**Solutions to Assignment 2 for StatØk2, Block 1, 2021/2022 by Jorge Yslas**

**(1)** a) $(Z_t)$ is iid white noise, hence strictly stationary, ergodic and mixing.
Consider the function

$$g\left(x_0, \ldots, x_q\right) = x_0 + \theta_1 x_1 + \cdots + \theta_q x_q \,.$$

Then

$$X_t = g\left(Z_t, \ldots, Z_{t-q}\right) = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q} \,.$$

It follows from Proposition 2.16 that $(X_t)$ is strictly stationary, from Theorem 2.30 that $(X_t)$ is ergodic, and from Theorem 2.38 that $(X_t)$ is mixing.
Now note that the random vectors $(\ldots, X_{-1}, X_0)$ and $(X_n, X_{n+1}, \ldots)$ are independent for $n > q$. This implies that the strong mixing coefficients $\alpha_n = 0$ for $n > q$, and consequently $(X_t)$ is strongly mixing.

b) We observe that

$$X_n + \cdots + X_1 = (Z_n - Z_{n-1}) + (Z_{n-1} - Z_{n-2}) + \cdots + (Z_1 - Z_0) = Z_n - Z_0.$$

Hence

$$\sqrt{n}\,\overline{X}_n = n^{-1/2}(Z_n - Z_0) \stackrel{d}{=} n^{-1/2}(Z_1 - Z_0) \xrightarrow{\mathbb{P}} 0 \,.$$

In particular, $\mathrm{var}(\sqrt{n}\overline{X}_n) = 2n^{-1}\mathrm{var}(Z) \to 0$. Thus all conditions of Ibragimov's CLT are satisfied but the condition on the positivity of $\lim_{n\to\infty} \mathrm{var}(\sqrt{n}\overline{X}_n)$ and therefore the CLT fails.
Following the same arguments as in part a), $(X_t) = ((Z_t - Z_{t-1})^2)$ is strictly stationary, ergodic, mixing and also strongly mixing with rate function $\alpha_n^X = 0$ for $n > 2$. Then Ibragimov's CLT is applicable if we have $E[(Z_t - Z_{t-1})^{2(2+\delta/2)}] < \infty$ (but we assume this condition!) and if we can show that the variance of $n^{-1/2}\sum_{t=1}^n (Z_t - Z_{t-1})^2$ has a positive limit. We know from Corollary 2.41 and 1-dependence of $(X_t)$ that

$$\mathrm{var}\left(\sqrt{n}\,\overline{X}_n\right) \to \sigma^2 = \gamma_X(0) + 2\sum_{h=1}^{\infty} \gamma_X(h) = \gamma_X(0) + 2\gamma_X(1) \,.$$

We have

$$X_t = \left(Z_t - Z_{t-1}\right)^2 = Z_t^2 - 2Z_t Z_{t-1} + Z_{t-1}^2.$$

Calculation yields

$$\begin{aligned}
\gamma_X(0) &= \mathrm{cov}\left(Z_t^2, Z_t^2\right) - 2\mathrm{cov}\left(Z_t^2, Z_t Z_{t-1}\right) + \mathrm{cov}\left(Z_t^2, Z_{t-1}^2\right) \\
&\quad - 2\mathrm{cov}\left(Z_t Z_{t-1}, Z_t^2\right) + 4\mathrm{cov}\left(Z_t Z_{t-1}, Z_t Z_{t-1}\right) - 2\mathrm{cov}\left(Z_t Z_{t-1}, Z_{t-1}^2\right) \\
&\quad + \mathrm{cov}\left(Z_{t-1}^2, Z_t^2\right) - 2\mathrm{cov}\left(Z_{t-1}^2, Z_t Z_{t-1}\right) + \mathrm{cov}\left(Z_{t-1}^2, Z_{t-1}^2\right) \\
&= \mathrm{cov}\left(Z_t^2, Z_t^2\right) + 4\mathrm{cov}\left(Z_t Z_{t-1}, Z_t Z_{t-1}\right) + \mathrm{cov}\left(Z_{t-1}^2, Z_{t-1}^2\right) \\
&= 2\left(\mathbb{E}\left[Z_0^4\right] - \left(\mathbb{E}\left[Z_0^2\right]\right)^2\right) + 4\left(\mathbb{E}\left[Z_t^2 Z_{t-1}^2\right] - \left(\mathbb{E}\left[Z_t Z_{t-1}\right]\right)^2\right) \\
&= 2\mathbb{E}\left[Z_0^4\right] + 2\sigma_Z^4
\end{aligned}$$

