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Lecture 08
Potential games

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Potential games

Fictitious Play

- In fictitious play (G.W. Brown, 1951), regrets become actual changes of move
    - Each player i assumes the (possibly mixed) strategies played by $-i$ to be fixed
    - If i gets a chance to play again, it plays the best response to what the other players' previous move
    - Somehow, "full rationality" is denied (we acknowledge predictions might be incorrect)
- How does a fictitious game evolve?
    - Nash equilibrium points are **absorbing** states. So, are they always convergence points?

Fictitious Play

- Not always! Players can also keep "cycling"
  - In Rock-Paper-Scissors, FP does not converge
- FP converges to a NE in some relevant cases:
  - The game can be solved via IESDS
  - **Potential games**
  - Other cases such as $2 \times N$ games where every outcome has a different payoff for all players)

Potential games

- Consider $\mathbb{G} = (S_1, \ldots, S_n; u_1, \ldots, u_n)$ and $S = S_1 \times \cdots \times S_n$
- Function $\Omega : S \to \mathbb{R}$ is an (**exact**) **potential** for $\mathbb{G}$ if:

$$\Omega(s_i', s_{-i}) - \Omega(s_i, s_{-i}) = u_i(s_i', s_{-i}) - u_i(s_i, s_{-i}) = \Delta u_i$$

- Function $\Omega : S \to \mathbb{R}$ is a **weighted potential** with weight $w = \{w_i > 0\}$ for $\mathbb{G}$ if:

$$\Omega(s_i', s_{-i}) - \Omega(s_i, s_{-i}) = w_i \Delta u_i$$

- Function $\Omega : S \to \mathbb{R}$ is an **ordinal potential** for $\mathbb{G}$ if:

$$\Omega(s_i', s_{-i}) > \Omega(s_i, s_{-i}) \iff u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$$

- If $\mathbb{G}$ admits a potential (resp., ordinal potential), it is called a **potential** (resp., **ordinal potential**) **game**

Potential games

- Potential games have nice properties
- If $\mathbb{G} = (S_1, \dots, S_n; u_1, \dots, u_n)$ has an ordinal potential Ω, its set of NE is the same as $\mathbb{G} = (S_1, \dots, S_n; \Omega, \dots, \Omega)$
- I.e., a game where all players want to maximize the potential
  - Game becomes a simple single-objective optimization problem
  - To some extent, it enables distributed optimization (player's decision is still independent)

Examples of potential

- The Prisoner's dilemma is a potential game

		B	
		M	F
A	M	-1, -1	-9, 0
	F	0, -9	-6, -6

		B	
		M	F
A	M	0	1
	F	1	4

potential Ω

- This potential is exact
- However, the players are not very smart (they do not maximize the global welfare)
- There must be some "dummy" somewhere

Examples of potential

- The Cournot duopoly is an ordinal potential game
  - Recall that firms choose quantities q_1 and q_2
  - the market clearing price is $a - q_1 - q_2$
  - the unit production cost is c (so the cost for producing q_i is cq_i)
- Therefore, $u_i(q_i, q_j) = q_i(a - q_i - q_j - c)$ and an ordinal potential function is

$$\Omega(q_1, q_2) = q_1 q_2 (a - q_1 - q_2 - c)$$

Potential games

- **Theorem**: Every finite ordinal potential game has (at least) one NE in pure strategies
  - This NE can be found deterministically (without using probabilities)
- *Proof (Sketch)*: a consequence of fictitious play
  - All players move, one at a time, to maximize their utility $\rightarrow$ they also maximize the potential
  - Continue fictitious play until a local maximum of Ω is found

Congestion games

- Congestion games are a special case of potential game, in which players aim to choose the “least congested” resource
    - Largely applied to network problems (finding the least congested route on a graph)
    - Or in resource allocation (minority games)
- It can be shown that:
    - congestion games are potential games
    - for every potential game, the is a congestion game with the same potential

Coordination games

- A **coordination game** models situations where players have incentive to coordinate their actions
- Players get a higher payoff when they choose the same strategy
- Example: Battle of the sexes
- Example: "Stag Hunt" (proposed by Rousseau). Two haunters they can decide to haunt together and aim for a bigger prey (sharing a payoff of 20); or they can hunt a smaller one separately (payoff 7).

