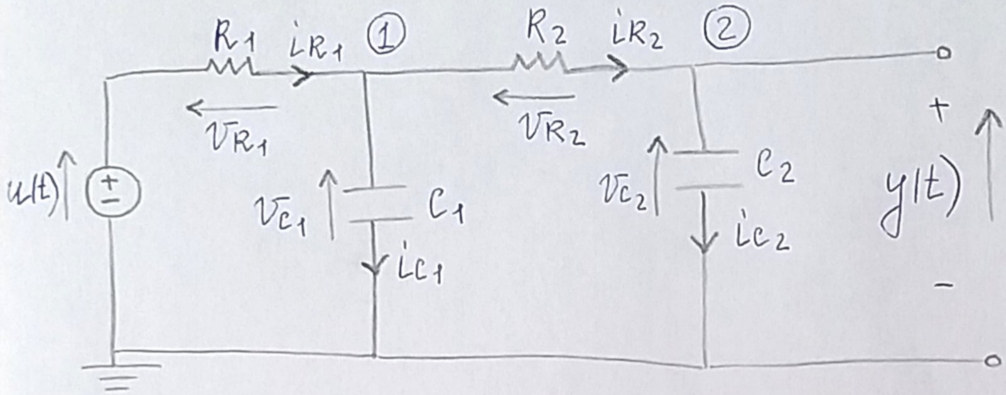
Zarema Balgabekova HW5



State variables:

Output: $y = v_{C_2} = x_2$

$x_1 = v_{C_1}$

$x_2 = v_{C_2}$

KCL ①: $i_{R_1} = i_{C_1} + i_{R_2}$

$$\frac{u - v_{C_1}}{R_1} = C_1 \frac{d v_{C_1}}{dt} + \frac{v_{C_1} - v_{C_2}}{R_2}$$

$$C_1 \frac{d v_{C_1}}{dt} = \frac{u}{R_1} - v_{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \frac{v_{C_2}}{R_2}$$

$$\frac{d v_{C_1}}{dt} = \frac{u}{C_1 R_1} - v_{C_1}\left(\frac{R_1 + R_2}{C_1 R_1 R_2}\right) + \frac{v_{C_2}}{C_1 R_2}$$

KCL ②: $i_{R_2} = i_{C_2}$

$$\frac{v_{C_1} - v_{C_2}}{R_2} = C_2 \frac{dv_{C_2}}{dt}$$

$$\frac{dv_{C_2}}{dt} = \frac{v_{C_1}}{C_2 R_2} - \frac{v_{C_2}}{C_2 R_2}$$

State-space representation (vector-matrix form):

$$\dot{x}(t) = \begin{bmatrix} -\frac{R_1 + R_2}{C_1 R_1 R_2} & \frac{1}{C_1 R_2} \\ \frac{1}{C_2 R_2} & -\frac{1}{C_2 R_2} \end{bmatrix} x(t) + \begin{bmatrix} \frac{1}{C_1 R_1} \\ 0 \end{bmatrix} u(t)$$

$$y(t) = [0 \quad 1] \, x(t)$$

$$\Rightarrow A = \begin{bmatrix} -\frac{R_1 + R_2}{C_1 R_1 R_2} & \frac{1}{C_1 R_2} \\ \frac{1}{C_2 R_2} & -\frac{1}{C_2 R_2} \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{C_1 R_1} \\ 0 \end{bmatrix}$$

$$C = [0 \quad 1] \qquad D = 0$$

if $R_1 = R_2 = C_1 = C_2 = 1 \Rightarrow$

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C = [0 \quad 1] \qquad D = 0$$

$$|\lambda I - A| = \begin{vmatrix} \lambda+2 & -1 \\ -1 & \lambda+1 \end{vmatrix} = \lambda^2 + 3\lambda + 1 = 0$$

$$\Rightarrow \lambda_1 = \frac{-3+\sqrt{5}}{2} \approx -0.382 \qquad \lambda_2 = \frac{-3-\sqrt{5}}{2} \approx -2.618$$

