Задача №8

N m i n

$x(k+1) = Ax(k) + v(k),$
$x(0) = x0,\ v(k) \in \mathrm{conv}\{b; -b\},$
$x(k), v(k) \in \mathbb{R}^2$
$k \in N \cup \{0\}.$

$$real^2 \quad (1)$$

:

$x(k+1) = Ax(k) + u(k)b$
$x(0) = x0,\ u(k) \in [-1,1]$

:

$$A := \left\langle \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle \middle| \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle \right\rangle$$

$$A := \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \quad (2)$$

b :

$$b := \langle 1, 1 \rangle$$

$$b := \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (3)$$

x 0 :

$$x_0 := \langle 3, 0 \rangle$$

$$x_0 := \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad (4)$$

:

```
with(ArrayTools) :
with(LinearAlgebra) :
with(Optimization) :
Minkowsky := proc(X, bx)
local lambdas, sumlam, n, constraint, solved, bounds, lam;
lambdas := Vector(numelems(X), symbol = λ) :
```

```
bounds := Vector( ) :
for lam in lambdas do
 Append(bounds, 0 ≤ lam) :
 end do:
 bounds := convert(bounds, set) :
sumlam := add(lambdas(n), n = 1 ..numelems(X) ) :
 constraint := simplify(VectorMatrixMultiply(Transpose(lambdas), X) ) :
 bounds := bounds union {constraint(1) = bx[1], constraint(2) = bx[2]} :
  solved := LPSolve(sumlam, bounds, assume = nonnegative)
 end proc:
```

X (1)

```
with(ComputationalGeometry) :

list_of_points1 := [convert(MatrixMatrixMultiply(MatrixInverse(A), b), list), convert(
     -MatrixMatrixMultiply(MatrixInverse(A), b), list) ]
```

$$list\_of\_points1 := [[0, \sqrt{2}], [0, -\sqrt{2}]] \quad (5)$$

```
X_1 := ConvexHull(list_of_points1)
```

Error, (in ComputationalGeometry:-ConvexHull) from Qhull: QH6214 qhull input error: not enough points(2) to construct initial simplex (need 3)

,

.

```
with(geometry) :
list_of_points1_alt := [point(Ab1, convert(MatrixMatrixMultiply(MatrixInverse(A), b), list) ),
     point(Ab2, convert( -MatrixMatrixMultiply(MatrixInverse(A), b), list) ) ]
```

$$list\_of\_points1\_alt := [Ab1, Ab2] \quad (6)$$

```
X_1 := convexhull(list_of_points1_alt)
```

$$X\_1 := [Ab1, Ab2] \quad (7)$$

(p o i n t s) ,

.

.

```
X_1 := [coordinates(X_1[1]), coordinates(X_1[2]) ]
```

$$X\_1 := [[0, \sqrt{2}], [0, -\sqrt{2}]] \quad (8)$$

( )

.

```
X_1_vec := Vector( ) :
for i from 1 to numelems(X_1) do
 X_1_vec(i) := convert(X_1(i), Vector)
 end do:

X_1_ZLP := Minkowsky(convert(X_1_vec, Vector), x_0)
```

Error, (in Optimization:-LPSolve) no feasible solution found

```
list_of_points2_1 := [convert(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixInverse(A),
    MatrixInverse(A)), b), list), convert(
    -MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixInverse(A), MatrixInverse(A)), b), list)]
```

$$list\_of\_points2\_1 := [[-1, 1], [1, -1]] \quad (9)$$

```
list_of_points2_2 := [ ]
```

$$list\_of\_points2\_2 := [\,] \quad (10)$$

```
for p1 in list_of_points1 do
 for p2 in list_of_points2_1 do
 list_of_points2_2 := [op(list_of_points2_2), p1 + p2]
 end do
 end do
```

```
list_of_points2_2
```

$$[[-1, 1+\sqrt{2}], [1, \sqrt{2}-1], [-1, 1-\sqrt{2}], [1, -1-\sqrt{2}]] \quad (11)$$

```
X_2 := list_of_points2_2
```

$$X\_2 := [[-1, 1+\sqrt{2}], [1, \sqrt{2}-1], [-1, 1-\sqrt{2}], [1, -1-\sqrt{2}]] \quad (12)$$

```
hull := ConvexHull(list_of_points2_2)
```

