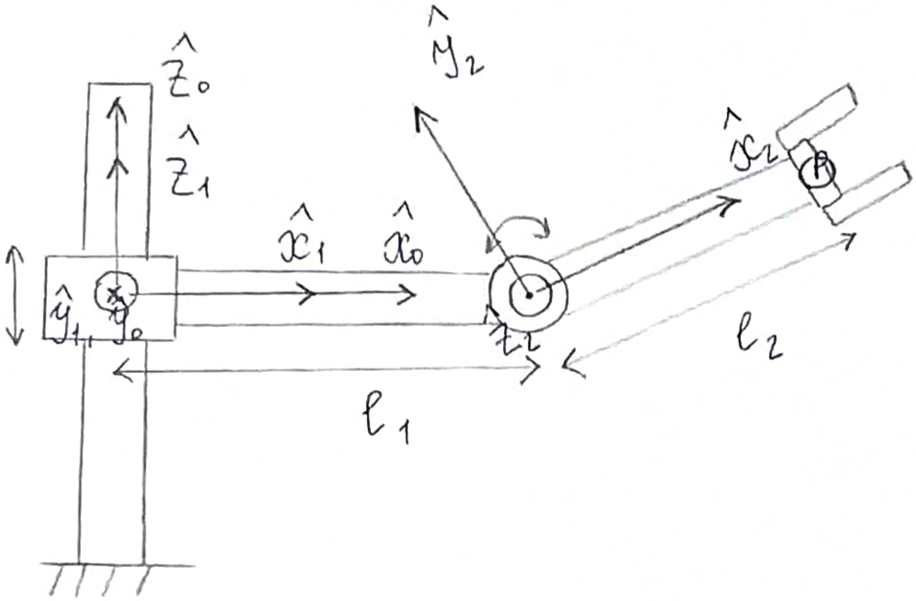
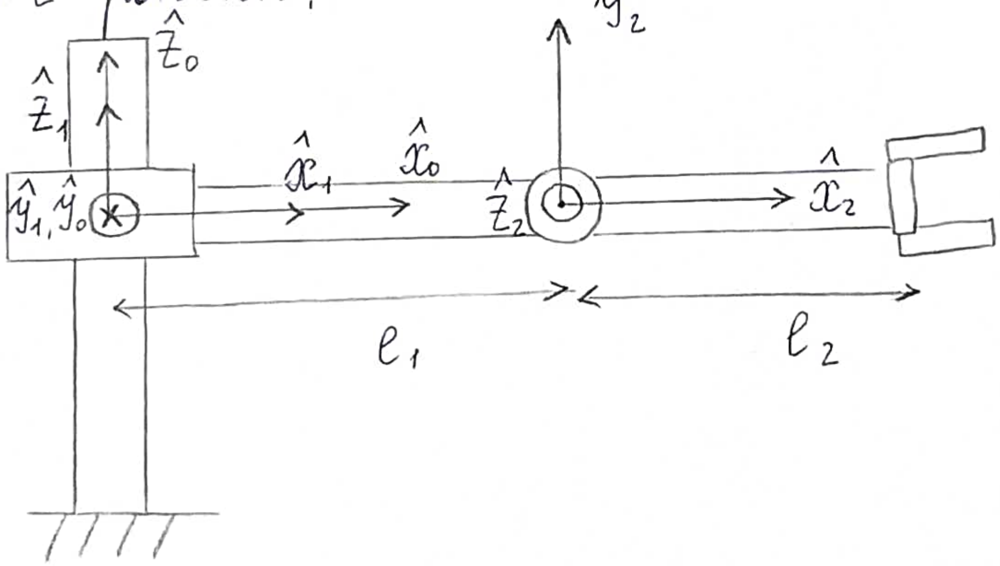
Robt 312 Zarema Balgabekova HW3



Home position:



DH parameters

Joint i	α_{i-1}	a_{i-1}	d_i	θ_i
1	$\alpha_0 = 0^\circ$	$a_0 = 0$	$d_1 = d_1$	$\theta_1 = 0^\circ$
2	$\alpha_1 = 90^\circ$	$a_1 = l_1$	$d_2 = 0$	$\theta_2 = \theta_2$

Forward kinematics:

We know that

$$ {}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ \Rightarrow {}^{0}_{1}T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & a_0 \\ s\theta_1 c\alpha_0 & c\theta_1 c\alpha_0 & -s\alpha_0 & -s\alpha_0 d_1 \\ s\theta_1 s\alpha_0 & c\theta_1 s\alpha_0 & c\alpha_0 & c\alpha_0 d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = $$

$$ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$
{}^{1}_{2}T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_1 \\ s\theta_2 c\alpha_1 & c\theta_2 c\alpha_1 & -s\alpha_1 & -s\alpha_1 d_2 \\ s\theta_2 s\alpha_1 & c\theta_2 s\alpha_1 & c\alpha_1 & c\alpha_1 d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} =
$$

$$
\begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & l_1 \\ 0 & 0 & -1 & 0 \\ s\theta_2 & c\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}_{2}T = {}^{0}_{1}T \, {}^{1}_{2}T \qquad \text{(post - multiplication)}
$$

$$
{}^{0}_{2}T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & l_1 \\ 0 & 0 & -1 & 0 \\ s\theta_2 & c\theta_2 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Inverse Kinematics:

From the manipulator, it is seen that

$$^{2}p = \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix}$$

$$^{0}p = {}^{0}_{2}T \cdot {}^{2}p = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & l_1 \\ 0 & 0 & -1 & 0 \\ s\theta_2 & c\theta_2 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_2 \\ 0 \\ 0 \\ 1 \end{bmatrix} =$$

$$\begin{bmatrix} c\theta_2 l_2 + l_1 \\ 0 \\ s\theta_2 l_2 + d_1 \\ 1 \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad p_x = c\theta_2 l_2 + l_1 \qquad [1]$$

$$p_z = s\theta_2 l_2 + d_1 \qquad [2]$$

From [1], $c\theta_2 = \dfrac{p_x - l_1}{l_2}$

Using trigonometric identity, $s\theta_2 = \pm\sqrt{1 - c^2\theta_2}$

$$\Rightarrow \quad s\theta_2 = \pm\sqrt{1 - \left(\frac{p_x - l_1}{l_2}\right)^2}$$

$$\Rightarrow \theta_2 = \operatorname{Atan2}(s\theta_2, c\theta_2) =$$

$$\operatorname{Atan2}\left(\pm\sqrt{1-\left(\frac{p_x-l_1}{l_2}\right)^2}, \frac{p_x-l_1}{l_2}\right)$$

Knowing $s\theta_2$, we can find d_1 from [2]:

$$d_1 = p_z - s\theta_2 \, l_2$$

For "arm stretched" configuration,

$$^0p = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} l_1 + l_2 \\ 0 \\ 0 \end{bmatrix}$$

$d_1 = 0 \qquad \theta_2 = 0^\circ$

Checking the inverse kinematics solution:

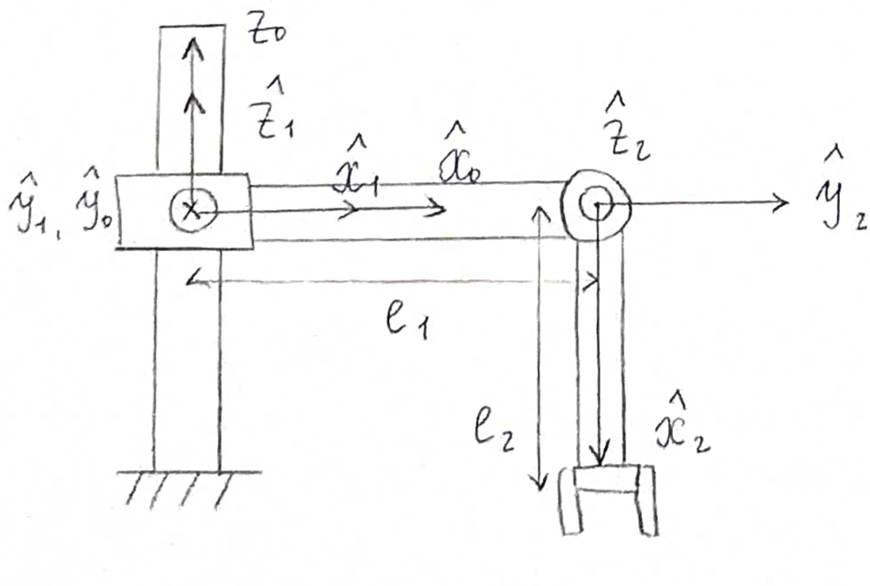
$$\theta_2 = \text{Atan2}\left(\pm\sqrt{1-\left(\frac{\cancel{\ell_1}+\ell_2-\cancel{\ell_1}}{\ell_2}\right)^2},\ \frac{\cancel{\ell_1}+\ell_2-\cancel{\ell_1}}{\ell_2}\right) =$$

$$\text{Atan2}(0, 1) = 0°$$

$$d_1 = 0 - 0 \cdot \ell_2 = 0$$

$\Rightarrow$ The solution is correct for "Arm Stretched"

