Written Examination: Statistical Analysis of Econometric Time Series
November 6, 2020
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- This is an open book examination. You may use any book, the lecture notes, assignments and their solutions, pocket calculator, computer, etc. In the solution to each problem you have to give reasons either by providing a proof or by referring to a result in the lecture notes, the assignments, any book, etc. In the latter case, give an *exact* reference.
- Use your time in a reasonable way. Do not copy the text of the problems. Refer to the lecture notes whenever possible. Do not re-prove results from the lecture notes.
- You may write with a pencil.
- This examination paper consists of 2 pages.
- You may write in English or Danish.
- The distribution of the points over the problems is as follows.

Problem	1 a	b	c	2 a	b	c	d	3 a	b	c	d	sum
# points	2	2	1	3	2	2	2	2	3	1	2	22

1. Consider the ARMA$(2,1)$ equations

$$X_t + 2X_{t-1}/3 + X_{t-2}/9 = Z_t + 0.5\, Z_{t-1}\,, \qquad t \in \mathbb{Z}\,,$$

where (Z_t) is iid white noise with variance σ^2.

(a) Show that these ARMA(2,1) equations have a causal stationary solution (X_t). Is the solution also invertible?

(b) Which of the following properties does the time series $(\cos(X_t))$ have?

(i) Strict stationarity

(ii) Ergodicity

(c) Determine the spectral density of (X_t).

Give arguments for each of your answers.

2. Consider two mutually independent stationary processes (X_t) and (Y_t) with mean zero, i.e., $\mathbb{E}[X_t] = \mathbb{E}[Y_t] = 0$ for all $t \in \mathbb{Z}$, and autovariance functions γ_X and γ_Y, respectively.

(a) Show that the time series $Z_t = X_t Y_t$, $t \in \mathbb{Z}$, constitutes a stationary time series and determine its autocovariance function γ_Z.

(b) Assume that $\sum_{h=0}^{\infty} |\gamma_X(h)\gamma_Y(h)| < \infty$. Argue that (Z_t) has a spectral density and give an expression of this spectral density.

(c) Consider the special case that $X_t = X$ for all $t \in \mathbb{Z}$ and an $N(0,1)$-distributed random variable X which is independent of (Y_t) satisfying $\sum_{h=0}^{\infty} |\gamma_Y(h)| < \infty$. Argue that (Z_t) and (Y_t) have the same spectral density.

(d) Consider the situation of (c) and assume that (Y_t) is strictly stationary ergodic and has finite variance. Argue that (Z_t) cannot be ergodic.

3. We consider a strictly stationary ARCH(2) process

$$X_t = \sigma_t \, Z_t \,, \qquad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \alpha_2 X_{t-2}^2 \,, \qquad t \in \mathbb{Z} \,,$$

for iid standard normal noise (Z_t). We assume that we observed the sample $X_1, \ldots, X_n$, $n \geq 3$. The best predictor $\widehat{W}$ of a random variable W, given the sample, is given by

$$\widehat{W} = \mathbb{E}(W \mid X_1, \ldots, X_n) \,.$$

The prediction error of $\widehat{W}$ is given by $\mathbb{E}[(\widehat{W} - W)^2]$.

(a) Verify that the condition $\alpha_1 + \alpha_2 < 1$ is sufficient for $\mathbb{E}[\sigma_0^2] < \infty$ and for stationarity of (X_t). Calculate $\mathbb{E}[\sigma_0^2]$.

(b) Calculate the best predictors of X_{n+1}, X_{n+1}^2 and σ_{n+1}^2 given the sample $X_1, \ldots, X_n$.

(c) Determine the prediction error of $\widehat{X}_{n+1}$ from (b) under the assumption $\alpha_1 + \alpha_2 < 1$.

(d) Let q_β be the β-quantile of the standard normal distribution function Φ, i.e., $\Phi(q_\beta) = \beta$. For $\alpha \in (0, 0.5)$ calculate the best predictor of the indicator function

$$\mathbf{1}_{[q_\alpha \sigma_{n+1}, q_{1-\alpha}\sigma_{n+1}]}(X_{n+1}) = \begin{cases} 1 & \text{if } X_{n+1} \in [q_\alpha \sigma_{n+1}, q_{1-\alpha}\sigma_{n+1}] \\ 0 & \text{otherwise}\,. \end{cases}$$

given the sample $X_1, \ldots, X_n$.

