Solutions to Assignment 4 for StatØk2, Block 1, 2021/2022 by Jorge Yslas

(1) a) Since we are considering a causal AR(1) model X_t has the linear process representation

$$X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}\,.$$

Now, since (Z_t) is an iid sequence it is strictly stationary and ergodic. Hence, (X_t) is also strictly stationary and ergodic.
We have that the Yule-Walker estimators $\hat{\phi}$ of ϕ and $\hat{\sigma}^2$ of σ^2 are given by

$$\begin{aligned}\hat{\phi} &= \gamma_{n,X}(1)/\gamma_{n,X}(0) = \rho_{n,X}(1)\,,\\ \hat{\sigma}^2 &= \gamma_{n,X}(0)\left[1-\rho_{n,X}^2(1)\right]\,.\end{aligned}$$

In Assignment 2 exercise 2 we proved that for a strictly stationary and ergodic sequence the sample autocovariances and the sample autocorrelations are consistent estimators of their deterministic counterparts:

$$\gamma_{n,X}(h) \overset{\text{a.s.}}{\to} \gamma_X(h) \quad \text{and} \quad \rho_{n,X}(h) \overset{\text{a.s.}}{\to} \rho_X(h)\,.$$

This implies that

$$\hat{\phi} = \rho_{n,X}(1) \overset{\text{a.s.}}{\to} \rho_X(1) = \phi$$

and

$$\hat{\sigma}^2 = \gamma_{n,X}(0)\left[1-\rho_{n,X}^2(1)\right] \overset{\text{a.s.}}{\to} \gamma_X(0)\left[1-\rho_X^2(1)\right] = \left(\sigma^2(1-\varphi^2)^{-1}\right)(1-\varphi^2) = \sigma^2\,.$$

b) **Note.** For a causal AR(1) process (X_t) we have that $\rho_X(h) = \phi^{|h|}$.
From Theorem 4.19 we know that

$$\sqrt{n}\left(\rho_{n,X}(1)-\rho_X(1)\right) \overset{\text{d}}{\to} Y_1\,,$$

where Y_1 is $N(0,w)$ and w is given by Bartlett's formula

$$\begin{aligned}w = w_{11} &= \sum_{k=1}^{\infty}\left(\rho_X(k+1)+\rho_X(k-1)-2\rho_X(1)\rho_X(k)\right)^2\\ &= \sum_{k=1}^{\infty}\left(\phi^{k+1}+\phi^{k-1}-2\phi^1\phi^k\right)^2\\ &= \sum_{k=1}^{\infty}\left(\phi^k\phi^{-1}-\phi\phi^k\right)^2\\ &= \left(\phi^{-1}-\phi\right)^2\sum_{k=1}^{\infty}\phi^{2k}\\ &= \left(\frac{1-\phi^2}{\phi}\right)^2\left(\frac{\phi^2}{1-\phi^2}\right)\\ &= 1-\phi^2\,.\end{aligned}$$

Therefore

$$\sqrt{n}\left(\hat{\phi}-\phi\right) = \sqrt{n}\left(\rho_{n,X}(1)-\rho_X(1)\right) \overset{\text{d}}{\to} Y_1 \sim N(0,1-\phi^2)\,.$$

(2) We have that

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2$$

and

$$\theta(z) = 1 + \theta_1 z + \theta_2 z^2 \,.$$

Let's first determine for which values of (ϕ_1, ϕ_2) there exists a causal solution. Let $\lambda = z^{-1}$, then

$$\begin{aligned} & 1 - \phi_1 z - \phi_2 z^2 = 0 \\ \iff & z^{-2} - \phi_1 z^{-1} - \phi_2 = 0 \\ \iff & \lambda^2 - \phi_1 \lambda - \phi_2 = 0 \,. \end{aligned}$$

