Løsning af lineære ligninger

Von Neuman rækker

Mogens Bladt
bladt@math.ku.dk
Department of Mathematical Sciences

Von Neumann rækker

Sætning

Lad $\boldsymbol{A}$ *være en* $n \times n$ *matrix med* $\|\boldsymbol{A}\| < 1$. *Da er* $\boldsymbol{I} - \boldsymbol{A}$ *invertibel med*

$$(\boldsymbol{I} - \boldsymbol{A})^{-1} = \sum_{k=0}^{\infty} \boldsymbol{A}^k.$$

Von Neumann rækker

Sætning

Lad $\boldsymbol{A}$ være en $n \times n$ matrix med $\|\boldsymbol{A}\| < 1$. Da er $\boldsymbol{I} - \boldsymbol{A}$ invertibel med

$$(\boldsymbol{I} - \boldsymbol{A})^{-1} = \sum_{k=0}^{\infty} \boldsymbol{A}^k.$$

Bevis:

Von Neumann rækker

Sætning

Lad $\boldsymbol{A}$ være en $n \times n$ matrix med $\|\boldsymbol{A}\| < 1$. Da er $\boldsymbol{I} - \boldsymbol{A}$ invertibel med

$$(\boldsymbol{I} - \boldsymbol{A})^{-1} = \sum_{k=0}^{\infty} \boldsymbol{A}^k.$$

Bevis: Først sikrer vi os, at $\boldsymbol{I} - \boldsymbol{A}$ er invertibel.

Von Neumann rækker

Sætning

Lad $\boldsymbol{A}$ være en $n \times n$ matrix med $\|\boldsymbol{A}\| < 1$. Da er $\boldsymbol{I} - \boldsymbol{A}$ invertibel med

$$(\boldsymbol{I} - \boldsymbol{A})^{-1} = \sum_{k=0}^{\infty} \boldsymbol{A}^k.$$

Bevis: Først sikrer vi os, at $\boldsymbol{I} - \boldsymbol{A}$ er invertibel. Hvis den ikke var det, så ville der findes et $\boldsymbol{x} \neq \boldsymbol{0}$ således at

$$(\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}.$$

Von Neumann rækker

Sætning

Lad $\boldsymbol{A}$ være en $n \times n$ matrix med $\|\boldsymbol{A}\| < 1$. Da er $\boldsymbol{I} - \boldsymbol{A}$ invertibel med

$$(\boldsymbol{I} - \boldsymbol{A})^{-1} = \sum_{k=0}^{\infty} \boldsymbol{A}^k.$$

Bevis: Først sikrer vi os, at $\boldsymbol{I} - \boldsymbol{A}$ er invertibel. Hvis den ikke var det, så ville der findes et $\boldsymbol{x} \neq \boldsymbol{0}$ således at

$$(\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}.$$

Vi kan uden tab af generalitet normalisere til $\|\boldsymbol{x}\| = 1$.

Von Neumann rækker

Sætning

Lad $\boldsymbol{A}$ være en $n \times n$ matrix med $\|\boldsymbol{A}\| < 1$. Da er $\boldsymbol{I} - \boldsymbol{A}$ invertibel med

$$(\boldsymbol{I} - \boldsymbol{A})^{-1} = \sum_{k=0}^{\infty} \boldsymbol{A}^k.$$

Bevis: Først sikrer vi os, at $\boldsymbol{I} - \boldsymbol{A}$ er invertibel. Hvis den ikke var det, så ville der findes et $\boldsymbol{x} \neq \boldsymbol{0}$ således at

$$(\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}.$$

Vi kan uden tab af generalitet normalisere til $\|\boldsymbol{x}\| = 1$. Men så vil

$$1 = \|\boldsymbol{x}\| = \|\boldsymbol{A}\boldsymbol{x}\| \leq \|\boldsymbol{A}\|\|\boldsymbol{x}\| = \|\boldsymbol{A}\| \implies \# \text{ (modstrid).}$$