$$hull := [2, 1, 3, 4] \quad (13)$$

```
hivec := convert(hull, Vector) :
hiset := convert(hivec, set)
```

$$hiset := \{1, 2, 3, 4\} \quad (14)$$

```
X_2_vec := Vector( ) :
for i from 1 to numelems(X_2) do
 X_2_vec(i) := convert(X_2[i], Vector)
 end do:
```

```
                -    ,
  :
```

```
for i from numelems(X_2_vec) to 1 by -1 do
 if not(member(i, hiset))
 then Remove(X_2_vec, i);
 end if
 end do
X_2_vec :
```

```
    :
```

```
X_2_ZLP := Minkowsky(convert(X_2_vec, Vector), x_0)
```

$$X\_2\_ZLP := [3.00000000000000, [\lambda_1 = 0., \lambda_2 = 2.56066017177982, \lambda_3 = 0., \lambda_4 = 0.439339828220182]] \quad (15)$$

```
list_of_points3_1 :=
    [convert(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixInverse(A),
```

```
MatrixInverse(A)), MatrixInverse(A)), b), list), convert(
-MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixInverse(A),
MatrixInverse(A)), MatrixInverse(A)), b), list)]
```

$$list\_of\_points3\_1 := [[-\sqrt{2}, 0], [\sqrt{2}, 0]] \quad (16)$$

```
list_of_points3_2 := [ ]
```

$$list\_of\_points3\_2 := [\,] \quad (17)$$

```
for p1 in list_of_points2_2 do
 for p2 in list_of_points3_1 do
 list_of_points3_2 := [op(list_of_points3_2), p1 + p2]
 end do
 end do
```

```
list_of_points3_2
```

$$[[-1-\sqrt{2}, 1+\sqrt{2}], [\sqrt{2}-1, 1+\sqrt{2}], [1-\sqrt{2}, \sqrt{2}-1], [1+\sqrt{2}, \sqrt{2}-1], [-1-\sqrt{2}, 1-\sqrt{2}], [\sqrt{2}-1, 1-\sqrt{2}], [1-\sqrt{2}, -1-\sqrt{2}], [1+\sqrt{2}, -1-\sqrt{2}]] \quad (18)$$

```
X_3 := list_of_points3_2
```

$$X\_3 := [[-1-\sqrt{2}, 1+\sqrt{2}], [\sqrt{2}-1, 1+\sqrt{2}], [1-\sqrt{2}, \sqrt{2}-1], [1+\sqrt{2}, \sqrt{2}-1], [-1-\sqrt{2}, 1-\sqrt{2}], [\sqrt{2}-1, 1-\sqrt{2}], [1-\sqrt{2}, -1-\sqrt{2}], [1+\sqrt{2}, -1-\sqrt{2}]] \quad (19)$$

```
hull := ConvexHull(list_of_points3_2)
```

$$hull := [8, 4, 2, 1, 5, 7] \quad (20)$$

```
hivec := convert(hull, Vector) :
hiset := convert(hivec, set)
```

$$hiset := \{1, 2, 4, 5, 7, 8\} \quad (21)$$

```
X_3_vec := Vector( ) :
for i from 1 to numelems(X_3) do
X_3_vec(i) := convert(X_3[i], Vector)
end do:

for i from numelems(X_3_vec) to 1 by -1 do
 if not(member(i, hiset))
 then Remove(X_3_vec, i);
 end if
 end do
```

```
X_3_ZLP := Minkowsky(convert(X_3_vec, Vector), x0)
```

$$X\_3\_ZLP := [1.24264068711928, [\lambda_1 = 0., \lambda_2 = 0., \lambda_3 = 1.06066017177982, \lambda_4 = 0., \lambda_5 = 0., \lambda_6 = 0.181980515339465]] \quad (22)$$

```
list_of_points4_1 :=
    [convert(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
    MatrixMatrixMultiply(MatrixInverse(A), MatrixInverse(A)), MatrixInverse(A)),
     MatrixInverse(A)), b), list), convert(
```

$$-MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixInverse(A), MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)), b), list)]$$

$$list\_of\_points4\_1 := [[-1, -1], [1, 1]] \quad (23)$$

$list\_of\_points4\_2 := [\,]$

$$list\_of\_points4\_2 := [\,] \quad (24)$$

for $p1$ **in** $list\_of\_points3\_2$ **do**
for $p2$ **in** $list\_of\_points4\_1$ **do**
$list\_of\_points4\_2 := [op(list\_of\_points4\_2), p1 + p2]$
end do
end do

