Problema 1

Parte 1

$$[\hat{n}(\vec{r},t),\hat{n}(\vec{r}',t)] = 0$$

$$\begin{aligned}
&\left(\sum_{\sigma}\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma}(\vec{r},t)\right)\left(\sum_{\sigma'}\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma'}(\vec{r}',t)\right)-\left(\sum_{\sigma'}\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma'}(\vec{r}',t)\right)\left(\sum_{\sigma}\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma}(\vec{r},t)\right)=\\
&=\sum_{\sigma,\sigma'}\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma}(\vec{r},t)\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma'}(\vec{r}',t)-\sum_{\sigma,\sigma'}\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma'}(\vec{r}',t)\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma}(\vec{r},t)=\\
&=\sum_{\sigma,\sigma'}\Big(\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma}(\vec{r},t)\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma'}(\vec{r}',t)\\
&\qquad-\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma'}(\vec{r}',t)\\
&\qquad+\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma'}(\vec{r}',t)\\
&\qquad-\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma'}(\vec{r}',t)\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma}(\vec{r},t)\Big)
\end{aligned}$$

$$\left[\hat{\psi}_{\sigma}(\vec{r},t),\hat{\psi}_{\sigma'}(\vec{r}',t)\right]=0 \qquad\qquad \left[\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t),\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\right]=0$$

$$\begin{aligned}
&\sum_{\sigma,\sigma'}\Big(\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma}(\vec{r},t)\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma'}(\vec{r}',t)\\
&\qquad-\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma'}(\vec{r}',t)\\
&\qquad+\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma}(\vec{r},t)\\
&\qquad-\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma'}(\vec{r}',t)\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma}(\vec{r},t)\Big)\\
&=\sum_{\sigma,\sigma'}\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\left[\hat{\psi}_{\sigma}(\vec{r},t),\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\right]\hat{\psi}_{\sigma'}(\vec{r}',t)+\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\left[\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t),\hat{\psi}_{\sigma'}(\vec{r}',t)\right]\hat{\psi}_{\sigma}(\vec{r},t)\\
&=\sum_{\sigma,\sigma'}\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\left[\hat{\psi}_{\sigma}(\vec{r},t),\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\right]\hat{\psi}_{\sigma'}(\vec{r}',t)-\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\left[\hat{\psi}_{\sigma'}(\vec{r}',t),\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\right]\hat{\psi}_{\sigma}(\vec{r},t)\\
&=\sum_{\sigma,\sigma'}\delta_{\sigma\sigma'}\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma'}(\vec{r}',t)-\delta_{\sigma\sigma'}\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\hat{\psi}_{\sigma}(\vec{r},t)=\\
&=\sum_{\sigma}\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma}(\vec{r},t)-\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\hat{\psi}_{\sigma}(\vec{r},t)=0
\end{aligned}$$

$$\left[\hat{\psi}_{\sigma}(\vec{r},t),\hat{\psi}^{\dagger}_{\sigma'}(\vec{r}',t)\right]=\left[\hat{\psi}_{\sigma'}(\vec{r}',t),\hat{\psi}^{\dagger}_{\sigma}(\vec{r},t)\right]=\delta_{\sigma\sigma'}$$

Parte 2

$$\hat{H}=\hat{T}+\hat{V}$$

$$\hat{T} = \sum_{\sigma} \int_{\Omega} d^3 r \hat{\psi}^{\dagger}_{\sigma}(\vec{r}) \left(-\frac{\hbar^2}{2m} \nabla^2 \right) \hat{\psi}_{\sigma}(\vec{r})$$

$$\hat{V} = \frac{1}{2} \sum_{\sigma,\sigma'} \int_{\Omega} d^3 r d^3 r' v(\vec{r} - \vec{r}\,') \hat{\psi}^{\dagger}_{\sigma}(\vec{r}) \hat{\psi}^{\dagger}_{\sigma'}(\vec{r}\,') \hat{\psi}_{\sigma'}(\vec{r}\,') \hat{\psi}_{\sigma}(\vec{r})$$

Calcoliamo $\partial_t \hat{\psi}^{\dagger}_{\sigma}(\vec{r}, t)$ utilizzando l'eq. di Heisenberg

$$\partial_t \hat{\psi}^{\dagger}_{\sigma}(\vec{r}, t) = \frac{i}{\hbar} \left[\hat{H}, \hat{\psi}^{\dagger}_{\sigma}(\vec{r}, t) \right]$$

Possiamo calcolare il commutatore in due pezzi, dato che

$$\left[\hat{H}, \hat{\psi}^{\dagger}_{\sigma}(\vec{r}, t) \right] = \left[\hat{T}, \hat{\psi}^{\dagger}_{\sigma}(\vec{r}, t) \right] + \left[\hat{V}, \hat{\psi}^{\dagger}_{\sigma}(\vec{r}, t) \right]$$

Per il primo troviamo

$$\begin{aligned} \hat{T} = \int_{\Omega} d^3 r \hat{\psi}^{\dagger}_{\sigma}(\vec{r}) \left(-\frac{\hbar^2}{2m} \nabla^2 \right) \hat{\psi}_{\sigma}(\vec{r}) = \\ = \int_{\Omega} d^3 r \hat{\psi}^{\dagger}_{\sigma}(\vec{r}) \left(-\frac{\hbar^2}{2m} \nabla \cdot \nabla \right) \hat{\psi}_{\sigma}(\vec{r}) \end{aligned}$$

