Solutions to Assignment 6 for StatØk2, Block 1, 2021/2022 by Jorge Yslas

(1) By exercise 3 b) in Assignment 5, we have an AR(p) equation for $(Y_t) = (X_t^2 - \mathbb{E}[X_0^2])$.

$$(X_t^2 - \mathbb{E}\left[X_0^2\right]) - \sum_{i=1}^{p} \alpha_i (X_{t-i}^2 - \mathbb{E}\left[X_0^2\right]) = \nu_t \,,$$

where $(\nu_t) = (X_t^2 - \sigma_t^2)$ is white noise. This equation can be written as

$$Y_t - \sum_{i=1}^{p} \alpha_i Y_{t-i} = \nu_t \,. \tag{1}$$

Since the Yule-Walker estimator is defined for an AR(p) process driven by white noise, we can use the same form of the estimator which is based on the ACVF of (Y_t).
To be precise, we calculate these covariances. Observe that (for $i > 0$)

$$\mathbb{E}\left[\nu_t X_{t-i}^2\right] = \mathbb{E}\left[\sigma_t^2 (Z_t^2 - 1) \sigma_{t-i}^2 Z_{t-i}^2\right] = \mathbb{E}\left[(Z_t^2 - 1)\right] \mathbb{E}\left[\sigma_t^2 \sigma_{t-i}^2 Z_{t-i}^2\right] = 0$$

and

$$\begin{aligned}\mathbb{E}\left[\nu_t X_t^2\right] &= \mathbb{E}\left[\sigma_t^2 (Z_t^2 - 1) \sigma_t^2 Z_t^2\right] \\ &= \mathbb{E}\left[\sigma_t^4 Z_t^4\right] - \mathbb{E}\left[\sigma_t^4 Z_t^2\right] = \mathbb{E}\left[\sigma_t^4\right] \mathbb{E}\left[Z_t^4\right] - \mathbb{E}\left[\sigma_t^4\right] \mathbb{E}\left[Z_t^2\right] = \mathbb{E}\left[\sigma_0^4\right] \left(\mathbb{E}\left[Z_0^4\right] - 1\right) \,.\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}\left[\nu_t Y_{t-i}\right] &= \mathbb{E}\left[\nu_t \left(X_{t-i}^2 - \mathbb{E}\left[X_0^2\right]\right)\right] \\ &= \mathbb{E}\left[\nu_t X_{t-i}^2\right] - \mathbb{E}\left[X_0^2\right] \mathbb{E}\left[\nu_t\right] \\ &= \mathbb{E}\left[\nu_t X_{t-i}^2\right] \\ &= \begin{cases} \mathbb{E}\left[\sigma_0^4\right] \left(\mathbb{E}\left[Z_0^4\right] - 1\right) & i = 0 \,, \\ 0 & i > 0 \,. \end{cases}\end{aligned}$$

We also have that

$$\mathbb{E}\left[Y_t\right] = \mathbb{E}\left[X_t^2 - \mathbb{E}\left[X_0^2\right]\right] = \mathbb{E}\left[X_0^2\right] - \mathbb{E}\left[X_0^2\right] = 0 \,.$$

Then

$$\gamma_Y(h) = \mathrm{cov}(Y_t, Y_{t+h}) = \mathbb{E}\left[Y_t Y_{t+h}\right] \,.$$

Observe also that

$$\gamma_Y(h) = \mathrm{cov}(Y_t, Y_{t+h}) = \mathrm{cov}(X_t^2 - \mathbb{E}\left[X_0^2\right], X_{t+h}^2 - \mathbb{E}\left[X_0^2\right]) = \mathrm{cov}(X_t^2, X_{t+h}^2) = \gamma_{X^2}(h) \,.$$

Now, let $\tilde{\sigma} = \mathbb{E}\left[\sigma_0^4\right] \left(\mathbb{E}\left[Z_0^4\right] - 1\right)$. Then, by multiplying (1) by Y_t, we get that

$$Y_t^2 - \sum_{i=1}^{p} \alpha_i Y_{t-i} Y_t = \nu_t Y_t$$

and by taking expected values on the expression above, it follows that

$$\gamma_Y(0) - \alpha_1 \gamma_Y(1) - \alpha_2 \gamma_Y(2) - \cdots - \alpha_p \gamma_Y(p) = \tilde{\sigma} \,.$$

