Solutions to Assignment 3 for StatØk2, Block 1, 2021/2022 by Jorge Yslas

(1) a) Remember that the autocovariance function of an MA(q) process

$$X_t = \sum_{j=0}^{q} \theta_j Z_{t-j},$$

where Z_t is a white noise with variance $\mathrm{var}(Z_t) = \sigma^2$, is given by

$$\gamma_X(h) = \begin{cases} \sigma^2 \sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|} & |h| \le q\,, \\ 0 & |h| > q\,. \end{cases}$$

Then, for the MA(1) process $X_t = Z_t + \theta Z_{t-1}$, we have that

$$\gamma_X(h) = \begin{cases} \sigma^2\left(1+\theta^2\right) & h = 0\,, \\ \sigma^2\,\theta & h = \pm 1\,, \\ 0 & |h| > 1\,. \end{cases}$$

Thus, its autocorrelation function is given by

$$\rho_X(h) = \begin{cases} 1 & h = 0\,, \\ \frac{\theta}{1+\theta^2} & h = \pm 1\,, \\ 0 & |h| > 1\,. \end{cases}$$

We notice that

$$|\rho| = \frac{|\theta|}{1+\theta^2} \le 0.5$$

since $0 \le 1 + \theta^2 - 2|\theta| = (1-|\theta|)^2$. The continuous function $\rho = \rho(\theta)$ may assume any value between $\rho(0) = 0$ and $\rho(1) = 0.5$. Therefore for any $\rho(\theta) \in [-0.5, 0.5]$ there exists a stationary MA(1) process with ACF $\rho_X(h)$.

b) In view of a) it remains to show that

$$\rho(h) = \begin{cases} 1 & h = 0\,, \\ \rho & h = \pm 1\,, \\ 0 & |h| > 1\,, \end{cases}$$

is not the ACF of any stationary process (X_t) if $|\rho| > 0.5$. This means it is not a non-negative definite function. Fix ρ such that $|\rho| > 1/2$. Observe that

$$\Gamma_2 = (\rho(i-j))_{i,j=1,2} = \begin{pmatrix} \rho(1-1) & \rho(1-2) \\ \rho(2-1) & \rho(2-2) \end{pmatrix} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

$$\Gamma_3 = (\rho(i-j))_{i,j=1,2,3} = \begin{pmatrix} \rho(1-1) & \rho(1-2) & \rho(1-3) \\ \rho(2-1) & \rho(2-2) & \rho(2-3) \\ \rho(3-1) & \rho(3-2) & \rho(3-3) \end{pmatrix} = \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{pmatrix} = \begin{pmatrix} \Gamma_2 & \mathbf{0}_\rho(2) \\ \mathbf{0}_\rho(2)' & 1 \end{pmatrix},$$

where $\mathbf{0}_\rho(2) = (0,\rho)'$. In general

$$\Gamma_{n+1} = \begin{pmatrix} \Gamma_n & \mathbf{0}_\rho(n) \\ \mathbf{0}_\rho(n)' & 1 \end{pmatrix},$$

where $\mathbf{0}_\rho(n) = (0,\ldots,0,\rho)'$ ($n-1$ times 0). We will show that Γ_n is not non-negative definite for some n.
Let $\mathbf{1}_n = (1,\ldots,1)'$ be an n-dimensional vector of 1's.
Proposition. $\mathbf{1}_n'\Gamma_n\mathbf{1}_n = n + (2n-2)\rho$.
We prove the proposition by induction.
$n = 2$

$$\mathbf{1}_2'\Gamma_2\mathbf{1}_2 = (1,1)\begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = (1,1)\begin{pmatrix}1+\rho\\ \rho+1\end{pmatrix} = 2+2\rho = 2+(2(2)-2)\rho\,.$$