Possiamo utilizzare l'integrazione per parti nel caso a più variabili, che sfrutta il teorema della divergenza:

$$\int_{\Omega} u \nabla \cdot \vec{v} = \int_{\Gamma} u \vec{v} \cdot \hat{n} - \int_{\Omega} \nabla u \cdot \vec{v} \tag{1}$$

Dove Γ è la superficie del volume Ω. Siccome le particelle sono vincolate a rimanere nella scatola, la loro funzione d'onda deve annullarsi sulla superficie della scatola, quindi il contributo dell'integrale di superficie è nullo.

$$\begin{aligned} &\int_{\Omega} d^3 r \hat{\psi}^{\dagger}_{\sigma}(\vec{r}) \left(-\frac{\hbar^2}{2m} \nabla \cdot \nabla \right) \hat{\psi}_{\sigma}(\vec{r}) = \\ &= \cancel{-\frac{\hbar^2}{2m} \int_{\Gamma} \hat{\psi}^{\dagger}_{\sigma}(\vec{r}) \nabla \hat{\psi}_{\sigma}(\vec{r})} + \frac{\hbar^2}{2m} \int_{\Omega} d^3 r \nabla \hat{\psi}^{\dagger}_{\sigma}(\vec{r}) \cdot \nabla \hat{\psi}_{\sigma}(\vec{r}) \end{aligned}$$

$$\begin{aligned} &\left[\hat{T}, \hat{\psi}^{\dagger}_{\sigma}(\vec{r}, t) \right] = \\ &= \left[\sum_{\sigma'} \frac{\hbar^2}{2m} \int_{\Omega} d^3 r' \nabla \hat{\psi}^{\dagger}_{\sigma'}(\vec{r}\,', t) \cdot \nabla \hat{\psi}_{\sigma'}(\vec{r}\,', t), \hat{\psi}^{\dagger}_{\sigma}(\vec{r}, t) \right] = \\ &= \frac{\hbar^2}{2m} \sum_{\sigma'} \int_{\Omega} d^3 r' \left[\nabla \hat{\psi}^{\dagger}_{\sigma'}(\vec{r}\,', t) \cdot \nabla \hat{\psi}_{\sigma'}(\vec{r}\,', t), \hat{\psi}^{\dagger}_{\sigma}(\vec{r}, t) \right] = \\ &= \frac{\hbar^2}{2m} \sum_{\sigma'} \int_{\Omega} d^3 r' \nabla \hat{\psi}^{\dagger}_{\sigma'}(\vec{r}\,', t) \cdot \nabla \hat{\psi}_{\sigma'}(\vec{r}\,', t) \hat{\psi}^{\dagger}_{\sigma}(\vec{r}, t) - \hat{\psi}^{\dagger}_{\sigma}(\vec{r}, t) \nabla \hat{\psi}^{\dagger}_{\sigma'}(\vec{r}\,', t) \cdot \nabla \hat{\psi}_{\sigma'}(\vec{r}\,', t) \end{aligned}$$

Il gradiente agisce solo sulle coordinate $\vec{r}'$, quindi posso portare $\hat{\psi}^\dagger_\sigma(\vec{r},t)$ dentro il gradiente.

$$\begin{aligned}
&\frac{\hbar^2}{2m}\sum_{\sigma'}\int_\Omega d^3r'\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\cdot\nabla\hat{\psi}_{\sigma'}(\vec{r}',t)\hat{\psi}^\dagger_\sigma(\vec{r},t)-\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\cdot\hat{\psi}^\dagger_\sigma(\vec{r},t)\nabla\hat{\psi}_{\sigma'}(\vec{r}',t)\\
&\qquad+\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\cdot\hat{\psi}^\dagger_\sigma(\vec{r},t)\nabla\hat{\psi}_{\sigma'}(\vec{r}',t)-\hat{\psi}^\dagger_\sigma(\vec{r},t)\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\cdot\nabla\hat{\psi}_{\sigma'}(\vec{r}',t)=\\
&=\frac{\hbar^2}{2m}\sum_{\sigma'}\int_\Omega d^3r'\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\cdot\nabla\left(\hat{\psi}_{\sigma'}(\vec{r}',t)\hat{\psi}^\dagger_\sigma(\vec{r},t)-\hat{\psi}^\dagger_\sigma(\vec{r},t)\hat{\psi}_{\sigma'}(\vec{r}',t)\right)\\
&\qquad+\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\cdot\hat{\psi}^\dagger_\sigma(\vec{r},t)\nabla\hat{\psi}_{\sigma'}(\vec{r}',t)-\hat{\psi}^\dagger_\sigma(\vec{r},t)\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\cdot\nabla\hat{\psi}_{\sigma'}(\vec{r}',t)=\\
&=\frac{\hbar^2}{2m}\sum_{\sigma'}\int_\Omega d^3r'\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\cdot\nabla\left[\hat{\psi}_{\sigma'}(\vec{r}',t)\hat{\psi}^\dagger_\sigma(\vec{r},t)\right]\\
&\qquad+\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\hat{\psi}^\dagger_\sigma(\vec{r},t)\cdot\nabla\hat{\psi}_{\sigma'}(\vec{r}',t)-\nabla\left(\hat{\psi}^\dagger_\sigma(\vec{r},t)\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\right)\cdot\nabla\hat{\psi}_{\sigma'}(\vec{r}',t)=\\
&=\frac{\hbar^2}{2m}\sum_{\sigma'}\int_\Omega d^3r'\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\cdot\nabla\left[\hat{\psi}_{\sigma'}(\vec{r}',t),\hat{\psi}^\dagger_\sigma(\vec{r},t)\right]+\nabla\left[\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t),\hat{\psi}^\dagger_\sigma(\vec{r},t)\right]\cdot\nabla\hat{\psi}_{\sigma'}(\vec{r}',t)=
\end{aligned}$$

