

COMMONWEALTH OF AUSTRALIA

Copyright Regulations 1969

WARNING

This material has been reproduced and communicated to you by or on behalf of the University of Sydney pursuant to Part VB of the Copyright Act 1968 (**the Act**). The material in this communication may be subject to copyright under the Act. Any further copying or communication of this material by you may be the subject of copyright protection under the Act.

Do not remove this notice.

COMP2123

Data structures and Algorithms

Lecture 11: Divide and Conquer

[GT 11 and 9]

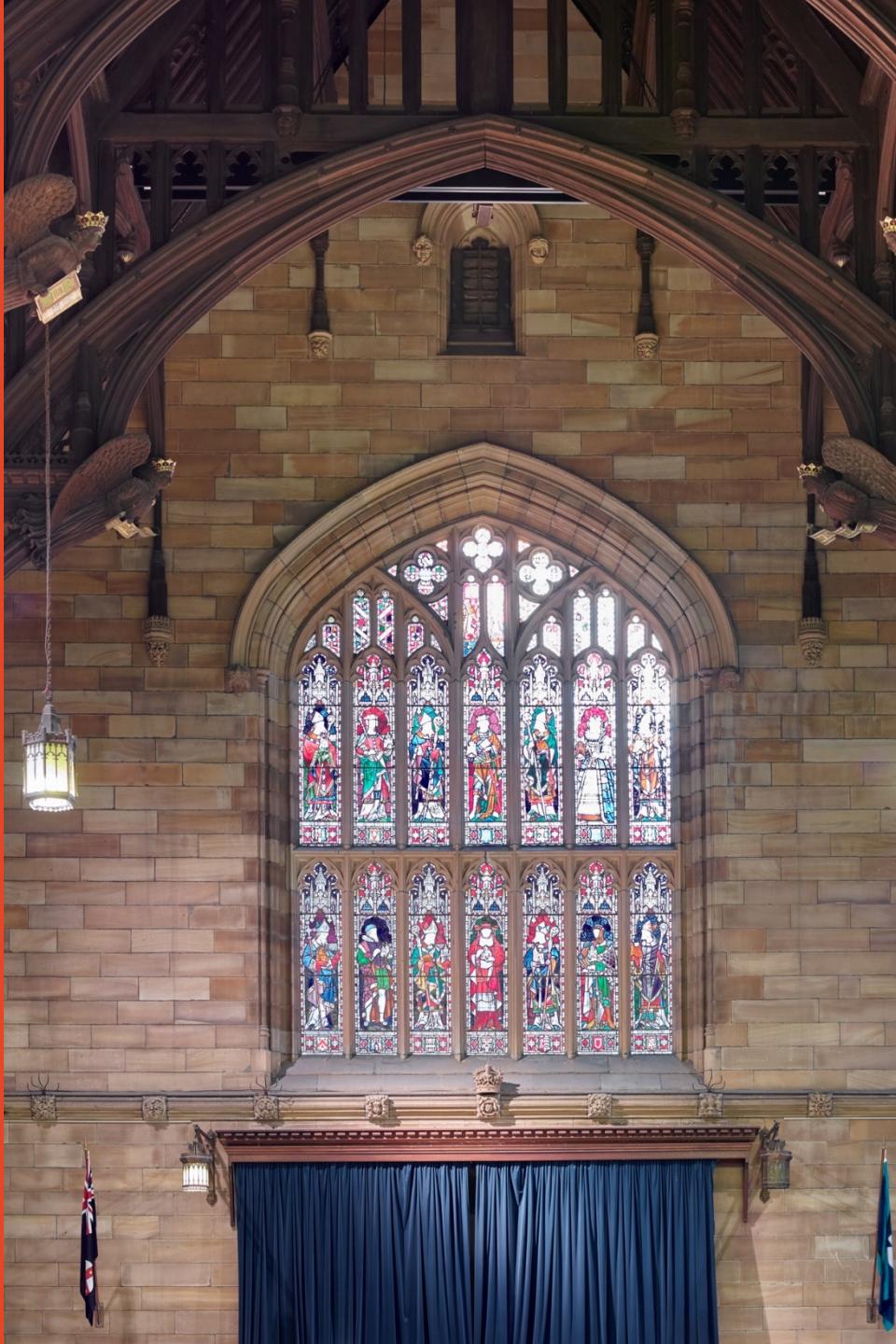
André van Renssen

School of Computer Science

*Some content is taken from material
provided by the textbook publisher Wiley.*



THE UNIVERSITY OF
SYDNEY



Divide and Conquer

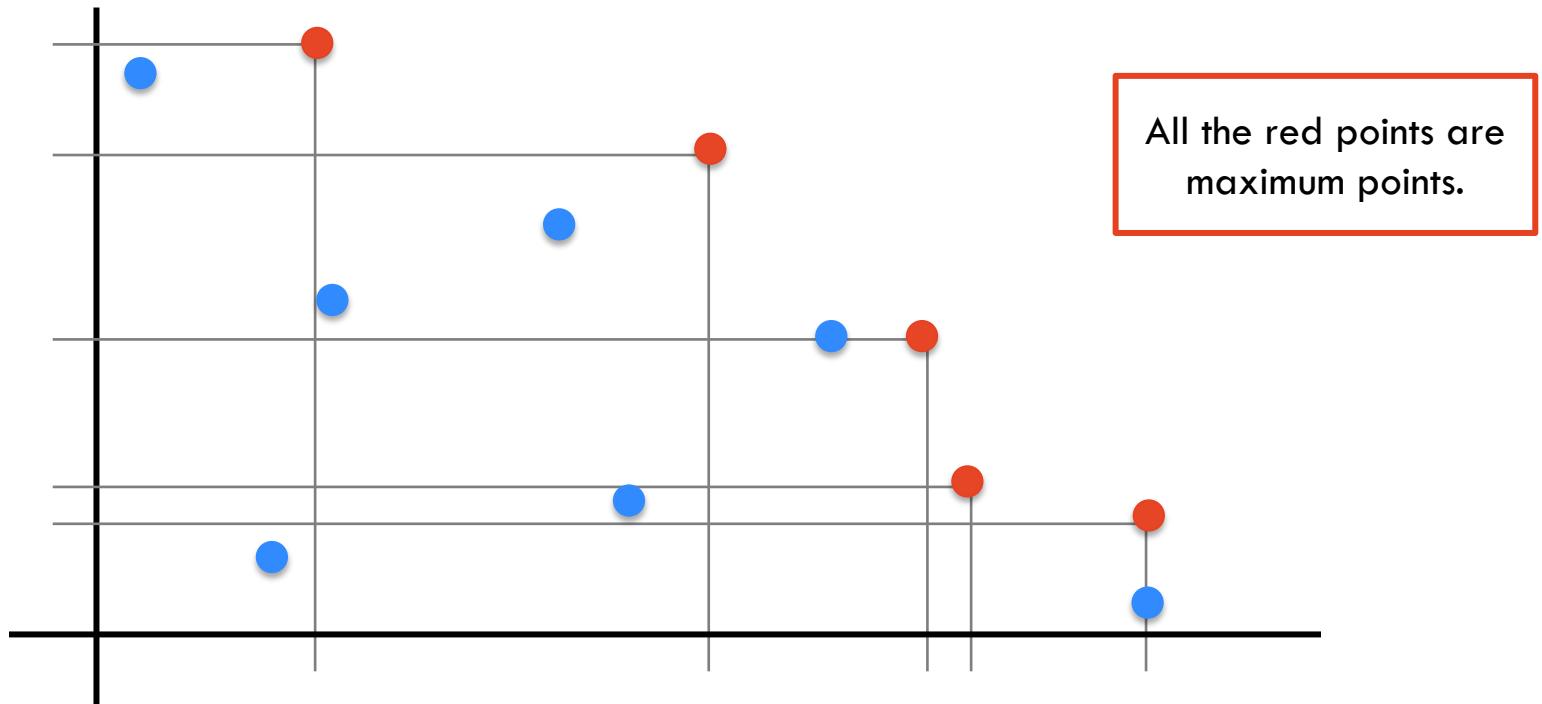
Divide and Conquer algorithms can normally be broken into these three parts:

1. **Divide** If it is a base case, solve directly, otherwise break up the problem into several parts.
2. **Recur/Delegate** Recursively solve each part [each sub-problem].
3. **Conquer** Combine the solutions of each part into the overall solution.

