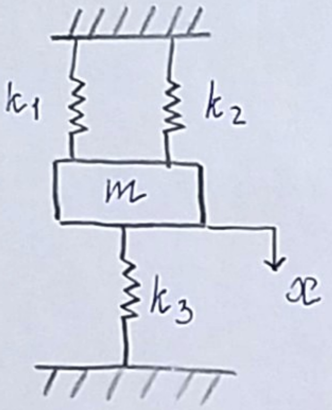
Zarema Balgabekova HW2

1. B-3-10



The equation of motion:

$$m\ddot{x}(t) = -(k_1+k_2)x(t) - k_3 x(t)$$

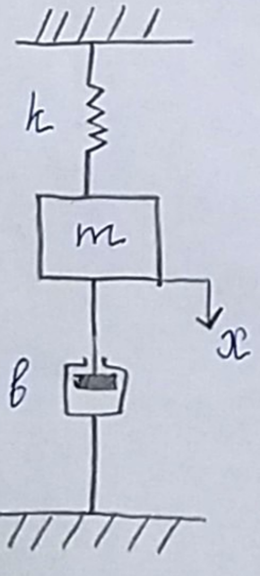
$$m\ddot{x}(t) + (k_1+k_2+k_3)x(t) = 0$$

$$\ddot{x}(t) + \frac{k_1+k_2+k_3}{m}x(t) = 0$$

$$\ddot{x}(t) + \omega_n^2 x(t) = 0$$

=> Natural frequency: $\omega_n = \sqrt{\frac{k_1+k_2+k_3}{m}}$

2. B-3-15



$m = 2\,kg$, $b = 4\,\frac{Ns}{m}$, $k = 20\,\frac{N}{m}$

$x(0) = 0.1\,m$ $\dot{x}(0) = 0$

The equation of motion

$$m\ddot{x}(t) = -kx(t) - b\dot{x}(t)$$

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = 0$$

– mathematical model of the system

$$2\ddot{x}(t) + 4\dot{x}(t) + 20x(t) = 0$$

$$\ddot{x}(t) + 2\dot{x}(t) + 10x(t) = 0$$

The Laplace transform of the equation:

$$s^2X(s) - sx(0) - \dot{x}(0) + 2sX(s) - 2x(0) + 10X(s) = 0$$

$$s^2X(s) - 0.1s + 2sX(s) - 0.2 + 10X(s) = 0$$

$$(s^2 + 2s + 10)X(s) = 0.1s + 0.2$$

$$X(s) = \frac{0.1s + 0.2}{s^2 + 2s + 10} = \frac{0.1(s+1) + 0.1}{(s+1)^2 + 3^2} = \frac{0.1(s+1)}{(s+1)^2 + 3^2} + \frac{1}{30} \cdot \frac{3}{(s+1)^2 + 3^2}$$

$$\Rightarrow x(t) = \mathcal{L}^{-1}[X(s)] = \frac{1}{10}e^{-t}\cos(3t) + \frac{1}{30}e^{-t}\sin(3t)$$

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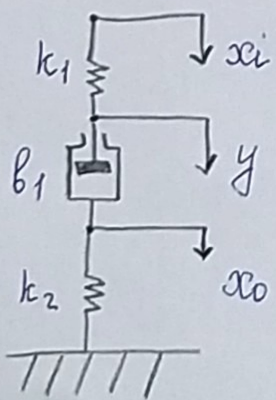
3. B-4-1



$$x_i(t) = X_i \, \mathrm{step}(t) \qquad x_o(0^-) = 0$$

The equations of motion:

$$0 = -k_1(y - x_i) - b_1(\dot{y} - \dot{x}_o)$$

$$k_1(x_i - y) = b_1(\dot{y} - \dot{x}_o) \qquad [1]$$

$$0 = -b_1(\dot{x}_o - \dot{y}) - k_2 x_o$$

$$k_2 x_o = b_1(\dot{y} - \dot{x}_o) \qquad [2]$$

As $x_i(0^-) = x_o(0^-) = y(0^-) = 0$, $\mathcal{L}$- transform of [1] and [2] equals:

