Written Examination Statistical Analysis of Econometric Time Series
November 10, 2017
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- This is an open book examination. You may use any book, the lecture notes, assignments and their solutions, pocket calculator, etc. In the solution to each problem you have to give reasons either by providing a proof or by referring to a result in the lecture notes, the assignments, any book, etc. In the latter case, give an *exact* reference.
- Use your time in a reasonable way. Do not copy the text of the problems. Refer to the lecture notes whenever possible. Do not re-prove results from the lecture notes.
- You may write with a pencil.
- This examination paper consists of 2 pages.
- You may write in English or Danish.
- The distribution of the points over the problems is as follows.

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Problem   1 a i ii iii iv v vi vii  b  2 a b c d     sum
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# points      1  1  1  1  1  3  2   2    2 2 2 2      20
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1. Consider the MA(2) process

$$X_t = Z_t + 2Z_{t-1} + Z_{t-2}\,, \qquad t \in \mathbb{Z}\,,$$

where (Z_t) is iid white noise with variance σ^2 and a finite fourth moment.

(a) Verify the following properties of the process (X_t):

(i) Strict stationarity

(ii) Stationarity

(iii) Ergodicity

(iv) Mixing

(v) Strong mixing

(vi) (X_t) satisfies the conditions of Ibragimov's central limit theorem

(vii) Non-invertibility

Give arguments for each of your answers.

(b) Calculate the spectral density of (X_t).

2. Consider a strictly stationary ARCH(2) process $X_t = \sigma_t Z_t$, $t \in \mathbb{Z}$, with iid standard normal noise (Z_t) where $\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \alpha_2 X_{t-2}^2$, $\alpha_i > 0$, $i = 0, 1, 2$.

(a) Assume that $E[\sigma_0^4] < \infty$. Show that $\nu_t = X_t^2 - \sigma_t^2$, $t \in \mathbb{Z}$, constitutes white noise.

(b) Under the conditions in **(a)** derive the spectral density of (X_t^2).
Hints: (i) The process (X_t^2) has an AR(2) representation. (ii) You are not required to determine the values of $\mathrm{var}(X_0^2)$ or $\mathrm{var}(\nu_0)$.

(c) Give a condition on the parameters $\alpha_i, i = 1, 2$, that ensures $E[X_t^2] = \infty$.

(d) Assume $n \geq 3$. Predict the 5%-quantile of the distribution of X_{n+1} conditional on the past $X_1, \ldots, X_n$, i.e., determine x such that

$$P(X_{n+1} \leq x \mid X_n, X_{n-1}, \ldots, X_1) = 0.05.$$

Notice: the 5%-quantile of the normal distribution is approximately -1.65.

(e) Assume $n \geq 3$. Calculate the best prediction $\widehat{X_{n+1}^2}$ of X_{n+1}^2 (in the mean square sense) given the past $X_1, \ldots, X_n$, i.e.,

$$\widehat{X_{n+1}^2} = E\left[X_{n+1}^2 \mid X_n, \ldots, X_1\right],$$

and the corresponding mean square prediction error, i.e.,

$$E\left[(\widehat{X_{n+1}^2} - X_{n+1}^2)^2\right].$$

Hint: It is useful to know that $E[Z_0^4] = 3$. We also assume that $E[\sigma_0^4]$ is known.

End of Examination Paper

Solutions to the written examination Statistical Analysis of Econometric Time Series, 10 November, 2017

1. **(a)**

   (i) **1 point** The process has the structure $X_t = f(Z_t, Z_{t-1}, Z_{t-2})$ for an iid sequence and a deterministic function. Proposition 2.14 yields strict stationarity.

   (ii) **1 point** The process is strictly stationary and has finite variance. Proposition 2.19 yields that (X_t) is stationary.

   (iii) **1 point** (Z_t) is iid, hence ergodic. Then $X_t = f(Z_t, Z_{t-1}, Z_{t-2})$ is ergodic in view of Theorem 2.25.

   (iv) **1 point** (Z_t) is iid, hence mixing. Then $X_t = f(Z_t, Z_{t-1}, Z_{t-2})$ is mixing in view of Theorem 2.38. Alternatively, one can use the fact that $(X_s)_{s\leq 0}$ and $(X_s)_{s>2}$ are independent, i.e., (X_t) is 2-dependent; see Remark 2.37.

   (v) **1 point** Since $\sigma(\ldots, X_{-1}, X_0)$ and $\sigma(X_3, X_4, \ldots)$ are independent the strong mixing coefficients $\alpha_h = 0$ for $h > 3$. Hence (X_t) is strongly mixing.

