Mathematical optimization

We have a vector $x \in \mathbb{R}^n$ and an objective function $f : \mathbb{R}^n \to \mathbb{R}$

$X \subseteq \mathbb{R}^n$ is our feasible solutions set

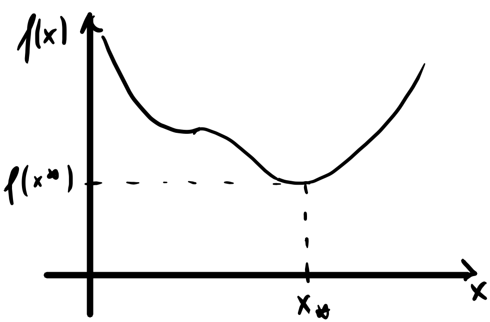
Our aim is to minimize $f(x)$, and we can find three answers:

1) $X = \emptyset$ → the problem is unfeasible

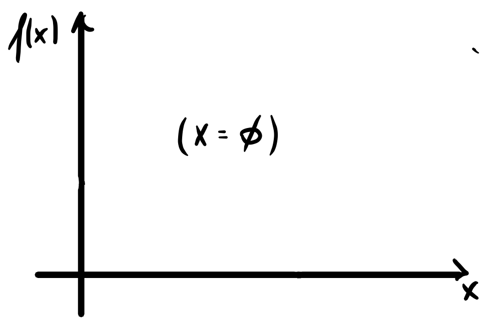
2) f is unbounded from below → the problem is unbounded

3) we find an optimal solution → the problem is feasible

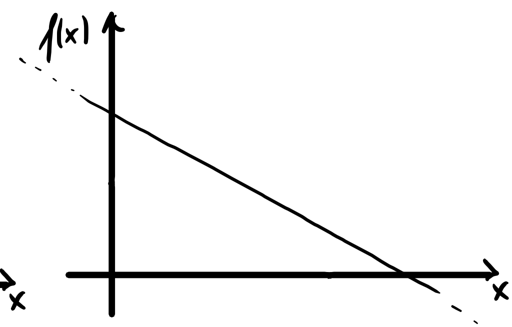
$$x^* \in X \mid f(x^*) \leq f(x) \quad \forall x \in X$$



feasible problem



unfeasible problem



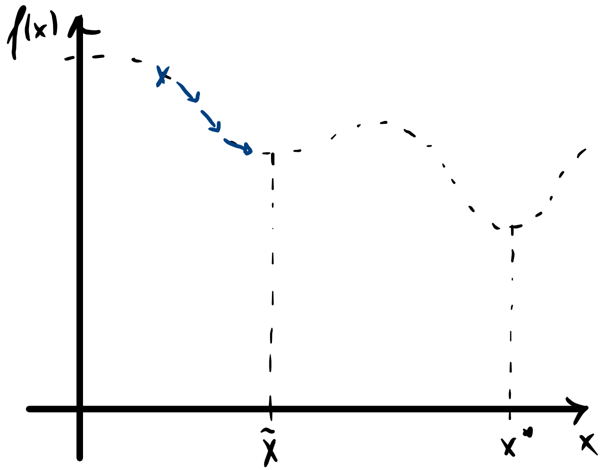
unbounded problem

Finding the optimal solution is tricky, we could find a local minimum instead of the global minimum

A solution $\tilde{x} \in X$ is a local minimum if $f(\tilde{x}) \leq f(x) \quad \forall x \in I_\varepsilon(\tilde{x})$

↳ small interval centered on $\tilde{x}$

By searching a solution iterating and computing the gradient of the function, we might end up on a local minimum.



