Written Examination: Statistical Analysis of Econometric Time Series
November 8, 2019
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- This is an open book examination. You may use any book, the lecture notes, assignments and their solutions, pocket calculator, computer, etc. In the solution to each problem you have to give reasons either by providing a proof or by referring to a result in the lecture notes, the assignments, any book, etc. In the latter case, give an *exact* reference.
- Use your time in a reasonable way. Do not copy the text of the problems. Refer to the lecture notes whenever possible. Do not re-prove results from the lecture notes.
- You may write with a pencil.
- This examination paper consists of 2 pages.
- You may write in English or Danish.
- The distribution of the points over the problems is as follows.

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Problem   1 a b  2 a b c   3 a b c d    sum
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# points   2 2    2 1 2     2 3 1 6    21
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1. Consider the ARMA$(1,2)$ equations

$$X_t - 0.9\, X_{t-1} = Z_t - 2\, Z_{t-1} + Z_{t-2}\,, \qquad t \in \mathbb{Z}\,,$$

where (Z_t) is iid white noise with variance σ^2.

(a) Show that these ARMA(1,2) equations have a stationary causal solution (X_t). Is the solution also invertible?

(b) Which of the following properties does (X_t^2) have?

(i) Strict stationarity

(ii) Ergodicity

Give arguments for each of your answers.

2. Consider two uncorrelated stationary processes (X_t) and (Y_t) with mean zero. This means that $\mathbb{E}[X_t] = \mathbb{E}[Y_t] = 0$ for all $t \in \mathbb{Z}$ and $\text{cov}(X_t, Y_s) = 0$ for all $s, t \in \mathbb{Z}$. We assume that the autocovariance functions

$$\gamma_X(h) = \text{cov}(X_0, X_h) \quad \text{and} \quad \gamma_Y(h) = \text{cov}(Y_0, Y_h)\,, \qquad h \in \mathbb{Z}\,,$$

satisfy the condition $\sum_{h\in\mathbb{Z}}(|\gamma_X(h)| + |\gamma_Y(h)|) < \infty$.

(a) Calculate the autocovariance function of the stationary process $Z_t = X_t + Y_t$, $t \in \mathbb{Z}$. (You may assume that the stationarity is proved.)

(b) Argue that both (X_t) and (Y_t) have a spectral density.

(c) Calculate the spectral density of (Z_t).

3. Consider the AR(1) process (X_t) which is the causal solution to the equations

$$X_t - \varphi X_{t-1} = Z_t\,, \qquad t \in \mathbb{Z}\,,$$

where $|\varphi| < 1$ and (Z_t) is iid normal $N(0, \sigma^2)$.

(a) Derive the best prediction for X_{n+2} given that you know

(i) X_n

(ii) X_n and X_{n-1}.

(b) Derive the best prediction for X_{n+1}^2 given that you know X_n.

(c) Show that $s^2 = \mathrm{var}(X_t) = \sigma^2(1-\varphi^2)^{-1}$.

(d) Show that $Y_t = X_t^2 - s^2$, $t \in \mathbb{Z}$, satisfies the AR(1) equation

$$Y_t = \varphi^2\, Y_{t-1} + \nu_t\,, \qquad t \in \mathbb{Z}\,,$$

for some white noise sequence (ν_t). Do these equations have a causal solution?

End of Examination Paper

Solutions to the written examination Statistical Analysis of Econometric Time Series, November 8, 2019

1. (a) **2 points** The ARMA(1,2) equation has a causal stationary solution if $\varphi(z) = 1-0.9z$ does not have a root for $|z|\le 1$. But $\varphi(z_0)=0$ for $z_0 = 1/0.9 = 10/9 > 1$. Therefore a causal stationary solution exists.
   The process is not invertible since $\theta(z) = 1-2z+z^2 = (1-z)^2$ has the root $z_0 = 1$ which is on the unit circle.

