Problema 1

Parte 1

$$[\hat{n}(\vec{r},t),\hat{n}(\vec{r}',t)]=0$$

$$\begin{aligned}
&\left(\sum_{\sigma}\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma}(\vec{r},t)\right)\left(\sum_{\sigma'}\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma'}(\vec{r}',t)\right)-\left(\sum_{\sigma'}\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma'}(\vec{r}',t)\right)\left(\sum_{\sigma}\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma}(\vec{r},t)\right)=\\
&=\sum_{\sigma,\sigma'}\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma}(\vec{r},t)\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma'}(\vec{r}',t)-\sum_{\sigma,\sigma'}\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma'}(\vec{r}',t)\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma}(\vec{r},t)=\\
&=\sum_{\sigma,\sigma'}\Big(\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma}(\vec{r},t)\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma'}(\vec{r}',t)\\
&\qquad-\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma'}(\vec{r}',t)\\
&\qquad+\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma'}(\vec{r}',t)\\
&\qquad-\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma'}(\vec{r}',t)\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma}(\vec{r},t)\Big)
\end{aligned}$$

$$\left[\hat{\psi}_{\sigma}(\vec{r},t),\hat{\psi}_{\sigma'}(\vec{r}',t)\right]=0 \qquad\qquad \left[\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t),\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\right]=0$$

$$\begin{aligned}
&\sum_{\sigma,\sigma'}\Big(\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma}(\vec{r},t)\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma'}(\vec{r}',t)\\
&\qquad-\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma'}(\vec{r}',t)\\
&\qquad+\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma}(\vec{r},t)\\
&\qquad-\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma'}(\vec{r}',t)\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma}(\vec{r},t)\Big)\\
&=\sum_{\sigma,\sigma'}\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\left[\hat{\psi}_{\sigma}(\vec{r},t),\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\right]\hat{\psi}_{\sigma'}(\vec{r}',t)+\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\left[\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t),\hat{\psi}_{\sigma'}(\vec{r}',t)\right]\hat{\psi}_{\sigma}(\vec{r},t)\\
&=\sum_{\sigma,\sigma'}\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\left[\hat{\psi}_{\sigma}(\vec{r},t),\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\right]\hat{\psi}_{\sigma'}(\vec{r}',t)-\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\left[\hat{\psi}_{\sigma'}(\vec{r}',t),\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\right]\hat{\psi}_{\sigma}(\vec{r},t)\\
&=\sum_{\sigma,\sigma'}\delta_{\sigma\sigma'}\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma'}(\vec{r}',t)-\delta_{\sigma\sigma'}\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma}(\vec{r},t)=\\
&=\sum_{\sigma}\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma}(\vec{r},t)-\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma}(\vec{r},t)=0
\end{aligned}$$

$$\left[\hat{\psi}_{\sigma}(\vec{r},t),\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\right]=\left[\hat{\psi}_{\sigma'}(\vec{r}',t),\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\right]=\delta_{\sigma\sigma'}$$

Parte 2

$$\hat{H}=\hat{T}+\hat{V}$$

$$\hat{T} = \sum_\sigma \int_\Omega d^3r \hat{\psi}^\dagger_\sigma(\vec{r}) \left(-\frac{\hbar^2}{2m}\nabla^2\right) \hat{\psi}_\sigma(\vec{r})$$

$$\hat{V} = \frac{1}{2} \sum_{\sigma,\sigma'} \int_\Omega d^3r d^3r' v(\vec{r}-\vec{r}\,') \hat{\psi}^\dagger_\sigma(\vec{r}) \hat{\psi}^\dagger_{\sigma'}(\vec{r}\,') \hat{\psi}_{\sigma'}(\vec{r}\,') \hat{\psi}_\sigma(\vec{r})$$

$$\begin{aligned}
&\left[\hat{T}, \hat{\psi}^\dagger_\sigma(\vec{r},t)\right] = \\
&= \int_\Omega d^3r \hat{\psi}^\dagger_\sigma(\vec{r}) \left(-\frac{\hbar^2}{2m}\nabla^2\right) \hat{\psi}_\sigma(\vec{r}) = \\
&= \int_\Omega d^3r \hat{\psi}^\dagger_\sigma(\vec{r}) \left(-\frac{\hbar^2}{2m}\nabla\cdot\nabla\right) \hat{\psi}_\sigma(\vec{r})
\end{aligned}$$

$$\int_\Omega u \nabla\cdot\vec{v} = \int_\Gamma u\vec{v}\cdot\hat{n} - \int_\Omega \nabla u\cdot\vec{v}$$