Maxima-Set (Pareto frontier)

Definition A point is maximum in a set if all other points in the set have either a smaller x- or smaller y-coordinate.

Problem Given a set S of n distinct points in the plane (2D), find the set of all maximum points.



Maxima-Set: Naïve Solution

Idea: Check every point (one at a time) to see if it is a maximum point in the set S .

To check if point p is a maximum point in S :

```
for q in S do
    if q ≠ p and q.x ≥ p.x and q.y ≥ p.y then
        return "No"
return "Yes"
```



There is a point q that dominates p

Maxima-Set: Naïve Solution

Idea: Check every point (one at a time) to see if it is a maximum point in the set S.

To check if point p is a maximum point in S:

```
for q in S do
    if q ≠ p and q.x ≥ p.x and q.y ≥ p.y then
        return "No"
return "Yes"
```

Naïve algorithm to find the maxima-set of S:

```
maximaSet ← empty list
for p in S do
    if p is a maximum point in S then
        add p to the maximaSet
return maximaSet
```

Maxima-Set: Naïve Solution

Idea: Check every point (one at a time) to see if it is a maximum point in the set S .

To check if point p is a maximum point in S :

```
for q in S do
    if q ≠ p and q.x ≥ p.x and q.y ≥ p.y then
        return "No"
return "Yes"
```



$O(n)$

Naïve algorithm to find the maxima-set of S :

```
maximaSet ← empty list
for p in S do
    if p is a maximum point in S then
        add p to the maximaSet
return maximaSet
```

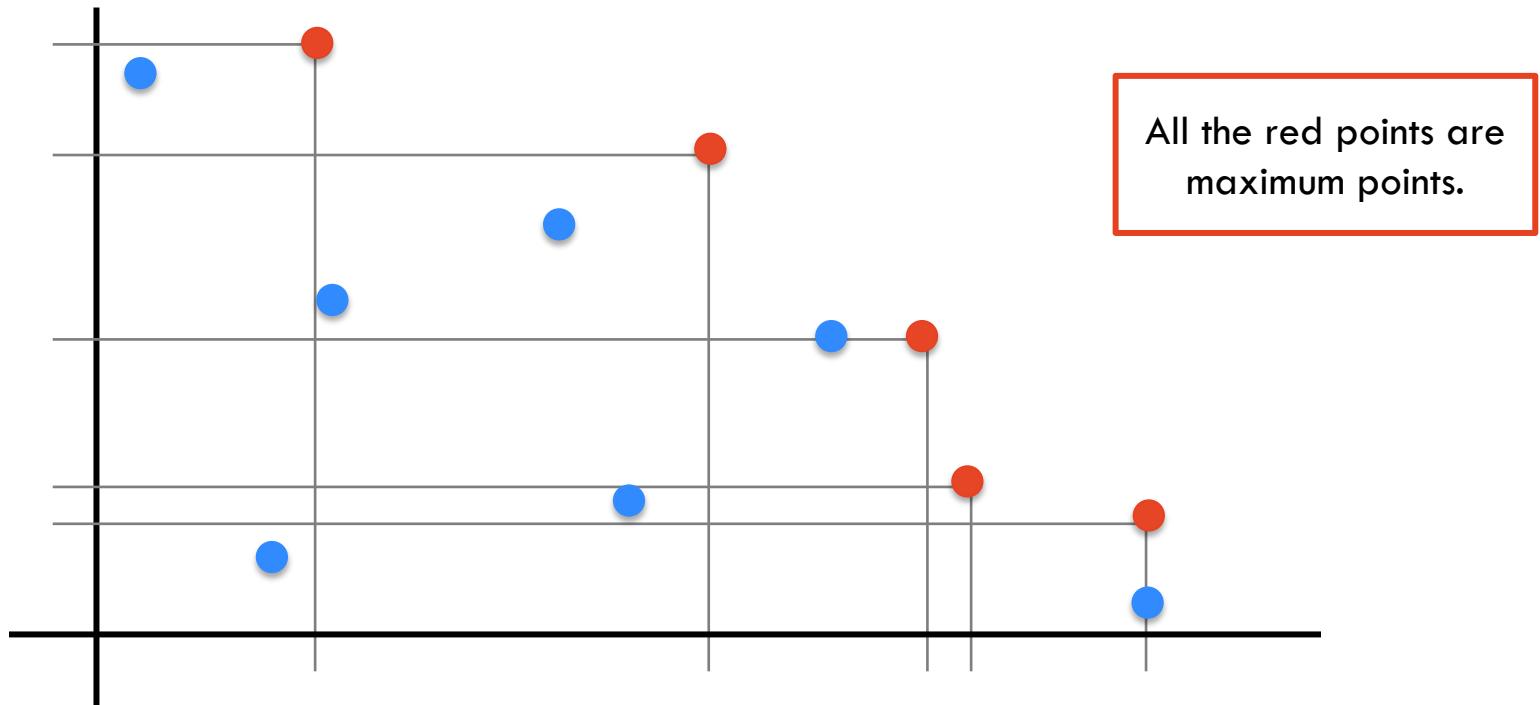


$O(n^2)$

Maxima-Set

Definition A point is maximum in a set if all other points in the set have either a smaller x or smaller y coordinate.

Problem Given a set S of n distinct points in the plane (2D), find the set of all maximum points.



Maxima-Set

Preprocessing Sort the points by increasing x coordinate and store them in an array. Note: we only do this once. Break ties in x by sorting by increasing y coordinate.

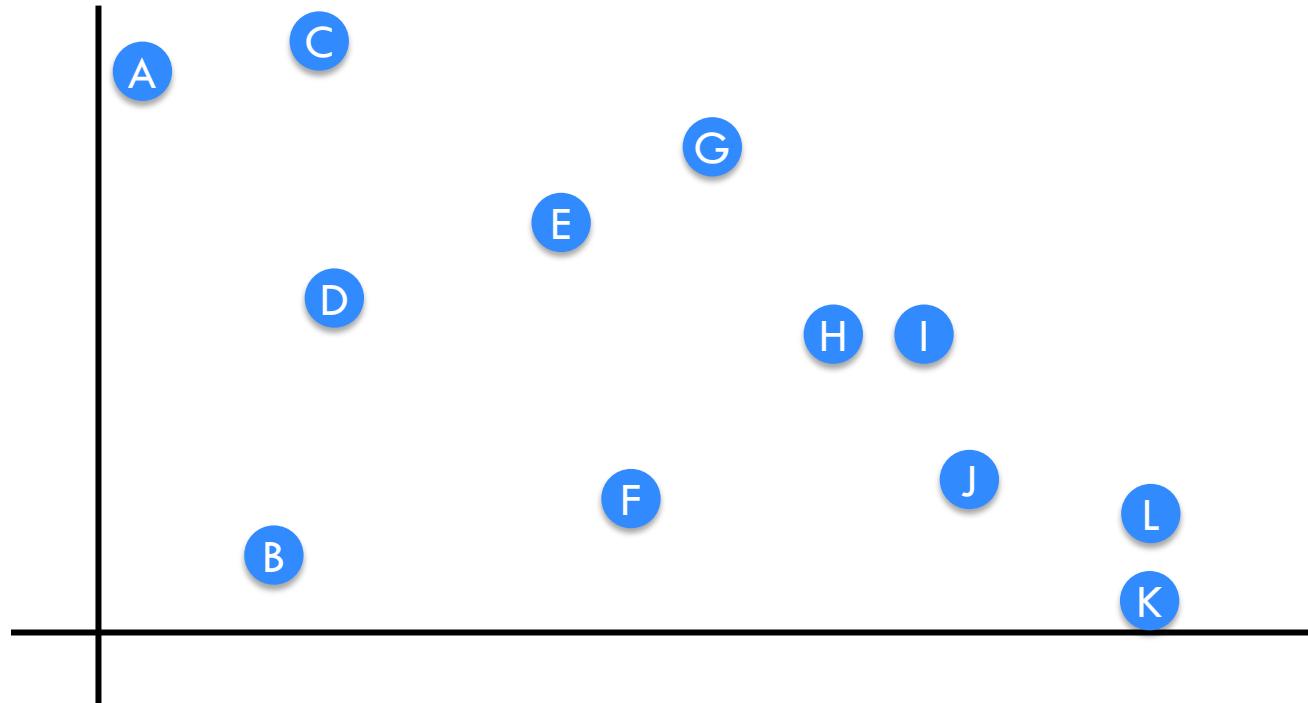
Divide sorted array into two halves.

