Lecture 02
Lotteries

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Game Theory 2023/24

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Random outcomes

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- Assume payoffs are affected by random outcomes
  - At the cafeteria, the food quality may vary
  - On one day, the fish might be rotten
  - How can we tell if beef is preferable?
- Rational players and randomness do not mix well together
- To make rational decisions involving random outcomes, we need to incorporate them into the utility function
  - How can we do that? By using the outcomes' **probability distribution**

Lotteries

[estrazione]

- **Definition:** A **lottery** over outcomes $X = \{x_1, \dots, x_n\}$ is a probability distribution p over X
  - $p(x_k) \geq 0,\ \ k = 1, \dots, n$
  - $\sum_{k=1}^{n} p(x_k) = 1$
- If actions are involved, p is conditional on the action
  - For $a \in A$, we consider $p(x_k|a)$ with the above properties
- A certain outcome can also be seen as a **degenerate** lottery: $p(x_k|a) = 1$ for some k and 0 for all $k' \neq k$

Nature

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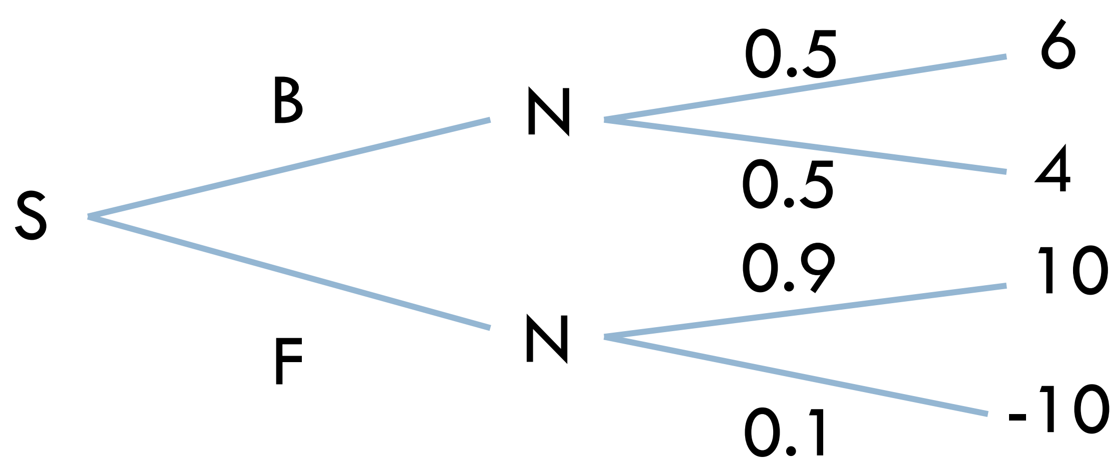
- In game theory jargon, random events are the consequences of the choices made by another player, called "Nature" (N)
  - Nature chooses between outcomes $x_1, \ldots, x_n$ according to a lottery p
  - This can be represented in the decision tree as follows



Continuous lotteries

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- Lotteries can also describe probabilities over a continuous space of events
- Probability of each specific outcome is zero
- Probability mass distribution → Probability density
- Still possible to represent it using the decision framework, however it become a bit scuffed
  - Nature's choice cannot be represented in the decision tree

Expected utility

- Usual way of comparing random outcomes: taking the expectation
  - Also works for "degenerate" lotteries (1 outcome with 100% probability)
  - "Expected utility theory" by von Neumann and Morgenstern
  - **Intuition:** if you repeat the same choice for N trials, for $N \to \infty$ average utility = expectation
- Expected payoff for lottery p
  - $\mathbb{E}_{x\sim p}[u(x)] = \sum_{k=1}^{n} p(x_k) \cdot u(x_k)$

multiply probability and utility

Expected utility

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- Von Neumann – Morgenstern (VNM) framework to define preferences among lotteries
- We write $p \succcurlyeq q$ to say "lottery p is preferred to q"
- Under VNM framework, preferences must satisfy:
  - Rationality (completeness and transitivity)
  - Continuity axiom
  - Independence axiom