End of Examination Paper

Solutions

1. Consider the ARMA$(2,1)$ equations

$$\varphi(B)X_t = X_t + 2\,X_{t-1}/3 + X_{t-2}/9 = Z_t + 0.5\,Z_{t-1} = \theta(B)Z_t\,, \qquad t \in \mathbb{Z}\,,$$

where (Z_t) is iid white noise with variance σ^2

(a) 2 points Then

$$\varphi(z) = 1 + 2z/3 + z^2/9\,, \qquad \theta(z) = 1 + 0.5\,z\,.$$

The polynomial $\varphi(z) = (1 + z/3)^2$ has root $z = -3$ which is outside the unit circle. Therefore there is a causal solution to the ARMA(2,1) equations. The polynomial θ has root $z = -2$, i.e., it is outside the unit circle. The polynomials do not have joint roots. Therefore these ARMA(2,1) equations have a stationary causal invertible solution (X_t).

(b) 2 points A causal ARMA process has representation as a linear process:

$$X_t = \frac{\theta(B)}{\varphi(B)} Z_t = \sum_{i=0}^{\infty} \psi_i\, Z_{t-i} = g(Z_t, Z_{t-1}, \ldots)\,, \qquad t \in \mathbb{Z}\,.$$

Since (Z_t) is iid, hence strictly stationary and ergodic, (X_t) inherits these properties. Similarly, $h(X_t) = \cos(X_t)$ inherits these properties from (X_t).

(c) 1 point The spectral density of (X_t) is given by

$$f_X(\lambda) = \frac{\sigma^2}{2\pi}\frac{|\theta(\mathrm{e}^{-i\lambda}|^2}{|\varphi(\mathrm{e}^{-i\lambda})|^2} = \frac{\sigma^2}{2\pi}\frac{|1 + 0.5\mathrm{e}^{-i\lambda}|^2}{|(1 + \mathrm{e}^{-i\lambda}/3)^2|^2} = \frac{\sigma^2}{2\pi}\frac{|1 + 0.5\mathrm{e}^{-i\lambda}|^2}{|1 + \mathrm{e}^{-i\lambda}/3|^4}\,, \qquad \lambda \in [0, \pi]\,.$$

2. Consider two mutually independent stationary processes (X_t) and (Y_t) with mean zero, i.e., $\mathbb{E}[X_t] = \mathbb{E}[Y_t] = 0$ for all $t \in \mathbb{Z}$, and autovariance functions γ_X and γ_Y, respectively.

(a) 3 points We have by independence of X_t and Y_t,

$$\mathbb{E}[Z_t] = \mathbb{E}[X_t]\,\mathbb{E}[Y_t] = 0\,, \quad t \in \mathbb{Z}\,.$$

Moreover,

$$\mathbb{E}[Z_t^2] = \mathbb{E}[X_t^2]\,\mathbb{E}[Y_t^2] < \infty\,, \qquad t \in \mathbb{Z}\,,$$

since both X_t and Y_t have finite variance by the stationarity assumption. We also have by independence,

$$\begin{aligned}
\gamma_Z(h) &= \mathrm{cov}(Z_t, Z_{t+h}) \\
&= \mathbb{E}[(X_tY_t)(X_{t+h}Y_{t+h})] - (\mathbb{E}[X_tY_t]\,\mathbb{E}[X_{t+h}Y_{t+h}])^2 \\
&= \mathbb{E}[X_tX_{t+h}]\mathbb{E}[Y_tY_{t+h}] - (\mathbb{E}[X_t]\mathbb{E}[Y_t]\,\mathbb{E}[X_{t+h}]\mathbb{E}[Y_{t+h}])^2 \\
&= \gamma_X(h)\,\gamma_Y(h)\,.
\end{aligned}$$

(b) 2 points Since $\sum_{h=0}^{\infty} |\gamma_X(h)\gamma_Y(h)| = \sum_{h=0}^{\infty} |\gamma_Z(h)| < \infty$ we know that the spectral density of (Z_t) exists and is given by

$$f_Z(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \mathrm{e}^{-i\lambda h} \gamma_Z(h) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \mathrm{e}^{-i\lambda h} \gamma_X(h)\gamma_Y(h) . \tag{0.1}$$

(c) 2 points Assume $X_t = X$ for all $t \in \mathbb{Z}$ and an $N(0,1)$-distributed random variable X which is independent of (Y_t). Then $\gamma_X(h) = \mathbb{E}[X^2] = 1$ for any h, hence $\gamma_X(h)\gamma_Y(h) = \gamma_Y(h)$. Then (**??**) turns into

$$f_Z(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \mathrm{e}^{-i\lambda h} \gamma_Y(h) = f_Y(\lambda) .$$

Hence (Z_t) and (Y_t) have the same spectral density.

