$(B^{-1}b)_i = 0 \implies$ DEGENERACY

$\hookrightarrow$ at least one $\exists\, i \in \{1 \dots n\}$

$(B^{-1}b)_i = 0 \Rightarrow$ DEGENERACY

$\rightarrow$ at least one $\exists i \in \{1 \dots n\}$

(Popular in exams $\rightarrow$ even the proof!!!)

Theorem (pag 37)

$\rightarrow$ std. form

A point $x \in P$ is a vertex of $P = \{x \geq 0 \mid Ax = b\} \neq \emptyset$

iff

x is a basic feasible solution associated to the system $Ax = b$

(1 to 1 correspondence between vertices and basic feasible solutions)

(not counting degenerate solutions)

Proof:

- "x is a bfs $\Rightarrow$ x is a vertex"

$$x = [\underbrace{x_1, \dots, x_k}_{>0}, 0, \dots, 0]^T \qquad (x \in \mathbb{R}^n) \qquad \text{for } k \in [0, m]$$

$\Downarrow$

$A_1, \dots, A_k$ are independent

k can be smaller than rank(A) due to some degeneracy

Assume that x is not a vertex $\Rightarrow \exists y, z \in P, y \neq z, \lambda \in]0,1[\mid x = \lambda y + (1-\lambda) z$

$$y = [y_1 \dots y_k \, 0 \dots 0]^T \qquad z = [z_1 \dots z_k \, 0 \dots 0]^T$$

If y, z didn't have the same "form" of x:

$$y = [y_1 \dots y_k \; 0 \dots \overset{\neq 0}{\alpha} \dots 0]$$

$$z = [z_1 \dots z_k \; 0 \dots \dots 0]$$

$$\Downarrow \qquad \qquad \neq 0 \Rightarrow \text{impossible}$$

$$x = [x_1 \dots x_k \; 0 \dots \lambda\alpha \dots 0]$$

Since $y \in P$, $\underbrace{A_1 y_1 + \dots + A_k y_k = b}_{(\#)}$ (same for z) $(\#\#)$

$$(\#) - (\#\#) = A_1 (\underbrace{y_1 - z_1}_{\alpha_1}) + \dots + A_k (\underbrace{y_k - z_k}_{\alpha_k}) = b - b = 0$$

$\alpha_1, \dots, \alpha_k$ are ≥ 0, but since $y \neq z$, α_i can't be $\emptyset$ $\forall i \dots$

$$\exists \alpha_j \neq 0 \Rightarrow (\#) - (\#\#) \neq 0$$

$$\Downarrow$$

$A_1, \dots, A_k$ must be linearly dependent, which is a contraddiction.

$$\Downarrow$$

Our assumption must be wrong

$$\Downarrow$$

x must be a vertex

- "X is a vertex $\Rightarrow$ X is a bfs"

$$x = [\underbrace{x_1 \dots x_k}_{>0} \underbrace{0 \dots 0}_{n-k}]^T \in \mathbb{R}^n \qquad k \in [0, n]$$

$x \in P \Rightarrow A_1 x_1 + \dots + A_k x_k = b \quad (*)$

Two possible cases:

i) $A_1 \dots A_k$ are linear independent

since k cannot be larger than m, $x_{k+1}, \dots, x_m$ can be $\emptyset$

$$x = [\underbrace{\underbrace{x_1 \dots x_k}_{>0} \; 0 \dots 0}_{m} \underbrace{\dots 0}_{n-m}]$$

$$A = [\underbrace{A_1 \dots A_k \underbrace{\dots A_m}_{\text{pick those columns st. } A_1 \dots A_m \text{ are lin. independent}}}_{B} \dots]$$

And since $x = \begin{bmatrix} B^{-1}b \\ \hline 0 \end{bmatrix}$ is satisfied $\Rightarrow$ x is a bfs

ii) $A_1 \dots A_k$ are linearly dependent

$\Downarrow$

$\exists\, \alpha_1, \dots, \alpha_k$ not all equal to 0 s.t. ($\gtrless 0$)

$$A_1\alpha_1 + A_2\alpha_2 + \dots + A_k\alpha_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \quad (\#\#)$$

Given $\varepsilon \gtrsim 0$ (close to 0)

$$(\#) + \varepsilon(\#\#) = (\underbrace{x_1 + \varepsilon\alpha_1}_{z_1})A_1 + \dots + (\underbrace{x_k + \varepsilon\alpha_k}_{z_k})A_k = b + 0 = b$$

$$(\#) - \varepsilon(\#\#) = (\underbrace{x_1 - \varepsilon\alpha_1}_{y_1})A_1 + \dots + (\underbrace{x_k - \varepsilon\alpha_k}_{y_k})A_k = b - 0 = b$$

$$y = [\underbrace{y_1 \dots y_k}_{>0}\ 0 \dots 0]^T \in \mathbb{R}^n \qquad z = [\underbrace{z_1 \dots z_k}_{>0}\ 0 \dots 0]^T \in \mathbb{R}^n$$