Nel penultimo passaggio ho sfruttato il fatto che

$$\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\cdot\hat{\psi}^\dagger_\sigma(\vec{r},t)\nabla\hat{\psi}_{\sigma'}(\vec{r}',t)=\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\hat{\psi}^\dagger_\sigma(\vec{r},t)\cdot\nabla\hat{\psi}_{\sigma'}(\vec{r}',t)=\nabla\left(\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\hat{\psi}^\dagger_\sigma(\vec{r},t)\right)\cdot\nabla\hat{\psi}_{\sigma'}(\vec{r}',t)$$

Ora possiamo sfruttare i due commutatori

$$\left[\hat{\psi}_{\sigma'}(\vec{r}',t),\hat{\psi}^\dagger_\sigma(\vec{r},t)\right]=\delta_{\sigma\sigma'}\delta(\vec{r}'-\vec{r})\qquad\qquad\left[\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t),\hat{\psi}^\dagger_\sigma(\vec{r},t)\right]=0\tag{2}$$

$$\begin{aligned}
&\frac{\hbar^2}{2m}\sum_{\sigma'}\int_\Omega d^3r'\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\cdot\nabla\delta_{\sigma\sigma'}\delta(\vec{r}'-\vec{r})=\\
&=\frac{\hbar^2}{2m}\sum_{\sigma'}\int_\Omega d^3r'\nabla\cdot\nabla\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\delta_{\sigma\sigma'}\delta(\vec{r}'-\vec{r})=\\
&=\frac{\hbar^2}{2m}\sum_{\sigma'}\delta_{\sigma\sigma'}\int_\Omega d^3r'\nabla^2\hat{\psi}^\dagger_{\sigma'}(\vec{r}',t)\delta(\vec{r}'-\vec{r})=\\
&=\frac{\hbar^2}{2m}\sum_{\sigma'}\delta_{\sigma\sigma'}\nabla^2\hat{\psi}^\dagger_{\sigma'}(\vec{r},t)=\\
&=\frac{\hbar^2}{2m}\nabla^2\hat{\psi}^\dagger_\sigma(\vec{r},t)
\end{aligned}$$

Nel secondo passaggio si è usata ancora (1) (ma al contrario), trascurando sempre l'integrale di superficie. Per il termine di interazione abbiamo

$$\begin{aligned}
&\left[\hat{V},\hat{\psi}^\dagger_\sigma(\vec{r},t)\right]=\left[\frac{1}{2}\sum_{\sigma',\sigma''}\int_\Omega d^3r'd^3r''v(\vec{r}'-\vec{r}'')\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}^\dagger_{\sigma''}(\vec{r}'')\hat{\psi}_{\sigma''}(\vec{r}'')\hat{\psi}_{\sigma'}(\vec{r}'),\hat{\psi}^\dagger_\sigma(\vec{r},t)\right]=\\
&=\frac{1}{2}\sum_{\sigma',\sigma''}\int_\Omega d^3r'd^3r''\left[v(\vec{r}'-\vec{r}'')\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}^\dagger_{\sigma''}(\vec{r}'')\hat{\psi}_{\sigma''}(\vec{r}'')\hat{\psi}_{\sigma'}(\vec{r}'),\hat{\psi}^\dagger_\sigma(\vec{r},t)\right]
\end{aligned}$$

Grazie all'identità deil commutatori

$$[AB,C]=A\,[B,C]+[A,C]\,B\tag{3}$$

troviamo

$$\frac{1}{2}\sum_{\sigma',\sigma''}\int_\Omega d^3r' d^3r'' v(\vec r' - \vec r'')\hat\psi^\dagger_{\sigma'}(\vec r')\hat\psi^\dagger_{\sigma''}(\vec r'')\left[\hat\psi_{\sigma''}(\vec r'')\hat\psi_{\sigma'}(\vec r'), \hat\psi^\dagger_\sigma(\vec r, t)\right]$$
$$+\left[v(\vec r' - \vec r'')\hat\psi^\dagger_{\sigma'}(\vec r')\hat\psi^\dagger_{\sigma''}(\vec r''), \hat\psi^\dagger_\sigma(\vec r, t)\right]\hat\psi_{\sigma''}(\vec r'')\hat\psi_{\sigma'}(\vec r') =$$
$$= \frac{1}{2}\sum_{\sigma',\sigma''}\int_\Omega d^3r' d^3r'' v(\vec r' - \vec r'')\hat\psi^\dagger_{\sigma'}(\vec r')\hat\psi^\dagger_{\sigma''}(\vec r'')\left[\hat\psi_{\sigma''}(\vec r'')\hat\psi_{\sigma'}(\vec r'), \hat\psi^\dagger_\sigma(\vec r, t)\right]$$
$$+\left[v(\vec r' - \vec r'')\hat\psi^\dagger_{\sigma'}(\vec r')\hat\psi^\dagger_{\sigma''}(\vec r''), \hat\psi^\dagger_{\sigma'}(\vec r, t)\right]\hat\psi_{\sigma''}(\vec r'')\hat\psi_{\sigma'}(\vec r') =$$
$$= \frac{1}{2}\sum_{\sigma',\sigma''}\int_\Omega d^3r' d^3r'' v(\vec r' - \vec r'')\hat\psi^\dagger_{\sigma'}(\vec r')\hat\psi^\dagger_{\sigma''}(\vec r'')\left[\hat\psi_{\sigma''}(\vec r'')\hat\psi_{\sigma'}(\vec r'), \hat\psi^\dagger_\sigma(\vec r, t)\right]$$
$$+ v(\vec r' - \vec r'')\left[\hat\psi^\dagger_{\sigma'}(\vec r')\hat\psi^\dagger_{\sigma''}(\vec r''), \hat\psi^\dagger_\sigma(\vec r, t)\right]\hat\psi_{\sigma''}(\vec r'')\hat\psi_{\sigma'}(\vec r')$$

