Assignment 2 for StatØk2, Block 1, 2021/2022

(1) (a) Show that an MA(q) process with iid noise (Z_t) is strictly stationary, ergodic, mixing and strongly mixing. Hint: use the results in the lecture notes.
(b) Show that the sample mean of the strongly mixing ergodic MA(1) process $X_t = Z_t - Z_{t-1}$ with iid white noise (Z_t) does not satisfy the central limit theorem with a normal limit but the sample mean of $Y_t = X_t^2$ satisfies the central limit theorem with a normal limit if $E[|Z_0|^{4+\delta}] < \infty$ for some $\delta > 0$.

(2) Let (X_t) be a strictly stationary ergodic sequence with finite variance. Show that for every $h \geq 0$, the sample autocovariances

$$\gamma_{n,X}(h) = \frac{1}{n} \sum_{t=1}^{n-h} (X_t - \overline{X}_n)\,(X_{t+h} - \overline{X}_n)$$

and the sample autocorrelations

$$\rho_{n,X}(h) = \frac{\gamma_{n,X}(h)}{\gamma_{n,X}(0)}$$

are consistent estimators of their deterministic counterparts:

$$\gamma_{n,X}(h) \stackrel{\text{a.s.}}{\to} \gamma_X(h) \quad \text{and} \quad \rho_{n,X}(h) \stackrel{\text{a.s.}}{\to} \rho_X(h)\,.$$

(3) Consider the (relative) returns

$$Y_t = \frac{X_t - X_{t-1}}{X_{t-1}}$$

of a price series (X_t) and the corresponding log-returns $\log(1+Y_t)$. Find a bound for the distance

$$|\Delta_t| = \Big|Y_t - \log(1+Y_t)\Big|\,,$$

assuming that Y_t is small. The largest daily return values observed in modern history of developed industrial countries were about -20%. How far do Y_t and $\log(1+Y_t)$ deviate in this case?

(4) Consider the non-stationary time series

$$X_t = m_t + Y_t\,, \qquad t = 0, 1, 2, \ldots\,,$$

where (Y_t) is a stationary time series and

$$m_t = \sum_{j=0}^{k} a_j t^j\,, \qquad t = 0, 1, 2, \ldots\,.$$

(a) Show by induction that

$$\Delta^k(X_t) = k!\, a_k + \Delta^k(Y_t)\,,$$

where Δ is the difference operator $\Delta Y_t = Y_t - Y_{t-1}$ acting on (Y_t).
(b) Argue that $(\Delta^k X_t)$, $k \geq 1$, is stationary (strictly stationary, ergodic, mixing) if (Y_t) is stationary (strictly stationary, ergodic, mixing), respectively.

(5) Consider the deterministic time series

$$X_t = c\cos(t\omega)\,,$$

where $c \neq 0$, $\omega \in (-\pi, \pi)$. Show that for each fixed h, the sample autocorrelation function

$$\rho_{n,X}(h) \to \cos(\omega h)\,, \qquad n \to \infty\,.$$

This fact indicates that a periodic term in the time series can be detected by studying the sample autocorrelation function.

(6) Show that $\gamma(h) = \cos(\theta h)$, $h \in \mathbb{Z}$, for some real θ is an autocovariance function

(a) by finding a stationary process with autocovariance function γ,

(b) by directly showing that γ is a non-negative definite function.

(c) Are the functions $\gamma(h) = \sin(\theta h)$ and $\gamma(h) = \sum_{i=1}^{n} b_i \cos(\theta_i h)$, $h \in \mathbb{Z}$, for given positive $b_1, \ldots, b_n$ and real $\theta, \theta_1, \ldots, \theta_n$ autocovariance functions?