Written Examination Statistical Analysis of Econometric Time Series
November 10, 2017
Lecturer: Thomas Mikosch

- This is an open book examination. You may use any book, the lecture notes, assignments and their solutions, pocket calculator, etc. In the solution to each problem you have to give reasons either by providing a proof or by referring to a result in the lecture notes, the assignments, any book, etc. In the latter case, give an *exact* reference.
- Use your time in a reasonable way. Do not copy the text of the problems. Refer to the lecture notes whenever possible. Do not re-prove results from the lecture notes.
- You may write with a pencil.
- This examination paper consists of 2 pages.
- You may write in English or Danish.
- The distribution of the points over the problems is as follows.

```
Problem  1 a i ii iii iv v vi vii  b  2 a b c d    sum
---------------------------------------------------
# points     1  1  1  1  1  3  2   2    2 2 2 2     20
```

1. Consider the MA(2) process

$$X_t = Z_t + 2Z_{t-1} + Z_{t-2}\,, \qquad t \in \mathbb{Z}\,,$$

where (Z_t) is iid white noise with variance σ^2 and a finite fourth moment.

(a) Verify the following properties of the process (X_t):

(i) Strict stationarity

(ii) Stationarity

(iii) Ergodicity

(iv) Mixing

(v) Strong mixing

(vi) (X_t) satisfies the conditions of Ibragimov's central limit theorem

(vii) Non-invertibility

Give arguments for each of your answers.

(b) Calculate the spectral density of (X_t).

2. Consider a strictly stationary ARCH(2) process $X_t = \sigma_t Z_t$, $t \in \mathbb{Z}$, with iid standard normal noise (Z_t) where $\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \alpha_2 X_{t-2}^2$, $\alpha_i > 0$, $i = 0, 1, 2$.

(a) Assume that $E[\sigma_0^4] < \infty$. Show that $\nu_t = X_t^2 - \sigma_t^2$, $t \in \mathbb{Z}$, constitutes white noise.

(b) Under the conditions in **(a)** derive the spectral density of (X_t^2).
Hints: (i) The process (X_t^2) has an AR(2) representation. (ii) You are not required to determine the values of $\mathrm{var}(X_0^2)$ or $\mathrm{var}(\nu_0)$.

(c) Give a condition on the parameters $\alpha_i, i = 1, 2$, that ensures $E[X_t^2] = \infty$.

(d) Assume $n \geq 3$. Predict the 5%-quantile of the distribution of X_{n+1} conditional on the past $X_1, \ldots, X_n$, i.e., determine x such that

$$P(X_{n+1} \leq x \mid X_n, X_{n-1}, \ldots, X_1) = 0.05.$$

Notice: the 5%-quantile of the normal distribution is approximately -1.65.

(e) Assume $n \geq 3$. Calculate the best prediction $\widehat{X_{n+1}^2}$ of X_{n+1}^2 (in the mean square sense) given the past $X_1, \ldots, X_n$, i.e.,

$$\widehat{X_{n+1}^2} = E\left[X_{n+1}^2 \mid X_n, \ldots, X_1\right],$$

and the corresponding mean square prediction error, i.e.,

$$E\left[(\widehat{X_{n+1}^2} - X_{n+1}^2)^2\right].$$

Hint: It is useful to know that $E[Z_0^4] = 3$. We also assume that $E[\sigma_0^4]$ is known.

End of Examination Paper