Note that the roots of these equations satisfy $|z| > 1$ if and only if $|\lambda| = |z^{-1}| < 1$. Therefore we will consider the equation

$$\lambda^2 - \phi_1 \lambda - \phi_2 = 0$$

and determine the range of values of (ϕ_1, ϕ_2) under which the roots of this equation satisfies $|\lambda| < 1$. We observe 2 cases:

i) *Real roots* ($\phi_1^2 + 4\phi_2 \geq 0 \iff \phi_2 \geq -\phi_1^2/4$).
We have that

$$|\lambda_{1,2}| < 1 \iff -1 < \lambda_{1,2} < 1 \iff -1 < \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2} < 1 \iff -2 < \phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2} < 2$$

Note that

$$\phi_1 - \sqrt{\phi_1^2 + 4\phi_2} \leq \phi_1 + \sqrt{\phi_1^2 + 4\phi_2} \,.$$

Thus, we need

$$\begin{aligned} & \phi_1 + \sqrt{\phi_1^2 + 4\phi_2} < 2 \\ \iff & \sqrt{\phi_1^2 + 4\phi_2} < 2 - \phi_1 \\ \iff & \phi_1^2 + 4\phi_2 < 4 - 4\phi_1 + \phi_1^2 \\ \iff & \phi_2 < 1 - \phi_1 \\ \iff & \phi_2 + \phi_1 < 1 \end{aligned}$$

and

$$\begin{aligned} & -2 < \phi_1 - \sqrt{\phi_1^2 + 4\phi_2} \\ \iff & \sqrt{\phi_1^2 + 4\phi_2} < 2 + \phi_1 \\ \iff & \phi_1^2 + 4\phi_2 < 4 + 4\phi_1 + \phi_1^2 \\ \iff & \phi_2 < 1 + \phi_1 \\ \iff & \phi_2 - \phi_1 < 1 \,. \end{aligned}$$

Note that a direct consequence is that $\phi_2 < 1$.

ii) *Complex roots* ($\phi_1^2 + 4\phi_2 < 0 \iff \phi_2 < -\phi_1^2/4$).
The roots are given by

$$\lambda_{1,2} = \frac{\phi_1}{2} \pm i\frac{\sqrt{-\phi_1^2 - 4\phi_2}}{2}\,.$$

Now

$$|\lambda_{1,2}|^2 = \frac{\phi_1^2}{4} + \frac{-\phi_1^2 - 4\phi_2}{4} = -\phi_2 < 1\,.$$

Thus, we need $\phi_2 > -1$.
Combining both cases we have the following conditions:

$$\begin{gathered} \phi_2 + \phi_1 < 1\,, \\ \phi_2 - \phi_1 < 1\,, \\ |\phi_2| < 1\,. \end{gathered}$$

To determine the set of parameter where we have an invertible solution we need to consider the equation

$$1 + \theta_1 z + \theta_2 z^2 = 0 \iff 1 - (-\theta_1)z - (-\theta_2)z^2 = 0\,,$$

which is exactly of the same form as the previous case. Then we have the following conditions:

$$\begin{gathered} (-\theta_2) + (-\theta_1) < 1\,, \\ (-\theta_2) - (-\theta_1) < 1\,, \\ |(-\theta_2)| < 1\,, \end{gathered}$$

or equivalently,

$$\begin{gathered} \theta_2 + \theta_1 > -1\,, \\ \theta_2 - \theta_1 > -1\,, \\ |\theta_2| < 1\,. \end{gathered}$$

(3) a) We sum $X_t = \phi X_{t-1} + Z_t$ for $t = 1, \dots, n$ and divide by n:

$$\begin{aligned} &\overline{X}_n - \phi\frac{1}{n}\sum_{t=1}^{n} X_{t-1} = \overline{Z}_n \\ \iff &\overline{X}_n - \phi\left(\overline{X}_n - \frac{1}{n}\left(X_n - X_0\right)\right) = \overline{Z}_n \\ \iff &\left(1-\phi\right)\overline{X}_n = \overline{Z}_n + \frac{\phi}{n}\left(X_0 - X_n\right) \\ \iff &\overline{X}_n = (1-\phi)^{-1}\,\overline{Z}_n + \left(\frac{\phi}{1-\phi}\right)\frac{1}{n}\left(X_0 - X_n\right) \\ \iff &\sqrt{n}\,\overline{X}_n = (1-\phi)^{-1}\sqrt{n}\,\overline{Z}_n + \left(\frac{\phi}{1-\phi}\right)\frac{1}{\sqrt{n}}\left(X_0 - X_n\right)\,. \end{aligned}$$

