HW #4 - due on February 26, in class

Parameter estimation in a discrete dynamical system

the logistic population growth model

Consider the logistic population growth model

$$x'(t) = x(t) * [1 - x(t)] + p, \quad t > 0 \tag{1}$$
$$x(0) = \bar{x}_0 \tag{2}$$

where $p \in \mathbb{R}$ denotes a control parameter and the initial condition $\bar{x}_0$ is specified. A discrete system is obtained by applying Euler's method to the initial-value problem (1-2)

$$x_j = x_{j-1} + h\,[x_{j-1} * (1 - x_{j-1}) + p]\,, \quad j = 1:m \tag{3}$$
$$x_0 = \bar{x}_0 \tag{4}$$

with the specified initial value $\bar{x}_0$ and a time step h. Notice that if we take $h = 1/m$, then x_m is the discrete version of the solution to (1)-(2) at time $t = 1$.

The goal of a parameter optimization problem is formulated as follows: through the control parameter p, we want to bring the final state of the system x_m as close as possible to a given target value (desired state, data) $\bar{y}_m$ provided at time $t_m = m * h$.

Given the initial state $x_0 = \bar{x}_0$, the optimization problem is: find an optimal value $p^\star$ to the parameter p that minimizes the cost functional

$$f(x, p) = \frac{1}{2}(x_m - \bar{y}_m)^2 \tag{5}$$

subject to the model constraints (3).

Your job:

- **Task 1 (10 points)** Write the first-order optimality conditions for the minimization of the cost (5) subject to the constraints (3).
- **Task 2 (20 points)** Write a code for a function $[f, p, x, \lambda] = optimpmodel(\bar{x}_0, \bar{y}_m, m, h)$ that takes as input the initial state $\bar{x}_0$, the target state $\bar{y}_m$, the number of time steps m, and the step size h and returns the minimum value of the cost function f, the optimal parameter value p, the time series of model states $x = [x_1;\, x_2;\, \ldots; x_m]$ and the associated lagrange multipliers $\lambda = [\lambda_1;\, \lambda_2;, \ldots; \lambda_m]$.
- **Task 3 (20 points)** For each of the following sets of input values:

  (i) $\bar{x}_0 = 2, \bar{y}_m = 0.5, m = 100, h = 0.01$

  (ii) $\bar{x}_0 = 0.5, \bar{y}_m = 2, m = 100, h = 0.01$

  provide the following outcomes:

- the value of the optimal parameter $p^\star$
- plots of the time-series of the discrete states x_j and the Lagrange multipliers λ_j, $j = 1 : m$ (use separate figures)