   (b) **2 points** The solution to the AR(1) equations is a linear causal process $X_t = \sum_{j=0}^\infty \psi_j Z_{t-j}$. The process (X_t^2) has the structure $X_t^2 = f(Z_t, Z_{t-1},\ldots)$ for an iid sequence (Z_t) and a deterministic function f. Therefore this process is strictly stationary and inherits ergodicity from (Z_t).

2. (a) **2 points** For any t we have $\mathbb{E}[Z_t] = \mathbb{E}[X_t] + \mathbb{E}[Y_t] = 0$ and

   $$\mathrm{var}(Z_t) = \mathrm{var}(X_t) + \mathrm{var}(Y_t) + 2\mathrm{cov}(X_t, Y_t) = \mathrm{var}(X_t) + \mathrm{var}(Y_t) < \infty$$

   and for $h \ge 1$,

   $$\begin{aligned}\mathrm{cov}(Z_t, Z_{t+h}) &= \mathrm{cov}(X_t, Y_{t+h}) + \mathrm{cov}(X_t, X_{t+h}) + \mathrm{cov}(Y_t, Y_{t+h}) + \mathrm{cov}(Y_t, X_{t+h})\\ &= 0 + \gamma_X(h) + \gamma_Y(h) + 0\,.\end{aligned}$$

   Hence (Z_t) is stationary with $\gamma_Z = \gamma_X + \gamma_Y$.

   (b) **1 point** Since γ_X and γ_Y are absolutely summable so is γ_Z. Hence the spectral density f_Z exists; see Corollary 6.11.

   (c) **2 points** According to Corollary 6.11, f_Z is given by

   $$\begin{aligned}f_Z(\lambda) &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \mathrm{e}^{-i\lambda h}\,\gamma_Z(h)\\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \mathrm{e}^{-i\lambda h}\,\gamma_X(h) + \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \mathrm{e}^{-i\lambda h}\,\gamma_Y(h)\\ &= f_X(\lambda) + f_Y(\lambda)\,.\end{aligned}$$

3. (a) **2 points** The AR(1) process (X_n) is causal. Therefore the best prediction of

   $$X_{n+2} = \varphi X_{n+1} + Z_{n+1} = \varphi^2 X_n + \varphi Z_{n+1} + Z_n$$

   given X_n, X_{n-1} is

   $$\begin{aligned}\mathbb{E}[X_{n+2} \mid X_n] &= \mathbb{E}[\varphi^2 X_n + \varphi Z_{n+1} + Z_n \mid X_n, X_{n-1}]\\ &= \varphi^2 X_n + \varphi\mathbb{E}[Z_{n+1} \mid X_n, X_{n-1}] + \mathbb{E}[Z_n \mid X_n, X_{n-1}]\\ &= \varphi^2 X_n + \varphi\mathbb{E}[Z_{n+1}] + \mathbb{E}[Z_n]\\ &= \varphi^2 X_n\,.\end{aligned}$$

   since Z_n, Z_{n+1} are independent of X_n, X_{n-1} and have mean zero. **Notice:** Conditioning on X_{n-1} had no influence on the result.

(b) **3 points** The best prediction of

$$X_{n+1}^2 = (\varphi X_n + Z_{n+1})^2 = \varphi^2 X_n^2 + Z_{n+1}^2 + 2\varphi Z_{n+1} X_n$$

given X_n is

$$\begin{aligned} \mathbb{E}[X_{n+1}^2 \mid X_n] &= \mathbb{E}[\varphi^2 X_n^2 + Z_{n+1}^2 + 2\varphi Z_{n+1} X_n \mid X_n] \\ &= \varphi^2 X_n^2 + \mathbb{E}[Z_{n+1}^2 \mid X_n] + 2X_n \varphi \mathbb{E}[Z_{n+1} \mid X_n] \\ &= \varphi^2 X_n^2 + \mathbb{E}[Z_{n+1}^2] + 2X_n \varphi \mathbb{E}[Z_{n+1}] \\ &= \varphi^2 X_n^2 + \sigma^2 + 0\,. \end{aligned}$$