Recur recursively find the MS of each half.

Conquer compute the MS of the union of Left and Right MS

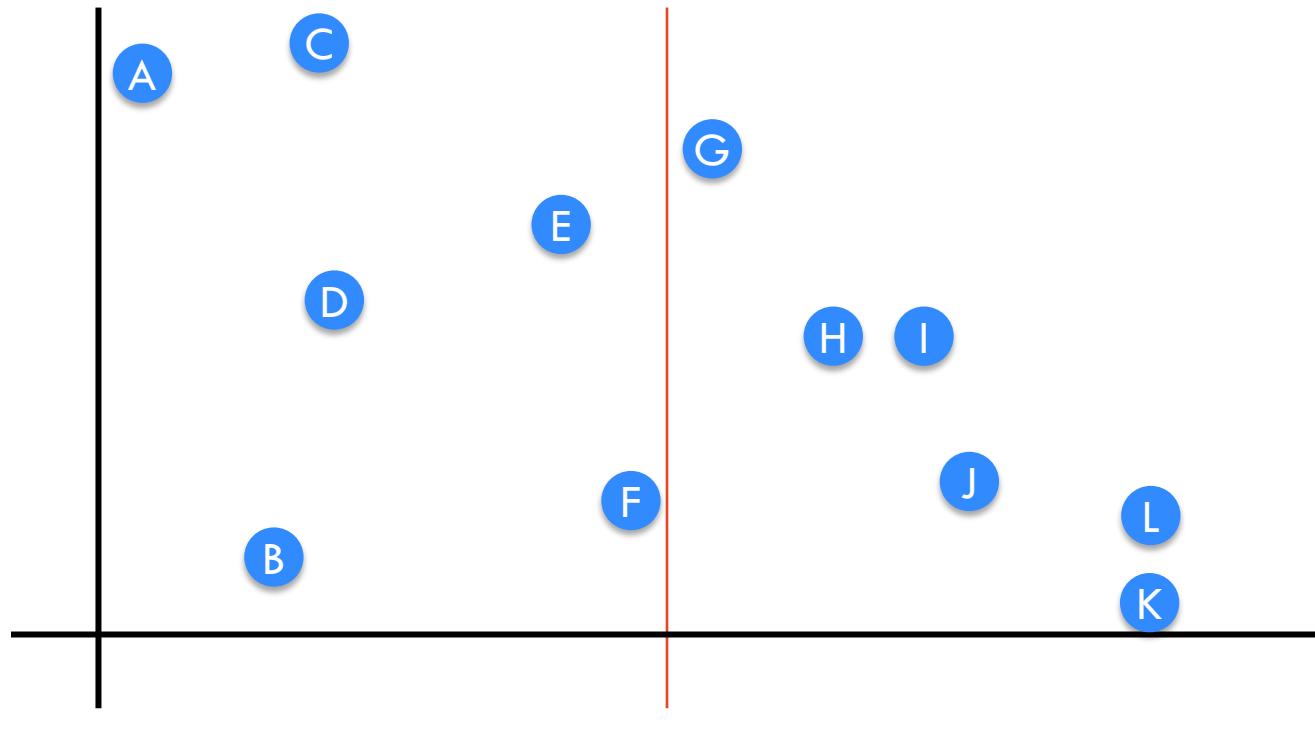
Maxima-Set

Preprocessing Sort the points by increasing x coordinate and store them in an array. Note: we only do this once.
Break ties in x by sorting by increasing y coordinate.



Maxima-Set

Divide array into two halves.