For så vidt angår rækkeudviklingen,

$$(\boldsymbol{I} - \boldsymbol{A}) \sum_{k=0}^{m} \boldsymbol{A}^k = \boldsymbol{I} - \boldsymbol{A}^{m+1}$$

For så vidt angår rækkeudviklingen,

$$(\boldsymbol{I}-\boldsymbol{A})\sum_{k=0}^{m}\boldsymbol{A}^k=\boldsymbol{I}-\boldsymbol{A}^{m+1}$$

$$\Downarrow$$

$$\sum_{k=0}^{m}\boldsymbol{A}^k=(\boldsymbol{I}-\boldsymbol{A})^{-1}-(\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{A}^{m+1}$$

For så vidt angår rækkeudviklingen,

$$(\boldsymbol{I}-\boldsymbol{A})\sum_{k=0}^{m}\boldsymbol{A}^k=\boldsymbol{I}-\boldsymbol{A}^{m+1}$$

$$\Downarrow$$

$$\sum_{k=0}^{m}\boldsymbol{A}^k=(\boldsymbol{I}-\boldsymbol{A})^{-1}-(\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{A}^{m+1}$$

$$\Downarrow$$

$$(\boldsymbol{I}-\boldsymbol{A})^{-1}-\sum_{k=0}^{m}\boldsymbol{A}^k=(\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{A}^{m+1}$$

For så vidt angår rækkeudviklingen,

$$(\boldsymbol{I}-\boldsymbol{A})\sum_{k=0}^{m}\boldsymbol{A}^k=\boldsymbol{I}-\boldsymbol{A}^{m+1}$$

$$\Downarrow$$

$$\sum_{k=0}^{m}\boldsymbol{A}^k=(\boldsymbol{I}-\boldsymbol{A})^{-1}-(\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{A}^{m+1}$$

$$\Downarrow$$

$$(\boldsymbol{I}-\boldsymbol{A})^{-1}-\sum_{k=0}^{m}\boldsymbol{A}^k=(\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{A}^{m+1}$$

$$\Downarrow$$

$$\left\|(\boldsymbol{I}-\boldsymbol{A})^{-1}-\sum_{k=0}^{m}\boldsymbol{A}^k\right\| \quad \leq \quad \left\|(\boldsymbol{I}-\boldsymbol{A})^{-1}\right\|\left\|\boldsymbol{A}^{m+1}\right\|$$

For så vidt angår rækkeudviklingen,

$$(\boldsymbol{I}-\boldsymbol{A})\sum_{k=0}^{m}\boldsymbol{A}^k = \boldsymbol{I}-\boldsymbol{A}^{m+1}$$

$$\Downarrow$$

$$\sum_{k=0}^{m}\boldsymbol{A}^k = (\boldsymbol{I}-\boldsymbol{A})^{-1}-(\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{A}^{m+1}$$

$$\Downarrow$$

$$(\boldsymbol{I}-\boldsymbol{A})^{-1}-\sum_{k=0}^{m}\boldsymbol{A}^k = (\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{A}^{m+1}$$

$$\Downarrow$$

$$\begin{aligned}\left\|(\boldsymbol{I}-\boldsymbol{A})^{-1}-\sum_{k=0}^{m}\boldsymbol{A}^k\right\| &\leq \left\|(\boldsymbol{I}-\boldsymbol{A})^{-1}\right\|\left\|\boldsymbol{A}^{m+1}\right\| \\ &\leq \left\|(\boldsymbol{I}-\boldsymbol{A})^{-1}\right\|\left\|\boldsymbol{A}\right\|^{m+1}\end{aligned}$$

