Written Examination: Statistical Analysis of Econometric Time Series
November 8, 2019
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- This is an open book examination. You may use any book, the lecture notes, assignments and their solutions, pocket calculator, computer, etc. In the solution to each problem you have to give reasons either by providing a proof or by referring to a result in the lecture notes, the assignments, any book, etc. In the latter case, give an *exact* reference.
- Use your time in a reasonable way. Do not copy the text of the problems. Refer to the lecture notes whenever possible. Do not re-prove results from the lecture notes.
- You may write with a pencil.
- This examination paper consists of 2 pages.
- You may write in English or Danish.
- The distribution of the points over the problems is as follows.

Problem	1 a	1 b	2 a	2 b	2 c	3 a	3 b	3 c	3 d	sum
# points	2	2	2	1	2	2	3	1	6	21

1. Consider the ARMA$(1,2)$ equations

$$X_t - 0.9\, X_{t-1} = Z_t - 2\, Z_{t-1} + Z_{t-2}\,, \qquad t \in \mathbb{Z}\,,$$

where (Z_t) is iid white noise with variance σ^2.

(a) Show that these ARMA(1,2) equations have a stationary causal solution (X_t). Is the solution also invertible?

(b) Which of the following properties does (X_t^2) have?

(i) Strict stationarity

(ii) Ergodicity

Give arguments for each of your answers.

2. Consider two uncorrelated stationary processes (X_t) and (Y_t) with mean zero. This means that $\mathbb{E}[X_t] = \mathbb{E}[Y_t] = 0$ for all $t \in \mathbb{Z}$ and $\mathrm{cov}(X_t, Y_s) = 0$ for all $s, t \in \mathbb{Z}$. We assume that the autocovariance functions

$$\gamma_X(h) = \mathrm{cov}(X_0, X_h) \quad \text{and} \quad \gamma_Y(h) = \mathrm{cov}(Y_0, Y_h)\,, \qquad h \in \mathbb{Z}\,,$$

satisfy the condition $\sum_{h\in\mathbb{Z}}(|\gamma_X(h)| + |\gamma_Y(h)|) < \infty$.

(a) Calculate the autocovariance function of the stationary process $Z_t = X_t + Y_t$, $t \in \mathbb{Z}$. (You may assume that the stationarity is proved.)

(b) Argue that both (X_t) and (Y_t) have a spectral density.

(c) Calculate the spectral density of (Z_t).

3. Consider the AR(1) process (X_t) which is the causal solution to the equations

$$X_t - \varphi X_{t-1} = Z_t\,, \qquad t \in \mathbb{Z}\,,$$

where $|\varphi| < 1$ and (Z_t) is iid normal $N(0, \sigma^2)$.

(a) Derive the best prediction for X_{n+2} given that you know

(i) X_n

(ii) X_n and X_{n-1}.

(b) Derive the best prediction for X_{n+1}^2 given that you know X_n.

(c) Show that $s^2 = \mathrm{var}(X_t) = \sigma^2(1 - \varphi^2)^{-1}$.

(d) Show that $Y_t = X_t^2 - s^2$, $t \in \mathbb{Z}$, satisfies the AR(1) equation

$$Y_t = \varphi^2\, Y_{t-1} + \nu_t\,, \qquad t \in \mathbb{Z}\,,$$

for some white noise sequence (ν_t). Do these equations have a causal solution?

End of Examination Paper