

## Change of Basis

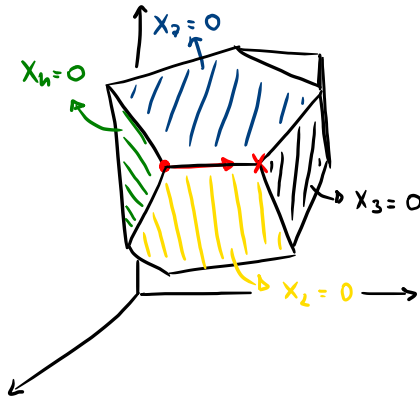
We're in a basis  $B$  and the optimality test has failed

Our cost function is  $\bar{c}^T = c^T - \underbrace{c_B^T B^{-1} A}_{\mu^T}$

Choose  $h \in [1, n]$  |  $x_h$  is non-basic:  $\bar{c}_h = c_h - \mu^T A_h < 0$

$$c^T x = c_B^T B^{-1} b + \underbrace{c_h x_h + \dots}_{\bar{c}_F^T x_F} \quad \nearrow < 0$$

Our goal is to increase  $x_h$  since  $c_h x_h$  would decrease:



following the red arrow will increase the value of  $x_h$  since we will stay inside the polyhedron

We also need to know when to stop, or we'll end up in a place with non-feasible solutions

$$x_B = \underbrace{B^{-1}b}_{\bar{b}} - \underbrace{B^{-1}A_h}_{\bar{A}_h} x_h \quad \left( \text{assume all others } x_F \neq x_h \text{ stay at } 0 \Rightarrow \text{traveling on the edge between the two vertices} \right)$$

$$\begin{bmatrix} x_{p[1]} \\ \vdots \\ x_{p[m]} \end{bmatrix} = \begin{bmatrix} \bar{b}_1 \\ \vdots \\ \bar{b}_m \end{bmatrix} - \begin{bmatrix} -\bar{a}_{1h} \\ \vdots \\ -\bar{a}_{mh} \end{bmatrix} x_h \quad \left( \text{Remember that } p[i] \text{ is the index of the column in } A \text{ that's placed in } B_i \right)$$

$$x_{p[i]} = \underbrace{\bar{b}_i - \bar{a}_{ih} x_h}_{\geq 0} \quad \forall i = 1, \dots, m \quad \left( \text{If } x_{p[i]} \text{ increases too much we end up outside of the polyhedron} \right)$$

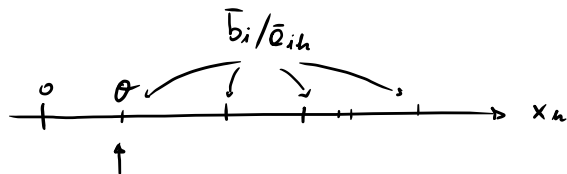
$$\bar{a}_{ih} x_h \leq \bar{b}_i$$

$\hookrightarrow \geq 0$  (since it's part of a bps  $x_B = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix}$ )  
 $\hookrightarrow > 0$  (increasing from 0)

•  $\bar{a}_{ih} \leq 0 \Rightarrow x_h$  can grow forever  $\rightarrow$  unbounded

•  $\bar{a}_{ih} < 0 \Rightarrow x_h \leq \frac{\bar{b}_i}{\bar{a}_{ih}}$

Going through all  $i$ , we get different  $x_h$  values:



we want the smallest one s.t. we won't get other variables negative (exiting the polyhedron)

$$\theta = \min \left\{ \frac{\bar{b}_i}{\bar{a}_{ih}} : \bar{a}_{ih} > 0 \right\}$$

Once  $x_h$  has reached  $\theta$ , there will be another variable reaching 0

$$x_{B[t]} = 0 \quad \text{with } t = \operatorname{argmin} \left\{ \frac{\bar{b}_i}{\bar{a}_{ih}} : \bar{a}_{ih} > 0 \right\}$$

We can now change basis to represent the vertex we've reached:

$$B = \left[ \begin{array}{c|c|c|c} | & & & \\ A_{B[t+1]} & \dots & A_{B[t]} & \dots & A_{B[m]} \\ | & & | & & | \end{array} \right] \Rightarrow B' = \left[ \begin{array}{c|c|c|c} | & & & \\ A_{B[t+1]} & \dots & \cancel{A_{B[t]}} & A_{B[h]} & \dots & A_{B[m]} \\ | & & | & | & & | \end{array} \right]$$

## Pseudocode for the simplex method

1) Initialization: find a starting feasible basis  $B = [A_{B(1)} \dots A_{B(m)}]$

2) Optimality test:

$$\text{with } \mu^T := c_B^T B^{-1}$$

if  $\bar{c}^T = c^T - \mu^T A \geq 0$ , then we found an optimal solution  $x = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix}$

else change basis

3) Change of basis:

choose  $\bar{c}_h$  from  $\bar{c}^T$  s.t.  $\bar{c}_h = c_h - \mu^T A_h < 0$

" $x_h$  wants to enter the basis"

$$t := \operatorname{argmin} \left\{ \frac{\bar{b}_i}{\bar{a}_{ih}} : \bar{a}_{ih} > 0 \right\}, \quad \bar{b} = B^{-1}b, \quad \bar{A}_h = B^{-1}A_h$$

