Mathematical Optimization

objective function

we have a point $x \in \mathbb{R}^n$ and a function $f: \mathbb{R}^n \to \mathbb{R}$

$X \subseteq \mathbb{R}^n$ is called feasible solution set

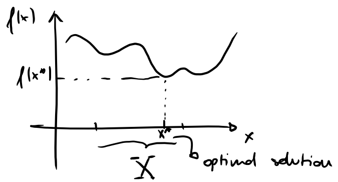
we have to minimize $f(x)$, $x \in X$

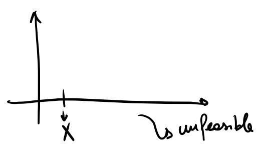
Solving this prob produces 3 answers:

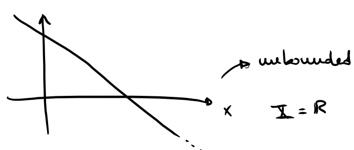
1) $X = \emptyset$ → problem is unfeasible
↳ difficult to prove

2) f is unbounded from below (goes to $-\infty$)
↳ the problem is unbounded

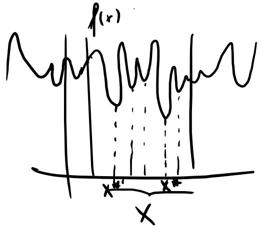
3) optimal solution → - feasible $x^* \in X : f(x^*) \leq f(x) \; \forall x \in X$







finding an opt sol. is tricky



x^* : absolute minimum

$x^{*'}$ = local minimum

↳ useful too

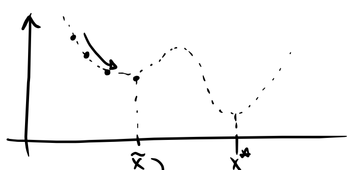
$x^* = \min\{x^{*'}\}$

A solution $\tilde{x} \in X$ is a local minimum if $f(\tilde{x}) \leq f(x)$ $\forall x \in X$:

$$\|x - \tilde{x}\| < \varepsilon \text{ for a given } \varepsilon > 0$$

A way of solving problems is usually an iterative way by searching a better solution with a gradient

↳ find <u>local minimum</u>



$f'(\tilde{x}) = 0 \Rightarrow$ found solution

In mathematical opt we want <u>global minimum</u>.

Time complexity is important

which are the regularity to escape local minimum ?

1) for X : the set is convex

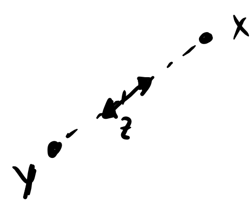
2) for objective function : the function is convex

if both are meeted, all local minimum is global

Def: of a ~~convex~~ combination of 2 points

$x, y \in \mathbb{R}^n$

$z = \lambda x + (1-\lambda) y$ for $\lambda \in [0,1]$



→ if $\lambda = 1$, $z = x$

if $\lambda = 0$, $z = y$

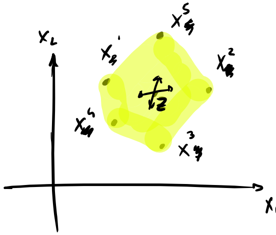
A combination with $\lambda \in]0,1[$ is strict

A strict combination is convex ?

combination of n points

Def: $x^1, x^2, \dots, x^k \in \mathbb{R}^n$

$$z = \sum_{i=1}^{k} \lambda_i x^i \qquad \lambda_1, \dots, \lambda_k \geq 0 \mid \sum_{i=1}^{k} \lambda_i = 1$$



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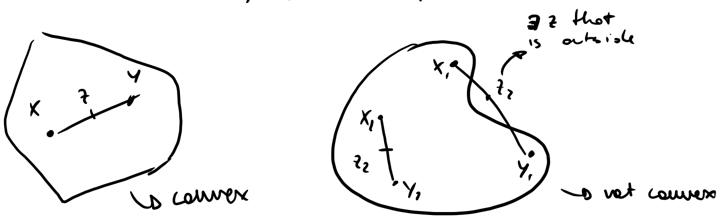
Def <u>convex set</u>