Continuity axiom

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- For lotteries p, q, r over action space A the following sets must be **closed**:
  - $\{a \in [0,1]: ap + (1-a)q \succcurlyeq r\}$
  - $\{a \in [0,1]: r \succcurlyeq ap + (1-a)q\}$
- This means that arbitrarily small variations in the gamble does not change preferred lotteries
  - If I prefer fish which is 100% not rotten to beef, I will still prefer fish if it has an arbitrarily small probability $\varepsilon > 0$ of being rotten

Independence axiom

- For lotteries $p, q, r, \forall a \in [0, 1]$
  - $p \succcurlyeq q \Rightarrow (1-a)p + ar \succcurlyeq (1-a)q + ar$
- This means that if we mix the same amount of another lottery into two lotteries, the preference remains unchanged
  - If I like betting on soccer more than betting on horse races, then I prefer the lottery "if heads bet on soccer, if tails play roulette" to "if heads bet on horse races, if tails play roulette"

VNM utility theorem

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between lotteries

- **Theorem:** If $\succcurlyeq$ satisfies the rationality, continuity and independence axioms, it can be mapped to u such that

$$p \succcurlyeq q \Rightarrow \mathbb{E}_{x \sim p}[u(x)] \geq \mathbb{E}_{x \sim q}[u(x)]$$

- Remark: If u is a suitable utility function to describe the preference $\succcurlyeq$, any affine (linear) transformation of u is also suitable

Continuous lotteries

- Same as discrete case (only the diagram is harder)
- **Example:** We are digging a dwell and need to decide how deep should it be (d=dwell's depth). Digging has a cost of $d^2/$(2 meters) and the amount of extracted water is $W(d) \sim \mathcal{U}([0,\ 20d])$.
- Utility = extracted water – cost.

$E(W(d)) = \frac{a+b}{2}$

- $\mathbb{E}[u(d)] = \mathbb{E}\left[W(d) - \frac{d^2}{2\text{ m}}\right] = 10d - \frac{d^2}{2\text{ m}}$
- $\mathbb{E}[u(5\text{ m})] = 10 \times 5\text{ m} - \frac{\overset{25}{\cancel{100}}\text{ m}^2}{2\text{ m}} = 50 - \overset{12.5}{\cancel{50}} = \overset{37.5}{\cancel{0}}$
- Best choice: $d = 10\text{ m}$ with $\mathbb{E}[u(10\text{ m})] = 50$

Ordinal vs absolute

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- When randomness is not involved, the payoff values don't matter as long as they reflect preferences
  - If we have A≽B, then we can set u(A)=1 and u(B)=0 or u(A)=100 and u(B)=-π
- However, changing payoffs in lotteries may affect the preferred lottery
  - In the cafeteria example, suppose we assign -100 to the rotten fish instead of -10

Risk attitude

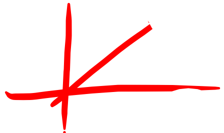
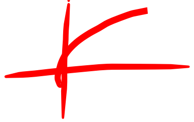
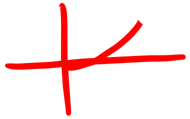
- Consider the following lotteries, where the possible outcomes are to with 0, 1, or 20 euros
  - $p_A = (0, 1, 0)$, i.e., we receive 1 euro 100% guaranteed
  - $p_B = (0.95,0,0.05)$, i.e., with 95% probability we get nothing but with 5% probability we get 20 euros
- u(1 euro) or 0.95× u(0 euros)+0.05× u(20 euros)?
- Depends on how much a player values gaining X euros

Risk attitude

- For a **risk-neutral** player, lotteries p_A and p_B are interchangeable
- For a **risk-averse** player $p_A \succcurlyeq p_B$ (prefers 1 euro guaranteed)
- For a **risk-loving** player $p_B \succcurlyeq p_A$ (prefers a 5% chance to get 20 euros)

$(\not\preccurlyeq)$

Risk attitude

- **Remark:** Monotonic utility functions such as $u(x) = x$, $u(x) = x^2$, and $u(x) = \log x$ do not affect preferences but they do affect risk attitude
  - Linear utility → risk-neutral 
  - Concave utility → risk-averse 
  - Convex utility → risk-loving 