Similarly by multiplying (1) by Y_{t-1} and taking expected values, we get

$$\gamma_Y(1) - \alpha_1\gamma_Y(0) - \alpha_2\gamma_Y(1) - \cdots - \alpha_p\gamma_Y(p-1) = 0\,.$$

In general, if we multiply (1) by Y_{t-i} and take expected values, we get the following equations

$$\begin{aligned}
\tilde{\sigma} &= \gamma_Y(0) - \alpha_1\gamma_Y(1) - \alpha_2\gamma_Y(2) - \cdots - \alpha_p\gamma_Y(p)\,,\\
0 &= \gamma_Y(1) - \alpha_1\gamma_Y(0) - \alpha_2\gamma_Y(1) - \cdots - \alpha_p\gamma_Y(p-1)\,,\\
\vdots &= \vdots\\
0 &= \gamma_Y(p) - \alpha_1\gamma_Y(p-1) - \alpha_2\gamma_Y(p-2) - \cdots - \alpha_p\gamma_Y(0)\,.
\end{aligned}$$

Then, we can replace $\gamma_Y(h)$ by $\gamma_{n,Y}(h)$ (or $\gamma_{n,X^2}(h)$) in order to estimate the parameters α_i.

(2) The general form of the Gaussian log-likelihood for a GARCH(p,q) process is

$$L_n(\theta)(X_1,\ldots,X_n) = -\frac{1}{2n}\sum_{t=1}^{n}\left(2\log\sigma_t(\theta) + \frac{X_t^2}{\sigma_t^2(\theta)}\right)\,.$$

Now, for an ARCH(1) process we have that (X_t) satisfies the equations

$$\begin{aligned}
X_t &= \sigma_t Z_t\,,\\
\sigma_t^2 &= \sigma^2(\theta) = \alpha_0 + \alpha_1 X_{t-1}^2\,,
\end{aligned}$$

where $\theta = (\alpha_0, \alpha_1)$ is any value in the parameter space, $\theta_0 = (\alpha_0^{(0)}, \alpha_1^{(0)})$ is the value of θ underlying the observations $X_1,\ldots,X_n$ (true paramater), $\sigma_i(\theta)$ is obtained by the ARCH(1) equations with parameter θ, using the observations X_i.
Then, even if the values σ_t are unobservable we can use the second equation to express σ_t in terms of the observed values X_t. Note also that we need to choose an initial value X_0, since $\sigma_1^2 = \alpha_0 + \alpha_1 X_0^2$. Thus, the log-likelihood takes the following form

$$L_n(\theta)(X_1,\ldots,X_n) = -\frac{1}{2n}\sum_{t=1}^{n}\left(\log\left(\alpha_0 + \alpha_1 X_{t-1}^2\right) + \frac{X_t^2}{\alpha_0 + \alpha_1 X_{t-1}^2}\right)\,.$$

On the other hand, in a GARCH(1,1) process we have that (X_t) satisfies the equations

$$\begin{aligned}
X_t &= \sigma_t Z_t\,,\\
\sigma_t^2 &= \sigma_t^2(\theta) = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1\sigma_{t-1}^2\,,
\end{aligned}$$

where $\theta = (\alpha_0, \alpha_1, \beta_1)$ is an element of the parameter space, $X_1,\ldots,X_n$ are the observations which correspond to the true parameter θ_0 to be estimated.
Again, we can use the second equation to express σ_t in terms of the X_t's. However, this equation involves previous values of σ_t, then we have to give an initial value σ_0 in order to be able to compute all consecutive values of σ_t. We also need an initial value for X_0 to be able to caluclate σ_1^2. For instance, if we take $\sigma_0 = X_0 = 0$ we have

$$\begin{aligned}
\sigma_1^2 &= \alpha_0\,,\\
\sigma_2^2 &= \alpha_0 + \alpha_1 X_1^2 + \beta_1\sigma_1^2\,,\\
&\vdots\\
\sigma_n^2 &= \alpha_0 + \alpha_1 X_{n-1}^2 + \beta_1\sigma_{n-1}^2\,.
\end{aligned}$$

