Solutions to Assignment 5 for StatØk2, Block 1, 2020/2021 by Jorge Yslas

(1) a) σ_t is finite a.s. if $\log \sigma_t$ is finite. But $\log \sigma_t = \sum_{j=0}^{\infty} \psi_j \eta_{t-j}$ is an infinite series of independent random variables. A consequence of Khinchin-Kolmogorov convergence theorem is that $\sum_{j=0}^{\infty} \mathrm{var}(\psi_j \eta_{t-j}) = \sum_{j=0}^{\infty} \psi_j^2 \cdot 1 < \infty$ implies the a.s. convergence of this infinite series.

b) Since (η_t) and (Z_t) are iid sequences then both are strictly stationary. Note that

$$\sigma_t = e^{\sum_{j=0}^{\infty} \psi_j \eta_{t-j}} = g(\eta_t, \eta_{t-1}, \dots) .$$

Therefore (σ_t) is also strictly stationary. We have that

$$X_t = \sigma_t Z_t$$

then, exploiting the mutual independence of (σ_t) and (Z_t),

$$\begin{aligned}
\begin{pmatrix} X_{t+1} \\ \vdots \\ X_{t+d} \end{pmatrix} &= \begin{pmatrix} \sigma_{t+1} Z_{t+1} \\ \vdots \\ \sigma_{t+d} Z_{t+d} \end{pmatrix} \\
&= \begin{pmatrix} \sigma_{t+1} & 0 & \dots & 0 \\ 0 & \sigma_{t+2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_{t+d} \end{pmatrix} \begin{pmatrix} Z_{t+1} \\ \vdots \\ Z_{t+d} \end{pmatrix} \\
&\stackrel{d}{=} \begin{pmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_d \end{pmatrix} \begin{pmatrix} Z_1 \\ \vdots \\ Z_d \end{pmatrix} \\
&= \begin{pmatrix} X_1 \\ \vdots \\ X_d \end{pmatrix} .
\end{aligned}$$

Therefore (X_t) is strictly stationary.

c) Using the mutual independence of (σ_t) and (Z_t), we have that

$$\mathbb{E}\left[X_0\right] = \mathbb{E}\left[\sigma_0 Z_0\right] = \mathbb{E}\left[\sigma_0\right] \mathbb{E}\left[Z_0\right] = 0$$

and since $\mathbb{E}[e^{s\eta_1}] = e^{s^2/2}$ for an $N(0,1)$ variable η_1,

$$\begin{aligned}
\mathbb{E}\left[X_0^2\right] &= \mathbb{E}\left[\sigma_0^2 Z_0^2\right] = \mathbb{E}\left[\sigma_0^2\right] \mathbb{E}\left[Z_0^2\right] \\
&= \mathbb{E}\left[\sigma_0^2\right] = \mathbb{E}\left[e^{2\sum_{j=0}^{\infty} \psi_j \eta_{-j}}\right] \\
&= \mathbb{E}\left[\lim_{n\to\infty} e^{2\sum_{j=0}^{n} \psi_j \eta_{-j}}\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[e^{2\sum_{j=0}^{n} \psi_j \eta_{-j}}\right] \\
&= \lim_{n\to\infty} \prod_{j=0}^{n} \mathbb{E}\left[e^{2\psi_j \eta_{-j}}\right] \\
&= \lim_{n\to\infty} \prod_{j=0}^{n} e^{2\psi_j^2}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} e^{2\sum_{j=0}^{n}\psi_j^2} \\
&= e^{2\sum_{j=0}^{\infty}\psi_j^2} .
\end{aligned}
$$

Therefore

$$\operatorname{var}(X_0) = e^{2\sum_{j=0}^{\infty}\psi_j^2} .$$

d) Observe that the solution to the equation

$$\log\sigma_t = \phi\log\sigma_{t-1} + \eta_t$$

has representation

$$\log\sigma_t = \sum_{j=0}^{\infty}\phi^j\eta_{t-j}$$

and

$$\sum_{j=0}^{\infty}\left(\phi^j\right)^2 = \sum_{j=0}^{\infty}\phi^{2j} = \left(1-\phi^2\right)^{-1} .$$

