Assignment 6 for StatØk2, Block 1, 2021/2022

(1) Argue that the parameters α_j, $j \geq 1, \ldots, p$, of an ARCH(p) process can be estimated by the Yule-Walker estimates based on the sample $Y_1 = X_1^2 - E[X_0^2], \ldots, Y_n = X_n^2 - E[X_0^2]$ provided $E[X_0^4] < \infty$.

(2) Write down the Gaussian log-likelihood for an ARCH(1) and a GARCH(1,1) process. Argue that you need an initial condition for σ_0, X_0 to define this function in the case of a GARCH$(1,1)$ process. In software one typically chooses $\sigma_0 = X_0 = 0$.

(3) Consider a stationary GARCH$(1,1)$ process (X_t) with iid standard normal noise and assume $\mathbb{E}[\sigma_0^4] < \infty$. Write $\phi = \alpha_1 + \beta_1$. We know (Lecture Notes) that the condition $\phi < 1$ ensures finite variance of X_0.

(a) Calculate $\mathrm{var}(X_0^2)$ and $\mathrm{cov}(X_0^2, X_1^2)$.

(b) Show that the autocorrelation function of (X_t^2) is given by

$$\rho_{X^2}(h) \;=\; \rho_{X^2}(1)\phi^{h-1}\,, \qquad h \geq 1\,.$$

Hint: use induction on h.

(4) Give reasons why it suffices to consider the spectral density f_X of a real-valued stationary process only on $[0, \pi]$.

(5) Consider a strictly stationary ARCH(1) process $X_t = \sigma_t Z_t$ with iid standard normal noise (Z_t) where $\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2$, $\alpha_0, \alpha_1 > 0$. Find a condition ensuring $E[X_0^4] < \infty$. Under this condition, derive the spectral density of (X_t^2).

Hint: the process (X_t^2) has an AR(1) representation.

(6) Determine the spectral densities for a stationary GARCH$(1,1)$ process (X_t) and for (X_t^2) given that $E[X_0^4] < \infty$.

Hint. Use the fact that $(X_t^2 - \mathbb{E}[X_0^2])$ satisfies an ARMA(1,1) equation.

(7) Let (Z_t) be iid Gaussian white noise with variance σ^2.

(a) Prove that the periodogram $2\pi I_{n,Z}(\lambda_j)$ at the Fourier frequencies $\lambda_j = 2\pi j/n \in (0, \pi)$ is an iid exponential $\sigma^2\mathrm{Exp}(1)$ sequence.

Hint: since the random variables $n^{-1/2}\sum_{t=1}^n Z_t \cos(\lambda_j t)$, $n^{-1/2}\sum_{t=1}^n Z_t \sin(\lambda_j t)$, $1 \leq j < n/2$, are jointly Gaussian (they are linear combinations of an iid Gaussian sequence (Z_t)) it suffices to show that these random variables are iid Gaussian $\mathrm{N}(0, \sigma^2/2)$ random variables.

(b) Show that with $q = [n/2]$,

$$\frac{\max_{1\leq j\leq q} I_{n,Z}(\lambda_j)}{\sigma^2} - \log q \xrightarrow{d} Y$$

where Y is Gumbel distributed with distribution function $P(Y \leq x) = \exp(-\mathrm{e}^{-x})$.

The iid property of the periodogram at the Fourier frequencies has been used by R.A. Fisher to construct his g–test for Gaussian white noise. He used the test statistic

$$g_n = \max_{1\leq j\leq q} I_{n,Z}(\lambda_j) \Big/ \left(q^{-1}\sum_{j=1}^q I_{n,Z}(\lambda_j)\right)$$

with $q = [n/2]$. *Its asymptotic distribution can be calculated under the null hypothesis that* (Z_t) *is iid; see e.g. Brockwell and Davis (1991), p. 339. Notice that* $q^{-1}\sum_{j=1}^q I_{n,Z}(\lambda_j) \xrightarrow{P} \sigma^2$.