Then we can put these expressions into the log-likelihood

$$L_n(\theta)(X_1, \dots, X_n) = -\frac{1}{2n} \sum_{t=1}^{n} \left(2 \log \sigma_t + \frac{X_t^2}{\sigma_t^2} \right) .$$

(3) a) We have that (X_t) satisfies the equations

$$\begin{aligned} X_t &= \sigma_t Z_t \,, \\ \sigma_t^2 &= \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \,. \end{aligned}$$

Then

$$\sigma_t^2 = \alpha_0 + \sigma_{t-1}^2 \left(\alpha_1 Z_{t-1}^2 + \beta_1 \right) . \tag{2}$$

Let's first compute $\mathrm{var}(X_0)$. We have that

$$\mathbb{E}\left[X_0^2\right] = \mathbb{E}\left[\sigma_0^2 Z_0^2\right] = \mathbb{E}\left[\sigma_0^2\right]$$

since we assume $\mathbb{E}[Z_0^2] = 1$. Now, by taking expected values in (2), we get that

$$\mathbb{E}\left[\sigma_t^2\right] = \alpha_0 + \mathbb{E}\left[\sigma_{t-1}^2\right] (\alpha_1 + \beta_1) \,.$$

It follows that

$$\mathbb{E}\left[\sigma_0^2\right] = \frac{\alpha_0}{1 - (\alpha_1 + \beta_1)} = \frac{\alpha_0}{1 - \phi} \,,$$

where $\phi = \alpha_1 + \beta_1$. We also have that

$$\mathbb{E}\left[X_0^4\right] = \mathbb{E}\left[\sigma_0^4 Z_0^4\right] = 3\mathbb{E}\left[\sigma_0^4\right]$$

since for standard normal Z_0, $\mathbb{E}\left[Z_0^4\right] = 3$. Then, by using (2), we get that

$$\begin{aligned} \mathbb{E}\left[\sigma_t^4\right] &= \mathbb{E}\left[\left(\alpha_0 + \sigma_{t-1}^2 \left(\alpha_1 Z_{t-1}^2 + \beta_1\right)\right)^2\right] \\ &= \mathbb{E}\left[\alpha_0^2 + 2\alpha_0 \sigma_{t-1}^2 \left(\alpha_1 Z_{t-1}^2 + \beta_1\right) + \sigma_{t-1}^4 \left(\alpha_1 Z_{t-1}^2 + \beta_1\right)^2\right] \\ &= \alpha_0^2 + 2\alpha_0 \mathbb{E}\left[\sigma_{t-1}^2\right] (\alpha_1 + \beta_1) + \mathbb{E}\left[\sigma_{t-1}^4\right] \mathbb{E}\left[\left(\alpha_1 Z_{t-1}^2 + \beta_1\right)^2\right] \\ &= \alpha_0^2 + 2\alpha_0 \phi \mathbb{E}\left[\sigma_{t-1}^2\right] + \mathbb{E}\left[\sigma_{t-1}^4\right] \mathbb{E}\left[\alpha_1^2 Z_{t-1}^4 + 2\alpha_1 \beta_1 Z_{t-1}^2 + \beta_1^2\right] \\ &= \alpha_0^2 + 2\alpha_0 \phi \frac{\alpha_0}{1 - \phi} + \mathbb{E}\left[\sigma_{t-1}^4\right] \left(3\alpha_1^2 + 2\alpha_1 \beta_1 + \beta_1^2\right) \\ &= \alpha_0^2 + 2\alpha_0^2 \frac{\phi}{1 - \phi} + \mathbb{E}\left[\sigma_{t-1}^4\right] \left(2\alpha_1^2 + \phi^2\right) , \end{aligned}$$

which implies that (here we need that σ_0 has finite 4th moment)

$$\mathbb{E}\left[\sigma_0^4\right] = \frac{\alpha_0^2 \left(\frac{1+\phi}{1-\phi}\right)}{1 - (2\alpha_1^2 + \phi^2)} \,.$$

Then

$$\mathbb{E}\left[X_0^4\right] = \frac{3\alpha_0^2 \,(1 + \phi)}{(1 - (2\alpha_1^2 + \phi^2))(1 - \phi)} \,.$$