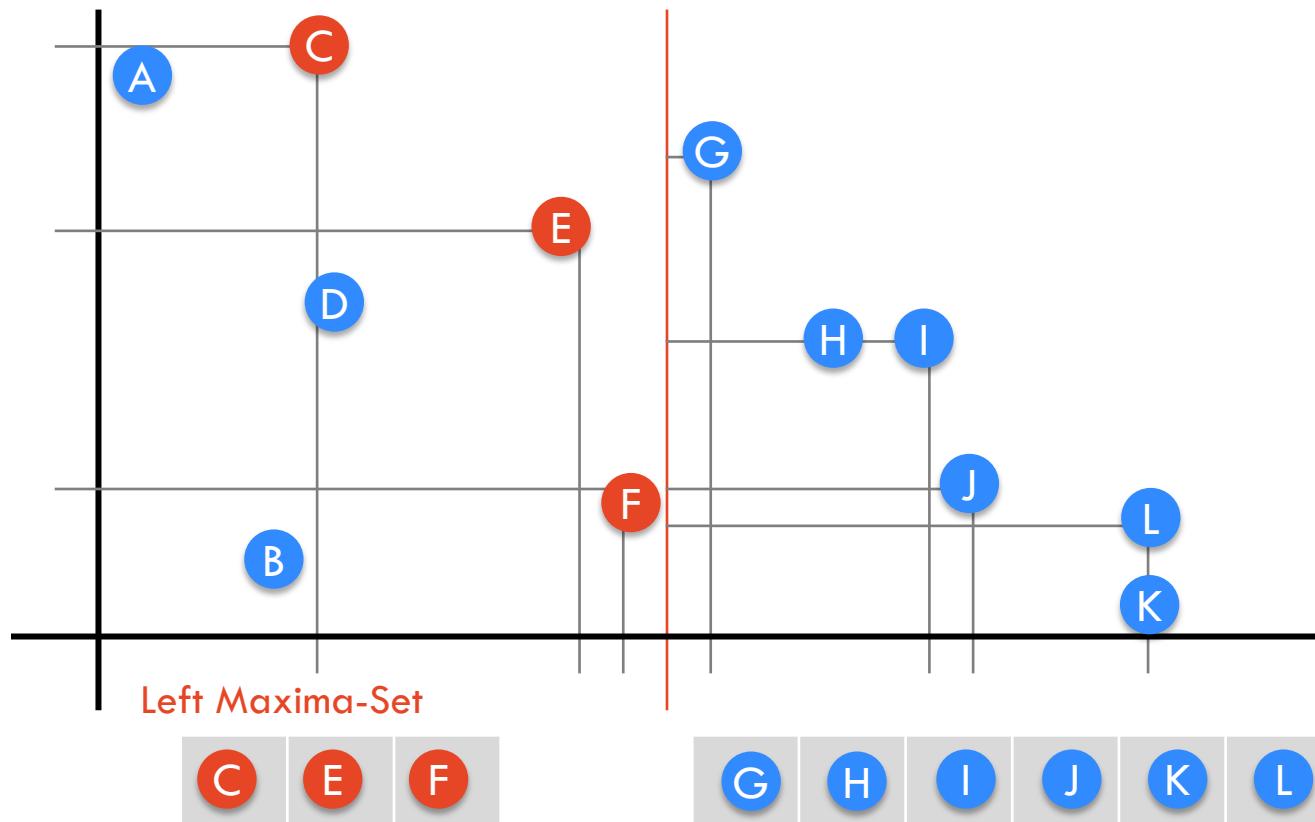


Sorted Points



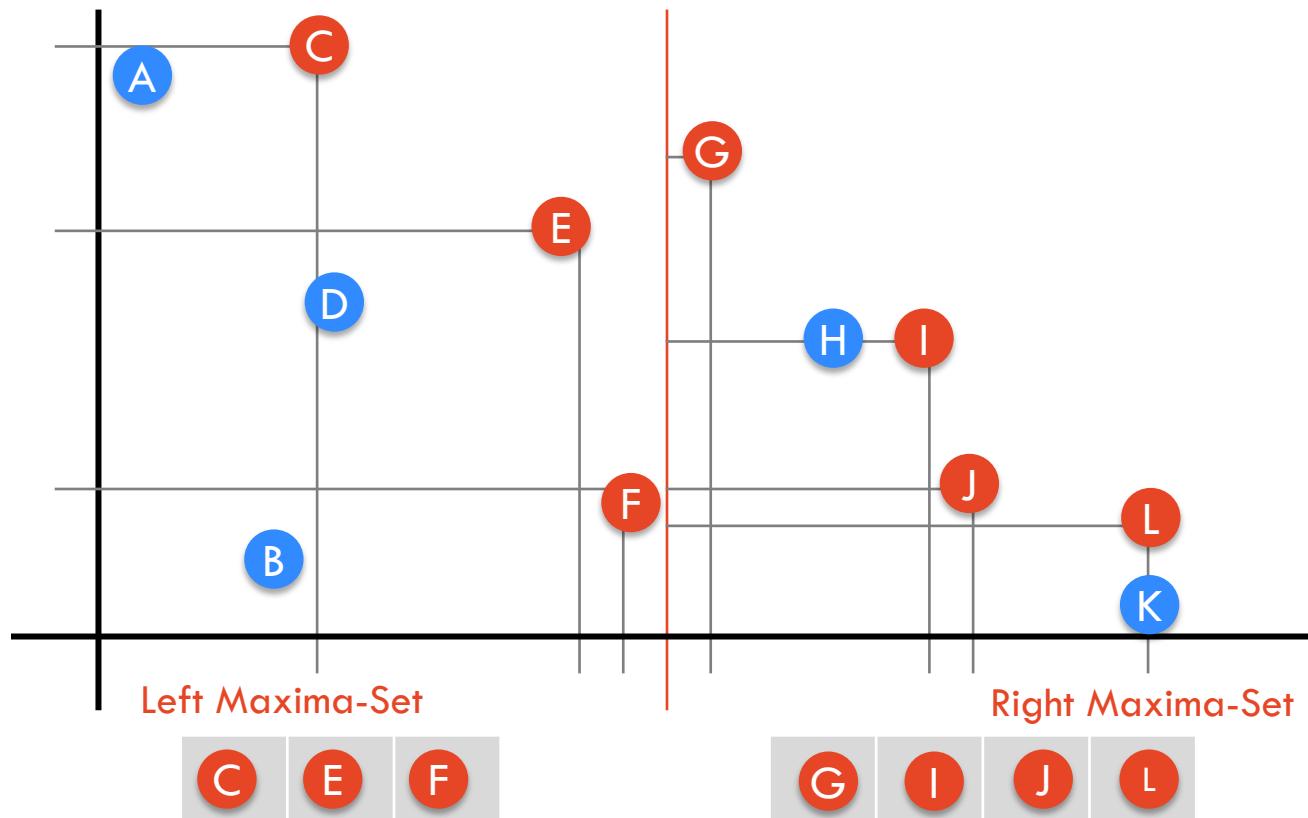
Maxima-Set

Recur recursively find the Maxima-Set of each half.



Maxima-Set

Recur recursively find the Maxima-Set of each half.



Maxima-Set

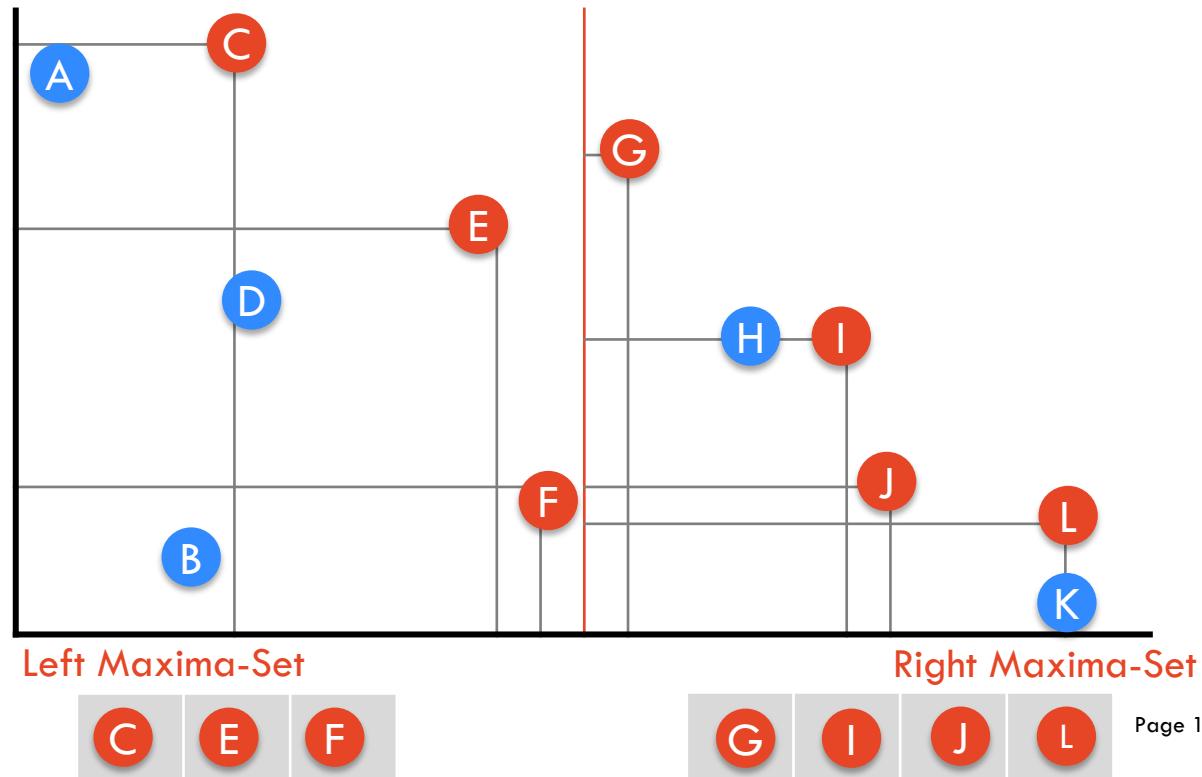
Conquer

1. Find the highest point p in the Right MS

$$p = G$$

Observations:

1. Every point in MS of the whole is in Left MS or Right MS
2. Every point in Right MS is in MS of the whole
3. Every point in Left MS is either in MS of the whole or is dominated by p