$list\_of\_points4\_2$

$$[[-2-\sqrt{2}, \sqrt{2}], [-\sqrt{2}, 2+\sqrt{2}], [-2+\sqrt{2}, \sqrt{2}], [\sqrt{2}, 2+\sqrt{2}], [-\sqrt{2}, -2+\sqrt{2}], [2-\sqrt{2}, \sqrt{2}], [\sqrt{2}, -2+\sqrt{2}], [2+\sqrt{2}, \sqrt{2}], [-2-\sqrt{2}, -\sqrt{2}], [-\sqrt{2}, 2-\sqrt{2}], [-2+\sqrt{2}, -\sqrt{2}], [\sqrt{2}, 2-\sqrt{2}], [-\sqrt{2}, -2-\sqrt{2}], [2-\sqrt{2}, -\sqrt{2}], [\sqrt{2}, -2-\sqrt{2}], [2+\sqrt{2}, -\sqrt{2}]] \quad (25)$$

$X\_4 := list\_of\_points4\_2$:
$hull := ConvexHull(list\_of\_points4\_2)$

$$hull := [2, 1, 9, 13, 15, 16, 8, 4] \quad (26)$$

$hivec := convert(hull, Vector)$:
$hiset := convert(hivec, set)$

$$hiset := \{1, 2, 4, 8, 9, 13, 15, 16\} \quad (27)$$

$X\_4\_vec := Vector(\,)$:
for i **from** 1 **to** $numelems(X\_4)$ **do**
$X\_4\_vec(i) := convert(X\_4[i], Vector)$
end do:

for i **from** $numelems(X\_4\_vec)$ **to** 1 **by** -1 **do**
if not$(member(i, hiset))$
then $Remove(X\_4\_vec, i)$;
end if
end do

X_4_vec :

$X\_4\_ZLP := Minkowsky(convert(X\_4\_vec, Vector), x_0)$

$$X\_4\_ZLP := [0.878679656440356, [\lambda_1 = 0., \lambda_2 = 0., \lambda_3 = 0., \lambda_4 = 0.439339828220178, \lambda_5 = 0., \lambda_6 = 0., \lambda_7 = 0., \lambda_8 = 0.439339828220178]] \quad (28)$$

$list\_of\_points5\_1 :=$
$[convert(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixInverse(A), MatrixInverse(A)),$

$MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)), b), list), convert($
$-MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply($
$MatrixMatrixMultiply(MatrixInverse(A), MatrixInverse(A)), MatrixInverse(A)),$
$MatrixInverse(A)), MatrixInverse(A)), b), list)]$

$$list\_of\_points5\_1 := [[0, -\sqrt{2}], [0, \sqrt{2}]] \quad (29)$$

$list\_of\_points5\_2 := [\,]$

$$list\_of\_points5\_2 := [\,] \quad (30)$$

for $p1$ **in** $list\_of\_points4\_2$ **do**
for $p2$ **in** $list\_of\_points5\_1$ **do**
$list\_of\_points5\_2 := [op(list\_of\_points5\_2), p1 + p2]$
end do
end do

$list\_of\_points5\_2$

$$[[-2-\sqrt{2}, 0], [-2-\sqrt{2}, 2\sqrt{2}], [-\sqrt{2}, 2], [-\sqrt{2}, 2\sqrt{2}+2], [-2+\sqrt{2}, 0], [-2+\sqrt{2}, 2\sqrt{2}], [\sqrt{2}, 2], [\sqrt{2}, 2\sqrt{2}+2], [-\sqrt{2}, -2], [-\sqrt{2}, 2\sqrt{2}-2], [2-\sqrt{2}, 0], [2-\sqrt{2}, 2\sqrt{2}], [\sqrt{2}, -2], [\sqrt{2}, 2\sqrt{2}-2], [2+\sqrt{2}, 0], [2+\sqrt{2}, 2\sqrt{2}], [-2-\sqrt{2}, -2\sqrt{2}], [-2-\sqrt{2}, 0], [-\sqrt{2}, -2\sqrt{2}+2], [-\sqrt{2}, 2], [-2+\sqrt{2}, -2\sqrt{2}], [-2+\sqrt{2}, 0], [\sqrt{2}, -2\sqrt{2}+2], [\sqrt{2}, 2], [-\sqrt{2}, -2\sqrt{2}-2], [-\sqrt{2}, -2], [2-\sqrt{2}, -2\sqrt{2}], [2-\sqrt{2}, 0], [\sqrt{2}, -2\sqrt{2}-2], [\sqrt{2}, -2], [2+\sqrt{2}, -2\sqrt{2}], [2+\sqrt{2}, 0]] \quad (31)$$