Thus

$$\begin{aligned}
\mathrm{var}(X_0^2) &= \mathbb{E}\left[X_0^4\right] - (\mathbb{E}\left[X_0^2\right])^2 \\
&= \frac{3\alpha_0^2\,(1+\phi)}{(1-(2\alpha_1^2+\phi^2))(1-\phi)} - \frac{\alpha_0^2}{(1-\phi)^2} \\
&= \frac{3\alpha_0^2\,(1+\phi)\,(1-\phi) - \alpha_0^2(1-(2\alpha_1^2+\phi^2))}{(1-(2\alpha_1^2+\phi^2))(1-\phi)^2} \\
&= \frac{\alpha_0^2\left(3(1-\phi^2) - 1 + 2\alpha_1^2 + \phi^2\right)}{(1-(2\alpha_1^2+\phi^2))(1-\phi)^2} \\
&= \frac{2\alpha_0^2\left(1+\alpha_1^2-\phi^2\right)}{(1-(2\alpha_1^2+\phi^2))(1-\phi)^2}\,.
\end{aligned}$$

We now compute $\mathrm{cov}(X_0^2, X_1^2)$. We have that

$$\begin{aligned}
\mathbb{E}\left[X_0^2X_1^2\right] &= \mathbb{E}\left[\sigma_0^2Z_0^2\sigma_1^2Z_1^2\right] \\
&= \mathbb{E}\left[\sigma_0^2Z_0^2\sigma_1^2\right] \\
&= \mathbb{E}\left[\sigma_0^2Z_0^2\left(\alpha_0+\sigma_0^2\left(\alpha_1Z_0^2+\beta_1\right)\right)\right] \\
&= \alpha_0\mathbb{E}\left[\sigma_0^2\right] + \mathbb{E}\left[\sigma_0^4\right](3\alpha_1+\beta_1) \\
&= \frac{\alpha_0^2}{1-\phi} + \frac{\alpha_0^2\,(1+\phi)}{(1-(2\alpha_1^2+\phi^2))(1-\phi)}\,(2\alpha_1+\phi) \\
&= \frac{\alpha_0^2\left(1-2\alpha_1^2-\phi^2+2\alpha_1+\phi+2\alpha_1\phi+\phi^2\right)}{(1-(2\alpha_1^2+\phi^2))(1-\phi)} \\
&= \frac{\alpha_0^2\left(1+\phi+2\alpha_1(1+\phi)-2\alpha_1^2\right)}{(1-(2\alpha_1^2+\phi^2))(1-\phi)} \\
&= \frac{\alpha_0^2\left((1+\phi)(1+2\alpha_1)-2\alpha_1^2\right)}{(1-(2\alpha_1^2+\phi^2))(1-\phi)}\,.
\end{aligned}$$

Consequently

$$\begin{aligned}
\mathrm{cov}(X_0^2, X_1^2) &= \mathbb{E}\left[X_0^2X_1^2\right] - (\mathbb{E}\left[X_0^2\right])^2 \\
&= \frac{\alpha_0^2\left((1+\phi)(1+2\alpha_1)-2\alpha_1^2\right)}{(1-(2\alpha_1^2+\phi^2))(1-\phi)} - \frac{\alpha_0^2}{(1-\phi)^2} \\
&= \frac{\alpha_0^2\left((1+\phi)(1+2\alpha_1)-2\alpha_1^2\right)(1-\phi) - \alpha_0^2(1-(2\alpha_1^2+\phi^2))}{(1-(2\alpha_1^2+\phi^2))(1-\phi)^2} \\
&= \frac{\alpha_0^2\left((1-\phi^2)(1+2\alpha_1)-2\alpha_1^2(1-\phi)-1+2\alpha_1^2+\phi^2\right)}{(1-(2\alpha_1^2+\phi^2))(1-\phi)^2} \\
&= \frac{\alpha_0^2\left(2\alpha_1(1-\phi^2)+2\alpha_1^2\phi\right)}{(1-(2\alpha_1^2+\phi^2))(1-\phi)^2} \\
&= \frac{2\alpha_0^2\alpha_1\left(1+\alpha_1\phi-\phi^2\right)}{(1-(2\alpha_1^2+\phi^2))(1-\phi)^2}\,.
\end{aligned}$$

b) We have that

$$\rho_{X^2}(1) = \rho_{X^2}(1)\phi^{1-1}\,.$$

We can calculate $\rho_{X^2}(1)$ by using the values $\gamma_{X^2}(i)$, $i = 0, 1$, from a).