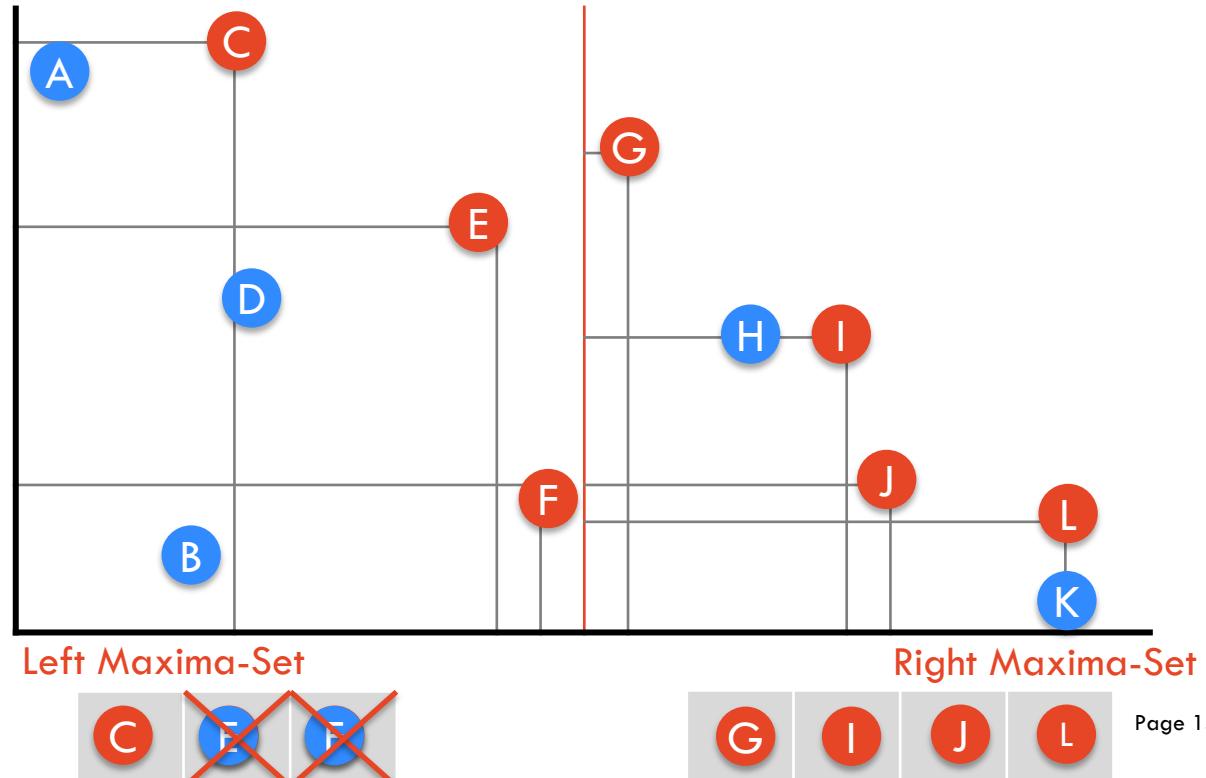
Maxima-Set

Conquer

1. Find the highest point p in the Right MS
2. Compare every point q in the Left MS to this point.
If $q.y > p.y$, add q to the Merged MS
3. Add every point in the Right MS to the Merged MS

$$p = G$$

Merged Maxima-Set



Maxima-Set

Base case a single point.

The MS of a single point is the point itself.



Maxima-Set: Analysis

Preprocessing Sort the points by increasing x coordinate and store them in an array. Note: we only do this once.
Break ties in x by sorting by increasing y coordinate.

$O(n \log n)$

Divide sorted array into two halves.

$O(n)$

Recur recursively find the MS of each half.

$2T(n/2)$

Conquer compute the MS of the union of Left and Right MS

$O(n)$

1. Find the highest point p in the Right MS
2. Compare every point q in the Left MS to this point.
If $q.y > p.y$, add q to the Merged MS
3. Add every point in the Right MS to the Merged MS

$$T(n) = 2T(n/2) + O(n) = O(n \log n)$$

Overall Running Time: pre-processing + $T(n) = O(n \log n)$

Maxima-Set: Correctness

Preprocessing Sort the points by increasing x coordinate and store them in an array. Note: we only do this once.
Break ties in x by sorting by increasing y coordinate.

Divide sorted array into two halves.

Recur recursively find the MS of each half.

Conquer compute MS of union of Left/Right MS

1. Find the highest point p in the Right MS
2. Compare every point q in the Left MS to this point.
If $q.y > p.y$, add q to the Merged MS
3. Add every point in the Right MS to the Merged MS

Observations:

1. Every point in MS of the whole is in Left MS or Right MS
2. Every point in Right MS is in MS of the whole
3. Every point in Left MS is either in MS of the whole or is dominated by p

Integer multiplication

Given two n-digit integers x and y

Problem compute the product $x \cdot y$

While this seems like recreational mathematics, it does have real applications: Public key encryption is based on manipulating integers with thousands of bits.

Integer multiplication: Naïve approach

Given two n -digit integers x and y

Problem compute the product $x \cdot y$

Suppose we wanted to do it by hand. We assume that two digits can be multiplied or added in constant time

In primary school we all learn an algorithm for this problem that performs $\Theta(n^2)$ operations

Integer multiplication: Divide and conquer

Let $x = x_1 \cdot 2^{n/2} + x_0$ and $y = y_1 \cdot 2^{n/2} + y_0$

Then $x \cdot y = x_1 \cdot y_1 \cdot 2^n + x_1 \cdot y_0 \cdot 2^{n/2} + x_0 \cdot y_1 \cdot 2^{n/2} + x_0 \cdot y_0$

We can compute the product of two n -digit numbers by making 4 recursive calls on $n/2$ -digit numbers and then combining the solutions to the subproblems.

Integer multiplication: Divide and conquer

```
def multiply(x, y):
    // x and y are positive integers represented in binary
    if x = 0 or y = 0 then return 0
    if x = 1 then return y
    if y = 1 then return x

    // recursive case
    let x1 and x0 be such that x = x1 2n/2 + x0
    let y1 and y0 be such that y = y1 2n/2 + y0

    return multiply(x1, y1) 2n +
           (multiply(x1, y0) + multiply(x0, y1)) 2n/2 +
           multiply(x0, y0)
```

Integer multiplication: Correctness

Let $x = x_1 \cdot 2^{n/2} + x_0$ and $y = y_1 \cdot 2^{n/2} + y_0$

Then $x \cdot y = x_1 \cdot y_1 \cdot 2^n + x_1 \cdot y_0 \cdot 2^{n/2} + x_0 \cdot y_1 \cdot 2^{n/2} + x_0 \cdot y_0$

Straight forward application of induction to prove
that $\text{multiply}(x, y) = x \cdot y$

Integer multiplication: Complexity analysis

Recall $x \cdot y = x_1 \cdot y_1 \cdot 2^n + (x_1 \cdot y_0 + x_0 \cdot y_1) \cdot 2^{n/2} + x_0 \cdot y_0$

Divide step (produce halves) takes $O(n)$

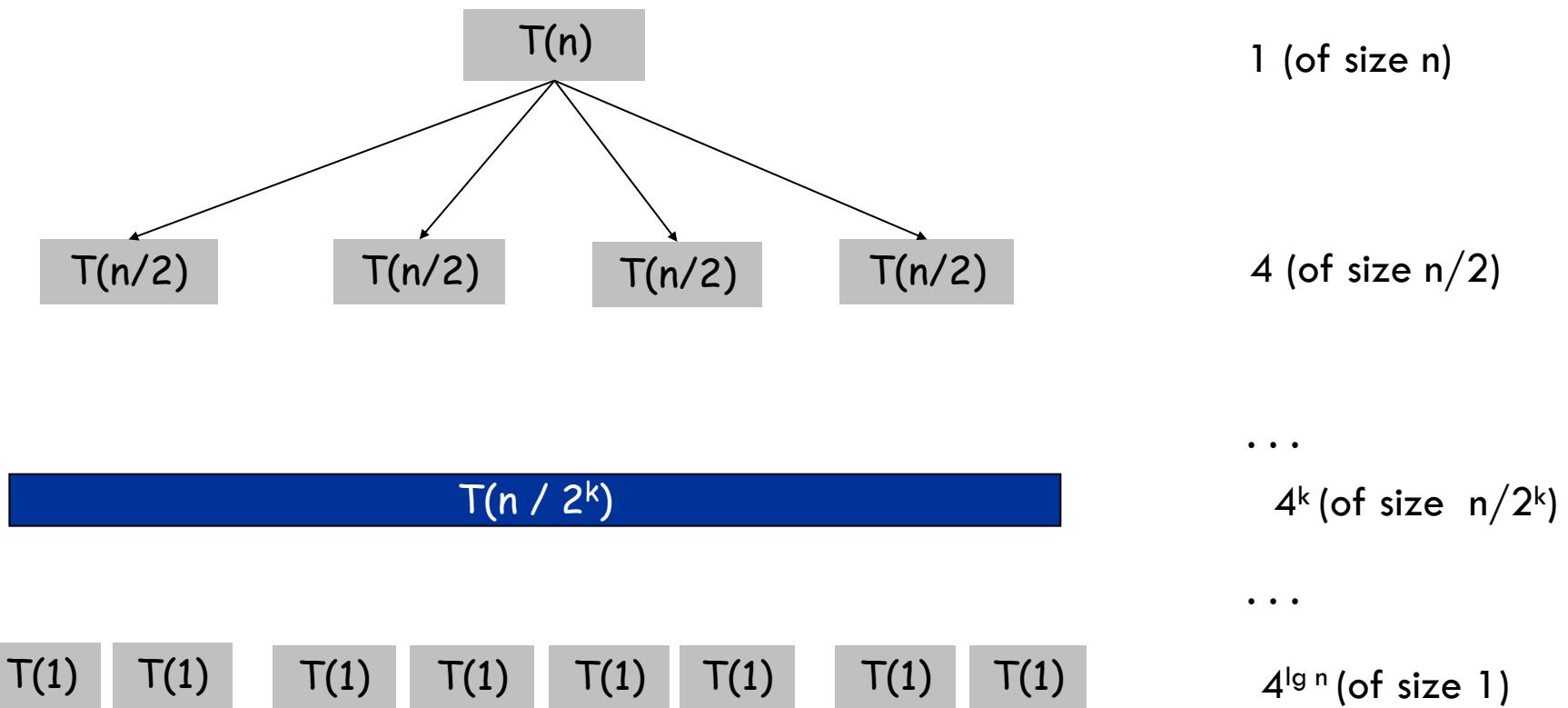
Recur step (solve subproblems) takes $4 T(n/2)$

