Robt 312 Zarema Balgabekova HW 2

Exercise 1

x - y - x Fixed Angles convention

Let us describe the orientation of an arbitrary frame {B} relative to the frame {A}. Assuming that in the first instance the rotation frame {B} coincides with the frame {A}, this could be done by performing three rotations about the principal axes of {A}:

1) rotate by an angle γ about $\hat{X}_A$
2) rotate by an angle β about $\hat{Y}_A$
3) rotate by an angle α about $\hat{X}_A$

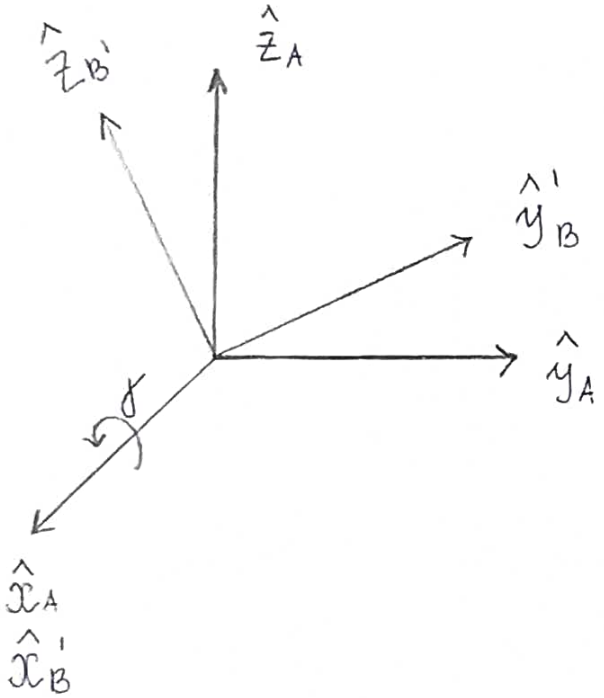
As we rotate relative to a fixed frame, the Pre-multiplication rule is used:

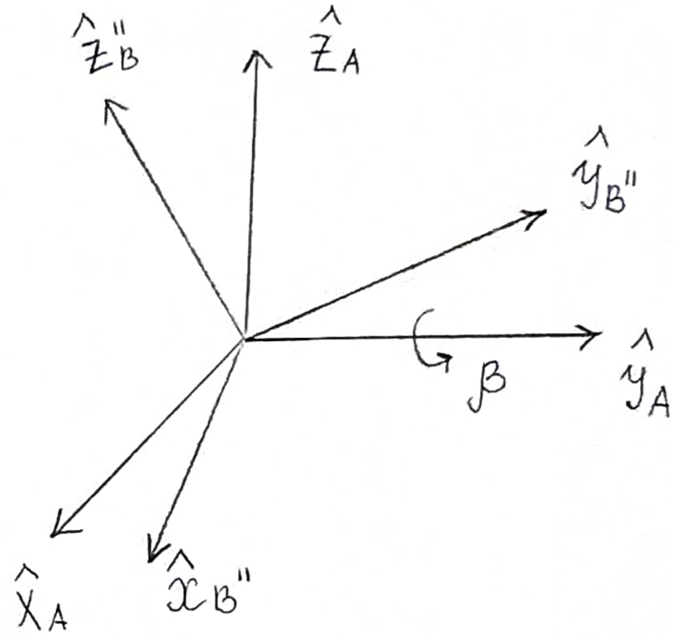
$$ {}^A_B R_{xyx}(\gamma,\beta,\alpha) = R_x(\alpha)\, R_y(\beta)\, R_x(\gamma) = $$

