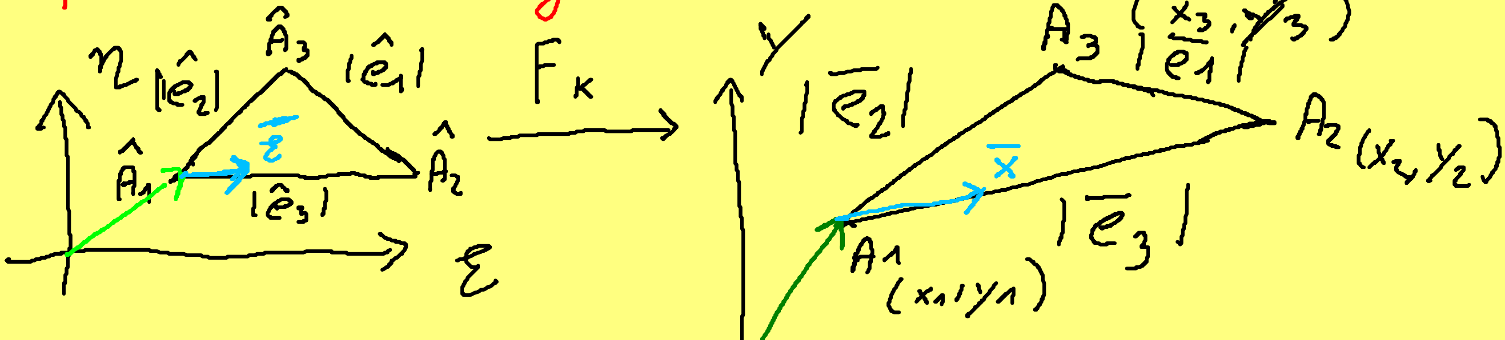
Map between 2 triangles



F_K: affine mapping
$\hat{K} \to K$

Express F_K using barycentric coords on $\hat{K}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = F_K(\xi, \eta) = \sum_{i=1}^{3} \hat{\phi}_i(\xi, \eta) \begin{bmatrix} x_i \\ y_i \end{bmatrix} \qquad \bar{x} = F_K(\bar{\varepsilon})$$

$\bar{x}_i$ lies on $\bar{e}_i \Rightarrow \bar{x}_i = \phi_i(s) = \overline{OA}_i + s\,\bar{e}_i \qquad s \in [0,1]$

$\bar{\varepsilon}_i$ lies on $\hat{e}_i \Rightarrow \bar{\varepsilon}_i = \hat{\phi}_i(s) = \overline{O\hat{A}_i} + s\,\hat{e}_i$

$$\bar{e}_i = \frac{d}{ds} F_K(\hat{\phi}_i(s)) = F_K' \frac{d}{ds}\hat{\phi}_i(s) = F_K' \hat{e}_i$$

$F_K' = J_K$ Jacobian of transformation

(1) size: $|K| = \int_K d\bar{x} = \int_{\hat{K}} \det(J_K)\, d\xi = \det(J_K)\,|\hat{K}|$

If J_K is not symmetric $J_K = U \Sigma V^T$ SVD decomposition

$U = [\bar{u}_1\ \bar{u}_2] \quad \Sigma = \mathrm{diag}(\sigma_1, \sigma_2) \quad V = [\bar{v}_1\ \bar{v}_2]$

$\det(J_K) = \sigma_1 \cdot \sigma_2$ as U, V orthogonal

$|K| = \sigma_1 \cdot \sigma_2\, |\hat{K}|$

(2) Shape : $|\bar{e}_i|^2 = \bar{e}_i^T \bar{e}_i = \hat{e}_i^T J_K^T J_K \hat{e}_i = \hat{e}_i^T V \Sigma^2 V^T \hat{e}_i$

Relate (1), (2) to shape regularity for $\mathcal{T}_h = \{K_i\}$:

given $\mathcal{F} = \{\mathcal{T}_h\}_{h>0}$ simplicial partition over polygon $\overline{\Omega}\subset\mathbb{R}^2$ (conforming) we have equivalence between

