Finding a Good Hypothesis

Linear classification with hypothesis set $\mathcal{H}$ = halfspaces.

How do we find a good hypothesis?

Good = minimizes the training error (ERM)

→ we know just the training error, but we aim to lower the generalization error

$\Rightarrow$ Perceptron Algorithm (Rosenblatt, 1958)

Note:

≥ 0

if $y_i \langle \mathbf{w}, \mathbf{x}_i \rangle > 0$ for all $i = 1, \ldots, m \Rightarrow$ all points are classified correctly by model $\mathbf{w} \Rightarrow$ *realizability assumption* for training set

+/-

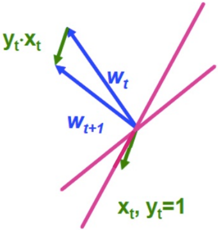
$\forall i$

Linearly separable data: there exists $\mathbf{w}$ such that: $y_i \langle \mathbf{w}, \mathbf{x}_i \rangle > 0$

Perceptron

Input: training set $(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_m, y_m)$
initialize $\mathbf{w}^{(1)} = (0, \ldots, 0)$;
for $t = 1, 2, \ldots$ **do** → If we find an error
  if $\exists i$ *s.t.* $y_i \langle \mathbf{w}^{(t)}, \mathbf{x}_i \rangle \leq 0$ **then** $\mathbf{w}^{(t+1)} \leftarrow \mathbf{w}^{(t)} + y_i \mathbf{x}_i$; → we update the hypotesis
  else return $\mathbf{w}^{(t)}$; → $\forall i$, x_i is correctly classified

Interpretation of update:



Note that:

$$
\begin{aligned}
y_i \langle \mathbf{w}^{(t+1)}, \mathbf{x}_i \rangle &= y_i \langle \mathbf{w}^{(t)} + y_i \mathbf{x}_i, \mathbf{x}_i \rangle \\
&= y_i \langle \mathbf{w}^{(t)}, \mathbf{x}_i \rangle + ||\mathbf{x}_i||^2
\end{aligned}
$$

$\Rightarrow$ update guides $\mathbf{w}$ to be "more correct" on $(\mathbf{x}_i, y_i)$.
→ not to be immediately correct

Termination? Depends on the realizability assumption!

Perceptron with Linearly Separable Data

If data is linearly separable one can prove that the perceptron terminates.

Proposition

Assume that $(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_m, y_m)$ is linearly separable, let:

- $B = \min\{||\mathbf{w}|| : y_i\langle \mathbf{w}, \mathbf{x}_i\rangle \geq 1 \ \ \forall i, i = 1, \ldots, m, \}$, and
- $R = \max_i ||\mathbf{x}_i||$.

Then the Perceptron algorithm stops after at most $(RB)^2$ iterations (and when it stops it holds that $\forall i, i \in \{1, \ldots, m\} : y_i\langle \mathbf{w}^{(t)}, \mathbf{x}_i\rangle > 0$).

Perceptron: Notes

- simple to implement (but some details are not described in the pseudocode...)
- for separable data
  - termination is guaranteed
  - may require a number of iterations that is exponential in d...
    $\Rightarrow$ other approaches (e.g., ILP - Integer Linear Programming) may be better to find ERM solution in such cases
  - potentially multiple solutions, which one is picked depends on starting values
- non separable data?
  - run for some time and keep best solution found up to that point (*pocket algorithm*)

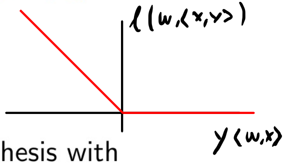
Perceptron: A Modern View

The previous presentation of the Perceptron is the standard one.

However, we can derive the Perceptron in a different way...

Assume you want to solve a:

- binary classification problem: $\mathcal{X} = \mathbb{R}^d$, $\mathcal{Y} = \{-1, 1\}$
- with linear models
- with loss $\ell(\mathbf{w}, (\mathbf{x}, y)) = \max\{0, -y\langle \mathbf{w}, \mathbf{x}\rangle\}$.



Approach: ERM $\Rightarrow$ need to find the model/hypothesis with smallest training error

How? (SGD)

Note: this is a common framework in all of machine learning!

Gradient Descent (GD)

General approach for *minimizing* a differentiable convex function $f(\mathbf{w})$

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a differentiable function

Definition

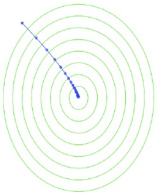
The *gradient* $\nabla f(\mathbf{w})$ of f at $\mathbf{w} = (w_1, \ldots, w_d)$ is

$$\nabla f(\mathbf{w}) = \left(\frac{\partial f(\mathbf{w})}{\partial w_1}, \ldots, \frac{\partial f(\mathbf{w})}{\partial w_d} \right)$$

Intuition: the gradient points in the direction of the greatest rate of increase of f around $\mathbf{w}$

Let $\eta \in \mathbb{R}, \eta > 0$ be a parameter.

