Bland's Rule

Choosing always the first non-basic variable with negative reduced cost will prevent any loop.

Th.
If we follow bland's rule we will not end up in loops

Proof:

Consider an instance of the tableau (T) in which the worst possible choice following Bland's rule

$$
\begin{array}{c|c|cccc|l}
 & & \cdots & x_{\beta[i]} & x_h & x_n & \\
\hline
 & = & & 0 & - & 0 & \\
\hline
 & 0 & & 0 & - & 0 & \\
x_{\beta[i]} & \vdots & & 1 & - & \vdots & \\
 & \vdots & & 0 & \vdots & \vdots & \\
 & \vdots & & \vdots & \vdots & 0 & \\
x_{\beta[t]} & & & \vdots & + & 1 & \text{row } t \\
 & 0 & & 0 & - & 0 & \\
\hline
\end{array}
$$

$\bar{b} = 0$ for the objective function to be constant

$\bar{c}_h < 0$, $\bar{c}_{\beta[i]} = 0 \ \forall i$, $\bar{a}_{ih} \leq 0 \ \forall i \neq t$, $\bar{a}_{th} > 0$

Let's assume that this table will loop: at some point x_h will enter the basis ($\tilde{T}$):

	$\cdots$ $x_{\beta[i]}$ $\cdots$ x_h $\cdots$ x_n
=	$+$ 0 $+$ $\cdots$ $\cdots$ $-$

$\rightarrow$ $\tilde{c}_i \geq 0 \ \forall i \neq h$, $\tilde{c}_h < 0$

(this must happen to choose x_h as the new basis variable)

[row 0 of $\tilde{T}$] = [row 0 of T] + $\sum_{i=1}^{m} \mu_i$ [row i of T]

- $\underset{\geq 0}{\tilde{c}_{\beta[i]}} = \overset{0}{\bar{c}_{\beta[i]}} + \mu_i$ $\rightarrow$ elements of the basis belong to the identity matrix

$$\Downarrow$$

$$\mu_i \geq 0 \quad \forall i \neq t$$

- $\tilde{c}_{\beta[t]} = \underset{<0}{\tilde{c}_h} = \underset{0}{\bar{c}_{\beta[t]}} + \mu_t$

$$\Downarrow$$

$$\mu_t < 0$$

- $\overset{\geq 0}{\tilde{c}_h} = \bar{c}_h + \sum_{i=1}^{m} \mu_i \bar{a}_{ih} = \overset{<0}{\bar{c}_h} + \sum_{i \neq t}^{m} \overset{\geq 0}{\mu_i} \underset{<0}{\bar{a}_{ih}} + \overset{<0}{\mu_t} \underset{>0}{\bar{a}_{th}} \Rightarrow \begin{cases} \tilde{c}_h \geq 0 \\ \tilde{c}_h < 0 \end{cases}$ $\rightarrow$ impossible to reach $\tilde{T}$ from T, hence following Bland's rule prevents any loop

Dealing with bounded variables

Always worked with: $0 \le x_i \quad \forall i$

We might have an upper bound: $0 \le x_i \le q_i \quad \forall i$

or also a different lower bound: $lb_i \le x_i \le ub_i \quad \forall i$

$$(lb_i \le x_i \le ub_i) - lb_i$$

$$0 \le x_i' \le q_i \quad , \quad \begin{aligned} x_i' &= x_i - lb_i \\ q_i &= ub_i - lb_i \end{aligned}$$

We could add this as new constraints to the matrix

$\Downarrow$

HUGE MATRIX

Those costraints can be implicitly checked while producing the theta using complemented variables, that make an upper bound a non-negativity constraint

$$x_j^c = q_j - x_j \geq 0$$ (yet no one implemented this)

$$x = \begin{bmatrix} x_B \\ x_F \end{bmatrix} = \begin{bmatrix} x_B \\ x_L \\ x_U \end{bmatrix} \begin{array}{l} \\ \rightarrow \text{at lower bound} \\ \rightarrow \text{at upper bound} \end{array}$$

$$\Downarrow$$

$$Ax = b \rightsquigarrow \begin{bmatrix} B & \underbrace{L \quad U}_{F} \end{bmatrix} \begin{bmatrix} x_B \\ x_L \\ x_U \end{bmatrix} = Bx_B + Lx_L + Ux_U$$

Basic solutions: $x_L = 0, \; x_U = q_U \Rightarrow x_B = B^{-1}b - B^{-1}Uq_U$

Optimality test

$\bar{c}_h \geq 0 \quad \forall x_h$ non-basic at the lower bound

$\bar{c}_h \leq 0 \quad \forall x_h$ non-basic at the upper bound

Optimality test fails in two cases:

1) $\bar{c}_h < 0$ and x_h is non-basic at the lower bound

$$x_h : 0 \rightarrow \min \{\vartheta, \vartheta'\}$$

$$\vartheta = \min \left\{ \frac{\bar{b}_i}{\bar{a}_{ih}} : \bar{a}_{ih} > 0 \right\}$$

$\vartheta' = \quad \ldots \quad$ (to avoid going over the upper bound)

2) $\bar{c}_h > 0$ and x_h is non-basic at the upper bound

$$x_h : q_h \rightarrow \omega_h$$

↳ similar procedure than ϑ