Assume that

$$\rho_{X^2}(h) = \rho_{X^2}(1)\phi^{h-1}$$

for some $h \geq 1$. Then

$$\begin{aligned}
\rho_{X^2}(h+1) &= \frac{\gamma_{X^2}(h+1)}{\gamma_{X^2}(0)} \\
&= \frac{\gamma_{X^2}(h+1)}{\gamma_{X^2}(0)} \frac{\gamma_{X^2}(h)}{\gamma_{X^2}(h)} \\
&= \frac{\gamma_{X^2}(h)}{\gamma_{X^2}(0)} \frac{\gamma_{X^2}(h+1)}{\gamma_{X^2}(h)} \\
&= \rho_{X^2}(h) \frac{\gamma_{X^2}(h+1)}{\gamma_{X^2}(h)} \\
&= \rho_{X^2}(1)\phi^{h-1} \frac{\gamma_{X^2}(h+1)}{\gamma_{X^2}(h)} \,.
\end{aligned}$$

We know that for $s > 0$

$$\gamma_{X^2}(s) = \mathbb{E}\left[X_0^2 X_s^2\right] - \left(\mathbb{E}\left[X_0^2\right]\right)^2$$

and

$$\mathbb{E}\left[X_0^2 X_s^2\right] = \mathbb{E}\left[X_0^2 \sigma_s^2 Z_s^2\right] = \mathbb{E}\left[X_0^2 \sigma_s^2\right] \,.$$

Now

$$\begin{aligned}
\mathbb{E}\left[X_0^2 X_{h+1}^2\right] &= \mathbb{E}\left[X_0^2 \sigma_{h+1}^2 Z_{h+1}^2\right] \\
&= \mathbb{E}\left[X_0^2 \sigma_{h+1}^2\right] \\
&= \mathbb{E}\left[X_0^2 \left(\alpha_0 + \alpha_1 X_h^2 + \beta_1 \sigma_h^2\right)\right] \\
&= \alpha_0 \mathbb{E}\left[X_0^2\right] + \alpha_1 \mathbb{E}\left[X_0^2 X_h^2\right] + \beta_1 \mathbb{E}\left[X_0^2 \sigma_h^2\right] \\
&= \alpha_0 \mathbb{E}\left[X_0^2\right] + \alpha_1 \mathbb{E}\left[X_0^2 X_h^2\right] + \beta_1 \mathbb{E}\left[X_0^2 X_h^2\right] \\
&= \alpha_0 \mathbb{E}\left[X_0^2\right] + \phi \mathbb{E}\left[X_0^2 X_h^2\right] \,.
\end{aligned}$$

Then

$$\begin{aligned}
\gamma_{X^2}(h+1) &= \mathbb{E}\left[X_0^2 X_{h+1}^2\right] - \left(\mathbb{E}\left[X_0^2\right]\right)^2 \\
&= \alpha_0 \mathbb{E}\left[X_0^2\right] + \phi \mathbb{E}\left[X_0^2 X_h^2\right] - \left(\mathbb{E}\left[X_0^2\right]\right)^2 \\
&= \alpha_0 \mathbb{E}\left[X_0^2\right] + \phi\left(\mathbb{E}\left[X_0^2 X_h^2\right] - \left(\mathbb{E}\left[X_0^2\right]\right)^2\right) + \phi\left(\mathbb{E}\left[X_0^2\right]\right)^2 - \left(\mathbb{E}\left[X_0^2\right]\right)^2 \\
&= \alpha_0 \mathbb{E}\left[X_0^2\right] + \phi \gamma_{X^2}(h) - (1-\phi)\left(\mathbb{E}\left[X_0^2\right]\right)^2 \\
&= \frac{\alpha_0^2}{1-\phi} + \phi \gamma_{X^2}(h) - (1-\phi)\left(\frac{\alpha_0}{1-\phi}\right)^2 \\
&= \phi \gamma_{X^2}(h) \,.
\end{aligned}$$