$$\begin{aligned}
\int_\Omega d^3r \hat{\psi}^\dagger_\sigma(\vec{r}) \left(-\frac{\hbar^2}{2m}\nabla\cdot\nabla\right) \hat{\psi}_\sigma(\vec{r}) = \\
\cancel{-\frac{\hbar^2}{2m}\int_\Gamma \hat{\psi}^\dagger_\sigma(\vec{r})\nabla\hat{\psi}_\sigma(\vec{r})} + \frac{\hbar^2}{2m}\int_\Omega d^3r \nabla\hat{\psi}^\dagger_\sigma(\vec{r})\cdot\nabla\hat{\psi}_\sigma(\vec{r})
\end{aligned}$$

$$\begin{aligned}
\left[\sum_{\sigma'} \frac{\hbar^2}{2m}\int_\Omega d^3r' \nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}\,',t)\cdot\nabla\hat{\psi}_{\sigma'}(\vec{r}\,',t), \hat{\psi}^\dagger_\sigma(\vec{r},t)\right] = \\
\frac{\hbar^2}{2m}\sum_{\sigma'}\int_\Omega d^3r' \left[\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}\,',t)\cdot\nabla\hat{\psi}_{\sigma'}(\vec{r}\,',t), \hat{\psi}^\dagger_\sigma(\vec{r},t)\right] =
\end{aligned}$$

$$\begin{aligned}
&\frac{\hbar^2}{2m}\sum_{\sigma'}\int_\Omega d^3r' \left[\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\cdot\nabla\hat{\psi}_{\sigma'}(\vec{r}',t),\hat{\psi}^\dagger_\sigma(\vec{r},t)\right] = \\
&= \frac{\hbar^2}{2m}\sum_{\sigma'}\int_\Omega d^3r'\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\cdot\nabla\hat{\psi}_{\sigma'}(\vec{r}',t)\hat{\psi}^\dagger_\sigma(\vec{r},t) - \hat{\psi}^\dagger_\sigma(\vec{r},t)\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\cdot\nabla\hat{\psi}_{\sigma'}(\vec{r}',t) = \\
&= \frac{\hbar^2}{2m}\sum_{\sigma'}\int_\Omega d^3r'\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\cdot\nabla\hat{\psi}_{\sigma'}(\vec{r}',t)\hat{\psi}^\dagger_\sigma(\vec{r},t) - \nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\cdot\hat{\psi}^\dagger_\sigma(\vec{r},t)\nabla\hat{\psi}_{\sigma'}(\vec{r}',t) \\
&\qquad + \nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\cdot\hat{\psi}^\dagger_\sigma(\vec{r},t)\nabla\hat{\psi}_{\sigma'}(\vec{r}',t) - \hat{\psi}^\dagger_\sigma(\vec{r},t)\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\cdot\nabla\hat{\psi}_{\sigma'}(\vec{r}',t) = \\
&= \frac{\hbar^2}{2m}\sum_{\sigma'}\int_\Omega d^3r'\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\cdot\nabla\left(\hat{\psi}_{\sigma'}(\vec{r}',t)\hat{\psi}^\dagger_\sigma(\vec{r},t) - \hat{\psi}^\dagger_\sigma(\vec{r},t)\hat{\psi}_{\sigma'}(\vec{r}',t)\right) \\
&\qquad + \nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\cdot\hat{\psi}^\dagger_\sigma(\vec{r},t)\nabla\hat{\psi}_{\sigma'}(\vec{r}',t) - \hat{\psi}^\dagger_\sigma(\vec{r},t)\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\cdot\nabla\hat{\psi}_{\sigma'}(\vec{r}',t) = \\
&= \frac{\hbar^2}{2m}\sum_{\sigma'}\int_\Omega d^3r'\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\cdot\nabla\left[\hat{\psi}_{\sigma'}(\vec{r}',t)\hat{\psi}^\dagger_\sigma(\vec{r},t)\right] \\
&\qquad + \nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\hat{\psi}^\dagger_\sigma(\vec{r},t)\cdot\nabla\hat{\psi}_{\sigma'}(\vec{r}',t) - \nabla\left(\hat{\psi}^\dagger_\sigma(\vec{r},t)\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\right)\cdot\nabla\hat{\psi}_{\sigma'}(\vec{r}',t) = \\
&= \frac{\hbar^2}{2m}\sum_{\sigma'}\int_\Omega d^3r'\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\cdot\nabla\left[\hat{\psi}_{\sigma'}(\vec{r}',t),\hat{\psi}^\dagger_\sigma(\vec{r},t)\right] + \nabla\left[\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t),\hat{\psi}^\dagger_\sigma(\vec{r},t)\right]\cdot\nabla\hat{\psi}_{\sigma'}(\vec{r}',t) =
\end{aligned}$$

