§3.6 Problem 2

Consider the homotopy $h(t,x) = tf(x) + (1-t)g(x)$ in which

$$f(x) = x^2 - 5x + 6, \quad g(x) = x^2 - 1.$$

Show that there is no path connecting a root of g to a root of f.

<u>Proof</u>. Ther roots of $f(x)$ are 2 and 3
the roots of $g(x)$ are ± 1.

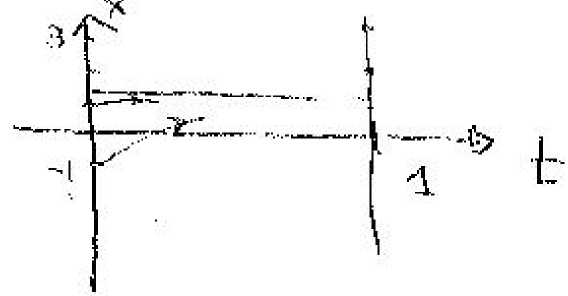
If there is a path $x = x(t)$, then

$$x' = -\frac{h_t}{h_x} = -\frac{-5x+7}{2x-5t}$$

or

$$x' = \frac{5x-7}{2x-5t}$$

Note that $x = \frac{7}{5}$ is an equilibrium solution of the ODE. Thus, any integral curve passing $(0,1)$ or $(0,-1)$ cannot reach the points $(1,2)$ and $(1,3)$.



Hence, there is no path connecting a root of g to a root of f.

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§6.1 Problem 1

Find the polynomials of least degree that interpolate these sets of data.

Solution

1a.

x	3	7
y	5	-1

$$p = \frac{x-7}{3-7}5 + \frac{x-3}{7-3}(-1) = \boxed{-\frac{5}{4}(x-7) - \frac{1}{4}(x-3)}$$

1b

x	7	1	2
y	146	2	1

$$p(x) = \frac{(x-1)(x-2)}{(7-1)(7-2)}146 + \frac{(x-7)(x-2)}{(1-7)(1-2)}2 + \frac{(x-7)(x-1)}{(2-7)(2-1)}\cdot 1$$

$$= \boxed{\frac{73}{15}(x-1)(x-2) + \frac{1}{3}(x-7)(x-2) - \frac{1}{5}(x-7)(x-1)}$$

#

§6.1 Problem 5

Prove that $\sum_{i=0}^{n} l_i(x) = 1$ for all x.

Proof: From the Theorem on polynomial interp. error, we have

$$f(x) = \sum_{i=0}^{n} f(x_i) l_i(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\cdots(x-x_n)$$

Taking $f(x) = 1$, we have

$$1 = \sum_{i=0}^{n} 1 \cdot l_i(x) + 0$$

or $$\sum_{i=0}^{n} l_i(x) = 1$$ #

§6.1 Problem 8

Discuss the problem of determining a polynomial of degree at most 2 for which $p(0)$, $p(1)$, and $p'(\xi)$ are prescribed, ξ being any preassigned point.

<u>Solution</u> Interpolation conditions are:

$$p(0) = y_0, \quad p(1) = y_1, \quad p'(\xi) = y'_\xi$$

where y_0, y_1, y'_ξ are given numbers.

Use Lagrange form.

$$p(x) = y_0 \ell_0(x) + y_1 \ell_1(x) + y'_\xi \ell_\xi(x)$$

where $\ell_0(x)$, $\ell_1(x)$, $\ell_\xi(x)$ satisfy

$$\ell_0(0)=1, \quad \ell_0(1) = 0, \quad \ell_0'(\xi) = 0$$

$$\ell_1(0)=0, \quad \ell_1(1)=1, \quad \ell_1'(\xi) = 0$$

$$\ell_\xi(0) = 0, \quad \ell_\xi(1) = 0, \quad \ell_\xi'(\xi) = 1$$

Find $\ell_0(x)$ (quadratic polynomial):

Let $\ell_0(x) = (x-1)(ax+b)$

$$\ell_0(0) = 1 \Rightarrow -b = 1 \Rightarrow b = -1$$

$$\ell_0'(\xi) = 0 \Rightarrow 2a\xi - 1 - a = 0$$

$$a(2\xi - 1) = 1$$

In order to determine a, ξ must not be $\frac{1}{2}$. Thus,

$$a = \frac{1}{2\xi - 1}$$

$$\Rightarrow \ell_0(x) = \frac{(x-1)(x-2\xi+1)}{2\xi-1}$$

Similarly:

$$\Rightarrow \ell_1(x) = \frac{x(-x+2\xi)}{2\xi-1}$$

$$\ell_\xi(x) = \frac{1}{2\xi-1}\, x(x-1)$$

Summary:

If $\xi \neq \frac{1}{2}$, then

$$p(x) = y_0 \ell_0(x) + y_1 \ell_1(x) + y'_\xi \ell_\xi(x).$$

otherwise, no solution.

#

36.1 Problem 9

Prove that if g interpolates the function f at $x_0, x_1, \cdots, x_{n-1}$ and if h interpolates at $x_1, x_2, \cdots, x_n$, then the function

$$g(x) + \frac{x_0 - x}{x_n - x_0}\left[g(x) - h(x)\right]$$

interpolates f at $x_0, x_1, \cdots, x_n$.

Proof: g interpolates f at $x_0, x_1, \cdots, x_{n-1}$:

$$g(x_i) = f(x_i), \quad i = 0, 1, \cdots, n-1$$

h interpolates f at $x_1, x_2, \cdots, x_n$:

$$h(x_i) = f(x_i), \quad i = 1, 2, \cdots, n$$

Let $\psi(x) = g(x) + \frac{x_0 - x}{x_n - x_0}\left[g(x) - h(x)\right]$.

Then, for $i = 1, \cdots, n-1$,

$$\psi(x_i) = f(x_i) + \frac{x_0 - x_i}{x_n - x_0}\left[f(x_i) - f(x_i)\right] = f(x_i)$$

$$\psi(x_0) = g(x_0) + \frac{x_0 - x_0}{x_n - x_n}\left[g(x_0) - h(x_0)\right] = g(x_0) = f(x_0)$$

$$\psi(x_n) = g(x_n) + \frac{x_0 - x_n}{x_n - x_0}\left[g(x_n) - h(x_n)\right] = h(x_n) = f(x_n)$$

Thus, $\psi(x)$ interpolates f at $x_0, x_1, \cdots, x_n$

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