Transformation of a r.v.

Applying a function g to a r.v. X, leads to another r.v. $Y = g(X)$.

If X is discrete, the Y is discrete and

$$f_Y(y) = P(Y = y) = P(\{s : g(X(s)) = y\}).$$

I can't always invert g(x)

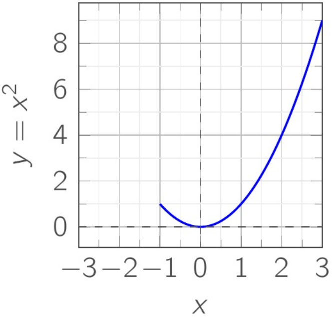
If X is continuous the procedure is more difficult, we have to:

- find $B_y = \{x : g(x) \leq y\}$, for each y in the range of Y;
- find the df
  $F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P(B_y) = \int_{B_y} f(x)\mathrm{d}x;$
- take the derivative, i.e. $f_Y(y) = F'_Y(y)$.

Transformation of a rv

Example 11

Let $X \sim \mathrm{Unif}(-1,3)$, we compute the pdf of $Y = X^2$. Consider the graph of $g(x) = x^2$.



Now $Y \in [0,9]$, for $x \in [-1,1]$, $g(x) \in [0,1]$, whereas for $x > 1$ $g(x)$ is bijective. We have two cases:

(1) $0 \le y \le 1$: $B_y = (-\sqrt{y}, \sqrt{y})$ and
$F_Y(y) = (1/4)\int_{B_y} \mathbf{1}_{[-1,3]}\mathrm{d}x = \sqrt{y}/2$;

(2) $y > 1$: $B_y = (-1, \sqrt{y})$,
$F_Y(y) = (\sqrt{y} + 1)/4$.

The pdf can be found by differentiating F_Y, taking care of the two cases.

Example 12 (Example 11 cont'd)

Let's compute $E(Y)$ and $\text{var}Y$. We have that

$$f_Y(y) = \begin{cases} 1/(4\sqrt{y}) & \text{if } 0 < y \leq 1 \\ 1/(8\sqrt{y}) & \text{if } 1 < y \leq 9. \end{cases}$$

Then

$$\begin{aligned} E(Y) &= \int_0^9 y f_Y(y) \mathrm{d}y = \int_0^1 y/(4\sqrt{y}) \mathrm{d}y + \int_1^9 y/(8\sqrt{y}) \mathrm{d}y \\ &= \int_0^1 \sqrt{y}/4 \mathrm{d}y + \int_1^9 \sqrt{y}/8 \mathrm{d}y = 7/3. \end{aligned}$$

$$\begin{aligned} E(Y^2) &= \int_0^9 y^2 f_Y(y) \mathrm{d}y = \int_0^1 y^2/(4\sqrt{y}) \mathrm{d}y + \int_1^9 y^2/(8\sqrt{y}) \mathrm{d}y \\ &= \int_0^1 y^{3/2}/4 \mathrm{d}y + \int_1^9 y^{3/2}/8 \mathrm{d}y = 61/5. \end{aligned}$$

Thus $\text{var}(Y) = E(Y^2) - E(Y)^2 = 61/5 - 49/9 = 794/45$.

Inverse tranform

Suppose Y is a continuous rv with distribution F_Y.

It can be shown that F_Y is continuous and bijective with inverse $F^{-1} = Q$.

Furthermore, if $X = \text{Unif}(0, 1)$, then $Q(X) \sim F_Y$.
This fact is useful when we want to draw random values from F_Y.
Indeed, if we

(a) draw a number p uniformly in (0,1)

(b) set $y = Q(p)$,

y is a random value from F_Y. This is knwon as the **inverse tranfrom sampling** method.

Inferential Statistics
L1 - Introduction to probability: part II

Erlis Ruli (erlis.ruli@unipd.it)

Department of Statistics, University of Padova

Contents

Random vectors

k-dimensional random vector[1]: is a mapping $X : \mathcal{S} \to \mathbb{R}^k$ which assigns a real vector $X(s) = (X_1(s), \ldots, X_k(s))$ to every $s \in \mathcal{S}$.

The df of an rve X:

now it's a vector

$$F(x) = P(X \le x) = P(X_1 \le x_1, \ldots, X_k \le x_k) \text{ for all } x \in \mathbb{R}^k.$$

Random vectors also can be:
discrete if $P(X = x) > 0$, for all x in the range of X or
continuous if there exist a function $f(x) : \mathbb{R}^k \to \mathbb{R}_{\ge 0}$, s.t. $\int f(x)\mathrm{d}x = 1$ and

$$P(X \in \text{cube}) = \int_{cube} f(x)\mathrm{d}x.$$

"shape"

[1] rve for short.

Distributions

For rve $X = (X_1, X_2)$ with pdf $f(x_1, x_2)$, marginal pdf of X_1:

$$f_{X_1}(x_1) = \int_{t \in \mathbb{R}} f(x_1, t)\mathrm{d}t.$$

conditional pdf of X_2 given X_1

$$f_{X_2|X_1}(x_2|x_1) = f(x_1, x_2)/f_{X_1}(x_1),$$

provided $f_{X_1}(x_1) > 0$; also written $X_2|X_1 \sim F_{X_2|X_1}$.

X_1 is independent of X_2 iff ($\iff$)

$$f(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2), \quad or \quad F(x_1, x_2) = F_{X_1}(x_1) F_{X_2}(x_2),$$

for all $(x_1, x_2) \in \mathbb{R}^2$.