We also know that

$$\rho_{\log\sigma_X}(h) = \phi^{|h|} .$$

We first compute $\rho_X(h)$. From c) we know that $\mathbb{E}\left[X_0\right] = 0$. Then

$$\gamma_X(h) = \mathbb{E}\left[X_tX_{t+h}\right] = \mathbb{E}\left[\sigma_t\sigma_{t+h}Z_tZ_{t+h}\right] = \mathbb{E}\left[\sigma_t\sigma_{t+h}\right]\mathbb{E}\left[Z_tZ_{t+h}\right] = \begin{cases} 0 & h\neq 0\,, \\ \mathbb{E}\left[\sigma_0^2\right] & h=0\,, \end{cases}$$

which implies that

$$\rho_X(h) = \begin{cases} 0 & h\neq 0\,, \\ 1 & h=0\,. \end{cases}$$

We now compute $\rho_{|X|}(h)$. We have

$$
\begin{aligned}
\mathbb{E}\left[|X_0|\right] &= \mathbb{E}\left[|\sigma_0Z_0|\right] = \mathbb{E}\left[|\sigma_0|\right]\mathbb{E}\left[|Z_0|\right] \\
&= \mathbb{E}\left[|Z_0|\right]\mathbb{E}\left[e^{\sum_{j=0}^{\infty}\phi^j\eta_{-j}}\right] \\
&= \mathbb{E}\left[|Z_0|\right]e^{\frac{1}{2}\sum_{j=0}^{\infty}\phi^{2j}} \\
&= \mathbb{E}\left[|Z_0|\right]e^{\frac{1}{2}\left(1-\phi^2\right)^{-1}}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}\left[|X_0|^2\right] &= \mathbb{E}\left[|\sigma_0Z_0|^2\right] = \mathbb{E}\left[\sigma_0^2\right]\mathbb{E}\left[Z_0^2\right] \\
&= \mathbb{E}\left[e^{2\sum_{j=0}^{\infty}\phi^j\eta_{-j}}\right] \\
&= e^{2\sum_{j=0}^{\infty}\phi^{2j}} \\
&= e^{2\left(1-\phi^2\right)^{-1}} .
\end{aligned}
$$

Then

$$
\begin{aligned}
\gamma_{|X|}(0) &= \operatorname{var}(|X_0|) \\
&= e^{\frac{2}{1-\phi^2}} - (\mathbb{E}\left[|Z_0|\right])^2 e^{\frac{1}{1-\phi^2}} \\
&= e^{\frac{1}{1-\phi^2}} \left(e^{\frac{1}{1-\phi^2}} - (\mathbb{E}\left[|Z_0|\right])^2 \right) .
\end{aligned}
$$

Now consider (with $h > 0$)

