Change of Basis

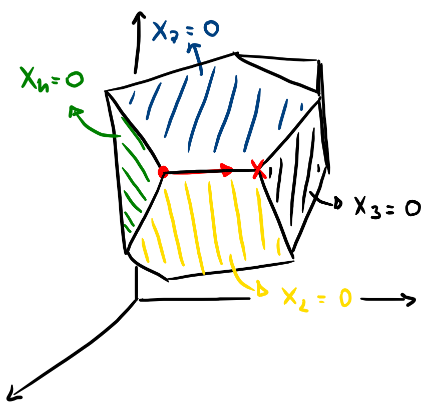
We're in a basis B and the optimality test has failed

Our cost function is $\bar{c}^T = c^T - \underbrace{c_B^T B^{-1}}_{\mu^T} A$

Choose $h \in [1, n]$ | x_h is non-basic : $\bar{c}_h = c_h - \mu^T A_h < 0$

$$c^T x = c_B^T B^{-1} b + \underbrace{\overset{<0}{c_h} x_h + \dots}_{\bar{c}_F^T x_F}$$

Our goal is to increase x_h since $c_h x_h$ would decrease :



following the red arrow will increase the value of x_h since we will stay inside the polyhedron

We also need to know when to stop, or we'll end up in a place with non-feasible solutions

$$x_B = \underbrace{B^{-1}b}_{\bar{b}} - \underbrace{B^{-1}A_h}_{\bar{A}_h} x_h$$

(assume all others $x_F \neq x_h$ stay at 0 $\Rightarrow$ traveling on the edge between the two vertices)

$\downarrow$

$$\begin{bmatrix} x_{\beta[1]} \\ \vdots \\ x_{\beta[m]} \end{bmatrix} = \begin{bmatrix} \bar{b}_1 \\ \vdots \\ \bar{b}_m \end{bmatrix} - \begin{bmatrix} \bar{a}_{1h} \\ \vdots \\ \bar{a}_{mh} \end{bmatrix} x_h$$

(Remember that $\beta[i]$ is the index of the column in A that's placed in B_i)

$\downarrow$

$$x_{\beta[i]} = \underbrace{\bar{b}_i - \bar{a}_{ih} x_h}_{\geq 0} \quad \forall i = 1, \dots, m$$

(If x_h increases too much we end up outside of the polyhedron)

$\downarrow$

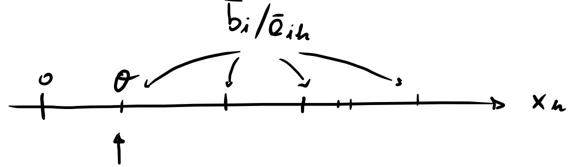
$$\bar{a}_{ih} x_h \leq \bar{b}_i$$

$\bar{b}_i \to \geq 0$ (since it's part of a bfs $x_B = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix}$

$x_h \to > 0$ (increasing from 0)

- $\bar{a}_{ih} \leq 0 \Rightarrow x_h$ can grow forever $\to$ unbounded
- $\bar{a}_{ih} < 0 \Rightarrow x_h \leq \dfrac{\bar{b}_i}{\bar{a}_{ih}}$

Going through all i, we get different x_h values:



we want the smallest one s.t. we won't get other variables negative (exiting the polyhedron)

$$\theta = \min \left\{ \frac{\bar{b}_i}{\bar{a}_{ih}} : \bar{a}_{ih} > 0 \right\}$$

Once x_h has reached θ, there will be another variable reaching 0

$$x_{B[t]} = 0 \quad \text{with } t = \operatorname{argmin} \left\{ \frac{\bar{b}_i}{\bar{a}_{ih}} : \bar{a}_{ih} > 0 \right\}$$

We can now change basis to represent the vertex we've reached:

$$B = \begin{bmatrix} | & & | & & | \\ A_{B[1]} & \cdots & A_{B[t]} & \cdots & A_{B[m]} \\ | & & | & & | \end{bmatrix} \Rightarrow B' = \begin{bmatrix} | & & | & | & & | \\ A_{B[1]} & \cdots & \cancel{A_{B[t]}} & A_{B[h]} & \cdots & A_{B[m]} \\ | & & | & | & & | \end{bmatrix}$$

Pseudocode for the simplex method

1) Initialization : find a starting feasible basis $B = [A_{\beta[1]} \cdots A_{\beta[m]}]$

2) Optimality test :

with $\mu^T := c_B^T B^{-1}$

if $\bar{c}^T = c^T - \mu^T A \geq 0$, then we found an optimal solution $x = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix}$

else change basis

3) Change of basis :

choose $\bar{c}_h$ from $\bar{c}^T$ s.t. $\bar{c}_h = c_h - \mu^T A_h < 0$

"x_h wants to enter the basis"

$t := \text{argmin} \left\{ \frac{\bar{b}_i}{\bar{a}_{ih}} : \bar{a}_{ih} > 0 \right\}$, $\bar{b} = B^{-1} b$, $\bar{A}_h = B^{-1} A_h$