For så vidt angår rækkeudviklingen,

$$(\boldsymbol{I}-\boldsymbol{A})\sum_{k=0}^{m}\boldsymbol{A}^k=\boldsymbol{I}-\boldsymbol{A}^{m+1}$$

$$\Downarrow$$

$$\sum_{k=0}^{m}\boldsymbol{A}^k=(\boldsymbol{I}-\boldsymbol{A})^{-1}-(\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{A}^{m+1}$$

$$\Downarrow$$

$$(\boldsymbol{I}-\boldsymbol{A})^{-1}-\sum_{k=0}^{m}\boldsymbol{A}^k=(\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{A}^{m+1}$$

$$\Downarrow$$

$$\begin{aligned}\left\|(\boldsymbol{I}-\boldsymbol{A})^{-1}-\sum_{k=0}^{m}\boldsymbol{A}^k\right\| &\leq \left\|(\boldsymbol{I}-\boldsymbol{A})^{-1}\right\|\left\|\boldsymbol{A}^{m+1}\right\| \\ &\leq \left\|(\boldsymbol{I}-\boldsymbol{A})^{-1}\right\|\left\|\boldsymbol{A}\right\|^{m+1} \\ &\to 0 \quad \text{når} \quad m\to\infty.\end{aligned}$$

Hvis vi er interesserede i at udregne den inverse til en matrix $\boldsymbol{A}$ bytter vi blot om på rollerne af $\boldsymbol{A}$ og $\boldsymbol{I} - \boldsymbol{A}$: Hvis $\|\boldsymbol{I} - \boldsymbol{A}\| < 1$ så vil

$$\boldsymbol{A}^{-1} = \sum_{k=0}^{\infty} (\boldsymbol{I} - \boldsymbol{A})^k \approx \sum_{k=0}^{N} (\boldsymbol{I} - \boldsymbol{A})^k,$$

for passende stort N.

Hvis vi er interesserede i at udregne den inverse til en matrix $\boldsymbol{A}$ bytter vi blot om på rollerne af $\boldsymbol{A}$ og $\boldsymbol{I}-\boldsymbol{A}$: Hvis $\|\boldsymbol{I}-\boldsymbol{A}\|<1$ så vil

$$\boldsymbol{A}^{-1}=\sum_{k=0}^{\infty}(\boldsymbol{I}-\boldsymbol{A})^k \approx \sum_{k=0}^{N}(\boldsymbol{I}-\boldsymbol{A})^k,$$

for passende stort N. Lad os tage et eksempel: Find den inverse matrix til

$$\boldsymbol{A}=\begin{pmatrix} 0.6 & 0.2 & 0.1 \\ 0.1 & 0.5 & 0.3 \\ 0.5 & 0.1 & 0.9 \end{pmatrix}.$$

Hvis vi er interesserede i at udregne den inverse til en matrix $\boldsymbol{A}$ bytter vi blot om på rollerne af $\boldsymbol{A}$ og $\boldsymbol{I}-\boldsymbol{A}$: Hvis $\|\boldsymbol{I}-\boldsymbol{A}\|<1$ så vil

$$\boldsymbol{A}^{-1}=\sum_{k=0}^{\infty}(\boldsymbol{I}-\boldsymbol{A})^k \approx \sum_{k=0}^{N}(\boldsymbol{I}-\boldsymbol{A})^k,$$

for passende stort N. Lad os tage et eksempel: Find den inverse matrix til

$$\boldsymbol{A}=\begin{pmatrix}0.6 & 0.2 & 0.1\\ 0.1 & 0.5 & 0.3\\ 0.5 & 0.1 & 0.9\end{pmatrix}.$$

Da $\|\boldsymbol{I}-\boldsymbol{A}\|=0.8<1$ fås for $N=20$,

$$\boldsymbol{A}^{-1}\approx\sum_{i=0}^{N}(\boldsymbol{I}-\boldsymbol{A})^i=\begin{pmatrix}1.74995445 & -0.70834605 & 0.04169061\\ 0.25003989 & 2.04162268 & -0.70832529\\ -0.99997278 & 0.16670018 & 1.16663872\end{pmatrix},$$