Applicando nuovamente (3) si nota che dal secondo commutatore si ottengono due commutatori che danno entrambi 0. Applichiamo (3) anche al primo, trovando

$$\frac{1}{2}\sum_{\sigma',\sigma''}\int_\Omega d^3r' d^3r'' v(\vec r' - \vec r'')\hat\psi^\dagger_{\sigma'}(\vec r')\hat\psi^\dagger_{\sigma''}(\vec r'')\left[\hat\psi_{\sigma''}(\vec r'')\hat\psi_{\sigma'}(\vec r'), \hat\psi^\dagger_\sigma(\vec r, t)\right]$$
$$= \frac{1}{2}\sum_{\sigma',\sigma''}\int_\Omega d^3r' d^3r'' v(\vec r' - \vec r'')\hat\psi^\dagger_{\sigma'}(\vec r')\hat\psi^\dagger_{\sigma''}(\vec r'')\left(\hat\psi_{\sigma''}(\vec r'')\left[\hat\psi_{\sigma'}(\vec r'), \hat\psi^\dagger_\sigma(\vec r, t)\right] + \left[\hat\psi_{\sigma''}(\vec r''), \hat\psi^\dagger_\sigma(\vec r, t)\right]\hat\psi_{\sigma'}(\vec r')\right)$$
$$= \frac{1}{2}\sum_{\sigma',\sigma''}\int_\Omega d^3r' d^3r'' v(\vec r' - \vec r'')\hat\psi^\dagger_{\sigma'}(\vec r')\hat\psi^\dagger_{\sigma''}(\vec r'')\left(\hat\psi_{\sigma''}(\vec r'')\delta_{\sigma'\sigma}\delta(\vec r' - \vec r) + \delta_{\sigma''\sigma}\delta(\vec r'' - \vec r)\hat\psi_{\sigma'}(\vec r')\right) =$$
$$= \frac{1}{2}\sum_{\sigma',\sigma''}\int_\Omega d^3r' d^3r'' v(\vec r' - \vec r'')\hat\psi^\dagger_{\sigma'}(\vec r')\hat\psi^\dagger_{\sigma''}(\vec r'')\hat\psi_{\sigma''}(\vec r'')\delta_{\sigma'\sigma}\delta(\vec r' - \vec r)$$
$$+ \frac{1}{2}\sum_{\sigma',\sigma''}\int_\Omega d^3r' d^3r'' v(\vec r' - \vec r'')\hat\psi^\dagger_{\sigma'}(\vec r')\hat\psi^\dagger_{\sigma''}(\vec r'')\delta_{\sigma''\sigma}\delta(\vec r'' - \vec r)\hat\psi_{\sigma'}(\vec r') =$$
$$= \frac{1}{2}\sum_{\sigma''}\int_\Omega d^3r'' v(\vec r - \vec r'')\hat\psi^\dagger_{\sigma}(\vec r)\hat\psi^\dagger_{\sigma''}(\vec r'')\hat\psi_{\sigma''}(\vec r'') + \frac{1}{2}\sum_{\sigma'}\int_\Omega d^3r' v(\vec r' - \vec r)\hat\psi^\dagger_{\sigma'}(\vec r')\hat\psi^\dagger_{\sigma}(\vec r)\hat\psi_{\sigma'}(\vec r') =$$
$$= \frac{1}{2}\sum_{\sigma'}\int_\Omega d^3r' v(\vec r - \vec r')\hat\psi^\dagger_{\sigma}(\vec r)\hat\psi^\dagger_{\sigma'}(\vec r')\hat\psi_{\sigma'}(\vec r') + \frac{1}{2}\sum_{\sigma'}\int_\Omega d^3r' v(\vec r' - \vec r)\hat\psi^\dagger_{\sigma'}(\vec r')\hat\psi^\dagger_{\sigma}(\vec r)\hat\psi_{\sigma'}(\vec r') =$$

Sfruttando $[\hat\psi^\dagger_\sigma(\vec r), \hat\psi^\dagger_{\sigma'}(\vec r')] = 0$ per scambiare i due operatori centrali e $v(\vec r - \vec r') = v(\vec r' - \vec r)$ possiamo sommare i due termini. Infine otteniamo

$$\left[\hat V, \hat\psi^\dagger_\sigma(\vec r, t)\right] = \sum_{\sigma'}\int_\Omega d^3r' v(\vec r - \vec r')\hat\psi^\dagger_{\sigma}(\vec r)\hat\psi^\dagger_{\sigma'}(\vec r')\hat\psi_{\sigma'}(\vec r')$$