Assume that $\mathbf{1}_n'\Gamma_n\mathbf{1}_n = n + (2n-2)\rho$.
Consider $n+1$. Observe that $\mathbf{1}_{n+1} = (\mathbf{1}_n', 1)'$, then

$$\begin{aligned}
\mathbf{1}_{n+1}'\Gamma_{n+1}\mathbf{1}_{n+1} &= (\mathbf{1}_n',1)\begin{pmatrix}\Gamma_n & \mathbf{0}_\rho(n)\\ \mathbf{0}_\rho(n)' & 1\end{pmatrix}\begin{pmatrix}\mathbf{1}_n\\1\end{pmatrix}\\
&= (\mathbf{1}_n',1)\begin{pmatrix}\Gamma_n\mathbf{1}_n + \mathbf{0}_\rho(n)\\ 1+\rho\end{pmatrix}\\
&= \mathbf{1}_n'\Gamma_n\mathbf{1}_n + \mathbf{1}_n'\mathbf{0}_\rho(n) + 1 + \rho\\
&= n + (2n-2)\rho + 1 + 2\rho\\
&= (n+1) + (2(n+1)-2)\rho\,.
\end{aligned}$$

In a similar way it can be shown that $(\mathbf{1}_n^*)'\Gamma_n\mathbf{1}_n^* = n-(2n-2)\rho$, where $\mathbf{1}_n^* = (1,-1,1,\ldots,(-1)^{n+1})'$.
$n = 2$

$$(\mathbf{1}_2^*)'\Gamma_2\mathbf{1}_2^* = (1,-1)\begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = (1,-1)\begin{pmatrix}1-\rho\\ \rho-1\end{pmatrix} = 2-2\rho = 2-(2(2)-2)\rho\,.$$

Assume that $(\mathbf{1}_n^*)'\Gamma_n\mathbf{1}_n^* = n-(2n-2)\rho$.
Consider $n+1$. Observe that $\mathbf{1}_{n+1}^* = ((\mathbf{1}_n^*)', (-1)^{n+2})'$, then

$$\begin{aligned}
(\mathbf{1}_{n+1}^*)'\Gamma_{n+1}\mathbf{1}_{n+1}^* &= ((\mathbf{1}_n^*)',(-1)^{n+2})\begin{pmatrix}\Gamma_n & \mathbf{0}_\rho(n)\\ \mathbf{0}_\rho(n)' & 1\end{pmatrix}\begin{pmatrix}\mathbf{1}_n^*\\(-1)^{n+2}\end{pmatrix}\\
&= ((\mathbf{1}_n^*)',(-1)^{n+2})\begin{pmatrix}\Gamma_n\mathbf{1}_n^* + (-1)^{n+2}\mathbf{0}_\rho(n)\\ (-1)^{n+2}+(-1)^{n+1}\rho\end{pmatrix}\\
&= (\mathbf{1}_n^*)'\Gamma_n\mathbf{1}_n^* + (-1)^{n+2}(\mathbf{1}_n^*)'\mathbf{0}_\rho(n) + 1 - \rho\\
&= n - (2n-2)\rho + 1 - 2\rho\\
&= (n+1) - (2(n+1)-2)\rho\,.
\end{aligned}$$

Some extra comments:

1. $|\rho| > 1/2$ if and only if $\rho > 1/2$ or $\rho < -1/2$.
2. $\frac{n}{2n-2} \to \frac{1}{2}$ as $n\to\infty$ and $\frac{n}{2n-2} > \frac{1}{2}$ for all n.

Then there exists $n_0 \in \mathbb{N}$ such that $|\rho| > \frac{n_0}{2n_0-2} > \frac{1}{2}$.
If $\rho < -1/2$, so $\rho < -\frac{n_0}{2n_0-2} < -\frac{1}{2}$, take $\mathbf{a} = \mathbf{1}_{n_0}$. Then

$$\mathbf{1}_{n_0}'\Gamma_{n_0}\mathbf{1}_{n_0} = n_0 + (2n_0-2)\rho\,.$$

Note that

$$n_0 + (2n_0-2)\rho < 0 \iff \rho < -\frac{n_0}{2n_0-2}\,.$$

Hence Γ_{n_0} is not non-negative definite.
If $\rho > 1/2$, so $\rho > \frac{n_0}{2n_0-2} > \frac{1}{2}$, take $\mathbf{a} = \mathbf{1}^*_{n_0}$. Then

$$(\mathbf{1}^*_{n_0})'\Gamma_{n_0}\mathbf{1}^*_{n_0} = n_0 - (2n_0 - 2)\rho \,.$$

Note that

$$n_0 - (2n_0 - 2)\rho < 0 \iff \rho > \frac{n_0}{2n_0 - 2} \,.$$

Hence Γ_{n_0} is not non-negative definite.
In any case we have that Γ_{n_0} is not non-negative definite and consequently $\rho(h)$ is not an autocorrelation function.