"Elbow at 90 degrees with end-effector pointing the floor" looks as follows:



$\Rightarrow$ For this configuration,

$$ {}^{0}p = \begin{bmatrix} l_1 \\ 0 \\ -l_2 \end{bmatrix} $$

$\theta_2 = -90^\circ \qquad d_1 = 0$

Checking IK solution:

$$ \theta_2 = A\tan 2 \left(\pm \sqrt{1 - \left(\frac{l_1 - l_1}{l_2}\right)^2}, \frac{l_1 - l_1}{l_2} \right) = $$

$$ A\tan 2 (\pm 1, 0) = \frac{\pi}{2} \text{ and } -\frac{\pi}{2} $$

For $\theta_2 = \frac{\pi}{2}$, $\quad d_1 = -l_2 - l_2 = -2l_2$

For $\theta_2 = -\frac{\pi}{2}$, $\quad d_1 = -l_2 - (-1)\, l_2 = 0$

$\Rightarrow$ The proposed solution $(\theta_2 = -90^\circ, d_1 = 0)$ is verified, and there is one more solution $(\theta_2 = 90^\circ, d_1 = -2l_2)$

Exercise 2

Eq. 4.47 :
$$g_1 = c_2 f_1 - s_2 f_2 + a_1$$
$$g_2 = s_2 c\alpha_1 f_1 + c_2 c\alpha_1 f_2 - s\alpha_1 f_3 - d_2 s\alpha_1$$
$$g_3 = s_2 s\alpha_1 f_1 + c_2 s\alpha_1 f_2 + c\alpha_1 f_3 + d_2 c\alpha_1$$

From Pieper's Solution, we know that when the last three axes of a manipulator intersect in a single point, frames {4}, {5}, and {6} have the same origin given as:

$$^0P_{4ORG} = \begin{bmatrix} c_1 g_1 - s_1 g_2 \\ s_1 g_1 + c_1 g_2 \\ g_3 \\ 1 \end{bmatrix}$$

Calculating the squared magnitude of $^0P_{4ORG}$ and denoting it as r,

$$r = (c_1 g_1 - s_1 g_2)^2 + (s_1 g_1 + c_1 g_2)^2 + g_3^2 =$$

$$c_1^2 g_1^2 - \cancel{2 c_1 s_1 g_1 g_2} + s_1^2 g_2^2 + s_1^2 g_1^2 + \cancel{2 c_1 s_1 g_1 g_2} +$$

$$c_1^2 g_2^2 + g_3^2 = g_1^2 (c_1^2 + s_1^2) + g_2^2 (s_1^2 + c_1^2) + g_3^2 =$$

$$g_1^2 + g_2^2 + g_3^2$$

Substituting the values from Eq. 4.47,

$$r = (c_2 f_1 - s_2 f_2 + a_1)^2 + (s_2 c\alpha_1 f_1 + c_2 c\alpha_1 f_2 - s\alpha_1 f_3 - d_2 s\alpha_1)^2$$

$$+ (s_2 s\alpha_1 f_1 + c_2 s\alpha_1 f_2 + c\alpha_1 f_3 + d_2 c\alpha_1)^2 =$$

$$c_2^2 f_1^2 + s_2^2 f_2^2 + a_1^2 - 2 c_2 s_2 f_1 f_2 - 2 s_2 f_2 a_1 + 2 c_2 f_1 a_1$$

$$+ s_2^2 c\alpha_1^2 f_1^2 + c_2^2 c\alpha_1^2 f_2^2 + s\alpha_1^2 f_3^2 + d_2^2 s\alpha_1^2 + 2 s_2 c_2 c\alpha_1^2 f_1 f_2$$

$$- \cancel{2 s_2 c\alpha_1 s\alpha_1 f_1 f_3} - \cancel{2 s_2 c\alpha_1 s\alpha_1 f_1 d_2} - \cancel{2 c_2 c\alpha_1 s\alpha_1 f_2 f_3}$$

$$- \cancel{2 c_2 c\alpha_1 s\alpha_1 f_2 d_2} + 2 s\alpha_1^2 f_3 d_2 + s_2^2 s\alpha_1^2 f_1^2 +$$

$$c_2^2 s\alpha_1^2 f_2^2 + c\alpha_1^2 f_3^2 + d_2^2 c\alpha_1^2 + 2 s_2 c_2 s\alpha_1^2 f_1 f_2 +$$

$$\cancel{2 s_2 s\alpha_1 c\alpha_1 f_1 f_3} + \cancel{2 s_2 s\alpha_1 c\alpha_1 f_1 d_2} + \cancel{2 c_2 s\alpha_1 c\alpha_1 f_2 f_3}$$