   (vi) **3 point** Since $\alpha_h = 0$ for $h > 3$ the condition $\sum_h \alpha_h^{\delta/(2+\delta)} < \infty$ is satisfied. Moreover, by Minkowski's inequality

$$(E[X_0^4])^{1/4} = \big(E\big[Z_0 + 2Z_{-1} + Z_{-2}\big]^4\big)^{1/4} \leq 2(E(Z_0^4))^{1/4} + (E((2Z_0)^4))^{1/4} < \infty\,.$$

The same bound can be derived by direct calculation of $E[X_0^4]$ or by some bounds for $E[|X_0|^{2+\delta}]$, some $\delta > 0$, with the C_r-inequality. We also have

$$\lim_{n\to\infty} (\sqrt{n}\overline{X}_n) = \gamma_X(0) + 2\sum_{h=1}^{\infty} \gamma_X(h)$$

Since $\gamma_X(h) = \sum_{j=0}^{2-|h|} \theta_j\theta_{j+|h|} \geq 0$ $(\theta_0 = 1, \theta_1 = 2, \theta_2 = 1)$ and $\gamma_X(0) = \mathrm{var}(X_0) > 0$ (X_0 is not a constant) we have

$$\gamma_X(0) + 2\sum_{h=1}^{\infty} \gamma_X(h) > \gamma_X(0) > 0\,.$$

   (vii). **2 points** We have $\phi(z) \equiv 1$ and

$$\theta(z) = 1 + 2z + z^2 = (1+z)^2$$

which has the root $z_0 = -1$. It lies on the unit circle and therefore the process is not invertible; see Theorem 4.14.

   (b) 2 points The MA(2) process (X_t) is causal. Therefore the spectral density has the form (see Theorem 6.15)

$$f_X(\lambda) = \frac{\sigma^2}{2\pi}|\theta(\mathrm{e}^{-i\lambda})|^2 = \frac{\sigma^2}{2\pi}\big|1 + 2\mathrm{e}^{-i\lambda} + \mathrm{e}^{-i2\lambda}\big|^2$$

2. **(a) 2 points** Since X_t^2 has finite second moment and $\sigma_t^2 = f(Z_{t-1}, Z_{t-2}, \ldots)$ the random variables $Z_t^2 - 1$ and σ_t^2 are independent, hence the random variables

$$\nu_t = \sigma_t^2 (Z_t^2 - 1)$$

have finite 2nd moment and are uncorrelated. Of course, $E[\nu_t] = E[\sigma_t^2] E[Z_t^2 - 1] = 0$ Hence (ν_t) constitutes white noise.

(b) 2 points In view of **(a)**, $(X_t^2 - E[X_0^2])$ has the AR(2) representation (see Section 5.2)

$$\begin{aligned} & X_t^2 - \alpha_1 X_{t-1}^2 - \alpha_2 X_{t-2}^2 - \alpha_0 \\ = \ & (X_t^2 - E[X_0^2]) - \alpha_1 (X_{t-1}^2 - E[X_0^2]) - \alpha_2 (X_{t-2}^2 - E[X_0^2]) \\ = \ & X_t^2 - \sigma_t^2 = \nu_t \,. \end{aligned}$$

Hence (see Theorem 6.15)

$$f_{X^2}(\lambda) = \frac{\mathrm{var}(\nu_0)}{2\pi} \frac{1}{\left|1 - \alpha_1 \mathrm{e}^{-i\lambda} - \alpha_2 \mathrm{e}^{-2i\lambda}\right|^2}, \qquad \lambda \in [0, \pi],$$

where $\mathrm{var}(\nu_0) = E[\nu_0^2] = E[\sigma_0^4] E[(Z_0^2 - 1)^2]$.

(c) 2 points We have $E[X_t^2] = E[\sigma_0^2] E[Z_0^2] = E[\sigma_0^2]$ and since (σ_t) is strictly stationary

$$\begin{aligned} E[\sigma_t^2] &= \alpha_0 + \alpha_1 E[\sigma_{t-1}^2] E[Z_{t-1}^2] + \alpha_2 E[\sigma_{t-2}^2] E[Z_{t-2}^2] \\ &= \alpha_0 + (\alpha_1 + \alpha_2) E[\sigma_t^2] \,. \end{aligned}$$

Since $\alpha_0 > 0$ the relation $E[\sigma_t^2] < \infty$ is possible only if $\alpha_1 + \alpha_2 < 1$. Hence, $\alpha_1 + \alpha_2 \geq 1$ implies $E[\sigma_t^2] = \infty$.

(d) 2 points Given the past values $X_n, X_{n-1}, \ldots, X_1$, the random variable X_{n+1} has a normal distribution with variance $\sigma_{n+1}^2 = \alpha_0 + \alpha_1 X_n^2 + \alpha_2 X_{n-1}^2$. Hence we have

$$P(X_{n+1} \leq x \mid X_n, X_{n-1}, \ldots, X_1) = P(Z_{n+1} \leq x/\sigma_{n+1} \mid X_n, X_{n-1}) = 0.05$$

and $x/\sigma_{n+1} = q_{0.05} = -1.65$, i.e., $x = -1.65\sigma_{n+1}$ is a prediction of the 5% quantile of X_{n+1} given $X_1, \ldots, X_n$.

(e) 2 points The prediction of X_{n+1}^2 given $X_n, X_{n-1}, \ldots, X_1$ is given by

$$E\big[X_{n+1}^2 \mid X_1, \ldots, X_n\big] = \sigma_{n+1}^2 E[Z_{n+1}^2] = \sigma_{n+1}^2 \,.$$

The mean square prediction error is given by

$$\begin{aligned} E\big[(X_{n+1}^2 - \sigma_{n+1}^2)^2\big] &= E[\sigma_{n+1}^4] E[(Z_0^2 - 1)^2] \\ &= E[\sigma_0^4]\big(E[Z_0^4] - 2E[Z_0^2] + 1\big) \\ &= E[\sigma_0^4]\big(3 - 2 + 1\big) = 2\,E[\sigma_0^4] \,. \end{aligned}$$