Num intro - ekstra øvelse

1. a), 1. c), 2 ?, 4, 6.b), 2.d, 2.e), 3

1.a) $x_n = \alpha n^2 (3^{-n})$

Pf

(i) Lad $\alpha \in \mathbb{R}$, så gælder der at

$$\lim_{n\to\infty} \frac{n^2}{3^n} = 0$$, så stabil

$$\left|\frac{n^2}{3^n}\right| \leq 1$$

$$x = \alpha\, x(\tfrac{1}{3}) + \beta x(\lambda) x$$

$$x_n = \alpha n^2 (\tfrac{1}{3})^n$$

(ii) hvis $n^2(\frac{1}{3})^n$ er en rod

så medfører dette at,

$$(\lambda - \tfrac{1}{3})^3 = \lambda^3 - \lambda^2 + \tfrac{1}{3}\lambda - \tfrac{1}{27}, = 0$$

$$x_{n+3} - x_{n+2} + \tfrac{1}{3} x_{n+1} - \tfrac{1}{27} x_n = 0$$

1.c) for 1.b) fik vi

$$\lambda \in \{1, -0.4\}$$

$$5\lambda^2 - 3\lambda - 2 = 0$$

$$5\lambda^3 - 3\lambda - 2 = (\lambda - 1)(5\lambda^2 + \lambda^2 + 2) = 0$$

Stabil da

$$\lambda \in \{1, \tfrac{1}{10}(-5 \pm \sqrt{15}\, i)\}, \quad |\lambda| \leq 1$$

2.a) $y = \sqrt{e^{x} - e^{-3x}}$

$y = e^{\frac{x}{2}} \cdot \sqrt{1 - e^{-4x}}$ $\qquad e^{0.0004} = 1.00000004 \approx 1$

$\approx e^{\frac{x}{2}} \cdot \sqrt{1 - (1 - 4x + 8x^2 - \frac{32}{3}x^3)}$

$= e^{\frac{x}{2}} \cdot \sqrt{4x - 8x^2 + \frac{32}{3}x^3}$

$= 2e^{\frac{x}{2}} \sqrt{x} \cdot \sqrt{1 - 2x + \frac{8}{3}x^2}$

$\sqrt{1+z} = 1 + \frac{z}{2}$ for $z = o(1)$

$\approx 2e^{\frac{x}{2}} \sqrt{x} \cdot (1 - x + \frac{4}{3}x^2)$

$= 2e^{\frac{x}{2}} \sqrt{x} (1 - x(1 - \frac{4}{3}x))$

2.d) $y = \frac{1-x}{1+x} - \frac{1}{2x+1}$

$\cancel{\sqrt{1+x^2+x^3} \ldots + o(x^3)}$

$= \frac{(1-x)(2x+1) - (1+x)}{(1+x)(2x+1)}$

$= \frac{-2x^2}{(1+x)(2x+1)}$ $\qquad \sqrt{1+z} = 1 + \frac{z}{2}$

f_0

2.e) $y = \log(\sqrt{1+x^3} - 1)$

$\approx \log(1 + \frac{x^3}{2} - 1) = 3\log(x) - \log(2)$

4) $$f'(x) = \frac{-f(x+2h) + 4f(x+h) - 3f(x)}{2h} + O(h^2)$$

Pf

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + f^{(3)}(x)\frac{h^3}{3!} + \dots$$

$$f(x+2h) = f(x) + f'(x)2h + f''(x)\frac{4h^2}{2!} + \dots$$

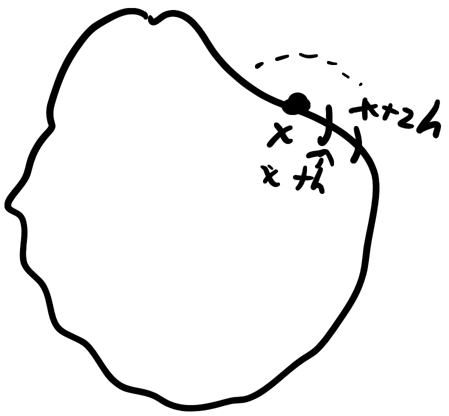
Så

$$4f(x+h) - f(x+2h) = 3f(x) + 2hf'(x) + O(h^3)$$

Så

$$f'(x) = \frac{4f(x+h) - f(x+2h) - 3f(x)}{2h} + \frac{O(h^3)}{2h}$$

$$= \frac{4f(x+h) - f(x+2h) - 3f(x)}{2h} + O(h^2)$$



$$\frac{f(x+h) - f(x-h)}{2h}$$

$$\text{"}f(x-h)\text{"} := f(x+2h)$$

6. a)

Pf
for $\frac{\pi}{4}$ fås:

$$I(h) := \frac{h}{2}\left(f(0) + 2f(h) + f(2h)\right) = \frac{\pi}{8}\left(f(0) + f\left(\frac{\pi}{2}\right) + 2f\left(\frac{\pi}{4}\right)\right)$$

$= 1.21000$ (eventuelt gang 4)

$$I\left(\frac{h}{2}\right) = \frac{\pi}{16}\left(f(0) + 2f\left(\frac{\pi}{8}\right) + 2\cdot f\left(3\cdot\frac{\pi}{8}\right) + f\left(\frac{\pi}{2}\right)\right)$$

$= 1.2111$ (eventuelt gang med 4)

b) $\frac{4}{3}\varphi\left(\frac{h}{2}\right) - \frac{1}{3}\varphi(h)$ Richardson

$I(h) := \varphi(h)$, $I\left(\frac{h}{2}\right) := \varphi\left(\frac{h}{2}\right)$

$\Rightarrow \frac{4}{3}I\left(\frac{h}{2}\right) - \frac{1}{3}I(h) = 1.2114$

(eventuelt gang med 4)

c)

Pf

$$E = -\frac{\frac{\pi}{2} - 0}{12}h^2 f''(\xi)$$

$$|E| < \frac{\pi}{24}h^2 \cdot \frac{3}{2}$$

$$-\frac{\pi}{24}h^2\frac{3}{2} < E < \frac{\pi}{24}h^2 \cdot \frac{3}{2} < 0.0001$$

$$h < \sqrt{\frac{0.0016}{\pi}} = \frac{0.04}{\sqrt{\pi}}$$

$$\frac{\frac{\pi}{2}}{\frac{0.04}{\sqrt{\pi}}} = \frac{\pi^{\frac{3}{2}}}{0.08} \approx 70$$