Assignment 4 for StatØk2, Block 1, 2021/2022

(1) Consider a causal stationary AR(1) model $X_t = \phi X_{t-1} + Z_t$ with iid white noise (Z_t).

(a) Prove the consistency of the Yule-Walker estimators $\widehat{\phi}$ of ϕ and $\widehat{\sigma}^2$ of $\sigma^2 = \mathrm{var}(Z_0)$.

(b) Prove asymptotic normality for $\hat{\phi}$, assuming $E[Z_0^4] < \infty$. Hint: use Bartlett's central limit theorem in the Lecture Notes.

(2) Let (X_t) be an ARMA(2,2) process given by the equation

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}\,, \quad t \in \mathbb{Z}\,.$$

Determine the parameter space C where Gaussian maximum likelihood estimation is possible for this process, i.e., determine the set of parameters $\beta = (\theta_1, \theta_2, \phi_1, \phi_2)$ where the ARMA(2,2) equations have a causal invertible solution.

(3) Let (X_t) be an AR(1) process with iid white noise (Z_t), i.e., $X_t = \phi X_{t-1} + Z_t$, $t \in \mathbb{Z}$, and $0 < \mathrm{var}(Z_0) = \sigma^2$.

(a) Assme that (X_t) is causal, i.e., $|\phi| < 1$. Prove the central limit theorem

$$\sqrt{n}\overline{X}_n \xrightarrow{d} N(0, (1-\phi)^{-2}\sigma^2)\,, \qquad n \to \infty\,,$$

by using the central limit theorem $\sqrt{n}\overline{Z}_n \xrightarrow{d} N(0, \sigma^2)$ for the noise sequence.
Hint: use the relation

$$\sqrt{n}\Big(\overline{X}_n - \phi \frac{1}{n}\sum_{t=1}^{n} X_{t-1}\Big) = \sqrt{n}\,\overline{Z}_n\,. \tag{0.1}$$

(b) Assume that (X_t) is non-causal, i.e., $|\phi| > 1$. Use the idea of the approach in (a) to prove a central limit theorem for $\sqrt{n}\,\overline{X}_n$.

(4) Consider an AR(1) process given by the AR(1) equation $X_t = \phi X_{t-1} + Z_t$, $|\phi| < 1$, and assume that (Z_t) is iid with a symmetric α-stable distribution for some $\alpha \in (1, 2)$, i.e., the characteristic function of Z_0 is given by

$$E\big[\mathrm{e}^{\,itZ_0}\big] = \mathrm{e}^{-c\,|t|^\alpha}\,, \quad t \in \mathbb{R}\,,$$

for some $c > 0$. These distributions have an infinite variance but a finite mean.

(a) Show that

$$\frac{1}{n^{1/\alpha}}\sum_{t=1}^{n} Z_t \stackrel{d}{=} Z_1\,.$$

(b) The AR(1) equation $X_t = \phi X_{t-1} + Z_t$, $t \in \mathbb{R}$, has a causal solution $X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}$, $t \in \mathbb{Z}$, where the infinite series converge a.s. Show that $\overline{X}_n \xrightarrow{\text{a.s.}} 0$ and

$$n^{1-1/\alpha}\overline{X}_n \xrightarrow{d} (1-\phi)^{-1} Z_0\,, \quad n \to \infty\,. \tag{0.2}$$

Hint: use (0.1).

Relation (0.2) shows that heavy tails of the univariate marginal distributions of (X_t) make estimation of the expectation of EX_0 a difficult problem: the closer α to one the larger the confidence bands for the sample mean. Indeed, choosing q as the 97.5% -quantile of the distribution of $(1-\phi)^{-1}Z_0$, an asymptotic 95% confidence band for $\overline{X}_n$ is given by $\pm q/n^{1-1/\alpha}$ which is asymptotically much wider than $\pm 1.96/\sqrt{n}$ prescribed by the central limit theorem.

(5) Let (X_t) be the unique stationary solution of the non-causal AR(1) equations $X_t = \phi X_{t-1} + Z_t$, $t \in \mathbb{Z}$, for $|\phi| > 1$, for white noise with variance σ^2. Calculate the ACF of this time series and compare it with the ACF of the causal AR(1) process given by the equations $X_t = \phi^{-1} X_{t-1} + Z_t$, $t \in \mathbb{Z}$.
If the two ACFs were the same then, from a second order perspective, the two time series would not be distinguishable and therefore it would be reasonable to restrict oneself to the study of causal AR(1) processes.