Hvis vi er interesserede i at udregne den inverse til en matrix $\boldsymbol{A}$ bytter vi blot om på rollerne af $\boldsymbol{A}$ og $\boldsymbol{I}-\boldsymbol{A}$: Hvis $\|\boldsymbol{I}-\boldsymbol{A}\|<1$ så vil

$$\boldsymbol{A}^{-1}=\sum_{k=0}^{\infty}(\boldsymbol{I}-\boldsymbol{A})^k \approx \sum_{k=0}^{N}(\boldsymbol{I}-\boldsymbol{A})^k,$$

for passende stort N. Lad os tage et eksempel: Find den inverse matrix til

$$\boldsymbol{A}=\begin{pmatrix}0.6 & 0.2 & 0.1\\ 0.1 & 0.5 & 0.3\\ 0.5 & 0.1 & 0.9\end{pmatrix}.$$

Da $\|\boldsymbol{I}-\boldsymbol{A}\|=0.8<1$ fås for $N=20$,

$$\boldsymbol{A}^{-1}\approx\sum_{i=0}^{N}(\boldsymbol{I}-\boldsymbol{A})^i=\begin{pmatrix}1.74995445 & -0.70834605 & 0.04169061\\ 0.25003989 & 2.04162268 & -0.70832529\\ -0.99997278 & 0.16670018 & 1.16663872\end{pmatrix},$$

hvor den eksakte værdi er givet ved

$$\boldsymbol{A}^{-1}=\begin{pmatrix}1.750000000 & -0.7083333333 & 0.04166666667\\ 0.2500000000 & 2.041666667 & -0.7083333333\\ -1.000000000 & 0.1666666667 & 1.166666667\end{pmatrix}.$$

Implementering

Algoritme 1 $\boldsymbol{A}^{-1}$ via Von Neuman's metode

input $\boldsymbol{A}$,ϵ
$\boldsymbol{B} = \boldsymbol{I} - \boldsymbol{A}$
$\boldsymbol{P} = \boldsymbol{B}$
$\boldsymbol{S} = \boldsymbol{I}$
repeat
    $\boldsymbol{S}_1 = \boldsymbol{S}$
    $\boldsymbol{S} = \boldsymbol{S} + \boldsymbol{P}$
    $\boldsymbol{P} = \boldsymbol{P} * \boldsymbol{B}$
until $\|\boldsymbol{S} - \boldsymbol{S}_1\| < \epsilon$

Von Neumann: Løsning af ligningssystem

På en helt tilsvarende måde kan vi løse ligningssystemet

$$\boldsymbol{Bx} = \boldsymbol{b}$$

iterativt.

Von Neumann: Løsning af ligningssystem

På en helt tilsvarende måde kan vi løse ligningssystemet

$$\boldsymbol{B}\boldsymbol{x} = \boldsymbol{b}$$

iterativt. Lad

$$\boldsymbol{B} = \boldsymbol{I} - \boldsymbol{A}.$$

Så er

$$\boldsymbol{B}\boldsymbol{x} = \boldsymbol{b}$$

Von Neumann: Løsning af ligningssystem

På en helt tilsvarende måde kan vi løse ligningssystemet

$$\boldsymbol{Bx} = \boldsymbol{b}$$

iterativt. Lad

$$\boldsymbol{B} = \boldsymbol{I} - \boldsymbol{A}.$$

Så er

$$\boldsymbol{Bx} = \boldsymbol{b}$$

$$\Updownarrow$$

$$(\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{b}$$

Von Neumann: Løsning af ligningssystem

På en helt tilsvarende måde kan vi løse ligningssystemet

$$\boldsymbol{B}\boldsymbol{x} = \boldsymbol{b}$$

iterativt. Lad

$$\boldsymbol{B} = \boldsymbol{I} - \boldsymbol{A}.$$

Så er

$$\boldsymbol{B}\boldsymbol{x} = \boldsymbol{b}$$
$$\Updownarrow$$
$$(\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{b}$$
$$\Updownarrow$$
$$\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}.$$