In mathematical optimization we want to find the global minimum

We need to find regularities to find the global minimum

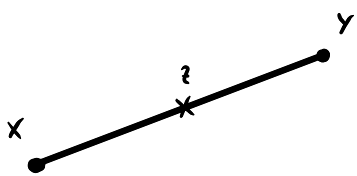
1) X must be convex
2) the objective function must be convex

If both situations are met, <u>all</u> local minimum found are global

Convex combination of 2 points

given two points $x, y \in \mathbb{R}^n$, the convex combination of them is

$$z = \lambda x + (1-\lambda) y \quad \text{with} \quad \lambda \in [0,1]$$



with $\lambda = 1$, $z \equiv x$

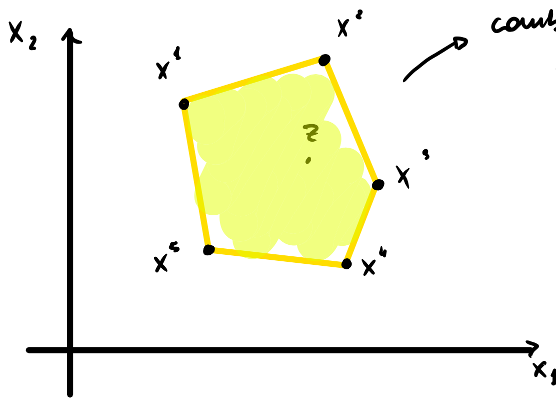
with $\lambda = 0$, $z \equiv y$

when $\lambda \in]0,1[$, it's called strict convex combination

Convex combination of k points

given m points $x^1, \dots, x^k \in \mathbb{R}^n$, the convex combination of those points is:

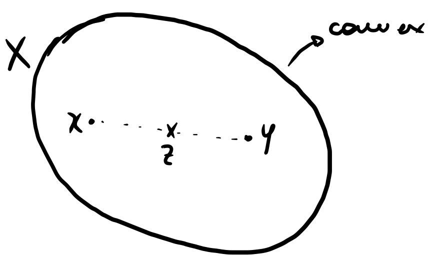
$$z = \sum_{i=1}^{k} \lambda_i x^i, \qquad \lambda_1, \dots, \lambda_k \geq 0 \mid \sum_{i=1}^{k} \lambda_k = 1$$

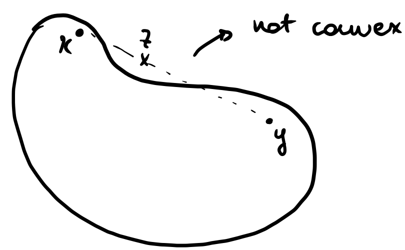


combining all points found by changing λ, we get the convex hull

Convex set

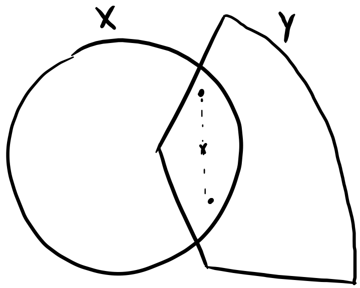
A set $X \subseteq \mathbb{R}^n$ is convex if $\forall x,y \in X$, $z = \lambda x + (1-\lambda) y \in X \quad \forall \lambda \in [0,1]$





Intersection of convex sets

Given two sets $X, Y \in \mathbb{R}^n$ which are convex, then $X \cup Y$ is convex
(<u>not</u> true for union)

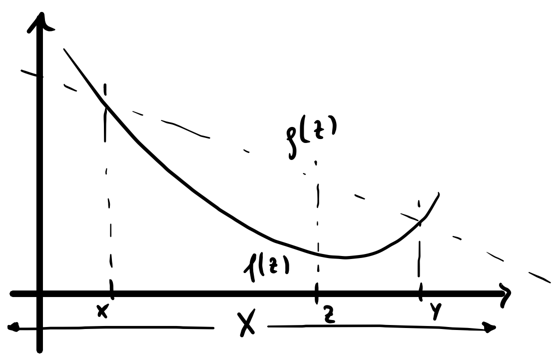


$$\left.\begin{array}{l} x \in X,\ y \in X \rightarrow z \in X \\ x \in Y,\ y \in Y \rightarrow z \in Y \end{array}\right\} z \in X \cap Y$$