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BACKWARD INDUCTION

Backward induction

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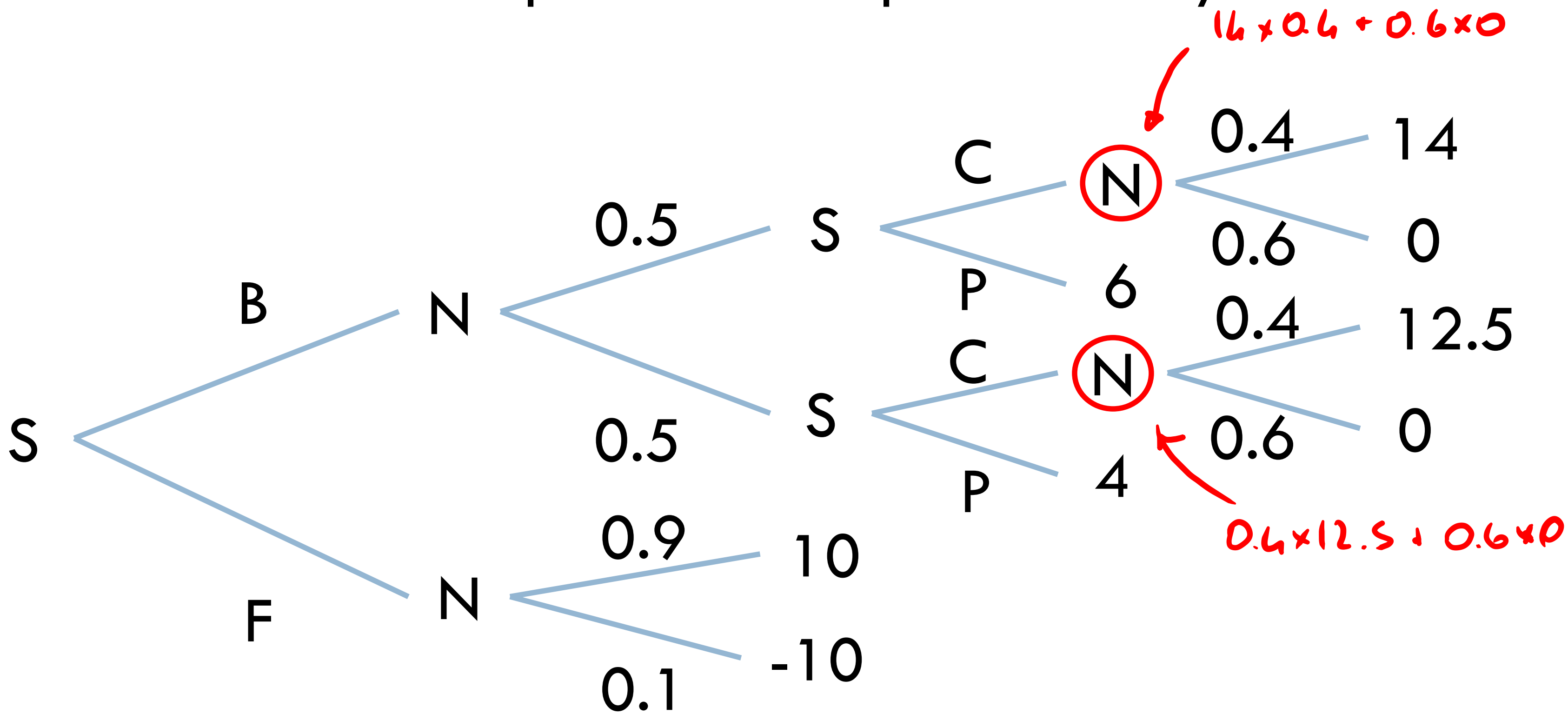
- Begin with nodes (not leaves) at the last level of the tree
- If it is Nature's move, replace the node with a leaf containing the average payoff
- If it is the player's move, replace the node with the payoff of the best choice (i.e., the payoff yielding highest utility)
- Repeat the process for the "pruned" tree

