Written Examination: Statistical Analysis of Econometric Time Series
November 9, 2018
Lecturer: Thomas Mikosch

- This is an open book examination. You may use any book, the lecture notes, assignments and their solutions, pocket calculator, computer, etc. In the solution to each problem you have to give reasons either by providing a proof or by referring to a result in the lecture notes, the assignments, any book, etc. In the latter case, give an *exact* reference.
- Use your time in a reasonable way. Do not copy the text of the problems. Refer to the lecture notes whenever possible. Do not re-prove results from the lecture notes.
- You may write with a pencil.
- This examination paper consists of 2 pages.
- You may write in English or Danish.
- The distribution of the points over the problems is as follows.

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Problem   1 a i ii iii iv v vi   b i ii   c   2 a b c d     sum
-----------------------------------------------------------------
# points      1  1  1  1  1  1     3  3   2    4 4 2         24
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1. Consider the ARMA$(1,1)$ equations

$$X_t - 0.5X_{t-1} = Z_t + 0.5Z_{t-1}\,, \qquad t \in \mathbb{Z}\,,$$

where (Z_t) is iid white noise with variance σ^2. We assume that these equations have a solution (X_t).

(a) Which of the following properties does the solution (X_t) have?

(i) Stationarity

(ii) Strict stationarity

(iii) Ergodicity

(iv) Mixing

(v) Causality

(vi) Invertibility

Give arguments for each of your answers.

(b)

(i) Show that the autocorrelation of (X_t) at lag 1 is given by

$$\rho_X(1) = \frac{1}{2} + \frac{1}{2}\frac{\sigma^2}{\gamma_X(0)}\,.$$

Hint. Use the linear process representation $X_t = Z_t + \sum_{i=1}^{\infty} \psi_i Z_{t-i}$.

(ii) Show that the corresponding sample autocovariance

$$\gamma_{n,X}(1) = \frac{1}{n}\sum_{t=1}^{n-1}(X_t - \overline{X}_n)(X_{t+1} - \overline{X}_n)\,, \qquad \overline{X}_n = \frac{1}{n}\sum_{t=1}^{n} X_t\,,$$

is a consistent estimator of $\gamma_X(1)$, i.e.,

$$\gamma_{n,X}(1) \stackrel{\text{a.s.}}{\to} \gamma_X(1)\,, \qquad n \to \infty\,.$$

(c) Calculate the spectral density of (X_t).

2. Consider a strictly stationary GARCH(1,1) process $X_t = \sigma_t Z_t$, $t \in \mathbb{Z}$, with iid standard normal noise (Z_t) where $\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2$, $\alpha_i > 0$, $i = 0, 1$, $\beta_1 > 0$.

(a) Which conditions on the parameters α_1, β_1 ensure that X_t has a finite variance? Calculate the variance of X_t under these conditions.

(b) Assume a finite fourth moment for (X_t). Derive the spectral densities of (X_t) and (X_t^2).
Hints: (i) The process $(X_t^2 - \mathbb{E}[X_0^2])$ has an ARMA(1,1) representation with white noise. (ii) We assume that $\mathrm{var}(\sigma_0^2(Z_0^2 - 1)) = 2\mathbb{E}[\sigma_0^4]$ is known.

(c) Predict the 97.5%-quantile x of the distribution of X_{n+1} conditional on the past $X_1, \ldots, X_n, \sigma_1, \ldots, \sigma_n$ for $n \geq 2$, i.e., determine x such that

$$P(X_{n+1} \leq x \mid X_n, X_{n-1}, \ldots, X_1, \sigma_n, \ldots, \sigma_1) = 0.975.$$

End of Examination Paper

Solutions to the written examination Statistical Analysis of Econometric Time Series, November 9, 2018

1. **(a)**

(i,v,vi) **3 points** We have $(1-0.5B)X_t = (1+0.5B)Z_t$. Therefore $\varphi(z) = 1-0.5z$, $\theta(z) = 1+0.5z$, and these polynomials are not zero for $|z| \le 1$. Therefore the equations have a *stationary causal and invertible* solution.

(ii,iii,iv) **3 points** The solution is a linear causal process $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$. The process has the structure $X_t = f(Z_t, Z_{t-1}, \ldots)$ for an iid sequence (Z_t) and a deterministic function f. Therefore the process is also strictly stationary, mixing and ergodic.

(b)

(i) **3 points** We have the causal representation $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ with $\psi_0 = 1$.

$$\begin{aligned}
\gamma_X(1) &= \operatorname{cov}(X_0, X_1) = \operatorname{cov}(X_0, 0.5X_0 + Z_1 + 0.5Z_0) \\
&= 0.5\gamma_X(0) + \operatorname{cov}(X_0, Z_1) + 0.5\operatorname{cov}(X_0, Z_0) \\
&= 0.5\gamma_X(0) + 0 + 0.5\operatorname{cov}(0.5X_{-1} + Z_0 + 0.5Z_{-1}, Z_0) \\
&= 0.5\gamma_X(0) + 0.5\sigma^2 .
\end{aligned}$$

Here we used the independence of X_0, Z_1, of X_{-1}, Z_0, of Z_{-1}, Z_0. Hence

$$\rho_X(1) = 0.5 + 0.5\frac{\sigma^2}{\gamma_X(0)} .$$

(ii) **3 points** We have

$$\begin{aligned}
\gamma_{n,X}(1) &= \frac{1}{n}\sum_{t=1}^{n-1}(X_t - \overline{X}_n)(X_{t+1} - \overline{X}_n) \\
&= \frac{n-1}{n}\Big(\frac{1}{n-1}\sum_{t=1}^{n-1} X_t X_{t+1} - (\overline{X}_n)^2\Big) \\
&\stackrel{\text{a.s.}}{\to} \mathbb{E}[X_0 X_1] - (\mathbb{E}[X_0])^2 = \mathbb{E}[X_0 X_1] = \gamma_X(1) .
\end{aligned}$$

Here we used the ergodic theorem for $(X_t X_{t+1})$ and (X_t).

