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Master Degree in Computer Engineering

Exact and heuristic approaches to the Travelling Salesman Problem (TSP)

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Chapter 1

Introduction

The *Travelling Salesman Problem*, also known as ***TSP***, is one of the most famous and studied optimization problems in the Computer Science and Operations Research fields. Although its first mathematical formulation was proposed in the 19th century by the mathematicians **William Rowan Hamilton** and **Thomas Kirkman**, it received scientific attention from the 1950s onwards.
In 1972, **Richard M. Karp** proved the **NP-hard nature** of the TSP; this meant that the computation time for any solving algorithm can grow exponentially with the input size. Despite this, many different approaches have been developed over the years, yielding both exact and approximate solutions.

In this paper, several algorithms are explained, developed and tested against each other, both exact and approximate.

1.1 Problem formulation

In this thesis, we will consider an **undirected graph** $G = (V, E)$, where V is a set of $|V| = N$ *nodes* (or *vertices*) and E is a set of $|E| = M$ *edges*.
We define a **Hamiltonian cycle** of G, $G^* = (V, E^*)$, as a graph whose edges form a cycle going through each node $v \in V$ exactly once.
We also define a **cost function** for the edges $c : E \to \mathbb{R}^+$, $c_e := c(e)\ \forall\ e \in E$.

The target of the TSP is finding an Hamiltonian cycle of G of minimum total cost, obtained by summing the costs of all edges in the cycle.
We can formulate this problem through *Integer Linear Programming (ILP)*. First, we define the following decision variables to represent whether or not a certain edge is included in the Hamiltonian cycle:

$$x_e = \begin{cases} 1 & \text{if } e \in E^* \\ 0 & \text{otherwise} \end{cases} \qquad \forall\ e \in E$$

The ILP model is the following:

$$
\begin{cases}
\min \sum_{e \in E} c_e x_e & (1.1) \\
\sum_{e \in \delta(h)} x_e = 2 \quad \forall\, h \in V & (1.2) \\
\sum_{e \in \delta(S)} x_e \leq |S| - 1 \quad \forall\, S \subset V : v_1 \in S & (1.3) \\
0 \leq x_e \leq 1 \quad \text{integer} \quad \forall\, e \in E & (1.4)
\end{cases}
$$

Constraints 1.2 impose that every node of the graph must be touched by exactly two edges of the cycle. This group of contraints alone isn't enough to guarantee to find a valid Hamiltonian Cycle: we could find lots of isolated cycles.

Constraints 1.3, called **Subtour Elimination Constraints (SEC)**, guarantee that any solution found through this model is made up of only one connected component: every vertex $v \neq v_1$ must be reachable from v_1.

Despite their importance, their number is exponential in N, thus, considering all of them at once is computationally expensive.

Chapter 2

Project Setup

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2.1 Performance profiler

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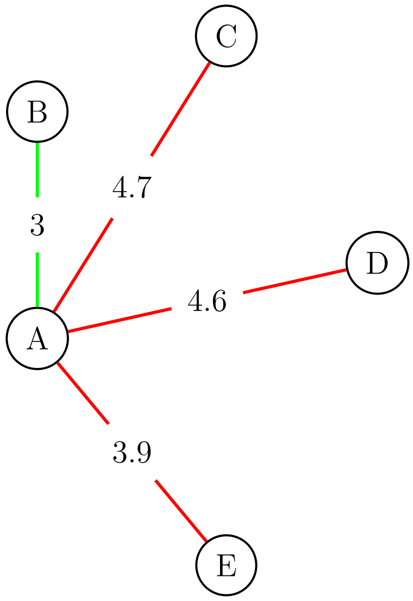
Chapter 3

Heuristics

An heuristic is any approach to problem solving that employs a practical method that's not fully optimized but it's sufficient to reach an immediate short-term goal or approximation.
Given the NP-Hard nature of the Travelling Salesman Problem, finding the optimal solution may require a long time, hence the need to have heuristics method to find solutions that are close to the optimal.