$$
\begin{aligned}
\mathbb{E}\left[|X_0||X_h|\right] &= \mathbb{E}\left[|\sigma_0 Z_0||\sigma_h Z_h|\right] \\
&= \mathbb{E}\left[|Z_0||Z_h|\right] \mathbb{E}\left[\sigma_0 \sigma_h\right] \\
&= (\mathbb{E}\left[|Z_0|\right])^2 \mathbb{E}\left[e^{\sum_{j=0}^{\infty} \phi^j \eta_{-j}} e^{\sum_{j=0}^{\infty} \phi^j \eta_{h-j}} \right] \\
&= (\mathbb{E}\left[|Z_0|\right])^2 \mathbb{E}\left[e^{\sum_{j=0}^{\infty} \phi^j \eta_{-j} + \sum_{j=0}^{\infty} \phi^j \eta_{h-j}} \right] \\
&= (\mathbb{E}\left[|Z_0|\right])^2 \mathbb{E}\left[e^{\sum_{j=0}^{\infty} \phi^j \eta_{-j} + \sum_{j=0}^{h-1} \phi^j \eta_{h-j} + \sum_{j=h}^{\infty} \phi^j \eta_{h-j}} \right] \\
&= (\mathbb{E}\left[|Z_0|\right])^2 \mathbb{E}\left[e^{\sum_{j=0}^{\infty} \phi^j \eta_{-j} + \sum_{j=0}^{h-1} \phi^j \eta_{h-j} + \phi^h \sum_{j=0}^{\infty} \phi^j \eta_{-j}} \right] \\
&= (\mathbb{E}\left[|Z_0|\right])^2 \mathbb{E}\left[e^{\left(1+\phi^h\right) \sum_{j=0}^{\infty} \phi^j \eta_{-j} + \sum_{j=0}^{h-1} \phi^j \eta_{h-j}} \right] \\
&= (\mathbb{E}\left[|Z_0|\right])^2 \mathbb{E}\left[e^{\left(1+\phi^h\right) \sum_{j=0}^{\infty} \phi^j \eta_{-j}} \right] \mathbb{E}\left[e^{\sum_{j=0}^{h-1} \phi^j \eta_{h-j}} \right] \\
&= (\mathbb{E}\left[|Z_0|\right])^2 e^{\frac{1}{2}\left(1+\phi^h\right)^2 \sum_{j=0}^{\infty} \phi^{2j}} e^{\frac{1}{2} \sum_{j=0}^{h-1} \phi^{2j}} \\
&= (\mathbb{E}\left[|Z_0|\right])^2 e^{\frac{1}{2}\left(1+\phi^h\right)^2 \left(1-\phi^2\right)^{-1}} e^{\frac{1}{2}\left(1-\phi^{2h}\right)\left(1-\phi^2\right)^{-1}} \\
&= (\mathbb{E}\left[|Z_0|\right])^2 e^{\frac{\left(1+\phi^h\right)^2}{2\left(1-\phi^2\right)} + \frac{1-\phi^{2h}}{2\left(1-\phi^2\right)}} \\
&= (\mathbb{E}\left[|Z_0|\right])^2 e^{\frac{1+2\phi^h+\phi^{2h}+1-\phi^{2h}}{2\left(1-\phi^2\right)}} \\
&= (\mathbb{E}\left[|Z_0|\right])^2 e^{\frac{1+\phi^h}{1-\phi^2}} .
\end{aligned}
$$

Thus

$$
\begin{aligned}
\gamma_{|X|}(h) &= (\mathbb{E}\left[|Z_0|\right])^2 e^{\frac{1+\phi^h}{1-\phi^2}} - (\mathbb{E}\left[|Z_0|\right])^2 e^{\frac{1}{1-\phi^2}} \\
&= (\mathbb{E}\left[|Z_0|\right])^2 e^{\frac{1}{1-\phi^2}} \left(e^{\frac{\phi^h}{1-\phi^2}} - 1 \right) .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\rho_{|X|}(h) &= \frac{\gamma_{|X|}(h)}{\gamma_{|X|}(0)} \\
&= \frac{(\mathbb{E}\left[|Z_0|\right])^2 e^{\frac{1}{1-\phi^2}} \left(e^{\frac{\phi^h}{1-\phi^2}} - 1 \right)}{e^{\frac{1}{1-\phi^2}} \left(e^{\frac{1}{1-\phi^2}} - (\mathbb{E}\left[|Z_0|\right])^2 \right)} \\
&= \frac{(\mathbb{E}\left[|Z_0|\right])^2}{e^{\frac{1}{1-\phi^2}} - (\mathbb{E}\left[|Z_0|\right])^2} \left(e^{\frac{\phi^h}{1-\phi^2}} - 1 \right) \\
&= \tilde{c} \left(e^{\frac{\phi^h}{1-\phi^2}} - 1 \right) ,
\end{aligned}
$$

where

$$\tilde{c} = \frac{(\mathbb{E}[|Z_0|])^2}{e^{\frac{1}{1-\phi^2}} - (\mathbb{E}[|Z_0|])^2}\,.$$

Using Taylor expansion of e^x, we have that

$$\rho_{|X|}(h) = \tilde{c}\left(1 + \frac{\phi^h}{1-\phi^2} + o(\phi^h) - 1\right) = \tilde{c}\left(\frac{\phi^h}{1-\phi^2} + o(\phi^h)\right)$$

and consequently

$$\rho_{|X|}(h)/\rho_{\log\sigma_X}(h) = \frac{\tilde{c}\left(\frac{\phi^h}{1-\phi^2} + o(\phi^h)\right)}{\phi^h} \to \frac{\tilde{c}}{1-\phi^2} = c\,,\quad h \to \infty\,.$$

(2) a) Note that

$$\begin{aligned}\nu_t &= X_t^2 - \sigma_t^2\\ &= \sigma_t^2 Z_t^2 - \sigma_t^2\\ &= \sigma_t^2\left(Z_t^2 - 1\right)\,.\end{aligned}$$

