Finding the first basis

We can create an artificial problem to help us find the first basis

$$\begin{cases} \min \quad w = \sum_{i=1}^{m} y_i \\ Ax + Iy = b \\ x, y \geq 0 \end{cases}$$

Whenever we find $w^* = 0$, $y_1^*, \dots, y_m^* = 0$ (by applying the simplex method), so we found a feasible solution $\rightarrow$ we can use this as the initial problem

Es)

$$\begin{cases} \min \quad z = x_1 + x_3 \\ x_1 + 2x_2 + x_4 = 5 \\ x_2 + 2x_3 = 6 \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$

Canonical tableau $\rightsquigarrow$

		x_1	x_2	x_3	x_4	y_1	y_2
$(w = \sum y_i =)$	0	0	0	0	0	1	1
$(y_1 =)$	5	1	2	0	1	1	0
$(y_2 =)$	6	0	1	2	0	0	1

Since y_1, y_2 belong to the basis, we want their costs to be 0

We can achieve that by $\left[\text{new row } 0\right] = \left[\text{old row } 0\right] - \sum_{i=1}^{m} \left[\text{row } i\right]$

0	0	0	0	0	1	1
5	1	2	0	1	1	0
6	0	1	2	0	0	1

(3) $\Rightarrow$

-11	-1	-3	-2	-1	0	0
5	1	2	0	1	1	0
6	0	1	2	0	0	1

simplex method

$-w$

	0	0	0	0	0	1	1
$x_2 =$	5/2	1/2	1	0	1/2	1/2	0
$x_3 =$	7/4	-1/4	0	1	-1/4	-1/4	1/2

$\bar{b} \geq 0$

$w^* = 0 \Rightarrow \underbrace{y_1^*, y_2^* = 0}_{\text{non-basic}}$

$$x^* = \begin{bmatrix} 0 \\ 5/2 \\ 7/4 \\ 0 \end{bmatrix} \rightarrow \text{feasible for the original problem}$$

Phase II:

$\rightarrow$ must be 0 to have a canonical form

0	1	0	1	0
5/2	1/2	1	0	1/2
7/4	-1/4	0	1	-1/4

$\rightarrow$ original obj function

pivot operation over x_3

-7/4	5/4	0	0	1/4
5/2	1/2	1	0	1/2
7/4	-1/4	0	1	-1/4

$\rightarrow$ simplex method

Particular cases of "phase I":

1) $w^* < 0$ $\Rightarrow$ The (original) problem is unfeasible

2) $w^* = 0$ with y_i basic for some i

$\hookrightarrow$ If $\exists\ \bar{a}_{th} \neq 0$ then I make a pivot operation on it (divide the whole row by it)

else that row is redundant : it can simply be removed since it's a linear combination of two other rows

ES)

$$\begin{cases} \min \ z = x_1 + x_3 \\ x_1 + 2x_2 \leq -5 \\ x_2 + 2x_3 = 6 \\ x_1, x_2, x_3 \geq 0 \end{cases} \Rightarrow \begin{cases} \min \ z = x_1 + x_3 \\ x_1 + 2x_2 + x_4 = -5 \\ x_2 + 2x_3 = 6 \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases} \Rightarrow \underline{\text{Impossible}}$$

(assume we don't realize)

		x_1	x_2	x_3	x_4	y_1	y_2
$-w =$	-11	1	1	-2	1	0	0
$y_1 =$	$+5$	-1	-2	0	-1	1	0
$y_2 =$	6	0	1	2	0	0	1

$\rightarrow$ canonical form

I can start phase I

" x_3 enters the basis "

" y_2 leaves the basis "

		x_1	x_2	x_3	x_4	y_1	y_2
$-w =$	-5	1	2	0	1	0	1
$y_1 =$	5	-1	-2	0	-1	1	0
$x_3 =$	3	0	$1/2$	1	0	0	$1/2$

$\rightarrow$ optimal : $w = 5$

$\Downarrow$

Unfeasible

ES)

$$\begin{cases} \min & z = x_1 + x_2 + 10x_3 \\ & x_2 + 4x_3 = 2 \\ & -2x_1 + x_2 - 6x_3 = 2 \\ & x_1, x_2, x_3 \geq 0 \end{cases} \longrightarrow$$

		x_1	x_2	x_3	y_1	y_2
$-w =$	-4	2	-2	2	0	0
$y_1 =$	2	0	1	4	1	0
$y_2 =$	2	-2	1	-6	0	1

"x_2 enters the basis"
"y_2 leaves the basis"

phase I

		x_1	x_2	x_3	y_1	y_2
$-w =$	0	2	0	10	2	0
$x_2 =$	2	0	1	4	1	0
$y_2 =$	0	-2	0	-10	-1	1

$\to$ y_2 is basic

pivot operation

0	2	0	10	~~2~~	~~0~~
2	0	1	4	~~1~~	~~0~~
0	1	0	5	~~$1/2$~~	~~$1/2$~~

$\to$ we can now remove y_1, y_2

phase II

change obj f.

0	1	1	10
2	0	1	4
0	1	0	5

pivot on those to get the costs to 0

$\Rightarrow$

-2	0	0	1
2	0	1	4
0	1	0	5

$\to$ Canonical form and $\bar{c}_F^T \geq 0$

$\Downarrow$

Optimal solution

$\not\to$ simplex method