$$k_1 X_i(s) - k_1 Y(s) = b_1 s\, Y(s) - b_1 s\, X_o(s) \qquad [3]$$

$$k_2 X_o(s) = b_1 s\, Y(s) - b_1 s\, X_o(s) \qquad [4]$$

From [4]: $Y(s) = \dfrac{(k_2 + b_1 s)\, X_o(s)}{b_1 s} = \left(\dfrac{k_2}{b_1 s} + 1\right) X_o(s)$

Substituting $Y(s)$ into [3]:

$$k_1 X_i(s) + b_1 s\, X_o(s) = (k_1 + b_1 s) \cdot \left(\frac{k_2}{b_1 s} + 1\right) X_o(s)$$

$$k_1 X_i(s) = \left(\frac{k_1 k_2}{b_1 s} + k_1 + k_2 + \cancel{b_1 s} - \cancel{b_1 s}\right) X_o(s)$$

$$\frac{X_o(s)}{X_i(s)} = \frac{k_1}{k_1 + k_2 + \dfrac{k_1 k_2}{b_1 s}} = \frac{k_1 b_1 s}{b_1 s(k_1 + k_2) + k_1 k_2}$$

$\Rightarrow$ <u>Transfer function</u> $G(s) = \dfrac{X_o(s)}{X_i(s)} = \dfrac{k_1 b_1 s}{b_1 s(k_1 + k_2) + k_1 k_2}$

$$X_i(s) = \frac{x_i}{s}$$

$$\Rightarrow X_o(s) = \frac{k_1 b_1 \cancel{s}}{b_1 s(k_1 + k_2) + k_1 k_2} \cdot \frac{x_i}{\cancel{s}} = \frac{\dfrac{k_1 \cancel{b_1}}{\cancel{b_1}(k_1 + k_2)}\, x_i}{s + \dfrac{k_1 k_2}{(k_1 + k_2) b_1}} =$$

$$\frac{k_1 x_i}{k_1 + k_2} \cdot \frac{1}{s + \frac{k_1 k_2}{B_1 (k_1 + k_2)}}$$

$$\Rightarrow x_0(t) = \frac{k\, x_i}{k_1 + k_2}\, e^{-\frac{k_1 k_2}{B_1 (k_1 + k_2)} t}$$

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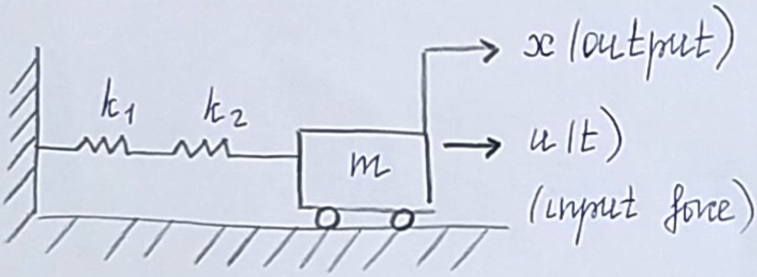
4 B-4-3



k_1 and k_2 are in series

$$\Rightarrow k_{eq} = \frac{k_1 k_2}{k_1 + k_2}$$

The equation of motion:

$$m \ddot{x}(t) = - k_{eq}\, x(t) + u(t)$$

$$m \ddot{x}(t) = - \frac{k_1 k_2}{k_1 + k_2} x(t) + u(t)$$

$$m \ddot{x}(t) + \frac{k_1 k_2}{k_1 + k_2} x(t) = u(t)$$

Assuming zero initial conditions, the Laplace transform of the equation

$$m s^2 X(s) + \frac{k_1 k_2}{k_1 + k_2} X(s) = U(s)$$

$$\frac{x(s)}{u(s)} = \frac{1}{ms^2 + \frac{k_1 k_2}{k_1+k_2}} = \frac{k_1+k_2}{m(k_1+k_2)s^2 + k_1 k_2}$$

