Solutions to Assignment 1 for StatØk2, Block 1, 2020/2021, by Jorge Yslas

(1) Recall that (X_t) is stationary if:

i) $\mathbb{E}[X_t] = m$.

ii) $\mathbb{E}[X_t^2] < \infty$.

iii) $\gamma_X(t, t+h) = \gamma_X(0, h)$.

Thus, we need to show i)-iii) for $Z_t = X_t + Y_t$.
i) Let $\mathbb{E}[X_t] = m_X$ and $\mathbb{E}[Y_t] = m_Y$, then

$$\mathbb{E}[Z_t] = \mathbb{E}[X_t + Y_t] = m_X + m_Y \quad \text{(constant)}\,.$$

ii)

$$\mathbb{E}\left[Z_t^2\right] = \mathbb{E}\left[(X_t + Y_t)^2\right] = \mathbb{E}\left[X_t^2 + 2X_tY_t + Y_t^2\right] = \mathbb{E}\left[X_t^2\right] + 2m_Xm_Y + \mathbb{E}\left[Y_t^2\right] < \infty\,.$$

iii)

$$\begin{aligned}
\gamma_Z(t, t+h) &= \operatorname{cov}(Z_t, Z_{t+h}) = \operatorname{cov}(X_t + Y_t, X_{t+h} + Y_{t+h}) \\
&= \operatorname{cov}(X_t, X_{t+h}) + \operatorname{cov}(X_t, Y_{t+h}) + \operatorname{cov}(Y_t, X_{t+h}) + \operatorname{cov}(Y_t, Y_{t+h}) \\
&= \gamma_X(t, t+h) + \gamma_Y(t, t+h) \\
&= \gamma_X(0, h) + \gamma_Y(0, h) \\
&= \gamma_Z(0, h)\,.
\end{aligned}$$

Therefore (Z_t) is stationary.

(2) We need to show i)-iii) for $Z_t = X_tY_t$.
i) Let $\mathbb{E}[X_t] = m_X$ and $\mathbb{E}[Y_t] = m_Y$, then

$$\mathbb{E}[Z_t] = \mathbb{E}[X_tY_t] = m_Xm_Y \quad \text{(constant)}\,.$$

ii)

$$\mathbb{E}\left[Z_t^2\right] = \mathbb{E}\left[X_t^2Y_t^2\right] = \mathbb{E}\left[X_t^2\right]\mathbb{E}\left[Y_t^2\right] < \infty\,.$$

iii)

$$\begin{aligned}
\gamma_Z(t, t+h) &= \operatorname{cov}(Z_t, Z_{t+h}) = \operatorname{cov}(X_tY_t, X_{t+h}Y_{t+h}) \\
&= \mathbb{E}[X_tY_tX_{t+h}Y_{t+h}] - \mathbb{E}[X_tY_t]\,\mathbb{E}[X_{t+h}Y_{t+h}] \\
&= \mathbb{E}[X_tX_{t+h}]\,\mathbb{E}[Y_tY_{t+h}] - m_X^2m_Y^2 \\
&= \left(\mathbb{E}[X_tX_{t+h}] - m_X^2 + m_X^2\right)\left(\mathbb{E}[Y_tY_{t+h}] - m_Y^2 + m_Y^2\right) - m_X^2m_Y^2 \\
&= \left(\gamma_X(t, t+h) + m_X^2\right)\left(\gamma_Y(t, t+h) + m_Y^2\right) - m_X^2m_Y^2 \\
&= \gamma_X(t, t+h)\gamma_Y(t, t+h) + m_X^2\gamma_Y(t, t+h) + m_Y^2\gamma_X(t, t+h) \\
&= \gamma_X(0, h)\gamma_Y(0, h) + m_X^2\gamma_Y(0, h) + m_Y^2\gamma_X(0, h) \\
&= \gamma_Z(0, h)\,.
\end{aligned}$$

Therefore (Z_t) is stationary.