$$\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\cdot\hat{\psi}^\dagger_\sigma(\vec{r},t)\nabla\hat{\psi}_{\sigma'}(\vec{r}',t) = \nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\hat{\psi}^\dagger_\sigma(\vec{r},t)\cdot\nabla\hat{\psi}_{\sigma'}(\vec{r}',t) = \nabla\left(\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\hat{\psi}^\dagger_\sigma(\vec{r},t)\right)\cdot\nabla\hat{\psi}_{\sigma'}(\vec{r}',t)$$

$$\left[\hat{\psi}_{\sigma'}(\vec{r}',t),\hat{\psi}^\dagger_\sigma(\vec{r},t)\right] = \delta_{\sigma\sigma'}\delta(\vec{r}'-\vec{r}) \qquad\qquad \left[\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t),\hat{\psi}^\dagger_\sigma(\vec{r},t)\right] = 0$$

Uso ancora integrazione per parti (ma al contrario) e trascuro integrale di superficie

$$\begin{aligned}
&\frac{\hbar^2}{2m}\sum_{\sigma'}\int_\Omega d^3r'\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\cdot\nabla\delta_{\sigma\sigma'}\delta(\vec{r}'-\vec{r}) = \\
&= \frac{\hbar^2}{2m}\sum_{\sigma'}\int_\Omega d^3r'\nabla\cdot\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\delta_{\sigma\sigma'}\delta(\vec{r}'-\vec{r}) = \\
&= \frac{\hbar^2}{2m}\sum_{\sigma'}\delta_{\sigma\sigma'}\int_\Omega d^3r'\nabla^2\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\delta(\vec{r}'-\vec{r}) = \\
&= \frac{\hbar^2}{2m}\sum_{\sigma'}\delta_{\sigma\sigma'}\nabla^2\hat{\psi}^\dagger_{\sigma'}(\vec{r},t) = \\
&= \frac{\hbar^2}{2m}\nabla^2\hat{\psi}^\dagger_\sigma(\vec{r},t)
\end{aligned}$$

$$\begin{aligned}
&\left[\frac{1}{2}\sum_{\sigma',\sigma''}\int_\Omega d^3r'd^3r''v(\vec{r}'-\vec{r}'')\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}^\dagger_{\sigma''}(\vec{r}'')\hat{\psi}_{\sigma''}(\vec{r}'')\hat{\psi}_{\sigma'}(\vec{r}'),\hat{\psi}^\dagger_\sigma(\vec{r},t)\right] = \\
&\frac{1}{2}\sum_{\sigma',\sigma''}\int_\Omega d^3r'd^3r''\left[v(\vec{r}'-\vec{r}'')\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}^\dagger_{\sigma''}(\vec{r}'')\hat{\psi}_{\sigma''}(\vec{r}'')\hat{\psi}_{\sigma'}(\vec{r}'),\hat{\psi}^\dagger_\sigma(\vec{r},t)\right]
\end{aligned}$$

$$[AB, C] = A\,[B, C] + [A, C]\,B$$

$$\frac{1}{2}\sum_{\sigma',\sigma''}\int_\Omega d^3r'd^3r''v(\vec{r}'-\vec{r}'')\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}^\dagger_{\sigma''}(\vec{r}'')\left[\hat{\psi}_{\sigma''}(\vec{r}'')\hat{\psi}_{\sigma'}(\vec{r}'), \hat{\psi}^\dagger_\sigma(\vec{r},t)\right]$$
$$+\left[v(\vec{r}'-\vec{r}'')\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}^\dagger_{\sigma''}(\vec{r}''), \hat{\psi}^\dagger_\sigma(\vec{r},t)\right]\hat{\psi}_{\sigma''}(\vec{r}'')\hat{\psi}_{\sigma'}(\vec{r}') =$$
$$= \frac{1}{2}\sum_{\sigma',\sigma''}\int_\Omega d^3r'd^3r''v(\vec{r}'-\vec{r}'')\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}^\dagger_{\sigma''}(\vec{r}'')\left[\hat{\psi}_{\sigma''}(\vec{r}'')\hat{\psi}_{\sigma'}(\vec{r}'), \hat{\psi}^\dagger_\sigma(\vec{r},t)\right]$$
$$+\left[v(\vec{r}'-\vec{r}'')\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}^\dagger_{\sigma''}(\vec{r}''), \hat{\psi}^\dagger_{\sigma'}(\vec{r},t)\right]\hat{\psi}_{\sigma''}(\vec{r}'')\hat{\psi}_{\sigma'}(\vec{r}') =$$

