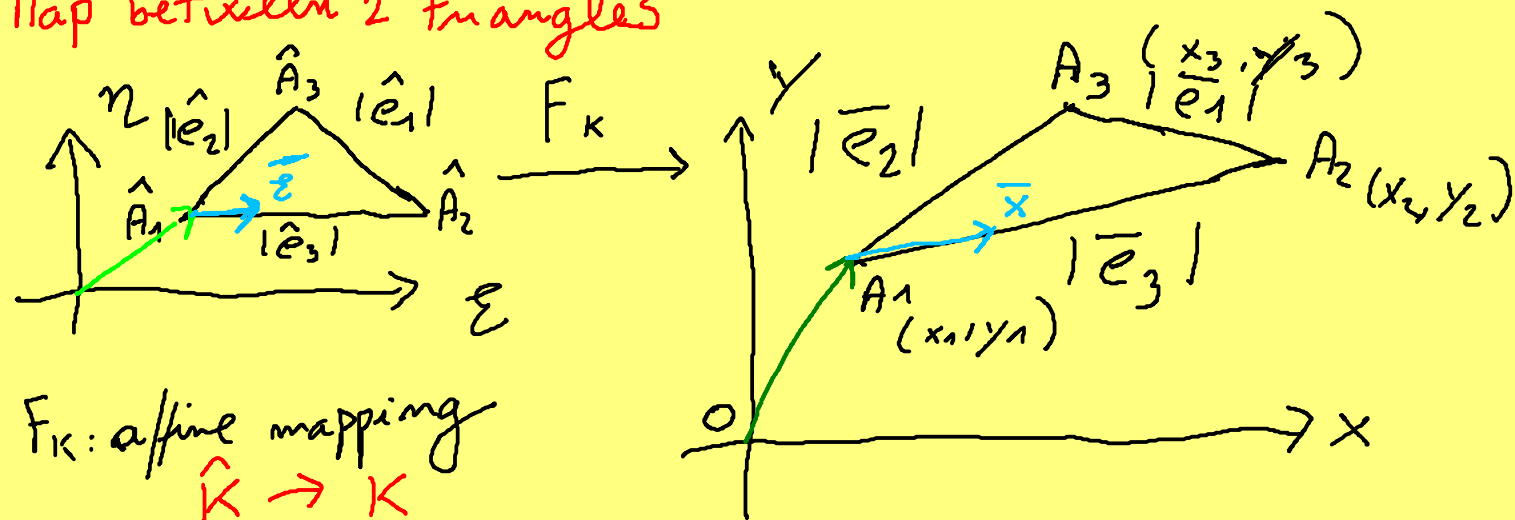


Map between 2 triangles



Express F_K using barycentric coords on \hat{K}

$$\begin{bmatrix} x \\ y \end{bmatrix} = F_K(\xi, \eta) = \sum_{i=1}^3 \hat{\phi}_i(\xi, \eta) \begin{bmatrix} x_i \\ y_i \end{bmatrix} \quad \bar{x} = F_K(\bar{e})$$

\bar{x}_i lies on $\bar{e}_i \Rightarrow \bar{x}_i = \phi_i(s) = \overline{OA_i} + s \bar{e}_i, \quad s \in [0, 1]$

\bar{e}_i lies on $\hat{e}_i \Rightarrow \bar{e}_i = \hat{\phi}_i(s) = \overline{OA_i} + s \hat{e}_i$

$$\bar{e}_i = \frac{d}{ds} F_K(\hat{\phi}_i(s)) = F_K' \frac{d}{ds} \hat{\phi}_i(s) = F_K' \hat{e}_i$$

$F_K' = J_K$ Jacobian of transformation

$$\textcircled{1} \text{ Size: } |K| = \int_K d\bar{x} = \int_K \det(J_K) d\xi = \det(J_K) |\hat{K}|$$

If J_K is not symmetric $J_K = U \Sigma V^T$ SVD decomposition

$$U = [\bar{u}_1 \ \bar{u}_2] \quad \Sigma = \text{diag}(\sigma_1, \sigma_2) \quad V = [\bar{v}_1 \ \bar{v}_2]$$

$$\det(J_K) = \sigma_1 \cdot \sigma_2 \text{ as } U, V \text{ orthogonal}$$

$$|K| = \sigma_1 \cdot \sigma_2 |\hat{K}|$$

$$\textcircled{2} \text{ Shape: } |\bar{e}_i|^2 = \bar{e}_i^T \bar{e}_i = \hat{e}_i^T J_K^T J_K \hat{e}_i = \hat{e}_i^T V \Sigma^2 V^T \hat{e}_i$$

