Assignment 5 for StatØk2, Block 1, 2021/2022

(1) One of the standard models in financial econometrics is the Gaussian stochastic volatility model given by

$$X_t = \sigma_t\, Z_t\,, \qquad t \in \mathbb{Z}\,,$$

where (Z_t) is an iid centered unit variance sequence and (σ_t) is a strictly stationary sequence of positive random variables satisfying

$$\log \sigma_t = \sum_{j=0}^{\infty} \psi_j \eta_{t-j}\,, \qquad t \in \mathbb{Z}$$

where (η_t) is iid standard normal and (ψ_j) are real numbers such that $\sum_{j=0}^{\infty} \psi_j^2 < \infty$, and (η_t) and (Z_t) are mutually independent.

(a) Argue that σ_t is finite with probability 1 for all t.

(b) Show that (σ_t) and (X_t) are strictly stationary sequences.

(c) Determine EX_0, $\mathrm{var}(X_0)$.

From now on, we assume that $(\log \sigma_t)$ *satisfies the AR(1) equation*

$$\log \sigma_t = \phi \log \sigma_{t-1} + \eta_t\,, \qquad t \in \mathbb{Z}\,,$$

for some $\phi \in (-1, 1)$.

(d) Determine the ACFs ρ_X and $\rho_{|X|}$ and show that $\rho_{|X|}(h)/\rho_{\log \sigma X}(h) \to c$ as $h \to \infty$ for some positive constant c.

(2) Let (X_t) be a GARCH$(1,1)$ process, i.e., $X_t = \sigma_t Z_t$, (Z_t) is iid centered unit variance, $\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2$ for positive $\alpha_0, \alpha_1, \beta_1$. Assume that (σ_t) is strictly stationary, $E\sigma_0^4 < \infty$, $EZ_0^4 < \infty$ and that we know that $\sigma_t = f(Z_{t-1}, Z_{t-2}, \ldots)$ for some deterministic function f.

(a) Show that these conditions imply that $\nu_t = X_t^2 - \sigma_t^2$, $t \in \mathbb{Z}$, constitutes white noise.

(b) Show that (X_t) constitutes a martingale difference sequence with respect to the filtration of σ-fields $\mathcal{F}_t = \sigma(Z_t, Z_{t-1}, \ldots)$, $t \in \mathbb{Z}$. See the lecture notes for a definition. Show that (X_t^2) is not a martingale difference sequence with respect to $(\mathcal{F}_t)$. How would one have to modify (X_t^2) to make it a martingale difference sequence with respect to $(\mathcal{F}_t)$?

(c) For real-life log-returns one often observes that the estimated GARCH parameters sum up to a value close to 1:

$$\widehat{\alpha}_1 + \widehat{\beta}_1 \approx 1\,.$$

This observation led Engle and Bollerslev (1986) to the introduction of the *integrated* GARCH$(1,1)$ *process* (IGARCH$(1,1)$) by requiring

$$\alpha_1 + \beta_1 = 1\,.$$

A strictly stationary version of an IGARCH process has the undesirable and empirically not observed property that both σ_t and X_t have infinite variance. Verify this property by assuming that (X_t) and (σ_t) are both strictly stationary. Also show that σ_t and X_t have infinite variance if

$$\alpha_1 + \beta_1 > 1\,.$$

(d) If (Z_t) is iid standard normal $(|X_t|)$ has power-law tails in the sense that

$$P(|X_0| > x) \sim c\, x^{-\kappa}\,, \qquad x \to \infty\,,$$

where $c > 0$ and κ is the unique positive solution of the equation $E[(\alpha_1 Z_0^2 + \beta_1)^{\kappa/2}] = 1$. Show that $\kappa = 2$ for IGARCH(1,1).

(3) Let (X_t) be a causal stationary solution to the ARCH(p) equations with iid $N(0,1)$ white noise such that $E[X_0^4] < \infty$.

(a) Show that an ARCH(1) process X_t has a finite fourth moment if and only if $3\alpha_1^2 < 1$.

(b) Show that $Y_t = X_t^2/\alpha_0$ satisfies the equations

$$Y_t = Z_t^2\Big(1 + \sum_{i=1}^{p} \alpha_i Y_{t-i}\Big)$$

and deduce that (Y_t) has the same autocorrelation function as the AR(p) process

$$W_t = \sum_{i=1}^{p} \alpha_i W_{t-i} + Z_t'$$

for some white noise process (Z_t').