Conquer step (add up results) takes $O(n)$

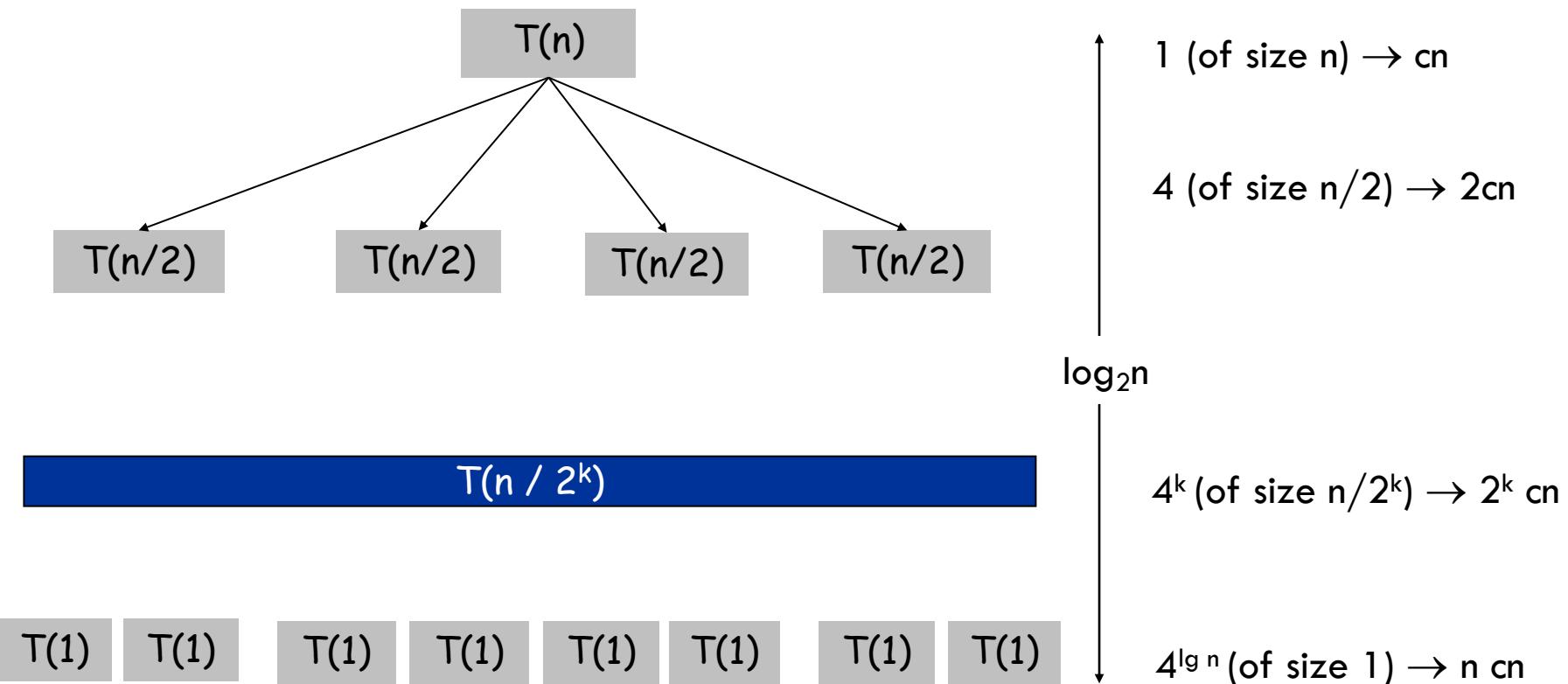
$$T(n) = \begin{cases} 4 T(n/2) + O(n) & \text{for } n > 1 \\ O(1) & \text{for } n = 1 \end{cases}$$

This solves to $T(n) = O(n^2)$. No better than naïve!!!

Proof by unrolling: $T(n) = 4 T(n/2) + O(n)$



Proof by unrolling: $T(n) = 4 T(n/2) + O(n)$



Integer multiplication: Divide and conquer v2.0

Let $x = x_1 \cdot 2^{n/2} + x_0$ and $y = y_1 \cdot 2^{n/2} + y_0$

$$\begin{aligned}x \cdot y &= x_1 \cdot y_1 \cdot 2^n + (x_1 \cdot y_0 + x_0 \cdot y_1) \cdot 2^{n/2} + x_0 \cdot y_0 \\(x_1 + x_0)(y_1 + y_0) &= x_1 \cdot y_1 + x_1 \cdot y_0 + x_0 \cdot y_1 + x_0 \cdot y_0\end{aligned}$$

We can compute the product of two n -digit numbers by making 3 recursive calls on $n/2$ -digit numbers and then combining the solutions to the subproblems.

Integer multiplication: Divide and conquer

```
def multiply(x, y):
    // base case
    :
    // recursive case
    let  $x_1$  and  $x_0$  be such that  $x = x_1 \cdot 2^{n/2} + x_0$ 
    let  $y_1$  and  $y_0$  be such that  $y = y_1 \cdot 2^{n/2} + y_0$ 

    first_term ← multiply( $x_1$ ,  $y_1$ )
    last_term ← multiply( $x_0$ ,  $y_0$ )
    other_term ← multiply( $x_1 + x_0$ ,  $y_1 + y_0$ )

    return first_term  $\cdot 2^n$  +
                    (other_term - first_term - last_term)  $\cdot 2^{n/2}$  +
                    last_term
```

Integer multiplication: Complexity analysis

Divide step (produce halves) takes $O(n)$

Recur step (solve subproblems) takes $3 T(n/2)$

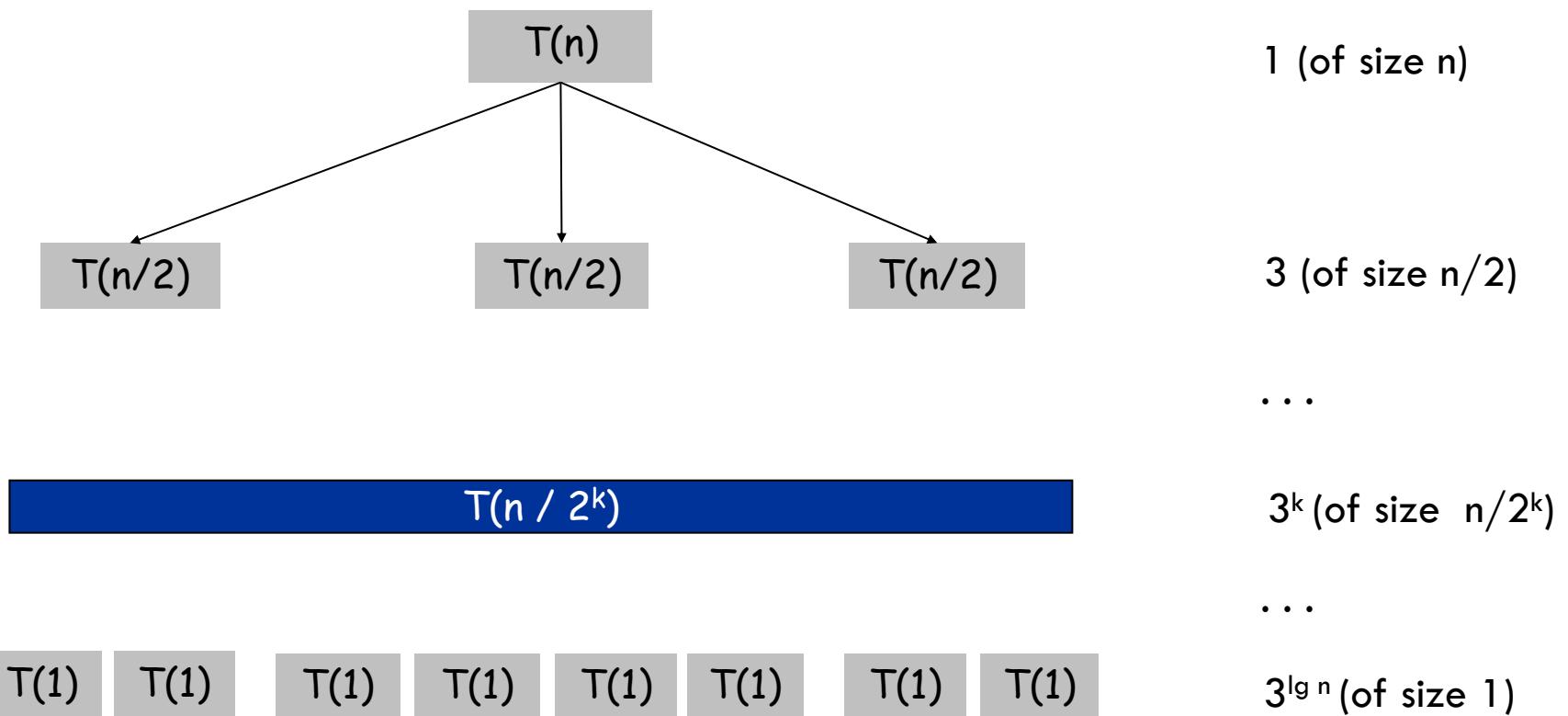
Conquer step (add up results) takes $O(n)$