and

$$\begin{aligned}
\gamma_X(1) &= \mathrm{cov}\left(Z_t^2, Z_{t+1}^2\right) - 2\mathrm{cov}\left(Z_t^2, Z_{t+1} Z_t\right) + \mathrm{cov}\left(Z_t^2, Z_t^2\right) \\
&\quad - 2\mathrm{cov}\left(Z_t Z_{t-1}, Z_{t+1}^2\right) + 4\mathrm{cov}\left(Z_t Z_{t-1}, Z_{t+1} Z_t\right) - 2\mathrm{cov}\left(Z_t Z_{t-1}, Z_t^2\right) \\
&\quad + \mathrm{cov}\left(Z_{t-1}^2, Z_{t+1}^2\right) - 2\mathrm{cov}\left(Z_{t-1}^2, Z_{t+1} Z_t\right) + \mathrm{cov}\left(Z_{t-1}^2, Z_t^2\right) \\
&= \mathrm{cov}\left(Z_t^2, Z_t^2\right) \\
&= \mathbb{E}\left[Z_0^4\right] - \sigma_Z^4 \,.
\end{aligned}$$

Then

$$\gamma_X(0) + 2\gamma_X(1) = 2\mathbb{E}\left[Z_0^4\right] + 2\sigma_Z^4 + 2\left(\mathbb{E}\left[Z_0^4\right] - \sigma_Z^4\right) = 4\mathbb{E}\left[Z_0^4\right] .$$

$\mathbb{E}\left[Z_0^4\right] > 0$? Yes, by Jensen's inequality

$$\mathbb{E}\left[Z_0^4\right] = \mathbb{E}\left[\left(Z_0^2\right)^2\right] \geq \left(\mathbb{E}\left[Z_0^2\right]\right)^2 = \sigma_Z^4 > 0 .$$

**(2)** Since $(X_t)$ is strictly stationary and ergodic, we have that

$$\overline{X}_n = \frac{1}{n}\sum_{t=1}^{n} X_t \to \mathbb{E}\left[X_0\right]$$

and

$$\frac{1}{n}\sum_{t=1}^{n} X_t X_{t+h} \to \mathbb{E}\left[X_0 X_h\right] .$$

Note also that

$$\frac{1}{n}\sum_{t=1}^{n} X_{t+h} \to \mathbb{E}\left[X_0\right] .$$

Now

$$\begin{aligned}
\gamma_{n,X}(h) &= \frac{1}{n}\sum_{t=1}^{n-h}\left(X_t - \overline{X}_n\right)\left(X_{t+h} - \overline{X}_n\right) \\
&= \frac{1}{n}\sum_{t=1}^{n-h}\left(X_t X_{t+h} - \overline{X}_n X_{t+h} - \overline{X}_n X_t + \overline{X}_n^2\right) \\
&= \frac{1}{n}\sum_{t=1}^{n-h} X_t X_{t+h} - \overline{X}_n\left(\frac{1}{n}\sum_{t=1}^{n-h} X_{t+h}\right) - \overline{X}_n\left(\frac{1}{n}\sum_{t=1}^{n-h} X_t\right) + \frac{n-h}{n}\overline{X}_n^2 .
\end{aligned}$$

We look at each of the terms

$$\frac{1}{n}\sum_{t=1}^{n-h} X_t X_{t+h} = \left(\frac{n-h}{n}\right)\left(\frac{1}{n-h}\sum_{t=1}^{n-h} X_t X_{t+h}\right) \to (1)\left(\mathbb{E}\left[X_0 X_h\right]\right) = \mathbb{E}\left[X_0 X_h\right]$$

$$\overline{X}_n\left(\frac{1}{n}\sum_{t=1}^{n-h} X_{t+h}\right) = \overline{X}_n\left(\frac{n-h}{n}\right)\left(\frac{1}{n-h}\sum_{t=1}^{n-h} X_{t+h}\right) \to \left(\mathbb{E}\left[X_0\right]\right)(1)\left(\mathbb{E}\left[X_0\right]\right) = \left(\mathbb{E}\left[X_0\right]\right)^2$$

$$\overline{X}_n\left(\frac{1}{n}\sum_{t=1}^{n-h} X_t\right) = \overline{X}_n\left(\frac{n-h}{n}\right)\left(\frac{1}{n-h}\sum_{t=1}^{n-h} X_t\right) \to \left(\mathbb{E}\left[X_0\right]\right)(1)\left(\mathbb{E}\left[X_0\right]\right) = \left(\mathbb{E}\left[X_0\right]\right)^2$$

$$\frac{n-h}{n}\overline{X}_n^2 \to \left(\mathbb{E}\left[X_0\right]\right)^2$$

Hence

$$\gamma_{n,X}(h) \to \mathbb{E}[X_0 X_h] - (\mathbb{E}[X_0])^2 = \gamma_X(h) .$$

