Geometric Descriptors of Curvature

Mario Zarco

Maximum Perpendicular Deviation

Maximum Perpendicular Deviation is a measure of the divergence of the reaching trajectory away from an ideal direct path trajectory. This metric has been defined as "the length of a perpendicular line between the idealized straight-line trajectory [drawn from the start and end points] and farthest point from that straight line in the observed trajectory" (Hehman, Stolier, & Freeman, 2015, p. 6). Maximum deviation is a signed index, such that the sign is positive if the deviation is largest in the direction of the non-chosen target and negative otherwise (Kieslich, Henninger, Wulff, Haslbeck, & Schulte-Mecklenbeck, 2019). Although this is the most widespread definition in mouse-tracking studies, this measure has been also construed adimensionally through scaling the maximum perpendicular distance by the length of the idealized trajectory (Erb & Marcovitch, 2019). The latter operationalization implicitly assumes that the trajectories have common start and end points. Otherwise, the order of the values in the set of maximum perpendicular distances would change after they are divided by the length of idealized trajectories of different lengths. I stick to the first definition.

Maximum Deviation for 2D time series

Maximum perpendicular deviation is the signed length of the perpendicular line between an ideal straight trajectory – defined from the sample corresponding to movement onset to the sample corresponding to movement termination – and the farthest point from such an idealized line in the reaching movement trajectory. The value of the sign of an outward trajectory with respect to the ideal trajectory toward the chosen response is negative.

Let $\vec{x}$ be a time series of dimension $(N, 2)$, i.e. a two-dimensional time series in which each dimension has N samples. Figure 1 shows a schematic of a simulated trajectory and the vectors defined below. A vector along the idealized straight trajectory, $\vec{u}_l$, is defined by

$$\vec{u}_l = \vec{x}_N - \vec{x}_1 \tag{1}$$

where x_1 and x_N are the initial and final samples of the time series, respectively. As shown in the schematic, a vector $\vec{u}_s$ can be defined from the initial sample to a sample on the trajectory. $N-2$ vectors are then calculated from the initial sample to each sample x_s except the first and last ones, such that

$$\vec{u}_s = \vec{x}_s - \vec{x}_1 \tag{2}$$

for $s = 2, \dots, N-1$. Let ϕ_l be the angle between the horizontal axis (x in Figure 1) and $\vec{u}_l$, such that $\phi_l = \text{atan}(u_{ly}/u_{lx})$. Let ϕ_s be the angle between the same horizontal axis and $\vec{u}_s$, such that $\phi_s = \text{atan}(u_{sy}/u_{sx})$. The angle between $\vec{u}_l$ and $\vec{u}_s$ is $\theta_s = \phi_l - \phi_s$. Using a trigonometric identity[1],

[1] $\text{atan}(x) - \text{atan}(y) = \text{atan}(\frac{x-y}{1+xy})$

$\theta_s = \text{atan}\left((u_{ly}u_{sx} - u_{sy}u_{lx})/(u_{lx}u_{sx} + u_{ly}u_{sy})\right)$. Thus, the angle between u_l and each vector u_l which is calculated as

$$\theta_s = \text{atan}\left(\frac{u_{ly}u_{sx} - u_{sy}u_{lx}}{u_{lx}u_{sx} + u_{ly}u_{sy}}\right) = \text{atan2}(||\vec{u}_s \times \vec{u}_l||,\ \vec{u}_s \bullet \vec{u}_l) \tag{3}$$

where $\times$ is the cross products, $||\ ||$ is the norm of a vector, and $\bullet$ is the dot product. To properly calculate the value of the sign according to the convention followed here, it is only necessary to change the order of the cross product in Equation (3) based on the location of the reached target. In the case of the Simon task in virtual reality, when the value of the horizontal component, i.e. x, of the target location is positive, the cross product is to be computed as $\vec{u}_l \times \vec{u}_s$.

The perpendicular distance between the idealized straight trajectory and the sample x_s, for $s = 2, ..., N-1$, is then calculated as

$$d_s = ||u_s||\ sin(|\theta_s|) \tag{4}$$

where $|\ |$ is the absolute value. Finally, let $\{d_s\}$ be the set of the $N-2$ perpendicular distances, maximum perpendicular deviation is then defined by

$$MaxDev = \text{sgn}(\theta_s(d_s^{max}))\ d_s^{max} \tag{5}$$

where sgn is the sign function, $d_s^{max} = \text{Argmax}(\{d_s\})$ and, accordingly, $\theta_s(d_s^{max})$ is the angle corresponding to the value of maximum perpendicular distance.

Maximum Deviation for 3D time series

Maximum deviation has been only defined for two-dimensional time series to the best knowledge of the author of this work. In the case of three-dimensional time series, maximum perpendicular deviation is then defined as specified above and, following a similar logic, I defined an idealized plane to calculate the value of the sign for this metric.

