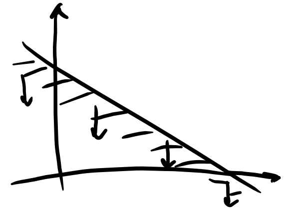
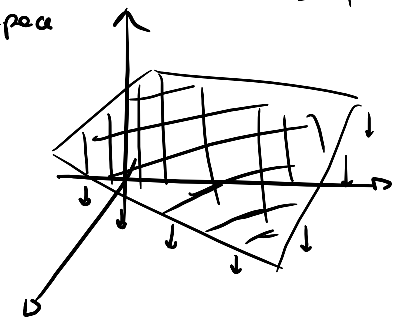
Simplex Method

linear + constant

Def: $\{x \in \mathbb{R}^n : \alpha^T x \le \alpha_0\}$ $\Rightarrow$ Affine half-space



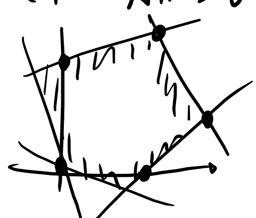
if $\alpha_0 = 0$ it's an half-space



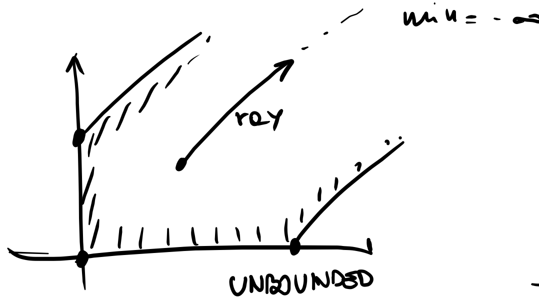
Def: $\{x \in \mathbb{R}^n : \alpha^T x = \alpha_0\}$ $\Rightarrow$ Hyperplane → generalization of a 3D plane

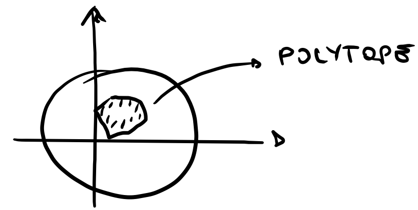
Def: The intersection of a finite number of affine half-spaces and hyperplanes gives us a polyhedron

$$P = \{x \in \mathbb{R}^n : Ax \geq b\}$$



We might find an unbounded polyhedron:





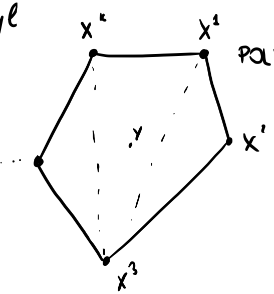
If we consider $z \in$ POLYHEDRON: $\quad z = \lambda x + (1-\lambda) y \Rightarrow$ either $x, y \leq z$, so the opt solution is either x or y, not z.

$\Downarrow$

Every point in the polyhedron can be described as a convex comb. of the vertices

A point $x \in P$ is said to be a <u>vertex</u> of P if it <u>cannot</u> be expressed as the <u>STRICT</u> convex combination of two <u>distinct</u> points $y, z \in P$

Theorem of Minkowski-Weyl



POLYTOPE

bounded

If we have a polytope, we can enumerate the vertices $x^1, \dots, x^k \in P$, and every point y inside the polytope can be obtained as a convex combination of the vertices

$$\forall y \in P, \quad \exists \ \lambda_1, \dots, \lambda_k \geq 0 \ : \quad 1) \ \sum_{i=1}^{k} \lambda_i = 1$$

$$2) \ y = \sum_{i=1}^{k} \lambda_i x^i$$

Th. consider P bounded polyhedron then $\Rightarrow$ an opt. problem $\min_{x \in P} c^T x$ has <u>an</u> opt solution on a vertex of P

→ not strictly the only one

Proof: let $x^1, \dots, x^k$ be the vertices of P. Compute $z^* = \min \{ c^T x^i : i = 1, \dots, k \}$