(2) $\Rightarrow$) Assume that Σ is the covariance matrix of $\mathbf{X} = (X_1, \ldots, X_n)$. We need to prove that Σ is non-negative definite and symmetric.
Symmetric. $\Sigma = \Sigma'$
Remember that the transpose matrix of any matrix A is given by

$$\left[A'\right]_{ij} = A_{ji} \,.$$

Now

$$\Sigma_{ij} = \mathrm{cov}(X_i, X_j) = \mathrm{cov}(X_j, X_i) = \Sigma_{ji} = \left[\Sigma'\right]_{ij} \,.$$

Therefore $\Sigma = \Sigma'$.
Non-negative definite. $\mathbf{a}'\Sigma\mathbf{a} \geq 0$

$$\begin{aligned}
\mathbf{a}'\Sigma\mathbf{a} &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \Sigma_{ij} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \mathrm{cov}(X_i, X_j) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \mathbb{E}\left[X_i X_j\right] \\
&= \mathbb{E}\Big[\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j X_i X_j\Big] \\
&= \mathbb{E}\Big[\Big(\sum_{i=1}^{n} a_i X_i\Big)^2\Big] \\
&\geq 0 \,.
\end{aligned}$$

$\Leftarrow$) Assume that Σ is non-negative definite and symmetric, i.e., $\mathbf{a}'\Sigma\mathbf{a} \geq 0$ and $\Sigma = \Sigma'$. We need to show that there exists a random vector $\mathbf{X}$ with covariance matrix Σ.
Since Σ is non-negative definite and symmetric then it admits the decomposition $\Sigma = O'\Lambda O$, where O is orthogonal, i.e., $OO' = O'O = I$ and Λ is a diagonal matrix whose diagonal entries are non-negative and the eigenvalues of Σ. Moreover, there exists a matrix $\Sigma^{1/2}$ such that $\Sigma = \Sigma^{1/2}\Sigma^{1/2}$ which can be defined as $\Sigma^{1/2} = O'\Lambda^{1/2}O$, where $\Lambda^{1/2}$ is the diagonal matrix whose diagonal entries are the square root of the eigenvalues of Σ.

Consider $\mathbf{Z} \sim N_n(\mathbf{0}, I)$ and let $\mathbf{X} = \Sigma^{1/2}\mathbf{Z}$. By properties of the multivariate normal distribution, we know that $\mathbf{X}$ is also multivariate normal distributed with mean $\mathbb{E}[\mathbf{X}] = \Sigma^{1/2}\mathbb{E}[\mathbf{Z}] = \mathbf{0}$ and covariance matrix

$$\operatorname{var}(\mathbf{X}) = \operatorname{var}\left(\Sigma^{1/2}\mathbf{Z}\right) = \Sigma^{1/2}\operatorname{var}(\mathbf{Z})\left(\Sigma^{1/2}\right)' = \Sigma^{1/2}I\Sigma^{1/2} = \Sigma\,.$$

Note. $\operatorname{var}(\mathbf{AX} + \mathbf{a}) = \mathbf{A}\operatorname{var}(\mathbf{X})\,\mathbf{A}'$.

(3) a) We know that the defining difference equation of an AR(1) process

$$X_t = \phi X_{t-1} + Z_t\,, \tag{1}$$

with $|\phi| < 1$ and (Z_t) white noise, has a solution given by

$$X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}\,. \tag{2}$$

Suppose that there is another stationary solution $(\tilde{X}_t)$ to (1) which does not have the form (2). Since $(\tilde{X}_t)$ is solution to (1), then

$$\begin{aligned}
\tilde{X}_t &= \phi\tilde{X}_{t-1} + Z_t \\
&= \phi\left(\phi\tilde{X}_{t-2} + Z_{t-1}\right) + Z_t \\
&= \phi^2\tilde{X}_{t-2} + \phi Z_{t-1} + Z_t \\
&\ \ \vdots \\
&= \phi^n\tilde{X}_{t-n} + \sum_{j=0}^{n-1}\phi^j Z_{t-j}
\end{aligned}$$

for all n.
Result.(Lyapunov inequality) For $0 < r < s < \infty$ and any random variable X we have

$$(\mathbb{E}[|X|^r])^{1/r} \leq (\mathbb{E}[|X|^s])^{1/s}\,.$$

Then

$$\mathbb{E}[|Z_t|] \leq (\mathbb{E}\left[Z_t^2\right])^{1/2} = \sigma$$

and

$$\mathbb{E}[|\tilde{X}_t|] \leq (\mathbb{E}[\tilde{X}_t^2])^{1/2} = (\mathbb{E}[\tilde{X}_0^2])^{1/2} \quad \text{(by stationarity)}\,.$$

