Transformations of a k-dimensional rve

Let $g : \mathbb{R}^k \to \mathbb{R}^p$, and consider $Y = g(X)$.

Under appropriate conditions on g, Y is a p-dimensional rve.

The pdf of Y it's easier to compute when: (i) g is bijective, (ii) $p = 1$.

In (i), necessarily $k = p$, and f is computed following a change-of-variable argument;
in (ii) we can use the same steps as in L0, slide 37/39, paying careful attention to the set B_y which now is a hypercube.

First, let's see (ii) by an example. Then we'll turn back to (i).

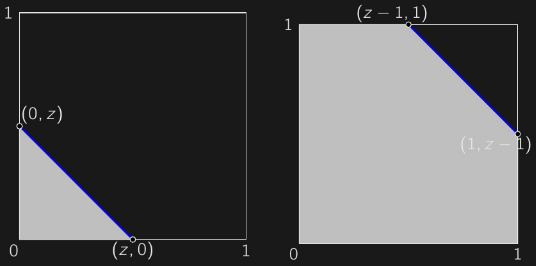
Example 5

Let the rve (X, Y) have joint pdf $f(x, y) = 1$ when $0 \leq x, y \leq 1$ and $f(x, y) = 0$ otherwise and compute the distribution of $Z = X + Y$. We have $z = g(x, y) = x + y$ and $p = 1$. Now for all $z \in [0, 2]$,

$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P(g(X, Y) \leq z) \\ &= P(\{X, Y : g(X, Y) \leq z\}) = \int\int_{B_z} f(x, y)\mathrm{d}x\mathrm{d}y. \end{aligned}$$

We consider two cases: (1) $z \leq 1$ and (2) $z > 1$; to see why these cases are useful, consider the following picture.

The probability involved in the last integral is the area of the unit square below the line $x + y = z$.
Left: $z \leq 1$ and area $= z^2/2$; **Right**: $z > 1$ and area $= F_Z(z) = 1 - (2-z)^2/2$.



Thus

$$F_Z(z) = \begin{cases} 0 & \text{if } z < 0 \\ z^2/2 & \text{if } 0 \leq z \leq 1, \\ 1 - (2-z)^2/2 & \text{if } 1 < z \leq 2 \\ 1 & \text{otherwise.} \end{cases}, \text{ and } f_Z(z) = \begin{cases} z & \text{if } 0 \leq z \leq 1, \\ 2 - z & \text{if } 1 < z \leq 2 \\ 0 & \text{otherwise .} \end{cases}$$

The change-of-variable argument

Formally, let $g : \mathbb{R}^k \to \mathbb{R}^k$ with $g(x) = (g_1(x), \dots, g_k(x))$ differentiable and bijective with inverse $g^{-1}(y) = (g_1^{-1}(y), \dots, g_k^{-1}(y))$.
Then $Y = g(X)$ is a rve with pdf

$$f_Y(y) = f_X(g^{-1}(y))|\det(J(y))|,$$

where

$$\det(J(y)) = \det\left(\left[\frac{dg^{-1}(y)}{dy_1}\middle|\frac{dg^{-1}(y)}{dy_2}\middle|\dots\middle|\frac{dg^{-1}(y)}{dy_k}\right]\right),$$

is called the Jacobian of the transformation and

$$\frac{dg^{-1}(y)}{dy_i} = (g_1^{-1}(y)/\partial y_i, \dots, g_k^{-1}(y)/\partial y_i)$$

is a column vector-valued function.

Let's put it in action with an example...

The change-of-variable argument

Example 6

Let X_1, X_2 be independent with $X_i \sim \mathrm{N}(0,1)$. We show that if $Y = X_1 + X_2$, then $Y \sim \mathrm{N}(0,2)$.
For, let $Z = X_1 - X_2$, thus $g = (g_1, g_2)$, with

$$g_1(x_1, x_2) = x_1 + x_2, \quad g_2 = x_1 - x_2$$

and inverse g^{-1} with

$$x_1 = g_1^{-1}(y, z) = (y+z)/2, \quad x_2 = g_2^{-1} = (y-z)/2.$$

Thus

$$J(y,z) = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}.$$

And the density is thus

$$f_{Y,Z}(y,z) = f_{X_1,X_2}\left(\frac{y+z}{2}, \frac{y-z}{2}\right) |-1/2|$$

$$
\begin{aligned}
f_{YZ}(y,z) &= f_{X_1}\left(\tfrac{y+z}{2}\right) f_{X_2}\left(\tfrac{y-z}{2}\right)\tfrac{1}{2} \\
&= \tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y+z}{2}\right)^2} \tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-z}{2}\right)^2} \tfrac{1}{2} \\
&= \tfrac{1}{\sqrt{2\pi\cdot 2}} e^{-\frac{1}{2}\left(\frac{y}{2}\right)^2} \tfrac{1}{\sqrt{2\pi\cdot 2}} e^{-\frac{1}{2}\left(\frac{z}{2}\right)^2}.
\end{aligned}
$$

And thus

$$
f_Y(y) = \int_{-\infty}^{\infty} f_{YZ}(y,z)\mathrm{d}z = \tfrac{1}{\sqrt{2\pi\cdot 2}} e^{-\frac{1}{2}\left(\frac{y}{2}\right)^2},
$$

so $Y \sim \mathrm{N}(0,2)$.

The multivariate normal distribution

If $(X_1, \dots, X_k)$ are independent $\mathrm{N}(0,1)$, then their **joint pdf**

$$f_{X_1,\dots,X_k}(x_1, \dots, x_k) = \prod_{i=1}^{k} f_{X_i}(x_i) = \frac{1}{(2\pi)^{-k/2}} e^{-\frac{1}{2}(x_1^2+\dots+x_k^2)},$$

is a **special case** of the multivariate normal distribution.

General case: let $E(X_i) = \mu_i$, $\mathrm{var} X_i = \sigma_i^2$ and $\mathrm{cov}(X_i, X_j) = \sigma_{ij}$, thus set

$$\mu = (\mu_1, \dots, \mu_k), \quad \Sigma = \begin{pmatrix} \sigma_1^2 & \dots & \sigma_{1k} \\ \vdots & \vdots & \vdots \\ \sigma_{1k} & \dots & \sigma_k^2 \end{pmatrix}.$$

The joint pdf is

$$f_{X_1,\dots,X_k}(x_1, \dots, x_k) = \frac{1}{(2\pi)^{-k/2} \det(\Sigma)^{1/2}} e^{-\frac{1}{2} x^{\mathrm{T}} \Sigma^{-1} x},$$

for all $x = (x_1, \dots, x_k)$.

Properties

Here are some important properties about the multivariate normal distribution.

If $(X_1, \ldots, X_p)$ are $\mathrm{N}_p(\mu, \Sigma)$, then

(i) If $Y = BX + b$, with B a $q \times p$ matrix and $b \in \mathbb{R}^q$, then $Y \sim \mathrm{N}_q(B\mu + b, B\Sigma B^{\mathrm{T}})$.

(ii) $X_i \sim \mathrm{N}(\mu_i, \sigma_i^2)$, where $\mu_i = E(X_i)$ and $\sigma_i^2 = \mathrm{var}(X_i)$, $i = 1, \ldots, p$.

(iii) All the conditional distributions involving components of X are normal with suitable parameters.

As in the univariate case, probabilities under the multivariate normal have to be computed by numerical methods.

The `mvtnorm` package of `R` provides an excellent implementation.