In particular

$$\gamma_{n,X}(0) \to \mathbb{E}\left[X_0^2\right] - (\mathbb{E}[X_0])^2 = \gamma_X(0) .$$

Then

$$\rho_{n,X}(h) = \frac{\gamma_{n,X}(h)}{\gamma_{n,X}(0)} \to \frac{\gamma_X(h)}{\gamma_X(0)} = \rho_X(h) .$$

**(3)** A Taylor expansion of $f(x)$ around $a$ is given by

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x) ,$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

with $c$ between $a$ and $x$. Now consider $f(x) = \log(1+x)$, then

$$\begin{aligned} f^{(1)}(x) &= \frac{1}{1+x} \\ f^{(2)}(x) &= -\frac{1}{(1+x)^2} . \end{aligned}$$

Thus

$$\begin{aligned} \log(1+x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(c)}{2!}x^2 \\ &= \log(1+0) + \frac{1}{1+0}x - \frac{1}{2(1+c)^2}x^2 \\ &= x - \frac{x^2}{2(1+c)^2} , \end{aligned}$$

with $c$ between 0 and $x$. Then

$$|x - \log(1+x)| = \left| x - \left( x - \frac{x^2}{2(1+c)^2} \right) \right| = \frac{x^2}{2(1+c)^2} .$$

Thus, if we assume that $|x| \leq \epsilon$, then

$$|x - \log(1+x)| \leq \frac{\epsilon^2}{2(1-\epsilon)^2} = \frac{1}{2}\left(\frac{\epsilon}{1-\epsilon}\right)^2 .$$

In particular, with $-20\%$ we get that $|\Delta_t| \leq 0.02314355$.

**(4)** a) We first show by induction that $\Delta^k\left(t^k\right) = k!$.

$k = 1$

$$\Delta(t) = t - (t-1) = 1\,.$$

Assume that

$$\Delta^k\left(t^k\right) = k!\,.$$

Note that, under this assumption, for any $m < k$ we have that

$$\Delta^k(t^m) = \Delta^{k-m}\left(\Delta^m(t^m)\right) = \Delta^{k-m}(m!) = 0\,.$$

Now consider $k+1$, then

$$\begin{aligned}
\Delta^{k+1}\left(t^{k+1}\right) &= \Delta^k\left(\Delta\left(t^{k+1}\right)\right) = \Delta^k\left(t^{k+1} - (t-1)^{k+1}\right) \\
&= \Delta^k\left(t^{k+1} - \sum_{j=0}^{k+1}\binom{k+1}{j}t^{k+1-j}(-1)^j\right) \\
&= \Delta^k\left((k+1)t^k - \binom{k+1}{2}t^{k-1} + \cdots + (-1)^{k+1}\right) \\
&= (k+1)\Delta^k\left(t^k\right) \\
&= (k+1)k! \\
&= (k+1)!\,.
\end{aligned}$$

Now we prove the main result by induction.

$k = 1$. We have that

$$X_t = m_t + Y_t = a_0 + a_1 t + Y_t\,.$$

Then

$$\begin{aligned}
\Delta(X_t) &= (1-B)X_t = X_t - X_{t-1} \\
&= (a_0 + a_1 t + Y_t) - (a_0 + a_1(t-1) + Y_{t-1}) \\
&= a_1 + Y_t - Y_{t-1} \\
&= a_1 + \Delta(Y_t)\,.
\end{aligned}$$