Which implies that

$$\frac{\gamma_{X^2}(h+1)}{\gamma_{X^2}(h)} = \phi$$

and consequently

$$\rho_{X^2}(h+1) = \rho_{X^2}(1)\phi^{(h+1)-1}.$$

(4) The spectral density of a real-valued stationary process is given by

$$f(\lambda) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{-in\lambda}\gamma(n), \quad \lambda \in [-\pi, \pi].$$

We know that $\gamma(h) = \gamma(-h)$ for all h, then

$$\begin{aligned}
f(-\lambda) &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{-in(-\lambda)}\gamma(n) \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{-i(-n)\lambda}\gamma(-n) \\
&= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{-ik\lambda}\gamma(k) \\
&= f(\lambda).
\end{aligned}$$

Meaning that the spectral density is even. Then it is enough to compute the density only on $[0, \pi]$.

(5) From exercise 3. a) in assignment 5, we know that $\mathbb{E}\left[X_0^4\right] < \infty$ if and only if $3\alpha_1^2 < 1$. We also know by exercise 1 that $(X_t^2 - \mathbb{E}\left[X_0^2\right])$ satisfies the AR(1) equation

$$(X_t^2 - \mathbb{E}\left[X_0^2\right]) - \alpha_1(X_{t-1}^2 - \mathbb{E}\left[X_0^2\right]) = \nu_t,$$

where $\nu_t = X_t^2 - \sigma_t^2$ is white noise with $\mathrm{var}(\nu_t) = \tilde{\sigma}^2 = \mathbb{E}\left[\sigma_0^4\right]\left(\mathbb{E}\left[Z_0^4\right] - 1\right) = 2\mathbb{E}\left[\sigma_0^4\right] = 2\alpha_0^2(1+\alpha_1)/((1-3\alpha_1^2)(1-\alpha_1))$. Then, the spectral density of (X_t^2) is given by

$$f_{X^2}(\lambda) = \frac{\tilde{\sigma}^2}{2\pi}\left|1 - \alpha_1 e^{-i\lambda}\right|^{-2} = \frac{\tilde{\sigma}^2}{2\pi}\left(1 - 2\alpha_1\cos(\lambda) + \alpha_1^2\right).$$

(6) For (X_t), we have that

$$\mathbb{E}\left[X_t\right] = \mathbb{E}\left[\sigma_t Z_t\right] = 0.$$

Also

$$\mathbb{E}\left[X_t X_{t+h}\right] = \mathbb{E}\left[\sigma_t Z_t \sigma_{t+h} Z_{t+h}\right] = 0$$

and

$$\mathbb{E}\left[X_t^2\right] = \mathbb{E}\left[\sigma_t^2 Z_t^2\right] = \mathbb{E}\left[\sigma_0^2\right] = \frac{\alpha_0}{1-\alpha_1-\beta_1} = \frac{\alpha_0}{1-\phi}.$$

Then

$$\gamma_X(h) = \begin{cases} \frac{\alpha_0}{1-\phi} & h = 0, \\ 0 & h \neq 0, \end{cases}$$

i.e., (X_t) is white noise. Therefore its spectral density is given by

$$f(\lambda) = \frac{\mathbb{E}\left[\sigma_0^2\right]}{2\pi} = \frac{\alpha_0}{2\pi(1-\phi)}\,.$$

We know that the general form of the spectral density of a causal ARMA(p,q) process is

$$f(\lambda) = \frac{\sigma^2}{2\pi}\frac{|\theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2}\,.$$

For an ARMA$(1,1)$ causal process we have that $\theta(z) = 1+\theta z$ and $\phi(z) = 1-\phi z$, then the spectral density is given by

$$\begin{aligned} f(\lambda) &= \frac{\sigma^2}{2\pi}\frac{|1+\theta e^{-i\lambda}|^2}{|1-\phi e^{-i\lambda}|^2} \\ &= \frac{\sigma^2}{2\pi}\frac{1+2\theta\cos(\lambda)+\theta^2}{1-2\phi\cos(\lambda)+\phi^2}\,. \end{aligned}$$

We can use this result to find the spectral density of (X_t^2). We have that (X_t) satisfies the GARCH(1,1) equations

$$\begin{aligned} X_t &= \sigma_t Z_t\,, \\ \sigma_t^2 &= \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1\sigma_{t-1}^2\,. \end{aligned}$$