(1): $\exists C_1>0$ s.t. $\forall \mathcal{T}_h \ \forall K\in\mathcal{T}_h$

$|K| \ge C_1 h_K^2$ $\quad h_K$: diameter simplex

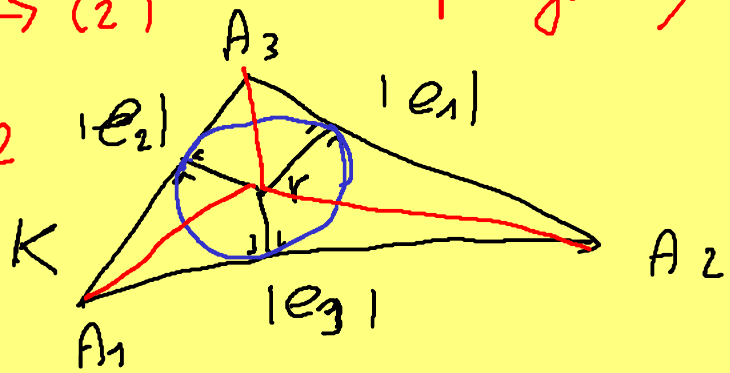
(2): $\exists C_2>0$ s.t. $\forall \mathcal{T}_h \ \forall K \ \exists B_r\subset S$ s.t.

$r \ge C_2 h_K \Rightarrow \mu_S = \inf \frac{r}{h_K} \ge C_2$

Shape regularity

(1) $\leftrightarrow$ (2)

$1 \Rightarrow 2$



$$|K| = \frac{1}{2}\sum_{i=1}^{3} r|e_i| \Rightarrow 2|K| \le 3 r h_K$$

Using 1 $\quad 3 r h_K \ge 2|K| \ge 2 C_1 h_K^2$

$$\Rightarrow \frac{3}{2}\frac{r}{h_K} \ge C_1 \Rightarrow \frac{r}{h_K} \ge C_2 = \frac{2}{3} C_1$$

$2 \Rightarrow 1$ As $B\subset K$

$$|K| \ge |B| \ge \pi r^2 \overset{(2)}{\ge} \underbrace{\pi C_2^2}_{C_1} h_K^2$$

(3): $\exists C_3>0$ s.t. $\forall \mathcal{T}_h \ \forall K\in\mathcal{T}_h \ \exists B^s\supset K$ s.t.

$|K| \ge C_3 |B^s|$

(1) $\leftrightarrow$ (2)

$\searrow$ (3) $\nearrow$

$1 \Rightarrow 3$ $\quad h_K^2 \geq |K| \geq c_1 h_K^2$

$$|B^K| = \pi R^2 \qquad R = \frac{|e_1||e_2||e_3|}{4|K|} \leq \frac{h_K^3}{4 c_1 h_K^2} = c_g h_K$$

$$|K| \geq c_1 h_K^2 \geq c_1 \frac{R^2}{c_g^2} = \underbrace{\frac{c_1}{\pi c_g^2}}_{c_3} |B^K|$$

$3 \Rightarrow 1$

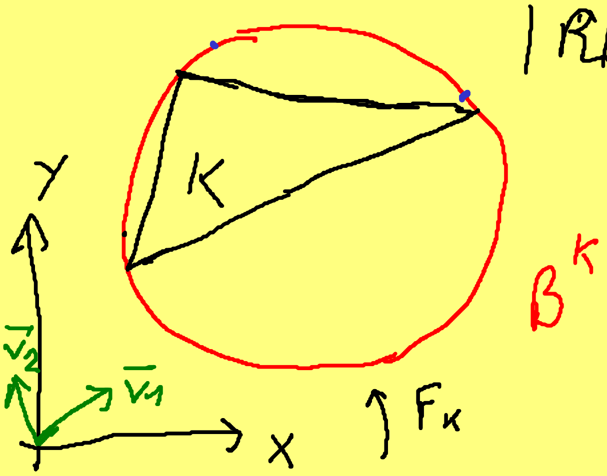
$$|K| \geq c_3 |B^K| = c_3 \pi R^2 \geq c_3 \pi \left(\frac{h_K}{2}\right)^2$$

$$= \underbrace{\frac{c_3 \pi}{4}}_{c_1} h_K^2 \qquad\qquad c_3 = \frac{4}{\pi} c_1$$