Per $\hat\psi_\sigma(\vec r, t)$ troviamo

$$\partial_t\hat\psi_\sigma(\vec r, t) = \left(\partial_t\hat\psi^\dagger_\sigma(\vec r, t)\right)^\dagger = -\frac{i}{\hbar}\left[\hat H, \hat\psi^\dagger_\sigma(\vec r, t)\right]^\dagger$$

$$-\frac{i}{\hbar}\left[\hat T, \hat\psi^\dagger_\sigma(\vec r, t)\right]^\dagger = -\frac{i}{\hbar}\left(\frac{\hbar^2}{2m}\nabla^2\hat\psi_\sigma(\vec r, t)\right) = -\frac{i\hbar}{2m}\nabla^2\hat\psi_\sigma(\vec r, t)$$

$$\begin{aligned}
&-\frac{i}{\hbar}\left[\hat{V},\hat{\psi}^\dagger_\sigma(\vec{r},t)\right]^\dagger = \\
&= -\frac{i}{\hbar}\sum_{\sigma'}\int_\Omega d^3r' v(\vec{r}-\vec{r}')\left(\hat{\psi}^\dagger_\sigma(\vec{r})\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}_{\sigma'}(\vec{r}')\right)^\dagger = \\
&= -\frac{i}{\hbar}\sum_{\sigma'}\int_\Omega d^3r' v(\vec{r}-\vec{r}')\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}_{\sigma'}(\vec{r}')\hat{\psi}_\sigma(\vec{r})
\end{aligned}$$

$$\begin{aligned}
&\partial_t\hat{n} = \hat{\psi}^\dagger_\sigma(\vec{r},t)\partial_t\hat{\psi}_\sigma(\vec{r},t) + \partial_t\hat{\psi}^\dagger_\sigma(\vec{r},t)\hat{\psi}_\sigma(\vec{r},t) = \\
&= \hat{\psi}^\dagger_\sigma(\vec{r},t)\left(-\frac{i\hbar}{2m}\nabla^2\hat{\psi}_\sigma(\vec{r},t)\right) + \left(\frac{i\hbar}{2m}\nabla^2\hat{\psi}^\dagger_\sigma(\vec{r},t)\right)\hat{\psi}_\sigma(\vec{r},t) \\
&= -\frac{i}{\hbar}\sum_{\sigma'}\int_\Omega d^3r' v(\vec{r}-\vec{r}')\hat{\psi}^\dagger_\sigma(\vec{r},t)\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}_{\sigma'}(\vec{r}')\hat{\psi}_\sigma(\vec{r}) \\
&= +\frac{i}{\hbar}\sum_{\sigma'}\int_\Omega d^3r' v(\vec{r}-\vec{r}')\hat{\psi}^\dagger_\sigma(\vec{r})\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}_{\sigma'}(\vec{r}')\hat{\psi}_\sigma(\vec{r},t) = \\
&= \frac{i\hbar}{2m}\left[\nabla^2\hat{\psi}^\dagger_\sigma(\vec{r},t)\hat{\psi}_\sigma(\vec{r},t) - \hat{\psi}^\dagger_\sigma(\vec{r},t)\nabla^2\hat{\psi}_\sigma(\vec{r},t)\right] = \\
&= \frac{i\hbar}{2m}\left[\nabla^2\hat{\psi}^\dagger_\sigma(\vec{r},t)\hat{\psi}_\sigma(\vec{r},t) - \hat{\psi}^\dagger_\sigma(\vec{r},t)\nabla^2\hat{\psi}_\sigma(\vec{r},t) + \nabla\hat{\psi}^\dagger_\sigma(\vec{r},t)\cdot\nabla\hat{\psi}_\sigma(\vec{r},t) - \nabla\hat{\psi}^\dagger_\sigma(\vec{r},t)\cdot\nabla\hat{\psi}_\sigma(\vec{r},t)\right] = \\
&= \frac{i\hbar}{2m}\nabla\cdot\left(\nabla\hat{\psi}^\dagger_\sigma(\vec{r},t)\hat{\psi}_\sigma(\vec{r},t) - \hat{\psi}^\dagger_\sigma(\vec{r},t)\nabla\hat{\psi}_\sigma(\vec{r},t)\right) \\
&= -\frac{i\hbar}{2im}\nabla\cdot\left(\hat{\psi}^\dagger_\sigma(\vec{r},t)\nabla\hat{\psi}_\sigma(\vec{r},t) - \nabla\hat{\psi}^\dagger_\sigma(\vec{r},t)\hat{\psi}_\sigma(\vec{r},t)\right)
\end{aligned}$$

Definendo

$$\hat{j} \equiv \frac{\hbar}{2im}\left(\hat{\psi}^\dagger_\sigma(\vec{r},t)\nabla\hat{\psi}_\sigma(\vec{r},t) - \nabla\hat{\psi}^\dagger_\sigma(\vec{r},t)\hat{\psi}_\sigma(\vec{r},t)\right)$$

Troviamo

$$\partial_t\hat{n} = -\nabla\cdot\hat{j}$$

Parte 3

Per $\vec{k} = \vec{k}'$

$$\int_\Omega d^3r \exp\left(i0\cdot\vec{r}\right) = \int_\Omega d^3r = \Omega$$

Per $\vec{k} \neq \vec{k}'$

$$\begin{aligned}
&\int_\Omega d^3r \exp\left(i(\vec{k}-\vec{k}')\cdot\vec{r}\right) \\
&= \int_{-L/2}^{L/2}\int_{-L/2}^{L/2}\int_{-L/2}^{L/2} dxdydz \exp\left(i(k_x-k'_x)x + i(k_y-k'_y)y + i(k_z-k'_z)z\right) \\
&= \left(\int_{-L/2}^{L/2} dx \exp\left(i(k_x-k'_x)x\right)\right)^3
\end{aligned}$$

$$\begin{aligned}
&\int_{-L/2}^{L/2} dx \exp\left(i(k_x-k'_x)x\right) = \left[\frac{\exp(i(k_x-k'_x)x)}{i(k_x-k'_x)}\right]_{-L/2}^{L/2} = \\
&= \frac{\exp(i(k_x-k'_x)\frac{L}{2)}) - \exp(-i(k_x-k'_x)\frac{L}{2})}{i(k_x-k'_x)}
\end{aligned}$$