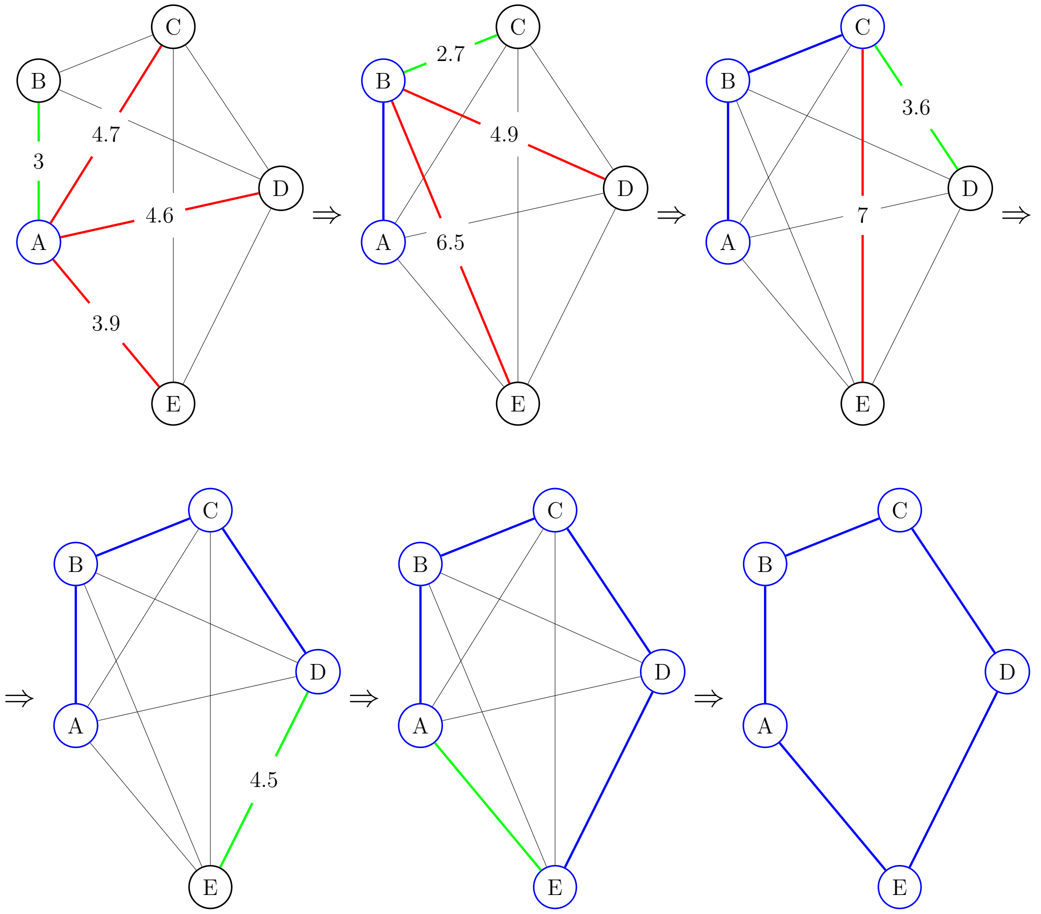
3.1 Nearest Neighbor

A first approach to the TSP, is to consider the edges with the **smallest weights first** when choosing the next node in the cycle.
This type of logic is called **greedy**: a greedy algorithm chooses the locally optimal choice at each stage.



In this example, among the edges connected to the node A, the edge (A, B) is the one with the smallest weight, so node B should be A's successor.

By repeating this process for each new node added to the path and connecting the last node (E) to the starting node (A):



3.1.1 Pseudocode

Algorithm 1 Greedy algorithm for the TSP

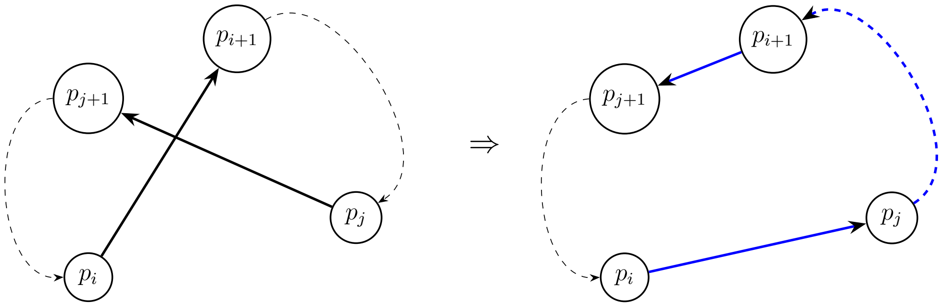
```
    Input Starting node ($s \in V$), Set of nodes ($V$)
    Output List of $n := |V|$ nodes forming an Hamiltonian cycle, Cost of the cycle
cycle $\leftarrow [s]$
cost $\leftarrow 0$
for $i = 0$ to $n - 2$ do
    next $\leftarrow \text{argmin}_v \{c_{cycle[i],v} \mid v \in V \land v \notin \text{cycle}\}$
    cost $\leftarrow$ cost $+ c_{cycle[i],next}$
    cycle$[i+1] \leftarrow$ next
end for
cost $\leftarrow$ cost $+ c_{cycle[n-1],s}$

return cycle, cost
```