$X\_5 := list\_of\_points5\_2:$
$hull := ConvexHull(list\_of\_points5\_2)$

$$hull := [4, 2, 17, 25, 29, 31, 16, 8] \quad (32)$$

$hivec := convert(hull, Vector):$
$hiset := convert(hivec, set)$

$$hiset := \{2, 4, 8, 16, 17, 25, 29, 31\} \quad (33)$$

$X\_5\_vec := Vector(\,):$
for i **from** 1 **to** $numelems(X\_5)$ **do**
$X\_5\_vec(i) := convert(X\_5[i], Vector)$
end do:

for i **from** $numelems(X\_5\_vec)$ **to** 1 **by** -1 **do**
if not$(member(i, hiset))$
then $Remove(X\_5\_vec, i)$;
end if
end do

$X\_5\_vec:$

$X\_5\_ZLP := Minkowsky\left(convert(X\_5\_vec, Vector),\ x_0\right)$

$$X\_5\_ZLP := \left[0.878679656440356, \left[\lambda_1 = 0., \lambda_2 = 0., \lambda_3 = 0., \lambda_4 = 0.439339828220178, \lambda_5 = 0., \lambda_6 = 0., \lambda_7 = 0., \lambda_8 = 0.439339828220178\right]\right] \quad (34)$$

```
list_of_points6_1 :=
    [convert(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
    MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixInverse(A),
     MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)),
     MatrixInverse(A)), b), list), convert(
    -MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
    MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixInverse(A), MatrixInverse(A)),
     MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)), b), list)]
```

$$list\_of\_points6\_1 := [[1, -1], [-1, 1]] \quad (35)$$

```
list_of_points6_2 := [ ]
```

$$list\_of\_points6\_2 := [\,] \quad (36)$$

```
for p1 in list_of_points5_2 do
 for p2 in list_of_points6_1 do
 list_of_points6_2 := [op(list_of_points6_2), p1 + p2]
 end do
 end do
```

```
list_of_points6_2
```

$$[[-1-\sqrt{2}, -1], [-3-\sqrt{2}, 1], [-1-\sqrt{2}, -1+2\sqrt{2}], [-3-\sqrt{2}, 1+2\sqrt{2}], [1-\sqrt{2}, 1], [-1-\sqrt{2}, 3], [1-\sqrt{2}, 1+2\sqrt{2}], [-1-\sqrt{2}, 3+2\sqrt{2}], [\sqrt{2}-1, -1], [-3+\sqrt{2}, 1], [\sqrt{2}-1, -1+2\sqrt{2}], [-3+\sqrt{2}, 1+2\sqrt{2}], [1+\sqrt{2}, 1], [\sqrt{2}-1, 3], [1+\sqrt{2}, 1+2\sqrt{2}], [\sqrt{2}-1, 3+2\sqrt{2}], [1-\sqrt{2}, -3], [-1-\sqrt{2}, -1], [1-\sqrt{2}, -3+2\sqrt{2}], [-1-\sqrt{2}, -1+2\sqrt{2}], [3-\sqrt{2}, -1], [1-\sqrt{2}, 1], [3-\sqrt{2}, -1+2\sqrt{2}], [1-\sqrt{2}, 1+2\sqrt{2}], [1+\sqrt{2}, -3], [\sqrt{2}-1, -1], [1+\sqrt{2}, -3+2\sqrt{2}], [\sqrt{2}-1, -1+2\sqrt{2}], [3+\sqrt{2}, -1], [1+\sqrt{2}, 1], [3+\sqrt{2}, -1+2\sqrt{2}], [1+\sqrt{2}, 1+2\sqrt{2}], [-1-\sqrt{2}, -1-2\sqrt{2}], [-3-\sqrt{2}, 1-2\sqrt{2}], [-1-\sqrt{2}, -1], [-3-\sqrt{2}, 1], [1-\sqrt{2}, 1-2\sqrt{2}], [-1-\sqrt{2}, 3-2\sqrt{2}], [1-\sqrt{2}, 1], [-1-\sqrt{2}, 3], [\sqrt{2}-1, -1-2\sqrt{2}], [-3+\sqrt{2}, 1-2\sqrt{2}], [\sqrt{2}-1, -1], [-3+\sqrt{2}, 1], [1+\sqrt{2}, 1-2\sqrt{2}], [\sqrt{2}-1, 3-2\sqrt{2}], [1+\sqrt{2}, 1], [\sqrt{2}-1, 3], [1-\sqrt{2}, -3-2\sqrt{2}], [-1-\sqrt{2}, -1-2\sqrt{2}], [1-\sqrt{2}, -3], [-1-\sqrt{2}, -1], [3-\sqrt{2}, -1-2\sqrt{2}], [1-\sqrt{2}, 1-2\sqrt{2}], [3-\sqrt{2}, -1], [1-\sqrt{2}, 1], [1+\sqrt{2}, -3-2\sqrt{2}], [\sqrt{2}-1, -1-2\sqrt{2}], [1+\sqrt{2}, -3], [\sqrt{2}-1, -1], [3+\sqrt{2}, -1-2\sqrt{2}], [1+\sqrt{2}, 1-2\sqrt{2}], [3+\sqrt{2}, -1], [1+\sqrt{2}, 1]] \quad (37)$$

```
X_6 := list_of_points6_2 :
hull := ConvexHull(list_of_points6_2)
```

