可分离变量方程

定义. 称$y' = f(y)g(x)$的方程为可分离变量的方程.

解法：

分离变量后再积分，

$$y' = \frac{dy}{dx} = f(y)g(x)$$

$$\Rightarrow \frac{dy}{f(y)} = g(x)\mathrm{d}x$$

$$\Rightarrow \int \frac{\mathrm{d}y}{f(y)} = \int g(x)\mathrm{d}x$$为其通解.

例1. 求解 $(yy')^2 + y^2 = 1.$

解： $y' = \pm\dfrac{\sqrt{1-y^2}}{y} \quad (y \neq 0)$

$$\Rightarrow \frac{y\mathrm{d}y}{\sqrt{1-y^2}} = \pm \mathrm{d}x \quad (y^2 \neq 1)$$

$$\Rightarrow \int \frac{y\mathrm{d}y}{\sqrt{1-y^2}} = \pm\int \mathrm{d}x$$

$$\Rightarrow -\sqrt{1-y^2} = \pm x + C$$

$\Rightarrow (x+C)^2 + y^2 = 1$ 为方程的通解.

验证 $y=0$ 不是方程的解， 而 $y=\pm 1$ 是方程的奇解.

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例2. 求$xy' = \sqrt{1+y^2}$满足$y\big|_{x=1} = 0$的特解.

解: $\dfrac{\mathrm{d}y}{\sqrt{1+y^2}} = \dfrac{\mathrm{d}x}{x} \Rightarrow \displaystyle\int \frac{\mathrm{d}y}{\sqrt{1+y^2}} = \int \frac{\mathrm{d}x}{x}$

$$\Rightarrow \ln\left|y+\sqrt{1+y^2}\right| = \ln|x| + C$$

由$y\big|_{x=1} = 0 \Rightarrow C = 0$

$$\left.\begin{array}{l} \Rightarrow y+\sqrt{1+y^2} = \pm x \\ \sqrt{1+y^2} - y = \pm\dfrac{1}{x} \end{array}\right\}$$

两式相减，得$y = \pm\dfrac{1}{2}\left(x - \dfrac{1}{x}\right)$

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齐次方程

定义. 称$y' = f\left(\dfrac{y}{x}\right)$为齐次方程.

解法： 设$\dfrac{y}{x} = u$, $\Rightarrow y = ux$

$$\Rightarrow y' = (ux)' \Rightarrow f(u) = u + xu'$$

例1. 求解$xy' = y - \sqrt{x^2 + y^2}\ (x > 0)$.

解： $y' = \dfrac{y}{x} - \sqrt{1 + \dfrac{y^2}{x^2}}$, 设$\dfrac{y}{x} = u$, $y = ux$

$\Rightarrow y' = xu' + u$, 代入得$xu' + u = u - \sqrt{1 + u^2}$

$$\Rightarrow \frac{\mathrm{d}u}{\sqrt{1+u^2}} = -\frac{\mathrm{d}x}{x} \quad \Rightarrow \ln\left|u + \sqrt{1+u^2}\right| = -\ln|x| + C$$

$\Rightarrow \ln\left|\dfrac{y}{x} + \sqrt{1 + \dfrac{y^2}{x^2}}\right| = -\ln|x| + C$为方程通解.

例2. 求解$y' = \dfrac{1-x+y}{x-y}$.

解: 令$x-y=u$, $\Rightarrow y' = 1-u'$

代入得,$1-u' = \dfrac{1-u}{u}$

$$\Rightarrow u' = 2-\frac{1}{u} \quad \Rightarrow \frac{u\mathrm{d}u}{2u-1} = \mathrm{d}x\,(2u-1\neq 0)$$

$$\Rightarrow \int \frac{u\mathrm{d}u}{2u-1} = \int \mathrm{d}x \Rightarrow \frac{1}{2}\int \frac{2u-1+1}{2u-1}\mathrm{d}u = \int \mathrm{d}x$$

$$\Rightarrow u+\frac{1}{2}\ln|2u-1| = 2x+C$$

$\Rightarrow (x-y)+\dfrac{1}{2}\ln|2(x-y)-1| = 2x+C$为方程的通解.

奇解为$y = x-\dfrac{1}{2}$.

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一阶线性微分方程（I）

定义. 称$y' = p(x)y + q(x)$为一阶线性方程.

当$q(x) = 0$时，方程$y' = p(x)y$为一阶线性齐次方程.

当$q(x) \neq 0$时，方程为一阶线性非齐次方程.

方程$y' = p(x)y$.

$$\Rightarrow \frac{\mathrm{d}y}{y} = p(x)\mathrm{d}x\,(y \neq 0) \Rightarrow \int \frac{\mathrm{d}y}{y} = \int p(x)\mathrm{d}x$$

$$\Rightarrow \ln|y| = \int p(x)\mathrm{d}x + C \Rightarrow |y| = e^{C_1} e^{\int p(x)\mathrm{d}x}$$

$$\Rightarrow |y| = \pm e^{C_1} e^{\int p(x)\mathrm{d}x}$$ 又$y = 0$是方程的解,

故$y = Ce^{\int p(x)\mathrm{d}x}$为通解公式.