$k_x - k'_x = \frac{2\pi}{L}(n_x - n'_x) = \frac{2\pi}{L}\Delta n,\ \Delta n \in \mathbb{Z} - \{0\}$

$$\begin{aligned}
&\frac{\exp(i\frac{2\pi}{L}\Delta n\frac{L}{2}) - \exp(-i\frac{2\pi}{L}\Delta n\frac{L}{2})}{i\frac{2\pi}{L}\Delta n} = \\
&= \frac{L}{\pi\Delta n}\frac{\exp(i\pi\Delta n) - \exp(-i\pi\Delta n)}{2i} = \\
&= L\frac{\sin(\pi\Delta n)}{\pi\Delta n} = 0
\end{aligned}$$

Parte 4

$$\begin{aligned}
&\sum_\sigma \int_\Omega d^3r \hat{\psi}^\dagger_\sigma(\vec{r})\hat{\psi}_\sigma(\vec{r}) = \\
&= \frac{1}{\Omega}\sum_\sigma \int_\Omega d^3r \left(\sum_{\vec{k}'} \exp\left(-i\vec{k}'\cdot\vec{r}\right)\hat{c}^\dagger_{\vec{k}',\sigma}\right)\left(\sum_{\vec{k}} \exp\left(i\vec{k}\cdot\vec{r}\right)\hat{c}_{\vec{k},\sigma}\right) = \\
&= \frac{1}{\Omega}\sum_\sigma \int_\Omega d^3r \sum_{\vec{k},\vec{k}'} \exp\left(i\left(\vec{k}-\vec{k}'\right)\cdot\vec{r}\right)\hat{c}^\dagger_{\vec{k}',\sigma}\hat{c}_{\vec{k},\sigma} = \\
&= \frac{1}{\Omega}\sum_\sigma \sum_{\vec{k},\vec{k}'} \hat{c}^\dagger_{\vec{k}',\sigma}\hat{c}_{\vec{k},\sigma}\int_\Omega d^3r \exp\left(i\left(\vec{k}-\vec{k}'\right)\cdot\vec{r}\right) = \\
&= \frac{1}{\Omega}\sum_\sigma \sum_{\vec{k},\vec{k}'} \hat{c}^\dagger_{\vec{k}',\sigma}\hat{c}_{\vec{k},\sigma}\Omega\delta_{\vec{k},\vec{k}'} = \sum_{\vec{k},\sigma}\hat{c}^\dagger_{\vec{k},\sigma}\hat{c}_{\vec{k},\sigma}
\end{aligned}$$

Parte 5

$$\hat{c}_{\vec{k},\sigma} = \frac{1}{\sqrt{\Omega}}\int_\Omega d^3r \exp(-i\vec{k}\cdot\vec{r})\hat{\psi}_\sigma(\vec{r}) \qquad \hat{c}^\dagger_{\vec{k},\sigma} = \frac{1}{\sqrt{\Omega}}\int_\Omega d^3r \exp(i\vec{k}\cdot\vec{r})\hat{\psi}^\dagger_\sigma(\vec{r})$$

$$\begin{aligned}
&\left[\hat{c}_{\vec{k},\sigma}, \hat{c}^\dagger_{\vec{k},\sigma'}\right] = \\
&= \frac{1}{\Omega}\left[\int_\Omega d^3r \exp(-i\vec{k}\cdot\vec{r})\hat{\psi}_\sigma(\vec{r}), \int_\Omega d^3r' \exp(i\vec{k}'\cdot\vec{r}')\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\right] = \\
&= \frac{1}{\Omega}\int_\Omega d^3r d^3r' \exp\left(i\vec{k}'\cdot\vec{r}' - i\vec{k}\cdot\vec{r}\right)\left[\hat{\psi}_\sigma(\vec{r}), \hat{\psi}^\dagger_{\sigma'}(\vec{r}')\right] = \\
&= \frac{1}{\Omega}\int_\Omega d^3r d^3r' \exp\left(i\vec{k}'\cdot\vec{r}' - i\vec{k}\cdot\vec{r}\right)\delta_{\sigma\sigma'}\delta(\vec{r}-\vec{r}') = \\
&= \frac{\delta_{\sigma\sigma'}}{\Omega}\int_\Omega d^3r d^3r' \exp\left(i\vec{k}'\cdot\vec{r} - i\vec{k}\cdot\vec{r}\right) = \frac{\delta_{\sigma\sigma'}}{\Omega}\Omega\delta_{\vec{k},\vec{k}'} = \delta_{\sigma\sigma'}\delta_{\vec{k},\vec{k}'}
\end{aligned}$$

Parte 6

$$\begin{aligned}\langle SF \,|\, SF\rangle &= \left\langle 0 \left| \left(\prod_{\vec{k}\leq k_F} \hat{c}^\dagger_{\vec{k},\uparrow}\hat{c}^\dagger_{\vec{k},\downarrow} \right)^\dagger \prod_{\vec{k}\leq k_F} \hat{c}^\dagger_{\vec{k},\uparrow}\hat{c}^\dagger_{\vec{k},\downarrow} \right| 0 \right\rangle = \\ &= \left\langle 0 \left| \prod_{\vec{k}\leq k_F} \hat{c}_{\vec{k},\downarrow}\hat{c}_{\vec{k},\uparrow}\hat{c}^\dagger_{\vec{k},\uparrow}\hat{c}^\dagger_{\vec{k},\downarrow} \right| 0 \right\rangle =\end{aligned}$$

