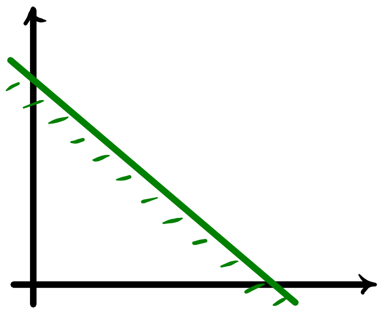
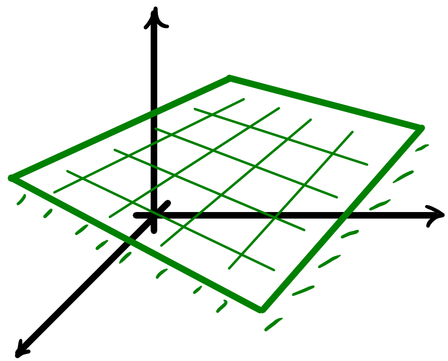
Simplex Method

Affine Half-Space : $\{x \in \mathbb{R}^n \mid \alpha^T x \leq \alpha_0\}$

$\downarrow$

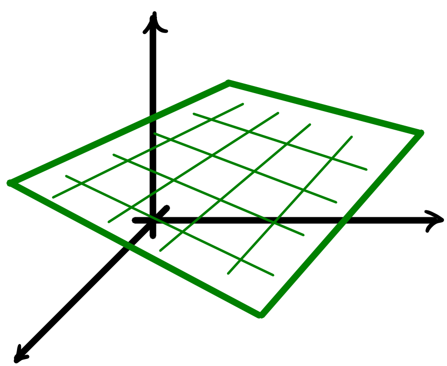
if $\alpha_0 = 0$, then it's an half-space





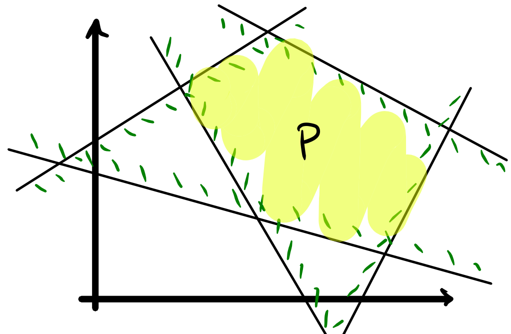
Hyperplane : $\{x \in \mathbb{R}^n \mid \alpha^T x = \alpha_0\}$

$\hookrightarrow$ Generalization of a plane (in $\mathbb{R}^3$)

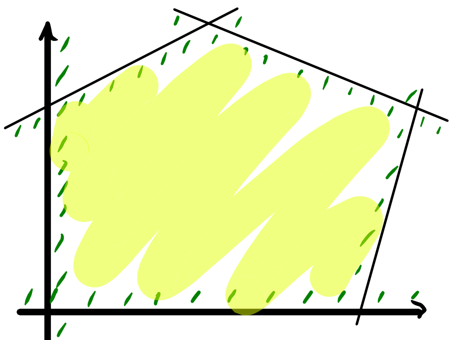


Polyhedron: The intersection of a finite amount of affine half-spaces and hyperplanes creates a polyhedron

$$P = \{ x \in \mathbb{R}^n \mid Ax \leq b \}$$

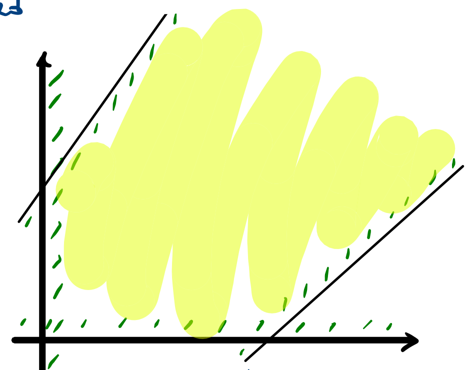


The polyhedron could be bounded or unbounded



Bounded

↳ Also called POLYTOPE



Unbounded

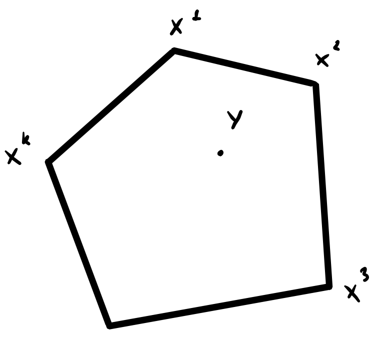
When we consider a polyhedron, given a $z \in P \subseteq \mathbb{R}^n$, $z = \lambda x + (1-\lambda) y$, $\lambda \in [0,1]$, $x, y \in P$

Since z is the convex combination of x and y, $\min\{x, y\} \leq z$

Vertex: A point $x \in P$ is said to be a **vertex** of P if it cannot be expressed as the STRICT convex combination of two DISTINCT points $y, z \in P$

