Theorem : (Important for the exam)

A point x of a polyhedron P $(x \in P = \{x \geq 0 \mid Ax = b\} \neq \emptyset)$ is a vertex of P $\iff$ x is a basic feasible solution (**bfs**) associated to the system $Ax = b$

$\hookrightarrow$ There's a 1-1 corrispondence between vertices and bfs (Ignoring degenerate solutions)

Proof :

- "x is a bfs $\Rightarrow$ x is a vertex of P"

$$x = [x_1, \dots, x_m, 0, \dots, 0]^T$$
$$\Downarrow$$
$$x = [x_1, \dots, x_k, 0, \dots, 0]^T$$

$k \in [0, m]$ since x is a bfs and we also must consider degenerate cases: arrange the elements of x s.t. $x_1, \dots, x_k > 0$ and $x_k, \dots, x_n = 0$

Let's assume that x is not a vertex of P:

$$\exists\, y, z \in P,\ y \neq z,\ \exists \lambda \in]0,1[\mid x = \lambda y + (1-\lambda) z$$

Since x is a convex combination of y and z and $y_i, z_i > 0$ $\forall i = 1, \dots n$, y and z must have the same form of x : $y = [y_1, \dots, y_k, 0, \dots, 0]^T$
$z = [z_1, \dots, z_k, 0, \dots, 0]^T$

Since $y, z \in P$:

A's columns are ordered following x's elements re-arrangement

$$A_1 y_1 + \dots + A_k y_k + A_{k+1} 0 + \dots = b \qquad (\mathrm{I})$$
$$A_1 z_1 + \dots + A_k z_k + \dots = b \qquad (\mathrm{II})$$

Let's now consider $(\mathrm{I}) - (\mathrm{II}) = A_1 (y_1 - z_1) + \dots + A_k (y_k - z_k) \overset{!}{=} b - b = 0$

let $\alpha_i = y_i - z_i \quad \forall i = 1, \dots, k$

Since $y \neq z$, $\exists i$, $i = 1, \dots, k \mid \alpha_i \neq 0 \Rightarrow (\mathrm{I}) - (\mathrm{II}) \neq 0$

This means that $A_1, \dots, A_k$ must be linear dependent, which is impossible since $A_1, \dots, A_k \in B$ (so they must be lin. independent)

Our only assumption was that x was not a vertex, hence this assumption must be wrong $\Rightarrow$ x is a vertex.

- "x is a vertex of P $\Rightarrow$ x is a bfs"

$$x = [x_1, x_2, \dots, x_n] \in P$$
$$\Downarrow$$
$$x = [x_1, \dots, x_k, 0, \dots] \in P$$

re-arrange the elements of x s.t. $x_1, \dots x_k > 0$, $x_{k+1}, \dots, x_n = 0$, with $k \in [0, n]$

A's columns are ordered following x's elements re-arrangement

Since $x \in P \Rightarrow A_1 x_1 + \dots + A_k x_k = b \quad (\mathrm{I})$

We now have 2 possible cases:

- $A_1, \dots, A_k$ are linear independent $\rightarrow k \leq m$

$$x = [\underbrace{x_1, \dots, x_k, 0, \dots, 0}_{m}, 0, \dots]$$

Arbitrarily pick the $A_{k+1}, \dots, A_m$ columns s.t. $A_1, \dots, A_m$ are lin. independent

$$A = [\underbrace{A_1 \dots A_k \dots A_m}_{B} \quad \dots \quad]$$

$$Ax = b \rightarrow [B \mid \dots] x = b \rightarrow x = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix} \quad (\text{bfs})$$

- $A_1, \dots, A_k$ are linear dependent (no limitations on k)

consider $\alpha_1, \dots, \alpha_k \in \mathbb{R}^n$ s.t. $A\alpha = 0 \quad (\mathrm{II})$
Given an $\varepsilon \gtrsim 0$:

$$(\mathrm{I}) + \varepsilon(\mathrm{II}) = (x_1 + \varepsilon\alpha_1)A_1 + \dots + (x_k + \varepsilon\alpha_k)A_k = b$$
$$(\mathrm{I}) - \varepsilon(\mathrm{II}) = (x_1 - \varepsilon\alpha_1)A_1 + \dots + (x_k - \varepsilon\alpha_k)A_k = b$$

let $x_i + \varepsilon\alpha_i = y_i$, $x_i - \varepsilon\alpha_i = z_i$, $\forall i = 1, \dots, k$

With ε arbitrarily small, $y = [y_1, \dots, y_k\, 0 \dots]^T$ and $z = [z_1, \dots, z_k\, 0 \dots]^T$ have all positive values ($y_i, z_i > 0 \;\; \forall i = 1, \dots, k$)

We can rewrite the equations as follows

$$Ay = Az = b$$

Since not all α_i are 0 and $\varepsilon \neq 0$, $y \neq z$, also x can be written as the convex combination of y and z

$$\begin{cases} y = x + \varepsilon\alpha \\ z = x - \varepsilon\alpha \end{cases} \longrightarrow x = \frac{1}{2}y + \frac{1}{2}z$$

This means x can't be a vertex, which is impossible
Hence $A_1, \dots, A_k$ must be linear independent

As we've stated before, getting all the vertices would require computing $\binom{n}{m}$ bfs

Solution:
1) Choose a random vertex and check it's optimality
2) If it's not an optimal solution go to a neighbour and check its optimality
3) Repeat step 2 till we find an optimal solution

Optimality test

Given an optimization problem $\begin{cases} Ax = b \\ x \geq 0 \end{cases}$ and a basis B:

$$Ax = [B \; F] \begin{bmatrix} x_B \\ x_F \end{bmatrix} = B x_B + F x_F = b$$

$$\downarrow$$

$$x_B = B^{-1} b - B^{-1} F x_F$$

the cost function can be written as follows:

$$c^T x = [c_B^T \; c_F^T] \begin{bmatrix} x_B \\ x_F \end{bmatrix}$$

$$= c_B^T x_B + c_F^T x_F$$

$$= c_B^T (B^{-1} b - B^{-1} F x_F) + c_F^T x_F$$

$$= \underbrace{c_B^T B^{-1} b}_{c_0} + \underbrace{(c_F^T - c_B^T B^{-1} F)}_{\bar{c}_F^T} x_F$$

$\bar{c}_F^T \rightarrow$ reduced cost vector

if we're on a vertex, $x_F = 0 \Rightarrow c^T x = c_0$

if $\bar{c}_F \geq 0$ then $c^T x \geq c_0$, then the vertex obtained by B is the optimal solution

$\hookrightarrow \bar{c}_{F_i}^T \geq 0 \quad \forall i$

$\hookrightarrow$ no need to check more vertices