(3)

$$X_1 = \frac{W_1+W_2}{\sqrt{2}}\,, X_2 = \frac{W_1-W_2}{\sqrt{2}}\,, X_3 = \frac{W_3+W_4}{\sqrt{2}}\,, X_3 = \frac{W_3-W_4}{\sqrt{2}}\,, \ldots$$

White noise?
The sequence (X_t) is generated from the iid $N(0,1)$ sequence (W_t) by linear combinations. Therefore the joint distribution of $(X_0,\ldots,X_h)$ is Gaussian and mean-zero for any $h \geq 0$. Moreover, $(W_t+W_{t+1})/\sqrt{2}, (W_t-W_{t+1})/\sqrt{2}$ are jointly Gaussian, uncorrelated and have mean zero and variance 1, hence they are iid, and since the sequence of the pairs $(X_1,X_2),(X_3,X_4),\ldots$ is iid by construction, (X_t) is iid $N(0,1)$. Therefore (X_t) is iid white noise and strictly stationary.

$$X_1 = \operatorname{sign}(W_2)\,|W_1|\;, X_2 = \operatorname{sign}(W_1)\,|W_2|\;, X_3 = \operatorname{sign}(W_4)\,|W_3|\;, X_4 = \operatorname{sign}(W_3)\,|W_4|\;,\ldots$$

White noise?
For an iid sequence (W_t) of symetric variables, the sequences $(\operatorname{sign}(W_t))$ and $(|W_t|)$ are independent and each of them is iid. Hence a permutation of $(\operatorname{sign}(W_t))$ does not change the distribution and it is independent of $(|W_t|)$. Therefore (X_t) has the same distribution as $(\operatorname{sign}(W_t)|W_t|) = (W_t)$. Hence (X_t) is iid $N(0,1)$, hence iid white noise and strictly stationary.

$$X_t = \operatorname{sign}(W_t)\,|W_1|$$

White noise?
Since (W_t) is iid symmetric, the iid sequence $(\operatorname{sign}(W_t))$ has mean zero and is independent of $(|W_t|)$, in particular of $|W_1|$. Hence all X_t have the same distribution as W_1. They are also uncorrelated since for $t \neq s$,

$$\operatorname{cov}(X_t,X_s) = \mathbb{E}(\operatorname{sign}(W_t)\operatorname{sign}(W_s)W_1^2) = \mathbb{E}(\operatorname{sign}(W_t)\operatorname{sign}(W_s))\,\mathbb{E}(W_1^2) = 0\cdot 1 = 0\,.$$

Hence (X_t) is white noise, hence stationary.

iid white noise? No, the sequence is not independent:

$$\mathbb{E}\left[X_1^2X_2^2\right] = \mathbb{E}\left[W_1^4\right] = 3 \neq 1 = \mathbb{E}\left[X_1^2\right]\mathbb{E}\left[X_2^2\right]\,.$$

Strictly stationary? Yes. Since (W_t) is an iid sequence so is $(\operatorname{sign}(W_t))$ and consequently it is strictly stationary, meaning that

$$(\operatorname{sign}(W_t),\ldots,\operatorname{sign}(W_{t+h})) \stackrel{\mathrm{d}}{=} (\operatorname{sign}(W_1),\ldots,\operatorname{sign}(W_{1+h}))$$

for all t. By independence of $|W_1|$ and $(\operatorname{sign}(W_t))$ we also have

$$\begin{aligned}(X_t,\ldots,X_{t+h}) &= |W_1|\,(\operatorname{sign}(W_t),\ldots,\operatorname{sign}(W_{t+h}))\\ &\stackrel{\mathrm{d}}{=} |W_1|\,(\operatorname{sign}(W_1),\ldots,\operatorname{sign}(W_h))\\ &= (X_1,\ldots,X_{1+h})\,.\end{aligned}$$

Thus, (X_t) is strictly stationary.

$$X_t = W_tW_{t-1}$$

White noise?

