Written Examination: Statistical Analysis of Econometric Time Series
November 9, 2018
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- This is an open book examination. You may use any book, the lecture notes, assignments and their solutions, pocket calculator, computer, etc. In the solution to each problem you have to give reasons either by providing a proof or by referring to a result in the lecture notes, the assignments, any book, etc. In the latter case, give an *exact* reference.
- Use your time in a reasonable way. Do not copy the text of the problems. Refer to the lecture notes whenever possible. Do not re-prove results from the lecture notes.
- You may write with a pencil.
- This examination paper consists of 2 pages.
- You may write in English or Danish.
- The distribution of the points over the problems is as follows.

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Problem   1 a i ii iii iv v vi   b i ii   c   2 a b c d     sum
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# points      1  1  1  1  1  1     3  3  2    4 4 2          24
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1. Consider the ARMA$(1,1)$ equations

$$X_t - 0.5X_{t-1} = Z_t + 0.5Z_{t-1}\,, \qquad t \in \mathbb{Z}\,,$$

where (Z_t) is iid white noise with variance σ^2. We assume that these equations have a solution (X_t).

(a) Which of the following properties does the solution (X_t) have?

(i) Stationarity

(ii) Strict stationarity

(iii) Ergodicity

(iv) Mixing

(v) Causality

(vi) Invertibility

Give arguments for each of your answers.

(b)

(i) Show that the autocorrelation of (X_t) at lag 1 is given by

$$\rho_X(1) = \frac{1}{2} + \frac{1}{2}\frac{\sigma^2}{\gamma_X(0)}\,.$$

Hint. Use the linear process representation $X_t = Z_t + \sum_{i=1}^{\infty} \psi_i Z_{t-i}$.

(ii) Show that the corresponding sample autocovariance

$$\gamma_{n,X}(1) = \frac{1}{n}\sum_{t=1}^{n-1}(X_t - \overline{X}_n)(X_{t+1} - \overline{X}_n)\,, \qquad \overline{X}_n = \frac{1}{n}\sum_{t=1}^{n} X_t\,,$$

is a consistent estimator of $\gamma_X(1)$, i.e.,

$$\gamma_{n,X}(1) \overset{\text{a.s.}}{\to} \gamma_X(1)\,, \qquad n \to \infty\,.$$

(c) Calculate the spectral density of (X_t).

2. Consider a strictly stationary GARCH(1,1) process $X_t = \sigma_t Z_t$, $t \in \mathbb{Z}$, with iid standard normal noise (Z_t) where $\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2$, $\alpha_i > 0$, $i = 0, 1$, $\beta_1 > 0$.

   (a) Which conditions on the parameters α_1, β_1 ensure that X_t has a finite variance? Calculate the variance of X_t under these conditions.

   (b) Assume a finite fourth moment for (X_t). Derive the spectral densities of (X_t) and (X_t^2).
   Hints: (i) The process $(X_t^2 - \mathbb{E}[X_0^2])$ has an ARMA(1,1) representation with white noise. (ii) We assume that $\mathrm{var}(\sigma_0^2(Z_0^2 - 1)) = 2\mathbb{E}[\sigma_0^4]$ is known.

   (c) Predict the 97.5%-quantile x of the distribution of X_{n+1} conditional on the past $X_1, \ldots, X_n, \sigma_1, \ldots, \sigma_n$ for $n \geq 2$, i.e., determine x such that

$$P(X_{n+1} \leq x \mid X_n, X_{n-1}, \ldots, X_1, \sigma_n, \ldots, \sigma_1) = 0.975.$$

End of Examination Paper