linear optimization

we are currently in the following situation:

$$\begin{cases} \min f(x) \\ g_i(x) \leq 0 \quad \forall i = 1, \dots, m \end{cases}$$

$f, g_1, \dots, g_m$ are linear functions:

$$f(x) = \sum_{j=1}^{n} c_j x_j, \qquad g_i(x) = b_i - \sum_{j=1}^{n} a_{ij} x_j \leq 0 \quad \forall i = 1, \dots, m$$

so we are working with a linear optimization model

MATRIX NOTATIONS:

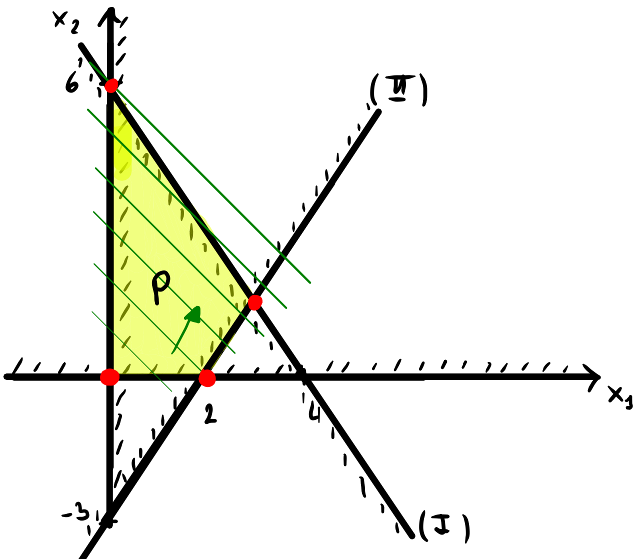
- $\sum_{i=1}^{n} a_i x_i = \langle a, x \rangle = a^T x$ (scalar product) $\qquad a, x \in \mathbb{R}^n$: $a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

- $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \\ a_{m1} & & a_{mn} \end{bmatrix} = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} — & a_1^T & — \\ & \vdots & \\ — & a_m^T & — \end{bmatrix}$

$$-\sum_{i=1}^{m} e_{ij} x_i \geq b_j, \quad \forall j = 1, \dots, n \quad \Rightarrow \quad Ax \geq b, \quad x, b \in \mathbb{R}^n : \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Es:
$$\begin{cases} \min \; f(x) = -x_1 - x_2 \\ 6x_1 + 4x_2 \leq 24 & (\mathrm{I}) \\ 3x_1 - 2x_2 \leq 6 & (\mathrm{II}) \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases}$$

implicit constraints



by iterating $f(x) = k$ to find graphically to find the minimal solution, we get
$$x^* = (0, 6)$$

By changing the objective function, we can easily find out that an optimal solution will always end up on one of the vertices of the polyhedron P

$\Downarrow$

Not all solutions are on the vertices of the polyhedron, but one of the optimal solution can always be found on one of them.

Idea: iterate through all vertices till I find the optimal solution
Problem: the number of vertices grows exponentially with n # vertices = $\binom{n}{m}$

A better solution is the simplex method: perform a gradient descent starting from a rnd vertex till I find a local solution
$\downarrow$
Invoke convexity so the local solution is also global

Standard and Canonical Form

$$\begin{cases} \min \ c^T x \\ Ax \geq b \\ x \geq 0 \end{cases} \longrightarrow \text{canonical form (used for visualisation)}$$

$\neq$

$$\begin{cases} \min \ c^T x \\ Ax = b \\ x \geq 0 \end{cases} \longrightarrow \text{standard form (used for algebraic equations)}$$

To convert one form to another we can use some tricks:

surplus variable

- $a_i^T x \geq b_i \rightsquigarrow a_i^T x - s_i = b_i, \quad s_i \geq 0$

slack variable

- $a_i^T x \leq b_i \rightsquigarrow a_i^T x + s_i = b_i, \quad s_i \geq 0$
- $a_i^T x = b_i \rightsquigarrow \begin{cases} a_i^T x \leq b_i \\ a_i^T x \geq b_i \end{cases}$

"free"

- $x_i \lesseqgtr 0 \rightsquigarrow x_i = x_i^+ - x_i^-, \quad x_i^+ \geq 0, \; x_i^- \geq 0$
- $a_i^T x < b_i$! __NO__ conversion in a continuous space to "$\leq$"

$\dashrightarrow a_i^T x \leq b_i - 1$ __ONLY__ in an INTEGER space