x_2 - x_1$同号

而$x_2 - x_1 = 3 - 7 = -4 < 0 \Rightarrow x_{n+1} < x_n$，即数列$\{x_n\}$单调递减.

又$0 < x_n \le x_1 = 7$，故数列有界.

由单调有界原理，极限存在，记$\lim\limits_{n\to\infty} x_n = A$.

对$x_{n+1} = \sqrt{2 + x_n}$两端取极限，得$A = \sqrt{2 + A}$

得$A = 2$(或-1，舍)，由极限保序性，$\lim\limits_{n\to\infty} x_n = 2$.