反函数的求导法则

定理. 设$x=g(y)$在某区间内单调连续，在该区间内点y处可导，且$g'(y)\neq 0$, 则其反函数$y=f(x)$在y的对应点x处亦可导，且

$$f'(x)=\frac{1}{g'(y)}.$$

分析. $f'(x)=\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim\limits_{\Delta y\to 0}\frac{1}{\frac{\Delta x}{\Delta y}}=\frac{1}{g'(y)}$

例. $(\arcsin x)'=\frac{1}{(\sin y)'}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-\sin^2 y}}=\frac{1}{\sqrt{1-x^2}}$

$y\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

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复合函数的求导法则

定理. 设$y = f(u)$可导，$u = g(x)$可导，且$f[g(x)]$在x邻域内有定义，则

$$\frac{dy}{dx} = (f[g(x)])' = f'(g(x))g'(x).$$

分析. $\frac{dy}{dx} = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(g(x)) \cdot g'(x)$

例. 设$y = a^{\sin \ln \arctan x^2}$, 求$y'$.

$$y' = \ln a \cdot a^{\sin \ln \arctan x^2} \cdot (\cos \ln \arctan x^2) \cdot \frac{1}{\arctan x^2} \cdot \frac{1}{1 + x^4} \cdot 2x$$

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导数计算的辅助公式

例1. 设$y=x^{x}$,求y'.

辅助公式1. $\left(f(x)^{g(x)}\right)'=\left(e^{g(x)\ln f(x)}\right)'=f^{g}\cdot(g\ln f)'$.

解. $\left(x^{x}\right)'=\ln x\cdot(x\ln x)'=x^{x}(\ln x+1)$

例2. 设$y=\sqrt[5]{\dfrac{(x+1)(x^{2}+1)2x}{(x^{5}+1)(x^{2}+x+1)}}$,求$y'$.

解. $y'=\sqrt[5]{\dfrac{(x+1)(x^{2}+1)2x}{(x^{5}+1)(x^{2}+x+1)}}\cdot\dfrac{1}{5}\left(\ln\dfrac{(x+1)(x^{2}+1)2x}{(x^{5}+1)(x^{2}+x+1)}\right)'$

$$=\frac{1}{5}\left(\frac{1}{x+1}+\frac{2x}{x^{2}+1}+\frac{1}{x}-\frac{5x^{4}}{x^{5}+1}-\frac{2x+1}{x^{2}+x+1}\right)\cdot\sqrt[5]{\frac{(x+1)(x^{2}+1)2x}{(x^{5}+1)(x^{2}+x+1)}}$$

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辅助公式2. $(C\cdot f(x))' = C\cdot f'(x).$

辅助公式3. $\left(\dfrac{1}{g(x)}\right)' = -\dfrac{g'(x)}{g^2(x)}.$

辅助公式4. $(f(x)g(x)h(x))' = f'\cdot g\cdot h + f\cdot g'\cdot h + f\cdot g\cdot h'.$

参数方程式函数求导法则

定理. 设$x=\varphi(t), y=\psi(t)$都在t点可导，且$\varphi'(t)\neq 0$, $x=\varphi(t)$在t的某邻域内是单调的连续函数，则参数方程$\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases}$确定的函数在点$x$处亦可导，且

$$\frac{dy}{dx}=\frac{\psi'(t)}{\varphi'(t)}.$$

分析. $\frac{dy}{dx}=\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim\limits_{\Delta t\to 0}\frac{\Delta y/\Delta t}{\Delta x/\Delta t}=\frac{\psi'(t)}{\varphi'(t)}.$

例1. 设 $\begin{cases} x = \arctan t^2 \\ y = t^2 + 2t \end{cases}$，求 $\dfrac{dy}{dx}$.

解. $$\frac{dy}{dx} = \frac{\left(t^2 + 2t\right)'}{\left(\arctan t^2\right)'} = \frac{2t+2}{\dfrac{2t}{1+t^4}} = \frac{(t+1)\left(1+t^4\right)}{t}$$

隐函数求导法则

方法. 方程两端关于x求导.

例2. 求隐函数$xe^{\sin y}=e^y$的导数y'.

解. 方程两端关于x求导, $\left(xe^{\sin y}\right)_x'=\left(e^y\right)_x'$

得$e^{\sin y}+xe^{\sin y}\cdot\cos y\cdot y'=e^y\cdot y'$

故$y'=\dfrac{e^{\sin y}}{e^y-xe^{\sin y}\cdot\cos y}$