Convex function

given a function $f: X \rightarrow \mathbb{R}$ where X is convex,

the function f is convex if $f(x) \leq$ linear interpolation between x, y, $\forall x, y \in X$



$$z = \lambda x + (1-\lambda) y$$

$$g(z) = \lambda f(x) + (1-\lambda) f(y)$$

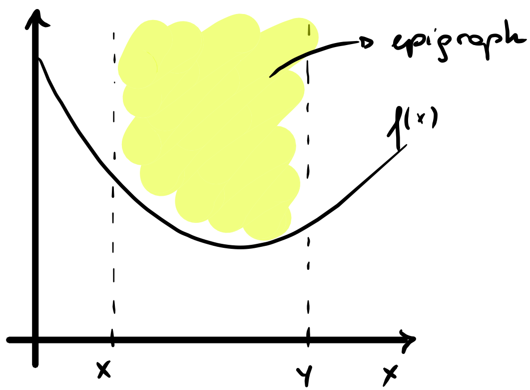
$$f(z) \leq g(z) \quad \forall_z \in [x,y]$$

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

↳ similar definition to the linear function definition

Alternative definition:

given a function $f: X \to \mathbb{R}$ where X is a convex set, f is convex if its epigraph in $[x,y]$ is convex $\forall_{x,y} \in X$



(We can now see how the convexity is not a function's property, but its epigraph)

Theorem 1.1.1

Given a set $X = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0,\ i=1,\dots,m\}$ with its constraints $g_i(x)$,

if the constraints $g_1, \dots, g_m$ are all convex, then X is convex too

<u>Proof</u>:

$$X = \bigcap_{i=1}^{m} X_i = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0\}$$

We have to prove that X_i is convex.

$\forall x,y \in X_i$, $z = \lambda x + (1-\lambda) y \quad \forall \lambda \in [0,1]$,

$$g_i(z) = g_i(\lambda x + (1-\lambda) y) \leq \lambda g_i(x) + (1-\lambda) g_i(y)$$

↳ convexity of g_i

$(\lambda \geq 0,\ g_i \leq 0,\ 1-\lambda \geq 0)$

≤ 0

Then $z \in X_i$ since $g_i(z) \leq 0$, so X_i is convex

Optimum solution for a convex set and a convex objective function

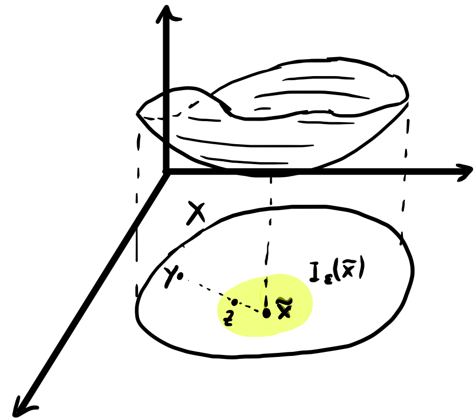
Consider an optimization problem where both f and X are convex, then every local optimal solution is also globally optimal

<u>Proof</u>:

Let $\tilde{x} \in X$ be a locally optimal solution, then

$$\exists \varepsilon > 0 \mid f(\tilde{x}) \leq f(z) \quad \forall x \in I_\varepsilon(\tilde{x})$$

$\forall y \in X$ we consider the convex combination between $\tilde{x}$ and y:



$$z = \lambda \tilde{x} + (1-\lambda) y, \quad \lambda \in [0,1[\mid z \in I_\varepsilon(\tilde{x})$$

$f(\tilde{x}) \leq f(z)$ since $z \in I_\varepsilon(\tilde{x})$ and $\tilde{x}$ is a local minimum

$\leq f(\lambda \tilde{x} + (1-\lambda) y)$

(convexity)

$\leq \lambda f(\tilde{x}) + (1-\lambda) f(y)$

$$(1-\lambda) f(\tilde{x}) \leq (1-\lambda) f(y)$$

$f(\tilde{x}) \leq f(y) \quad \forall y \in X$ since $1-\lambda > 0$

↳ $\tilde{x}$ is a global optimal solution