Let $\vec{x}$ be a time series of dimension $(N, 3)$, i.e. a three-dimensional time series in which each dimension has N points. Figure **??** shows a schematic of a simulated trajectory and the vectors defined below. A vector along an idealized straight line, $\vec{u}_l$, is defined by

$$\vec{u}_1 = \vec{x}_N - \vec{x}_1 \tag{6}$$

where x_1 and x_N are the initial and final samples of the time series, respectively. As shown in the schematic, a vector $\vec{u}_s$ can be defined from the initial sample to a sample on the trajectory. $N-2$ vectors are then calculated from the initial sample to each sample x_s except the first and last one, such that

$$\vec{u}_s = \vec{x}_s - \vec{x}_1 \tag{7}$$

where $s = 2, \ldots, N-1$. As $\vec{u}_l \bullet \vec{u}_s = ||\vec{u}_l||||\vec{u}_s|| \cos(\theta_s)$, the angle between the vector $\vec{u}_l$ and each vector $\vec{u}_s$ is calculated as

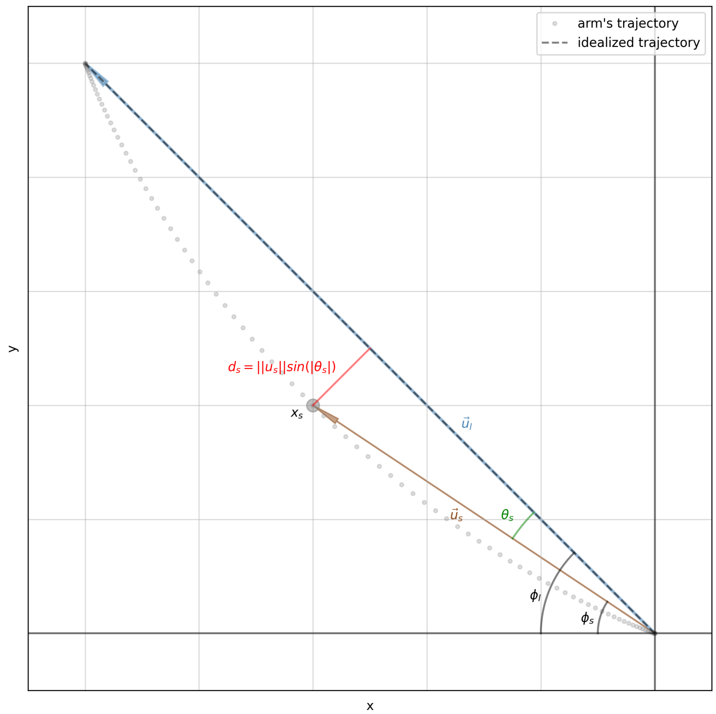

Figure 1: Schematic of a simulated trajectory and the defined vectors used to calculate maximum perpendicular deviation with two-dimensional data.

$$\theta_s = \text{acos}\left(\frac{\vec{u}_1 \bullet \vec{u}_s}{||\vec{u}_1||\ ||\vec{u}_s||}\right). \tag{8}$$

The value of the sign for a sample x_s on the trajectory depends upon the value of the angle between the normal vector to an idealized plane P, specified according to the experimental setup, and the vector u_s. For instance, plane P is perpendicular to the plane xz in Figure **??**. The idealized plane contains the initial and final sampled of the time series, as well as the two vectors $\vec{u}_a$ and $\vec{u}_b$ defined as

$$\vec{u}_a = \vec{x}_p - \vec{x}_1 \tag{9}$$

and

$$\vec{u}_b = \vec{x}_f - \vec{x}_1 \tag{10}$$

where $\vec{x}_p$ is the projection of $\vec{x}_f$ onto the plane xz. The normalized perpendicular vector to plane P is then determined by

$$\hat{u}_n = \frac{\vec{u}_a \times \vec{u}_b}{||\vec{v}_a \times \vec{v}_b||} \tag{11}$$

where $\hat{}$ indicates a normalized vector. Let β_s be the angle between $\hat{u}_n$ and $\hat{u}_s$, such that $\hat{u}_n \bullet \hat{u}_s = \cos(\beta_s)$, where $\hat{u}_s$ is the normalized vector of $\vec{u}_s$. If $\cos(\beta_s) = \sin(\frac{\pi}{2} - \beta_s)$, an angle α_s can be then defined as $\alpha_s = \frac{\pi}{2} - \beta_s$. The angle α_s is calculated as

$$\alpha_s = \mathrm{asin}\,(\hat{u}_n \bullet \hat{u}_s) = \frac{\pi}{2} - \beta_s, \tag{12}$$

such that $\alpha_s < 0$ when $\beta_s > \frac{\pi}{2}$. That is, the value of α_s is negative for samples x_s belonging to an outward trajectory with respect to the idealized trajectory towards the chosen response. Thus, to properly calculate the angle α_s, the only condition is to change the order of the cross product in Equation (11) based on the location of the reached target. In the specific case of the Simon task, if the value of the horizontal component, i.e. x, of the target location is positive, the cross product is to be computed as $\vec{v}_b \times \vec{v}_a$.

