Solutions to Assignment 7 for StatØk2, Block 1, 2021/2022 by Jorge Yslas

1. a) In view of part c) we may assume without loss of generality that $\mu = 0$. Indeed, $\mathbb{E}[X_{n+h}] = \mathbb{E}[P_n X_{n+h}] = \mu$ and therefore $\mathbb{E}[X_{n+h} - P_n X_{n+h}] = \mu - \mu = 0$.

We have that

$$\begin{aligned}
&\mathbb{E}\Big[(X_{n+h} - P_n X_{n+h})^2\Big] \\
&= \mathbb{E}\Big[\Big(X_{n+h} - \Big(a_0 + \sum_{i=1}^{n} a_i X_{n+1-i}\Big)\Big)^2\Big] \\
&= \mathbb{E}\Big[\Big(X_{n+h} - \underbrace{\mu\Big(1 - \sum_{i=1}^{n} a_i\Big)}_{a_0=0} - \sum_{i=1}^{n} a_i X_{n+1-i}\Big)^2\Big] \\
&= \mathbb{E}\Big[X_{n+h}^2 - 2X_{n+h} \sum_{i=1}^{n} a_i X_{n+1-i} + \Big(\sum_{i=1}^{n} a_i X_{n+1-i}\Big)^2\Big] \\
&= Var\,(X_{n+h})^2 - 2\sum_{i=1}^{n} a_i Cov(X_{n+h}\,, X_{n+1-i}) + \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j Cov(X_{n+1-i}\,, X_{n+1-j}) \\
&= \gamma_X(0) - 2\sum_{i=1}^{n} a_i \gamma_X(h+i-1) + \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \gamma_X(i-j) \\
&= \gamma_X(0) - 2\mathbf{a}_n' \gamma_n(h) + \underbrace{\mathbf{a}_n' \Gamma_n \mathbf{a}_n}_{=\mathbf{a}_n' \gamma_n(h)} = \gamma_X(0) - \mathbf{a}_n' \gamma_n(h)
\end{aligned}$$

b) Assume that $\mathbf{a}_n^{(1)}$ and $\mathbf{a}_n^{(2)}$ are solutions to the equation $\Gamma_n \mathbf{a}_n = \gamma_n(h)$, i.e. $\Gamma_n \mathbf{a}_n^{(1)} = \gamma_n(h)$ and $\Gamma_n \mathbf{a}_n^{(2)} = \gamma_n(h)$. We also have that $a_0^{(j)} = \mu\left(1 - \sum_{i=1}^{n} a_i^{(j)}\right)$ for $j = 1, 2$. As in part a) we may assume without loss of generality that $\mu = 0$ and therefore $a_0^{(j)} = 0$, $j = 1, 2$. Consider the random variable

$$Z = \underbrace{(a_0^{(1)} - a_0^{(2)})}_{=0} + \sum_{j=1}^{n} (a_j^{(1)} - a_j^{(2)})\, X_{n+1-j}$$

Then

$$\begin{aligned}
\mathbb{E}[Z^2] &= \mathbb{E}\Big[\Big(\sum_{j=1}^{n} (a_j^{(1)} - a_j^{(2)}) X_{n+1-j}\Big)^2\Big] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} (a_i^{(1)} - a_i^{(2)})\,(a_j^{(1)} - a_j^{(2)}) Cov(X_{n+1-i}, X_{n+1-j}) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} (a_i^{(1)} - a_i^{(2)})\,(a_j^{(1)} - a_j^{(2)})\, \gamma_X(i-j) \\
&= (\mathbf{a}_n^{(1)})' \Gamma_n \mathbf{a}_n^{(1)} - (\mathbf{a}_n^{(2)})' \Gamma_n \mathbf{a}_n^{(1)} - (\mathbf{a}_n^{(1)})' \Gamma_n \mathbf{a}_n^{(2)} + (\mathbf{a}_n^{(2)})' \Gamma_n \mathbf{a}_n^{(2)} \\
&= (\mathbf{a}_n^{(1)})' \gamma_n(h) - (\mathbf{a}_n^{(2)})' \gamma_n(h) - (\mathbf{a}_n^{(1)})' \gamma_n(h) + (\mathbf{a}_n^{(2)})' \gamma_n(h) = 0
\end{aligned}$$

Hence $Z = 0$ a.s.

c) We have that

$$\mathbb{E}[X_{n+h}] = \mu$$

On the other hand,

$$\begin{aligned}\mathbb{E}[P_n X_{n+1}] &= \mathbb{E}\Big[a_0 + \sum_{i=1}^{n} a_i X_{n+1-i}\Big] \\ &= a_0 + \sum_{i=1}^{n} a_i \mathbb{E}[X_{n+1-i}] = a_0 + \sum_{i=1}^{n} a_i \mu \\ &= \mu\Big(1 - \sum_{i=1}^{n} a_i\Big) + \sum_{i=1}^{n} a_i \mu = \mu\end{aligned}$$

Therefore $\mathbb{E}\left[X_{n+h}\right] = \mathbb{E}\left[P_n X_{n+1}\right]$.

d) Note first that proving

$$\mathbb{E}\left[\left(X_{n+h} - P_n X_{n+h}\right) X_j\right] = 0, \qquad j = 1, \ldots, n$$

is the same as proving

$$\mathbb{E}\left[\left(X_{n+h} - P_n X_{n+h}\right) X_{n+1-j}\right] = 0, \qquad j = 1, \ldots, n$$

We may also assume without loss of generality that $\mu = 0$. Indeed,

$$\begin{aligned}&\mathbb{E}[(X_{n+h} - P_n X_{n+h})X_{n+1-j}] \\ = \;& \mathbb{E}\big[(X_{n+h} - \mu) - (P_n X_{n+h} - \mu)(X_{n+1-j} - \mu)] + \underbrace{\mu\mathbb{E}[X_{n+h} - P_n X_{n+h}]}_{=0}.\end{aligned}$$