i)

$$\mathbb{E}(X_t) = \mathbb{E}(W_t W_{t-1}) = \mathbb{E}(W_t)\,\mathbb{E}(W_{t-1}) = 0\,.$$

ii)

$$\mathbb{E}\left(X_t^2\right) = \mathbb{E}\left(W_t^2\right)\mathbb{E}\left(W_{t-1}^2\right) = 1\,.$$

iii) $h = 1$

$$\mathbb{E}(X_t X_{t+1}) = \mathbb{E}(W_t W_{t-1} W_{t+1} W_t) = \mathbb{E}\left(W_t^2\right)\mathbb{E}(W_{t-1})\,\mathbb{E}(W_{t+1}) = 0\,.$$

$h > 1$ $W_t W_{t-1}$ and $W_{t+h} W_{t+h-1}$ are independent. Hence

$$\mathbb{E}(X_t X_{t+h}) = \mathbb{E}(W_t W_{t-1} W_{t+h} W_{t+h-1}) = \mathbb{E}(W_t)\,\mathbb{E}(W_{t-1})\,\mathbb{E}(W_{t+h})\,\mathbb{E}(W_{t+h-1}) = 0\,.$$

Therefore (X_t) is white noise.
iid white noise? No, the sequence is not independent:

$$\mathbb{E}\left[X_1^2 X_2^2\right] = \mathbb{E}\left[W_1^2 W_0^2 W_2^2 W_1^2\right] = \mathbb{E}\left[W_1^4\right]\mathbb{E}\left[W_0^2\right]\mathbb{E}\left[W_2^2\right] = 3 \neq 1 = \mathbb{E}\left[X_1^2\right]\mathbb{E}\left[X_2^2\right]\;.$$

Stationary? Yes, every white noise process is stationary.
Strictly stationary? Yes. It follows from the fact that (W_t) is an iid sequence and the definition of $X_t = f(W_t, W_{t-1})$ as a measurable function of W_t, W_{t-1}.

$$X_t = W_t \cos(t) + W_{t-1}\sin(t)$$

White noise?
i)

$$\mathbb{E}[X_t] = \mathbb{E}[W_t \cos(t) + W_{t-1}\sin(t)] = 0\,.$$

ii)

$$\begin{aligned}
\mathbb{E}\left[X_t^2\right] &= \mathbb{E}\left[(W_t\cos(t) + W_{t-1}\sin(t))^2\right] \\
&= \mathbb{E}\left[W_t^2\cos^2(t) + 2W_tW_{t-1}\cos(t)\sin(t) + W_{t-1}^2\sin^2(t)\right] \\
&= \cos^2(t) + \sin^2(t) \\
&= 1\,.
\end{aligned}$$

Actually $X_t \sim N(0,1)$.
iii)

$$\begin{aligned}
\mathbb{E}[X_tX_{t+1}] &= \mathbb{E}[(W_t\cos(t) + W_{t-1}\sin(t))(W_{t+1}\cos(t+1) + W_t\sin(t+1))] \\
&= \mathbb{E}[W_tW_{t+1}\cos(t)\cos(t+1)] + \mathbb{E}\left[W_t^2\cos(t)\sin(t+1)\right] \\
&\quad + \mathbb{E}[W_{t-1}W_{t+1}\sin(t)\cos(t+1)] + \mathbb{E}[W_{t-1}W_t\sin(t)\sin(t+1)] \\
&= \cos(t)\sin(t+1)\;.
\end{aligned}$$

Therefore it is not white noise.
Stationary? No. $\gamma_X(t, t+1)$ depends on t.
Strictly stationary? No. Strictly stationary $\implies$ stationary.

