Artificial problem

$$\begin{cases} \min \; z = \sum_{i=1}^{m} y_i \\ Ax + Iy = b \\ x, y \geq 0 \end{cases}$$

$y \in \mathbb{R}^m \rightarrow$ must all go to $\emptyset$ so $\min \; z = 0$

If opt $w^* = 0$, then $y_1^* \dots y_m^* = 0$, then x^* is a feasible solution of the original problem

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$$\begin{cases} \min \; z = x_1 + x_3 \\ x_1 + 2x_2 + x_4 = 5 \\ x_2 + 2x_3 = 6 \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$

Σy_i

		x_1	x_2	x_3	x_4	y_1	y_2
$w =$	0	0	0	0	0	1	1
$y_1 =$	5	1	2	0	1	1	0
$y_2 =$	6	0	1	2	0	0	1

$\rightarrow b \geq 0$

1) $Ax + Iy = b \longrightarrow a_i^T x + y_i = b_i$

$$w = \sum_{i=1}^{m} \left(b_i - a_i^T x \right)$$

2) mini-pivot over the y_i values to bring the artificial cost to 0

3) $\begin{bmatrix} \text{new} \\ \text{row } 0 \end{bmatrix} \leftarrow \begin{bmatrix} \text{original} \\ \text{row } 0 \end{bmatrix} - \sum_{i=1}^{m} [\text{row } i]$

Finding the first basis

We can create an artificial problem to help us find the first basis

$$\begin{cases} \min \quad w = \sum_{i=1}^{m} y_i \\ Ax + Iy = b \\ x, y \geq 0 \end{cases}$$

by applying the simplex method

Whenever we find $w^* = 0$, $y_1^*, \dots, y_m^* = 0$, so we found a feasible solution $\to$ we can use this as the initial problem

Es)

$$\begin{cases} \min \quad z = x_1 + x_3 \\ x_1 + 2x_2 + x_4 = 5 \\ x_2 + 2x_3 = 6 \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$

Canonical tableau

		x_1	x_2	x_3	x_4	y_1	y_2
$(w = \sum y_i =)$	0	0	0	0	0	1	1
$(y_1 =)$	5	1	2	0	1	1	0
$(y_2 =)$	6	0	1	2	0	0	1

Since y_1, y_2 belong to the basis, we want their costs to be 0

We can achieve that by $\left[\text{new row } 0\right] = \left[\text{old row } 0\right] - \sum_{i=1}^{m} [\text{row } i]$

0	0	0	0	0	1	1
5	1	2	0	1	1	0
6	0	1	2	0	0	1

$\overset{(3)}{\Rightarrow}$

-11	-1	-3	-2	-1	0	0
5	1	2	0	1	1	0
6	0	1	2	0	0	1

simplex method

$-w$

	0	0	0	0	0	1	1
$x_2 =$	5/2	1/2	1	0	1/2	1/2	0
$x_3 =$	7/4	-1/4	0	1	-1/4	-1/4	1/2

$\bar{b} \geq 0$

$w^* = 0 \Rightarrow \underbrace{y_1^*, y_2^*}_{\text{non-basic}} = 0$

$$x^* = \begin{bmatrix} 0 \\ 5/2 \\ 7/4 \\ 0 \end{bmatrix} \rightarrow \text{feasible for the original problem}$$

Phase II:

→ must be 0 to have a canonical form

0	1	0	1	0
5/2	1/2	1	0	1/2
7/4	-1/4	0	1	-1/4

→ original obj function

pivot operation over x_3

-7/4	5/4	0	0	1/4
5/2	1/2	1	0	1/2
7/4	-1/4	0	1	-1/4

⇝ simplex method

Particular cases of "phase I":

1) $w^* < 0$ $\Rightarrow$ The (original) problem is unfeasible

2) $w^* = 0$ with y_i basic for some i

$\hookrightarrow$ If $\exists\ \bar{a}_{th} \neq 0$ then I make a pivot operation on it (divide the whole row by it)

else that row is redundant: it can simply be removed since it's a linear combination of two other rows

ES)

$$\begin{cases} \min\ z = x_1 + x_3 \\ x_1 + 2x_2 \leq -5 \\ x_2 + 2x_3 = 6 \\ x_1, x_2, x_3 \geq 0 \end{cases} \Rightarrow \begin{cases} \min\ z = x_1 + x_3 \\ x_1 + 2x_2 + x_4 = -5 \\ x_2 + 2x_3 = 6 \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases} \Rightarrow$$ <u>Impossible</u> (assume we don't realize)

		x_1	x_2	x_3	x_4	y_1	y_2
$-w =$	-11	1	1	-2	1	0	0
$y_1 =$	$+5$	-1	-2	0	-1	1	0
$y_2 =$	6	0	1	2	0	0	1

$\rightarrow$ canonical form

I can start phase I

"x_3 enters the basis"

"y_2 leaves the basis"

		x_1	x_2	x_3	x_4	y_1	y_2
$-w =$	-5	1	2	0	1	0	1
$y_1 =$	5	-1	-2	0	-1	1	0
$x_3 =$	3	0	$1/2$	1	0	0	$1/2$

$\rightarrow$ optimal : $w = 5$

$\Downarrow$

Unfeasible

ES)

$$\begin{cases} \min & z = x_1 + x_2 + 10x_3 \\ & x_2 + 4x_3 = 2 \\ & -2x_1 + x_2 - 6x_3 = 2 \\ & x_1, x_2, x_3 \geq 0 \end{cases} \longrightarrow$$

$$\begin{array}{r|c|ccc|cc} & & x_1 & x_2 & x_3 & y_1 & y_2 \\ \hline -w = & -4 & 2 & -2 & 2 & 0 & 0 \\ \hline y_1 = & 2 & 0 & 1 & 4 & 1 & 0 \\ y_2 = & 2 & -2 & 1 & -6 & 0 & 1 \end{array}$$

"x_2 enters the basis"

"y_2 leaves the basis"

phase I

$$\begin{array}{r|c|ccc|cc} & & x_1 & x_2 & x_3 & y_1 & y_2 \\ -w = & 0 & 2 & 0 & 10 & 2 & 0 \\ \hline x_2 = & 2 & 0 & 1 & 4 & 1 & 0 \\ y_2 = & 0 & -2 & 0 & -10 & -1 & 1 \end{array}$$

→ y_2 is basic

pivot operation

$$\begin{array}{c|ccc|cc} 0 & 2 & 0 & 10 & \cancel{2} & \cancel{0} \\ \hline 2 & 0 & 1 & 4 & \cancel{1} & \cancel{0} \\ 0 & 1 & 0 & 5 & \cancel{1/2} & \cancel{1/2} \end{array}$$

→ we can now remove y_1, y_2

phase II

change obj f.

$$\begin{array}{c|ccc} 0 & 1 & 1 & 10 \\ \hline 2 & 0 & 1 & 4 \\ 0 & 1 & 0 & 5 \end{array} \Rightarrow \begin{array}{c|ccc} -2 & 0 & 0 & 1 \\ \hline 2 & 0 & 1 & 4 \\ 0 & 1 & 0 & 5 \end{array}$$

pivot on those to get the costs to 0

→ Canonical form and $\bar{c}_F^T \geq 0$

$\Downarrow$

Optimal solution

→ simplex method