$$= \frac{1}{2}\sum_{\sigma',\sigma''}\int_\Omega d^3r'd^3r''v(\vec{r}'-\vec{r}'')\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}^\dagger_{\sigma''}(\vec{r}'')\left[\hat{\psi}_{\sigma''}(\vec{r}'')\hat{\psi}_{\sigma'}(\vec{r}'), \hat{\psi}^\dagger_\sigma(\vec{r},t)\right]$$
$$+\,v(\vec{r}'-\vec{r}'')\left[\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}^\dagger_{\sigma''}(\vec{r}''), \hat{\psi}^\dagger_\sigma(\vec{r},t)\right]\hat{\psi}_{\sigma''}(\vec{r}'')\hat{\psi}_{\sigma'}(\vec{r}')$$

Applicando nuovamente la formula (...) si nota che dal secondo commutatore si ottengono due commutatori che danno entrambi 0. Applichiamo (...) anche al primo, trovando

$$\frac{1}{2}\sum_{\sigma',\sigma''}\int_\Omega d^3r'd^3r''v(\vec{r}'-\vec{r}'')\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}^\dagger_{\sigma''}(\vec{r}'')\left[\hat{\psi}_{\sigma''}(\vec{r}'')\hat{\psi}_{\sigma'}(\vec{r}'), \hat{\psi}^\dagger_\sigma(\vec{r},t)\right]$$
$$= \frac{1}{2}\sum_{\sigma',\sigma''}\int_\Omega d^3r'd^3r''v(\vec{r}'-\vec{r}'')\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}^\dagger_{\sigma''}(\vec{r}'')\left(\hat{\psi}_{\sigma''}(\vec{r}'')\left[\hat{\psi}_{\sigma'}(\vec{r}'), \hat{\psi}^\dagger_\sigma(\vec{r},t)\right] + \left[\hat{\psi}_{\sigma''}(\vec{r}''), \hat{\psi}^\dagger_\sigma(\vec{r},t)\right]\hat{\psi}_{\sigma'}(\vec{r}')\right)$$
$$= \frac{1}{2}\sum_{\sigma',\sigma''}\int_\Omega d^3r'd^3r''v(\vec{r}'-\vec{r}'')\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}^\dagger_{\sigma''}(\vec{r}'')\left(\hat{\psi}_{\sigma''}(\vec{r}'')\delta_{\sigma'\sigma}\delta(\vec{r}'-\vec{r}) + \delta_{\sigma''\sigma}\delta(\vec{r}''-\vec{r})\hat{\psi}_{\sigma'}(\vec{r}')\right) =$$
$$= \frac{1}{2}\sum_{\sigma',\sigma''}\int_\Omega d^3r'd^3r''v(\vec{r}'-\vec{r}'')\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}^\dagger_{\sigma''}(\vec{r}'')\hat{\psi}_{\sigma''}(\vec{r}'')\delta_{\sigma'\sigma}\delta(\vec{r}'-\vec{r})$$
$$+\frac{1}{2}\sum_{\sigma',\sigma''}\int_\Omega d^3r'd^3r''v(\vec{r}'-\vec{r}'')\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}^\dagger_{\sigma''}(\vec{r}'')\delta_{\sigma''\sigma}\delta(\vec{r}''-\vec{r})\hat{\psi}_{\sigma'}(\vec{r}') =$$
$$= \frac{1}{2}\sum_{\sigma''}\int_\Omega d^3r''v(\vec{r}-\vec{r}'')\hat{\psi}^\dagger_{\sigma}(\vec{r})\hat{\psi}^\dagger_{\sigma''}(\vec{r}'')\hat{\psi}_{\sigma''}(\vec{r}'') + \frac{1}{2}\sum_{\sigma'}\int_\Omega d^3r'v(\vec{r}'-\vec{r})\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}^\dagger_{\sigma}(\vec{r})\hat{\psi}_{\sigma'}(\vec{r}') =$$

$$= \frac{1}{2}\sum_{\sigma'}\int_\Omega d^3r'v(\vec{r}-\vec{r}')\hat{\psi}^\dagger_{\sigma}(\vec{r})\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}_{\sigma'}(\vec{r}') + \frac{1}{2}\sum_{\sigma'}\int_\Omega d^3r'v(\vec{r}'-\vec{r})\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}^\dagger_{\sigma}(\vec{r})\hat{\psi}_{\sigma'}(\vec{r}') =$$

