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1. $f(t) = e^{-3t} + 3\,step(t)$

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[e^{-3t} + 3\,step(t)] = \mathcal{L}[e^{-3t}] + 3\mathcal{L}[step(t)] =$$

$$\int_0^\infty e^{-3t} e^{-st} dt + 3\int_0^\infty step(t) e^{-st} dt = \int_0^\infty e^{-(s+3)t} dt + 3\int_0^\infty e^{-st} dt =$$

$$\lim_{\tau \to \infty} \left(\left. \frac{e^{-(s+3)t}}{-(s+3)} \right|_0^\tau \right) + 3\lim_{\tau \to \infty} \left(\left. \frac{e^{-st}}{-s} \right|_0^\tau \right) =$$

$$\lim_{\tau \to \infty} \left(\frac{e^{-(s+3)\tau}}{-(s+3)} - \frac{e^{-(s+3)0}}{-(s+3)} \right) + 3\lim_{\tau \to \infty} \left(\frac{e^{-s\tau}}{-s} - \frac{e^{-s0}}{-s} \right)$$

Introduce $s = \sigma + j\omega$

$$F(s) = \lim_{\tau \to \infty} \left(\frac{e^{-(\sigma+3)\tau} e^{-j\omega\tau}}{-(s+3)} + \frac{1}{s+3} \right) + 3\lim_{\tau \to \infty} \left(\frac{e^{-\sigma\tau} e^{-j\omega\tau}}{-s} + \frac{1}{s} \right) =$$

$$\frac{1}{s+3} + \frac{3}{s}$$

The first Laplace integral converges if $\sigma > \sigma_c = -3$, the second Laplace integral converges if $\sigma > \sigma_c = 0$

$\Rightarrow$ the whole Laplace integral converges if $\sigma > \sigma_c = 0$.

2. The following functions are defined for $t \geq 0$

$$f_1(t) = e^{-3t} + \sin(\sqrt{3}\,t) \Rightarrow F_1(s) = \frac{1}{s+3} + \frac{\sqrt{3}}{s^2+3}$$

$$f_2(t) = -8 + \cos(t/2) \Rightarrow F_2(s) = -\frac{8}{s} + \frac{s}{s^2+\frac{1}{4}}$$

$$f_3(t) = 7e^{2t}\cos(3t) - 2e^{7t}\sin(5t)$$

$$\Rightarrow F_3(s) = \frac{7(s-2)}{(s-2)^2+9} - \frac{2\cdot 5}{(s-7)^2+25} = \frac{7(s-2)}{(s-2)^2+9} - \frac{10}{(s-7)^2+25}$$

$$f_4(t) = t^3 + t + te^{-t} + t^2e^{-3t} + te^{2t}$$

$$\Rightarrow F_4(s) = \frac{6}{s^4} + \frac{1}{s^2} + \frac{1}{(s+1)^2} + \frac{2}{(s+3)^3} + \frac{1}{(s-2)^2}$$

$$f_5(t) = 2e^{-t}\cos(t) + 3(t^4e^{-t} + e^{-3t}\sin(2t))$$

$$\Rightarrow F_5(s) = \frac{2(s+1)}{(s+1)^2+1} + \frac{3\cdot 24}{(s+1)^5} + \frac{3\cdot 2}{(s+3)^2+4} =$$

$$\frac{2(s+1)}{(s+1)^2+1} + \frac{72}{(s+1)^5} + \frac{6}{(s+3)^2+4}$$

$$f_6(t) = \mathrm{step}(t-1)\,e^{-(t-1)} + \mathrm{step}(t-3)\sin(t-3)$$

$$\Rightarrow F_6(s) = \frac{1}{s+1}e^{-s} + \frac{1}{s^2+1}e^{-3s} = \frac{e^{-s}}{s+1} + \frac{e^{-3s}}{s^2+1}$$

$f_7(t) = \text{step}(t-1) - 2\,\text{step}(t-2) + t$

$$\Rightarrow F_7(s) = \frac{1}{s}e^{-s} - \frac{2}{s}e^{-2s} + \frac{1}{s^2} = \frac{e^{-s}}{s} - \frac{2e^{-s}}{s} + \frac{1}{s^2}$$

$$f_8(t) = 2\,\text{step}(t-0.5)\,e^{-2t+1} = 2\,\text{step}(t-0.5)\,e^{-2(t-0.5)}$$

$$\Rightarrow F_8(s) = \frac{2e^{-0.5s}}{s+2}$$

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3. $f_i(t) = \mathcal{L}^{-1}[F_i(s)]$