Coordination games

- A coordination game has multiple pure-strategy NE
- In the Stag Hung, players can choose D = big prey (deer); or H = small prey (hare)

		Grunt	
		D	H
Brunt	D	10, 10	0, 7
	H	7, 0	7, 7

Coordination games

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- A coordination game can be seen as a potential game, with coordination points as potential maxima

		Grunt	
		D	H
Brunt	D	10, 10	0, 7
	H	7, 0	7, 7

payoffs

		Grunt	
		D	H
Brunt	D	-4	-7
	H	-7	0

potential Ω

Coordination games

- Another case is the **anti-coordination game**, where players get better payoffs for playing different strategies
  - For example, Hawk-and-Dove games, such as the Chicken game (H = hold the wheel and save your life; D = steer the wheel and be a chicken)

		B	
		H	D
A	H	-99, -99	10, -10
	D	-10, 10	0, 0

Potential=coordination+dummy

- A dummy (or pure-externality) game is such that for all s_{-i}, $u_i(s_i, s_{-i}) = u_i(s_i', s_{-i})$, i.e. player i's payoff depends only on s_{-i}
- Every potential game is a sum of a coordination game and a dummy game

$$\begin{array}{c|c|c|}
 & M & F \\ \hline
M & -1, -1 & -9, 0 \\ \hline
F & 0, -9 & -6, -6 \\ \hline
\end{array}
=
\begin{array}{c|c|c|}
 & M & F \\ \hline
M & -1, -1 & 0, 0 \\ \hline
F & 0, 0 & 3, 3 \\ \hline
\end{array}
+
\begin{array}{c|c|c|}
 & M & F \\ \hline
M & 0, 0 & -9, 0 \\ \hline
F & 0, -9 & -9, -9 \\ \hline
\end{array}$$

coordination

dummy

Computational complexity of NE

How easy is to find NE?

- Since Nash equilibria are considered a “natural” evolution of the game, one may wonder how much does it take to reach them
- Nash theorem guarantees existence
- Plus, there are notable results for specific types of game

A negative result

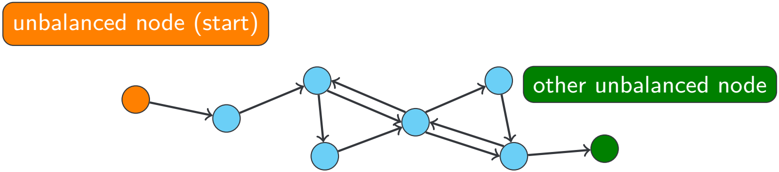
- Unfortunately, in the general case, finding a NE is **computationally hard**
- This has been proven by Papadimitriou in 2007 ("The Complexity of Finding Nash Equilibria")
- However, computationally hard does not mean NP-complete
- The search for a NE cannot be NP-complete as a solution *must* exist
    - There may even be multiple solutions, which further complicates things

The PPAD class

- The NE search problem is PPAD-complete
- PPAD = Polynomial Parity Arguments on Directed graphs (Papadimitriou, 1994)
- More or less, P<PPAD<NP, which means that PPAD is computationally hard unless P=NP
- This class includes the problem equivalent to the *end-of-line problem*

The PPAD class

- End-of-line problem: "Consider a directed graph with an unbalanced node (in-degree≠out-degree). There must be at least another one. Find it."



- This problem is bound to have a solution
- However, finding it without exploring the whole graph is far from trivial, and in some cases cannot be avoided

How is NE search a PPAD problem?

- The NE search problem corresponds to finding a fixed point of the **BR** function
- Finding a fixed point over a compact set can be shown to be equivalent to finding the end of a proper path on a directed graph
- There is an elegant (not difficult but very long) proof of it, involving graph coloring and compact partitioning

Consequences on NE?

- This may imply bad consequences on the practical usefulness of Nash Equilibrium
- To be optimistic:
  - Certain simple problems can be shown to have a NE which can be found through constructive steps (good for engineering)
  - One may be "close" to a NE (maybe it is enough)

→ relaxation: ϵ-**Nash Equilibrium**, i.e., rather than checking for "no unilateral improvement" ignore all improvements smaller than $\epsilon > 0$