Deadline 28 October.

(1) Recall the definition of a causal AR(1) process $X_t = \phi X_{t-1} + Z_t$ for (Z_t) iid with mean zero and variance $\sigma^2 < \infty$ and of the Yule-Walker estimator $\widehat{\phi}$ of the parameter ϕ.

(a) A natural predictor of the unknown (not yet observed) X_{n+1} is given by the quantity

$$\widehat{X}_{n+1} = E(X_{n+1} \mid X_1, \ldots, X_n) .$$

We proved in the Notes that

$$\widehat{X}_{n+1} = E(X_{n+1} \mid X_n) = \phi X_n$$

and that $\widehat{X}_{n+1}$ is the best approximation of X_{n+1} in the sense that it minimizes $E(X_{n+1} - f(X_1, \ldots, X_n))^2$ in the class of all (measurable) functions f of the data $X_1, \ldots, X_n$ with the property $Ef^2(X_1, \ldots, X_n) < \infty$. Argue that the "surrogate predictor" $\widetilde{X}_{n+1} = \widehat{\phi} X_n$ it is not the best approximation of X_{n+1} in the aforementioned sense.

(b) Simulate a sample of size 1000 of the process $X_t = 0.9X_{t-1} + Z_t$ with (Z_t) iid Student t-distributed with 10 degrees of freedom. Calculate the "surrogate predictor" $\widetilde{X}_{k+1} = \widehat{\phi}_k X_k$ for $k = 901, \ldots, 1000$, where $\widehat{\phi}_k$ is the Yule-Walker estimator based on the sample $X_1, \ldots, X_k$. Plot X_k and $\widetilde{X}_k$, $k = 901, \ldots, 1000$ in one graph.

(2) Recall that a stationary GARCH(1,1) process is given by the equations

$$\begin{aligned} X_t &= \sigma_t Z_t , \quad (Z_t) \text{ iid white noise with } \mathrm{var}(Z_0) = 1 \\ \sigma_t^2 &= \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2 , \qquad t \in \mathbb{Z} . \end{aligned}$$

(a) Either you find a program in R for the simulation of a GARCH(1,1) process (X_t) with iid standard normal noise (Z_t), $\alpha_0 = 10^{-6}$, $\alpha_1 = 0.9$, $\beta_1 = 0.1$, OR you write such a program yourself. In the latter case, choose the initial value $\sigma_0 = 0$.[1] Simulate a sample of size $n = 1000$ and show (X_t) and (σ_t) in different graphs.

(b) For the simulated data from (a) give conditional distributional forecasts for X_t given X_{t-1}, $t = 2, \ldots, n$. Show 95% and 99% conditional forecast intervals together with the data.

(c) In R (e.g. the package *tseries*) one can find estimators of the GARCH parameters. If one does not want to use such a package one can exploit the ARMA structure of the squares (X_t^2) with noise $(\nu_t) = (X_t^2 - \sigma_t^2)$.

Simulate a sample $X_1, \ldots, X_{1000}$ from an ARCH(1) process $X_t = \sigma_1 Z_t$ with iid standard normal noise and $\sigma_t^2 = 10^{-6} + 0.5X_{t-1}^2$ (choose $X_0 = 0$.) Estimate the parameter $\alpha_1 = 0.5$ by the Yule-Walker estimator. This means the following. The ARCH equalities imply that

$$(X_t^2 - \mathbb{E}[X_0^2]) - \alpha_1 (X_{t-1}^2 - \mathbb{E}[X_0^2]) = X_t^2 - \sigma_t^2 = \nu_t ,$$

where (ν_t) is white noise. Hence $Y_t = X_t^2 - \mathbb{E}[X_0^2]$, $t \in \mathbb{Z}$, has interpretation as an AR(1) process. Use the Yule-Walker estimator for α_1. Since $\mathbb{E}[X_0^2]$ is not known

[1] In software, one uses a burn-in period, i.e., one starts at $\sigma_{-1000} = 0$, calculates $(\sigma_s)_{s \le 0}$ and uses the resulting σ_0 as initial value. The reason for this is that $(\sigma_t)_{t \ge 1}$ is "closer" to a stationary GARCH process.

estimate it by $\gamma_{n,X}(0)$. Repeat the estimation procedure 100 times and show the results in a boxplot.

(d) The GARCH(1,1) process with standard normal noise is known to be heavy-tailed in the following sense. Let $\kappa > 0$ be the unique solution to the equation $E[(\alpha_1 Z_0^2 + \beta_1)^{\kappa/2}] = 1$. Then

$$x^\kappa P(\sigma_0 > x) \to c_0 \quad \text{and} \quad x^\kappa P(|X_0| > x) \to c_1$$

for some positive constants c_0, c_1.
Consider the model from (a) and determine the corresponding value $\kappa > 0$.

(e) The value κ from (c) can be estimated from data. A classical estimator is the Hill estimator. For a sample of positive observations $Y_1, \ldots, Y_n$ let $Y_{(1)} \leq \cdots \leq Y_{(n)}$ be the order statistics. Define the Hill estimator of κ as

$$\kappa_n^{(k)} = \Big(k^{-1} \sum_{i=1}^{k} \log(Y_{(n-i+1)}/Y_{(n-k)}) \Big)^{-1}, \quad k = 1, 2, \ldots, n-1\,.$$

The properties of the estimator $\kappa_n^{(k)}$ depend very much on the value k. This value has to be chosen "small" in comparison to n, but not "too small", in order to get a consistent estimator of κ. Simulate a sample of size $n = 10,000$ from σ_t for the model from (a). Calculate the corresponding Hill estimators $\kappa_n^{(k)}$, $k = 50, \ldots, 500$, and plot $\kappa_n^{(k)}$ against k (so-called Hill plot). An estimate of κ can be read off from this Hill plot where the graph is "relatively stable (constant)". Show the theoretical value κ in the same plot.

(f) Make a Hill plot for the S&P 500 returns from web.math.ku.dk/∼mikosch (OR for a return series of speculative prices of your own choice, but this series should have more than 1000 observations) for $k = 10, \ldots, 0.1 \times n$ separately for the gains $X_i > 0$ and the losses $X_i < 0$ multiplied by -1. Guess possible values of the tail parameters κ for the gains and the losses.