Sfruttiamo $\left[\hat{\psi}^\dagger_\sigma(\vec{r}), \hat{\psi}^\dagger_{\sigma'}(\vec{r}')\right] = 0$ e $v(\vec{r}-\vec{r}') = v(\vec{r}'-\vec{r})$

$$\left[\hat{V}, \hat{\psi}^\dagger_\sigma(\vec{r},t)\right] = \sum_{\sigma'}\int_\Omega d^3r'v(\vec{r}-\vec{r}')\hat{\psi}^\dagger_{\sigma}(\vec{r})\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}_{\sigma'}(\vec{r}')$$

$$\partial_t\hat{\psi}_\sigma(\vec{r},t) = \left(\partial_t\hat{\psi}^\dagger_\sigma(\vec{r},t)\right)^\dagger = -\frac{i}{\hbar}\left[\hat{H}, \hat{\psi}^\dagger_\sigma(\vec{r},t)\right]^\dagger$$

$$-\frac{i}{\hbar}\left[\hat{T},\hat{\psi}^\dagger_\sigma(\vec{r},t)\right]^\dagger = -\frac{i}{\hbar}\left(\frac{\hbar^2}{2m}\nabla^2\hat{\psi}_\sigma(\vec{r},t)\right) = -\frac{i\hbar}{2m}\nabla^2\hat{\psi}_\sigma(\vec{r},t)$$

$$\begin{aligned}
-\frac{i}{\hbar}\left[\hat{V},\hat{\psi}^\dagger_\sigma(\vec{r},t)\right]^\dagger &= \\
-\frac{i}{\hbar}\sum_{\sigma'}\int_\Omega d^3r' v(\vec{r}-\vec{r}')\left(\hat{\psi}^\dagger_\sigma(\vec{r})\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}_{\sigma'}(\vec{r}')\right)^\dagger &= \\
-\frac{i}{\hbar}\sum_{\sigma'}\int_\Omega d^3r' v(\vec{r}-\vec{r}')\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}_{\sigma'}(\vec{r}')\hat{\psi}_\sigma(\vec{r}) &=
\end{aligned}$$

$$\begin{aligned}
\partial_t\hat{n} = \hat{\psi}^\dagger_\sigma(\vec{r},t)\partial_t\hat{\psi}_\sigma(\vec{r},t) + \partial_t\hat{\psi}^\dagger_\sigma(\vec{r},t)\hat{\psi}_\sigma(\vec{r},t) &= \\
\hat{\psi}^\dagger_\sigma(\vec{r},t)\left(-\frac{i\hbar}{2m}\nabla^2\hat{\psi}_\sigma(\vec{r},t)\right) + \left(\frac{i\hbar}{2m}\nabla^2\hat{\psi}^\dagger_\sigma(\vec{r},t)\right)\hat{\psi}_\sigma(\vec{r},t) & \\
-\frac{i}{\hbar}\sum_{\sigma'}\int_\Omega d^3r' v(\vec{r}-\vec{r}')\hat{\psi}^\dagger_\sigma(\vec{r},t)\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}_{\sigma'}(\vec{r}')\hat{\psi}_\sigma(\vec{r}) & \\
+\frac{i}{\hbar}\sum_{\sigma'}\int_\Omega d^3r' v(\vec{r}-\vec{r}')\hat{\psi}^\dagger_\sigma(\vec{r})\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}_{\sigma'}(\vec{r}')\hat{\psi}_\sigma(\vec{r},t) &= \\
= \frac{i\hbar}{2m}\left[\nabla^2\hat{\psi}^\dagger_\sigma(\vec{r},t)\hat{\psi}_\sigma(\vec{r},t) - \hat{\psi}^\dagger_\sigma(\vec{r},t)\nabla^2\hat{\psi}_\sigma(\vec{r},t)\right] &= \\
= \frac{i\hbar}{2m}\left[\nabla^2\hat{\psi}^\dagger_\sigma(\vec{r},t)\hat{\psi}_\sigma(\vec{r},t) - \hat{\psi}^\dagger_\sigma(\vec{r},t)\nabla^2\hat{\psi}_\sigma(\vec{r},t) + \nabla\hat{\psi}^\dagger_\sigma(\vec{r},t)\cdot\nabla\hat{\psi}_\sigma(\vec{r},t) - \nabla\hat{\psi}^\dagger_\sigma(\vec{r},t)\cdot\nabla\hat{\psi}_\sigma(\vec{r},t)\right] &= \\
= \frac{i\hbar}{2m}\nabla\cdot\left(\nabla\hat{\psi}^\dagger_\sigma(\vec{r},t)\hat{\psi}_\sigma(\vec{r},t) - \hat{\psi}^\dagger_\sigma(\vec{r},t)\nabla\hat{\psi}_\sigma(\vec{r},t)\right) & \\
= -\frac{i\hbar}{2im}\nabla\cdot\left(\hat{\psi}^\dagger_\sigma(\vec{r},t)\nabla\hat{\psi}_\sigma(\vec{r},t) - \nabla\hat{\psi}^\dagger_\sigma(\vec{r},t)\hat{\psi}_\sigma(\vec{r},t)\right) &
\end{aligned}$$