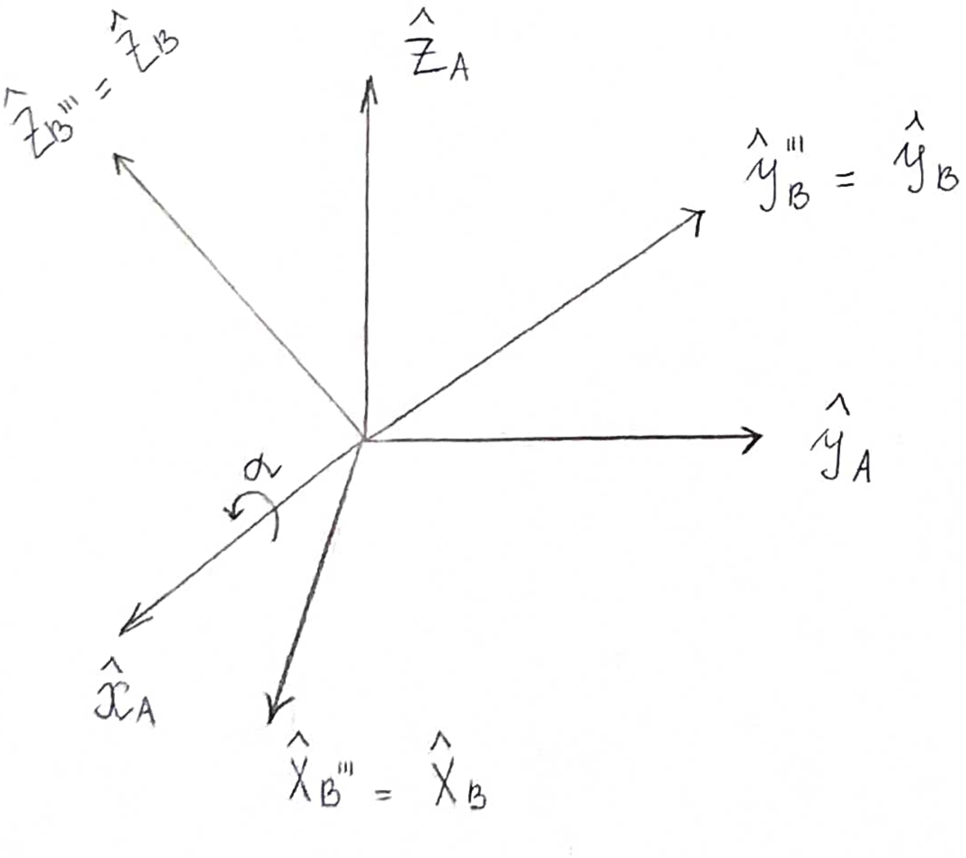
$$ \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} = $$

$$ \begin{bmatrix} c\beta & 0 & s\beta \\ s\alpha s\beta & c\alpha & -s\alpha c\beta \\ -c\alpha s\beta & s\alpha & c\alpha c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} = $$

$$\begin{bmatrix} c\beta & s\beta s\gamma & s\beta c\gamma \\ s\alpha s\beta & c\alpha c\gamma - s\alpha c\beta s\gamma & -c\alpha s\gamma - s\alpha c\beta c\gamma \\ -c\alpha s\beta & s\alpha c\gamma + c\alpha c\beta s\gamma & -s\alpha s\gamma + c\alpha c\beta c\gamma \end{bmatrix}$$







Inverse problem:

$${}^{A}_{B}R_{XYX}(\gamma, \beta, \alpha) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} =$$

$$\begin{bmatrix} c\beta & s\beta s\gamma & s\beta c\gamma \\ s\alpha s\beta & c\alpha c\gamma - s\alpha c\beta s\gamma & -c\alpha s\gamma - s\alpha c\beta c\gamma \\ -c\alpha s\beta & s\alpha c\gamma + c\alpha c\beta s\gamma & -s\alpha s\gamma + c\alpha c\beta c\gamma \end{bmatrix}$$

$$\Rightarrow \quad r_{11} = c\beta$$

$$r_{21}^2 + r_{31}^2 = s^2\alpha\, s^2\beta + c^2\alpha\, s^2\beta = s^2\beta\,\underbrace{(s^2\alpha + c^2\alpha)}_{1} = s^2\beta$$

$$s\beta = \pm\sqrt{r_{21}^2 + r_{31}^2}$$

$$\Rightarrow \quad \beta = \operatorname{Atan2}\left(\pm\sqrt{r_{21}^2 + r_{31}^2},\ r_{11}\right)$$