Example 1

For the bivariate pdf

$$f(x,y) = \begin{cases} k(x+2y) & \text{if } 0 < y < 1 \quad \text{and} \quad 0 < x < 2 \\ 0 & \text{otherwise,} \end{cases}$$

(a) Find the value of k.
(b) Find the marginal distribution of X.
(c) Find the joint df of X and Y
(d) Find the pdf of the rv $Z = 9/(X+1)^2$.

Solution

(a) Integrating the pdf over the domain gives

$$1 = \int_0^1 \left(\int_0^2 k(x+2y)dx \right) dy = \int_0^1 k(4y+2)dy = 4k,$$

so $k = 1/4$.

(b) The marginal distribution of X is obtained by integrating out Y,

$$f_X(x) = \int_{y \in \mathcal{Y}} f(x, y)dy = \int_0^1 (x + 2y)/4dy = (x + 1)/4,$$

if I don't specify, $\int f_X(x)\,dx \neq 1$

for $x \in (0, 2)$ and $f_X(x) = 0$ otherwise.

(c) The joint df of X and Y is

$$F(s, t) = \int_0^s \int_0^t \tfrac{1}{4}(x + 2y)dydx = (2st^2 + s^2 t)/8,$$

for $s \in (0, 2)$, $t \in (0, 1)$. !! → Always specify the bounds

(d) $Z = g(X) \in (1, 9)$ and g is bijective with inverse g^{-1}, so

$$F_Z(z) = P(Z \leq z) = P(g(X) \leq z) = P(X \leq g^{-1}(z)) = P\left(X \leq \tfrac{3}{\sqrt{z}} - 1\right)$$
$$= \tfrac{9-z}{8z}.$$

$$f_Z(z) = \tfrac{1}{8z} - \tfrac{z-9}{8z^2}.$$

Moments

Expectation of X:

$$E(X) = (E(X_1), E(X_2), \ldots, E(X_k))$$

Covariance and correlation between two rv's X_i and X_j:

$$\sigma_{ij} = \operatorname{cov}(X_i, X_j) = E(X_iX_j) - E(X_i)E(X_j), \quad \text{and} \quad \rho_{ij} = \cancel{\frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}}$$

Covariance and correlation matrices of X: both simmetric

$$\operatorname{cov}(X) = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1k} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{k1} & \sigma_{k2} & \cdots & \sigma_{kk} \end{pmatrix}, \quad \text{and} \quad \operatorname{cor}(X) = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1k} \\ \rho_{21} & 1 & \cdots & \rho_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ \rho_{k1} & \rho_{k2} & \cdots & 1 \end{pmatrix}.$$

Example 2

Let (X, Y) have density $f(x, y) = x + y$ if $0 \leq x, y \leq 1$ and zero otherwise. We see that

$$\int_0^1 \int_0^1 (x + y)\mathrm{d}x\mathrm{d}y = 1,$$

→ f is a valid pdf

thus f is a valid pdf. The marginal pdf of X is

$$f_X(x) = \int_0^1 (x + y)\mathrm{d}y = x + 1/2, \quad \text{for } 0 \leq x \leq 1,$$

$\int_0^1 (x+y)\,dx$

similarly the marginal pdf of Y is $f_Y(y) = y + 1/2$, for all $0 \leq y \leq 1$.

We have $E(X) = \int_0^1 x(x + 1/2)\mathrm{d}x = 7/12 = E(Y)$.

Example 1 (cont'd)

Furthermore

$$E(X^2) = \int_0^1 x^2(x + 1/2)\mathrm{d}x = 5/12 = E(Y^2),$$

so

$$\mathrm{var}(X) = \mathrm{var}(Y) = 5/12 - (7/12)^2 = 11/12^2,$$

and

$$\int_0^1\int_0^1 xy(x+y)\,dx\,dy$$

$$\mathrm{cov}(X, Y) = E(XY) - E(X)E(Y) = 1/3 - (7/12)^2 = -1/144.$$

The covariance matrix of the rve $Z = (X, Y)$ is

$$\overset{\text{cov}}{\cancel{\mathrm{var}}}(Z) = \frac{1}{12^2}\begin{pmatrix} 11 & -1 \\ -1 & 11 \end{pmatrix}.$$

compute the correlation matrix

Independence

For a rve $(X_1, \ldots, X_k)$, we say that X_i's are **fully independent** if for all $x_1, \ldots, x_k$,

$$F(x_1, \ldots, x_k) = F_{X_1}(x_1) \cdots F_{X_k}(x_k),$$

or

$$f(x_1, \ldots, x_k) = f_{X_1}(x_1) \cdots f_{X_k}(x_k).$$

Example 3 (Example 2 cont'd)

Since $f_X(x) f_Y(y) = (x + 1/2)(y + 1/2) \neq f(x, y) = (x + y)$, X and Y are not independent.

Conditional distributions

Given two continuous rv X, Y with joint pdf $f(x, y)$, the conditional distribution of Y given $X = x$ is defined by

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)},$$

whenever $f_X(x) > 0$. For Y, X discrete rv's, the above conditional distribution is defined by

$$p_{Y|X}(y|x) = \frac{P(X=x, Y=y)}{p_X(X=x)}.$$

Example 4 (Example 1 cont'd)

The conditional distribution of Y given $X = 3$ is

$$f_{Y|X}(y|3) = \cancel{\frac{(x+y)}{4}}. \quad \frac{3+2y}{4}$$