Definendo

$$\hat{j} \equiv \frac{\hbar}{2im}\left(\hat{\psi}^\dagger_\sigma(\vec{r},t)\nabla\hat{\psi}_\sigma(\vec{r},t) - \nabla\hat{\psi}^\dagger_\sigma(\vec{r},t)\hat{\psi}_\sigma(\vec{r},t)\right)$$

Troviamo

$$\partial_t\hat{n} = -\nabla\cdot\hat{j}$$

Parte 3
Per $\vec{k} = \vec{k}'$

$$\int_\Omega d^3r \exp(i0\cdot\vec{r}) = \int_\Omega d^3r = \Omega$$

Per $\vec{k} \neq \vec{k}'$

$$\begin{aligned}
&\int_\Omega d^3r \exp\left(i(\vec{k}-\vec{k}')\cdot\vec{r}\right) \\
&= \int_{-L/2}^{L/2}\int_{-L/2}^{L/2}\int_{-L/2}^{L/2} dxdydz \exp\left(i(k_x-k'_x)x + i(k_y-k'_y)y + i(k_z-k'_z)z\right) \\
&= \left(\int_{-L/2}^{L/2} dx \exp\left(i(k_x-k'_x)x\right)\right)^3
\end{aligned}$$

$$\begin{aligned}
\int_{-L/2}^{L/2} dx \exp\left(i(k_x - k'_x)x\right) &= \left[\frac{\exp(i(k_x - k'_x)x)}{i(k_x - k'_x)}\right]_{-L/2}^{L/2} = \\
&= \frac{\exp(i(k_x - k'_x)\frac{L}{2)} - \exp(-i(k_x - k'_x)\frac{L}{2})}{i(k_x - k'_x)}
\end{aligned}$$

$k_x - k'_x = \frac{2\pi}{L}(n_x - n'_x) = \frac{2\pi}{L}\Delta n,\ \Delta n \in \mathbb{Z} - \{0\}$

$$\begin{aligned}
&\frac{\exp(i\frac{2\pi}{L}\Delta n\frac{L}{2}) - \exp(-i\frac{2\pi}{L}\Delta n\frac{L}{2})}{i\frac{2\pi}{L}\Delta n} = \\
&= \frac{L}{\pi\Delta n}\frac{\exp(i\pi\Delta n) - \exp(-i\pi\Delta n)}{2i} = \\
&= L\frac{\sin(\pi\Delta n)}{\pi\Delta n} = 0
\end{aligned}$$

Parte 4

$$\begin{aligned}
&\sum_\sigma \int_\Omega d^3r \hat{\psi}^\dagger_\sigma(\vec{r})\hat{\psi}_\sigma(\vec{r}) = \\
&= \frac{1}{\Omega}\sum_\sigma \int_\Omega d^3r \left(\sum_{\vec{k}'} \exp\left(-i\vec{k}'\cdot\vec{r}\right)\hat{c}^\dagger_{\vec{k}',\sigma}\right)\left(\sum_{\vec{k}} \exp\left(i\vec{k}\cdot\vec{r}\right)\hat{c}_{\vec{k},\sigma}\right) = \\
&= \frac{1}{\Omega}\sum_\sigma \int_\Omega d^3r \sum_{\vec{k},\vec{k}'} \exp\left(i\left(\vec{k}-\vec{k}'\right)\cdot\vec{r}\right)\hat{c}^\dagger_{\vec{k}',\sigma}\hat{c}_{\vec{k},\sigma} = \\
&= \frac{1}{\Omega}\sum_\sigma \sum_{\vec{k},\vec{k}'} \hat{c}^\dagger_{\vec{k}',\sigma}\hat{c}_{\vec{k},\sigma}\int_\Omega d^3r \exp\left(i\left(\vec{k}-\vec{k}'\right)\cdot\vec{r}\right) = \\
&= \frac{1}{\Omega}\sum_\sigma \sum_{\vec{k},\vec{k}'} \hat{c}^\dagger_{\vec{k}',\sigma}\hat{c}_{\vec{k},\sigma}\Omega\delta_{\vec{k},\vec{k}'} = \sum_{\vec{k},\sigma}\hat{c}^\dagger_{\vec{k},\sigma}\hat{c}_{\vec{k},\sigma}
\end{aligned}$$