Assume that the result holds for $k$, i.e., with

$$X_t = m_t + Y_t = \sum_{j=0}^{k} a_j t^j + Y_t$$

we have

$$\Delta^k(X_t) = k!a_k + \Delta^k(Y_t)\,.$$

Now consider $k+1$. Then

$$X_t = m_t + Y_t = \sum_{j=0}^{k+1} a_j t^j + Y_t$$

and

$$
\begin{aligned}
\Delta^{k+1}\left(X_t\right) &= \Delta\left(\Delta^k\left(X_t\right)\right) = \Delta\left(\Delta^k\left(\left(X_t - a_{k+1}t^{k+1}\right) + a_{k+1}t^{k+1}\right)\right) \\
&= \Delta\left(\Delta^k\left(X_t - a_{k+1}t^{k+1}\right) + \Delta^k\left(a_{k+1}t^{k+1}\right)\right) \\
&= \Delta\left(\Delta^k\left(X_t - a_{k+1}t^{k+1}\right) + \Delta^k\left(a_{k+1}t^{k+1}\right)\right) \\
&= \Delta\left(\left(k!a_k + \Delta^k\left(Y_t\right)\right) + a_{k+1}\Delta^k\left(t^{k+1}\right)\right) \\
&= \Delta\left(k!a_k + \Delta^k\left(Y_t\right)\right) + a_{k+1}\Delta^{k+1}\left(t^{k+1}\right) \\
&= (k+1)!a_{k+1} + \Delta^{k+1}\left(Y_t\right)
\end{aligned}
$$

and the result follows.

b) Note that it is enough to show these properties for $\Delta\left(Y_t\right)$ and then use induction for $\Delta^k\left(Y_t\right) = \Delta(\Delta^{k-1}(Y_t))$. By definition, we have that

$$
\Delta\left(Y_t\right) = Y_t - Y_{t-1}\,.
$$

First, we prove that if $(Y_t)$ is stationary then also $(\Delta\left(Y_t\right))$ does. Thus, we need to show the 3 properties of a stationary processes.
i)

$$
\mathbb{E}\left[\Delta\left(Y_t\right)\right] = \mathbb{E}\left[Y_t - Y_{t-1}\right] = 0\,.
$$

ii)

$$
\begin{aligned}
\mathbb{E}\left[\left(\Delta\left(Y_t\right)\right)^2\right] &= \mathbb{E}\left[Y_t^2 - 2Y_tY_{t-1} + Y_{t-1}^2\right] \\
&= \mathbb{E}\left[Y_t^2\right] - 2\left(\mathbb{E}\left[Y_tY_{t-1}\right] - m_Y^2 + m_Y^2\right) + \mathbb{E}\left[Y_{t-1}^2\right] \\
&= \mathbb{E}\left[Y_t^2\right] - 2\left(\gamma_Y(1) + m_Y^2\right) + \mathbb{E}\left[Y_{t-1}^2\right] < \infty\,.
\end{aligned}
$$

iii)

$$
\begin{aligned}
\gamma_{\Delta(Y)}(t, t+h) &= \mathrm{cov}(Y_t - Y_{t-1}, Y_{t+h} - Y_{t+h-1}) \\
&= \mathrm{cov}(Y_t, Y_{t+h}) - \mathrm{cov}(Y_t, Y_{t+h-1}) - \mathrm{cov}(Y_{t-1}, Y_{t+h}) + \mathrm{cov}(Y_{t-1}, Y_{t+h-1}) \\
&= 2\gamma_Y(h) - \gamma_Y(h-1) - \gamma_Y(h+1)
\end{aligned}
$$

Therefore $(\Delta\left(Y_t\right))$ is stationary.
In order to show the other properties, consider the function

$$
g\left(x_0, x_1\right) = x_0 - x_1\,.
$$

Then

$$
\Delta\left(Y_t\right) = g\left(Y_t, Y_{t-1}\right) = Y_t - Y_{t-1}\,.
$$

Note that the right-hand side is finite a.s., since it is the finite sum of real-valued r.v.'s. Then it follows from Proposition 2.16 that $(\Delta\left(Y_t\right))$ is strictly stationary, from Theorem 2.30 that $(\Delta\left(Y_t\right))$ is ergodic and from Theorem 2.38 that $(\Delta\left(Y_t\right))$ is mixing.

**(5)** First note that

$$
\overline{X}_n = \frac{1}{n}\sum_{t=1}^{n} X_t = \frac{1}{n}\sum_{t=1}^{n} c\cos\left(t\omega\right) = \frac{c}{n}\sum_{t=1}^{n}\cos\left(t\omega\right) = \frac{c}{n}\left(\frac{\cos\left(\omega(n+1)/2\right)\sin\left(\omega n/2\right)}{\sin\left(\omega/2\right)}\right)
$$

