Change of Basis

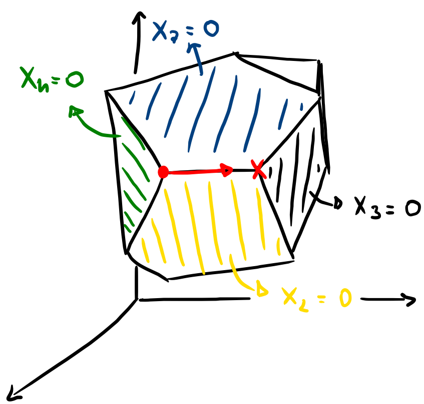
We're in a basis B and the optimality test has failed

Our cost function is $\bar{c}^T = c^T - \underbrace{c_B^T B^{-1}}_{\nu^T} A$

Choose $h \in [1, n]$ | x_h is non-basic : $\bar{c}_h = c_h - \nu^T A_h < 0$

$$c^T x = c_B^T B^{-1} b + \underbrace{\overset{<0}{c_h} x_h + \dots}_{\bar{c}_F^T x_F}$$

Our goal is to increase x_h since $c_h x_h$ would decrease :



following the red arrow will increase the value of x_h since we will stay inside the polyhedron

We also need to know when to stop, or we'll end up in a place with non-feasible solutions

$$x_B = \underbrace{B^{-1}b}_{\bar{b}} - \underbrace{B^{-1}A_h}_{\bar{A}_h} x_h$$

(assume all others $x_F \neq x_h$ stay at 0 ⇒ traveling on the edge between the two vertices)

$$\downarrow$$

$$\begin{bmatrix} x_{\beta[1]} \\ \vdots \\ x_{\beta[m]} \end{bmatrix} = \begin{bmatrix} \bar{b}_1 \\ \vdots \\ \bar{b}_m \end{bmatrix} - \begin{bmatrix} - \; \bar{a}_{1h} \; - \\ \vdots \\ - \; \bar{a}_{mh} \; - \end{bmatrix} x_h$$

(Remember that $\beta[i]$ is the index of the column in A that's placed in B_i)

$$\downarrow$$

$$x_{\beta[i]} = \underbrace{\bar{b}_i - \bar{a}_{ih} x_h}_{\geq 0} \quad \forall i = 1, \dots, m$$

(If $x_{\beta[i]}$ increases too much we end up outside of the polyhedron)

$$\downarrow$$

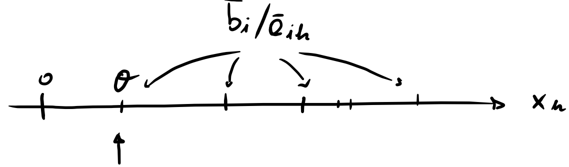
$$\bar{a}_{ih} x_h \leq \bar{b}_i$$

$\bar{b}_i$ → ≥ 0 (since it's part of a bfs $x_B = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix}$

x_h → > 0 (increasing from 0)

- $\bar{a}_{ih} \leq 0$ ⇒ x_h can grow forever → unbounded
- $\bar{a}_{ih} < 0$ ⇒ $x_h \leq \dfrac{\bar{b}_i}{\bar{a}_{ih}}$

Going through all i, we get different x_h values:



we want the smallest one s.t. we won't get other variables negative (exiting the polyhedron)

$$\theta = \min \left\{ \frac{\bar{b}_i}{\bar{a}_{ih}} : \bar{a}_{ih} > 0 \right\}$$

Once x_h has reached θ, there will be another variable reaching 0

$$x_{B[t]} = 0 \quad \text{with } t = \operatorname{argmin} \left\{ \frac{\bar{b}_i}{\bar{a}_{ih}} : \bar{a}_{ih} > 0 \right\}$$

We can now change basis to represent the vertex we've reached:

$$B = \begin{bmatrix} | & & | & & | \\ A_{\beta[1]} & \cdots & A_{\beta[t]} & \cdots & A_{\beta[m]} \\ | & & | & & | \end{bmatrix} \Rightarrow B' = \begin{bmatrix} | & & | & | & & | \\ A_{\beta[1]} & \cdots & \cancel{A_{\beta[t]}} & A_{\beta[h]} & \cdots & A_{\beta[m]} \\ | & & | & | & & | \end{bmatrix}$$

Pseudocode for the simplex method

1) Initialization : find a starting feasible basis $B = [A_{\beta[1]} \cdots A_{\beta[m]}]$

2) Optimality test :

with $\mu^T := c_B^T B^{-1}$

if $\bar{c}^T = c^T - \mu^T A \geq 0$, then we found an optimal solution $x = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix}$

else change basis

3) Change of basis :

choose $\bar{c}_h$ from $\bar{c}^T$ s.t. $\bar{c}_h = c_h - \mu^T A_h < 0$

"x_h wants to enter the basis"

$t := \operatorname{argmin} \left\{ \frac{\bar{b}_i}{\bar{a}_{ih}} : \bar{a}_{ih} > 0 \right\}$, $\bar{b} = B^{-1}b$, $\bar{A}_h = B^{-1}A_h$

$\hookrightarrow$ if $\emptyset \rightarrow$ unbounded problem

"$x_{\beta[t]}$ must leave the basis to let x_h enter"

repeat step 2