Assignment 7 for StatØk2, Block 1, 2021/2022

(1) Let (X_t) be a stationary process with mean μ.

(a) Consider the linear h-step prediction $P_n X_{n+h}$ based on $X_1, \ldots, X_n$. Show that the prediction error is given by

$$E\big[(X_{n+h} - P_n X_{n+h})^2\big] = \gamma_X(0) - \mathbf{a}_n^\top \gamma_n(h)\,,$$

where $\mathbf{a}_n = (a_1, \ldots, a_n)^\top$ solves the equation $\Gamma_n \mathbf{a}_n = \gamma_n(h)$ and $a_0 = \mu\big(1 - \sum_{i=1}^n a_i\big)$, $\Gamma_n = \big(\gamma_X(i-j)\big)_{i,j=1,\ldots,n}$ and $\gamma_n(h) = (\gamma_X(h), \ldots, \gamma_X(h+n-1))^\top$.

(b) Show that $P_n X_{n+h}$ is unique, i.e., if there are two solutions $\mathbf{a}_n^{(1)}$ and $\mathbf{a}_n^{(2)}$ to the system of equations $\Gamma_n \mathbf{a}_n = \gamma_n(h)$ then the random variable

$$Z = a_0^{(1)} - a_0^{(2)} + \sum_{j=1}^n (a_j^{(1)} - a_j^{(2)})\, X_{n+1-j}$$

is zero a.s.

(c) Show that

$$E X_{n+h} = E[P_n X_{n+h}]\,.$$

(d) Show that

$$E[(X_{n+h} - P_n X_{n+h}) X_j] = 0\,, \qquad j = 1, \ldots, n\,. \tag{0.1}$$

(e) Consider the Hilbert space of all linear combinations $a_0 + \sum_{i=1}^n a_i X_{n+1-i}$ equipped with the inner product $(X, Y) = \operatorname{cov}(X, Y)$. Show that the linear h-step predictor $P_n X_{n+h}$ can be derived from the projection theorem in Hilbert space (see Lecture Notes) by solving (0.1). (Note that one can always assume without loss of generality that $a_0 = 0$ and $\mu = 0$.)

(f) Determine the linear h-step prediction $P_n X_{n+h}$ of a causal AR(1) process $X_t = \phi_1 X_{t-1} + Z_t$, $t \in \mathbb{Z}$, with driving white noise (Z_t), by solving the prediction equation $\Gamma_n \mathbf{a}_n = \gamma_n(h)$. Compare with the best prediction of X_{n+h} in the class of all square integrable functions of $X_1, \ldots, X_n$.