Von Neumann: Løsning af ligningssystem

På en helt tilsvarende måde kan vi løse ligningssystemet

$$\boldsymbol{Bx} = \boldsymbol{b}$$

iterativt. Lad

$$\boldsymbol{B} = \boldsymbol{I} - \boldsymbol{A}.$$

Så er

$$\begin{aligned} & \boldsymbol{Bx} = \boldsymbol{b} \\ \Updownarrow \quad & \\ & (\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{b} \\ \Updownarrow \quad & \\ & \boldsymbol{x} = \boldsymbol{Ax} + \boldsymbol{b}. \end{aligned}$$

At løse

$$\boldsymbol{Bx} = \boldsymbol{b}$$

er dermed det samme som at finde fixpunkter for

$$\boldsymbol{x} \to \boldsymbol{Ax} + \boldsymbol{b}.$$

Sætning

Lad $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$. *Definer*

$$\boldsymbol{x}^{(k)} = \boldsymbol{A}\boldsymbol{x}^{(k-1)} + \boldsymbol{b}$$

hvor $\boldsymbol{b} \in \mathbb{R}^n$ *og* $\boldsymbol{A}$ *er en* $n \times n$ *matrix. Hvis* $\|\boldsymbol{A}\| < 1$ *så gælder, at*

$$\boldsymbol{x}^{(k)} \to \boldsymbol{x} \quad as \quad k \to \infty,$$

hvor $\boldsymbol{x}$ *er løsning til*

$$\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}.$$

Bevis:

$$\|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}\| \quad = \quad \|\boldsymbol{A}\boldsymbol{x}^{(k)} + \boldsymbol{b} - \boldsymbol{x}\|$$

Sætning

Lad $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$. *Definer*

$$\boldsymbol{x}^{(k)} = \boldsymbol{A}\boldsymbol{x}^{(k-1)} + \boldsymbol{b}$$

hvor $\boldsymbol{b} \in \mathbb{R}^n$ *og* $\boldsymbol{A}$ *er en* $n \times n$ *matrix. Hvis* $\|\boldsymbol{A}\| < 1$ *så gælder, at*

$$\boldsymbol{x}^{(k)} \to \boldsymbol{x} \quad as \quad k \to \infty,$$

hvor $\boldsymbol{x}$ *er løsning til*

$$\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}.$$

Bevis:

$$\begin{aligned} \|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}\| &= \|\boldsymbol{A}\boldsymbol{x}^{(k)} + \boldsymbol{b} - \boldsymbol{x}\| \\ &= \|\boldsymbol{A}\boldsymbol{x}^{(k)} + \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\| \end{aligned}$$

Sætning

Lad $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$. *Definer*

$$\boldsymbol{x}^{(k)} = \boldsymbol{A}\boldsymbol{x}^{(k-1)} + \boldsymbol{b}$$

hvor $\boldsymbol{b} \in \mathbb{R}^n$ *og* $\boldsymbol{A}$ *er en* $n \times n$ *matrix. Hvis* $\|\boldsymbol{A}\| < 1$ *så gælder, at*

$$\boldsymbol{x}^{(k)} \to \boldsymbol{x} \quad as \quad k \to \infty,$$

hvor $\boldsymbol{x}$ *er løsning til*

$$\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}.$$

Bevis:

$$\begin{aligned} \|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}\| &= \|\boldsymbol{A}\boldsymbol{x}^{(k)} + \boldsymbol{b} - \boldsymbol{x}\| \\ &= \|\boldsymbol{A}\boldsymbol{x}^{(k)} + \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\| \\ &= \|\boldsymbol{A}(\boldsymbol{x}^{(k)} - \boldsymbol{x})\| \end{aligned}$$

Sætning

Lad $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$. *Definer*

$$\boldsymbol{x}^{(k)} = \boldsymbol{A}\boldsymbol{x}^{(k-1)} + \boldsymbol{b}$$

hvor $\boldsymbol{b} \in \mathbb{R}^n$ *og* $\boldsymbol{A}$ *er en* $n \times n$ *matrix. Hvis* $\|\boldsymbol{A}\| < 1$ *så gælder, at*

$$\boldsymbol{x}^{(k)} \to \boldsymbol{x} \quad as \quad k \to \infty,$$

hvor $\boldsymbol{x}$ *er løsning til*

$$\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}.$$

Bevis:

$$\begin{aligned} \|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}\| &= \|\boldsymbol{A}\boldsymbol{x}^{(k)} + \boldsymbol{b} - \boldsymbol{x}\| \\ &= \|\boldsymbol{A}\boldsymbol{x}^{(k)} + \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\| \\ &= \|\boldsymbol{A}(\boldsymbol{x}^{(k)} - \boldsymbol{x})\| \\ &\leq \|\boldsymbol{A}\|\|\boldsymbol{x}^{(k)} - \boldsymbol{x}\| \end{aligned}$$