$Re(\lambda_1) < 0$, $Re(\lambda_2) < 0$ => the system is <u>asymptotically stable</u>

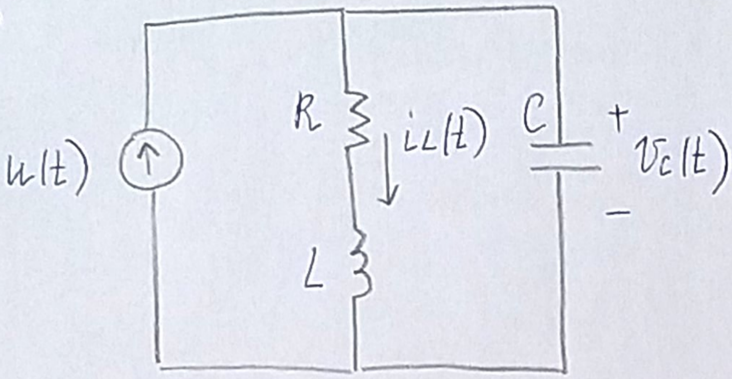
$$G(s) = C(sI-A)^{-1}B + D = [0 \quad 1] \begin{bmatrix} s+2 & -1 \\ -1 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

$$\frac{1}{s^2+3s+1} [0 \quad 1] \begin{bmatrix} s+1 & 1 \\ 1 & s+2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

$$\frac{1}{s^2+3s+1} [1 \quad s+2] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{s^2+3s+1}$$

=> <u>Transfer function</u>: $G(s) = \dfrac{1}{s^2+3s+1}$

2.



State variables

Output: $y = v_c = x_1$

$x_1 = v_c$

$x_2 = i_L$

KCL: $u(t) = i_L(t) + C \dfrac{dv_c(t)}{dt}$

$$\frac{dv_c(t)}{dt} = \frac{u(t)}{C} - \frac{i_L(t)}{C}$$

KVL: $v_c(t) = R\, i_L(t) + L \dfrac{di_L(t)}{dt}$

$$\frac{di_L(t)}{dt} = \frac{v_c(t)}{L} - \frac{R}{L} i_L(t)$$

State-space representation

$$\dot{x}(t) = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} x(t) + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

$$\Rightarrow A = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = 0$$

if $R = 2,\ C = 1,\ L = 0.5 \Rightarrow$

$$A = \begin{bmatrix} 0 & -1 \\ 2 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = 0$$

$$|\lambda I - A| = \begin{vmatrix} \lambda & 1 \\ -2 & \lambda+4 \end{vmatrix} = \lambda^2 + 4\lambda + 2 = 0$$

$$\lambda_1 = -2 + \sqrt{2} \approx -0.586 \qquad \lambda_2 = -2 - \sqrt{2} \approx -3.414$$

$Re(\lambda_1) < 0, \; Re(\lambda_2) < 0 \quad \Rightarrow$ the system is <u>asymptotically stable</u>

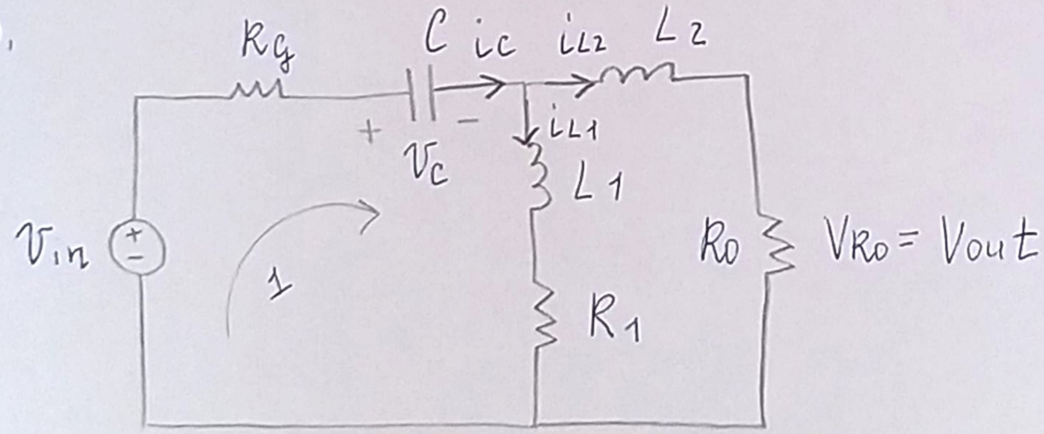
$$G(s) = C(sI - A)^{-1}B + D = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & 1 \\ -2 & s+4 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

