Assignment 3 for StatØk2, Block 1, 2021/2022

(1) Consider an MA(1) process $X_t = Z_t + \theta Z_{t-1}$ with white noise (Z_t), $\theta \neq 0$.

(a) Calculate the autocovariance function γ_X of this process.

(b) Show that the function

$$\gamma(h) = \begin{cases} 1 & \text{for } h = 0 \\ \rho & \text{for } h = \pm 1 \\ 0 & \text{otherwise}, \end{cases}$$

is the autocorrelation function of some stationary process if and only if $|\rho| \leq 1/2$. Hint: find a relationship between (a) and (b). To show that

$$\Gamma_n = \big(\gamma(i-j)\big)_{i,j=1,\ldots,n}$$

is not a covariance matrix (is not non-negative definite) it suffices to find a (simple) vector $\mathbf{a}$ such that $\mathbf{a}'\Gamma_n\mathbf{a} < 0$.

(2) Consider an $n \times n$ matrix Σ and a mean-zero random vector $\mathbf{X} = (X_1, \ldots, X_n)'$.

(a) Prove that Σ is the covariance matrix of $\mathbf{X}$ if and only if it is non-negative definite and symmetric, i.e. $\mathbf{a}'\Sigma\mathbf{a} \geq 0$ for any $\mathbf{a} \in \mathbb{R}^n$ and $\Sigma = \Sigma'$.
Hints: A symmetric non-negative definite matrix Σ has the decomposition $\Sigma = O'\Lambda O$, where O is orthonormal, i.e., $OO' = O'O = I$, and Λ is a diagonal matrix whose diagonal entries are the eigenvalues of Σ which are non-negative. Morover, there exists a matrix $\Sigma^{1/2}$ such that $\Sigma = \Sigma^{1/2}\Sigma^{1/2}$. It can be defined as $\Sigma^{1/2} = O'\Lambda^{1/2}O$, where $\Lambda^{1/2}$ is the diagonal matrix whose diagonal entries are the square roots of the eigenvalues of Σ. Which covariance matrix does the vector $\mathbf{X} = \Sigma^{1/2}\mathbf{Z}$ have if $\mathbf{Z} \in \mathbb{R}^n$ is $N(\mathbf{0}, I)$ distributed?

(3) Consider the defining difference equation of an AR(1) process

$$X_t = \phi\, X_{t-1} + Z_t\,, \qquad t \in \mathbb{Z}\,, \tag{0.1}$$

for a white noise process (Z_t).

(a) If $|\phi| < 1$ we know that (0.1) has a solution given by the infinite series

$$X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}\,, \qquad t \in \mathbb{Z}\,. \tag{0.2}$$

Show that the stationary solution (X_t) with this representation is a.s. unique.
Hint: suppose there is another stationary solution $(\widetilde{X}_t)$ to (0.1) which does not have the form (0.2). By iterating (0.1), show that $E[|X_t - \widetilde{X}_t|] = 0$, hence $X_t = \widetilde{X}_t$ a.s.

(b) Show that the AR(1) equation (0.1) does not have a stationary solution for $\phi = \pm 1$.

(4) Consider a sample $X_1, \ldots, X_n$, where we assume that for some $p \in (0,1)$,

$$X_i^{(1)} = X_i\,, \quad i = 1, \ldots, [np]\,,$$

comes from a strictly stationary ergodic model with finite variance and expectation $EX_1^{(1)}$ and

$$X_i^{(2)} = X_i\,, \quad i = [np]+1, \ldots, n\,,$$

comes from another strictly stationary ergodic model with finite variance and expectation $EX_1^{(2)}$. Apply the ergodic theorem to show that the sample autocovariances satisfy

$$\gamma_{n,X}(h) \stackrel{\text{a.s.}}{\to} p\,\gamma_{X^{(1)}}(h) + (1-p)\,\gamma_{X^{(2)}}(h) + p\,(1-p)\,|EX_0^{(1)} - EX_0^{(2)}|^2\,, \tag{0.3}$$

for $h = 0, 1, 2, \ldots$.

This means the following: if $\gamma_{X^{(i)}}(h) \to 0$ as $h \to \infty$ then we should see that the sample ACF tends to a positive constant for large h.

(5) Let (Z_t) be white noise.

(a) Show that the ARMA$(2,1)$ equations

$$(1 - B + \frac{1}{4}B^2)X_t = (1 + B)Z_t\ .$$

have a causal stationary solution.

(b) Do the ARMA$(1,2)$ equations $(1-0.5B)X_t = (1+0.5B)(1+0.7B)Z_t$ define a causal, stationary solution? If so, determine the first 5 coefficients ψ_j in the linear process representation $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$.