Decision over time

- Nature's move: replace with expected utility



Decision over time

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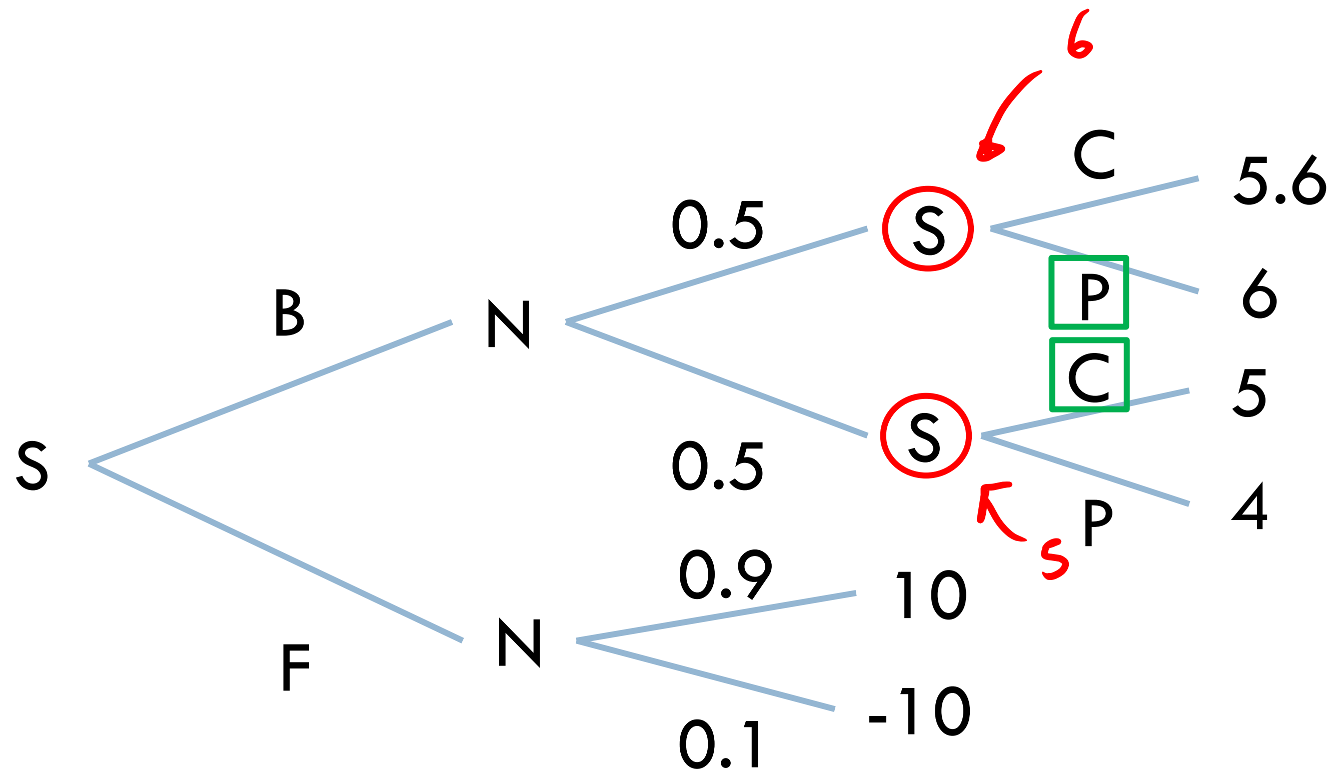
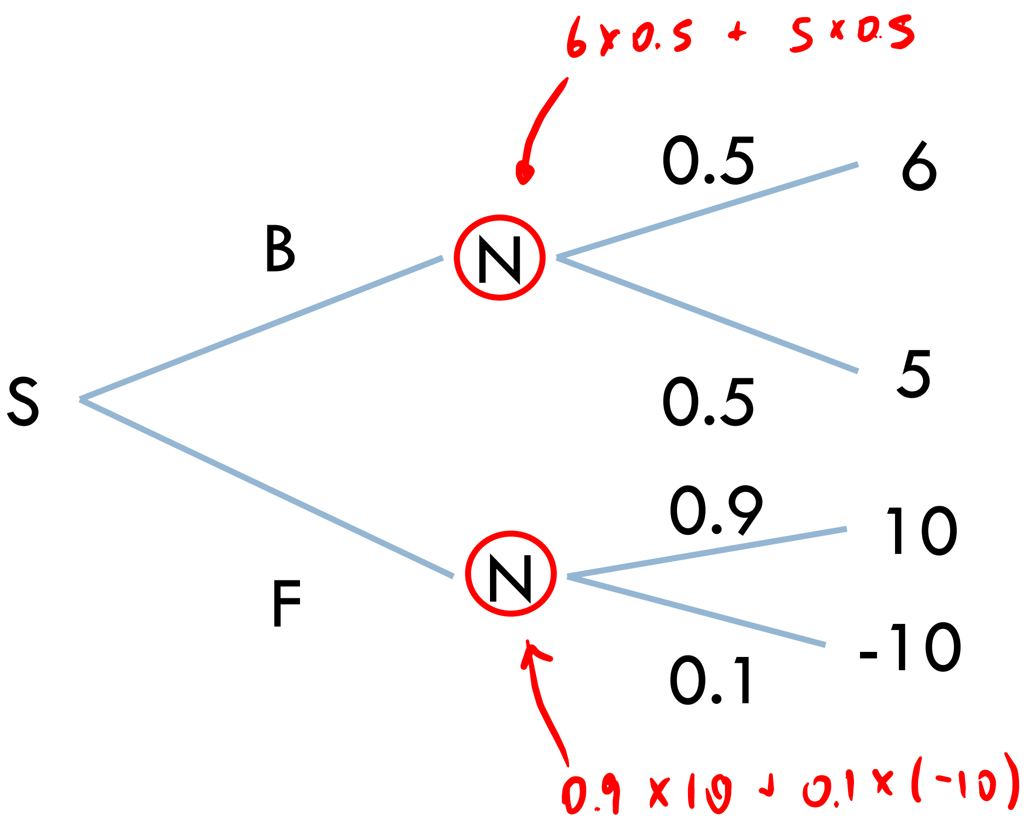
- Player's move: choose best option