and consequently $\overline{X}_n \to 0$.
Now

$$\begin{aligned}\gamma_{n,X}(h) &= \frac{1}{n}\sum_{t=1}^{n-h}\left(X_t - \overline{X}_n\right)\left(X_{t+h} - \overline{X}_n\right)\\ &= \frac{1}{n}\sum_{t=1}^{n-h}\left(X_t X_{t+h} - \overline{X}_n X_t - \overline{X}_n X_{t+h} + \overline{X}_n^2\right)\\ &= \frac{1}{n}\sum_{t=1}^{n-h} X_t X_{t+h} - \overline{X}_n\left(\frac{1}{n}\sum_{t=1}^{n-h} X_t\right) - \overline{X}_n\left(\frac{1}{n}\sum_{t=1}^{n-h} X_{t+h}\right) + \left(\frac{n-h}{n}\right)\overline{X}_n^2 .\end{aligned}$$

Let's look at each of the terms

$$\left(\frac{n-h}{n}\right)\overline{X}_n^2 \to (1)\,(0)^2 = 0$$

$$\overline{X}_n\left(\frac{1}{n}\sum_{t=1}^{n-h} X_t\right) = \overline{X}_n\left(\frac{n-h}{n}\right)\left(\frac{1}{n-h}\sum_{t=1}^{n-h} X_t\right) \to (0)(1)(0) = 0$$

For $\overline{X}_n\left(n^{-1}\sum_{t=1}^{n-h} X_{t+h}\right)$ we will use a different argument. Note that $|X_{t+h}| = |c\cos\left((t+h)\omega\right)| \leq |c|$, then

$$\left|\frac{1}{n}\sum_{t=1}^{n-h} X_{t+h}\right| \leq \frac{1}{n}\sum_{t=1}^{n-h}|X_{t+h}| \leq \frac{1}{n}\sum_{t=1}^{n-h}|c| = |c|\frac{n-h}{n} \leq |c|\,.$$

Meaning that $\left(n^{-1}\sum_{t=1}^{n-h} X_{t+h}\right)$ is bounded, and since $\overline{X}_n \to 0$ it follows that

$$\overline{X}_n\left(\frac{1}{n}\sum_{t=1}^{n-h} X_{t+h}\right) \to 0\,.$$

For the remaining term we will use the following trigonometric identities:

- $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$.
- $\sin(2x) = 2\sin(x)\cos(x)$.
- $\cos^2(x) = (1+\cos(2x))/2$.

Thus

$$
\begin{aligned}
\frac{1}{n}\sum_{t=1}^{n-h} X_t X_{t+h} &= \frac{1}{n}\sum_{t=1}^{n-h} c\cos(t\omega)\, c\cos((t+h)\omega) \\
&= \frac{c^2}{n}\sum_{t=1}^{n-h} \cos(t\omega)\cos(t\omega+h\omega) \\
&= \frac{c^2}{n}\sum_{t=1}^{n-h} \cos(t\omega)\left(\cos(t\omega)\cos(h\omega)-\sin(t\omega)\sin(h\omega)\right) \\
&= \frac{c^2}{n}\left(\cos(h\omega)\sum_{t=1}^{n-h}\cos^2(t\omega)-\sin(h\omega)\sum_{t=1}^{n-h}\cos(t\omega)\sin(t\omega)\right) \\
&= \frac{c^2}{n}\left(\cos(h\omega)\sum_{t=1}^{n-h}\frac{1+\cos(2\omega t)}{2}-\sin(h\omega)\sum_{t=1}^{n-h}\frac{\sin(2\omega t)}{2}\right) \\
&= \frac{c^2}{n}\left(\frac{\cos(h\omega)}{2}\left((n-h)+\sum_{t=1}^{n-h}\cos(2\omega t)\right)-\frac{\sin(h\omega)}{2}\sum_{t=1}^{n-h}\sin(2\omega t)\right) \\
&= \frac{c^2}{n}\Bigg(\frac{\cos(h\omega)}{2}\Big((n-h)+\frac{\cos(\omega(n-h+1))\sin(\omega(n-h))}{\sin(\omega)}\Big) \\
&\quad -\frac{\sin(h\omega)}{2}\frac{\sin(\omega(n-h+1))\sin(\omega(n-h))}{\sin(\omega)}\Bigg) \\
&= \frac{c^2\cos(h\omega)}{2}\left(\frac{n-h}{n}\right)+\frac{c^2\cos(h\omega)}{2}\left(\frac{1}{n}\right)\left(\frac{\cos(\omega(n-h+1))\sin(\omega(n-h))}{\sin(\omega)}\right) \\
&\quad -\frac{c^2\sin(h\omega)}{2}\left(\frac{1}{n}\right)\left(\frac{\sin(\omega(n-h+1))\sin(\omega(n-h))}{\sin(\omega)}\right) \\
&\to \frac{c^2\cos(h\omega)}{2}\,.
\end{aligned}
$$