(d) 2 points Now consider the situation of (c) and assume that (Y_t) is strictly stationary ergodic and has finite variance. Then $(Z_t) = X\,(Y_t)$ and the ergodic theorem fails e.g. for (Z_t^2):

$$\frac{1}{n} \sum_{t=1}^{n} Z_t^2 = X^2 \frac{1}{n} \sum_{t=1}^{n} Y_t^2 \stackrel{\text{a.s.}}{\to} X^2\, \mathbb{E}[Y_0^2] , \quad n \to \infty .$$

The right-hand side is finite and is random but does not equal the constant $\mathbb{E}[Z_0^2] = \mathbb{E}[X^2]\mathbb{E}[Y_0^2]$ which is required by the ergodic theorem.

3. We consider a strictly stationary ARCH(2) process

$$X_t = \sigma_t\, Z_t , \qquad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \alpha_2 X_{t-2}^2 , \qquad t \in \mathbb{Z} ,$$

for iid standard normal noise (Z_t). We assume that we observed the sample $X_1, \dots, X_n$, $n \geq 3$.

(a) 2 points We have to verify that $\mathbb{E}[X_0^2] = \mathbb{E}[\sigma_0^2] < \infty$. Then stationarity follows from strict stationarity. We have

$$\mathbb{E}[\sigma_0^2] = \alpha_0 + (\alpha_1 + \alpha_2)\mathbb{E}[\sigma_0^2] .$$

Therefore

$$\mathbb{E}[\sigma_0^2] = \frac{\alpha_0}{1 - (\alpha_1 + \alpha_2)}$$

provided $\alpha_1 + \alpha_2 < 1$.

(b) 3 points Since σ_{n+1} is independent of Z_{n+1} and a function of X_n, X_{n-1} the best predictors of X_{n+1}, X_{n+1}^2 and σ_{n+1}^2, given the sample $X_1, \dots, X_n$, are respectively given by

$$
\begin{aligned}
\widehat{X}_{n+1} &= \mathbb{E}\big(X_{n+1} \mid X_1, \ldots, X_n\big) \\
&= \mathbb{E}\big(X_{n+1} \mid X_{n-1}, X_n\big) \\
&= \sigma_{n+1}\, \mathbb{E}(Z_{n+1}) = 0\,, \\
\widehat{X^2_{n+1}} &= \mathbb{E}\big(X^2_{n+1} \mid X_1, \ldots, X_n\big) \\
&= \mathbb{E}\big(X^2_{n+1} \mid X_{n-1}, X_n\big) \\
&= \sigma^2_{n+1}\, \mathbb{E}(Z^2_{n+1}) = \sigma^2_{n+1}\,, \\
\widehat{\sigma^2_{n+1}} &= \mathbb{E}\big(\sigma^2_{n+1} \mid X_1, \ldots, X_n\big) \\
&= \mathbb{E}\big(\sigma^2_{n+1} \mid X_{n-1}, X_n\big) \\
&= \sigma^2_{n+1}\,.
\end{aligned}
$$

(b)1 point In view of (a) the prediction error of $\widehat{X}_{n+1}$ is given by

$$\mathbb{E}[(\widehat{X}_{n+1} - X_{n+1})^2] = \mathbb{E}[X^2_{n+1}] = \mathbb{E}[\sigma_0^2] = \frac{\alpha_0}{1-(\alpha_1+\alpha_2)}\,.$$

(c) 2 points Let q_β be the β-quantile of the standard normal distribution function Φ , i.e., $\Phi(q_\beta) = \beta$. For $\alpha \in (0, 0.5)$ the best predictor of the indicator function

$$\mathbf{1}_{[q_\alpha\sigma_{n+1}, q_{1-\alpha}\sigma_{n+1}]}(X_{n+1}) = \begin{cases} 1 & \text{if } X_{n+1} \in [q_\alpha\sigma_{n+1}, q_{1-\alpha}\sigma_{n+1}]\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

given the sample $X_1, \ldots, X_n$, is

$$
\begin{aligned}
\mathbb{E}\big(\mathbf{1}_{[q_\alpha\sigma_{n+1}, q_{1-\alpha}\sigma_{n+1}]}(X_{n+1}) \mid X_1, \ldots, X_n\big) &= \mathbb{P}\big(X_{n+1} \in [q_\alpha\sigma_{n+1}, q_{1-\alpha}\sigma_{n+1}] \mid X_1, \ldots, X_n\big) \\
&= \mathbb{P}\big(X_{n+1} \in [q_\alpha\sigma_{n+1}, q_{1-\alpha}\sigma_{n+1}] \mid X_{n-1}, X_n\big) \\
&= \mathbb{P}\big(Z_{n+1} \in [q_\alpha, q_{1-\alpha}]\big) \\
&= \Phi(q_{1-\alpha}) - \Phi(q_\alpha) = 1 - 2\alpha\,.
\end{aligned}
$$