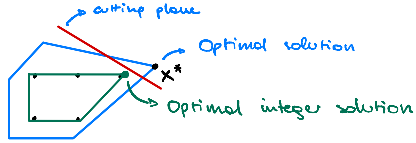
Cutting plane method



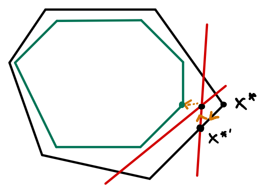
A cutting plane is defined as:

$\alpha^T x \leq \alpha_0$ s.t.

1) $\alpha^T x \leq \alpha_0$, $\forall x \in X$ → feasible integer solution

2) $\alpha^T x^* > \alpha_0$ → x^* is not feasible

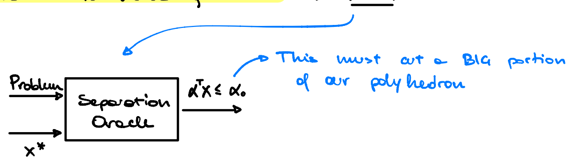
I will need to add the cut as a new constraint



The sequence of cutting planes could take very long

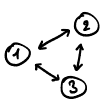
↳ And risk numerical issues, computer-wise

We need to be able to make good cuts ⇒ how?



Valid inequation for the integer points

Es:



Load the maximum number of objects considering:

- if 1 ⇒ no 2
- if 2 ⇒ no 3
- if 3 ⇒ no 1

⇓

Obvious solution : 1

Let's try to make an integer programming problem:

$$\begin{cases} -\min -x_1 - x_2 - x_3 \\ x_1 + x_2 \leq 1 \\ x_2 + x_3 \leq 1 \\ x_1 + x_3 \leq 1 \\ x_1, x_2, x_3 \in [0,1] \ \underline{\text{integer}} \end{cases}$$ → forget the integrality

$\Downarrow$

$x^* = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $c^T x^* = -\frac{3}{2}$ $<$ $c^T x_I^* = -1$ → integer solution

Standard form:

how to find those!

$\frac{1}{2} \cdot (x_1 + x_2 + x_4 = 1) +$

$\frac{1}{2} \cdot (x_2 + x_3 + x_5 = 1) +$

$\frac{1}{2} \cdot (x_1 + x_3 + x_6 = 1) =$

$\downarrow$

$$x_1 + x_2 + x_3 + \frac{1}{2} x_4 + \frac{1}{2} x_5 + \frac{1}{2} x_6 \underset{=}{\leq} \frac{3}{2}$$

reduce till integer

lowers the result

must be integer

$$\underbrace{x_1 + x_2 + x_3}_{} \leq 3/2$$

$$x_1 + x_2 + x_3 \leq 1 \leq 3/2$$

With integer values I cannot have values greater than 1

$$x_1 + x_2 + x_3 \leq 1$$

↳ Valid for the integer points, but violated by x^*

Define $\lfloor z \rfloor = \min v \in \mathbb{Z} : v \leq z$

$$\lfloor v \rfloor = \begin{bmatrix} \lfloor v_1 \rfloor \\ \vdots \\ \lfloor v_n \rfloor \end{bmatrix}$$

Vector

The general procedure is the following:

1) Choose $u \in \mathbb{R}^n$: $u^T A x = u^T b$
2) $\lfloor u^T A \rfloor x \leq u^T b$
3) $\lfloor u^T A \rfloor x \leq \lfloor u^T \rfloor b$ → due to integrality

⇒ Chvatal procedure

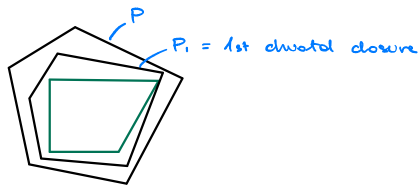
Sufficient condition for validity:

$\alpha^T x \leq \alpha_0$ valid for X if $\exists u \in \mathbb{R}^m$: $\alpha^T = \lfloor u^T A \rfloor$

$\alpha_0 = \lfloor u^T b \rfloor$

By adding all chvatal cuts to the original problem we get P_1 called 1st Chvatal closure

↳ only a finite amount of the useful cuts



By iterating this procedure → i^{th} chvatal closure

↳ We're always getting closer and closer to conv(X)

$P \supsetneq P_1 \supsetneq P_2 \supsetneq \dots \supsetneq P_K = \text{conv}(X)$ ⇒ Theoretical result

↳ Chvatal rank (finite)