(4) a) We know that the series (X_t) is stationary:
i)

$$\mathbb{E}[X_t] = \mathbb{E}[A\cos(\theta t) + B\sin(\theta t)] = \cos(\theta t)\mathbb{E}[A] + \sin(\theta t)\mathbb{E}[B] = 0\,.$$

ii)

$$\begin{aligned}
\mathbb{E}\left[X_t^2\right] &= \mathbb{E}\left[A^2\cos^2(\theta t) + 2A\cos(\theta t)B\sin(\theta t) + B^2\sin^2(\theta t)\right] \\
&= \cos^2(\theta t)\mathbb{E}\left[A^2\right] + 2\cos(\theta t)\sin(\theta t)\mathbb{E}[AB] + \sin^2(\theta t)\mathbb{E}\left[B^2\right] \\
&= \cos^2(\theta t) + \sin^2(\theta t) \\
&= 1\,.
\end{aligned}$$

iii) **Note.** $\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

$$\begin{aligned}
\gamma_X(t, t+h) &= \operatorname{cov}(X_t, X_{t+h}) = \mathbb{E}[X_t X_{t+h}] \\
&= \mathbb{E}\left[(A\cos(\theta t) + B\sin(\theta t))\,(A\cos(\theta(t+h)) + B\sin(\theta(t+h)))\right] \\
&= \cos(\theta t)\cos(\theta(t+h)) + \sin(\theta t)\sin(\theta(t+h)) \\
&= \cos(\theta h)\,.
\end{aligned}$$

Now, note that any vector $(X_0, \ldots, X_h)$ is obtained by a linear transformation of the iid $N(0,1)$ vector (A, B). More specifically,

$$(X_0, \ldots, X_h) = A(\cos(\theta 0), \ldots, \cos(\theta h)) + B(\sin(\theta 0), \ldots, \sin(\theta(h))\,.$$

Therefore $(X_0, \ldots, X_h)$ is multivariate normal with mean zero. Thus, (X_t) is mean-zero stationary Gaussian and, consequently, strictly stationary.

b) We have that

$$\begin{aligned}
\overline{X}_n &= \frac{1}{n}\sum_{t=1}^{n} X_t = \frac{1}{n}\sum_{t=1}^{n}(A\cos(\theta t) + B\sin(\theta t)) = \frac{1}{n}\left(A\sum_{t=1}^{n}\cos(\theta t) + B\sum_{t=1}^{n}\sin(\theta t)\right) \\
&= \frac{1}{n}\left(A\frac{\cos(\theta(n+1)/2)\sin(\theta n/2)}{\sin(\theta/2)} + B\frac{\sin(\theta(n+1)/2)\sin(\theta n/2)}{\sin(\theta/2)}\right)\,.
\end{aligned}$$

Then

$$\begin{aligned}
\left|\overline{X}_n\right| &\le \frac{1}{n}\left(|A|\left|\frac{\cos(\theta(n+1)/2)\sin(\theta n/2)}{\sin(\theta/2)}\right| + |B|\left|\frac{\sin(\theta(n+1)/2)\sin(\theta n/2)}{\sin(\theta/2)}\right|\right) \\
&\le \frac{1}{n}\left(\frac{|A| + |B|}{\sin(\theta/2)}\right) \\
&\to 0\,.
\end{aligned}$$

Which implies that $\overline{X}_n \to 0$ a.s.

c) **Note.** $\sin(2x) = 2\sin(x)\cos(x)$ and $\sin^2(x) = (1-\cos(2x))/2$.

Consider $f(x)=x^2$, then $\mathbb{E}[f(X_0)]=\mathbb{E}\left[X_0^2\right]=1$. Now,