Relate $\textcircled{1}, \textcircled{2}$ to shape regularity for $\mathcal{T}_h = \{K_i\}$

Given $\mathcal{T} = \{Z_h\}_{h>0}$ simplicial partition over polygon $\bar{\Omega} \subset \mathbb{R}^2$ (regular) we have equivalence between

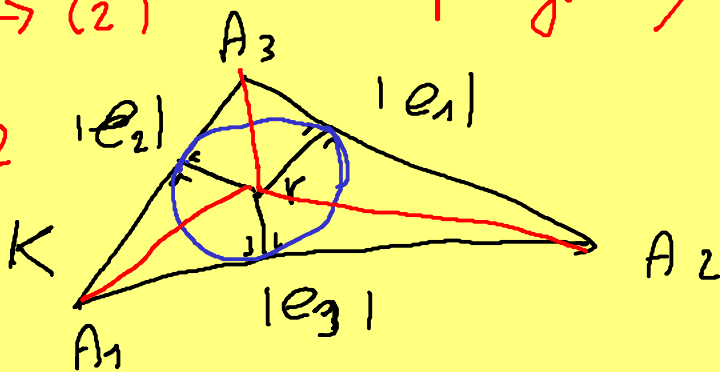
(1): $\exists C_1 > 0$ s.t. $\forall Z_h \quad \forall K \in Z_h$
 $|K| \geq C_1 h_K^2 \quad h_K: \text{diameter simplex}$

(2): $\exists C_2 > 0$ s.t. $\forall Z_h \quad \forall K \quad \exists B_r \subset K$ s.t.
 $r \geq C_2 h_K \Rightarrow \rho_S = \inf \frac{r}{h_K} \geq C_2$

(1) \leftrightarrow (2)

Shape regularity

1 \Rightarrow 2



$$|K| = \frac{1}{2} \sum_{i=1}^3 r |e_i| \Rightarrow 2|K| \leq 3r h_K$$

Using 1 $3r h_K \geq 2|K| \geq 2C_1 h_K^2$

$$\Rightarrow \frac{3}{2} \frac{r}{h_K} \geq C_1 \Rightarrow \frac{r}{h_K} \geq C_2 = \frac{2}{3} C_1$$

2 \Rightarrow 1 $A_S B \subset K$

$$|K| \geq |B| \geq \pi r^2 \stackrel{(2)}{\geq} \underbrace{\pi C_2^2}_{C_1} h_K^2$$

(3): $\exists C_3 > 0$ s.t. $\forall Z_h \quad \forall K \in Z_h \quad \exists B \supset K$ s.t.
 $|K| \geq C_3 |B^s|$

(1) \leftrightarrow (3)

\hookrightarrow (3) \hookleftarrow

1 \Rightarrow 3

$$h_K^2 \geq |K| \geq C_1 h_K^2$$

$$|B^K| = \pi R^2 \quad R = \frac{|e_1| |e_2| |e_3|}{4 |K|} \leq \frac{h_K^3}{4 C_1 h_K^2} = C_g h_K$$