We now show the properties of white noise.
i)

$$\mathbb{E}[\nu_t] = \mathbb{E}\left[\sigma_t^2\left(Z_t^2-1\right)\right] = \mathbb{E}\left[\sigma_t^2\right]\mathbb{E}\left[\left(Z_t^2-1\right)\right] = 0\,.$$

ii)

$$\begin{aligned}\mathbb{E}\left[\nu_t^2\right] &= \mathbb{E}\left[\sigma_t^4\left(Z_t^2-1\right)^2\right]\\ &= \mathbb{E}\left[\sigma_t^4\right]\mathbb{E}\left[\left(Z_t^2-1\right)^2\right]\\ &= \mathbb{E}\left[\sigma_0^4\right]\mathbb{E}\left[\left(Z_t^4 - 2Z_t^2 + 1\right)\right]\\ &= \mathbb{E}\left[\sigma_0^4\right]\left(\mathbb{E}\left[Z_0^4\right] - 1\right)]\,.\end{aligned}$$

iii) Assume $h > 0$. Since Z_{t+h} is independent of $(\sigma_{t+h}, Z_t, \sigma_t)$,

$$\begin{aligned}\mathbb{E}[\nu_t\nu_{t+h}] &= \mathbb{E}\left[\sigma_t^2\left(Z_t^2-1\right)\sigma_{t+h}^2\left(Z_{t+h}^2-1\right)\right]\\ &= \mathbb{E}\left[\left(Z_{t+h}^2-1\right)\right]\mathbb{E}\left[\sigma_t^2\left(Z_t^2-1\right)\sigma_{t+h}^2\right]\\ &= 0\,.\end{aligned}$$

Therefore (ν_t) is white noise.

b) (X_t) Martingale difference.
i)

$$\mathbb{E}[|X_t|] = \mathbb{E}[|\sigma_t Z_t|] = \mathbb{E}[|\sigma_t|]\,\mathbb{E}[|Z_t|] = \mathbb{E}[|\sigma_0|]\,\mathbb{E}[|Z_0|] < \infty\,.$$

ii) We have that

$$\sigma_t = f(Z_{t-1}, Z_{t-2}, \dots)\,.$$

Then

$$X_t = \sigma_t Z_t = g(Z_t, Z_{t-1}, \dots)$$

from where it follows that X_t is $\mathcal{F}_t$ measurable.
iii)

$$\mathbb{E}\left[X_t|\mathcal{F}_{t-1}\right] = \mathbb{E}\left[\sigma_t Z_t|\mathcal{F}_{t-1}\right] = \mathbb{E}\left[Z_t\right]\sigma_t = 0\,.$$

$\left(X_t^2\right)$ is not a Martingale difference.
iii)

$$\mathbb{E}\left[X_t^2|\mathcal{F}_{t-1}\right] = \mathbb{E}\left[\sigma_t^2 Z_t^2|\mathcal{F}_{t-1}\right] = \sigma_t^2\mathbb{E}\left[Z_t^2\right] = \sigma_t^2\,.$$

If we take $\nu_t = X_t^2 - \sigma_t^2$ then (ν_t) is a martingale difference with respect to $\mathcal{F}_t$.

c) Consider first the case $\alpha_1 + \beta_1 = 1$. Suppose that the variance of σ is finite. Since

$$\sigma_t^2 = \alpha_0 + \sigma_{t-1}^2\left(\alpha_1 Z_{t-1}^2 + \beta_1\right)\,,$$

we have by stationarity of (σ_t) that

$$\begin{aligned}\mathbb{E}\left[\sigma_t^2\right] &= \mathbb{E}\left[\alpha_0 + \sigma_{t-1}^2\left(\alpha_1 Z_{t-1}^2 + \beta_1\right)\right]\\ &= \alpha_0 + \mathbb{E}\left[\sigma_{t-1}^2\right](\alpha_1 + \beta_1)\\ &= \alpha_0 + \mathbb{E}\left[\sigma_t^2\right]\,.\end{aligned}$$