$$T(n) = \begin{cases} 3 T(n/2) + O(n) & \text{for } n > 1 \\ O(1) & \text{for } n = 1 \end{cases}$$

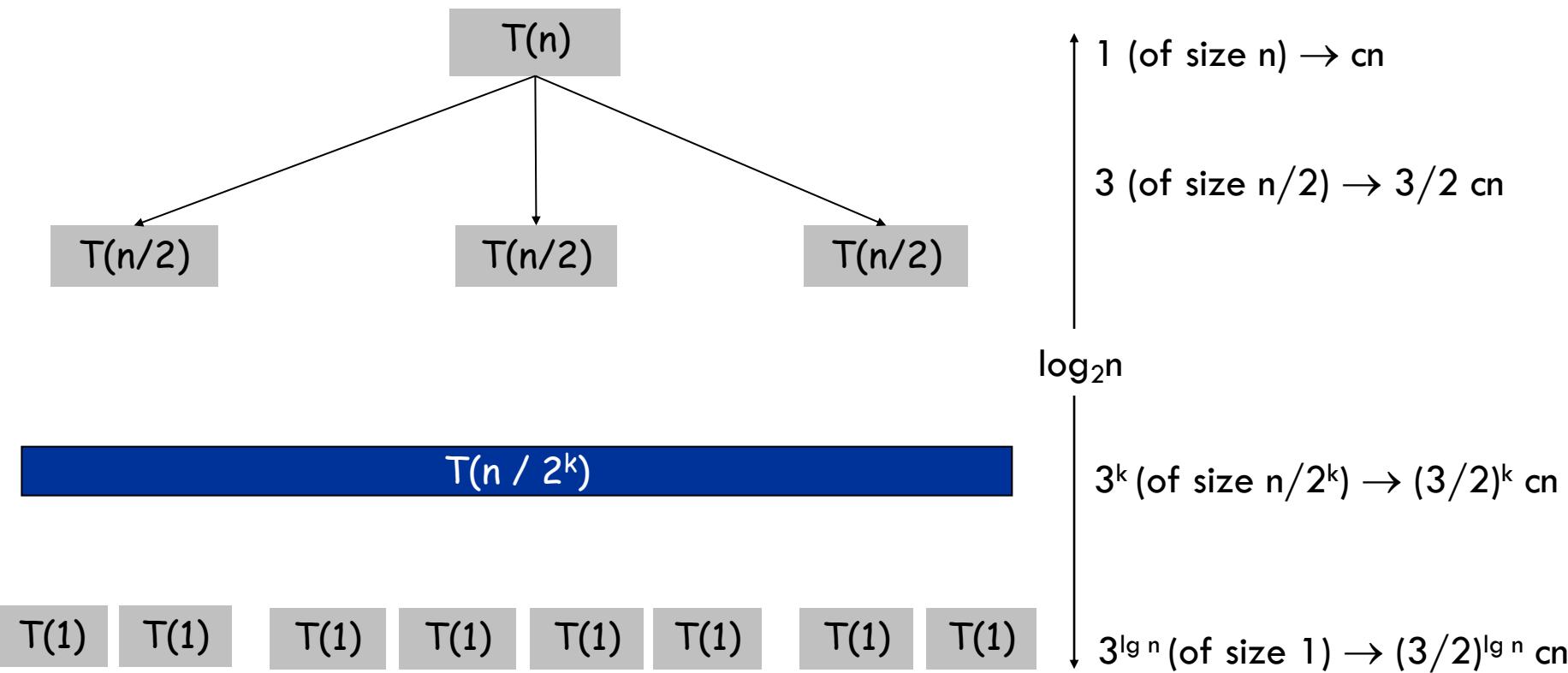
This solves to $T(n) = O(n^{\log_2 3})$, where $\log_2 3 \approx 1.6$

Better than naïve!!!

Proof by unrolling: $T(n) = 3 T(n/2) + O(n)$



Proof by unrolling: $T(n) = 3 T(n/2) + O(n)$



Geometric series facts

Let r be a positive real and k a positive integer then

$$1 + r + r^2 + \dots + r^k = (r^{k+1} - 1)/(r-1)$$

Consequently if $r > 1$ then

$$1 + r + r^2 + \dots + r^k < r^{k+1} / (r-1)$$

and if $r < 1$ then

$$1 + r + r^2 + \dots + r^k < 1 / (1-r)$$

Logarithms facts

Base exchange rule:

$$\log_a x = (\log_b x) / (\log_b a)$$

Product rule:

$$\log_a (xy) = \log_a x + \log_a y$$

Power rule:

$$\log_a x^b = b \log_a x$$

先看主定理的内容：

设某个递归算法的时间复杂度递归公式为： $T(n) = a * T(\frac{n}{b}) + n^d$, 其中 $a > 1, b > 1, d > 0$ 。则有：

- 当 $a < b^d$, 也就是 $\log_b a < d$ 时, $T(n) = O(n^d)$
- 当 $a = b^d$, 也就是 $\log_b a = d$ 时, $T(n) = O(n^{\log_b a} * \log n)$
- 当 $a > b^d$, 也就是 $\log_b a > d$ 时, $T(n) = O(n^{\log_b a})$

Master Theorem

Let $f(n)$ and $T(n)$ be defined as follows:

$$T(n) = \begin{cases} a T(n/b) + f(n) & \text{for } n \geq d \\ c & \text{for } n < d \end{cases}$$

Depending on a , b and $f(n)$ the recurrence solves to:

1. if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$ then $T(n) = \Theta(n^{\log_b a})$,
2. if $f(n) = \Theta(n^{\log_b a} \log^k n)$ for $k \geq 0$ then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$,
3. if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ and $a f(n/b) \leq \delta f(n)$ for $\varepsilon > 0$ and $\delta < 1$ then $T(n) = \Theta(f(n))$,

Note: If $f(n)$ is given as big-O, you can only conclude $T(n)$ as big-O (not Θ).

Note: You should be able to solve all recurrences in this class using unrolling, but if you are comfortable using the Master Theorem, go for it.