Initial value theorem

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$$

Final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$F_1(s) = \frac{1}{s-1}$$

pole at $s = 1 \Rightarrow \lim_{t\to\infty} f_1(t)$ cannot be computed

$$\lim_{t\to 0^+} f_1(t) = \lim_{s\to\infty} \frac{s}{s-1} = \lim_{s\to\infty} \frac{\frac{s}{s}}{\frac{s}{s} - \frac{1}{s}} = 1$$

$$F_2(s) = \frac{s-1}{s(s+1)}$$

poles at $s=0$ and $s=-1$

$$\lim_{t\to 0^+} f_2(t) = \lim_{s\to\infty} \frac{s-1}{s+1} = \lim_{s\to\infty} \frac{\frac{s}{s} - \frac{1}{s}}{\frac{s}{s} + \frac{1}{s}} = 1$$

$$\lim_{t \to \infty} f_2(t) = \lim_{s \to 0} \frac{s-1}{s+1} = -1$$

$$F_3(s) = \frac{s+1}{s^2+2s+2}$$

$s^2+2s+2=0$

$\mathcal{D} = 1-2 = -1 \; (j^2)$

poles at $s = -1 \pm j$

$$\lim_{t \to 0^+} f_3(t) = \lim_{s \to \infty} \frac{s^2+s}{s^2+2s+2} = \lim_{s \to \infty} \frac{\frac{s^2}{s^2} + \frac{s}{s^2}}{\frac{s^2}{s^2} + \frac{2s}{s^2} + \frac{2}{s^2}} = 1$$

$$\lim_{t \to \infty} f_3(t) = \lim_{s \to 0} \frac{s^2+s}{s^2+2s+2} = 0$$

$$F_4(s) = -\frac{2}{s^2}$$

2 poles at 0 $\Rightarrow$ $\lim_{t \to \infty} f_4(t)$ cannot be computed

$$\lim_{t \to 0^+} f_4(t) = \lim_{s \to \infty} -\frac{2}{s} = 0$$

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4. B-2-13

a) $$F_1(s) = \frac{s+5}{(s+1)(s+3)} = \frac{a_1}{s+1} + \frac{a_2}{s+3}$$

$$a_1 = \left[\frac{s+5}{s+3}\right]_{s=-1} = 2 \qquad a_2 = \left[\frac{s+5}{s+1}\right]_{s=-3} = -1$$

$$\Rightarrow F_1(s) = \frac{2}{s+1} - \frac{1}{s+3}$$

$$\Rightarrow f_1(t) = \mathcal{L}^{-1}[F_1(s)] = 2e^{-t} - e^{-3t} \qquad t \geq 0$$

b) $$F_2(s) = \frac{3(s+4)}{s(s+1)(s+2)} = \frac{a_1}{s} + \frac{a_2}{s+1} + \frac{a_3}{s+2}$$

$$a_1 = \left[\frac{3(s+4)}{(s+1)(s+2)}\right]_{s=0} = 6 \qquad a_2 = \left[\frac{3(s+4)}{s(s+2)}\right]_{s=-1} = -9$$

$$a_3 = \left[\frac{3(s+4)}{s(s+1)}\right]_{s=-2} = 3$$

$$\Rightarrow F_2(s) = \frac{6}{s} - \frac{9}{s+1} + \frac{3}{s+2}$$

$$\Rightarrow f_2(s) = \mathcal{L}^{-1}[F_2(s)] = 6 - 9e^{-t} + 3e^{-2t} \qquad t \geq 0$$

