Exercises L2

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2.1:

2.1.1:

Prove that

$$S^2 \leq \frac{1}{n-1}\sum_i (X_i - a)^2, \qquad \forall a \in \mathbb{R}$$

Since $S^2 = \frac{1}{n-1}\sum_i (X_i - \overline{X})^2$, we can rewrite the inequality as follows:

$$\cancel{\frac{1}{n-1}}\sum_i (X_i - \overline{X})^2 \leq \cancel{\frac{1}{n-1}}\sum_i (X_i - a)^2$$

$$\sum_i (X_i - \overline{X})^2 \leq \sum_i (X_i - a)^2$$

$$\sum_i (X_i^2 - 2X_i\overline{X} + \overline{X}^2) \leq \sum_i (X_i^2 - 2X_i a + a^2)$$

$$\cancel{\sum_i X_i^2} - 2\overline{X}\sum_i X_i + \overline{X}^2 \sum_i 1 \leq \cancel{\sum_i X_i^2} - 2a\sum_i X_i + a^2 \sum_i 1$$

$$-2\overline{X}\sum_i X_i + n\overline{X}^2 \leq -2a\sum_i X_i + na^2$$

$$2(\overline{X} - a)\sum_i X_i \geq n(\overline{X}^2 - a^2)$$

$$2(\overline{X} - a)\frac{1}{n}\sum_i X_i \geq \overline{X}^2 - a^2$$

$$2(\overline{X} - a)\overline{X} \geq \overline{X}^2 - a^2$$

$$\overline{X}^2 - 2\overline{X}a + a^2 \geq 0$$

$$(\overline{X} - a)^2 \geq 0 \rightarrow \text{True } \forall a \in \mathbb{R}$$

2.1.2:

Prove that

$$(n-1)\frac{S^2}{n} = \overline{X^2} - \overline{X}^2$$

$$\begin{aligned}
\frac{\cancel{n-1}}{n}\frac{1}{\cancel{n-1}}\sum_i (X_i-\overline{X})^2 &= \frac{1}{n}\sum_i X_i^2 - \overline{X}^2 \\
\frac{\cancel{1}}{\cancel{n}}\sum_i (X_i-\overline{X})^2 &= \frac{\cancel{1}}{\cancel{n}}\sum_i X_i^2 - n\overline{X}^2 \\
\cancel{\sum_i X_i^2} - 2\overline{X}\sum_i X_i + \overline{X}^2 &= \cancel{\sum_i X_i^2} - n\overline{X}^2 \\
-2n\overline{X}^2 + n\overline{X}^2 &= -n\overline{X}^2 \\
0 &= 0 \rightarrow \text{True always}
\end{aligned}$$

2.2:

Given $X_1, ..., X_n$ i.i.d. rv s.t. $X_i \sim F$

2.2.1:

Prove that

$$f_{X_{(1)}}(t) = n(1-F(t))^{n-1}f(t)$$

($f_{X_{(1)}}(t)$ is the pdf of the minimum among the X_i)

$$P[X_i = t] = f(t)$$

$$P[X_i > t] = 1 - P[X_i < t] = 1 - F(t)$$

To find the pdf of the minimum, we have to compute the probability of X_i being the smallest (say t), and all the other variabiles being greater than t and sum it for each X_i:

$$f_{X_{(1)}}(t) = \sum_{i=1}^{n}\left[f_{X_i}(t)\prod_{j=1, j\neq i}^{n}(1-F_{X_j}(t))\right]$$

Since the variables are i.i.d., we can simplify the equation:

$$f_{X_{(1)}}(t) = \sum^{n} f(t)\prod^{n-1}(1-F(t)) = nf(t)(1-F(t))^{n-1}$$

2.2.2:

Prove that

$$f_{X_{(n)}}(t) = n(F(t))^{n-1}f(t)$$

($f_{X_{(n)}}(t)$ is the pdf of the maximum among the X_i)

With the same logic as before, we have to find the probability of X_i being t and all other variables being smaller than t and sum it for each X_i:

$$f_{X_{(n)}} = \sum^{n} f(t)F(t)^{n-1} = nf(t)(F(t))^{n-1}$$

2.2.3:

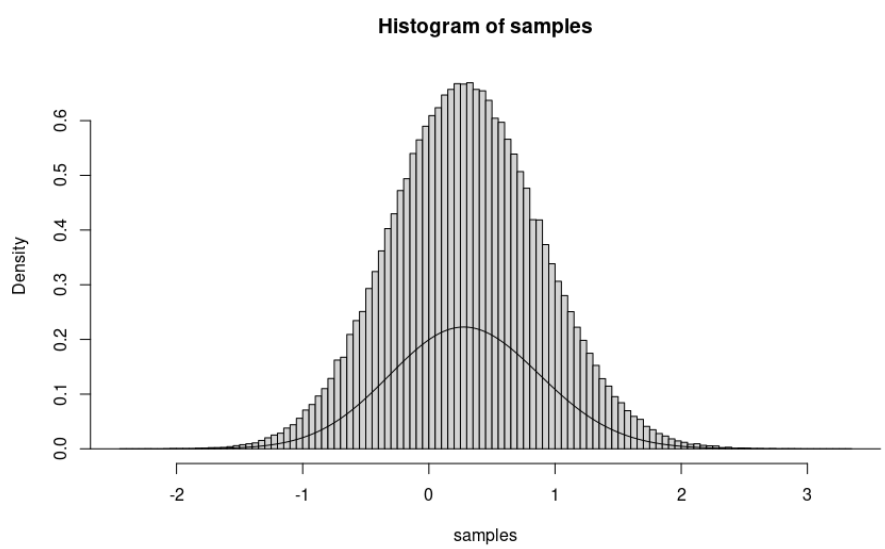
Since the variables are not independent anymore, we cannot simplify the formula seen before:

$$f_{X_{(1)}}(t) = \sum_{i=1}^{n}\left[P[X_i = t|X_k, \forall k \neq i] \prod_{j=1, j\neq i}^{n} (1 - P[X_j < t|X_k, \forall k \neq j])\right]$$

$$f_{X_{(n)}}(t) = \sum_{i=1}^{n}\left[P[X_i = t|X_k, \forall k \neq i] \prod_{j=1, j\neq i}^{n} (P[X_j < t|X_k, \forall k \neq j])\right]$$

2.2.4:

$$F_{X_{(3)}}(t) = 4f(t)(F(t))^2(1 - F(t))$$

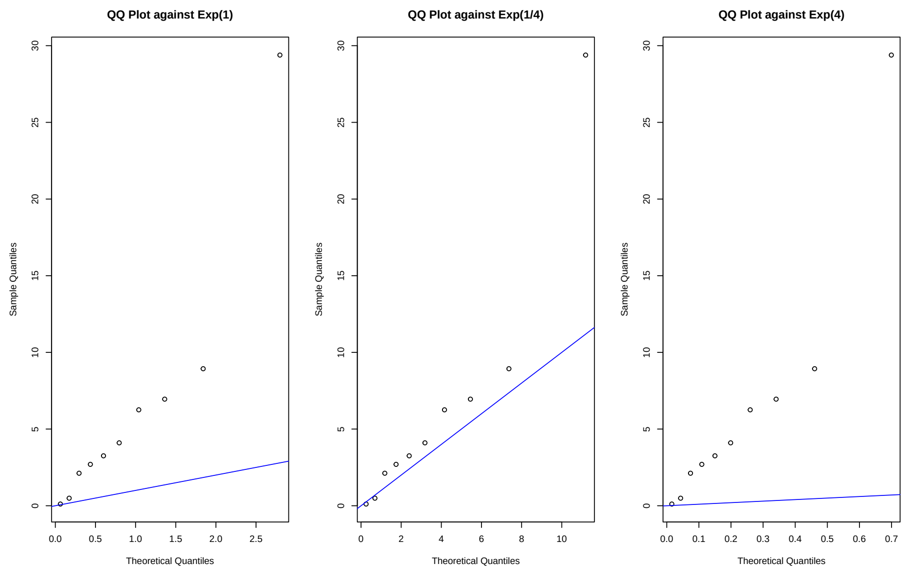


2.3:

For the observed sample:

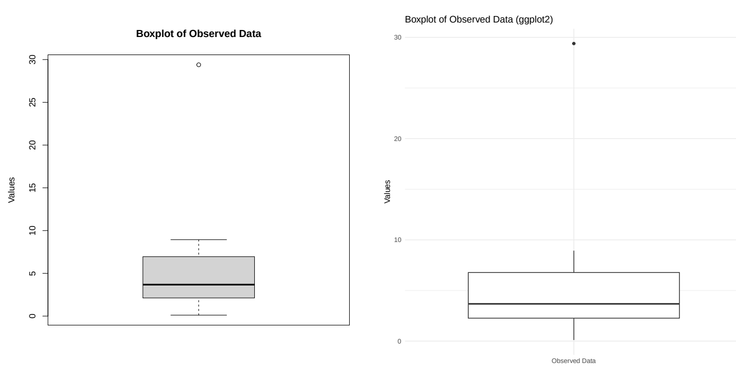
8.935, 0.492, 6.951, 4.102, 0.111, 2.699, 3.255, 6.254, 2.120, 29.389

2.3.1:



The most compatible is the $Exp(1/4)$

2.3.2:



We can't spot any differences and the boxplot tells us that the distribution is "roughly" symmetric, since the whiskers are kinda of the same length.
The quantiles are roughly $Q_1 = 2, Q_2 = 4, Q_3 = 7$.
We can also see only one sample outside the boxplot, wich means that there is only a value X_i s.t. $X_i \geq q_3 + 1.5 \cdot iqr$.

2.3.3:

$$\hat{F_n}(5.25) = \frac{1}{n} \sum_i \mathbf{1}_{X_i}(x) = \frac{6}{10} = 0.6$$

Running the following code in R, we get the same result:

```
data <- c(8.935, 0.492, 6.951, 4.102, 0.111, 2.699, 3.255, 6.254, 2.120, 29.389)
f <- ecdf(data)
f(5.25) # 0.6
```