$\Rightarrow$ Transfer function $G(s) = \frac{X(s)}{U(s)} = \frac{k_1+k_2}{m(k_1+k_2)s^2+k_1k_2}$

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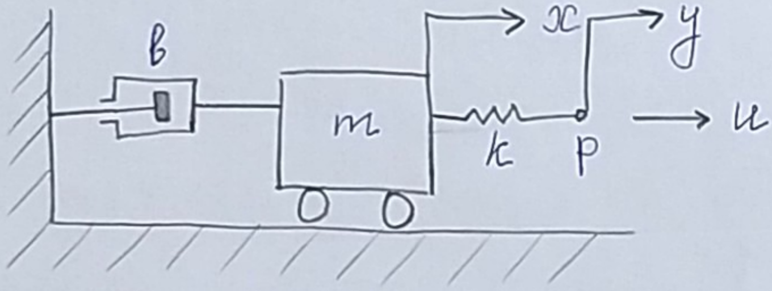
5. B-4-4

Define the point where force u is applied as P and the displacement of P as y



The equations of motion

$$0 = -k(y-x) + u \qquad u = k(y-x) \quad [1]$$

$$m\ddot{x} = -k(x-y) - b\dot{x}$$

$$m\ddot{x} + b\dot{x} = k(y-x) \quad [2]$$

From [1] and [2]

$$m\ddot{x}(t) + b\dot{x}(t) = u(t)$$

Assuming zero initial conditions, the Laplace transform of the equation:

$$m s^2 X(s) + b s X(s) = U(s)$$

$$\frac{X(s)}{U(s)} = \frac{1}{ms^2 + bs}$$

$\Rightarrow$ <u>Transfer function</u>: $G(s) = \dfrac{X(s)}{U(s)} = \dfrac{1}{ms^2+bs}$

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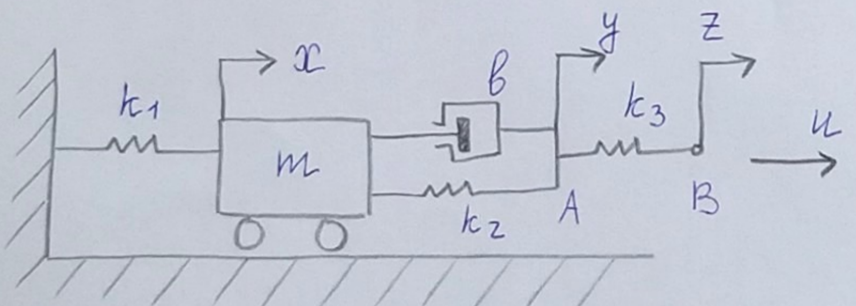
6. B-4-5



Define 2 point A and B and their displacements as y and z.

The equations of motion:

$$0 = -k_3(z-y) + u \qquad u = k_3(z-y) \qquad [1]$$

$$0 = -k_3(y-z) - b(\dot{y}-\dot{x}) - k_2(y-x)$$

$$b(\dot{y}-\dot{x}) + k_2(y-x) = k_3(z-y)$$

From [1]: $b(\dot{y}-\dot{x}) + k_2(y-x) = u \qquad [2]$

$$m\ddot{x} = -b(\dot{x}-\dot{y}) - k_2(x-y) - k_1 x$$

$$m\ddot{x} + k_1 x = b(\dot{y}-\dot{x}) + k_2(y-x)$$

From [2]: $m\ddot{x}(t) + k_1 x(t) = u(t)$ [3]