Decision over time

- Nature's move: replace with expected utility



Decision over time

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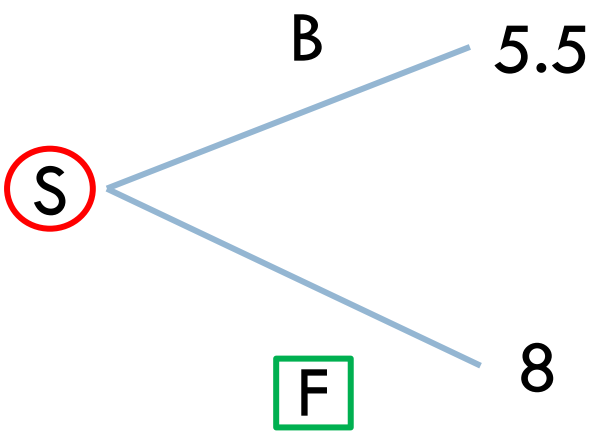
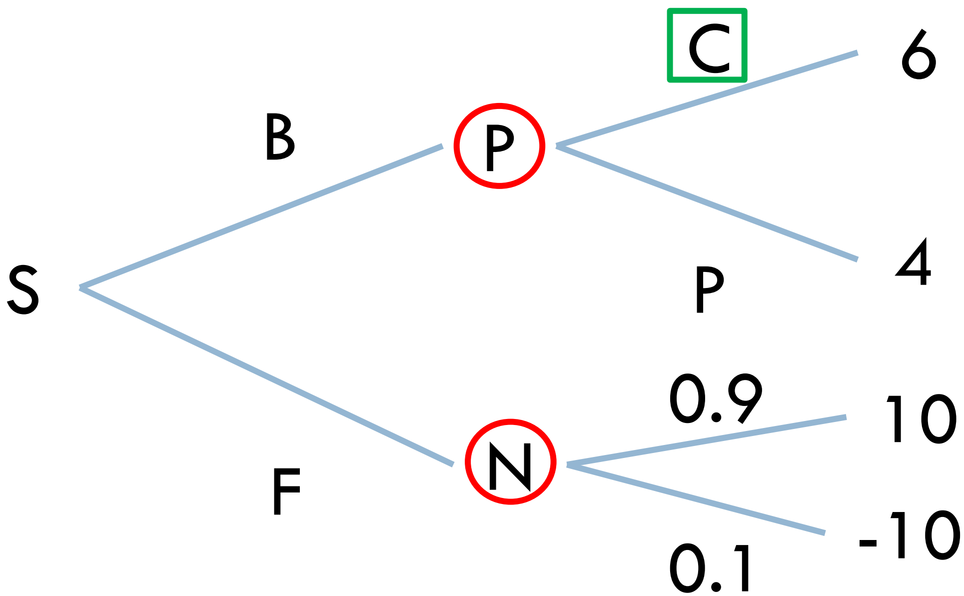
- Player's move: choose best option
- In conclusion, the player's best choice is to still take the fish