$Ay = Az = b \implies y, z \in P$, $y \neq z$ since not all α are 0 and $\varepsilon \neq 0$

Since $y = x + \begin{bmatrix} \varepsilon\alpha_1 \\ \vdots \\ \varepsilon\alpha_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $z = x - \begin{bmatrix} \varepsilon\alpha_1 \\ \vdots \\ \varepsilon\alpha_k \\ 0 \\ \vdots \\ 0 \end{bmatrix} \implies x = \frac{1}{2}x + \frac{1}{2}y$

$=$

$\Downarrow$

x is not a vertex

$\Downarrow$

impossible

ii) $A_1 \dots A_k$ are linearly dependent

$\Downarrow$

$\exists\, \alpha_1, \dots, \alpha_k$ not all equal to 0 s.t. $\quad \gtrless 0$

$$A_1\alpha_1 + A_2\alpha_2 + \dots + A_k\alpha_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \quad (\#\#)$$

Given $\varepsilon \geq 0$ (close to 0)

$$(\#) + \varepsilon(\#\#) = (\underbrace{x_1 + \varepsilon\alpha_1}_{z_1})A_1 + \dots + (\underbrace{x_k + \varepsilon\alpha_k}_{z_k})A_k = b + 0 = b$$

$$(\#) - \varepsilon(\#\#) = (\underbrace{x_1 - \varepsilon\alpha_1}_{y_1})A_1 + \dots + (\underbrace{x_k - \varepsilon\alpha_k}_{y_k})A_k = b - 0 = b$$

$$y = [\underbrace{y_1 \dots y_k}_{>0}\ 0 \dots 0]^T \in \mathbb{R}^n \qquad z = [\underbrace{z_1 \dots z_k}_{>0}\ 0 \dots 0]^T \in \mathbb{R}^n$$

$Ay = Az = b \implies y, z \in P$, $y \neq z$ since not all α are 0 and $\varepsilon \neq 0$

Since $y = x + \begin{bmatrix} \varepsilon\alpha_1 \\ \vdots \\ \varepsilon\alpha_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\quad z = x - \begin{bmatrix} \varepsilon\alpha_1 \\ \vdots \\ \varepsilon\alpha_k \\ 0 \\ \vdots \\ 0 \end{bmatrix} \implies x = \frac{1}{2}x + \frac{1}{2}y$

$=$

$\Downarrow$

x is not a vertex

$\Downarrow$

impossible

$$\# BFS = \binom{n}{m} = \frac{n!}{m!(n-m)!}$$

GEORGE DANTZIG:

1) choose a random vertex

2) check the optimality of the vertex

↳ locally optimal → opt. solution

↳ not locally optimal → travel to nearby vertex (change the basis)

Optimality test

$$Ax = b \qquad x \geq 0$$

Current basis: B

$$x_B = B^{-1}b - B^{-1}F x_F$$

compute
$$
\begin{aligned}
c^T x &= [c_B^T, c_F^T] \begin{bmatrix} x_B \\ x_F \end{bmatrix} \\
&= c_B^T x_B + c_F^T x_F \qquad x \in \mathbb{R}^n \\
&= c_B^T (B^{-1}b - B^{-1}F x_F) + c_F^T x_F \qquad x \in P \\
&= c_B^T B^{-1} b + \underbrace{(c_F^T - c_B^T B^{-1} F)}_{\bar{c}_F^T} x_F + 0^T x_B
\end{aligned}
$$

$\bar{c}_F^T \rightarrow$ reduced cost vector

Suppose $x_F = 0$ (we're on a vertex)

$\hookrightarrow \; c^T x = c_B^T B^{-1} b = c_0$

If $\bar{c}_F \geq 0$ then $c^T x \geq c_0 \Rightarrow$ optimal solution

$\hookrightarrow \; \bar{c}_{F_i}^T \geq 0 \quad \forall i$

The optimality test is SUFFICIENT to stop

$$\overline{c_F}^T = c_F^T - c_B^T B^{-1} F$$

if $\overline{c_F}_i^T \geq 0 \quad \forall i$ then <u>STOP</u>

$$\overline{c}^T = c^T - c_B^T B^{-1} A$$

$$[\overline{c_B}^T \ \overline{c_F}^T] = [c_B^T - c_B^T B^{-1} B \mid c_F^T - c_B^T B^{-1} F]$$

$$= [0 \mid \overline{c_F}^T]$$