Putting all together, we conclude that

$$
\gamma_{n,X}(h) \to \frac{c^2\cos(h\omega)}{2}\,.
$$

Particularly

$$
\gamma_{n,X}(0) \to \frac{c^2\cos(0\omega)}{2} = \frac{c^2}{2}\,.
$$

Implying

$$
\rho_{n,X}(h) = \frac{\gamma_{n,X}(h)}{\gamma_{n,X}(0)} \to \frac{c^2\cos(h\omega)/2}{c^2/2} = \cos(h\omega)\ .
$$

**(6)** a) Let $X_t = A\cos(\theta t)+B\sin(\theta t)$, where $A$ and $B$ are random variables such that $\mathbb{E}[A] = \mathbb{E}[B] = \mathbb{E}[AB] = 0$ and $\mathbb{E}\left[A^2\right] = \mathbb{E}\left[B^2\right] = 1$ (see Example 2.11). Then $\mathbb{E}[X_t] = 0$, $\mathrm{var}(X_t) = 1$, and

$$
\begin{aligned}
\gamma_X(t,t+h) &= \mathrm{cov}(X_t, X_{t+h}) \\
&= \mathbb{E}\left[(A\cos(\theta t)+B\sin(\theta t))\,(A\cos(\theta(t+h))+B\sin(\theta(t+h)))\right] \\
&= \cos(\theta t)\cos(\theta(t+h))+\sin(\theta t)\sin(\theta(t+h)) \\
&= \cos(\theta h)
\end{aligned}
$$

**Note**. $\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

b) Recall that $\gamma : \mathbb{Z} \to \mathbb{R}$ is non-negative definite if

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \gamma\left(t_i - t_j\right) \geq 0\,,$$

for all $n$, and all $a_1, \ldots, a_n \in \mathbb{R}$, $t_1, \ldots, t_n \in \mathbb{Z}$. Now, let $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{R}$, $t_1, \ldots, t_n \in \mathbb{Z}$, then

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \gamma\left(t_i - t_j\right) &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \cos\left(\theta(t_i - t_j)\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \cos\left(\theta t_i - \theta t_j\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \left(\cos\left(\theta t_i\right)\cos\left(\theta t_j\right) + \sin\left(\theta t_i\right)\sin\left(\theta t_j\right)\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \cos\left(\theta t_i\right)\cos\left(\theta t_j\right) + \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \sin\left(\theta t_i\right)\sin\left(\theta t_j\right) \\
&= \left(\sum_{i=1}^{n} a_i \cos\left(\theta t_i\right)\right)^2 + \left(\sum_{i=1}^{n} a_i \sin\left(\theta t_i\right)\right)^2 \\
&\geq 0\,.
\end{aligned}$$

Note also that $\gamma(h) = \cos(\theta h) = \cos(-\theta h) = \gamma(-h)$. Thus, it follows by Theorem 3.2 that $\gamma(h) = \cos(\theta h)$ is an autocovariance function.

c) $\gamma(h) = \sin(\theta h)$ autocovariance function? No, since $\gamma(h) = \sin(\theta h)$ is not even, actually $\gamma(-h) = \sin(-\theta h) = -\sin(\theta h) = -\gamma(h)$.

$\gamma(h) = \sum_{k=1}^{m} b_k \cos(\theta_k h)$ with $b_1, \ldots, b_m$ positive is autocovariance function? Yes. Let $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{R}$, $t_1, \ldots, t_n \in \mathbb{Z}$, then

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \gamma\left(t_i - t_j\right) &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \left(\sum_{k=1}^{m} b_k \cos(\theta_k(t_i - t_j))\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m} a_i a_j b_k \cos(\theta_k(t_i - t_j)) \\
&= \sum_{k=1}^{m} b_k \left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \cos(\theta_k(t_i - t_j))\right) \\
&\geq 0\,.
\end{aligned}$$

Since $\cos(\theta_k h) = \cos(-\theta_k h)$ for all $k = 1, \ldots, m$ it follows that $\gamma(h) = \gamma(-h)$ and therefore $\gamma(h) = \sum_{k=1}^{m} b_k \cos(\theta_k h)$ is an autocovariance function.