Theorem of Minkowski-Weyl

Given a polytope P, and it's vertices $x^1, x^2, \dots, x^k \in P$, we can obtain every point inside the polytope as a convex combination of the vertices



$\forall y \in R \ \exists \lambda_1, \dots, \lambda_k \in [0,1]$ so that:

- $\sum_{i=1}^{k} \lambda_i = 1$
- $y = \sum_{i=1}^{k} \lambda_i x^i$

not strictly the only one

Th. Given a polytope P, an optimization problem $\min_{x \in P} c^T x$ has an optimal solution among the vertices of P

Proof: let $x^1, \dots, x^k$ be the vertices of P and compute $z^* = \min \{ c^T x^i \mid i = 1, \dots, k \}$

$$\forall y \in P, \ \exists \lambda \in [0,1]^k \mid \quad \cdot \sum_{i=1}^{k} \lambda_i = 1$$

$$\cdot \ y = \sum_{i=1}^{k} \lambda_i x^i$$

$$c^T y = c^T \left(\sum_{i=1}^{k} \lambda_i x^i \right) = \sum_{j=1}^{k} \lambda_i c^T x^i \geq \sum_{i=1}^{k} \lambda_i z^* = z^*$$

$$\Downarrow$$

$$c^T y \geq z^* \quad \forall y \in P$$

Consider an optimization problem in standard form $\begin{cases} Ax = b \\ x \geq 0 \end{cases}$ and suppose

A is an $m \times n$ matrix $(n > m)$ with $\text{rank}(A) = m$

We can obtain a basis of A (set of m linear independent columns of A)
by picking them arbitrarily (as long they are independent)

$$A = \begin{bmatrix} | & | & & | \\ A_1 & A_2 & \dots & A_n \\ | & | & & | \end{bmatrix} = \Big[\underbrace{B}_{m \times m} \,\Big|\, \underbrace{F}_{m \times (n-m)} \Big]$$

$\in B$ $\qquad \notin B$

$$\begin{aligned} Ax &= A_1 x_1 + \dots + A_n x_n \\ &= A_1 x_1 + A_2 x_2 + \dots + A_n x_n \\ &= \left[B \mid F \right] \begin{bmatrix} x_B \\ \hline x_F \end{bmatrix} \\ &= B x_B + F x_F \end{aligned}$$

(change row and column order won't change the result)

For a given B, we can rewrite the initial equation as follows :

$$B x_B = b - F x_F$$

Since B is made of m linear independent columns, $\det(B) \neq 0$, so:

$$x_B = B^{-1}b - B^{-1}Fx_F$$ $\rightarrow$ Canonical form with respect to the basis B

We then set $x_F = 0$ to find the solution of the equation (view example to understand the intuition)

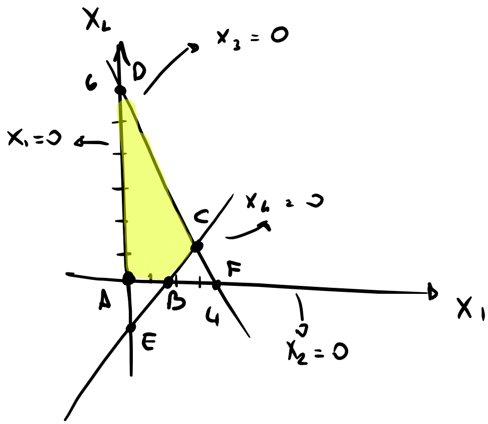
$$\begin{cases} x_F = 0 \\ x_B = B^{-1}b \end{cases} \rightarrow x = \begin{bmatrix} B^{-1}b \\ \hline 0 \end{bmatrix}$$ basic solution with respect to the basis B (feasible if $B^{-1}b \geq 0$)

if $B^{-1}b$ has a 0 $(\exists i \mid [B^{-1}b]_i = 0)$
we have a degeneracy

To get all the vertices I will have to compute all the $\binom{n}{m}$ basis.

Es:

$$\begin{cases} \min & -x_1 - x_2 \\ & 6x_1 + 4x_2 \leq 24 \implies 6x_1 + 4x_2 + x_3 = 24 \\ & 3x_1 - 2x_2 \leq 6 \implies 3x_1 + 2x_2 + x_4 = 6 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$



$A \to x_1, x_2 = 0$

$B \to x_2, x_4 = 0$

$C \to x_3, x_4 = 0$

$D \to x_1, x_3 = 0$

$E \to x_1, x_4 = 0$ — $x_2 < 0$

$F \to x_2, x_3 = 0$ — $x_4 < 0$

$\implies$!! not vertices

It may happen to have two parallel costraints $\implies$ no solutions

$\implies$ ∞ number of solutions — overlapping