Thus, we have that

$$\begin{aligned}
\mathbb{E}[|X_t - \tilde{X}_t|] &= \mathbb{E}\Big[\Big|\sum_{j=0}^{\infty}\phi^j Z_{t-j} - \phi^n\tilde{X}_{t-n} - \sum_{j=0}^{n-1}\phi^j Z_{t-j}\Big|\Big] \\
&= \mathbb{E}\Big[\Big|\sum_{j=n}^{\infty}\phi^j Z_{t-j} - \phi^n\tilde{X}_{t-n}\Big|\Big] \\
&\leq |\phi|^n\mathbb{E}[|\tilde{X}_{t-n}|] + \sum_{j=n}^{\infty}|\phi|^j\mathbb{E}\left[|Z_{t-j}|\right]
\end{aligned}$$

$$
\begin{aligned}
&\le |\phi|^n \mathbb{E}[|\tilde{X}_{t-n}|] + \sup_j \mathbb{E}\left[|Z_{t-j}|\right] \sum_{j=n}^{\infty} |\phi|^j \\
&\le |\phi|^n (\mathbb{E}[\tilde{X}_0^2])^{1/2} + \sigma \sum_{j=n}^{\infty} |\phi|^j \\
&= |\phi|^n (\mathbb{E}[\tilde{X}_0^2])^{1/2} + \sigma \frac{|\phi|^n}{1-|\phi|}
\end{aligned}
$$

for all n. Now, since

$$
|\phi|^n \mathbb{E}([\tilde{X}_0^2])^{1/2} + \sigma \frac{|\phi|^n}{1-|\phi|} \to 0 \quad \text{as} \quad n \to \infty,
$$

it follows that $\mathbb{E}[|X_t - \tilde{X}_t|] = 0$ and consequently $X_t = \tilde{X}_t$ a.s.

b) Suppose that there exists a stationary solution (X_t) to (1) when $|\phi| = 1$. We have that

$$
X_t = \phi^{n+1} X_{t-n-1} + \sum_{j=0}^{n} \phi^j Z_{t-j}
$$

which implies that

$$
X_t - \phi^{n+1} X_{t-n-1} = \sum_{j=0}^{n} \phi^j Z_{t-j}\,.
$$

Then

$$
\begin{aligned}
\operatorname{var}(X_t - \phi^{n+1} X_{t-n-1}) &= \operatorname{var}\left(\sum_{j=0}^{n} \phi^j Z_{t-j}\right) \quad \text{(Form of an MA(n) process)} \\
&= \sigma^2 \sum_{j=0}^{n} \left(\phi^j\right)^2 \\
&= \sigma^2 (n+1)\,.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\operatorname{var}(X_t - \phi^{n+1} X_{t-n-1}) &= \operatorname{var}(X_t) + \left(\phi^{n+1}\right)^2 \operatorname{var}(X_{t-n-1}) - 2\phi^{n+1} \operatorname{cov}(X_t, X_{t-n-1}) \\
&= 2\gamma_X(0) - 2\phi^{n+1}\gamma_X(n+1) \\
&\le 2\gamma_X(0) + 2|\phi|^{n+1}|\gamma_X(n+1)| \\
&\le 4\gamma_X(0)\,.
\end{aligned}
$$

Note. $|\gamma_X(h)| \le \gamma_X(0)$ for all h.
Then we have that

$$
\sigma^2(n+1) \le 4\gamma_X(0)
$$

for all n, implying that $\gamma_X(0) = \infty$ which is a contradiction to the assumption that (X_t) is stationary. Therefore there is no stationary solution to (1) when $|\phi| = 1$.