Possiamo usare la formula (2), e siccome $\vec{k} = \vec{k}'$ e $\sigma = \sigma'$ troviamo

$$\hat{c}_{\vec{k},\uparrow}\hat{c}^\dagger_{\vec{k},\uparrow} = 1 + \hat{c}^\dagger_{\vec{k},\uparrow}\hat{c}_{\vec{k},\uparrow}$$

Quindi

$$\left\langle 0 \left| \prod_{\vec{k}\leq k_F} \hat{c}_{\vec{k},\downarrow}\hat{c}^\dagger_{\vec{k},\downarrow} + \hat{c}_{\vec{k},\downarrow}\hat{c}^\dagger_{\vec{k},\uparrow}\hat{c}_{\vec{k},\uparrow}\hat{c}^\dagger_{\vec{k},\downarrow} \right| 0 \right\rangle$$

Dato che nel secondo termine $\hat{c}_{\vec{k},\uparrow}$ viene applicato a $|0\rangle$ prima di $\hat{c}^\dagger_{\vec{k},\uparrow}$, significa che stiamo cercando di distruggere una particella con vettore d'onda $\vec{k}$ e spin $\uparrow$ prima che questa venga creata, quindi questo termine applicato a $|0\rangle$ darà 0 e quindi possiamo ignorarlo.

$$\begin{aligned}&\left\langle 0 \left| \prod_{\vec{k}\leq k_F} \hat{c}_{\vec{k},\downarrow}\hat{c}^\dagger_{\vec{k},\downarrow} \right| 0 \right\rangle = \\ &= \left\langle 0 \left| \prod_{\vec{k}\leq k_F} 1 + \hat{c}^\dagger_{\vec{k},\downarrow}\hat{c}_{\vec{k},\downarrow} \right| 0 \right\rangle\end{aligned}$$

Qui possiamo riutilizzare lo stesso argomento precedente e troviamo

$$\left\langle 0 \left| \prod_{\vec{k}\leq k_F} 1 \right| 0 \right\rangle = \langle 0 \,|\, 0\rangle = 1$$

Parte 7

Nei calcoli successivi non è riportato il termine di normalizzazione $1/\sqrt{\Omega}$, ma dato che si semplificherebbero come nell'esercizio di prima possiamo ometterli senza problemi.

$$\begin{aligned}\hat{T} &= \sum_\sigma \int_\Omega d^3r \hat{\psi}^\dagger_\sigma(\vec{r}) \left(-\frac{\hbar^2}{2m}\nabla^2 \right) \hat{\psi}_\sigma(\vec{r}) = \\ &= \sum_\sigma \int_\Omega d^3r \left(\sum_{\vec{k}'} \exp\left(-i\vec{k}'\cdot\vec{r}\right) \hat{c}^\dagger_{\vec{k}',\sigma} \right) \left(-\frac{\hbar^2}{2m}\nabla^2 \right) \left(\sum_{\vec{k}} \exp\left(i\vec{k}\cdot\vec{r}\right) \hat{c}_{\vec{k},\sigma} \right) = \\ &= -\frac{\hbar^2}{2m} \sum_\sigma \sum_{\vec{k},\vec{k}'} \hat{c}^\dagger_{\vec{k}',\sigma}\hat{c}_{\vec{k},\sigma} \int_\Omega d^3r \exp\left(-i\vec{k}'\cdot\vec{r}\right) \nabla^2 \exp\left(i\vec{k}\cdot\vec{r}\right) = \\ &= -\frac{\hbar^2}{2m} \sum_\sigma \sum_{\vec{k},\vec{k}'} \hat{c}^\dagger_{\vec{k}',\sigma}\hat{c}_{\vec{k},\sigma} \int_\Omega d^3r \exp\left(-i\vec{k}'\cdot\vec{r}\right) \nabla^2 \exp\left(i\vec{k}\cdot\vec{r}\right)\end{aligned}$$

Per il laplaciano vale

$$\nabla^2 \exp\left(i\vec{k}\cdot\vec{r}\right) = \left((ik_x)^2 \exp\left(i\vec{k}\cdot\vec{r}\right), (ik_y)^2 \exp\left(i\vec{k}\cdot\vec{r}\right), (ik_y)^2 \exp\left(i\vec{k}\cdot\vec{r}\right)\right) = -\vec{k}^2 \exp\left(i\vec{k}\cdot\vec{r}\right)$$

Quindi troviamo

$$\begin{aligned}
&\frac{\hbar^2}{2m}\sum_\sigma\sum_{\vec{k},\vec{k}'}\vec{k}^2\hat{c}^\dagger_{\vec{k}',\sigma}\hat{c}_{\vec{k},\sigma}\int_\Omega d^3r \exp\left(-i\vec{k}'\cdot\vec{r}\right)\exp\left(i\vec{k}\cdot\vec{r}\right)\\
&= \frac{\hbar^2}{2m}\sum_\sigma\sum_{\vec{k},\vec{k}'}\vec{k}^2\hat{c}^\dagger_{\vec{k}',\sigma}\hat{c}_{\vec{k},\sigma}\delta_{\vec{k},\vec{k}'} = \frac{\hbar^2}{2m}\sum_\sigma\sum_{\vec{k}}\vec{k}^2\hat{c}^\dagger_{\vec{k},\sigma}\hat{c}_{\vec{k},\sigma}
\end{aligned}$$

$$\hat{V} = \frac{1}{2}\sum_{\sigma,\sigma'}\int_\Omega d^3r d^3r' v(\vec{r}-\vec{r}')\hat{\psi}^\dagger_\sigma(\vec{r})\hat{\psi}^\dagger_{\sigma'}(\vec{r}')\hat{\psi}_{\sigma'}(\vec{r}')\hat{\psi}_\sigma(\vec{r}) =$$

