ASSIGNMENT 1 FOR STATØK2, BLOCK 1, 2021/2022

(1) Let (X_t) and (Y_t) be uncorrelated stationary processes, i.e., X_s and Y_t are uncorrelated for every s and t. Show that $Z_t = X_t + Y_t, t \in \mathbb{Z}$, is stationary. Calculate the autocovariance function $\gamma_Z(h) = \mathrm{cov}(Z_0, Z_h)$, $h \in \mathbb{Z}$.

(2) Let (X_t) and (Y_t) be independent stationary processes. Does $Z_t = X_t Y_t, t \in \mathbb{Z}$, constitute a stationary process?

(3) Consider an iid sequence (W_i) of standard normal random variables and define the time series

$$X_1 = \frac{W_1 + W_2}{\sqrt{2}}\,, X_2 = \frac{W_1 - W_2}{\sqrt{2}}\,, X_3 = \frac{W_3 + W_4}{\sqrt{2}}\,, X_4 = \frac{W_3 - W_4}{\sqrt{2}}\,, \ldots\,,$$

$$X_1 = \mathrm{sign}(W_2)|W_1|\,, X_2 = \mathrm{sign}(W_1)|W_2|\,, X_3 = \mathrm{sign}(W_4)|W_3|\,, X_4 = \mathrm{sign}(W_3)|W_4|\,, \ldots\,,$$

$$X_1 = \mathrm{sign}(W_1)|W_1|\,, X_2 = \mathrm{sign}(W_2)|W_1|\,, X_3 = \mathrm{sign}(W_3)|W_1|\,, \ldots\,,$$

$$X_1 = W_1 W_0\,, X_2 = W_2 W_1\,, X_3 = W_3 W_2\,, \ldots\,,$$

$$X_1 = W_1 \cos(1) + W_0 \sin(1)\,, X_2 = W_2 \cos(2) + W_1 \sin(2)\,, X_3 = W_3 \cos(3) + W_2 \sin(3)\,, \ldots.$$

(a) Which of these time series models constitute white noise or iid white noise?
(b) Which of these models are stationary/strictly stationary?

Hint 1: If $(Y_1, \ldots, Y_t)$ is jointly Gaussian then any vector of linear combinations of $Y_1, \ldots, Y_n$ is jointly Gaussian as well.
Hint 2: For an iid sequence (Y_i) of symmetric random variables the sequences $(\mathrm{sign}(Y_i))$ and $(|Y_i|)$ are independent.

(4) Consider the sinusoid time series

$$X_t = A\cos(\theta t) + B\sin(\theta t)\,, \qquad t \in \mathbb{Z}\,,$$

for iid standard normal random variables A, B and some $\theta \in (0, \pi/2)$.
(a) Show that this is a strictly stationary time series.
(b) Show that the sample mean $\overline{X}_n \overset{\text{a.s.}}{\to} \mathbb{E}X_0 = 0$.
(c) Show that this time series is non-ergodic by finding a suitable function f such that

$$\frac{1}{n}\sum_{t=1}^{n} f(X_t) \overset{\text{a.s.}}{\not\to} \mathbb{E}[f(X_0)]\,.$$

Hint: recall that

$$\sum_{t=1}^{n} \sin(t\theta) = \frac{\sin(\theta(n+1)/2)\sin(\theta n/2)}{\sin(\theta/2)} \text{ and } \sum_{t=1}^{n} \cos(t\theta) = \frac{\cos(\theta(n+1)/2)\sin(\theta n/2)}{\sin(\theta/2)}\,.$$

(5) Show that an MA(q) process with white noise (Z_t) is stationary and calculate the autocorrelation function.

(6) A standard Cauchy random variable X has characteristic function [1]

$$E\big[\mathrm{e}^{\,itX}\big] = E[\cos(tX)] + iE[\sin(tX)] = \mathrm{e}^{\,-|t|}\,, \qquad t \in \mathbb{R}\,.$$

[1] The characteristic function of X determines its distribution, i.e., if X and Y have the same characteristic function, their distributions coincide.

(a) Show that for any real values (ψ_j) and iid copies (X_t) of X,

$$\sum_{j=1}^{k} \psi_j X_j \stackrel{d}{=} \sum_{j=1}^{k} |\psi_j| \, X_0 \, .$$

(b) Show that the sample mean $\overline{X}_n$ of $X_1, \ldots, X_n$ satisfies

$$\overline{X}_n \stackrel{d}{=} X_0 \, .$$

(c) Show that $E[|X_0|] = \infty$.

Hint: either use the form of the density of X_0 (the Cauchy distribution is the student distribution with one degree of freedom) or apply Kolmogorov's strong law of large numbers.

(d) Use the results in the lecture notes to conclude that (X_t) is an ergodic time series.