Sætning

Lad $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$. *Definer*

$$\boldsymbol{x}^{(k)} = \boldsymbol{A}\boldsymbol{x}^{(k-1)} + \boldsymbol{b}$$

hvor $\boldsymbol{b} \in \mathbb{R}^n$ *og* $\boldsymbol{A}$ *er en* $n \times n$ *matrix. Hvis* $\|\boldsymbol{A}\| < 1$ *så gælder, at*

$$\boldsymbol{x}^{(k)} \to \boldsymbol{x} \quad \text{as} \quad k \to \infty,$$

hvor $\boldsymbol{x}$ *er løsning til*

$$\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}.$$

Bevis:

$$\begin{aligned}
\|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}\| &= \|\boldsymbol{A}\boldsymbol{x}^{(k)} + \boldsymbol{b} - \boldsymbol{x}\| \\
&= \|\boldsymbol{A}\boldsymbol{x}^{(k)} + \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\| \\
&= \|\boldsymbol{A}(\boldsymbol{x}^{(k)} - \boldsymbol{x})\| \\
&\leq \|\boldsymbol{A}\|\|\boldsymbol{x}^{(k)} - \boldsymbol{x}\| \\
&\ldots
\end{aligned}$$

Sætning

Lad $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$. *Definer*

$$\boldsymbol{x}^{(k)} = \boldsymbol{A}\boldsymbol{x}^{(k-1)} + \boldsymbol{b}$$

hvor $\boldsymbol{b} \in \mathbb{R}^n$ *og* $\boldsymbol{A}$ *er en* $n \times n$ *matrix. Hvis* $\|\boldsymbol{A}\| < 1$ *så gælder, at*

$$\boldsymbol{x}^{(k)} \to \boldsymbol{x} \quad \textit{as} \quad k \to \infty,$$

hvor $\boldsymbol{x}$ *er løsning til*

$$\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}.$$

Bevis:

$$\begin{aligned}
\|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}\| &= \|\boldsymbol{A}\boldsymbol{x}^{(k)} + \boldsymbol{b} - \boldsymbol{x}\| \\
&= \|\boldsymbol{A}\boldsymbol{x}^{(k)} + \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\| \\
&= \|\boldsymbol{A}(\boldsymbol{x}^{(k)} - \boldsymbol{x})\| \\
&\leq \|\boldsymbol{A}\|\|\boldsymbol{x}^{(k)} - \boldsymbol{x}\| \\
&\quad \dots \\
&\leq \|\boldsymbol{A}\|^{(k+1)}\|\boldsymbol{x}^{(0)} - \boldsymbol{x}\|
\end{aligned}$$

Sætning

Lad $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$. *Definer*

$$\boldsymbol{x}^{(k)} = \boldsymbol{A}\boldsymbol{x}^{(k-1)} + \boldsymbol{b}$$

hvor $\boldsymbol{b} \in \mathbb{R}^n$ *og* $\boldsymbol{A}$ *er en* $n \times n$ *matrix. Hvis* $\|\boldsymbol{A}\| < 1$ *så gælder, at*

$$\boldsymbol{x}^{(k)} \to \boldsymbol{x} \quad as \quad k \to \infty,$$

hvor $\boldsymbol{x}$ *er løsning til*

$$\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}.$$

Bevis:

$$\begin{aligned}
\|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}\| &= \|\boldsymbol{A}\boldsymbol{x}^{(k)} + \boldsymbol{b} - \boldsymbol{x}\| \\
&= \|\boldsymbol{A}\boldsymbol{x}^{(k)} + \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\| \\
&= \|\boldsymbol{A}(\boldsymbol{x}^{(k)} - \boldsymbol{x})\| \\
&\leq \|\boldsymbol{A}\|\|\boldsymbol{x}^{(k)} - \boldsymbol{x}\| \\
& \quad \ldots \\
&\leq \|\boldsymbol{A}\|^{(k+1)}\|\boldsymbol{x}^{(0)} - \boldsymbol{x}\| \\
&\to 0 \quad \text{når} \quad k \to \infty
\end{aligned}$$

$\square$

Korollar

Hvis $\|\boldsymbol{I} - \boldsymbol{A}\| < 1$ *så kan det lineære ligningssystem*

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

løses ved iterationen

$$\boldsymbol{x}^{(k+1)} = (\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x}^{(k)} + \boldsymbol{b}.$$

Korollar

Hvis $\|\boldsymbol{I} - \boldsymbol{A}\| < 1$ *så kan det lineære ligningssystem*

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

løses ved iterationen

$$\boldsymbol{x}^{(k+1)} = (\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x}^{(k)} + \boldsymbol{b}.$$

En lettere udvidelse giver

Sætning

Hvis $\boldsymbol{A}$ *og* $\boldsymbol{B}$ *er* $n \times n$ *matricer med* $\|\boldsymbol{I} - \boldsymbol{A}\boldsymbol{B}\| < 1$, *så gælder der, at:*

$$\boldsymbol{A}^{-1} = \boldsymbol{B}\sum_{k=0}^{\infty}(\boldsymbol{I} - \boldsymbol{A}\boldsymbol{B})^k \quad \text{and} \quad \boldsymbol{B}^{-1} = \sum_{k=0}^{\infty}(\boldsymbol{I} - \boldsymbol{A}\boldsymbol{B})^k\boldsymbol{A}.$$

Bevis:

Bevis: Af Von Neumann's sætning følger det, at

$$(\boldsymbol{AB})^{-1} = \sum_{k=0}^{\infty} (\boldsymbol{I} - \boldsymbol{AB})^k,$$

Bevis: Af Von Neumann's sætning følger det, at

$$(\boldsymbol{AB})^{-1} = \sum_{k=0}^{\infty} (\boldsymbol{I} - \boldsymbol{AB})^k,$$

og resultatet følger så af

$$\boldsymbol{A}^{-1} \quad = \quad \boldsymbol{B}(\boldsymbol{AB})^{-1}$$

og

$$\boldsymbol{B}^{-1} \quad = \quad (\boldsymbol{AB})^{-1}\boldsymbol{A}.$$

□

Bevis: Af Von Neumann's sætning følger det, at

$$(\boldsymbol{AB})^{-1} = \sum_{k=0}^{\infty} (\boldsymbol{I} - \boldsymbol{AB})^k,$$

og resultatet følger så af

$$\boldsymbol{A}^{-1} = \boldsymbol{B}(\boldsymbol{AB})^{-1}$$

og

$$\boldsymbol{B}^{-1} = (\boldsymbol{AB})^{-1}\boldsymbol{A}.$$

$\square$

Dette kan være en smart måde at regne $\boldsymbol{A}^{-1}$ ud på hvis $\|\boldsymbol{I} - \boldsymbol{A}\| \geq 1$, dvs. $\boldsymbol{A}$ ikke opfylder kravet til direkte summation.