The perpendicular distance between the idealized straight line and the sample x_s, for $s = 2, ..., N-1$, is then calculated as

$$d_s = ||\vec{u}_s||\ sin(\theta_s). \tag{13}$$

Finally, let $\{d_s\}$ be the set of the $N-2$ perpendicular distances, maximum perpendicular deviation is then defined by

$$MaxDev = \mathrm{sgn}(\alpha_s(d_s^{max}))\ d_s^{max} \tag{14}$$

where $d_s^{max} = \mathrm{Argmax}(\{d_s\})$ and, accordingly, $\alpha_s(d_s^{max})$ is the angle corresponding to the value of maximum perpendicular distance.

Maximal Log Ratio

Maximal Log Ratio is a measure of deflection of the reaching trajectory towards the non-chosen target. This metric is defined as the point on the recorded trajectory that maximizes the logarithm of the ratio between the distance to the chosen target and the non-chosen target (Maldonado, Dunbar, & Chemla, 2019).

Let $\vec{x}$ be a time series of dimension (N, M), with $M = 2$ *or* 3. That is, the time series can be either two-dimensional or three-dimensional. The distance from any point on the time series, $\vec{x}_s$, to the non-chosen target, $\vec{tg}_1$ is determined as

$$d_{s \to tg_1} = |\vec{tg}_1 - \vec{x}_s|, \tag{15}$$

whereas the distance to the chosen target, $\vec{tg}_2$, is calculated as

$$d_{s \to tg_2} = |\vec{tg}_2 - \vec{x}_s| \tag{16}$$

where $s = 1, \ldots, N$ and $|\ |$ indicates the norm of the vector. The log distance ratio is computed as

$$LogR = \log\left(\frac{d_{s \to tg_2}}{d_{s \to tg_1}}\right) \tag{17}$$

Finally, let $\{LogR\}$ be the set of the N log distance ratios, the maximal log distance ratio is then calculated as

$$MaxLogR = \text{Argmax}\Big(\{LogR\}\Big). \tag{18}$$

Total Curvature

Total curvature is a measure of the central tendency of divergence of the reaching trajectory. This metric has been defined as the "average value of all distances from the points defining the trajectory and the straight line defining the minimum distance" (Zadravec & Matjačić, 2013, p. 111). This measure can be obtained from the procedure to calculate maximum perpendicular deviation, while maintaining the sign value convention. That is, the value of the sign of an outward trajectory with respect to the ideal trajectory toward the chosen target is negative.

In the case of two-dimensional time series, total curvature is calculated following Equations 1 - 4 and computing the arithmetic mean of the perpendicular distances d_s as indicated by

$$TC = \frac{1}{N}\sum_{s=1}^{N} \text{sgn}(\theta_s) \cdot d_s. \tag{19}$$

In the case of three-dimensional time series, total curvature is calculated following Equations 6 - 13 and computing the arithmetic mean of the perpendicular distances d_s as indicated by

$$TC = \frac{1}{N}\sum_{s=1}^{N} \text{sgn}(\alpha_s) \cdot d_s. \tag{20}$$

References

Erb, C. D., & Marcovitch, S. (2019). Tracking the within-trial, cross-trial, and developmental dynamics of cognitive control: Evidence from the simon task. *Child Development*, *90*(6), e831–e848. doi: 10.1111/cdev.13111

Hehman, E., Stolier, R. M., & Freeman, J. B. (2015). Advanced mouse-tracking analytic techniques for enhancing psychological science. *Group Processes & Intergroup Relations*, *18*(3), 384–401. doi: 10.1177/1368430214538325

Kieslich, P. J., Henninger, F., Wulff, D. U., Haslbeck, J. M., & Schulte-Mecklenbeck, M. (2019). Mouse-tracking: A practical guide to implementation and analysis 1. In *A handbook of process tracing methods* (pp. 111–130). Routledge.

Maldonado, M., Dunbar, E., & Chemla, E. (2019). Mouse tracking as a window into decision making. *Behavior research methods*, *51*, 1085–1101. doi: 10.3758/s13428-018-01194-x

Zadravec, M., & Matjačić, Z. (2013). Planar arm movement trajectory formation: an optimization based simulation study. *Biocybernetics and Biomedical Engineering*, *33*(2), 106–117. doi: 10.1016/j.bbe.2013.03.006