Initialization: $S = \{s\}$ $\qquad L[i] = c_{si} \; \Big\} \; \forall j \neq s$ ← complete incidence matrix

$L[s] = 0 \qquad pred[i] = s$

$pred[s] = s$

for $k = 1$ to $n-1$ do
  $h = \operatorname{argmin}\{L[i],\ i \notin S\}$
  $S = S \cup \{h\}$
  for all $j \notin S$ do
    if $L[h] + c[h,j] < L[j]$ then
      $L[j] = L[h] + c[h,j]$
      $pred[j] = h$

Floyd-Warshall method

↳ Finds the shortest path from all nodes to all nodes

Hp: no negative cycles

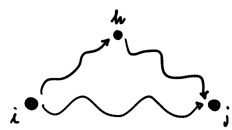
We will need:

d_{ij} = cost of the shortest path from i to j

$pred_{ij}$ = predecessor of j in the shortest path from i to j

How can we check if the D matrix is right? → Triangularity test

Triangularity test: $d_{ij} \leq d_{ih} + d_{hj}$

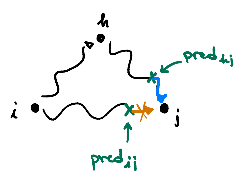


If D fails the triangularity test ⇒ D is wrong

Algorithm:

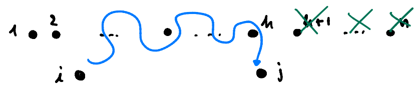
$O(n^2)$ $\begin{cases} d_{ij} = c_{ij} \\ pred_{ij} = i \end{cases} \; \forall i, j = 1, \dots, n$ ← if two edges are not present the cost is $+\infty$

$O(n^3)$
for $h = 1$ to n do
  for $i = 1$ to n do
    for $j = 1$ to n do
      if $d_{ih} + d_{hj} < d_{ij}$ then
        $d_{ij} = d_{ih} + d_{hj}$
        $pred_{ij} = pred_{hj}$



Proof:

Invariant: At the iteration k, d_{ij} = cost of the shortest path from i to j not visiting $k+1, k+2, \dots, n$



Iteration $k=0$: only path from i to j is the direct path $i \to j$: $d_{ij} = c_{ij}$

↳ Property holds after the initialisation

Inductive step: iteration k:
- P_{ij}' non visiting k $\Rightarrow$ min: d_{ij} → already calculated
- P_{ij}'' visiting k $\Rightarrow$ $P_{ij}'' = \underbrace{P_\alpha}_{i \to k} + \underbrace{P_\beta}_{k \to j}$

$$d_{ij} = d_\alpha + d_\beta = d_{ik} + d_{kj}$$

↳ Choose the best d_{ij}

Finding negative cycles

cycles: path $i \to j$, $i = j$

cost of this path $= d_{ii}$

$\Downarrow$

We can use the Floyd-Warshall algorithm and loop through the diagonal of D:

```
for i = 1 to n do
    if dii < 0 then
        Print ("Negative cycle found")
```

We can use pred[i] to backtrack the cycle.