$$hull := [34, 49, 57, 61, 31, 16, 8, 4] \quad (38)$$

```
hivec := convert(hull, Vector) :
hiset := convert(hivec, set)
```

$$hiset := \{4, 8, 16, 31, 34, 49, 57, 61\} \quad (39)$$

```
X_6_vec := Vector( ) :
for i from 1 to numelems(X_6) do
X_6_vec(i) := convert(X_6[i], Vector)
end do:

for i from numelems(X_6_vec) to 1 by -1 do
if not(member(i, hiset) )
then Remove(X_6_vec, i);
end if
end do

X_6_vec :

X_6_ZLP := Minkowsky(convert(X_6_vec, Vector), x_0)
```

$$X\_6\_ZLP := \left[0.679622758982958, \left[\lambda_1 = 0., \lambda_2 = 0., \lambda_3 = 0., \lambda_4 = 0.459952844872869, \lambda_5 = 0., \lambda_6 = 0., \lambda_7 = 0., \lambda_8 = 0.219669914110090\right]\right] \quad (40)$$

```
list_of_points7_1 :=
    [convert(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
    MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
    MatrixInverse(A), MatrixInverse(A) ), MatrixInverse(A) ), MatrixInverse(A) ), MatrixInverse(A) ),
     MatrixInverse(A) ), MatrixInverse(A) ), b), list), convert(
    -MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
    MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixInverse(A),
     MatrixInverse(A) ), MatrixInverse(A) ), MatrixInverse(A) ), MatrixInverse(A) ),
     MatrixInverse(A) ), MatrixInverse(A) ), b), list) ]
```

$$list\_of\_points7\_1 := \left[\left[\sqrt{2}, 0\right], \left[-\sqrt{2}, 0\right]\right] \quad (41)$$

```
list_of_points7_2 := [ ]
```

$$list\_of\_points7\_2 := [\,] \quad (42)$$

```
for p1 in list_of_points6_2 do
 for p2 in list_of_points7_1 do
list_of_points7_2 := [op(list_of_points7_2), p1 + p2]
 end do
 end do

list_of_points7_2 :

X_7 := list_of_points7_2 :
hull := ConvexHull(list_of_points7_2)
```

$$hull := [68, 98, 113, 121, 61, 31, 16, 8] \quad (43)$$

```
hivec := convert(hull, Vector) :
hiset := convert(hivec, set)
```

$$hiset := \{8, 16, 31, 61, 68, 98, 113, 121\} \quad (44)$$

```
X_7_vec := Vector( ) :
for i from 1 to numelems(X_7) do
X_7_vec(i) := convert(X_7[i], Vector)
end do:

for i from numelems(X_7_vec) to 1 by -1 do
if not(member(i, hiset) )
then Remove(X_7_vec, i);
end if
end do

X_7_vec :

X_7_ZLP := Minkowsky(convert(X_7_vec, Vector), x_0)
```