(c) **1 point** The easiest way for calculation of s^2 is to use the recursion $X_t = \varphi X_{t-1} + Z_t$ together with stationarity ($s^2 = \mathrm{var}(X_t) = \mathrm{var}(X_{t-1})$) and independence of Z_t and X_{t-1}:

$$\begin{aligned} s^2 &= \mathbb{E}[X_t^2] = \mathrm{var}(X_t) \\ &= \mathbb{E}[\varphi^2 X_{t-1}^2 + Z_t^2 + 2Z_t X_{t-1}] \\ &= \varphi^2 s^2 + \sigma^2 + 0 \end{aligned}$$

Hence $s^2 = \sigma^2(1-\varphi^2)^{-1}$; see also (4.6) in the Notes.

(d) (d) **6 points** We have

$$\begin{aligned} Y_{t+1} &= X_{t+1}^2 - s^2 \\ &= \varphi^2 X_t^2 + Z_{t+1}^2 - s^2 + 2\varphi Z_{t+1} X_t \\ &= \varphi^2 (X_t^2 - s^2) + \varphi^2 s^2 + (Z_{t+1}^2 - \sigma^2) + \sigma^2 - s^2 + 2\varphi Z_{t+1} X_t \\ &= \varphi^2 Y_t + (Z_{t+1}^2 - \sigma^2) + 2\varphi Z_{t+1} X_t \\ &= \varphi^2 Y_t + \nu_t\,. \end{aligned}$$

Hence (Y_t) obeys an AR(1) equation with parameter $\varphi^2 \in (0,1)$. It has a causal solution if we can also show that

$$\nu_t = (Z_{t+1}^2 - \sigma^2) + 2\varphi Z_{t+1} X_t\,, \qquad t \in \mathbb{Z}\,,$$

is white noise. Obviously, $\mathbb{E}[\nu_t] = 0$ and $\nu_t = f(Z_{t+1}, X_t) = g(Z_{t+1}, Z_t, \ldots)$ for deterministic functions f, g. Hence (ν_t) is strictly stationary and has finite variance. Moreover, for $h \geq 1$,

$$\mathrm{cov}(\nu_0, \nu_h) = \mathbb{E}\big[\big((Z_1^2 - \sigma^2) + 2\varphi Z_1 X_0\big)\big((Z_{h+1}^2 - \sigma^2) + 2\varphi Z_{h+1} X_h\big)\big] = 0$$

since Z_1, Z_{h+1} are iid and

$$\mathbb{E}[(Z_1^2 - \sigma^2)(Z_{h+1}^2 - \sigma^2)] = \mathbb{E}[Z_1^2 - \sigma^2]\mathbb{E}[Z_{h+1}^2 - \sigma^2] = 0\,,$$

since Z_{h+1} is independent of $Z_1, X_h = f(Z_h, Z_{h-1}, \ldots)$ and

$$\mathbb{E}[(Z_1^2 - \sigma^2) 2\varphi Z_{h+1} X_h] = 2\varphi \mathbb{E}[(Z_1^2 - \sigma^2) X_h]\mathbb{E}[Z_{h+1}] = 0$$

since $Z_1, Z_{h+1}, X_0 = g(Z_0, Z_{-1}, \ldots)$ are independent and

$$\mathbb{E}[2\varphi Z_1 X_0\ (Z_{h+1}^2 - \sigma^2)] = 2\varphi \mathbb{E}[Z_1]\mathbb{E}[X_0]\mathbb{E}[Z_{h+1}^2 - \sigma^2] = 0$$

since $Z_1 X_0 X_h = h(Z_h, Z_{h-1}, \ldots)$ is independent of Z_{h+1} and

$$\mathbb{E}[(2\varphi Z_1 X_0)(2\varphi Z_{h+1} X_h)] = (2\varphi)^2\, \mathbb{E}[Z_1 X_0 X_h]\mathbb{E}[Z_{h+1}] = 0\,.$$