(4) Observation. Since $np - 1 \le [np] \le np + 1$ we have that

$$
p - \frac{1}{n} = \frac{np-1}{n} \le \frac{[np]}{n} \le \frac{np+1}{n} = p + \frac{1}{n}
$$

from where it follows that $\frac{[np]}{n} \to p$ as $n \to \infty$.
First consider

$$\begin{aligned}\overline{X}_n &= \frac{1}{n}\sum_{t=1}^{n} X_t \\ &= \frac{1}{n}\left(\sum_{t=1}^{[np]} X_t + \sum_{t=[np]+1}^{n} X_t\right) \\ &= \frac{1}{n}\sum_{t=1}^{[np]} X_t^{(1)} + \frac{1}{n}\sum_{t=[np]+1}^{n} X_t^{(2)} .\end{aligned}$$

For the first term we have that (using that $[np] \to \infty$ as $n \to \infty$)

$$\frac{1}{n}\sum_{t=1}^{[np]} X_t^{(1)} = \left(\frac{[np]}{n}\right)\left(\frac{1}{[np]}\sum_{t=1}^{[np]} X_t^{(1)}\right) \to p\mathbb{E}\left[X_0^{(1)}\right] .$$

For the second term we have that

$$\begin{aligned}\frac{1}{n}\sum_{t=[np]+1}^{n} X_t^{(2)} &= \frac{1}{n}\left(\sum_{t=1}^{n} X_t^{(2)} - \sum_{t=1}^{[np]} X_t^{(2)}\right) \\ &= \frac{1}{n}\sum_{t=1}^{n} X_t^{(2)} - \frac{1}{n}\sum_{t=1}^{[np]} X_t^{(2)} \\ &= \frac{1}{n}\sum_{t=1}^{n} X_t^{(2)} - \left(\frac{[np]}{n}\right)\left(\frac{1}{[np]}\sum_{t=1}^{[np]} X_t^{(2)}\right) \\ &\to \mathbb{E}\left[X_0^{(2)}\right] - p\mathbb{E}\left[X_0^{(2)}\right] \\ &= (1-p)\mathbb{E}\left[X_0^{(2)}\right] .\end{aligned}$$

Hence

$$\overline{X}_n \to p\mathbb{E}\left[X_0^{(1)}\right] + (1-p)\mathbb{E}\left[X_0^{(2)}\right] .$$

Next, we have that

$$\begin{aligned}\gamma_{n,X}(h) &= \frac{1}{n}\sum_{t=1}^{n-h}\left(X_t - \overline{X}_n\right)\left(X_{t+h} - \overline{X}_n\right) \\ &= \frac{1}{n}\sum_{t=1}^{n-h}\left(X_t X_{t+h} - \overline{X}_n X_t - \overline{X}_n X_{t+h} + \left(\overline{X}_n\right)^2\right) \\ &= \frac{1}{n}\sum_{t=1}^{n-h} X_t X_{t+h} - \left(\overline{X}_n\right)\left(\frac{1}{n}\sum_{t=1}^{n-h} X_t\right) - \left(\overline{X}_n\right)\left(\frac{1}{n}\sum_{t=1}^{n-h} X_{t+h}\right) + \left(\frac{n-h}{n}\right)\left(\overline{X}_n\right)^2 .\end{aligned}$$

Now let's consider each term in the expression above

$$\left(\frac{n-h}{n}\right)\left(\overline{X}_n\right)^2 \to \left(p\mathbb{E}\left[X_0^{(1)}\right] + (1-p)\mathbb{E}\left[X_0^{(2)}\right]\right)^2$$

$$\left(\overline{X}_n\right)\left(\frac{1}{n}\sum_{t=1}^{n-h} X_t\right) = \left(\overline{X}_n\right)\left(\frac{n-h}{n}\right)\left(\frac{1}{n-h}\sum_{t=1}^{n-h} X_t\right) \to \left(p\mathbb{E}\left[X_0^{(1)}\right] + (1-p)\mathbb{E}\left[X_0^{(2)}\right]\right)^2 .$$

By using that $(X_t^{(1)})$ and $(X_t^{(2)})$ come from strictly stationary and ergodic models, it follows that

$$\left(\overline{X}_n\right)\left(\frac{1}{n}\sum_{t=1}^{n-h} X_{t+h}\right) = \left(\overline{X}_n\right)\left(\frac{n-h}{n}\right)\left(\frac{1}{n-h}\sum_{t=1}^{n-h} X_{t+h}\right) \to \left(p\mathbb{E}\left[X_0^{(1)}\right] + (1-p)\mathbb{E}\left[X_0^{(2)}\right]\right)^2 .$$