Implying that $\alpha_0 = 0$, which is a contradiction to the assumption that $\alpha_0 > 0$.
Similarly, for the case $\alpha_1 + \beta_1 > 1$, suppose that the variance of σ is finite. Since

$$\sigma_t^2 = \alpha_0 + \sigma_{t-1}^2\left(\alpha_1 Z_{t-1}^2 + \beta_1\right)\,,$$

we have that

$$\begin{aligned}\mathbb{E}\left[\sigma_t^2\right] &= \mathbb{E}\left[\alpha_0 + \sigma_{t-1}^2\left(\alpha_1 Z_{t-1}^2 + \beta_1\right)\right]\\ &= \alpha_0 + \mathbb{E}\left[\sigma_{t-1}^2\right](\alpha_1 + \beta_1)\,.\end{aligned}$$

Then

$$\mathbb{E}\left[\sigma_t^2\right](1 - \alpha_1 - \beta_1) = \alpha_0\,.$$

Implying that $\alpha_0 < 0$, which is a contradiction to the assumption that $\alpha_0 > 0$.

d) With $\kappa = 2$

$$\mathbb{E}\left[\left(\alpha_1 Z_0^2 + \beta_1\right)^{\kappa/2}\right] = \mathbb{E}\left[\alpha_1 Z_0^2 + \beta_1\right] = \alpha_1 + \beta_1 = 1\,.$$

(3) a) We have that (X_t) satisfies the ARCH(1) equations

$$\begin{aligned}X_t &= \sigma_t Z_t\,,\\ \sigma_t^2 &= \alpha_0 + \alpha_1 X_{t-1}^2\,.\end{aligned}$$

Furthermore, we know that $\left(\sigma_t^2\right)$ has representation

$$\sigma_t^2 = \alpha_0 \sum_{j=-\infty}^{t} \prod_{k=j+1}^{t} \alpha_1 Z_{k-1}^2 = f(Z_{t-1}, Z_{t-2}, \dots)\,.$$

Note that

$$\mathbb{E}\left[|X_0|^p\right] = \mathbb{E}\left[|\sigma_0|^p |Z_0|^p\right] = \mathbb{E}\left[|\sigma_0|^p\right] \mathbb{E}\left[|Z_0|^p\right]\,.$$

Then $\mathbb{E}\left[|X_0|^p\right] < \infty$ if and only if $\mathbb{E}\left[|\sigma_0|^p\right] < \infty$ since $\mathbb{E}[|Z_0|^p] < \infty$ for $Z_0 \sim N(0,1)$. Thus $\mathbb{E}\left[X_0^4\right] < \infty$ if and only if $\mathbb{E}\left[\sigma_0^4\right] < \infty$. We have that

$$\begin{aligned}
\sigma_t^2 &= \alpha_0 + \alpha_1 X_{t-1}^2 \\
&= \alpha_0 + \alpha_1 \sigma_{t-1}^2 Z_{t-1}^2\,.
\end{aligned}$$

Hence by stationarity

$$\begin{aligned}
\mathbb{E}\left[\sigma_t^2\right] &= \mathbb{E}\left[\alpha_0 + \alpha_1 \sigma_{t-1}^2 Z_{t-1}^2\right] \\
&= \alpha_0 + \alpha_1 \mathbb{E}\left[\sigma_{t-1}^2 Z_{t-1}^2\right] \\
&= \alpha_0 + \alpha_1 \mathbb{E}\left[\sigma_t^2\right]\,.
\end{aligned}$$

Both sides are finite or infinite at the same time. Hence $\mathbb{E}[\sigma_t^2] < \infty$ if and only if it has representation

$$\mathbb{E}\left[\sigma_0^2\right] = \frac{\alpha_0}{1-\alpha_1},$$

and this is possible if and only if $\alpha_1 < 1$ since $\alpha_0 > 0$. Now,

$$\begin{aligned}
\sigma_t^4 &= \left(\alpha_0 + \alpha_1 \sigma_{t-1}^2 Z_{t-1}^2\right)^2 \\
&= \alpha_0^2 + 2\alpha_0\alpha_1 \sigma_{t-1}^2 Z_{t-1}^2 + \alpha_1^2 \sigma_{t-1}^4 Z_{t-1}^4
\end{aligned}$$