Bevis: Af Von Neumann's sætning følger det, at

$$(\boldsymbol{AB})^{-1} = \sum_{k=0}^{\infty} (\boldsymbol{I} - \boldsymbol{AB})^k,$$

og resultatet følger så af

$$\boldsymbol{A}^{-1} = \boldsymbol{B}(\boldsymbol{AB})^{-1}$$

og

$$\boldsymbol{B}^{-1} = (\boldsymbol{AB})^{-1}\boldsymbol{A}.$$

$\square$

Dette kan være en smart måde at regne $\boldsymbol{A}^{-1}$ ud på hvis $\|\boldsymbol{I} - \boldsymbol{A}\| \geq 1$, dvs. $\boldsymbol{A}$ikke opfylder kravet til direkte summation.Lad os beregne $\boldsymbol{A}^{-1}$ baseret på et oprindeligt gæt $\boldsymbol{X}_0 = \boldsymbol{B}$, hvor vi nu antager at $\|\boldsymbol{I} - \boldsymbol{AX}_0\| < 1$.

Bevis: Af Von Neumann's sætning følger det, at

$$(\boldsymbol{AB})^{-1} = \sum_{k=0}^{\infty} (\boldsymbol{I} - \boldsymbol{AB})^k,$$

og resultatet følger så af

$$\boldsymbol{A}^{-1} \quad = \quad \boldsymbol{B}(\boldsymbol{AB})^{-1}$$

og

$$\boldsymbol{B}^{-1} \quad = \quad (\boldsymbol{AB})^{-1}\boldsymbol{A}.$$

$\square$

Dette kan være en smart måde at regne $\boldsymbol{A}^{-1}$ ud på hvis $\|\boldsymbol{I} - \boldsymbol{A}\| \geq 1$, dvs. $\boldsymbol{A}$ikke opfylder kravet til direkte summation.Lad os beregne $\boldsymbol{A}^{-1}$ baseret på et oprindeligt gæt $\boldsymbol{X}_0 = \boldsymbol{B}$, hvor vi nu antager at $\|\boldsymbol{I} - \boldsymbol{AX}_0\| < 1$. Lad

$$\boldsymbol{X}_n = \boldsymbol{B} \sum_{k=0}^{n} (\boldsymbol{I} - \boldsymbol{AB})^k.$$

Så er

$$
\begin{aligned}
\boldsymbol{X}_{n+1} &= \boldsymbol{B}\sum_{k=0}^{n+1}(\boldsymbol{I}-\boldsymbol{A}\boldsymbol{B})^k \\
&= \boldsymbol{B}+\boldsymbol{B}\sum_{k=1}^{n+1}(\boldsymbol{I}-\boldsymbol{A}\boldsymbol{B})^k \\
&= \boldsymbol{B}+\left(\boldsymbol{B}\sum_{k=0}^{n}(\boldsymbol{I}-\boldsymbol{A}\boldsymbol{B})^k\right)(\boldsymbol{I}-\boldsymbol{A}\boldsymbol{B}) \\
&= \boldsymbol{B}+\boldsymbol{X}_n(\boldsymbol{I}-\boldsymbol{A}\boldsymbol{B}) \\
&= \boldsymbol{X}_n+\boldsymbol{B}-\boldsymbol{X}_n\boldsymbol{A}\boldsymbol{B} \\
&= \boldsymbol{X}_n+(\boldsymbol{B}-\boldsymbol{X}_n+\boldsymbol{X}_n)-\boldsymbol{X}_n\boldsymbol{A}(\boldsymbol{B}-\boldsymbol{X}_n+\boldsymbol{X}_n) \\
&= 2\boldsymbol{X}_n-\boldsymbol{X}_n\boldsymbol{A}\boldsymbol{X}_n+(\boldsymbol{I}-\boldsymbol{X}_n\boldsymbol{A})(\boldsymbol{B}-\boldsymbol{X}_n).
\end{aligned}
$$

Da $\boldsymbol{X}_n \to \boldsymbol{A}^{-1}$ så vil

$$(\boldsymbol{I}-\boldsymbol{X}_n\boldsymbol{A})(\boldsymbol{B}-\boldsymbol{X}_n)$$

gå mod nulmatricen, og vi kunne forledes til at tro at

$$\boldsymbol{X}_{n+1} = 2\boldsymbol{X}_n-\boldsymbol{X}_n\boldsymbol{A}\boldsymbol{X}_n$$

også konvergerer mod den inverse. Dette er faktisk også tilfældet.