Decision over time

- **Remark:** It is possible to have Nature's and player's moves at the same tree level



Discount for future payoffs

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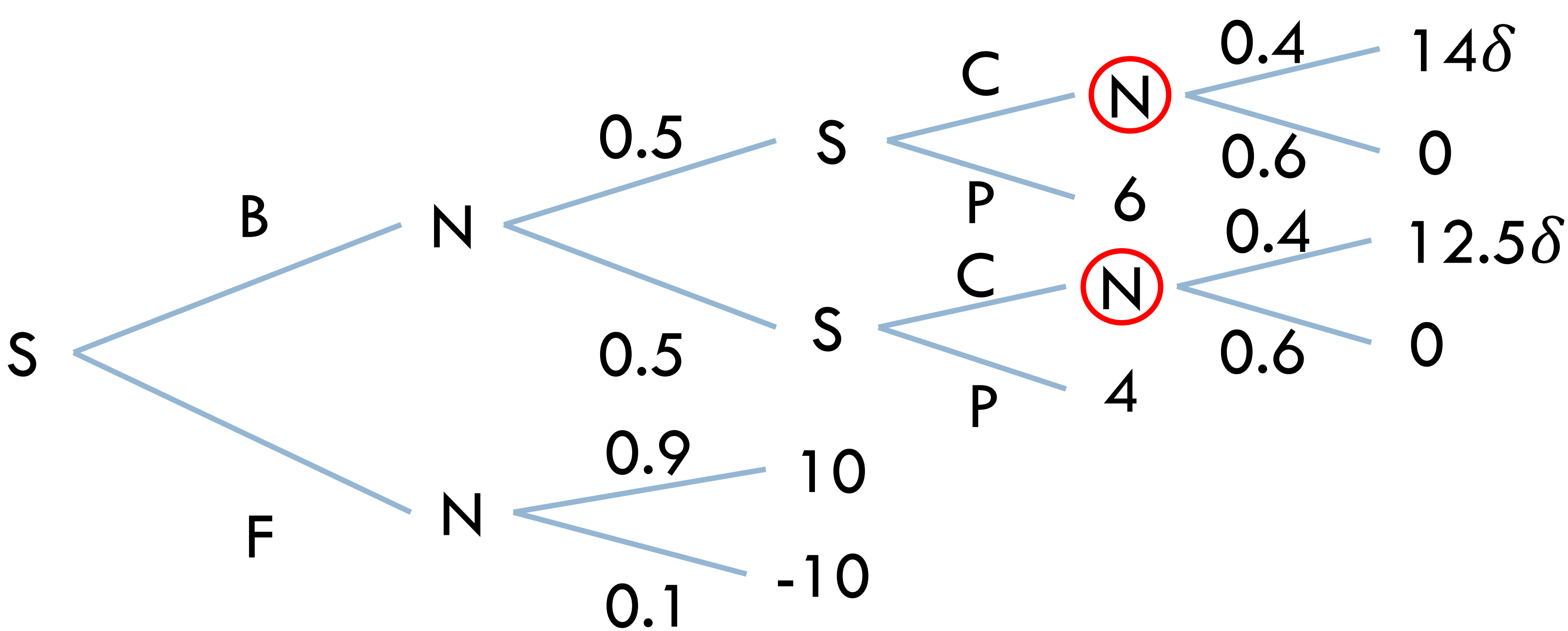
- If the player's decisions are made far apart, we (may) include a discount factor $0 < \delta < 1$
  - Likely, that's not the case for adding the chef's sauce