The Master Theorem

$$T(n) = \begin{cases} a T(n/b) + f(n) & \text{for } n \geq d \\ c & \text{for } n < d \end{cases}$$

Examples

1. $T(n) = 8T(n/2) + n^2$

$a=8, b=2, f(n)=n^2, \log_b(a) \rightarrow \log_2(8) = 3;$ so $f(n) = O(n^{\log_b a - \varepsilon})$ (case 1)

$T(n) \in \Theta(n^3)$

2. $T(n) = 2T(n/2) + O(n)$

$a=2, b=2, f(n)=n, \log_b(a) \rightarrow \log_2(2) = 1;$ so $f(n) = \Theta(n^{\log_b a} \log^k n)$ (case 2 with $k=0$)

$T(n) \in O(n \log(n))$

3. $T(n) = 2T(n/2) + O(n^2)$

$a=2, b=2, f(n)=n^2, \log_b(a) \rightarrow \log_2(2) = 1;$ so $f(n) = \Omega(n^{\log_b a + \varepsilon})$ (case 3)

$T(n) \in O(n^2)$

Selection

Given an unsorted array A holding n numbers and an integer k ,
find the k th smallest number in A

Trivial solution: Sort the elements and return k th element

Can we do better than $O(n \log n)$?

Yes, with divide and conquer!

First attempt

1. **Divide** find the median ($\lfloor n/2 \rfloor$ th element for simplicity) and split array on the halves, \leq and $>$ than the median
2. **Recur** if $k \leq \lfloor n/2 \rfloor$ find k th element on smaller half
if $k > \lfloor n/2 \rfloor$ find $(k - \lfloor n/2 \rfloor)$ th element on larger half
3. **Conquer** return value of the recursive call



$k = 6$
 $n = 10$



13
Conquer

Selection time complexity

Divide step (find median and split) takes at least $O(n)$

Recur step (solve left or right subproblem) takes $T(n/2)$

Conquer step (return recursive result) takes $O(1)$

If we could compute the median in $O(n)$ time then:

$$T(n) = \begin{cases} T(n/2) + O(n) & \text{for } n > 1 \\ O(1) & \text{for } n = 1 \end{cases}$$

This solves to $T(n) = O(n)$ but only if we can solve the median problem, which is in fact a special case of selection with $k=[n/2]$

Second attempt: Approximating the median

We don't need the exact median. Suppose we could find in $O(n)$ time an element x in A such that

$$|A| / 3 \leq \text{rank}(A, x) \leq 2 |A| / 3$$

Then we get the recurrence

$$T(n) = \begin{cases} T(2n/3) + O(n) & \text{for } n > 1 \\ O(1) & \text{for } n = 1 \end{cases}$$

Which again solves to $T(n) = O(n)$

To approximate the median we can use a recursive call!

Median of 3 medians

Consider the following procedure

- Partition A into $|A| / 3$ groups of 3
- For each group find the median (brute force)
- Let x be the median of the medians (computed recursively)

We claim that x has the desired property

$$|A| / 3 \leq \text{rank}(A, x) \leq 2|A| / 3$$

Half of the groups have a median that is smaller/larger than x , and each group has two elements smaller/larger than x , thus

$$\begin{aligned}\# \text{ elements smaller than } x &> 2(|A| / 6) = |A| / 3 \\ \# \text{ elements greater than } x &> 2(|A| / 6) = |A| / 3\end{aligned}$$

Median of 3 medians

Let x be the median of the medians, then

$$|A| / 3 \leq \text{rank}(A, x) \leq 2|A| / 3$$

1	12	5	16	19	7	23	6	13
---	----	---	----	----	---	----	---	----

1	12	5
---	----	---

16	19	7
----	----	---

23	6	13
----	---	----

1	5	12
---	---	----

7	16	19
---	----	----

6	13	23
---	----	----

1	5	12
---	---	----

6	13	23
---	----	----

7	16	19
---	----	----

elements smaller than $x > 2(|A| / 6) = |A| / 3$
elements greater than $x > 2(|A| / 6) = |A| / 3$

Median of 3 median time complexity

We don't need the exact median. With a recursive call on $n/3$ elements, we can find x in A such that

$$|A|/3 < \text{rank}(A, x) < 2|A|/3$$

Then we get the recurrence

$$T(n) = T(2n/3) + T(n/3) + O(n)$$

Which solves to $T(n) = O(n \log n)$

No better than sorting!

Median of 5 medians

We don't need the exact median. With a recursive call on $n/5$ elements, we can find x in A such that

$$3|A|/10 < \text{rank}(A, x) < 7|A|/10$$

Then we get the recurrence

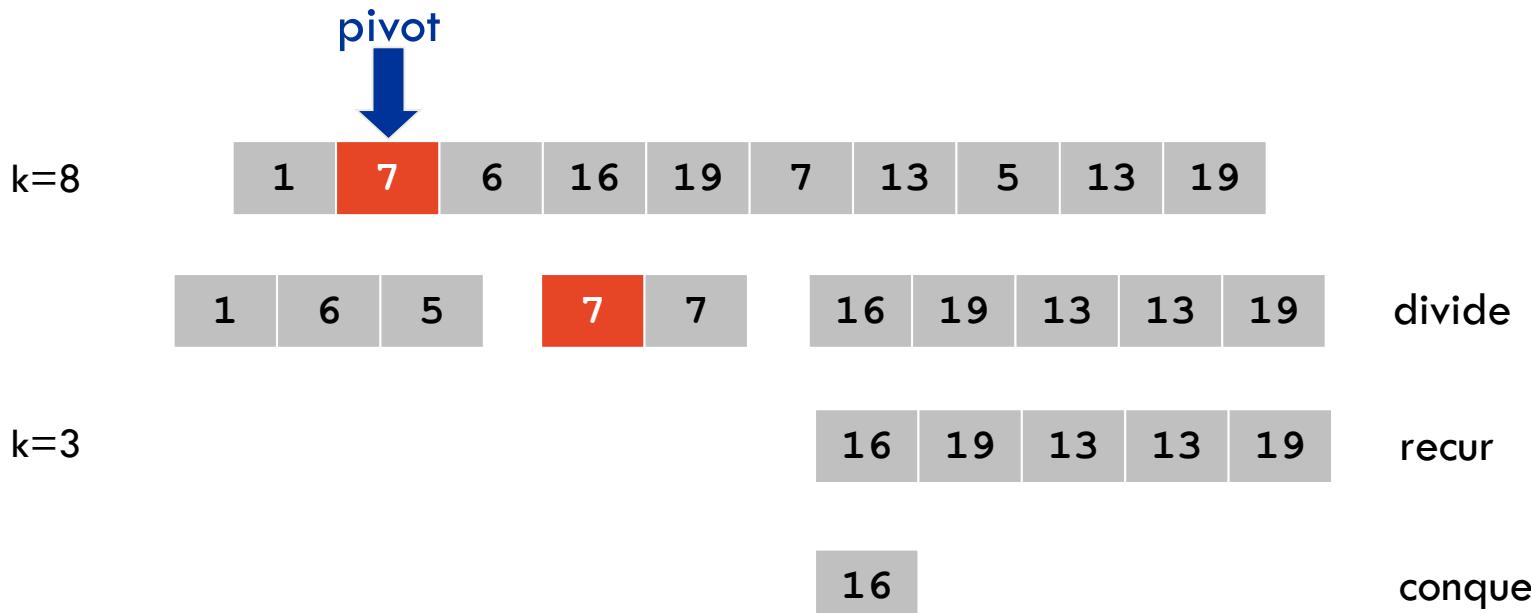
$$T(n) = T(7n/10) + T(n/5) + O(n)$$

Which solves to $T(n) = O(n)$

Asymptotically faster than sorting!

Quick selection

1. **Divide** Choose a random element from the list as the **pivot**
Partition the elements into 3 lists:
(i) less than, (ii) equal to and (iii) greater than the **pivot**
2. **Recur** Recursively select right element from correct list
3. **Conquer** Return solution to recursive problem



Quick selection complexity analysis

Divide step (pick pivot and split) takes $O(n)$

Recur step (solve left and right subproblem) takes $T(n')$

Conquer step (return solution) takes $O(1)$

Now we can set up the recurrence for $T(n)$:

$$E[T(n)] = \begin{cases} E[T(n')] + O(n) & \text{for } n > 1 \\ O(1) & \text{for } n = 1 \end{cases}$$

This solves to $E[T(n)] = O(n)$

(details available on the textbook but not examinable)