For the remaining term we have

$$\begin{aligned}
\frac{1}{n}\sum_{t=1}^{n-h} X_t X_{t+h} &= \frac{1}{n}\left(\sum_{t=1}^{[np]-h} X_t X_{t+h} + \sum_{t=[np]-h+1}^{[np]} X_t X_{t+h} + \sum_{t=[np]+1}^{n-h} X_t X_{t+h}\right) \\
&= \frac{1}{n}\left(\sum_{t=1}^{[np]-h} X_t^{(1)} X_{t+h}^{(1)} + \sum_{t=[np]-h+1}^{[np]} X_t^{(1)} X_{t+h}^{(2)} + \sum_{t=[np]+1}^{n-h} X_t^{(2)} X_{t+h}^{(2)}\right) \\
&= \frac{1}{n}\sum_{t=1}^{[np]-h} X_t^{(1)} X_{t+h}^{(1)} + \frac{1}{n}\sum_{t=[np]-h+1}^{[np]} X_t^{(1)} X_{t+h}^{(2)} + \frac{1}{n}\sum_{t=[np]+1}^{n-h} X_t^{(2)} X_{t+h}^{(2)} .
\end{aligned}$$

We now consider each term in the last expression. For the first term we have

$$\frac{1}{n}\sum_{t=1}^{[np]-h} X_t^{(1)} X_{t+h}^{(1)} = \left(\frac{[np]-h}{n}\right)\left(\frac{1}{[np]-h}\sum_{t=1}^{[np]-h} X_t^{(1)} X_{t+h}^{(1)}\right) \to p\mathbb{E}\left[X_0^{(1)} X_h^{(1)}\right] .$$

For the second term

$$\begin{aligned}
\frac{1}{n}\sum_{t=[np]-h+1}^{[np]} X_t^{(1)} X_{t+h}^{(2)} &= \frac{1}{n}\left(\sum_{t=1}^{[np]} X_t^{(1)} X_{t+h}^{(2)} - \sum_{t=1}^{[np]-h} X_t^{(1)} X_{t+h}^{(2)}\right) \\
&= \left(\frac{[np]}{n}\right)\left(\frac{1}{[np]}\sum_{t=1}^{[np]} X_t^{(1)} X_{t+h}^{(2)}\right) \\
&\quad - \left(\frac{[np]-h}{n}\right)\left(\frac{1}{[np]-h}\sum_{t=1}^{[np]-h} X_t^{(1)} X_{t+h}^{(2)}\right) \\
&\to 0 .
\end{aligned}$$

Finally, for the third term

$$\begin{aligned}
\frac{1}{n}\sum_{t=[np]+1}^{n-h} X_t^{(2)} X_{t+h}^{(2)} &= \frac{1}{n}\left(\sum_{t=1}^{n-h} X_t^{(2)} X_{t+h}^{(2)} - \sum_{t=1}^{[np]} X_t^{(2)} X_{t+h}^{(2)}\right) \\
&= \frac{1}{n}\sum_{t=1}^{n-h} X_t^{(2)} X_{t+h}^{(2)} - \frac{1}{n}\sum_{t=1}^{[np]} X_t^{(2)} X_{t+h}^{(2)}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{n-h}{n}\right)\left(\frac{1}{n-h}\sum_{t=1}^{n-h} X_t^{(2)}X_{t+h}^{(2)}\right)\\
&- \left(\frac{[np]}{n}\right)\left(\frac{1}{[np]}\sum_{t=1}^{[np]} X_t^{(2)}X_{t+h}^{(2)}\right)\\
&\to \mathbb{E}\left[X_0^{(2)}X_h^{(2)}\right] - p\mathbb{E}\left[X_0^{(2)}X_h^{(2)}\right]\\
&= (1-p)\mathbb{E}\left[X_0^{(2)}X_h^{(2)}\right].
\end{aligned}
$$