The solution found using the greedy algorithm is dependent to the starting node: a possible solution to this is to iterate through all possible starting nodes and keeping track of the best solution found so far.

3.2 2-opt

The 2-opt algorithm is an algorithm that takes an existing cycle and tries to improve it's cost by changing the edges that compose the cycle, without braking it.
The main idea behind the 2-opt algorithm is to find two edges that cross each other and fix them:



$$p_i \coloneqq \text{cycle}[i]$$

This process is then repeated until we can't find any pair of edges that, if rearranged, lead to an improvement to the cost of the cycle.
To find a pair of edges that can be changed to improve the cost we need to find a pair of nodes p_i, p_j, that satisfies the following inequality:

$$c_{p_i,p_{i+1}} + c_{p_j,p_{j+1}} > c_{p_i,p_j} + c_{p_{i+1},p_{j+1}}$$

Note that the cycle list is to be considered as a circular array: situations with indexes out of bounds won't be treated in this paper since the fix is trivial.

If this inequality holds then swapping the edges (p_i, p_j), (p_{i+1}, p_{j+1}) with (p_i, p_{i+1}), (p_j, p_{j+1}) will lower the cost of the cycle:

$$\begin{aligned}\text{cost}(\text{new cycle}) &= \text{cost}(\text{old cycle}) - (c_{p_i,p_{i+1}} + c_{p_j,p_{j+1}}) + (c_{p_i,p_j} + c_{p_{i+1},p_{j+1}}) \\ &\leq \text{cost}(\text{old cycle})\end{aligned}$$

Note that using this inequality to find suitable swaps in the cycle, we will consider also pair of edges that don't intersect, but whose change will improve the cost of the cycle.

After finding the nodes p_i, p_j the edges (p_i, p_j) and (p_{j+1}, p_{j+1}) will take the place of the edges (p_i, p_{i+1}) and (p_j, p_{j+1}), reversing the route connecting $p_{i+1} \to p_j$.
If we had our cycle stored as a list of nodes ordered following the order of the nodes in the cycle, then this step can be done simply by **reversing the list** from the index $i + 1$ to the index j:

$$[..., p_i, p_{i+1}, ..., p_j, p_{j+1}, ...] \Rightarrow [..., p_i, p_j, ...(\text{reversed})..., p_{i+1}, p_{j+1}, ...]$$

3.2.1 Pseudocode

Algorithm 2 2-opt algorithm for the TSP

Input List (cycle) of $n := |V|$ nodes forming an Hamiltonian cycle, Cost (cost) of the cycle

Output List of n nodes forming an Hamiltonian cycle (with 2-opt swaps applied), Cost of the new cycle

```
while *Exists a swap improving the cost* do
    $(i, j) \leftarrow$ find_swap(cycle)
    cost $\leftarrow$ cost $- (c_{p_i,p_{i+1}} + c_{p_j,p_{j+1}}) + (c_{p_i,p_j} + c_{p_{i+1},p_{j+1}})$
    reverse(cycle, $i + 1, j$)
end while

return cycle, cost
```

3.2.2 Swap policy

In this paper two different ways of finding a swap have been confronted:

- Returning the **first swap** found that improves the cost
- Looking among all possible pair of edges and returning the swap that improves the cost the most (the **best swap**)

Using a performance profiler, it's easy to see that the best_swap policy consistently finds better solutions than the first_swap policy:
... ADD A PERFORMANCE PROFILER FIRST_SWAP VS BEST_SWAP

Chapter 4

Metaheuristic

4.1 Tabu search

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4.2 Variable Neighborhood search (VNS)

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Chapter 5

Exact methods

Chapter 6

Bibliography