$$X\_7\_ZLP := \left[0.514718625761429, \left[\lambda_1 = 0., \lambda_2 = 0., \lambda_3 = 0., \lambda_4 = 0.348349570550446, \lambda_5 = 0., \lambda_6 = 0., \lambda_7 = 0., \lambda_8 = 0.166369055210983\right]\right] \quad (45)$$

```
list_of_points8_1 :=
[convert(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
MatrixMatrixMultiply(MatrixInverse(A), MatrixInverse(A)), MatrixInverse(A)),
MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)),
MatrixInverse(A)), b), list), convert(
-MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
MatrixInverse(A), MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)),
MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)), b), list) ]
```

$$list\_of\_points8\_1 := [[1, 1], [-1, -1]] \quad (46)$$

```
list_of_points8_2 := [ ]
```

$$list\_of\_points8\_2 := [\,] \quad (47)$$

```
for p1 in list_of_points7_2 do
for p2 in list_of_points8_1 do
list_of_points8_2 := [op(list_of_points8_2), p1 + p2]
end do
end do

list_of_points8_2 :

X_8 := list_of_points8_2 :
hull := ConvexHull(list_of_points8_2)
```

$$hull := [31, 16, 136, 196, 226, 241, 121, 61] \quad (48)$$

```
hivec := convert(hull, Vector) :
```

```
hiset := convert(hivec, set)
```

$$hiset := \{16, 31, 61, 121, 136, 196, 226, 241\} \quad (49)$$

```
X_8_vec := Vector( ) :
for i from 1 to numelems(X_8) do
 X_8_vec(i) := convert(X_8[i], Vector)
 end do:

for i from numelems(X_8_vec) to 1 by -1 do
 if not(member(i, hiset))
 then Remove(X_8_vec, i);
 end if
 end do

X_8_vec :

X_8_ZLP := Minkowsky(convert(X_8_vec, Vector), x_0)
```

$$X\_8\_ZLP := \left[0.439339828220178, \left[\lambda_1 = 0., \lambda_2 = 0., \lambda_3 = 0., \lambda_4 = 0.219669914110089, \lambda_5 = 0., \lambda_6 = 0., \lambda_7 = 0., \lambda_8 = 0.219669914110089\right]\right] \quad (50)$$

```
list_of_points9_1 :=
    [convert(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
    MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
    MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixInverse(A)), MatrixInverse(A)),
     MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)),
     MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)), b), list), convert(
    -MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
    MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
    MatrixMatrixMultiply(MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)),
     MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)),
     MatrixInverse(A)), MatrixInverse(A)), b), list)]
```

$$list\_of\_points9\_1 := \left[\left[0, \sqrt{2}\right], \left[0, -\sqrt{2}\right]\right] \quad (51)$$

```
list_of_points9_2 := [ ]
```

$$list\_of\_points9\_2 := [\,] \quad (52)$$

```
for p1 in list_of_points8_2 do
 for p2 in list_of_points9_1 do
 list_of_points9_2 := [op(list_of_points9_2), p1 + p2]
 end do
 end do

list_of_points9_2 :

X_9 := list_of_points9_2 :
hull := ConvexHull(list_of_points9_2)
```

$$hull := [272, 392, 452, 482, 241, 121, 61, 31] \quad (53)$$

```
hivec := convert(hull, Vector) :
hiset := convert(hivec, set)
```

$$hiset := \{31, 61, 121, 241, 272, 392, 452, 482\} \quad (54)$$

```
X_9_vec := Vector( ) :
for i from 1 to numelems(X_9) do
 X_9_vec(i) := convert(X_9[i], Vector)
 end do:

for i from numelems(X_9_vec) to 1 by -1 do
 if not(member(i, hiset))
 then Remove(X_9_vec, i);
 end if
 end do

X_9_vec :

X_9_ZLP := Minkowsky(convert(X_9_vec, Vector), x_0)
```

$$X\_9\_ZLP := \left[0.439339828220178, \left[\lambda_1 = 0., \lambda_2 = 0., \lambda_3 = 0., \lambda_4 = 0.219669914110089, \lambda_5 = 0., \lambda_6 = 0., \lambda_7 = 0., \lambda_8 = 0.219669914110089\right]\right] \quad (55)$$

```
list_of_points10_1 :=
    [convert(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
    MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
    MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixInverse(A),
     MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)),
     MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)),
     MatrixInverse(A)), b), list), convert(
    -MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
    MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
    MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixInverse(A), MatrixInverse(A)),
     MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)),
     MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)), b), list)]
```