The value of information

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→ nature "moves" after

- Expected utility implies that a rational player chooses its actions so as to make the right choice **on average**
- Suppose the player has the possibility so see Nature's choice in advance: how much is this information worth?

→ nature "moves" before

The value of information

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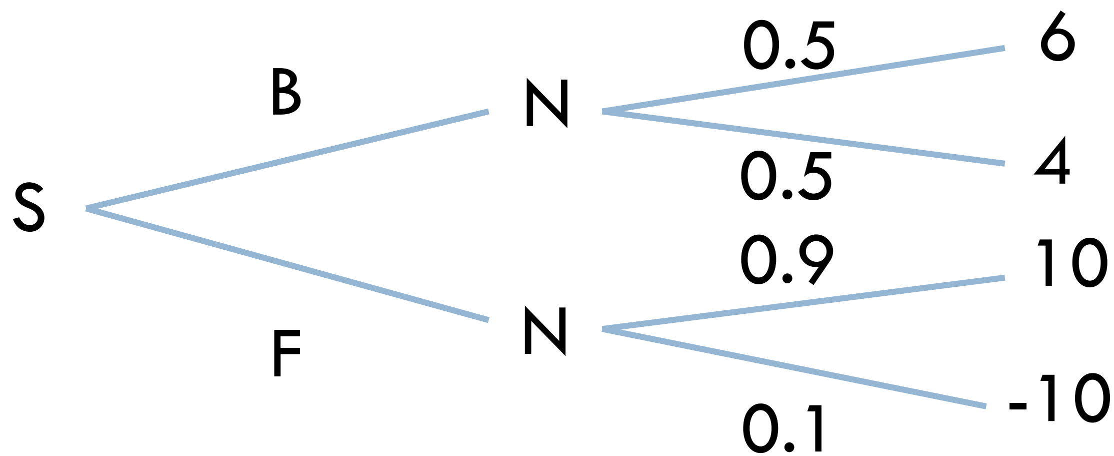

- Assume a friend of S knows how good is the cafeteria's food today and he is willing to tell her (under reasonable compensation)

The value of information

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- If S is notified in advance, her best option will change depending on the information received
- The possible outcomes are the same but the moves' order changes
- This situation can be modeled by making Nature move first