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5. B-2-14

a) $$F_1(s) = \frac{6s+3}{s^2} = \frac{6}{s} + \frac{3}{s^2}$$

$$\Rightarrow f_1(t) = \mathcal{L}^{-1}[F_1(s)] = 6 + 3t \qquad t \geq 0$$

b) $F_2(s) = \dfrac{5s+2}{(s+1)(s+2)^2} = \dfrac{a_1}{s+1} + \dfrac{a_2}{s+2} + \dfrac{a_3}{(s+2)^2} =$

$$\frac{(s+2)^2 a_1 + (s+1)(s+2) a_2 + (s+1) a_3}{(s+1)(s+2)^2}$$

$$a_1 = \left[\frac{5s+2}{(s+2)^2}\right]_{s=-1} = -3 \qquad a_3 = \left[\frac{5s+2}{s+1}\right]_{s=-2} = 8$$

$$a_1 + a_2 = 0 \qquad 4a_1 + 3a_2 + a_3 = 5 \qquad 4a_1 + 2a_2 + a_3 = 2$$

$$\Rightarrow a_2 = -a_1 = 3$$

$$\Rightarrow F_2(s) = \frac{-3}{s+1} + \frac{3}{s+2} + \frac{8}{(s+2)^2}$$

$$\Rightarrow f_2(t) = \mathcal{L}^{-1}[F_2(s)] = -3e^{-t} + 3e^{-2t} + 8te^{-2t} \qquad t \geq 0$$

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6 B-2-16

$$F(s) = \frac{s^2 + 2s + 4}{s^2} = 1 + \frac{2}{s} + \frac{4}{s^2}$$

$$\Rightarrow f(t) = \mathcal{L}^{-1}[F(s)] = \delta(t) + 2 + 4t \qquad t \geq 0^-$$

7. B-2-17

$$F(s) = \frac{s}{s^2+2s+10} = \frac{s}{(s+1)^2+3^2} = \frac{s+1-1}{(s+1)^2+3^2} =$$

$$\frac{s+1}{(s+1)^2+3^2} - \frac{1}{3}\,\frac{3}{(s+1)^2+3^2}$$

$$\Rightarrow f(t) = \mathcal{L}^{-1}[F(s)] = e^{-t}\cos(3t) - \frac{1}{3}e^{-t}\sin(3t) \qquad t \geq 0$$

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8. $\ddot{y}(t) - 2\dot{y}(t) - 3y(t) = 0 \qquad y(0) = 1\,,\ \dot{y}(0) = 0$

The Laplace transform of the differential equation;

$$s^2 Y(s) - s\,y(0) - \dot{y}(0) - 2\left(sY(s) - y(0)\right) - 3Y(s) = 0$$

$$s^2 Y(s) - s - 2s\,Y(s) + 2 - 3Y(s) = 0$$

$$Y(s) = \frac{s-2}{s^2-2s-3} = \frac{s-2}{(s+1)(s-3)} = \frac{a_1}{s+1} + \frac{a_2}{s-3}$$

$$a_1 = \left[\frac{s-2}{s-3}\right]_{s=-1} = \frac{3}{4} \qquad a_2 = \left[\frac{s-2}{s+1}\right]_{s=3} = \frac{1}{4}$$

$$\Rightarrow Y(s) = \frac{3}{4(s+1)} + \frac{1}{4(s-3)}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{3}{4}e^{-t} + \frac{1}{4}e^{3t} \qquad t \geq 0$$

9. $\ddot{y}(t) - \dot{y}(t) = e^t \quad , y(0) = \dot{y}(0) = 0$

The Laplace transform of the differential equation

$$s^2 Y(s) - sy(0) - \dot{y}(0) - sY(s) + y(0) = \frac{1}{s-1}$$

$$s^2 Y(s) - sY(s) = \frac{1}{s-1}$$

$$Y(s) = \frac{1}{(s-1)(s^2- \ )} = \frac{1}{s(s-1)^2} = \frac{a_1}{s} + \frac{a_2}{s-1} + \frac{a_3}{(s-1)^2} =$$

$$\frac{(s-1)^2 a_1 + s(s-1)a_2 + s a_3}{s(s-1)^2}$$

$$a_1 = \left[\frac{1}{(s-1)^2}\right]_{s=0} = 1 \qquad a_3 = \left[\frac{1}{s}\right]_{s=1} = 1$$

$$a_1 + a_2 = 0 \qquad -2a_1 - a_2 + a_3 = 0 \qquad a_1 = 1$$

$$\Rightarrow a_2 = -a_1 = -1$$

$$\Rightarrow Y(s) = \frac{1}{s} - \frac{1}{s-1} + \frac{1}{(s-1)^2}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}[Y(s)] = 1 - e^t + te^t \qquad t \geq 0$$