Parte 5

$$\hat{c}_{\vec{k},\sigma} = \frac{1}{\sqrt{\Omega}}\int_\Omega d^3r \exp(-i\vec{k}\cdot\vec{r})\hat{\psi}_\sigma(\vec{r}) \qquad \hat{c}^\dagger_{\vec{k},\sigma} = \frac{1}{\sqrt{\Omega}}\int_\Omega d^3r \exp(i\vec{k}\cdot\vec{r})\hat{\psi}^\dagger_\sigma(\vec{r})$$

$$\begin{aligned}
&\left[\hat{c}_{\vec{k},\sigma}, \hat{c}^\dagger_{\vec{k},\sigma'}\right] = \\
&= \frac{1}{\Omega}\left[\int_\Omega d^3r \exp(-i\vec{k}\cdot\vec{r})\hat{\psi}_\sigma(\vec{r}), \int_\Omega d^3r' \exp(i\vec{k}'\cdot\vec{r}')\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\right] = \\
&= \frac{1}{\Omega}\int_\Omega d^3r d^3r' \exp\left(i\vec{k}'\cdot\vec{r}' - i\vec{k}\cdot\vec{r}\right)\left[\hat{\psi}_\sigma(\vec{r}), \hat{\psi}^\dagger_{\sigma'}(\vec{r}')\right] = \\
&= \frac{1}{\Omega}\int_\Omega d^3r d^3r' \exp\left(i\vec{k}'\cdot\vec{r}' - i\vec{k}\cdot\vec{r}\right)\delta_{\sigma\sigma'}\delta(\vec{r}-\vec{r}') = \\
&= \frac{\delta_{\sigma\sigma'}}{\Omega}\int_\Omega d^3r d^3r' \exp\left(i\vec{k}'\cdot\vec{r} - i\vec{k}\cdot\vec{r}\right) = \frac{\delta_{\sigma\sigma'}}{\Omega}\Omega\delta_{\vec{k},\vec{k}'} = \delta_{\sigma\sigma'}\delta_{\vec{k},\vec{k}'}
\end{aligned}$$

Parte 6

$$
\begin{aligned}
\langle SF \,|\, SF\rangle &= \left\langle 0 \left| \left(\prod_{\vec{k}\le k_F} \hat{c}^\dagger_{\vec{k},\uparrow} \hat{c}^\dagger_{\vec{k},\downarrow} \right)^\dagger \prod_{\vec{k}\le k_F} \hat{c}^\dagger_{\vec{k},\uparrow} \hat{c}^\dagger_{\vec{k},\downarrow} \right| 0 \right\rangle = \\
&= \left\langle 0 \left| \prod_{\vec{k}\le k_F} \hat{c}_{\vec{k},\downarrow} \hat{c}_{\vec{k},\uparrow} \hat{c}^\dagger_{\vec{k},\uparrow} \hat{c}^\dagger_{\vec{k},\downarrow} \right| 0 \right\rangle =
\end{aligned}
$$

Possiamo usare la formula (...), e siccome $\vec{k} = \vec{k}'$ e $\sigma = \sigma'$ troviamo

$$
\hat{c}_{\vec{k},\uparrow} \hat{c}^\dagger_{\vec{k},\uparrow} = 1 + \hat{c}^\dagger_{\vec{k},\uparrow} \hat{c}_{\vec{k},\uparrow}
$$

Quindi

$$
\left\langle 0 \left| \prod_{\vec{k}\le k_F} \hat{c}_{\vec{k},\downarrow} \hat{c}^\dagger_{\vec{k},\downarrow} + \hat{c}_{\vec{k},\downarrow} \hat{c}^\dagger_{\vec{k},\uparrow} \hat{c}_{\vec{k},\uparrow} \hat{c}^\dagger_{\vec{k},\downarrow} \right| 0 \right\rangle
$$

Dato che nel secondo termine $\hat{c}_{\vec{k},\uparrow}$ viene applicato a $|0\rangle$ prima di $\hat{c}^\dagger_{\vec{k},\uparrow}$, significa che stiamo cercando di distruggere una particella con vettore d'onda $\vec{k}$ e spin $\uparrow$ prima che questo venga creato, quindi questo termina darà un contributo nullo.