$$list\_of\_points10\_1 := [[-1, 1], [1, -1]] \quad (56)$$

```
list_of_points10_2 := [ ]
```

$$list\_of\_points10\_2 := [\,] \quad (57)$$

```
for p1 in list_of_points9_2 do
 for p2 in list_of_points10_1 do
 list_of_points10_2 := [op(list_of_points10_2), p1 + p2]
 end do
 end do

list_of_points10_2 :

X_10 := list_of_points10_2 :
hull := ConvexHull(list_of_points10_2)
```

$$hull := [241, 121, 61, 543, 784, 904, 964, 482] \quad \textbf{(58)}$$

```
hivec := convert(hull, Vector) :
hiset := convert(hivec, set)
```

$$hiset := \{61, 121, 241, 482, 543, 784, 904, 964\} \quad \textbf{(59)}$$

```
X_10_vec := Vector( ) :
for i from 1 to numelems(X_10) do
X_10_vec(i) := convert(X_10[i], Vector)
end do:

for i from numelems(X_10_vec) to 1 by -1 do
if not(member(i, hiset))
then Remove(X_10_vec, i);
end if
end do

X_10_vec :

X_10_ZLP := Minkowsky(convert(X_10_vec, Vector), x_0)
```

$$X\_10\_ZLP := \left[0.383218742691848, \left[\lambda_1 = 0., \lambda_2 = 0., \lambda_3 = 0., \lambda_4 = 0.236772133285122, \lambda_5 = 0., \lambda_6 = 0., \lambda_7 = 0., \lambda_8 = 0.146446609406726\right]\right] \quad \textbf{(60)}$$

```
list_of_points11_1 :=
    [convert(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
    MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
    MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
    MatrixInverse(A), MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)),
     MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)),
     MatrixInverse(A)), MatrixInverse(A)), b), list), convert(
    -MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
    MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
    MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixInverse(A),
     MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)),
     MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)),
     MatrixInverse(A)), MatrixInverse(A)), b), list)]
```

$$list\_of\_points11\_1 := \left[\left[-\sqrt{2}, 0\right], \left[\sqrt{2}, 0\right]\right] \quad \textbf{(61)}$$

```
list_of_points11_2 := [ ]
```

$$list\_of\_points11\_2 := [\,] \quad \textbf{(62)}$$

```
for p1 in list_of_points10_2 do
 for p2 in list_of_points11_1 do
list_of_points11_2 := [op(list_of_points11_2), p1 + p2]
 end do
 end do

list_of_points11_2 :
```

```
X_11 := list_of_points11_2 :
hull := ConvexHull(list_of_points11_2)
```

$$hull := [482, 241, 121, 1085, 1567, 1808, 1928, 964] \quad (63)$$

```
hivec := convert(hull, Vector) :
hiset := convert(hivec, set)
```

$$hiset := \{121, 241, 482, 964, 1085, 1567, 1808, 1928\} \quad (64)$$

```
X_11_vec := Vector( ) :
for i from 1 to numelems(X_11) do
X_11_vec(i) := convert(X_11[i], Vector)
end do:

for i from numelems(X_11_vec) to 1 by -1 do
if not(member(i, hiset))
then Remove(X_11_vec, i);
end if
end do

X_11_vec :

X_11_ZLP := Minkowsky(convert(X_11_vec, Vector), x_0)
```

$$X\_11\_ZLP := \left[0.324582562663163, \left[\lambda_1 = 0., \lambda_2 = 0., \lambda_3 = 0., \lambda_4 = 0.200543703183920, \lambda_5 = 0., \lambda_6 = 0., \lambda_7 = 0., \lambda_8 = 0.124038859479243\right]\right] \quad (65)$$

```
list_of_points12_1 :=
[convert(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
MatrixMatrixMultiply(MatrixInverse(A), MatrixInverse(A)), MatrixInverse(A)),
MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)),
MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)),
MatrixInverse(A)), b), list), convert(
-MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
MatrixInverse(A), MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)),
MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)),
MatrixInverse(A)), MatrixInverse(A)), MatrixInverse(A)), b), list)]
```

$$list\_of\_points12\_1 := [[-1, -1], [1, 1]] \quad (66)$$

```
list_of_points12_2 := [ ]
```