10. B-2-22

$$\ddot{x} + 4x = 0 \qquad x(0) = 5 \qquad \dot{x}(0) = 0$$

The Laplace transform of the differential equation:

$$s^2 X(s) - s\,x(0) - \dot{x}(0) + 4X(s) = 0$$

$$s^2 X(s) - 5s + 4X(s) = 0$$

$$X(s) = \frac{5s}{s^2+4} = \frac{5s}{s^2+2^2}$$

$$\Rightarrow x(t) = \mathcal{L}^{-1}[X(s)] = 5\cos(2t) \qquad t \geq 0$$

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11. B-2-24

$$2\ddot{x} + 2\dot{x} + x = 1 \qquad x(0) = 0 \qquad \dot{x}(0) = 2$$

The Laplace transform of the differential equation

$$2s^2X(s) - 2s\,x(0) - 2\dot{x}(0) + 2s\,X(s) - 2x(0) + X(s) = \frac{1}{s}$$

$$2s^2 X(s) - 4 + 2s\,X(s) + X(s) = \frac{1}{s}$$

$$(2s^2 + 2s + 1)\,X(s) = \frac{1}{s} + 4$$

$$X(s) = \frac{4s+1}{s(2s^2+2s+1)} = \frac{2s+0.5}{s(s^2+s+0.5)} = \frac{a_1}{s} + \frac{a_2 s + a_3}{s^2+s+0.5}$$

$$= \frac{(s^2+s+0.5)a_1 + s(a_2 s + a_3)}{s(s^2+s+0.5)}$$

$$a_1 = \left[\frac{2s+0.5}{s^2+s+0.5}\right]_{s=0} = 1$$

$$a_1 + a_2 = 0 \qquad a_1 + a_3 = 2 \qquad 0.5a_1 = 0.5$$

$$\Rightarrow a_2 = -a_1 = -1 \qquad a_3 = 2 - a_1 = 1$$

$$\Rightarrow x(s) = \frac{1}{s} - \frac{1s-1}{s^2+s+0.5} = \frac{1}{s} - \frac{s-1}{(s+0.5)^2+0.5^2} =$$

$$\frac{1}{s} - \frac{s+0.5}{(s+0.5)^2+0.5^2} + \frac{3\cdot 0.5}{(s+0.5)^2+0.5^2}$$

$$\Rightarrow x(t) = \mathcal{L}^{-1}[X(s)] = 1 - e^{-0.5t}\cos(0.5t) + 3e^{-0.5t}\sin(0.5t)$$

$$t \geq 0$$

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12 B-2-25

$$\ddot{x} + x = \sin 3t \qquad x(0) = 0 \qquad \dot{x}(0) = 0$$

The Laplace transform of the differential equation:

$$s^2X(s) - sx(0) - \dot{x}(0) + x(s) = \frac{3}{s^2+9}$$

$$s^2X(s) + x(s) = \frac{3}{s^2+9}$$

$$x(s) = \frac{3}{(s^2+9)(s^2+1)} = \frac{a_1 s + a_2}{s^2+9} + \frac{a_3 s + a_4}{s^2+1} =$$

$$\frac{(s^2+1)(a_1 s+a_2)+(s^2+9)(a_3 s+a_4)}{(s^2+1)(s^2+9)} =$$

$$\frac{s^3 a_1 + s^2 a_2 + a_1 s + a_2 + s^3 a_3 + s^2 a_4 + 9 s a_3 + 9 a_4}{(s^2+1)(s^2+9)}$$

$\Rightarrow a_1 + a_3 = 0 \qquad a_2 + a_4 = 0 \qquad a_1 + 9a_3 = 0 \qquad a_2 + 9a_4 = 3$

$a_3 = -a_1 \qquad a_4 = -a_2 \qquad a_1 - 9a_1 = 0 \qquad a_2 - 9a_2 = 3$

$a_3 = 0 \qquad a_4 = \frac{3}{8} \qquad a_1 = 0 \qquad a_2 = -\frac{3}{8}$

$$\Rightarrow X(s) = -\frac{3}{8(s^2+9)} + \frac{3}{8(s^2+1)} = -\frac{3}{8(s^2+3^2)} + \frac{3 \cdot 1}{8(s^2+1^2)}$$

$$\Rightarrow x(t) = \mathcal{L}^{-1}[X(s)] = -\frac{1}{8}\sin(3t) + \frac{3}{8}\sin(t) \quad t \geq 0$$