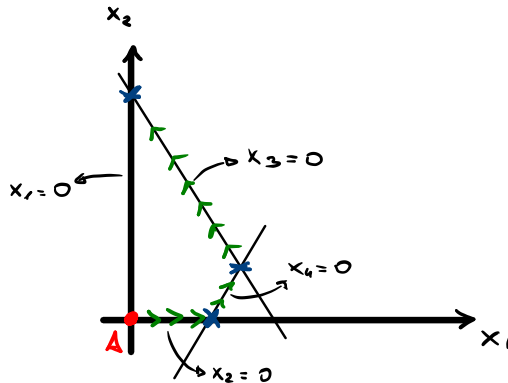
↳ if  $\emptyset \rightarrow$  unbounded problem

" $x_{B(t)}$  must leave the basis to let  $x_h$  enter"

repeat step 2

$E_s$  :

$$\begin{cases} \min & -x_1 - x_2 & := z \\ & 6x_1 + 4x_2 + x_3 & = 24 \\ & 3x_1 - 2x_2 + x_4 & = 6 \\ & x_1, x_2, x_3, x_4 & \geq 0 \end{cases}$$



First iteration: current vertex  $A = (0,0)$ , basic variables  $(x_3, x_4)$

Since the origin is feasible, we can choose it as the first vertex

$$\begin{cases} x_3 = 24 - 6x_1 - 4x_2 \\ x_4 = 6 - 3x_1 + 2x_2 \\ z = -x_1 - x_2 \end{cases}$$

$$X_0 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \quad B^T b = \bar{b} = \begin{bmatrix} 24 \\ 6 \end{bmatrix} \quad B^T F = \begin{bmatrix} 6 & 4 \\ 3 & -2 \end{bmatrix}$$

$$C_0^T B^T b = 0 \quad C_F^T - C_0^T B^T F = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad X_F = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

$$\begin{cases} x_b = B^{-1}b - B^{-1}F x_F \\ C^T x = C_b^T B^{-1}b + (C_F^T - C_b^T B^{-1}F) x_F \end{cases}$$

$$\bar{c}_P = \begin{bmatrix} -1 \\ -1 \end{bmatrix} < 0 \quad \Rightarrow \text{We're not in an optimal state}$$

We can choose between  $h=1$  or  $h=2$  to find a new basis

↓

we choose  $h_1 = "x_1 \text{ enters the basis}"$

We must find  $t = \argmin \left\{ \frac{\bar{b}_i}{\bar{a}_{ih}} : \bar{a}_{ih} > 0 \right\}, h=1$

$$\begin{aligned} \bullet i=1 : \frac{\bar{b}_1}{\bar{a}_{11}} &= 24/6 = 4 \\ \bullet i=2 : \frac{\bar{b}_2}{\bar{a}_{21}} &= 6/3 = 2 \end{aligned} \quad \left\{ \begin{array}{l} t=2 \Rightarrow B[t] = 4 \end{array} \right.$$

" $x_4$  leaves the basis"

Since we chose to increase  $x_1$ ,  $x_2$  is still 0, so now we're in B

Second iteration: current vertex  $B(2,0)$ , basic variables  $(x_1, x_3)$

Instead of computing  $B^{-1}$  with this new basis, we can rewrite our system:

$$\begin{cases} x_3 = 12 - 8x_2 + 2x_4 \\ x_1 = 2 + 2/3x_2 - 1/3x_4 \\ z = -2 - 5/3x_2 + 1/3x_4 \end{cases} \rightarrow \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{bmatrix} = \begin{bmatrix} 8 & -2 \\ -2/3 & 1/3 \end{bmatrix}$$

$$\bar{C}_F^T = \begin{bmatrix} -5/3 \\ 1/3 \end{bmatrix} \begin{matrix} \rightarrow \bar{C}_2 < 0 \\ \rightarrow \bar{C}_4 \end{matrix} \rightarrow "x_2 \text{ enters the basis}"$$

$$l = \operatorname{argmin} \left\{ \frac{\bar{b}_i}{\bar{a}_{ih}} : a_{ih} > 0 \right\} \quad h=2$$

$$l = 1 \Rightarrow \beta[1] = 3 \rightarrow "x_3 \text{ leaves the basis}"$$

$x_4$  stays to 0  $\rightarrow$  we're now at  $C$

Third iteration: current vertex  $C(3, 1.5)$ , basic variables  $(x_1, x_2)$

$$\begin{cases} "x_1 \text{ leaves the basis}" \\ "x_4 \text{ enters the basis}" \end{cases}$$

Fourth iteration: current vertex  $D(0, 6)$ , basic variables  $(x_2, x_4)$

$$\begin{cases} x_2 = 6 - 3/2 x_1 - 1/4 x_3 \\ x_4 = 18 - 6 x_1 - 1/2 x_3 \\ z = -6 + 1/2 x_1 + 1/4 x_3 \end{cases} \rightarrow \bar{C}_F^T = \begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix} > 0$$

$$x_D = \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \xrightarrow{x_2} \\ \xrightarrow{x_4} \end{matrix} \rightsquigarrow \begin{bmatrix} 0 \\ 6 \\ 0 \\ 18 \end{bmatrix} \rightarrow C^T x = -6 \quad \text{optimal solution}$$