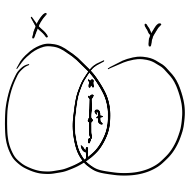
$X \subseteq \mathbb{R}^n$ is convex if

$\forall x, y \in X, \quad z = \lambda x + (1-\lambda) y \in X \quad \forall \lambda \in [0,1]$



Def if $X, Y \in \mathbb{R}^n$ are convex, then <u>$X \cap Y$ is convex</u>

↳ intersection, <u>NOT UNION</u>



$$\left.\begin{array}{l} x \in X,\ y \in X \to z \in X \\ x \in Y,\ y \in Y \to z \in Y \end{array}\right\} z \in X \cap Y$$

Def convex function:

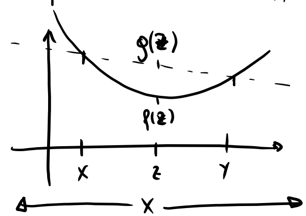
$f: X \to \mathbb{R}$ where <u>X is convex</u>

f is convex if $f(x) \leq$ linear interpolation between x, y $\forall\, x \leq y \in X$

↓

same term but different concept

(convex func $\neq$ convex set)



↓

$z = \lambda x + (1-\lambda) y$

$g(z) = \lambda f(x) + (1-\lambda) f(y)$

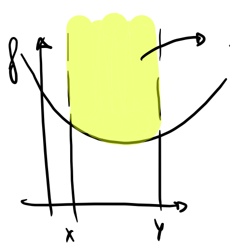
$\Rightarrow f(z) \leq g(z) \quad \forall z \in [x, y]$

$\Downarrow$

$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$

↓

(linearity requires =)



→ epigraph of the function f

f is convex if the epigraph of f (a set) is convex

↓

convexity is a property of the epigraph of f, not a property of f.

$\boxed{???}$

Th 1.1.1: $X = \{ x \in \mathbb{R}^n : g_i(x) \le 0,\ i = 1, \dots, m \}$

If $g_1, g_2, \dots, g_m$ are convex, then X is convex

Proof: $X = \bigcap_{i=1}^{m} X_i = \{ x \in \mathbb{R}^n : g_i(x) \le 0 \}$

prove X_i is convex

$\forall\, x, y \in X_i$ and $z = \lambda x + (1-\lambda) y \quad \forall \lambda \in [0,1]$

$$g_i(z) = g_i(\underbrace{\lambda x + (1-\lambda) y}_{z})$$

$$\le \lambda\, g_i(x) + (1-\lambda)\, g_i(y)$$

convexity of g_i

$$\ge 0 \quad \le 0 \quad \ge 0 \quad \le 0$$

$$\le 0 \Rightarrow z \in X_i \longrightarrow X_i \text{ is convex}$$

Consider a convex opt. problem (namely $\min f(x)$ $x \in X$) where f and X are convex, then every local optimal solution is also globally optimal

<u>concept of the proof:</u>

locally min (assumption)

$\forall y \in X$

if I move from x^* to y I only find that the obj funct can't go down (local minimum)

$\Downarrow$

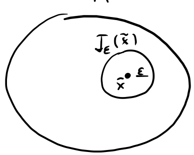
$f(y) \geq f(\tilde{x})$

Proof let $\tilde{x} \in X$ be a locally optimal solution

$\Downarrow$

$$\exists \varepsilon > 0 : f(\tilde{x}) \leq f(z), \ \underbrace{\forall z \in X : \| z - \tilde{x} \| \leq \varepsilon}_{\forall x \in I_\varepsilon(\tilde{x})}$$

X



smaller but close to 1 ($\neq 1$)

$\forall y \in X$ and define $z = \lambda \tilde{x} + (1-\lambda) y$ for $\lambda \lesssim 1 \mid z \in I_\varepsilon(\tilde{x})$

$f(\tilde{x}) \leq f(z)$ since $\tilde{x}$ is a local minimum

$\leq f(\lambda \tilde{x} + (1-\lambda) y)$

(convexity)

$\leq \lambda f(\tilde{x}) + (1-\lambda) f(y)$

$(1-\lambda) f(\tilde{x}) \leq (1-\lambda) f(y)$

$f(\tilde{x}) \leq f(y)$ (since $1-\lambda > 0$) $\Rightarrow \tilde{x}$ is globally optimal

$\hookrightarrow \forall y \in X$