1.11. Let $X \sim \mathcal{N}(0, 1)$ be a real-valued random variable.

1.11(a). Show that

$$E\, e^{bX} = e^{b^2/2}$$

for every $b \in \mathbb{R}$.

1, 11)

a)

See • Abstract change of variable - formula
• T 11.7 MI
• P305 MI (bottom)

Note that for $X \sim N(0,1)$ $X(P) \sim f \cdot m$ with

$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, we by definition of the expectation have

$$E e^{bX} \equiv \int_{\Omega} e^{bX} \, dP \underset{m}{=} \int_{-\infty}^{\infty} e^{bx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{bx - \frac{x^2}{2}} \, dx.$$

Note that we would like to use the fact that we almost have something N-density like integrated from $-\infty$ to ∞. In order to do this we would like to have x in only one format and not two (x and x^2 are present) in the exponential.

↳ The solution; completing the square;

$ax^2+bx+c=0 \iff a(x+d)^2+e=0$ $\left(\Rightarrow ax^2+bx+c \overset{*}{=} a(x+d)^2+e\right)$

for $d := \frac{b}{2a}$, $e := e - \frac{b^2}{2a}$.

↳ We may either insert into the formula or note that

$$a(x+d)^2+e \equiv a(x^2+d^2+2xd)+e = \overset{\boxtimes}{\overline{ax^2+ad^2+2axd+e}}.$$

As we have $a \doteq -\frac{1}{2}$, $2axd \doteq -xd$, so we need to have

$\circledast$ $d \doteq -b$ in our case in order to preserve $\circledast$.

$\overset{\boxtimes}{\Rightarrow} \doteq -\frac{1}{2}x^2 - \frac{1}{2}b^2 + bx + e \Rightarrow e = \frac{1}{2}b^2$. $\leadsto -\frac{x^2}{2} + bx = -\frac{1}{2}(x-b)^2 + \frac{1}{2}b^2$

And therefore

$$\Rightarrow \quad Ee^{bX} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{bx - \frac{x^2}{2}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-b)^2 + \frac{b^2}{2}} dx = e^{\frac{b^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-b)^2} dx.$$

Notice now that $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-b)^2}$ is the density of $N(b,1)$, so

$$Ee^{bX} = e^{\frac{b^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-b)^2} dx = e^{\frac{b^2}{2}} \cdot 1 = e^{\frac{b^2}{2}}.$$