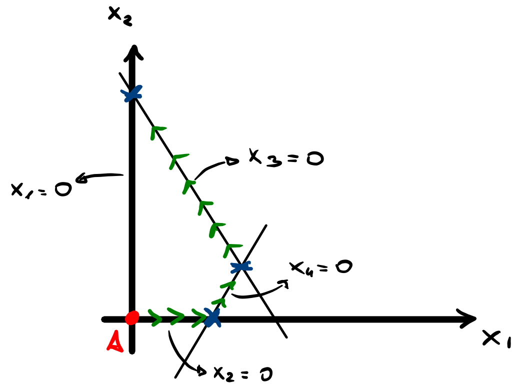
$\hookrightarrow$ if $\emptyset \to$ unbounded problem

"$x_{\beta[t]}$ must leave the basis to let x_h enter"

repeat step 2

Es :

$$\begin{cases} \min \ -x_1 - x_2 := z \\ 6x_1 + 4x_2 + x_3 = 24 \\ 3x_1 - 2x_2 + x_4 = 6 \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$



First iteration: current vertex $A = (0,0)$, basic variables (x_3, x_4)

Since the origin is feasible, we can choose it as the first vertex

$$x_B = \overbrace{\begin{bmatrix} x_3 \\ x_4 \end{bmatrix}}^{\begin{bmatrix} x_{\beta[1]} \\ x_{\beta[2]} \end{bmatrix}} \quad B^{-1}b = \bar{b} = \begin{bmatrix} 24 \\ 6 \end{bmatrix} \quad B^{-1}F = \overbrace{\begin{bmatrix} 6 & 4 \\ 3 & -2 \end{bmatrix}}^{\begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{bmatrix}}$$

$$\underbrace{c_B^T B^{-1} b}_{\bar{c}_0} = 0 \quad \underbrace{c_F^T - c_B^T B^{-1} F}_{\bar{c}_F^T} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad x_F = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{cases} x_3 = 24 - 6x_1 - 4x_2 \\ x_4 = 6 - 3x_1 + 2x_2 \\ z = \quad -x_1 - x_2 \end{cases}$$

$$\begin{cases} x_B = B^{-1}b - B^{-1}F x_F \\ c^T x = c_B^T B^{-1} b + (c_F^T - c_B^T B^{-1} F) x_F \end{cases}$$

$\bar{c}_F^T = \begin{bmatrix} -1 \\ -1 \end{bmatrix} < 0 \quad \Longrightarrow$ We're <u>not</u> in an optimal state

Whe can choose between $h=1$ or $h=2$ to find a new basis

$\Downarrow$

we choose h_1 = "x_1 enters the basis"

We must find $t = \operatorname{argmin} \left\{ \dfrac{\bar{b}_i}{\bar{a}_{ih}} : \bar{a}_{ih} > 0 \right\}, \quad h = 1$

$$\left.\begin{aligned} &\bullet\ i=1: \quad \frac{\bar{b}_1}{\bar{a}_{11}} = 24/6 = 4 \\ &\bullet\ i=2: \quad \frac{\bar{b}_2}{\bar{a}_{21}} = 6/3 = 2 \end{aligned}\right\} \quad t = 2 \Rightarrow B[t] = 4$$

"x_4 leaves the basis"

Since we chosed to increase x_1, x_2 is still 0, so now we're in B

Second iteration: current vertex $B(2,0)$, basic variables (x_1, x_3)

Instead of computing B^{-1} with this new basis, we can rewrite our system:

$$\begin{cases} x_3 = 12 - 8x_2 + 2x_4 \\ x_1 = 2 + 2/3\, x_2 - 1/3\, x_4 \\ z = -2 - 5/3\, x_2 + 1/3\, x_4 \end{cases} \quad \longrightarrow \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{bmatrix} = \begin{bmatrix} 8 & -2 \\ -2/3 & 1/3 \end{bmatrix}$$

$$\bar{c}_F^T = \begin{bmatrix} -5/3 \\ 1/3 \end{bmatrix} \begin{array}{l} \to \bar{c}_2 < 0 \\ \to \bar{c}_4 \end{array} \quad \to \text{" } x_2 \text{ enters the basis"}$$

$$l = \operatorname{argmin} \left\{ \frac{\bar{b}_i}{\bar{a}_{ih}} : a_{ih} > 0 \right\} \quad h = 2$$

$$= 1 \Rightarrow \beta[1] = 3 \quad \to \text{" } x_3 \text{ leaves the basis"}$$

x_4 stays to 0 → we're now on C

Third iteration: current vertex $C(3, 1.5)$, basic variables (x_1, x_2)

"x_1 leaves the basis"
"x_4 enters the basis"

Fourth iteration: current vertex $D(0,6)$, basic variables (x_2, x_4)

$$\begin{cases} x_2 = 6 - 3/2\, x_1 - 1/4\, x_3 \\ x_4 = 18 - 6x_1 - 1/2\, x_3 \\ z = -6 + 1/2\, x_1 + 1/4\, x_3 \end{cases} \quad \to \bar{c}_F^T = \begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix} > 0$$

$$x_B = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \\ 0 \\ 0 \end{bmatrix} \begin{array}{l} x_2 \\ x_4 \\ \\ \\ \end{array} \rightsquigarrow \begin{bmatrix} 0 \\ 6 \\ 0 \\ 18 \end{bmatrix} \quad \to c^T x = -6 \quad \text{optimal solution}$$