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一阶线性微分方程（I）

方程$y' = p(x)y$.

$$\Rightarrow y = Ce^{\int p(x)\mathrm{d}x}$$

注意到，若$p(x) = g'(x) / g(x)$

$$\Rightarrow \ln|y| = \int \frac{g'(x)}{g(x)}\mathrm{d}x + C$$

$$\Rightarrow |y| = e^{C_1}|g(x)|$$

$$\Rightarrow y = Cg(x)$$

若套公式，$y = Ce^{\int \frac{g'(x)}{g(x)}\mathrm{d}x} = C|g(x)|$

注：通解中，若出现$\ln|g(x)|$，可以去掉绝对值

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一阶线性微分方程（II）

方程$y' = p(x)y + q(x)$

(常数变易法)

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = p(x)y + q(x) \Rightarrow \frac{\mathrm{d}y}{y} = \left[p(x) + \frac{q(x)}{y}\right]\mathrm{d}x$$

$$\Rightarrow \int \frac{\mathrm{d}y}{y} = \int \left[p(x) + \frac{q(x)}{y}\right]\mathrm{d}x$$

$$\Rightarrow \ln|y| = \int p(x)\mathrm{d}x + \int \frac{q(x)}{y}\mathrm{d}x + C$$

$$\Rightarrow y = \left(C_1 e^{\int \frac{q(x)}{y}\mathrm{d}x}\right) e^{\int p(x)\mathrm{d}x}$$

$\Rightarrow$ 此方程的解为$y = C(x)e^{\int p(x)\mathrm{d}x}$

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一阶线性微分方程（II）

方程$y' = p(x)y + q(x)$

将$y = C(x)e^{\int p(x)\mathrm{d}x}$代入，

$$y' = C'(x)e^{\int p(x)\mathrm{d}x} + C(x)p(x)e^{\int p(x)\mathrm{d}x},$$

$$\Rightarrow C'(x)e^{\int p(x)\mathrm{d}x} + C(x)p(x)e^{\int p(x)\mathrm{d}x}$$

$$= p(x)C(x)e^{\int p(x)\mathrm{d}x} + q(x)$$

$$\Rightarrow C'(x) = q(x)e^{-\int p(x)\mathrm{d}x}$$

$$\Rightarrow C(x) = \int q(x)e^{-\int p(x)\mathrm{d}x}\mathrm{d}x + C$$

$$\Rightarrow y = e^{\int p(x)\mathrm{d}x}\left[C + \int q(x)e^{-\int p(x)\mathrm{d}x}\mathrm{d}x\right]$$

一阶线性非齐次方程的通解公式

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例1. 求方程$xy' = y + x^2 e^x$的通解.

解： $y' = \frac{1}{x} y + x e^x$， $p(x) = \frac{1}{x}, q(x) = x e^x$

$$\Rightarrow y = e^{\int p(x)\mathrm{d}x}\left[C + \int q(x) e^{-\int p(x)\mathrm{d}x} \mathrm{d}x\right]$$

$$= e^{\int \frac{1}{x}\mathrm{d}x}\left[C + \int x e^x e^{-\int \frac{1}{x}\mathrm{d}x} \mathrm{d}x\right]$$

$$= x\left[C + \int x e^x \frac{1}{x}\mathrm{d}x\right]$$

故通解为$y = x\left(C + e^x\right)$.

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一阶线性微分方程（II）

例2. 求方程$y' = \dfrac{y}{x + y^2 e^y}$的通解.

解：$\dfrac{\mathrm{d}x}{\mathrm{d}y} = \dfrac{1}{y}x + ye^y,$

$$\Rightarrow x = e^{\int p(y)\mathrm{d}y}\left[C + \int q(y)e^{-\int p(y)\mathrm{d}y}\mathrm{d}y\right]$$

$$= e^{\int \frac{1}{y}\mathrm{d}y}\left[C + \int ye^y e^{-\int \frac{1}{y}\mathrm{d}y}\mathrm{d}y\right]$$

$$= y\left[C + \int ye^y \frac{1}{y}\mathrm{d}y\right]$$

故通解为$x = y\left(C + e^y\right).$

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伯努利方程

定义. 称$y' = p(x)y + y^{\lambda}q(x)\left(\lambda \neq 0,1\right)$为伯努利方程.

$$\Rightarrow y^{-\lambda}y' = p(x)y^{1-\lambda} + q(x)$$

$$\Rightarrow \frac{1}{1-\lambda}\left(y^{1-\lambda}\right)' = p(x)y^{1-\lambda} + q(x)$$

$$\Rightarrow \left(y^{1-\lambda}\right)' = (1-\lambda)\,p(x)y^{1-\lambda} + (1-\lambda)q(x)$$

$$\Rightarrow y^{1-\lambda} = e^{(1-\lambda)\int p(x)\mathrm{d}x}\left[C + (1-\lambda)\int q(x)e^{-(1-\lambda)\int p(x)\mathrm{d}x}\mathrm{d}x\right]$$

伯努利方程

例. 求方程$y' = \dfrac{y}{x} + x^2 y^2$的通解.

解： $y^{-2} y' = \dfrac{1}{x} y^{-1} + x^2$

$$\Rightarrow -\left(y^{-1}\right)' = \frac{1}{x} y^{-1} + x^2 \quad \Rightarrow \left(y^{-1}\right)' = -\frac{1}{x} y^{-1} - x^2$$

$$\Rightarrow y^{-1} = e^{\int p(x)\mathrm{d}x}\left[C + \int q(x) e^{-\int p(x)\mathrm{d}x}\mathrm{d}x\right]$$

$$= e^{-\int \frac{1}{x}\mathrm{d}x}\left[C - \int x^2 e^{\int \frac{1}{x}\mathrm{d}x}\mathrm{d}x\right]$$

$$= \frac{1}{x}\left[C - \int x^3 \mathrm{d}x\right]$$

$$= \frac{1}{x}\left[C - \frac{1}{4} x^4\right]$$

故通解为$y^{-1} = \dfrac{1}{x}\left(C - \dfrac{1}{4} x^4\right)$

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