Note that

$$\mathbb{E}\Big[\Big|\frac{1}{\sqrt{n}}X_n\Big|^2\Big] = \frac{1}{n}\mathbb{E}[|X_n|^2] = \frac{1}{n}\mathbb{E}[|X_0|^2] \to 0\,,$$

where we used that for a stationary process the second moment is finite and does not depend on t. Then $\frac{1}{\sqrt{n}} X_n \xrightarrow{L^2} 0$ and consequently $\frac{1}{\sqrt{n}} X_n \xrightarrow{\mathrm{p}} 0$. Similarly $\frac{1}{\sqrt{n}} X_0 \xrightarrow{\mathrm{p}} 0$. Using that $\sqrt{n}\,\overline{Z}_n \xrightarrow{d} Y \sim N(0,\sigma^2)$, we get that

$$\sqrt{n}\,\overline{X}_n \xrightarrow{\mathrm{d}} (1-\phi)^{-1}\,Y \sim N(0,(1-\phi)^{-2}\,\sigma^2)\,.$$

b) It follows in exactly the same way as a). We know that there exists a unique stationary solution given by

$$X_t = -\sum_{j=1}^{\infty} \phi^{-j} Z_{t+j}\,.$$

Since X_t satisfies the AR(1) equation, we have that

$$\begin{aligned} &\overline{X}_n - \phi \frac{1}{n}\sum_{t=1}^{n} X_{t-1} = \overline{Z}_n \\ \iff &\sqrt{n}\,\overline{X}_n = (1-\phi)^{-1}\sqrt{n}\,\overline{Z}_n + \left(\frac{\phi}{1-\phi}\right)\frac{1}{\sqrt{n}}\,(X_0 - X_n) \end{aligned}$$

Then, using again the convergence of $\sqrt{n}\overline{Z}_n$ and that (X_t) is stationary, we get that

$$\sqrt{n}\,\overline{X}_n \xrightarrow{\mathrm{d}} (1-\phi)^{-1}\,Y \sim N(0,(1-\phi)^{-2}\,\sigma^2)\,.$$

(4) a) We have that

$$\begin{aligned} \mathbb{E}\left[\exp\left(is\left(\frac{1}{n^{1/\alpha}}\sum_{t=1}^{n} Z_t\right)\right)\right] &= \mathbb{E}\left[\exp\left(\sum_{t=1}^{n} i\left(\frac{s}{n^{1/\alpha}}\right) Z_t\right)\right] \\ &= \prod_{t=1}^{n} \mathbb{E}\left[\exp\left(i\left(\frac{s}{n^{1/\alpha}}\right) Z_t\right)\right] \\ &= \prod_{t=1}^{n} \exp\left(-c\left|\frac{s}{n^{1/\alpha}}\right|^{\alpha}\right) \\ &= \exp\left(-c\,|s|^{\alpha}\right) \\ &= \mathbb{E}\left[\exp\left(isZ_1\right)\right]\,, \quad s \in \mathbb{R}\,. \end{aligned}$$

Therefore

$$\frac{1}{n^{1/\alpha}}\sum_{t=1}^{n} Z_t \overset{\mathrm{d}}{=} Z_1\,.$$

b) We have that

$$X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}\,.$$

Then

$$\mathbb{E}\left[X_t\right] = \mathbb{E}\Big[\sum_{j=0}^{\infty} \phi^j Z_{t-j}\Big] = \sum_{j=0}^{\infty} \phi^j \mathbb{E}\left[Z_{t-j}\right] = 0\,.$$