(c) 2 points The ARMA$(1,1)$ process (X_t) is causal. Therefore the spectral density has the form

$$f_X(\lambda) = \frac{\sigma^2}{2\pi}\frac{\left|1 - 0.5\mathrm{e}^{-i\lambda}\right|^2}{\left|1 + 0.5\mathrm{e}^{-i\lambda}\right|^2}, \quad \lambda \in (0,\pi) .$$

2. **(a) 4 points** We have $\mathbb{E}[X_t] = \mathbb{E}[\sigma_t]\mathbb{E}[Z_t] = 0$ and $\mathbb{E}[X_t^2] = \mathbb{E}[\sigma_t^2]\mathbb{E}[Z_t^2] = \mathbb{E}[\sigma_t^2]$ since σ_t, Z_t are independent and $\mathbb{E}[Z_t^2] = 1$. Hence $\operatorname{var}(X_t)$ is finite if and only if

$\mathbb{E}[\sigma_t^2] < \infty$. We have by strict stationarity of (σ_t),

$$\begin{aligned}\mathbb{E}[\sigma_t^2] &= \alpha_0 + \alpha_1 \mathbb{E}[X_{t-1}^2] + \beta_1 \mathbb{E}[\sigma_{t-1}^2] \\ &= \alpha_0 + (\alpha_1 + \beta_1)\mathbb{E}[\sigma_t^2]\,.\end{aligned}$$

If $\alpha_1 + \beta_1 = 1$ and $\alpha_0 > 0$ this would lead to a contradiction, hence $\mathbb{E}[\sigma_t^2] = \infty$. Similarly, if $\alpha_1 + \beta_1 > 1$ then

$$\mathbb{E}[X_t^2] = \mathbb{E}[\sigma_t^2] = \frac{\alpha_1}{1 - (\alpha_1 + \beta_1)}\,.$$

This cannot be true since the right-hand side would be negative. Therefore the latter relation holds if and only if $\alpha_1 + \beta_1 < 1$.

(b) 4 points (X_t) is white noise. Therefore it has spectral density $\mathrm{var}(X_0)/(2\pi) = \mathbb{E}[\sigma_0^2]/(2\pi)$.

$(X_t^2 - \mathbb{E}[X_0^2])$ solves the ARMA(1,1) equation

$$(X_t^2 - \mathbb{E}[X_0^2]) - (\alpha_1 + \beta_1)(X_{t-1}^2 - \mathbb{E}[X_0^2]) = (X_t^2 - \sigma_t^2) - \beta_1(X_{t-1}^2 - \sigma_{t-1}^2)\,,$$

where $\nu_t = X_t^2 - \sigma_t^2 = \sigma_t^2(Z_t^2 - 1)$ is white noise. Therefore $\varphi(z) = 1 - (\alpha_1 + \beta_1)z$, $\theta(z) = 1 - \beta_1 z$. Since $\alpha_1 + \beta_1 < 1$ is necessary for a second moment this assumption ensures that $\varphi(z)\theta(z) \neq 0$, $|z| \leq 1$. Hence $(X_t^2 - \mathbb{E}[X_0^2])$ has spectral density

$$f_{X^2}(\lambda) = \frac{\mathrm{var}(\nu_0)}{2\pi}\frac{|1 - \beta_1 \mathrm{e}^{-iz}|^2}{|1 - (\alpha_1 + \beta_1)\mathrm{e}^{-i\lambda}|^2}\,, \qquad \lambda \in (0, \pi)\,.$$

Moreover, $\mathrm{var}(\nu_0) = \mathbb{E}[\sigma_0^4]\mathrm{var}(Z_0^2 - 1) = 2\,\mathbb{E}[\sigma_0^4]$.

(c) 2 points Given the past values $X_n, X_{n-1}, \ldots, X_1, \sigma_1, \ldots, \sigma_n$, the random variable X_{n+1} has a normal distribution with variance $\sigma_{n+1}^2 = \alpha_0 + \alpha_1 X_n^2 + \beta_1 \sigma_n^2$. Hence we have

$$P(X_{n+1} \leq x \mid X_n, X_{n-1}, \ldots, X_1, \sigma_1, \ldots, \sigma_n) = P(Z_{n+1} \leq x/\sigma_{n+1} \mid X_n, \sigma_n) = 0.975$$

for $x/\sigma_{n+1} = q_{0.975} = 1.96...$, i.e., $x = 1.96\sigma_{n+1}$ is a prediction of the 97.5%-quantile of X_{n+1} given the past.