$$\begin{aligned}
\frac{1}{n}\sum_{t=1}^{n} f(X_t) &= \frac{1}{n}\sum_{t=1}^{n} X_t^2 = \frac{1}{n}\sum_{t=1}^{n}(A\cos(\theta t)+B\sin(\theta t))^2 \\
&= \frac{1}{n}\sum_{t=1}^{n}\left(A^2\cos^2(\theta t)+2AB\cos(\theta t)\sin(\theta t)+B^2\sin^2(\theta t)\right) \\
&= \frac{1}{n}\sum_{t=1}^{n}\left(A^2\left(1-\sin^2(\theta t)\right)+AB\sin(2\theta t)+B^2\sin^2(\theta t)\right) \\
&= \frac{1}{n}\sum_{t=1}^{n}\left(A^2+\left(B^2-A^2\right)\sin^2(\theta t)+AB\sin(2\theta t)\right) \\
&= \frac{1}{n}\sum_{t=1}^{n}\left(A^2+\left(B^2-A^2\right)\left(\frac{1-\cos(2\theta t)}{2}\right)+AB\sin(2\theta t)\right) \\
&= \frac{1}{n}\sum_{t=1}^{n}\left(\frac{1}{2}\left(A^2+B^2\right)+\left(\frac{A^2-B^2}{2}\right)\cos(2\theta t)+AB\sin(2\theta t)\right) \\
&= \frac{1}{2}\left(A^2+B^2\right)+\frac{1}{n}\left(\left(\frac{A^2-B^2}{2}\right)\sum_{t=1}^{n}\cos(2\theta t)\right)+\frac{1}{n}\left(AB\sum_{t=1}^{n}\sin(2\theta t)\right) \\
&= \frac{1}{2}\left(A^2+B^2\right)+\frac{1}{n}\left(\left(\frac{A^2-B^2}{2}\right)\frac{\cos(\theta(n+1))\sin(\theta n)}{\sin(\theta)}\right)+\frac{1}{n}\left(AB\frac{\sin(\theta(n+1))\sin(\theta n)}{\sin(\theta)}\right) \\
&\to \frac{1}{2}\left(A^2+B^2\right) \neq \mathbb{E}[f(X_0)]\,.
\end{aligned}$$

In particular, the limit is random. Hence $(f(X_t))$ is non-ergodic.

(5) Since (Z_t) is white noise, it satisfies:

i) $\mathbb{E}[Z_t]=0$.

ii) $\mathbb{E}\left[Z_t^2\right]=\sigma^2$.

iii) $\mathbb{E}[Z_tZ_{t+h}]=0$ for $h\neq 0$.

We have that (with $\theta_0=1$)

$$\begin{aligned}
X_t &= Z_t+\theta_1 Z_{t-1}+\cdots+\theta_q Z_{t-q} \\
&= \sum_{j=0}^{q}\theta_j Z_{t-j}\,.
\end{aligned}$$

Then
i)

$$\mathbb{E}[X_t]=\mathbb{E}\left[\sum_{j=0}^{q}\theta_j Z_{t-j}\right]=\sum_{j=0}^{q}\theta_j\mathbb{E}[Z_{t-j}]=0$$

ii)

$$\begin{aligned}
\mathbb{E}\left[X_t^2\right] &= \mathbb{E}\left[\left(\sum_{j=0}^{q}\theta_j Z_{t-j}\right)^2\right] \\
&= \mathbb{E}\left[\sum_{j=0}^{q}\sum_{i=0}^{q}\theta_j Z_{t-j}\theta_i Z_{t-i}\right] \\
&= \sum_{j=0}^{q}\sum_{i=0}^{q}\theta_j\theta_i\mathbb{E}\left[Z_{t-j}Z_{t-i}\right] \\
&= \sigma^2\sum_{j=0}^{q}\theta_j^2
\end{aligned}$$

iii) Assume first that $h > 0$. We have that

$$\begin{aligned}
\operatorname{cov}\left(X_t, X_{t+h}\right) &= \operatorname{cov}\left(\sum_{j=0}^{q}\theta_j Z_{t-j}, \sum_{i=0}^{q}\theta_i Z_{t+h-i}\right) \\
&= \sum_{j=0}^{q}\sum_{i=0}^{q}\theta_j\theta_i\operatorname{cov}\left(Z_{t-j}, Z_{t+h-i}\right) .
\end{aligned}$$

Note that $\operatorname{cov}\left(X_t, X_{t+h}\right) = 0$ when $h > q$. Assume that $h \leq q$, then $\operatorname{cov}\left(Z_{t-j}, Z_{t+h-i}\right) \neq 0$ when $t - j = t + h - i$ which happens if and only if $i = j + h$ (note also that for $i \leq q$, then $j \leq q - h$), and consequently

$$\gamma_X\left(h\right) = \operatorname{cov}\left(X_t, X_{t+h}\right) = \sigma^2\sum_{j=0}^{q-h}\theta_j\theta_{j+h} .$$