$$list\_of\_points12\_2 := [\,] \quad (67)$$

```
for p1 in list_of_points11_2 do
for p2 in list_of_points12_1 do
list_of_points12_2 := [op(list_of_points12_2), p1 + p2]
end do
```

```
end do
```

list_of_points12_2 :

```
X_12 := list_of_points12_2 :
hull := ConvexHull(list_of_points12_2)
```

$$hull := [964, 482, 241, 2169, 3133, 3615, 3856, 1928] \quad \textbf{(68)}$$

```
hivec := convert(hull, Vector) :
hiset := convert(hivec, set)
```

$$hiset := \{241, 482, 964, 1928, 2169, 3133, 3615, 3856\} \quad \textbf{(69)}$$

```
X_12_vec := Vector( ) :
for i from 1 to numelems(X_12) do
 X_12_vec(i) := convert(X_12[i], Vector)
 end do:

for i from numelems(X_12_vec) to 1 by -1 do
 if not(member(i, hiset) )
 then Remove(X_12_vec, i);
 end if
 end do
```

X_12_vec :

```
X_12_ZLP := Minkowsky( convert(X_12_vec, Vector), x_0)
```

$$X\_12\_ZLP := \left[0.292893218813452, \left[\lambda_1 = 0., \lambda_2 = 0., \lambda_3 = 0., \lambda_4 = 0.146446609406726, \lambda_5 = 0., \lambda_6 = 0., \lambda_7 = 0., \lambda_8 = 0.146446609406726\right]\right] \quad \textbf{(70)}$$

```
list_of_points13_1 :=
    [convert(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
    MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
    MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
    MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixInverse(A), MatrixInverse(A) ),
     MatrixInverse(A) ), MatrixInverse(A) ), MatrixInverse(A) ), MatrixInverse(A) ),
     MatrixInverse(A) ), MatrixInverse(A) ), MatrixInverse(A) ), MatrixInverse(A) ),
     MatrixInverse(A) ), MatrixInverse(A) ), MatrixInverse(A) ), b), list), convert(
    -MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
    MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
    MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(MatrixMatrixMultiply(
    MatrixMatrixMultiply(MatrixInverse(A), MatrixInverse(A) ), MatrixInverse(A) ),
     MatrixInverse(A) ), MatrixInverse(A) ), MatrixInverse(A) ), MatrixInverse(A) ),
     MatrixInverse(A) ), MatrixInverse(A) ), MatrixInverse(A) ), MatrixInverse(A) ),
     MatrixInverse(A) ), MatrixInverse(A) ), b), list) ]
```

$$list\_of\_points13\_1 := \left[\left[0, -\sqrt{2}\right], \left[0, \sqrt{2}\right]\right] \quad \textbf{(71)}$$

```
list_of_points13_2 := [ ]
```

$$list\_of\_points13\_2 := [\,] \tag{72}$$

```
for p1 in list_of_points12_2 do
 for p2 in list_of_points13_1 do
list_of_points13_2 := [op(list_of_points13_2), p1 + p2]
 end do
 end do
```

```
list_of_points13_2 :
```

```
X_13 := list_of_points13_2 :
hull := ConvexHull(list_of_points13_2)
```

$$hull := [482, 4337, 6265, 7229, 7711, 3856, 1928, 964] \tag{73}$$

```
hivec := convert(hull, Vector) :
hiset := convert(hivec, set)
```

$$hiset := \{482, 964, 1928, 3856, 4337, 6265, 7229, 7711\} \tag{74}$$

```
X_13_vec := Vector( ) :
for i from 1 to numelems(X_13) do
X_13_vec(i) := convert(X_13[i], Vector)
 end do:

for i from numelems(X_13_vec) to 1 by -1 do
 if not(member(i, hiset))
 then Remove(X_13_vec, i);
 end if
 end do
```

```
X_13_vec :
```

```
X_13_ZLP := Minkowsky(convert(X_13_vec, Vector), x_0)
```

$$X\_13\_ZLP := \left[0.292893218813452, \left[\lambda_1 = 0., \lambda_2 = 0., \lambda_3 = 0., \lambda_4 = 0.146446609406726, \lambda_5 = 0., \lambda_6 = 0., \lambda_7 = 0., \lambda_8 = 0.146446609406726\right]\right] \tag{75}$$