$$|K| \geq C_1 h_K^2 \geq C_1 \frac{R^2}{C_g^2} = \frac{C_1}{\pi C_g^2} |B^K|$$

3 \Rightarrow 1

$$|K| \geq C_3 |B^K| = C_3 \pi R^2 \geq C_3 \pi \left(\frac{h_K}{2} \right)^2$$

$$= \frac{C_3 \pi}{4} h_K^2$$

$$C_3 = \frac{4}{\pi} C_1$$

Exploit (3) with Jacobian map

$$|K| = \sigma_1 \sigma_2 |\hat{K}|$$

$$|\bar{R}|^2 = \bar{R}^T \hat{K} V \Sigma^2 V^T \bar{R} \hat{K}$$

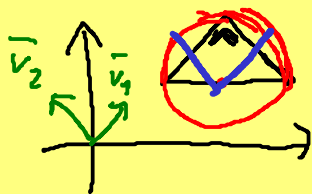
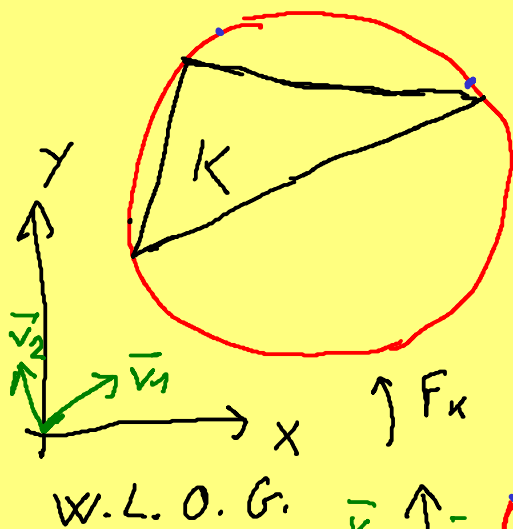
choose $\bar{R}^{\hat{K}}$ aligned with \bar{v}_1, \bar{v}_2

$$|\bar{R}^{\hat{K}}|^2 = [\bar{R}^{\hat{K}} \ 0] \Sigma^2 \begin{bmatrix} \bar{R}^{\hat{K}} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \bar{R}^{\hat{K}} \sigma_1^2 & 0 \end{bmatrix} \begin{bmatrix} \bar{R}^{\hat{K}} \\ 0 \end{bmatrix}$$

$$= \sigma_1^2 |\bar{R}^{\hat{K}}|^2$$

$$= \sigma_2^2 |\bar{R}^{\hat{K}}|^2$$



$$|K| = \sigma_1 \sigma_2 |\hat{K}| \quad |B^K| = \frac{1}{4} (R^K)^2 = \frac{\pi}{2} (\sigma_1^2 + \sigma_2^2) R^{\hat{K}^2} \quad \text{as } \hat{K} \text{ regular}$$

$$\frac{|K|}{|B^K|} = \left[\frac{2\sigma_1 \sigma_2}{\sigma_1^2 + \sigma_2^2} \right] \frac{|\hat{K}|}{\frac{\pi}{2} R^{\hat{K}^2}} \approx \frac{C_3}{Q_S(K)} \quad \text{it depends on } K$$

We require $\max_K Q_K \ll C_S$ s.t.

$$\frac{|K|}{|B^K|} \gg \frac{C_3}{C_S} > 0 \iff \text{shape regularity for } K$$

$$\forall K \forall \tau_n$$

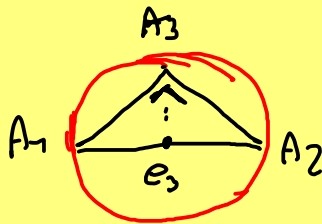
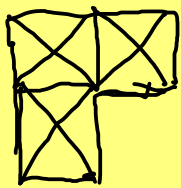
$$C_3(\hat{K}) = \frac{6}{h} C_2(\hat{K})$$

for OT mesh $J := H(P) = \begin{bmatrix} P_{1e} & P_{2e} \\ P_{1n} & P_{2n} \end{bmatrix}$

$$\text{as } \chi(e) = \nabla_e P(e) \triangle_e P(e) > 0$$

$$J = \sum \lambda_i e_i^T c_i$$

For criss-cross mesh
(or left/right biased)



$$|\bar{R}^{\hat{K}}| = |\bar{e}_3|$$

$$|\hat{K}| = \frac{|\bar{e}_3|^2}{2} \frac{|\bar{R}^{\hat{K}}|}{2} = \frac{|\bar{e}_3|^2}{4} = \frac{h^2}{4}$$

$$\frac{|K|}{|B^K|} = \frac{1}{Q_S(K)} \frac{|\hat{K}|}{|B^{\hat{K}}|} = \frac{h^2}{4 Q_S(K) \pi \left(\frac{h_K}{2}\right)^2} = \frac{1}{Q_S(K) \pi} \approx C_3?$$

Ex: L-shaped domain

$$\mu_s = 0.0607$$

$$Q_s = 2.8$$

$$C_2 = \frac{\pi}{6} C_3 \approx \frac{\cancel{\pi} 1}{6 Q_s \pi}$$

$$\frac{1}{6(2.8)} \approx 0.0595238 \approx \mu_s$$

Crack domain

$$Q_s = 4.514 \quad \mu_s = 0.03607$$

$$\frac{1}{6 \cdot Q_s} \approx 0.037 \approx \mu_s$$