$$\frac{1}{s^2+4s+2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+4 & -1 \\ 2 & s \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

$$\frac{1}{s^2+4s+2} \begin{bmatrix} s+4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{s+4}{s^2+4s+2}$$

$\Rightarrow$ <u>Transfer function</u>: $G(s) = \dfrac{s+4}{s^2+4s+2}$

3.



State variables:

Output $y = v_{out} = i_{L2} R_0$

$x_1 = i_{L1}$

$x_2 = i_{L2}$

$x_3 = v_C$

KCL: $i_C = i_{L1} + i_{L2}$

$$C\frac{dv_C}{dt} = i_{L1} + i_{L2}$$

$$\frac{dv_C}{dt} = \frac{i_{L1}}{C} + \frac{i_{L2}}{C}$$

KVL 1: $V_{in} = i_C R_g + v_C + L_1\frac{di_{L1}}{dt} + i_{L1}R_1$

$$L_1\frac{di_{L1}}{dt} = v_{in} - v_C - i_{L1}R_1 - (i_{L1}+i_{L2})R_g$$

$$\frac{di_{L1}}{dt} = \frac{v_{in}}{L_1} - \frac{v_C}{L_1} - i_{L1}\frac{(R_1+R_g)}{L_1} - i_{L2}\frac{R_g}{L_1}$$

KVL (outer loop): $v_{in} = i_C R_g + v_C + L_2\frac{di_{L2}}{dt} + v_{out}$

$$L_2\frac{di_{L2}}{dt} = v_{in} - v_C - i_{L2}R_o - (i_{L1}+i_{L2})R_g$$

$$\frac{di_{L2}}{dt} = \frac{v_{in}}{L_2} - \frac{v_C}{L_2} - i_{L1}\frac{R_g}{L_2} - i_{L2}\frac{(R_o+R_g)}{L_2}$$

State - space representation (vector - matrix form).

$$\dot{x}(t) = \begin{bmatrix} -\frac{R_1+R_g}{L_1} & -\frac{R_g}{L_1} & -\frac{1}{L_1} \\ -\frac{R_g}{L_2} & -\frac{R_0+R_g}{L_2} & -\frac{1}{L_2} \\ \frac{1}{C} & \frac{1}{C} & 0 \end{bmatrix} x(t) + \begin{bmatrix} \frac{1}{L_1} \\ \frac{1}{L_2} \\ 0 \end{bmatrix} u(t)$$

$$y(t) = [0 \quad R_0 \quad 0]\, x(t)$$

$$\Rightarrow \quad A = \begin{bmatrix} -\frac{R_1+R_g}{L_1} & -\frac{R_g}{L_1} & -\frac{1}{L_1} \\ -\frac{R_g}{L_2} & -\frac{R_0+R_g}{L_2} & -\frac{1}{L_2} \\ \frac{1}{C} & \frac{1}{C} & 0 \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{L_1} \\ \frac{1}{L_2} \\ 0 \end{bmatrix}$$

$$C = [0 \quad R_0 \quad 0] \qquad D = 0$$

if $R_g = R_1 = C = L_1 = L_2 = R_0 = 1 \Rightarrow$

$$A = \begin{bmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ 1 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad C = [0 \quad 1 \quad 0] \qquad D = 0$$

$$|\lambda I - A| = \begin{vmatrix} \lambda+2 & 1 & 1 \\ 1 & \lambda+2 & 1 \\ -1 & -1 & \lambda \end{vmatrix} = (\lambda+2)(\lambda(\lambda+2)+1) - (\cancel{\lambda}+1) + (-1+\cancel{\lambda}+2) =$$