$$= \frac{1}{2}\sum_{\sigma,\sigma'}\int_\Omega d^3r d^3r' \sum_{\vec{q}} v_{\vec{q}} \exp\left(i\vec{q}\cdot(\vec{r}-\vec{r}')\right)\left(\sum_{\vec{k}_1}\exp\left(-i\vec{k}_1\cdot\vec{r}\right)\hat{c}^\dagger_{\vec{k}_1,\sigma}\right)\left(\sum_{\vec{k}_2}\exp\left(-i\vec{k}_2\cdot\vec{r}'\right)\hat{c}^\dagger_{\vec{k}_2,\sigma'}\right)$$

$$\cdot\left(\sum_{\vec{k}_3}\exp\left(i\vec{k}_3\cdot\vec{r}'\right)\hat{c}_{\vec{k}_3,\sigma'}\right)\left(\sum_{\vec{k}_4}\exp\left(i\vec{k}_4\cdot\vec{r}\right)\hat{c}_{\vec{k}_4,\sigma}\right) =$$

$$= \frac{1}{2}\sum_{\sigma,\sigma'}\sum_{\vec{k}_{1,2,3,4}} v_{\vec{q}}\,\hat{c}^\dagger_{\vec{k}_1,\sigma}\hat{c}^\dagger_{\vec{k}_2,\sigma'}\hat{c}_{\vec{k}_3,\sigma'}\hat{c}_{\vec{k}_4,\sigma}$$

$$\int_\Omega d^3r d^3r' \exp\left(i\vec{q}\cdot(\vec{r}-\vec{r}')\right)\exp\left(-i\vec{k}_1\cdot\vec{r}\right)\exp\left(-i\vec{k}_2\cdot\vec{r}'\right)\exp\left(i\vec{k}_3\cdot\vec{r}'\right)\exp\left(i\vec{k}_4\cdot\vec{r}\right) =$$

$$= \frac{1}{2}\sum_{\sigma,\sigma'}\sum_{\vec{k}_{1,2,3,4}} v_{\vec{q}}\,\hat{c}^\dagger_{\vec{k}_1,\sigma}\hat{c}^\dagger_{\vec{k}_2,\sigma'}\hat{c}_{\vec{k}_3,\sigma'}\hat{c}_{\vec{k}_4,\sigma}\int_\Omega d^3r d^3r' \exp\left(i\left(\vec{q}-\vec{k}_1+\vec{k}_4\right)\cdot\vec{r}\right)\exp\left(i\left(\vec{k}_3-\vec{k}_2-\vec{q}\right)\cdot\vec{r}'\right) =$$

$$= \frac{1}{2}\sum_{\sigma,\sigma'}\sum_{\vec{k}_{1,2,3,4}} v_{\vec{q}}\,\hat{c}^\dagger_{\vec{k}_1,\sigma}\hat{c}^\dagger_{\vec{k}_2,\sigma'}\hat{c}_{\vec{k}_3,\sigma'}\hat{c}_{\vec{k}_4,\sigma}\int_\Omega d^3r d^3r' \exp\left(i\left(\vec{k}_4-\left(\vec{k}_1-\vec{q}\right)\right)\cdot\vec{r}\right)\exp\left(i\left(\vec{k}_3-\left(\vec{k}_2+\vec{q}\right)\right)\cdot\vec{r}'\right) =$$

$$= \frac{1}{2}\sum_{\sigma,\sigma'}\sum_{\vec{k}_{1,2,3,4}} v_{\vec{q}}\,\hat{c}^\dagger_{\vec{k}_1,\sigma}\hat{c}^\dagger_{\vec{k}_2,\sigma'}\hat{c}_{\vec{k}_3,\sigma'}\hat{c}_{\vec{k}_4,\sigma}\delta_{\vec{k}_4,\vec{k}_1-\vec{q}}\delta_{\vec{k}_3,\vec{k}_2+\vec{q}} =$$

$$= \frac{1}{2}\sum_{\sigma,\sigma'}\sum_{\vec{k}_{1,2}} v_{\vec{q}}\,\hat{c}^\dagger_{\vec{k}_1,\sigma}\hat{c}^\dagger_{\vec{k}_2,\sigma'}\hat{c}_{\vec{k}_2+\vec{q},\sigma'}\hat{c}_{\vec{k}_1-\vec{q},\sigma} = \frac{1}{2}\sum_{\sigma,\sigma'}\sum_{\vec{k},\vec{k}'} v_{\vec{q}}\,\hat{c}^\dagger_{\vec{k},\sigma}\hat{c}^\dagger_{\vec{k}'\sigma'}\hat{c}_{\vec{k}'+\vec{q},\sigma'}\hat{c}_{\vec{k}-\vec{q},\sigma}$$

Infine l'Hamiltoniana risulta

$$\hat{H} = \frac{\hbar^2}{2m}\sum_\sigma\sum_{\vec{k}}\vec{k}^2\hat{c}^\dagger_{\vec{k},\sigma}\hat{c}_{\vec{k},\sigma} + \frac{1}{2}\sum_{\sigma,\sigma'}\sum_{\vec{k},\vec{k}'} v_{\vec{q}}\,\hat{c}^\dagger_{\vec{k},\sigma}\hat{c}^\dagger_{\vec{k}'\sigma'}\hat{c}_{\vec{k}'+\vec{q},\sigma'}\hat{c}_{\vec{k}-\vec{q},\sigma}$$