Then also $a_0 = 0$. We have

$$\begin{aligned}&\mathbb{E}[(X_{n+h} - P_n X_{n+h})X_{n+1-j}] \\ &\quad = \mathbb{E}[\Big(X_{n+h} - \Big(\underbrace{a_0}_{=0} + \sum_{i=1}^{n} a_i X_{n+1-i}\Big)\Big)X_{n+1-j}\Big] \\ &\quad = \mathbb{E}[X_{n+h} X_{n+1-j}] - \sum_{i=1}^{n} a_i \mathbb{E}[X_{n+1-i} X_{n+1-j}] \\ &\quad = \gamma_X(h + j - 1) - \sum_{i=1}^{n} a_i \gamma_X(j - i)\end{aligned}$$

but the jth equation of the system $\Gamma_n \mathbf{a}_n = \gamma_n(h)$ is

$$\sum_{i=1}^{n} a_i \gamma_X(j - i) = \gamma_X(h + j - 1)$$

Therefore

$$\mathbb{E}\left[\left(X_{n+h} - P_n X_{n+h}\right) X_j\right] = 0, \qquad j = 1, \ldots, n$$

e) Let

$$M_n = \left\{\sum_{j=1}^{n} b_j X_j : b_j \in \mathbb{R},\, j = 1, \ldots, n\right\}$$

Let $P_n X_{n+h} = \sum_{i=1}^{n} a_i X_{n+1-i} \in M_n$ such that

$$\mathbb{E}\left[\left(X_{n+h} - P_n X_{n+h}\right) X_j\right] = 0$$

for all $j = 1, \ldots, n$. We saw in d) that this system is just $\Gamma_n \mathbf{a}_n = \gamma_n(h)$, i.e. $P_n X_{n+h}$ is determined by the equation $\Gamma_n \mathbf{a}_n = \gamma_n(h)$.

Let $Y \in M_n$, so $Y = \sum_{j=1}^{n} b_j X_j$. Observe that $\mathbb{E}[Y] = 0$. Now consider

$$\begin{aligned}
(X_{n+h} - P_n X_{n+h}, Y) &= Cov\left(X_{n+h} - P_n X_{n+h}, Y\right) \\
&= \mathbb{E}\left[\left(X_{n+h} - P_n X_{n+h}\right) Y\right] \\
&= \mathbb{E}\left[\left(X_{n+h} - P_n X_{n+h}\right)\left(\sum_{j=1}^{n} b_j X_j\right)\right] \\
&= \sum_{j=1}^{n} b_j \mathbb{E}\left[\left(X_{n+h} - P_n X_{n+h}\right)\left(\ X_j\right)\right] \\
&= 0
\end{aligned}$$

Therefore $(X_{n+h} - P_n X_{n+h})$ is orthogonal to M_n. It follows then, by the projection theorem, that $P_n X_{n+h}$ is such that

$$\mathbb{E}\left[\left(X_{n+h} - P_n X_{n+h}\right)^2\right] = \inf_{Y \in M_n} \mathbb{E}\left[\left(X_{n+h} - Y\right)^2\right]$$

f) We know that for an AR(1) process

$$\gamma_X(h) = \frac{\sigma^2 \phi^{|h|}}{1 - \phi^2}$$

for all h. Now consider the prediction equation $\Gamma_n \mathbf{a}_n = \gamma_n(h)$. Note that this represents the equations

$$\gamma_X(h + j - 1) = \sum_{i=1}^{n} a_i \gamma_X(j - i), \quad j = 1, \ldots, n$$

Using the specific form of $\gamma_X(h)$, we have that this system is equivalent to

$$\phi^{h+j-1} = \sum_{i=1}^{n} a_i \phi^{|j-i|}, \quad j = 1, \ldots, n$$

i.e.

$$\begin{aligned}
&\phi^0 a_1 + \phi^1 a_2 + \cdots + \phi^{n-1} a_n = \phi^h \\
&\phi^1 a_1 + \phi^0 a_2 + \cdots + \phi^{n-2} a_n = \phi^{h+1} \\
&\phi^2 a_1 + \phi^1 a_2 + \cdots + \phi^{n-3} a_n = \phi^{h+2} \\
&\vdots \\
&\phi^{n-1} a_1 + \phi^{n-2} a_2 + \cdots + \phi^0 a_n = \phi^{h+n-1}
\end{aligned}$$

Note that

$$(a_1, a_2, \ldots, a_n) = \left(\phi^h, 0, \ldots, 0\right)$$

is the solution to the system. Thus, the linear h-step prediction is given by

$$P_n X_{n+h} = \phi^h X_n$$

On the other hand, we have that

$$\begin{aligned}
&\mathbb{E}\left[X_{n+1} | X_1, \ldots, X_n\right] = \mathbb{E}\left[\phi X_n + Z_{n+1} | X_1, \ldots, X_n\right] = \phi X_n \\
&\mathbb{E}\left[X_{n+2} | X_1, \ldots, X_n\right] = \mathbb{E}\left[\phi X_{n+1} + Z_{n+2} | X_1, \ldots, X_n\right] = \phi \mathbb{E}\left[X_{n+1} | X_1, \ldots, X_n\right] = \phi^2 X_n \\
&\vdots \\
&\mathbb{E}\left[X_{n+h} | X_1, \ldots, X_n\right] = \phi^h X_n
\end{aligned}$$

Therefore, the best prediction of X_{n+h} in the class of all square integrable functions of $X_1, \ldots, X_n$ coincides with the linear h-step prediction.