If $h < 0$ (so, $-h > 0$) we have that

$$\operatorname{cov}\left(X_t, X_{t+h}\right) = \operatorname{cov}\left(X_{t+h}, X_t\right) = \operatorname{cov}\left(X_{t+h}, X_{(t+h)-h}\right) = \gamma_X\left(-h\right) .$$

Therefore, we have in general that

$$\gamma_X\left(h\right) = \begin{cases}\sigma^2\sum_{j=0}^{q-|h|}\theta_j\theta_{j+|h|} & |h| \leq q \\ 0 & |h| > q\end{cases} .$$

(6) We have that the characteristic function of X is given by

$$\varphi\left(t\right) = \mathbb{E}\left[e^{itX}\right] = e^{-|t|} .$$

Then

$$\begin{aligned}
\mathbb{E}\left[e^{it\left(\sum_{j=1}^{k}\psi_j X_j\right)}\right] &= \mathbb{E}\left[e^{\sum_{j=1}^{k} i(t\psi_j)X_j}\right] \\
&= \prod_{j=1}^{k} \mathbb{E}\left[e^{i(t\psi_j)X_j}\right] \\
&= \prod_{j=1}^{k} e^{-|t\psi_j|} \\
&= e^{-\sum_{j=1}^{k}|t||\psi_j|} \\
&= e^{-|t|\left(\sum_{j=1}^{k}|\psi_j|\right)} .
\end{aligned}$$

On the other hand

$$\begin{aligned}
\mathbb{E}\left[e^{it\left(\sum_{j=1}^{k}|\psi_j|\right)X_0}\right] &= e^{-|t|\left|\sum_{j=1}^{k}|\psi_j|\right|} \\
&= e^{-|t|\left(\sum_{j=1}^{k}|\psi_j|\right)} .
\end{aligned}$$

Hence

$$\sum_{j=1}^{k}\psi_j X_j \overset{\mathrm{d}}{=} \sum_{j=1}^{k}|\psi_j| X_0 .$$

b) Note that

$$\overline{X}_n = \frac{1}{n}\sum_{t=1}^{n} X_t = \sum_{t=1}^{n}\left(\frac{1}{n}\right) X_t$$

Then, it follows by a) that

$$\sum_{t=1}^{n}\left(\frac{1}{n}\right) X_t \overset{\mathrm{d}}{=} \sum_{t=1}^{n}\left|\frac{1}{n}\right| X_0 = X_0 .$$

Therefore

$$\overline{X}_n \overset{\mathrm{d}}{=} X_0 .$$

c) **Opt 1**. Using the density of X_0.
X_0 has density given by

$$f(x) = \frac{1}{\pi\left(1+x^2\right)}$$

Now, for $X_{0+} = \max(X_0, 0)$,

$$\mathbb{E}\left[X_{0+}\right] = \int_0^{\infty} \frac{x}{\pi\left(1+x^2\right)} dx = \frac{1}{2\pi}\int_0^{\infty} \frac{2x}{\left(1+x^2\right)} dx = \frac{1}{2\pi}\log\left(1+x^2\right)\Big|_0^{\infty} = \infty$$

and since $X_{0+} \leq |X_0|$ the result follows.
Opt 2. Using Kolmogorov's strong law of large numbers.

Suppose that $\mathbb{E}\left[|X_0|\right] < \infty$, then by Kolmogorov's strong law of large numbers we have that

$$\overline{X}_n \stackrel{\text{a.s.}}{\to} \mathbb{E}\left[X_0\right]$$

and since almost sure convergence implies convergence in distribution, it follows that

$$\overline{X}_n \stackrel{\text{d}}{\to} \mathbb{E}\left[X_0\right]$$

which is a contradiction to part b) of this exercise. Therefore $\mathbb{E}\left[|X_0|\right] = \infty$.

d) It follows from the fact that (Z_t) is an iid sequence, hence ergodic.