Mandelbrot and Julia sets

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When we dip into complex numbers or fractals, sooner or later we come across the Mandelbrot set, or even the Julia set.

1 Complex numbers

Before we start to talk about complex numbers, it is important to know what an imaginary number is.
We have the following equation:

$$x^2 + 1 = 0$$

If we try to get the value of x, we will find that $x = \sqrt{-1}$. Since this root has no real solution, we could "*imagine*" that this number exists and has the name i.
This might feel a bit artificial; we are actually making up a number that, at first sight, should not exist. Nonetheless, we can make sense from all of that:
Let us define a number $z = (a, b)$, where $a, b \in \mathbb{R}$ and operations such as addition and multiplication with these numbers:

$$z_0 + z_1 = (a, b) + (c, d) := (a + c, b + d)$$

$$z_0 z_1 = (a, b)(c, d) := (ac - bd, ad + bc)$$

We can think that, if the second number is zero, we can have something like: $(a, 0) = a$.
Now, we define another number like those i such us $i = (0, 1)$. If we try to multiply this number by another one y we obtain:

$$bi = (b, 0)(0, 1) = (b \cdot 0 - 0 \cdot 1, b \cdot 1 + 0 \cdot 0) = (0, b)$$

Now, we can rewrite the number $z = (a, b)$ we had before as:

$$z = (a, b) = (a, 0) + (0, b) = a + bi$$

If we deal to get the actual value of i by multiplying it by itself, we find the following result:

$$i = (0, 1)$$

$$i^2 = i \cdot i = (0, 1)(0, 1) = (-1, 0) = -1$$

Since $i^2 = -1$, the number i must be necessary $\sqrt{-1}$. But it does not end up here, we can find more values of it by keeping multiplying it by itself:

$$i^3 = i^2 \cdot i = (-1, 0)i = -1i = -i$$

$$i^4 = i^3 \cdot i = -i \cdot i = (0, -1)(0, 1) = (1, 0) = 1$$

Finally, we can say that the number z we have been working with is going to be called complex, defining the complex numbers as:
A complex number $z \in \mathbb{C}$ is a mix of a real number $\mathbb{R}$ and an imaginary one i of the form:

$$z = x + yi, \quad where \quad x, y \in \mathbb{R}, \quad i^2 = -1.$$

$$\mathbb{C} := \{a + bi : a, b \in \mathbb{R}, i^2 = -1\}$$

Next, we can define the basic operations between complex numbers as:

$$\bar{z} = z^* = (x + yi)^* = x - yi$$

$$|z| = \sqrt{x^2 + y^2}$$

$$z_0 + z_1 = (x_0 + y_0 i) + (x_1 + y_1 i) = x_0 + x_1 + y_o i + y_1 i = x_0 + x_1 + (y_0 + y_1)i$$

$$z_0 - z_1 = (x_0 + y_0 i) - (x_1 + y_1 i) = x_0 - x_1 + y_o i - y_1 i = x_0 - x_1 + (y_0 - y_1)i$$

$$z_0 z_1 = (x_0 + y_0 i)(x_1 + y_1 i) = x_0 x_1 + x_0 y_1 i + x_1 y_0 i + y_0 y_1 i^2 = x_0 x_1 - y_0 y_1 + (x_0 y_1 + x_1 y_0)i$$

$$\frac{z_0}{z_1} = \frac{z_0}{z_1}\frac{z_1^*}{z_1^*} = \frac{z_0 z_1^*}{z_1 z_1^*} = \frac{z_0 z_1^*}{(x_1 + y_1 i)(x_1 - y_1 i)} = \frac{z_0 z_1^*}{x_1^2 - x_1 y_1 i + x_1 y_1 i - y_1^2} = \frac{z_0 z_1^*}{x_1^2 + y_1^2} = \frac{z_0 z_1^*}{|z_1|^2}$$

Something interesting that I just proved on the division is that, if I multiply a complex z by its conjugate z^*, we obtain its magnitude $|z|$ squared:

$$zz^* = (x + yi)(x - yi) = x \cdot x - xyi + xyi - y^2 i^2 = x^2 + y^2 = |z|^2$$

Since complex numbers have two numbers to be defined, we can think they are also coordinates in a plane named "complex plane".

Complex numbers also have an argument denoted as $arg(z)$ that corresponds to the smallest angle:

$$arg(z) := arctg\left(\frac{y}{x}\right)$$

We can see more and more interesting properties of complex numbers, but this context is more than enough to understand Mandelbrot and Julia sets.

2 Mandelbrot set

First, let us what computing the square of a number is like. The first thing you come up with is that you are making that number bigger, but it actually depends.
Let's define the function:

$$f_{\mathbb{R}}(x) := x^2, \quad f_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$$

And we are computing it over and over:

$$F_{\mathbb{R}}(x) = f_{\mathbb{R}}(f_{\mathbb{R}}(f_{\mathbb{R}}(...f_{\mathbb{R}}(x))))$$

In this case, if $x > 1$, we can see it diverges, the function makes the number bigger and bigger until it becomes infinite. Nevertheless, if $0 < x < 1$, the function makes the number smaller and smaller, tending to 0. In the case that $-1 < x < 0$, the function makes x positive and tend to zero. And finally, when $x < -1$, the function makes it positive and bigger diverging.
This might sound a bit odd, but when $x \in (-1, 1)$, my function $F_{\mathbb{R}}(x)$ converges to 0. Otherwise, it diverges.
We can use something like this for complex numbers. We are going to use the same concept of multiplying a number by itself several time as:

$$z_{n+1} = z_n^2 + c, \quad z_0 = 0 \wedge z_k, c \in \mathbb{C}$$

Here, c is the complex number we want to see whether goes to infinite or not. At this point, this could be mind-blowing, but all the complex numbers that, after this process, do not go to infinite, belong to the Mandelbrot set; a set named after Benoît Mandelbrot, a polish mathematician known as "The father of fractals":

$$\mathcal{M} := \{c \in \mathbb{C} : (|z_n| \leq 2)_{n \geq 0}, z_{n+1} = z_n^2 + c, z_0 = 0\}, \quad \mathcal{M} \subset \mathbb{C}$$

The points of the Mandelbrot set, somehow, if we draw them on the complex plane, they make a quite beautiful fractal, which makes this set that special in complex analysis.

3 Julia set

The Julia set has a lot to do with the Mandelbrot set. Julia set has complex numbers that, like the Mandelbrot set, make fractals when you draw them on the complex plane, but also, somehow, all possible fractals in the Julia set shows up at some point in the Mandelbrot set.
The idea of these fractals are super similar to Mandelbrot's ones, but with slight changes.
First, the complex c is nothing but a constant that defines how the fractal will look like. But we must be extremely careful, because it must be $|c| \leq 1$. If it is not, there won't be any fractal. Additionally, the complex z will actually be the point per se to see whether goes to infinite.

$$f_c(z) := z^2 + c, \quad f_c : \mathbb{C} \to \mathbb{C}$$

As you can tell, the formula remains the same, but with different interpretation of the complexes involved. To define the Julia set, first we need to define the filled Julia set, which is:

$$K(f_c) := \{z \in \mathbb{C} : \sup_{n \geq 0} |f_c^n(z)| < \infty\}$$

Putting it into words, the filled Julia set is the set of complex numbers that once you apply the functions $f_c(z)$ any amount of times, it does not go to infinite. With this definition, we can finally define the Julia set as the bounds of the filled Julia set:

$$J_c = \partial K$$