The value of information

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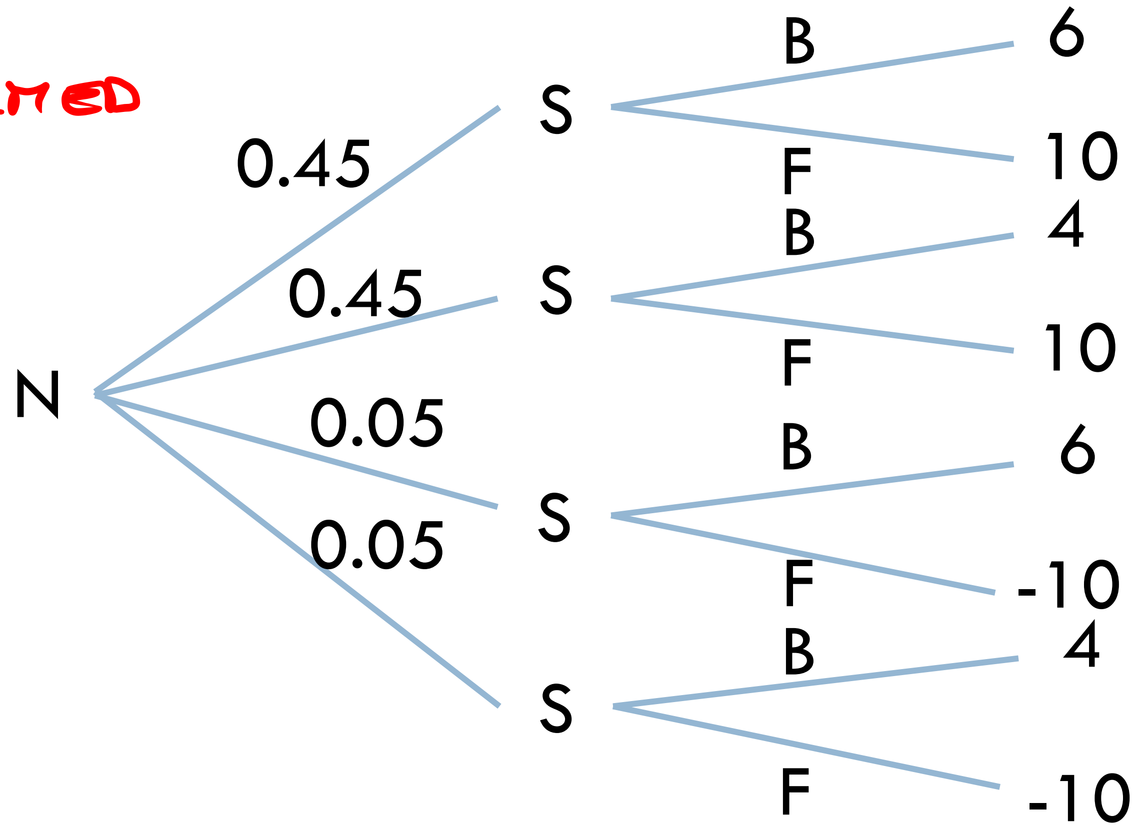
INFORMED



- S is always able to make the best choice, no gambling
- Expected utility: 0.9×10+ 0.05×6+ 0.05×4=9.25

The value of information

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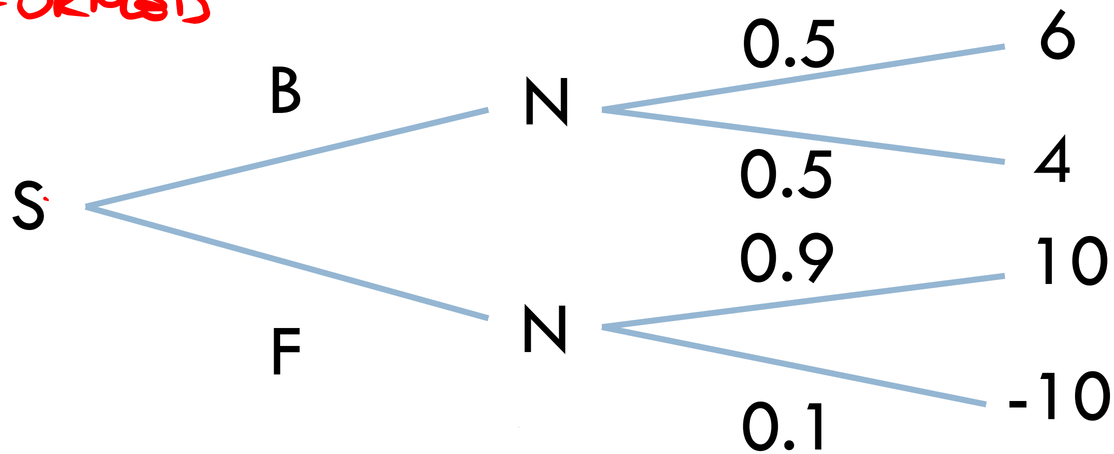
UNINFORMED



- Expected utility: 8 (choose fish with its expected payoff)
- Knowing Nature's choice is worth 1.25

↳ I can then choose wether to know the nature's "move" or not