Then

$$\begin{aligned} & X_t^2 - \alpha_1 X_{t-1}^2 = \alpha_0 + \beta_1\sigma_{t-1}^2 + X_t^2 - \sigma_t^2 \\ \iff & X_t^2 - (\alpha_1+\beta_1)X_{t-1}^2 = \alpha_0 - \beta_1(X_{t-1}^2 - \sigma_{t-1}^2) + (X_t^2 - \sigma_t^2) \\ \iff & X_t^2 - (\alpha_1+\beta_1)X_{t-1}^2 = \alpha_0 - \beta_1\nu_{t-1} + \nu_t \\ \iff & (X_t^2 - \mathbb{E}\left[X_0^2\right]) - (\alpha_1+\beta_1)(X_{t-1}^2 - \mathbb{E}\left[X_0^2\right]) = \nu_t - \beta_1\nu_{t-1}\,, \end{aligned}$$

where $\nu_t = X_t^2 - \sigma_t^2$ is a white noise with $\mathrm{var}(\nu_t) = \tilde{\sigma}^2 = \mathbb{E}\left[\sigma_0^4\right]\left(\mathbb{E}\left[Z_0^4\right]-1\right)$. Thus, $(X_t^2 - \mathbb{E}\left[X_0^2\right])$ satisfies an ARMA(1,1) equation with $\theta(z) = 1+(-\beta_1)z$ and $\phi(z) = 1-(\alpha_1+\beta_1)z$. Then the spectral density is given by

$$f_{X^2}(\lambda) = \frac{\tilde{\sigma}^2}{2\pi}\frac{1-2\beta_1\cos(\lambda)+\beta_1^2}{1-2(\alpha_1+\beta_1)\cos(\lambda)+(\alpha_1+\beta_1)^2}\,.$$

(7) a) Let $X_j := n^{-1/2}\sum_{t=1}^n Z_t\cos(\lambda_j t)$ and $Y_j := n^{-1/2}\sum_{t=1}^n Z_t\sin(\lambda_j t)$, $\lambda_j \in (0,\pi/2)$. Since these random variables have mean zero and are jointly Gaussian it is enough to compute their covariances which determine the joint distribution. Thus

$$\mathbb{E}\left[X_j\right] = n^{-1/2}\sum_{t=1}^n \mathbb{E}\left[Z_j\right]\cos(\lambda_j t) = 0$$

and

$$
\begin{aligned}
\operatorname{var}(X_j) &= n^{-1}\sum_{t=1}^{n}\operatorname{var}(Z_j)\cos^2(\lambda_j t)\\
&= \frac{\sigma^2}{n}\sum_{t=1}^{n}\frac{1+\cos(2\lambda_j t)}{2}\\
&= \frac{\sigma^2}{2n}\left(n+\sum_{t=1}^{n}\cos(2\lambda_j t)\right)\\
&= \frac{\sigma^2}{2n}\left(n+\frac{\cos(\lambda_j(n+1))\sin(\lambda_j n)}{\sin(\lambda_j)}\right)\\
&= \frac{\sigma^2}{2}\,.
\end{aligned}
$$

In the last step we used that $\sin(\lambda_j n) = \sin(2\pi j) = 0$. On the other hand,

$$
\mathbb{E}\left[Y_j\right] = n^{-1/2}\sum_{t=1}^{n}\mathbb{E}\left[Z_j\right]\sin(\lambda_j t) = 0
$$

and

$$
\begin{aligned}
\operatorname{var}(Y_j) &= n^{-1}\sum_{t=1}^{n}\operatorname{var}(Z_j)\sin^2(\lambda_j t)\\
&= \frac{\sigma^2}{n}\sum_{t=1}^{n}\frac{1-\cos(2\lambda_j t)}{2}\\
&= \frac{\sigma^2}{2n}\left(n-\sum_{t=1}^{n}\cos(2\lambda_j t)\right)\\
&= \frac{\sigma^2}{2n}\left(n-\frac{\cos(\lambda_j(n+1))\sin(\lambda_j n)}{\sin(\lambda_j)}\right)\\
&= \frac{\sigma^2}{2}\,.
\end{aligned}
$$