$$(\lambda+2)(\lambda+1)^2 = 0$$

$\lambda_1 = -2 \qquad \lambda_{2,3} = -1$

$Re(\lambda_1) < 0$, $Re(\lambda_{2,3}) < 0 \Rightarrow$ the system is <u>asymptotically stable</u>

$$G(s) = C(sI-A)^{-1}B + D = [0 \ 1 \ 0] \begin{bmatrix} s+2 & 1 & 1 \\ 1 & s+2 & 1 \\ -1 & -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} =$$

$$\frac{1}{(s+2)(s^2+2s+1)} [0 \ 1 \ 0] \begin{bmatrix} s(s+2)+1 & -(s+1) & -s-1 \\ -(s+1) & s(s+2)+1 & -s-1 \\ s+1 & s+1 & (s+2)^2-1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

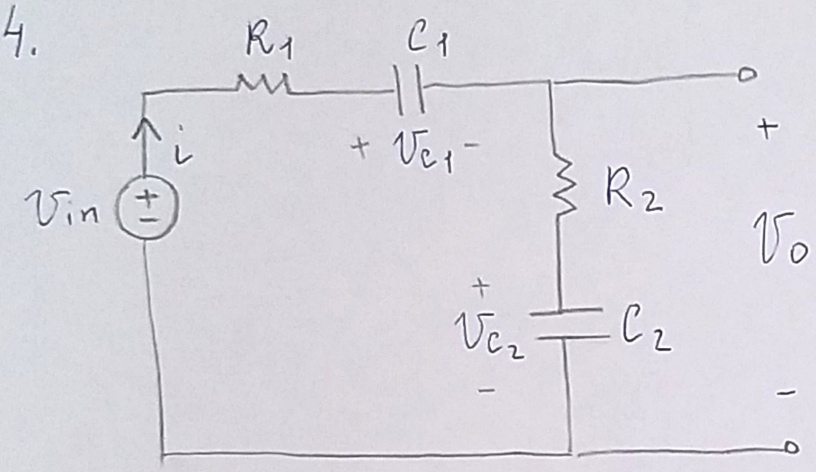
$$= \frac{1}{(s+2)(s^2+2s+1)} \begin{bmatrix} -s-1 & s^2+2s+1 & -s-1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} =$$

$$\frac{-s-1+s^2+2s+1}{(s+2)(s^2+2s+1)} = \frac{s^2+s}{(s+2)(s+1)^2} = \frac{s\cancel{(s+1)}}{(s+2)(s+1)^{\cancel{2}}} =$$

$$\frac{s}{s^2 + 3s + 2}$$

$\Rightarrow$ Transfer function: $G(s) = \dfrac{s}{s^2 + 3s + 2}$

4.



State variables

$x_1 = v_{c1}$

$x_2 = v_{c2}$

Output: $y = v_o = iR_2 + v_{c2}$

KCL (trivial): $i = C_1 \dfrac{dv_{c1}}{dt} = C_2 \dfrac{dv_{c2}}{dt}$

KVL: $v_{in} = iR_1 + v_{c_1} + iR_2 + v_{c_2}$

$$i = \frac{v_{in} - v_{c1} - v_{c2}}{R_1 + R_2}$$

$$\Rightarrow \frac{dv_{c1}}{dt} = \frac{v_{in}}{C_1(R_1+R_2)} - \frac{v_{c1}}{C_1(R_1+R_2)} - \frac{v_{c2}}{C_1(R_1+R_2)}$$

$$\frac{dv_{c2}}{dt} = \frac{v_{in}}{C_2(R_1+R_2)} - \frac{v_{c1}}{C_2(R_1+R_2)} - \frac{v_{c2}}{C_2(R_1+R_2)}$$

$$y = \frac{v_{in} \cdot R_2}{R_1+R_2} - \frac{v_{c1} R_2}{R_1+R_2} - \frac{v_{c2} R_2}{R_1+R_2} + v_{c2}$$

$$\Rightarrow y = \frac{v_{in} R_2}{R_1+R_2} - \frac{v_{c1} R_2}{R_1+R_2} + \frac{v_{c2} R_1}{R_1+R_2}$$