GD algorithm:



$\mathbf{w}^{(0)} \leftarrow \mathbf{0}$;
for $t \leftarrow 0$ *to* $T-1$ **do**
  $\mathbf{w}^{(t+1)} \leftarrow \mathbf{w}^{(t)} - \eta \nabla f(\mathbf{w}^{(t)})$;
return $\bar{\mathbf{w}} = \frac{1}{T}\sum_{t=1}^{T} \mathbf{w}^{(t)}$;

Notes:

- output vector could also be $\mathbf{w}^{(T)}$ or $\arg\min_{\mathbf{w}^{(t)} \in \{1,\dots,T\}} f(\mathbf{w}^{(t)})$
- returning $\bar{\mathbf{w}}$ is useful for nondifferentiable functions (using *subgradients* instead of gradients...) and for stochastic gradient descent...
- η: *learning rate*; sometimes a time dependent $\eta^{(t)}$ is used (e.g., "move" more at the beginning than at the end)

Note: there are guarantees on the number of iterations required by GD to return a *good* value of $\bar{\mathbf{w}}$ under some assumptions on f (see the book for details)

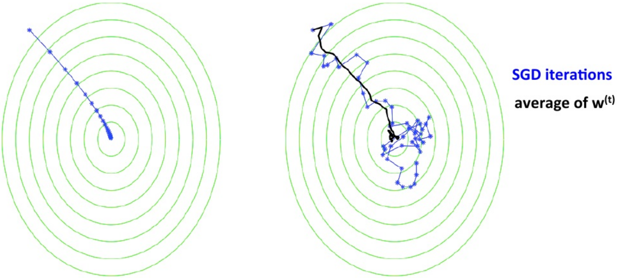
Stochastic Gradient Descent (SGD)

Idea: instead of using exactly the gradient, we take a (random) vector with *expected value* equal to the gradient direction.

SGD algorithm:

$\mathbf{w}^{(0)} \leftarrow \mathbf{0}$;
for $t \leftarrow 0$ *to* $T-1$ **do**
  choose $\mathbf{v}_t$ at random from distribution such that $\mathbf{E}[\mathbf{v}_t | \mathbf{w}^{(t)}] \in \nabla f(\mathbf{w}^{(t)})$;
  /* $\mathtt{v}_t$ has *expected value* equal to the gradient of $f(\mathtt{w}^{(t)})$ */
  $\mathbf{w}^{(t+1)} \leftarrow \mathbf{w}^{(t)} - \eta \mathbf{v}_t$;

return $\bar{\mathbf{w}} = \frac{1}{T}\sum_{t=1}^{T} \mathbf{w}^{(t)}$;



Why should we use SGD instead of GD?

Question: when do we use GD in the first place?

Answer: for example to find $\mathbf{w}$ that minimizes $L_S(\mathbf{w})$

That is: we use GD for $f(\mathbf{w}) = L_S(\mathbf{w})$
$\Rightarrow \nabla f(\mathbf{w})$ depends on all pairs $(\mathbf{x}_i, y_i) \in S, i = 1, \ldots, m$: may require long time to compute it!

What about SGD?

We need to pick $\mathbf{v}_t$ such that $\mathbf{E}[\mathbf{v}_t | \mathbf{w}^{(t)}] \in \nabla f(\mathbf{w}^{(t)})$: **how**?

Pick a random $(\mathbf{x}_i, y_i) \in S \Rightarrow$ pick $\mathbf{v}_t \in \nabla \ell(\mathbf{w}^{(t)}, (\mathbf{x}_i, y_i))$:

- satisfies the requirement!
- requires much less computation than GD

Analogously we can use SGD for regularized losses, etc.

Back to Our Linear Classification Problem

- binary classification problem: $\mathcal{X} = \mathbb{R}^d$, $\mathcal{Y} = \{-1, 1\}$
- with linear models
- with loss $\ell(\mathbf{w}, (\mathbf{x}, y)) = \max\{0, -y\langle \mathbf{w}, \mathbf{x}\rangle\}$.

How to find the ERM solution? SGD!

SGD for Linear Classification

SGD: Take i uniformly at random from $\{1, \dots, m\}$

Let $(\vec{x}', y')$ be the corresponding point in the training set and consider the vector $\nabla \ell(\vec{w}, (\vec{x}', y'))$

Note that the GD considers ℓ as gradient of the function to minimize f:

$$\nabla L_S(\vec{w}) = \frac{1}{m} \sum_{i=1}^{m} \nabla \ell(\vec{w}, (\vec{x}_i, y_i))$$

and for SGD we have:

$$\mathbb{E}[\nabla \ell(\vec{w}, (\vec{x}', y'))] = \sum_{i=1}^{m} \Pr[(\vec{x}', y') = (\vec{x}_i, y_i)] \nabla \ell(\vec{w}, (\vec{x}_i, y_i))$$

$$\overset{!}{=} \frac{1}{m} \sum_{i=1}^{m} \nabla \ell(\vec{w}, (\vec{x}_i, y_i))$$

$$\overset{!}{=} \nabla L_S(\vec{w})$$