$$
\begin{aligned}
&\left\langle 0 \left| \prod_{\vec{k}\le k_F} \hat{c}_{\vec{k},\downarrow} \hat{c}^\dagger_{\vec{k},\downarrow} \right| 0 \right\rangle = \\
&= \left\langle 0 \left| \prod_{\vec{k}\le k_F} 1 + \hat{c}^\dagger_{\vec{k},\downarrow} \hat{c}_{\vec{k},\downarrow} \right| 0 \right\rangle
\end{aligned}
$$

Qui possiamo riutilizzare lo stesso argomento precedente e troviamo

$$
\left\langle 0 \left| \prod_{\vec{k}\le k_F} 1 \right| 0 \right\rangle = \langle 0 \,|\, 0\rangle = 1
$$

Parte 7

$$
\begin{aligned}
\hat{T} &= \sum_\sigma \int_\Omega d^3r \hat{\psi}^\dagger_\sigma(\vec{r}) \left(-\frac{\hbar^2}{2m} \nabla^2 \right) \hat{\psi}_\sigma(\vec{r}) = \\
&= \sum_\sigma \int_\Omega d^3r \left(\sum_{\vec{k}'} \exp\left(-i\vec{k}' \cdot \vec{r} \right) \hat{c}^\dagger_{\vec{k}',\sigma} \right) \left(-\frac{\hbar^2}{2m} \nabla^2 \right) \left(\sum_{\vec{k}} \exp\left(i\vec{k} \cdot \vec{r} \right) \hat{c}_{\vec{k},\sigma} \right) = \\
&= -\frac{\hbar^2}{2m} \sum_\sigma \sum_{\vec{k},\vec{k}'} \hat{c}^\dagger_{\vec{k}',\sigma} \hat{c}_{\vec{k},\sigma} \int_\Omega d^3r \exp\left(-i\vec{k}' \cdot \vec{r} \right) \nabla^2 \exp\left(i\vec{k} \cdot \vec{r} \right) = \\
&= -\frac{\hbar^2}{2m} \sum_\sigma \sum_{\vec{k},\vec{k}'} \hat{c}^\dagger_{\vec{k}',\sigma} \hat{c}_{\vec{k},\sigma} \int_\Omega d^3r \exp\left(-i\vec{k}' \cdot \vec{r} \right) \nabla^2 \exp\left(i\vec{k} \cdot \vec{r} \right)
\end{aligned}
$$

$$\nabla^2 \exp\left(i\vec{k}\cdot\vec{r}\right) = \left((ik_x)^2 \exp\left(i\vec{k}\cdot\vec{r}\right),\, (ik_y)^2 \exp\left(i\vec{k}\cdot\vec{r}\right),\, (ik_y)^2 \exp\left(i\vec{k}\cdot\vec{r}\right)\right) = -\vec{k}^{\,2} \exp\left(i\vec{k}\cdot\vec{r}\right)$$

$$\begin{aligned}
&\frac{\hbar^2}{2m}\sum_{\sigma}\sum_{\vec{k},\vec{k}\,'}\vec{k}^{\,2}\hat{c}^{\dagger}_{\vec{k}\,',\sigma}\hat{c}_{\vec{k},\sigma}\int_{\Omega} d^3r \exp\left(-i\vec{k}\,'\cdot\vec{r}\right)\exp\left(i\vec{k}\cdot\vec{r}\right) \\
&= \frac{\hbar^2}{2m}\sum_{\sigma}\sum_{\vec{k},\vec{k}\,'}\vec{k}^{\,2}\hat{c}^{\dagger}_{\vec{k}\,',\sigma}\hat{c}_{\vec{k},\sigma}\delta_{\vec{k},\vec{k}\,'} = \frac{\hbar^2}{2m}\sum_{\sigma}\sum_{\vec{k}}\vec{k}^{\,2}\hat{c}^{\dagger}_{\vec{k},\sigma}\hat{c}_{\vec{k},\sigma}
\end{aligned}$$

$$\begin{aligned}
\hat{V} = \frac{1}{2}\sum_{\sigma,\sigma'}\int_{\Omega} d^3r d^3r' v(\vec{r}-\vec{r}')\hat{\psi}^{\dagger}_{\sigma}(\vec{r})\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}')\hat{\psi}_{\sigma'}(\vec{r}')\hat{\psi}_{\sigma}(\vec{r}) &= \\
&=
\end{aligned}$$