$$+ \cancel{2 c_2 s\alpha_1 c\alpha_1 f_2 d_2} + 2 c\alpha_1^2 f_3 d_2$$

$$= f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 - \cancel{2c2s2f_1f_2}$$

$$-2s2f_2a_1 + 2c2f_1a_1 + \cancel{2s2c2\,c^2\alpha_1 f_1f_2} +$$

$$\underline{2s^2\alpha_1 f_3d_2} + \cancel{2s2c2\,s^2\alpha_1 f_1f_2} + \underline{2c^2\alpha_1 f_3d_2} =$$

$$f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2a_1(c2f_1 - s2f_2)$$

$$+ 2f_3d_2(s^2\alpha_1 + c^2\alpha_1) = f_1^2 + f_2^2 + f_3^2 +$$

$$a_1^2 + d_2^2 + 2d_2f_3 + 2a_1(c2f_1 - s2f_2)$$

$\Rightarrow$ We obtained <u>Eq 4.49!</u>

$$r = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2f_3 + 2a_1(c2f_1 - s2f_2)$$

Excercise 3

Eq. 4.69 ; $[{}^{0}_{3}T(\theta_2)]^{-1}\,{}^{0}_{6}T = {}^{3}_{4}T(\theta_4)\,{}^{4}_{5}T(\theta_5)\,{}^{5}_{6}T(\theta_6)$

From the forward kinematics of PUMA 560, we know that :

$${}^{0}_{3}T(\theta_2) = {}^{0}_{1}T\,\underbrace{{}^{1}_{2}T\,{}^{2}_{3}T} = {}^{0}_{1}T\,{}^{1}_{3}T \qquad (\text{post-multiplication})$$

where $${}^{0}_{1}T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{1}_{3}T = \begin{bmatrix} c23 & -s23 & 0 & a_2c_2 \\ 0 & 0 & 1 & d_3 \\ -s23 & -c23 & 0 & -a_2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$c23 = \cos(\theta_2+\theta_3)$

$s23 = \sin(\theta_2+\theta_3)$

$$\Rightarrow {}^{0}_{3}T(\theta_2) = \left[\begin{array}{ccc|c} c_1c_{23} & -c_1s_{23} & -s_1 & a_2c_1c_2 - s_1d_3 \\ s_1c_{23} & -s_1s_{23} & c_1 & a_2s_1c_2 + c_1d_3 \\ -s_{23} & -c_{23} & 0 & -a_2s_2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Knowing that $\,{}^{A}_{B}T^{-1} = {}^{B}_{A}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R^{T} & & -{}^{A}_{B}R^{T}\,{}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$,

$$
{}^{0}_{3}R^{T} = \begin{bmatrix} C1C23 & S1C23 & -S23 \\ -C1S23 & -S1S23 & -C23 \\ -S1 & C1 & 0 \end{bmatrix}
$$

$$
{}^{0}_{3}R^{T}\,{}^{0}P_{3ORG} = \begin{bmatrix} C1C23 & S1C23 & -S23 \\ -C1S23 & -S1S23 & -C23 \\ -S1 & C1 & 0 \end{bmatrix} \begin{bmatrix} a_2C_1C2 - S1d_3 \\ a_2S1C2 + C1d3 \\ -a_2S2 \end{bmatrix}
$$

$$
= \begin{bmatrix} a_2(C2C23 + S2S23) \\ -a_2\,(S23C2 - S2C23) \\ d_3\,(S_1^2 + C_1^2) \end{bmatrix} = \begin{bmatrix} a_2C3 \\ -a_2S3 \\ d_3 \end{bmatrix}
$$

$$
\Rightarrow \left[{}^{0}_{3}T\,(\theta_2)\right]^{-1} = \begin{bmatrix} C1C23 & S1C23 & -S23 & -a_2C3 \\ -C1S23 & -S1S23 & -C23 & a_2S3 \\ -S1 & C1 & 0 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Denote ${}^{0}_{6}T$ as

$$ {}^{0}_{6}T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Also, ${}^{3}_{4}T(\theta_4)\,{}^{4}_{5}T(\theta_5)\,{}^{5}_{6}T(\theta_6) = {}^{3}_{6}T$

Substituting everything into Eq. 4.69, we obtain

Eq. 4.70 :

$$ \begin{bmatrix} C1C23 & S1C23 & -S23 & -a_2C3 \\ -C1S23 & -S1S23 & -C23 & a_2S3 \\ -S1 & C1 & 0 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^{3}_{6}T $$