and by stationarity,

$$\begin{aligned}
\mathbb{E}\left[\sigma_t^4\right] &= \mathbb{E}\left[\alpha_0^2 + 2\alpha_0\alpha_1 \sigma_{t-1}^2 Z_{t-1}^2 + \alpha_1^2 \sigma_{t-1}^4 Z_{t-1}^4\right] \\
&= \alpha_0^2 + 2\alpha_0\alpha_1 \mathbb{E}\left[\sigma_{t-1}^2 Z_{t-1}^2\right] + \alpha_1^2 \mathbb{E}\left[\sigma_{t-1}^4 Z_{t-1}^4\right] \\
&= \alpha_0^2 + 2\alpha_0^2 \left(\frac{\alpha_1}{1-\alpha_1}\right) + 3\alpha_1^2 \mathbb{E}\left[\sigma_{t-1}^4\right] \\
&= \alpha_0^2 \left(\frac{1+\alpha_1}{1-\alpha_1}\right) + 3\alpha_1^2 \mathbb{E}\left[\sigma_t^4\right]\,.
\end{aligned}$$

For finite $\mathbb{E}[\sigma_t^4]$ this equation holds if and only if

$$\mathbb{E}\left[\sigma_0^4\right] = \frac{\alpha_0^2 \left(\frac{1+\alpha_1}{1-\alpha_1}\right)}{1-3\alpha_1^2}\,.$$

Since we already know that $\alpha_1 < 1$ is necessary for $\mathbb{E}[\sigma_t^2] < \infty$ we need the even stronger condition $3\alpha_1^2 < 1$ for a finite 4th moment.

b) We have that (X_t) satisfies the ARCH(p) equations

$$X_t = \sigma_t Z_t\,,$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 \,.$$

Then

$$\begin{aligned}
Y_t &= \frac{X_t^2}{\alpha_0} \\
&= \frac{\sigma_t^2 Z_t^2}{\alpha_0} \\
&= \frac{Z_t^2}{\alpha_0} \left(\alpha_0 + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 \right) \\
&= Z_t^2 \left(1 + \sum_{i=1}^{p} \alpha_i \frac{X_{t-i}^2}{\alpha_0} \right) \\
&= Z_t^2 \left(1 + \sum_{i=1}^{p} \alpha_i Y_{t-i} \right) \,.
\end{aligned}$$

From this equation it follows that

$$\mathbb{E}\left[Y_0\right] = \frac{1}{1 - \sum_{i=1}^{p} \alpha_i} \,.$$

We also have that

$$\begin{aligned}
Y_t - \sum_{i=1}^{p} \alpha_i Y_{t-i} &= 1 + Y_t - \frac{\sigma_t^2}{\alpha_0} \\
&= 1 + \frac{X_t^2 - \sigma_t^2}{\alpha_0} \\
&= 1 + \tilde{\nu}_t \,,
\end{aligned}$$

where $\tilde{\nu}_t = (X_t^2 - \sigma_t^2)/\alpha_0$ is a white noise. Then

$$\begin{aligned}
(Y_t - \mathbb{E}\left[Y_0\right]) - \sum_{i=1}^{p} \alpha_i (Y_{t-i} - \mathbb{E}\left[Y_0\right]) &= 1 - \mathbb{E}\left[Y_0\right] + \mathbb{E}\left[Y_0\right] \sum_{i=1}^{p} \alpha_i + \tilde{\nu}_t \\
&= 1 - \mathbb{E}\left[Y_0\right] \left(1 - \sum_{i=1}^{p} \alpha_i \right) + \tilde{\nu}_t \\
&= 1 - \left(\frac{1}{1 - \sum_{i=1}^{p} \alpha_i} \right) \left(1 - \sum_{i=1}^{p} \alpha_i \right) + \tilde{\nu}_t \\
&= \tilde{\nu}_t \,.
\end{aligned}$$

Thus

$$(Y_t - \mathbb{E}\left[Y_0\right]) - \sum_{i=1}^{p} \alpha_i (Y_{t-i} - \mathbb{E}\left[Y_0\right]) = \tilde{\nu}_t \,.$$

Hence, $(Y_t - \mathbb{E}\left[Y_0\right])$ satisfies an AR(p) equation. Note also that

$$\operatorname{cov}((Y_t - \mathbb{E}\left[Y_0\right]), (Y_{t+h} - \mathbb{E}\left[Y_0\right])) = \operatorname{cov}(Y_t, Y_{t+h}) \,.$$

Therefore, (Y_t) has the same autocorrelation function as an AR(p) process.