2.1) i) $s^2 \le \frac{1}{n-1}\sum_i (x_i - a)^2 \qquad \forall a \in R$

$$\overbrace{\frac{1}{\cancel{n-1}} \sum_i (x_i - \bar{x})^2}^{s^2} \le \frac{1}{\cancel{n-1}} \sum_i (x_i - a)^2$$

$$\sum_i (x_i - \bar{x})^2 \le \sum_i (x_i - a)^2$$

$$\sum_i (x_i^2 - 2x_i\bar{x} + \bar{x}^2) \le \sum_i (x_i^2 - 2x_i a + a^2)$$

$$\cancel{\sum_i x_i^2} - 2\bar{x}\sum_i x_i + \bar{x}^2 \sum_i \le \cancel{\sum_i x_i^2} - 2a\sum_i x_i + a^2 \sum_i$$

$$-2\bar{x}\sum_i x_i + n\bar{x}^2 \le -2a\sum_i x_i + na^2$$

$$\sum_i x_i \,(2\bar{x} - 2a) \ge n(\bar{x}^2 - a^2)$$

$$\underbrace{\frac{1}{n}\sum_i x_i}_{\bar{x}} 2(\bar{x} - a) \ge \bar{x}^2 - a^2$$

$$2\bar{x}^2 - 2\bar{x}a - \bar{x}^2 + a^2 \ge 0$$

$$\bar{x}^2 - 2\bar{x}a + a^2 \ge 0$$

$$(\bar{x} - a)^2 \ge 0 \quad \rightarrow \quad \forall a \in R$$

ii) $(n-1)\frac{s^2}{n} = \overline{x^2} - \bar{x}^2$

$$\frac{\cancel{n-1}}{\cancel{n}} \frac{1}{\cancel{n-1}} \sum_i (x_i - \bar{x})^2 = \frac{1}{\cancel{n}} \sum_i (x_i)^2 - \overset{n}{\vee}\bar{x}^2$$

$$\cancel{\sum_i x_i^2} - 2\bar{x}\underbrace{\sum_i x_i}_{n\bar{x}} + \bar{x}^2 \sum_i = \cancel{\sum_i x_i^2} - n\bar{x}^2$$

$$-2\cancel{n}\bar{x}^2 + \cancel{n\bar{x}^2} + \cancel{n\bar{x}^2} = 0$$

$$0 = 0$$

2.2) $X_i \sim F$

i) pdf of the minimum $\quad f_{X_{(1)}}(t) = n(1 - F(t))^{n-1} f(t)$

$$P[X_{(1)} = t] = f(t)$$

$$P[X_i > t] = 1 - P[X_{(1)} < t] \;\Big|\; = 1 - F(t) \qquad \forall i \in [1, n-1]$$

$$f_{X_i}(t) = f(t) \prod^{n-1} (1 - F(t)) \;\Big|\; = (1 - F(t))^{n-1} f(t)$$

$$f_{X_{(1)}}(t) = \sum^{n} f_{X_i}(t) \;\Big|\; = \underbrace{n(1 - F(t))^{n-1} f(t)}_{\forall i \;\; f_{X_i}(t) \text{ are equal}}$$

i) $1 - F_{X_{(1)}}(t) = 1 - P(X_{(1)} \le t)$

$= P(X_{(1)} > t)$

$= P(\min_i X_i > t)$

$= P(X_1 > t, \ldots, X_n > t)$

$= \prod_{i=1}^{n} P(X_i > t)$

$= \prod_{i=1}^{n} (1 - P(X_i \le t))$

$= (1 - F(t))^n$

$$F_{X_{(1)}}(t) = 1 - (1 - F(t))^n$$

$$f_{X_{(1)}}(t) = n(1 - F(t))^{n-1} f(t)$$

ii) $P[X_{(n)} = t] = f(t)$

$P[X_{(i)} < t] = F(t)$

same concept as before

$$f_{X(n)}(t) = \sum^{n} f(t) \prod^{n-1} F(t) \overset{!}{=} n(F(t))^{n-1} f(t)$$

iii)

$$f_{X_{(1)}}(t) = \underbrace{\sum_{i=1}^{n}}_{\forall X_i}\Big[\underbrace{f_{X_i}(t)}_{P[X_i = t]} \prod_{j=1, j\neq i}^{n} \big(1 - \underbrace{F_{X_j}(t)}_{P[X_j < t]}\big)\Big]$$

$$f_{X(n)}(t) = \sum_{i=1}^{n}\Big[f_{X_i}(t) \prod_{j=1, j\neq i}^{n} \big(F_{X_j}(t)\big)\Big]$$

iv) $n = 4$

$$f_{X(3)}(t) = 4 f(t)(F(t))^2(1 - F(t))$$

ii)
$$F_{X(n)}(t) = P(X_{(n)} \leq t) \overset{!}{=} P(\max_i X_i \leq t) \overset{!}{=} P(X_1 \leq t, X_2 \leq t, \dots, X_n \leq t) \overset{!}{=} \prod P(X_i \leq t) \overset{!}{=} (F(t))^n$$

$$f_{X(n)} = \frac{d}{dt} F_{X(n)}(t) \overset{!}{=} n F(t)^{n-1} f(t)$$

iii) We must compute $P(X_1 < t, \dots, X_n < t)$ through integrals

iv)
$$F_{X(k)}(t) = P(X_{(k)} \leq t, X_{(k)} = X_1) + P(X_{(k)} \leq t, X_{(k)} = X_2) + \dots P(X_{(k)} \leq t, \underbrace{X_{(k)} = X_n}_{\text{disjoint events}})$$

– i.i.d.

$$= n P(X_{(k)} \leq t, X_{(k)} = X_1)$$

$= n P(X_1 \leq t$, k-1 r.v. out of $X_2, \dots, X_n$ are less than X_1, and the remaining being greater than X_1)

$<$ or $\leq$ doesn't change the result since it's continuous

Continue on R...

$$= n \binom{n-1}{k-1} \underbrace{P(X_1 \leq t, X_2 \leq X_1, \dots, X_k \leq X_1, X_{k+1} > X_1, \dots, X_n > X_1)}_{\text{since they are iid, this probability is the same for each combination}}$$

$$= n \binom{n-1}{k-1} \underbrace{\int_{-\infty}^{t}}_{X_1} \underbrace{\int\int\int_{-\infty}^{X_1}}_{X_2 \cdots X_k} \underbrace{\int\int\int_{X_1}^{+\infty}}_{X_{k+1} \cdots X_n} f(x_1) f(x_2) \dots f(x_n)\, dx_n \dots dx_k \dots dx_1$$

$$= n \binom{n-1}{k-1} \underbrace{\int_{-\infty}^{t}}_{X_1} \underbrace{\int_{-\infty}^{X_1} f(x_2)\, dx_2 \dots}_{X_2 \cdots X_k} \underbrace{\int_{X_1}^{+\infty} f(x_{k+1})\, dx_{k+1} \dots}_{X_{k+1} \cdots X_n} f(x_1)\, dx_1$$

$$= n \binom{n-1}{k-1} \int_{-\infty}^{t} (F(x_1))^{k-1} (1 - F(x_1))^{n-k} f(x_1)\, dx_1$$

$$f_{X(k)}(t) = n \binom{n-1}{k-1} (F(x_1))^{k-1} (1 - F(x_1))^{n-k} f(x_1)$$