We also have that (X_t) is strictly stationary and ergodic since (Z_t) is an iid sequence. It follows then that

$$\overline{X}_n \overset{\text{a.s.}}{\to} \mathbb{E}\left[X_0\right] = 0\,.$$

Now, since X_t satisfies the AR(1) equation, we have that

$$\begin{aligned}
&\overline{X}_n - \phi \frac{1}{n}\sum_{t=1}^{n} X_{t-1} = \overline{Z}_n \\
\iff &\overline{X}_n = (1-\phi)^{-1}\,\overline{Z}_n + \left(\frac{\phi}{1-\phi}\right)\frac{1}{n}\left(X_0 - X_n\right) \\
\iff &n^{1-1/\alpha}\overline{X}_n = (1-\phi)^{-1}\,n^{1-1/\alpha}\overline{Z}_n + \left(\frac{\phi}{1-\phi}\right)\frac{1}{n^{1/\alpha}}\left(X_0 - X_n\right)\,.
\end{aligned}$$

Note that

$$\mathbb{E}\Big[\Big|\frac{1}{n^{1/\alpha}}X_n\Big|\Big] = \frac{1}{n^{1/\alpha}}\mathbb{E}[|X_n|] = \frac{1}{n^{1/\alpha}}\mathbb{E}[|X_0|] \to 0\,,$$

implying that $\frac{1}{n^{1/\alpha}}X_n \overset{L^1}{\to} 0$ and consequently $\frac{1}{n^{1/\alpha}}X_n \overset{\text{p}}{\to} 0$. Similarly $\frac{1}{n^{1/\alpha}}X_0 \overset{\text{p}}{\to} 0$. Combining these results with a), we get that

$$n^{1-1/\alpha}\overline{X}_n \overset{\text{d}}{\to} (1-\phi)^{-1}\,Z_0\,.$$

(5) We know that for $|\phi| > 1$ the unique stationary solution to the AR(1) equation $X_t = \phi X_{t-1} + Z_t$ is given by

$$X_t = -\sum_{j=1}^{\infty}\phi^{-j}Z_{t+j}\,.$$

Then

$$\mathbb{E}\left[X_t\right] = -\sum_{j=1}^{\infty}\phi^{-j}\mathbb{E}\left[Z_{t+j}\right] = 0\,.$$

Thus

$$\begin{aligned}
\operatorname{cov}\left(X_t, X_{t+h}\right) &= \mathbb{E}\left[X_t X_{t+h}\right] \\
&= \mathbb{E}\Big[\Big(-\sum_{j=1}^{\infty}\phi^{-j}Z_{t+j}\Big)\Big(-\sum_{k=1}^{\infty}\phi^{-k}Z_{t+h+k}\Big)\Big] \\
&= \lim_{n\to\infty}\mathbb{E}\Big[\sum_{j=1}^{n}\phi^{-j}Z_{t+j}\sum_{k=1}^{n}\phi^{-k}Z_{t+h+k}\Big] \\
&= \lim_{n\to\infty}\sum_{j=1}^{n}\sum_{k=1}^{n}\phi^{-j}\phi^{-k}\mathbb{E}\left[Z_{t+j}Z_{t+h+k}\right] \quad (\neq 0 \text{ when } k = j-h) \\
&= \lim_{n\to\infty}\sigma^2\sum_{j=1+h}^{n}\phi^{-j}\phi^{-(j-h)}
\end{aligned}$$

$$
\begin{aligned}
&= \sigma^2 \phi^h \sum_{j=1+h}^{\infty} \phi^{-2j} \\
&= \sigma^2 \phi^h \phi^{-2h} \sum_{j=1}^{\infty} \phi^{-2j} \\
&= \phi^{-h} \sigma^2 \frac{\phi^{-2}}{1-\phi^{-2}} \,.
\end{aligned}
$$

Implying that

$$\rho_X(h) = \phi^{-h} \,.$$

On the other hand, since $|\phi| > 1$ then $|\phi|^{-1} < 1$, which implies that $X_t = \phi^{-1} X_{t-1} + Z_t$ is causal and we know that the correlation function of the stationary solution is given by

$$\rho_X(h) = (\phi^{-1})^h = \phi^{-h} \,.$$