Theorem on Nullspace: Multiple Roots

Theorem (Nullspace – Multiple Roots)

Consider the difference equation $(p(E)x)_n = 0$ with characteristic polynomial p, with $p(0) \neq 0$. Define $x(\lambda) = [\lambda, \lambda^2, \ldots]$. If λ is a root of p with multiplicity k, then the following sequences form a basis for the nullspace of $p(E)$:

$$x(\lambda), \frac{\mathrm{d}x(\lambda)}{\mathrm{d}\lambda}, \frac{\mathrm{d}^2 x(\lambda)}{\mathrm{d}\lambda^2}, \ldots, \frac{\mathrm{d}^{k-1} x(\lambda)}{\mathrm{d}\lambda^{k-1}}$$

Q: How to find $a_1, a_2, \ldots, a_m$?
A: Similarly to the case of simple roots.

Proof Sketch: Multiple Roots $(k > 2)$

If λ has multiplicity k, then $p(\lambda) = p'(\lambda) = \ldots = p^{(k-1)}(\lambda) = 0$. Using the same reasoning as for $k = 2$, we can show that the followings are solutions:

$$\begin{aligned}
x(\lambda) &= [\lambda,\, \lambda^2,\, \lambda^3,\, \ldots] \\
x'(\lambda) &= [1,\, 2\lambda,\, 3\lambda^2,\, \ldots] \\
x''(\lambda) &= [0,\, 2,\, 6\lambda,\, \ldots] \\
&\vdots \\
x^{(k-1)}(\lambda) &= \frac{\mathrm{d}^{k-1}}{\mathrm{d}\lambda^{k-1}}[\lambda,\, \lambda^2,\, \lambda^3,\, \ldots]
\end{aligned}$$

Moreover, they form a set of k independent sequences, and so are independent of each other.

Proof Sketch: Multiple Roots $(k = 2)$

Recall that $p(E)x(\lambda) = p(\lambda)x(\lambda)$. Taking derivative w.r.t. λ:

$$\Big(p(E)x(\lambda)\Big)' = \Big(p(\lambda)x(\lambda)\Big)' \implies p(E)x'(\lambda) = p'(\lambda)x(\lambda) + p(\lambda)x'(\lambda)$$

λ is a double root ($k = 2$), so $p(\lambda) = p'(\lambda) = 0$. Thus, $p(E)x'(\lambda) = 0$. So $x'(\lambda) = [1, 2\lambda, 3\lambda^2, \ldots]$ is a solution.

It remains to show that $x(\lambda)$ and $x'(\lambda)$ are independent (for $\lambda \neq 0$). Consider $x(\lambda)$ and $x'(\lambda)$ truncated at $n = 2$: i.e., $[\lambda, \lambda^2]$ and $[1, 2\lambda]$.

$$\text{Since} \qquad \det \begin{bmatrix} \lambda & \lambda^2 \\ 1 & 2\lambda \end{bmatrix} = 2\lambda^2 - \lambda^2 \neq 0$$

they are linealy independent in $\mathbb{R}^2 \implies x(\lambda)$ and $x'(\lambda)$ are independent.

Multiple Roots: Examples

Solve $4x_{n+2} - 4x_{n+1} + x_n = 0$.

Characteristic polynomial: $p(\lambda) = 4\lambda^2 - 4\lambda + 1$. Roots are $\lambda = \frac{1}{2}, \frac{1}{2}$. By Theorem 2:

- $x(\lambda) = [\lambda, \lambda^2, \lambda^3, \ldots]$ for $\lambda = \frac{1}{2}$ is a solution — Corresponding to $x_n = \left(\frac{1}{2}\right)^n$.
- $\frac{\mathrm{d}}{\mathrm{d}\lambda} x(\lambda) = [1, 2\lambda, 3\lambda^2, \ldots]$ for $\lambda = \frac{1}{2}$ is a solution — Corresponding to $x_n = n\left(\frac{1}{2}\right)^{n-1}$.

So, the general solution is:

$$x_n = a_1\left(\frac{1}{2}\right)^n + a_2 n\left(\frac{1}{2}\right)^{n-1} = (c_1 + c_2 n)\left(\frac{1}{2}\right)^n$$

for real numbers c_1, c_2 that depend on the initial condition.

Multiple Roots: Examples

Assume $x_n = \alpha(n+1)^2 2^n$ for some α is the solution to some difference equation. What can be said about the roots of its characteristic polynomial $p(\lambda)$?

Multiple Roots

Note that for $x(\lambda) = [\lambda,\, \lambda^2,\, \lambda^3, \ldots]$:

$$x'(\lambda) = [1,\, 2\lambda,\, 3\lambda^2,\, \ldots] \qquad \text{(corresponding to } x_n = n\lambda^{n-1}\text{)}$$

$$x''(\lambda) = [0,\, 2,\, 6\lambda,\, \ldots] \qquad \text{(corresponding to } x_n = n(n-1)\lambda^{n-2}\text{)}$$

$$\vdots$$

$$x^k(\lambda) = \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}[\lambda,\, \lambda^2,\, \lambda^3,\, \ldots]$$

(corresponding to $x_n = n(n-1)\ldots(n-k+1)\lambda^{n-k}$)

So, in general, if $n^k\lambda^n$ appears in the solution to some difference equation, λ is necessarily a multiple root of the characteristic polynomial with multiplicity $(k+1)$.

Multiple Roots: Examples

Assume $x_n = \alpha(n+1)^2 2^n$ for some α is the solution to some difference equation. What can be said about the roots of its characteristic polynomial $p(\lambda)$?

From the previous discussion:

- $\lambda = 2$ is a multiple root of $p(\lambda)$ with multiplicity 3.
- In other words, $(\lambda - 2)^3$ is a factor of $p(\lambda)$.

Stability

A sequence x is said to bounded if there exists a constant c such that

$$|x_n| \leq c, \quad \forall n \in \mathbb{N}.$$

Q: Which of the following sequences are bounded?

$$x_n = \cos(2^n), \quad \forall n$$
$$x_n = n(\tfrac{1}{2})^n, \quad \forall n$$
$$x_n = \alpha 2^n + \beta(\tfrac{1}{3})^n, \quad \forall n$$

Stable Difference Equations

A difference equation $p(E)x = 0$ is said to be stable if all of its solutions are bounded. It is otherwise called unstable.

Theorem on Stability

How to determine whether a difference equation is stable or not?

Theorem (Theorem on Stable Difference Equations)

Consider a polynomial p with $p(0) \neq 0$. Then, the following statements are equivalent:

1. $p(E)x = 0$ *is stable.*
2. *All roots of p satisfy $|\lambda| \leq 1$, and all multiple roots satisfy $|\lambda| < 1$.*

Note: $|\cdot|$ indicates the magnitude (λ could be a complex number).

Q: Is $x_{n+2} + 2x_{n+1} + 5x_n = 0$ stable?