State-space representation (vector-matrix form)

$$\dot{x}(t) = \begin{bmatrix} -\frac{1}{C_1(R_1+R_2)} & -\frac{1}{C_1(R_1+R_2)} \\ -\frac{1}{C_2(R_1+R_2)} & -\frac{1}{C_2(R_1+R_2)} \end{bmatrix} x(t) + \begin{bmatrix} \frac{1}{C_1(R_1+R_2)} \\ \frac{1}{C_2(R_1+R_2)} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} -\frac{R_2}{R_1+R_2} & \frac{R_1}{R_1+R_2} \end{bmatrix} x(t) + \frac{R_2}{R_1+R_2} u(t)$$

$$\Rightarrow A = \begin{bmatrix} -\dfrac{1}{C_1(R_1+R_2)} & -\dfrac{1}{C_1(R_1+R_2)} \\ -\dfrac{1}{C_2(R_1+R_2)} & -\dfrac{1}{C_2(R_1+R_2)} \end{bmatrix} \qquad \mathcal{B} = \begin{bmatrix} \dfrac{1}{C_1(R_1+R_2)} \\ \dfrac{1}{C_2(R_1+R_2)} \end{bmatrix}$$

$$C = \begin{bmatrix} -\dfrac{R_2}{R_1+R_2} & \dfrac{R_1}{R_1+R_2} \end{bmatrix} \qquad \mathcal{D} = \frac{R_2}{R_1+R_2}$$

$$G(s) = C(sI-A)^{-1}\mathcal{B} + \mathcal{D}$$

Let $a = \dfrac{1}{C_1(R_1+R_2)}$ $\quad b = \dfrac{1}{C_2(R_1+R_2)}$ $\quad c = \dfrac{R_2}{R_1+R_2}$

$$\Rightarrow A = \begin{bmatrix} -a & -a \\ -b & -b \end{bmatrix} \qquad \mathcal{B} = \begin{bmatrix} a \\ b \end{bmatrix} \qquad C = \begin{bmatrix} -c & (1-c) \end{bmatrix}$$

$$\mathcal{D} = c$$

$$G(s) = \begin{bmatrix} -c & (1-c) \end{bmatrix} \begin{bmatrix} s+a & a \\ b & s+b \end{bmatrix}^{-1} \begin{bmatrix} a \\ b \end{bmatrix} + c =$$

$$\frac{1}{s^2 + s(a+b)} \begin{bmatrix} -c & (1-c) \end{bmatrix} \begin{bmatrix} s+b & -a \\ -b & s+a \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + c =$$

$$\frac{1}{s^2+s(a+b)}\begin{bmatrix} -cs-b & s(1-c)+a \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} + c =$$

$$\frac{s(b-bc-ac)}{s^2+s(a+b)} + c = \frac{cs^2+sb}{s^2+s(a+b)} = \frac{\cancel{s}(cs+b)}{\cancel{s}(s+a+b)} =$$

$$\frac{cs+b}{s+a+b} = \frac{s\left(\frac{R_2}{R_1+R_2}\right) + \frac{1}{C_2(R_1+R_2)}}{s + \frac{1}{C_1(R_1+R_2)} + \frac{1}{C_2(R_1+R_2)}} =$$

$$\frac{\frac{sR_2C_2+1}{C_2(R_1+R_2)}}{\frac{C_1C_2(R_1+R_2)s + C_2 + C_1}{C_1C_2(R_1+R_2)}} = \frac{C_1(sR_2C_2+1)}{C_1C_2(R_1+R_2)s+C_1+C_2} =$$

$$\frac{sR_2C_2+1}{C_2(R_1+R_2)s + C_2/C_1 + 1}$$

$\Rightarrow$ <u>Transfer function:</u>

$$G(s) = \frac{R_2C_2s+1}{C_2(R_1+R_2)s + C_2/C_1 + 1}$$