Exploit (3) with Jacobian map

$$|K| = \sigma_1 \cdot \sigma_2 |\hat{K}|$$

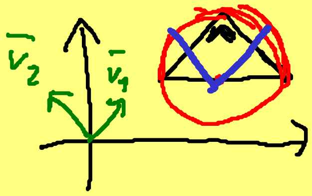
$$|\bar{R}|^2 = \bar{R}^{\hat{K}} V \Sigma^2 V^T \bar{R}^{\hat{K}}$$



choose $\bar{R}^{\hat{K}}$ aligned with $\bar{v}_1, \bar{v}_2$

$$|\bar{R}^K|^2 = \begin{bmatrix} \bar{R}^{\hat{K}} & 0 \end{bmatrix} \Sigma^2 \begin{bmatrix} \bar{R}^{\hat{K}} \\ 0 \end{bmatrix}$$

W.L.O.G.



$$= \begin{bmatrix} \bar{R}^{\hat{K}} \sigma_1^2 & 0 \end{bmatrix} \begin{bmatrix} \bar{R}^{\hat{K}} \\ 0 \end{bmatrix}$$

$$= \sigma_1^2 |\bar{R}^{\hat{K}}|^2$$

$$= \sigma_2^2 |\bar{R}^{\hat{K}}|^2$$

$|K| = \sigma_1\sigma_2|\hat K|$ $\quad |B^K| = \pi (R^K)^2 = \frac{\pi}{2}(\sigma_1^2+\sigma_2^2) R^{\hat K 2}$

as $\hat K$ regular

$$\frac{|K|}{|B^K|} = \left[\frac{2\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2}\right]\frac{|\hat K|}{\underbrace{\pi R^{\hat K 2}}_{=|B^{\hat K}|}} \geq \frac{C_3}{Q_s}$$

Q_s it depends on K

We require $\boxed{\max_K Q_K \ll C_s}$ s.t.

$$\frac{|K|}{|B^K|} \geq \frac{C_3}{C_S} > 0 \longleftrightarrow \text{shape regularity for } K$$

$\forall K \ \forall \mathcal{T}_h$

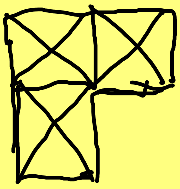
$$\boxed{C_3(\hat K) = \frac{6}{\pi} C_2(\hat K)}$$

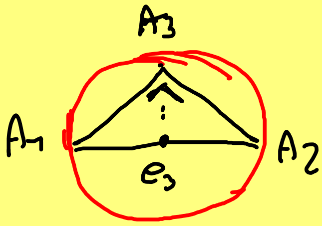
For oT mesh $J := H(p) = \begin{bmatrix} p_{\xi\xi} & p_{\xi\eta} \\ p_{\eta\xi} & p_{\eta\eta} \end{bmatrix}$

as $x(\xi) = \nabla_\xi P(\xi)$ $\quad \Delta_\xi P(\xi) > 0$

$$J = \sum \lambda_i e_i^T e_i$$

For criss-cross mesh (or left/right biased)





$$|\bar R^{\hat K}| = \frac{|\bar e_3|}{2}$$

$$|\hat K| = \frac{|\bar e_3|\,|R^{\hat K}|}{2} = \frac{|\bar e_3|^2}{4} = \frac{h_{\hat K}^2}{4}$$

$$\frac{|K|}{|B_K|} = \frac{1}{Q_s(K)}\frac{|\hat K|}{|B^{\hat K}|} = \frac{h_{\hat K}^2}{4\,Q_s(K)\,\pi\left(\frac{h_K}{2}\right)^2} = \frac{1}{Q_s(K)\pi} \simeq C_3 ?$$

Ex: L-shaped domain

$$\mu_s = 0.0607$$

$$Q_s = 2.8$$

$$C_2 = \frac{\pi}{6} C_3 \approx \frac{\cancel{\pi} 1}{6 Q_s \cancel{\pi}}$$

$$\frac{1}{6(2.8)} \approx 0.0595238 \approx \mu_s$$

Crack domain

$$Q_s = 4.514 \qquad \mu_s = 0.03607$$

$$\frac{1}{6 \cdot Q_s} \approx 0.037 \approx \mu_s$$