PROBABILITY REVIEW

(not all of it... just interesting parts)

$$E[X] = \begin{bmatrix} E[X_1] = m_{X_1} \\ \vdots \\ E[X_n] = m_{X_n} \end{bmatrix}$$

Covariance matrix

$$\Sigma = E\left[(X - m_X)(X - m_X)^T\right] = \begin{bmatrix} \sigma^2_{X_1} & \sigma_{X_1 X_2} & \cdots & \sigma_{X_1 X_n} \\ \vdots & \ddots & & \vdots \\ & & \sigma_{X_i X_j} & \vdots \\ \sigma_{X_n X_1} & \cdots & & \sigma^2_{X_n} \end{bmatrix}$$

$$\sigma_{X_i X_j} = Cov(X_i, X_j) = E\left[(X_i - m_{X_i})(X_j - m_{X_j})\right]$$

If X_i and X_j are independent, $\sigma_{X_i X_j} = 0$ (not true the other direction)

proof on moodle

Linearity of Expectation $E[X_1 + X_2] = E[X_1] + E[X_2]$

constants

$Var[aX + b] = a^2 Var[X]$

$Var[X_1 + X_2] = Var[X_1] + Var[X_2] + 2\sigma_{X_1 X_2}$

$\to$ If $\sigma_{X_1,X_2} = 0$ then $Var[X_1 + X_2] = Var[X_1] + Var[X_2]$

$\Downarrow$

If X_1, X_2 are independent then

$Var[X_1 + X_2] = Var[X_1] + Var[X_2]$

Relative frequency: $f_n(A) = \dfrac{S_n}{n}$ $\left(S_n = \sum_{i=1}^{n} X_i \quad , \quad X_i(z) = \begin{cases} 1 & \text{if } z \in A \\ 0 & \text{if } z \notin A \end{cases} \right)$

S_n: # of fav. outcomes

n: # of outcomes

z: outcome

A: event we're interested in

Each X_i is a **Bernulli r.v.** of parameter p: $X_i \sim B(p)$

$$p = P[X_i = 1] = P[z \in A]$$

Then $S_n = \sum_{i=1}^{n} X_i$ is a **Binomial r.v.** of parameters n, p

$$S_n \sim Bin(n, p)$$

$$P(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$E[Bin(n,p)] = np$ $\quad$ $Var[Bin(n,p)] = np(1-p)$

$$f_n(A) = \frac{S_n}{n} \quad , \quad S_n \sim Bin(n,p)$$

$$\underline{\underline{E[f(a)] = p}} \quad \left(= E\left[\frac{S_n}{n}\right] = \frac{1}{n} E[S_n] = \frac{\cancel{n}p}{\cancel{n}} \right)$$

$$\underline{\underline{Var[f(a)] = \frac{p(1-p)}{n}}} \quad \left(= Var\left[\frac{S_n}{n}\right] = \frac{1}{n^2} Var[S_n] = \frac{1}{n^{\cancel{2}}} \cancel{n} p(1-p) \right)$$

The more tries, the lower the variance

Chebyshev's inequality:

X is a v.v $\rightarrow E[X] = \mu \quad Var[X] = \sigma^2$ then:

$$P[|X-\mu| > \varepsilon] < \frac{\sigma^2}{\varepsilon^2}$$ $\rightarrow$ The further i want to be from the expectation, the more improbable it is.

$\Downarrow$

$$P[|f_n(A) - p| > \varepsilon] \leq \frac{p(1-p)}{n\varepsilon^2}$$ $\Rightarrow$ the larger n, the closer is my frequency to the true probability

$$\left(\lim_{n \to \infty} f_n(A) = p \right)$$

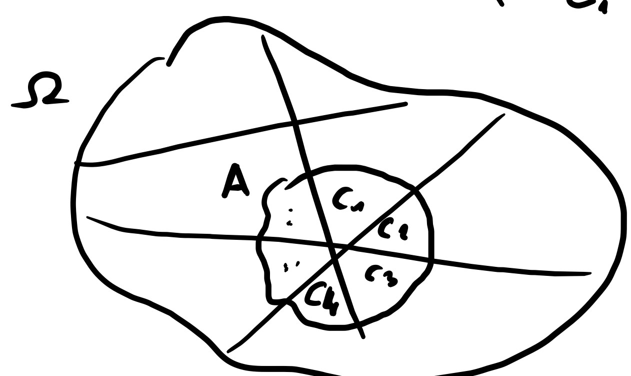
$$P[A|B] = \frac{P(A\cap B)}{P(B)} = \lim_{n\to\infty} \frac{f_n(A\cap B)}{f_n(B)} = \lim_{n\to\infty} \frac{S_n(A\cap B)}{S_n(B)}$$

$P[A|B]$ = fraction of times that A and B happens over the times B happens

<u>Bayes Rule</u> : $P(A|B) = \dfrac{P(B|A)P(A)}{P(B)}$

combined with the law of total probability

$$P(A) = \sum_{i=1}^{n} P(A|C_i)P(C_i) \quad \left(\begin{array}{l} \cdot \ \bigcup_{i=1}^{n} C_i = \Omega \\ \cdot \ C_i \cap C_j = \emptyset \quad \forall i,j \end{array} \right)$$



Es: $P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)$