Bland's Rule

Choosing always the first non-basic variable with negative reduced cost will prevent any loop.

Th.

If we follow Bland's rule we will not end up in loops

Proof:

Consider an instance of the tableau (T) in which the worst possible choice following Bland's rule

		$\dots$	$x_{\beta[i]}$	x_h	x_n	
	=		0	-	0	
	0		0	-	0	
$x_{\beta[i]}$	$\vdots$		1	-	$\vdots$	
	$\vdots$		0	$\vdots$		
			$\vdots$		0	
$x_{\beta[t]}$			$\vdots$	+	1	row t
	0		0	-	0	

→ $\bar b = 0$ for the objective function to be constant

$\bar c_h < 0$, $\bar c_{\beta[i]} = 0 \ \forall i$, $\bar a_{ih} \le 0 \ \forall i \ne t$, $\bar a_{th} > 0$

Let's assume that this table will loop: at some point x_n will enter the basis ($\tilde{T}$):

	$\cdots\ x_{\beta[i]}$	$\cdots\ x_h$	$\cdots\ x_n$
=	$+\ 0\ +\ \cdots$	$\cdots$	$-$

$\rightarrow$ $\tilde{c}_i \geq 0 \ \forall i \neq n$, $\tilde{c}_n < 0$ (this must happen to choose x_n as the new basis variable)

$$[\text{row } 0 \text{ of } \tilde{T}] = [\text{row } 0 \text{ of } T] + \sum_{i=1}^{m} \mu_i [\text{row } i \text{ of } T]$$

- $\underbrace{\tilde{c}_{\beta[i]}}_{\geq 0} = \overset{0}{\bar{c}_{\beta[i]}} + \mu_i$ $\rightarrow$ elements of the basis belong to the identity matrix

  $\Downarrow$

  $\mu_i \geq 0 \ \forall i \neq t$

- $\tilde{c}_{\beta[t]} = \underbrace{\tilde{c}_n}_{<0} = \underbrace{\bar{c}_{\beta[t]}}_{0} + \mu_t$

  $\Downarrow$

  $\mu_t < 0$

- $\overset{\geq 0}{\tilde{c}_h} = \bar{c}_h + \sum_{i=1}^{m} \mu_i \bar{a}_{ih} = \overset{<0}{\bar{c}_h} + \sum_{i \neq t}^{m} \overset{\geq 0}{\mu_i} \underset{<0}{\bar{a}_{ih}} + \overset{<0}{\mu_t} \underset{>0}{\bar{a}_{th}} \Rightarrow \begin{cases} \tilde{c}_h \geq 0 \\ \tilde{c}_h < 0 \end{cases}$ $\rightarrow$ impossible to reach $\tilde{T}$ from T, hence following Bland's rule prevents any loop

Dealing with bounded variables

Always worked with: $0 \leq x_i \quad \forall i$

We might have an upper bound: $0 \leq x_i \leq q_i \quad \forall i$

or also a different lower bound: $l_{bi} \leq x_i \leq u_{bi} \quad \forall i$

$$(l_{bi} \leq x_i \leq u_{bi}) - l_{bi}$$

$$0 \leq x_i' \leq q_i \quad , \quad x_i' = x_i - l_{bi}$$
$$q_i = u_{bi} - l_{bi}$$

We could add this as new costraints to the matrix

$\Downarrow$

<u>HUGE MATRIX</u>

Those constraints can be implicitly checked while producing the theta using complemented variables, that make an upper bound a non-negativity constraint

$$x_j^c = q_j - x_j \geq 0 \qquad \text{(yet no one implemented this)}$$

$$x = \begin{bmatrix} x_B \\ x_F \end{bmatrix} = \begin{bmatrix} x_B \\ x_L \\ x_U \end{bmatrix} \begin{matrix} \\ \to \text{at lower bound} \\ \to \text{at upper bound} \end{matrix}$$

$$\Downarrow$$

$$Ax = b \rightsquigarrow \begin{bmatrix} B & \underbrace{L \quad U}_{F} \end{bmatrix} \begin{bmatrix} x_B \\ x_L \\ x_U \end{bmatrix} = Bx_B + Lx_L + Ux_U$$

Basic solutions: $x_L = 0$, $x_U = q_U \Rightarrow x_B = B^{-1}b - B^{-1}Uq_U$

Optimality test

$\bar{c}_h \geq 0 \quad \forall x_h$ non-basic at the lower bound

$\bar{c}_h \leq 0 \quad \forall x_h$ non-basic at the upper bound

Optimality test fails in two cases:

1) $\bar{c}_h < 0$ and x_h is non-basic at the lower bound

$$x_h : 0 \to \min\{\vartheta, \vartheta'\}$$

$$\vartheta = \min\left\{\frac{\bar{b}_i}{\bar{a}_{ih}} : \bar{a}_{ih} > 0\right\}$$

$\vartheta' = \ \ldots$ (to avoid going over the upper bound)

2) $\bar{c}_h > 0$ and x_h is non-basic at the upper bound

$$x_h : q_h \to \omega_h$$

↳ similar procedure than ϑ