This shows that $X_j, Y_j \sim N(0, \sigma^2/2)$. Now the covariances

$$
\begin{aligned}
\mathbb{E}\left[X_j Y_j\right] &= \mathbb{E}\left[\left(n^{-1/2}\sum_{t=1}^{n}Z_t\cos(\lambda_j t)\right)\left(n^{-1/2}\sum_{t=1}^{n}Z_t\sin(\lambda_j t)\right)\right]\\
&= \frac{\sigma^2}{n}\sum_{t=1}^{n}\cos(\lambda_j t)\sin(\lambda_j t)\\
&= \frac{\sigma^2}{2n}\sum_{t=1}^{n}\sin(2\lambda_j t)\\
&= \frac{\sigma^2}{2n}\frac{\sin(\lambda_j(n+1))\sin(\lambda_j n)}{\sin(\lambda_j)}\\
&= 0\,.
\end{aligned}
$$

For the next calculations we assume $j \neq k$.

$$\begin{aligned}
\mathbb{E}\left[X_j Y_k\right] &= \frac{\sigma^2}{n} \sum_{t=1}^{n} \cos(\lambda_j t) \sin(\lambda_k t) \\
&= \frac{\sigma^2}{2n} \sum_{t=1}^{n} \left(\sin((\lambda_j + \lambda_k)t) + \sin((\lambda_k - \lambda_j)t)\right) \\
&= \frac{\sigma^2}{2n} \left(\frac{\sin((\lambda_j + \lambda_k)(n+1)/2) \sin((\lambda_j + \lambda_k)n/2)}{\sin((\lambda_j + \lambda_k)/2)} \right. \\
&\qquad \left. + \frac{\sin((\lambda_k - \lambda_j)(n+1)/2) \sin((\lambda_k - \lambda_j)n/2)}{\sin((\lambda_k - \lambda_j)/2)} \right) \\
&= 0\,,
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[X_j X_k\right] &= \frac{\sigma^2}{n} \sum_{t=1}^{n} \cos(\lambda_j t) \cos(\lambda_k t) \\
&= \frac{\sigma^2}{2n} \sum_{t=1}^{n} \left(\cos((\lambda_j + \lambda_k)t) + \cos((\lambda_k - \lambda_j)t)\right) \\
&= \frac{\sigma^2}{2n} \left(\frac{\cos((\lambda_j + \lambda_k)(n+1)/2) \sin((\lambda_j + \lambda_k)n/2)}{\sin((\lambda_j + \lambda_k)/2)} \right. \\
&\qquad \left. + \frac{\cos((\lambda_k - \lambda_j)(n+1)/2) \sin((\lambda_k - \lambda_j)n/2)}{\sin((\lambda_k - \lambda_j)/2)} \right) \\
&= 0\,,
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[Y_j Y_k\right] &= \frac{\sigma^2}{n} \sum_{t=1}^{n} \sin(\lambda_j t) \sin(\lambda_k t) \\
&= \frac{\sigma^2}{2n} \sum_{t=1}^{n} \left(\cos((\lambda_k - \lambda_j)t) - \cos((\lambda_j + \lambda_k)t)\right) \\
&= 0\,.
\end{aligned}$$

Hence all X_j, Y_j are uncorrelated and have the same distribution. Since they are Gaussian all X_j, Y_j are iid. Now,

$$2\pi I_{n,Z}(\lambda_j) = X_j^2 + Y_j^2 = \sigma^2(\xi_2^2/2) \stackrel{d}{=} \sigma^2\, Exp(1)$$

and hence $2pI_{n,Z}(\lambda_j)$, $1 \leq j < n/2$ are iid exponential.

b)

$$\begin{aligned}
\mathbb{P}\left(\max_{1\leq j\leq q} \frac{2\pi I_{n,Z}(\lambda_i)}{\sigma^2} - \log q \leq x \right) &= \mathbb{P}\left(\max_{1\leq j\leq q} \frac{2\pi I_{n,Z}(\lambda_i)}{\sigma^2} \leq x + \log q \right) \\
&= \left(\mathbb{P}\left(\frac{2\pi I_{n,Z}(\lambda_1)}{\sigma^2} \leq x + \log q \right) \right)^q \\
&= \left(1 - e^{-(x+\log q)} \right)^q \\
&= \left(1 - \frac{e^{-x}}{q} \right)^q \\
&\to \exp(-e^{-x})\,.
\end{aligned}$$