From the forward kinematics, we also know that:

$$ {}^{3}_{6}T = \begin{bmatrix} C_4C_5C_6 - S_4S_6 & -C_4C_5S_6 - S_4C_6 & -C_4S_5 & a_3 \\ S_5C_6 & -S_5S_6 & C_5 & d_4 \\ -S_4C_5C_6 - C_4S_6 & S_4C_5S_6 - C_4C_6 & S_4S_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Equating (1,4) and (2,4) elements from two sides of Eq. 4.70, we obtain Eq. 4.71:

$$C_1 C_{23} p_x + S_1 C_{23} p_y - S_{23} p_z - a_2 C_3 = a_3$$

$$-C_1 S_{23} p_x - S_1 S_{23} p_y - C_{23} p_z + a_2 S_3 = d_4$$

We can solve these equations simultaneously:

$$\begin{cases} S_{23} p_z - C_{23} (C_1 p_x + S_1 p_y) = -a_3 - a_2 C_3 \\ C_{23} p_z + S_{23} (C_1 p_x + S_1 p_y) = a_2 S_3 - d_4 \end{cases}$$

$$C_{23} = \frac{a_2 S_3 - d_4 - S_{23} (C_1 p_x + S_1 p_y)}{p_z}$$

$$S_{23} p_z - \frac{(a_2 S_3 - d_4)(C_1 p_x + S_1 p_y)}{p_z} + \frac{S_{23} (C_1 p_x + S_1 p_y)^2}{p_z} = -a_3 - a_2 C_3$$

$$S_{23} \left(\frac{p_z^2 + (C_1 p_x + S_1 p_y)^2}{p_z} \right) = \frac{(-a_3 - a_2 C_3) p_z + (C_1 p_x + S_1 p_y)(a_2 S_3 - d_4)}{p_z}$$

$$S_{23} = \frac{(-a_3 - a_2 C_3) p_z + (C_1 p_x + S_1 p_y)(a_2 S_3 - d_4)}{p_z^2 + (C_1 p_x + S_1 p_y)^2}$$ (Eq. 4.72)

(only S_{23})

Exercise 4

To obtain the homogeneous transform matrix ${}^{i-1}_{i}T$, we need to decompose the transformation in 2 basic rotations and 2 basic translations.

Define three intermediate frames $\{R\}$, $\{Q\}$, and $\{P\}$

$\{i-1\}$ initial frame

$\{R\}$ rotated of α_{i-1} degrees about axis $\hat{x}_{i-1}$

$\{Q\}$ translated of a_{i-1} along $\hat{x}_R$

$\{P\}$ rotated of θ_i degrees about $\hat{Z}_Q$

$\{i\}$ obtained from $\{P\}$ translating of d_i along $\hat{Z}_P$

$$\Rightarrow {}^{i-1}_{R}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha_{i-1} & -s\alpha_{i-1} & 0 \\ 0 & s\alpha_{i-1} & c\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{R}_{Q}T = \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$
{}^{Q}_{P}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{P}_{i}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

As we perform transformations relative to a moving frame, the Post-multiplication rule is used:

$$
{}^{i-1}_{i}T = R_x(\alpha_{i-1})\, D_x(a_{i-1})\, R_z(\theta_i)\, D_z(d_i) =
$$

$$
\underbrace{{}^{i-1}_{R}T \cdot {}^{R}_{Q}T} \cdot \underbrace{{}^{Q}_{P}T \cdot {}^{P}_{i}T} = {}^{i-1}_{Q}T \cdot {}^{Q}_{i}T
$$

$$ {}^{i-1}_{R}T = {}^{i-1}_{K}T \cdot {}^{K}_{R}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha_{i-1} & -S\alpha_{i-1} & 0 \\ 0 & S\alpha_{i-1} & C\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = $$

$$ \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & C\alpha_{i-1} & -S\alpha_{i-1} & 0 \\ 0 & S\alpha_{i-1} & C\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{R}_{i}T = {}^{R}_{P}T \cdot {}^{P}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} = $$

$$ \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$
{}^{i-1}_{i}T = {}^{i-1}_{R}T \cdot {}^{R}_{i}T = \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & c\alpha_{i-1} & -s\alpha_{i-1} & 0 \\ 0 & s\alpha_{i-1} & c\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ c\alpha_{i-1}s\theta_i & c\alpha_{i-1}c\theta_i & -s\alpha_{i-1} & -s\alpha_{i-1}d_i \\ s\alpha_{i-1}s\theta_i & s\alpha_{i-1}c\theta_i & c\alpha_{i-1} & c\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$