Assuming zero initial conditions, the Laplace transform of [3]:

$$ms^2 X(s) + k_1 X(s) = U(s)$$

$$\frac{X(s)}{U(s)} = \frac{1}{ms^2 + k_1}$$

=> Transfer function: $G(s) = \dfrac{X(s)}{U(s)} = \dfrac{1}{ms^2+k_1}$

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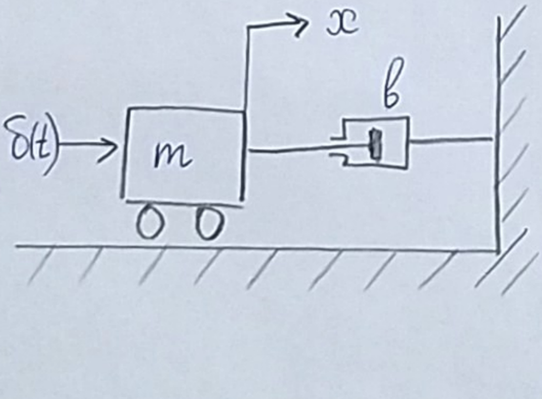
7. B-4-13

$m = 100\ kg$, $b = 200\ \frac{N\,s}{m}$



The equation of motion

$$m\ddot{x}(t) = -b\dot{x}(t) + \delta(t)$$

$$m\ddot{x}(t) + b\dot{x}(t) = \delta(t)$$

Assuming zero initial conditions, $\mathcal{L}$- transform of the equation:

$$ms^2 X(s) + bs X(s) = 1$$

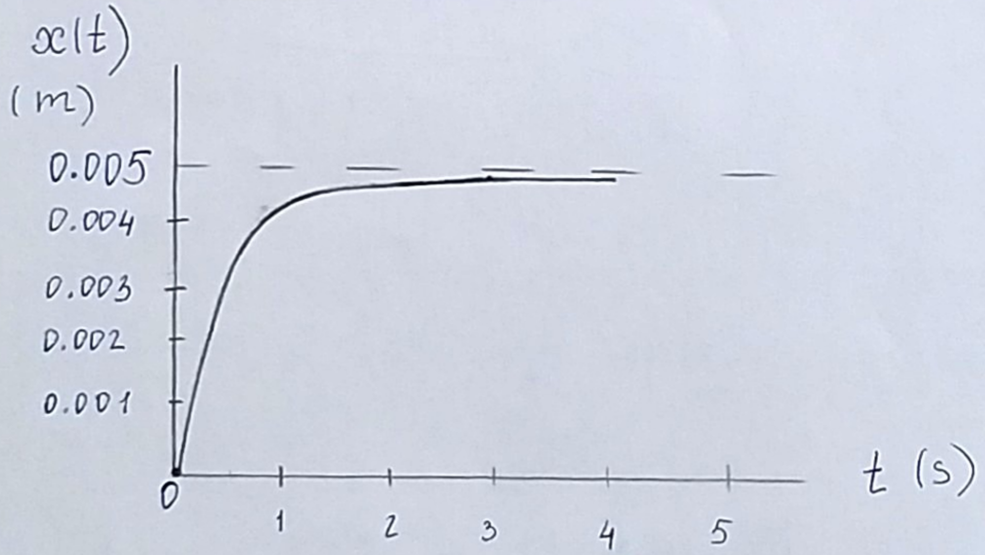
$$X(s) = \frac{1}{ms^2 + bs} = \frac{1}{100s^2 + 200s} = \frac{1}{100} \cdot \frac{1}{s(s+2)} =$$

$$\frac{a_1}{s} + \frac{a_2}{s+2}$$

$$a_1 = \left[\frac{1}{100} \cdot \frac{1}{s+2}\right]_{s=0} = \frac{1}{200} \qquad a_2 = \left[\frac{1}{100} \cdot \frac{1}{s}\right]_{s=-2} = -\frac{1}{200}$$

$$\Rightarrow X(s) = \frac{1}{200s} - \frac{1}{200(s+2)} = \frac{1}{200}\left(\frac{1}{s} - \frac{1}{s+2}\right)$$

$$\Rightarrow x(t) = \mathcal{L}^{-1}[X(s)] = \frac{1}{200}\left(1 - e^{-2t}\right) = 0.005\left(1 - e^{-2t}\right)$$