Putting all together, we have that

$$
\begin{aligned}
\gamma_{n,X}(h) &\to p\mathbb{E}\big[X_0^{(1)}X_h^{(1)}\big] + (1-p)\mathbb{E}\big[X_0^{(2)}X_h^{(2)}\big] - \Big(p\mathbb{E}\big[X_0^{(1)}\big] + (1-p)\mathbb{E}\big[X_0^{(2)}\big]\Big)^2\\
&= p\Big(\mathbb{E}\big[X_0^{(1)}X_h^{(1)}\big] - \big(\mathbb{E}\big[X_0^{(1)}\big]\big)^2\Big) + (1-p)\Big(\mathbb{E}\big[X_0^{(2)}X_h^{(2)}\big] - \big(\mathbb{E}\big[X_0^{(2)}\big]\big)^2\Big)\\
&\quad + p\big(\mathbb{E}\big[X_0^{(1)}\big]\big)^2 + (1-p)\big(\mathbb{E}\big[X_0^{(2)}\big]\big)^2 - \Big(p^2\big(\mathbb{E}\big[X_0^{(1)}\big]\big)^2 + (1-p)^2\big(\mathbb{E}\big[X_0^{(2)}\big]\big)^2\\
&\quad + 2p(1-p)\mathbb{E}\big[X_0^{(1)}\big]\mathbb{E}\big[X_0^{(2)}\big]\Big)\\
&= p\gamma_{X^{(1)}}(h) + (1-p)\gamma_{X^{(2)}}(h) + p(1-p)\Big(\big(\mathbb{E}\big[X_0^{(1)}\big]\big)^2 - 2\mathbb{E}\big[X_0^{(1)}\big]\mathbb{E}\big[X_0^{(2)}\big] + \big(\mathbb{E}\big[X_0^{(2)}\big]\big)^2\Big)\\
&= p\gamma_{X^{(1)}}(h) + (1-p)\gamma_{X^{(2)}}(h) + p(1-p)\left(\mathbb{E}\big[X_0^{(1)}\big] - \mathbb{E}\big[X_0^{(2)}\big]\right)^2.
\end{aligned}
$$

(5) a) We have

$$
\Big(1 - B + \frac{1}{4}B^2\Big)X_t = (1+B)\,Z_t\,.
$$

Then $\phi(z) = 1 - z + \frac{1}{4}z^2$ and $\theta(z) = 1 + z$. Note that

$$
\theta(z) = 0 \iff 1 + z = 0 \iff z = -1
$$

and

$$
\phi(z) = 0 \iff 1 - z + \frac{1}{4}z^2 = 0 \iff \left(1 - \frac{1}{2}z\right)^2 = 0 \iff 1 - \frac{1}{2}z = 0 \iff z = 2 > 1\,.
$$

Hence, $\phi(z)$ and $\theta(z)$ have no common zeros for all complex z and $\phi(z) \neq 0$, $z \in \mathbb{C}$, $|z| \leq 1$. Therefore there is a causal (Theorem 4.10) and stationary (Proposition 4.9) solution.

b) We have

$$
(1 - 0.5B)\,X_t = (1 + 0.5B)\,(1 + 0.7B)\,Z_t\,.
$$

Then $\phi(z) = 1 - 0.5z$ and $\theta(z) = (1 + 0.5z)\,(1 + 0.7z)$. Note that

$$
\theta(z) = 0 \iff (1 + 0.5z)\,(1 + 0.7z) \iff z = -2 \text{ or } z = -10/7
$$

and

$$
\phi(z) = 0 \iff 1 - 0.5z = 0 \iff z = 2 > 1\,.
$$

Hence, $\phi(z)$ and $\theta(z)$ have no common zeros for all complex z and $\phi(z) \neq 0$, $z \in \mathbb{C}$, $|z| \leq 1$. Therefore there is a causal and stationary solution. The coefficients in the linear process representation $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ are determined by the relation

$$\psi(z) = \sum_{j=0}^{\infty} \psi_j z^j = \frac{\theta(z)}{\phi(z)}$$

so

$$\phi(z)\psi(z) = \theta(z)\,,$$

that is,

$$(1 - 0.5z)\left(\psi_0 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \psi_4 z^4 + \cdots\right) = 1 + 1.2z + 0.35z^2\,.$$

Comparing the coefficients we obtain

$$\begin{aligned}
&1 = \psi_0\\
&1.2 = \psi_1 - 0.5\psi_0, \quad \psi_1 = 1.7\\
&0.35 = \psi_2 - 0.5\psi_1, \quad \psi_2 = 1.2\\
&0 = \psi_3 - 0.5\psi_2, \quad \psi_3 = 0.6\\
&0 = \psi_4 - 0.5\psi_3, \quad \psi_4 = 0.3
\end{aligned}$$