→ considers only vertices

$\forall y \in P, \quad \exists \lambda \in [0,1]^k \ | \ y = \sum_{i=1}^{k} \lambda_i x^i, \ \sum_{i=1}^{k} x^i = 1$

$$c^T y = c^T \underbrace{\left(\sum_{i=1}^{k} \lambda_i x^i \right)}_{y} = \sum_{i=1}^{k} \underbrace{\lambda_i}_{\geq 0} \underbrace{c^T x^i}_{\geq z^*} \geq \sum_{i=1}^{k} \lambda_i z^* = z^*$$

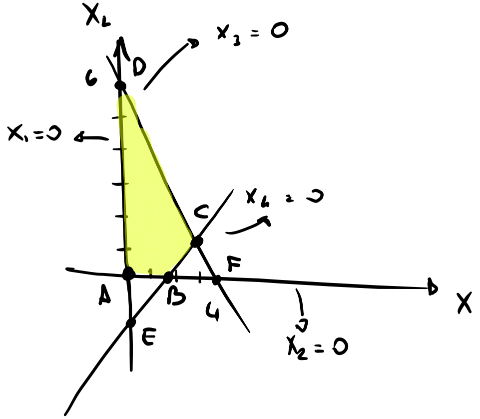
$$\Downarrow$$

$$c^T y \geq z^*$$

$\forall y$, y is not better than z^*

Es:

$$\begin{cases} \min & -x_1 - x_2 \\ & 6x_1 + 4x_2 \leq 24 \quad \Rightarrow \quad 6x_1 + 4x_2 + x_3 = 24 \\ & 3x_1 - 2x_2 \leq 6 \quad \Rightarrow \quad 3x_1 + 2x_2 + x_4 = 6 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$



$A \rightarrow x_1, x_2 = 0$

$B \rightarrow x_2, x_4 = 0$

$C \rightarrow x_3, x_4 = 0$

$D \rightarrow x_1, x_3 = 0$

$E \rightarrow x_1, x_4 = 0$ $\rightarrow x_2 < 0$

$F \rightarrow x_2, x_3 = 0$ $\rightarrow x_4 < 0$

$\Rightarrow$!! not vertices

It may happen to have two parallel costraints $\Rightarrow$ no solutions

$\Rightarrow \infty$ number of solutions $\leftarrow$ overlapping

In general: $Ax = b$ $(x \geq 0)$ A $m \times n$ matrix

Hp: All the rows of A are linear independent $\Leftrightarrow$ rank$(A) = m$

Def: A basis of matrix A is a set of m linear independent columns of A

$$A = \begin{bmatrix} | & & | \\ A_1 & \dots & A_n \\ | & & | \end{bmatrix} \qquad (n > m)$$

$$\Downarrow$$

$$\underset{(m \times m)}{B} = \begin{bmatrix} | & & | \\ A_{[\beta_1]} & \dots & A_{[\beta m]} \\ | & & | \end{bmatrix}, \quad \det(B) \neq 0$$

$\to$ column index of A

$Ax = b$ $\qquad A = [m \times n] \quad x = [n] \quad b = [m]$

$$\begin{bmatrix} | & & | \\ A_1 & \dots & A_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$\in B$ $\quad \in B$ $\quad \notin B$

$$A = [\underbrace{B}_{m \times m} \mid \underbrace{F}_{m \times (n-m)}]$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_B \\ \hline x_F \end{bmatrix}$$

$\to$ (ordering)

$$Ax = b \Rightarrow [B \mid F]\left[\frac{x_B}{x_F}\right] = Bx_B + Fx_F = b$$

for a given B we can always rewrite the equations in this form.

$$Bx_B = b - Fx_F$$

$\boxed{x_B = B^{-1}b - B^{-1}Fx_F} \rightarrow$ Canonical form with respect to the basis B

Then we set to 0 some variables to 0 to find the solution of the equation

$$\begin{cases} x_F = 0 \\ x_B = B^{-1}b \end{cases} \rightarrow \left[\frac{B^{-1}b}{0}\right] = \text{basic solution with respect to } B$$

↳ feasible if $B^{-1}b \geq 0$

I will thus have to compute $\binom{n}{m}$ basis to find all vertices.

↳ huge!