Initial value theorem:

$$x(0^+) = \lim_{s\to\infty} s\,X(s) = \lim_{s\to\infty} \left(\frac{1}{100} \cdot \frac{1}{s+2}\right) = 0$$

$$\mathcal{L}_+[\dot{x}(t)] = s\,X(s) - x(0^+) = \frac{1}{100} \cdot \frac{1}{s+2}$$

Initial velocity:

$$\dot{x}(0^+) = \lim_{s \to \infty} \left(\frac{1}{100} \cdot \frac{s}{s+2} \right) = \frac{1}{100} \lim_{s \to \infty} \left(\frac{\frac{s}{s}}{\frac{s}{s} + \frac{2}{s}} \right) =$$

$$\frac{1}{100} \cdot 1 = 0.01 \text{ m/s}$$

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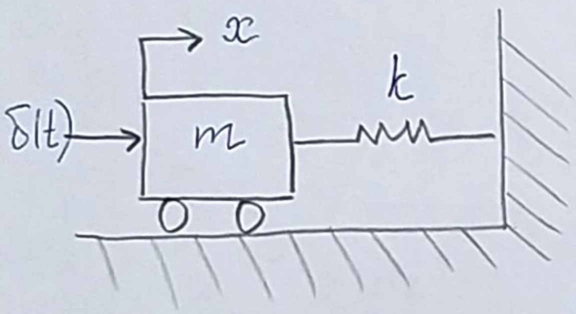
8. B-4-14

$x(0^-) = 0$, $\dot{x}(0^-) = 0$



Let $u(t) = \delta(t)$

The equation of motion:

$$m\ddot{x}(t) = -kx(t) + u(t)$$

$$m\ddot{x}(t) + kx(t) = u(t)$$

Assuming zero initial conditions, $\mathcal{L}$- transform of the equation:

$$ms^2 X(s) + kX(s) = U(s)$$

$$\frac{X(s)}{U(s)} = \frac{1}{ms^2 + k}$$

=> Transfer function: $G(s) = \frac{X(s)}{U(s)} = \frac{1}{ms^2+k}$

$$U(s) = 1$$

$$\Rightarrow X(s) = \frac{1}{ms^2+k} = \frac{1}{m} \cdot \frac{1}{s^2+\frac{k}{m}} =$$

$$\frac{1}{m} \cdot \frac{1}{s^2+\left(\sqrt{\frac{k}{m}}\right)^2} = \frac{1}{\sqrt{km}} \cdot \frac{\sqrt{\frac{k}{m}}}{s^2+\left(\sqrt{\frac{k}{m}}\right)^2}$$

$$\Rightarrow x(t) = \mathcal{L}^{-1}[X(s)] = \frac{1}{\sqrt{km}} \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

Initial value theorem:

$$x(0^+) = \lim_{s\to\infty} s X(s) = \lim_{s\to\infty}\left(\frac{1}{m} \cdot \frac{s}{s^2+\frac{k}{m}}\right) = 0$$

$$\mathcal{L}_+[\dot{x}(t)] = sX(s) - x(0^+) = \frac{1}{m} \cdot \frac{s}{s^2+\frac{k}{m}}$$

Initial velocity.

$$\dot{x}(0^+) = \lim_{s\to\infty}\left(\frac{1}{m}\frac{s^2}{s^2+\frac{k}{m}}\right) = \frac{1}{m}\lim_{s\to\infty}\left(\frac{\frac{s^2}{s^2}}{\frac{s^2}{s^2}+\frac{k}{ms^2}}\right) =$$

$$\frac{1}{m} \cdot 1 = \frac{1}{m}$$