HW #3 - due 02/14/2019, in class

Mth 410 students: Solve Problem 4 and one additional problem of your choice.
Mth 510 students: Solve all problems

Problem 1 (10 points) Use Lagrange multipliers theory to solve the problem:

$$\min_{(x_1,x_2)} (x_1-2)^2 + (x_2-2)^2$$

subject to the constraints

$$\begin{aligned} x_1^2 - x_2 &\le 0 \\ x_1 + x_2 &\le 2 \end{aligned}$$

Problem 2 (10 points) Consider the points O(0,0), A(a,0) and B(0,b) where $a > 0$ and $b > 0$ are given coordinates. Find the point $P(x,y)$ *inside the triangle* OAB (feasible points are interior or on the closed edges of the triangle) that minimizes the function

$$f(x,y) = 2x^2 + 2y^2 + (x-a)^2 + (y-b)^2$$

Problem 3 (10 points) Use Lagrange multipliers theory to solve the problem:

$$\min_{(x_1,x_2,\ldots,x_n)} \sum_{i=1}^{n} x_i, \quad \text{subject to } x_1x_2\ldots x_n = 1,\ x_i \ge 0,\ i = 1:n$$

Establish the arithmetic-geometric inequality: for any numbers $x_i \ge 0, i = 1:n$,

$$(x_1x_2\ldots x_n)^{1/n} \le \frac{1}{n}\sum_{i=1}^{n} x_i$$

Show that the second order sufficient conditions are satisfied at the optimal solution.

Problem 4 (30 points) In the context of the optimal portfolio selection problem discussed in class, consider a symmetric and positive definite covariance matrix $\mathbf{Q} \in \mathbb{R}^{n\times n}$ and the constrained optimization problem (optimal portfolio design): find $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ that minimizes the variance $\sigma^2(\mathbf{x})$:

$$\min_{\mathbf{x}\in\mathbb{R}^n} \mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} \tag{1}$$

subject to the constraints

$$\sum_{i=1}^{n} x_i = 1, \quad \sum_{i=1}^{n} \bar{r}_i x_i = l \tag{2}$$

where $r_i > 0, i = 1:n$ are the mean/expected rates of return and $l > 0$ is the mean/expected level of return. For simplicity, assume that the data $(\mathbf{Q}, \bar{\mathbf{r}}, l)$ is given such that the optimal solution satisfies $x_i^\star > 0, i = 1:n$.

(10 points) Find the solution to the equality-constrained problem (1-2). **MTH 510 students**: Show that the second-order sufficient conditions are satisfied at the optimal solution.

Use a computer to complete the following two tasks. Consider the data ($n = 3$)

$$\mathbf{Q} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & 2 & 0 \\ -\frac{1}{3} & 0 & 3 \end{bmatrix}; \quad \bar{\mathbf{r}} = \begin{bmatrix} 1.1 \\ 2 \\ 3 \end{bmatrix} \tag{3}$$

(10 points) We may view the solution $\sigma^2(\mathbf{x}^\star)$ as a function of the (hyper)parameter l. Provide the graph of $\sigma^2(\mathbf{x}^\star)$ as a function of l, for $1.2 \le l \le 2.5$.

(10 points) Provide an estimate of the value of $l^\star$ in [1.2, 2.5] for which the variance is minimized. What is the corresponding optimal portfolio distribution $\mathbf{x}^\star(l^\star)$?