Considering $s\beta \neq 0$,

$$r_{21} = s\alpha\, s\beta \qquad s\alpha = \frac{r_{21}}{s\beta}$$

$$r_{31} = -c\alpha\, s\beta \qquad c\alpha = -\frac{r_{31}}{s\beta}$$

$$\Rightarrow \quad \alpha = \operatorname{Atan2}\left(\frac{r_{21}}{s\beta},\ -\frac{r_{31}}{s\beta}\right)$$

$$r_{12} = s\beta s\gamma \qquad s\gamma = \frac{r_{12}}{s\beta}$$

$$r_{13} = s\beta c\gamma \qquad c\gamma = \frac{r_{13}}{s\beta}$$

$$\Rightarrow \quad \gamma = \text{Atan2}\left(\frac{r_{12}}{s\beta}, \frac{r_{13}}{s\beta}\right)$$

If $s\beta = 0 \Rightarrow \beta = 0^\circ$ or $\beta = 180^\circ$

In this case, α can be chosen by convention as $\alpha = 0^\circ$

For $\beta = 0^\circ$,
$$r_{32} = s\alpha c\gamma + c\alpha c\beta s\gamma = s\gamma$$
$$r_{33} = -s\alpha s\gamma + c\alpha c\beta c\gamma = c\gamma$$

$$\Rightarrow \quad \beta = 0^\circ, \quad \alpha = 0^\circ, \quad \gamma = \text{Atan2}(r_{32}, r_{33})$$

For $\beta = 180^\circ$,
$$r_{32} = -s\gamma \qquad s\gamma = -r_{32}$$
$$r_{33} = -c\gamma \qquad c\gamma = -r_{33}$$

$$\Rightarrow \beta = 180^\circ, \quad \alpha = 0^\circ, \quad \gamma = \text{Atan2}(-r_{32}, -r_{33})$$

Excercise 2.

As rotations are performed with respect the moving frame, the Post-multiplication rule is used.

$$R_y(90°) \cdot R_x(120°) = \left[\cos\left(\frac{90}{2}\right) + \sin\left(\frac{90}{2}\right)(0i + 1j + 0k)\right]$$

$$\left[\cos\left(\frac{120}{2}\right) + \sin\left(\frac{120}{2}\right)(1i + 0j + 0k)\right] =$$

$$[\cos(45) + \sin(45)j]\,[\cos(60) + \sin(60)i] =$$

$$\cos(45)\cos(60) + \cos(45)\sin(60)i + \sin(45)\cos(60)j -$$

$$\sin(45)\sin(60)k = \frac{\sqrt{2}}{2}\cdot\frac{1}{2} + \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}i + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}j -$$

$$\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}k = \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}i + \frac{\sqrt{2}}{4}j - \frac{\sqrt{6}}{4}k$$

$$\Rightarrow \cos\left(\frac{\theta}{2}\right) = \frac{\sqrt{2}}{4} \qquad \theta = 2\arccos\left(\frac{\sqrt{2}}{4}\right) = 138.6°$$

$$\sin\left(\frac{138.6}{2}\right) = 0.935$$

$$\Rightarrow R_y(90°) \cdot R_x(120°) = \left[\cos\left(\frac{138.6}{2}\right) + \sin\left(\frac{138.6}{2}\right)\cdot\frac{1}{0.935}\left(\frac{\sqrt{6}}{4}i + \frac{\sqrt{2}}{4}j\right.\right.$$

$$\left.\left. - \frac{\sqrt{6}}{4}k\right)\right] = \left[\cos\left(\frac{138.6}{2}\right) + \sin\left(\frac{138.6}{2}\right)\frac{1}{3.74}\left(\sqrt{6}i + \sqrt{2}j - \sqrt{6}k\right)\right]$$

$$= R\left(\frac{\sqrt{6}i + \sqrt{2}j - \sqrt{6}k}{3.74}, 138.6°\right)$$

Exercise 3

$^{B}P_{H} = [2 \quad 2 \quad 1 \quad 1]^T$

As both translation and rotation are performed relative to the frame {A}, the Pre-multiplication rule is used:

$$
{}^{A}_{B}T = \left[\begin{array}{c|c} {}^{A}_{B}R & \begin{matrix}0\\0\\0\end{matrix} \\ \hline 0\ 0\ 0 & 1 \end{array}\right] \left[\begin{array}{c|c} I_3 & {}^{A}P_{BORG} \\ \hline 0\ 0\ 0 & 1 \end{array}\right] =
$$

$$
\left[\begin{array}{c|c} {}^{A}_{B}R & {}^{A}_{B}R \cdot {}^{A}P_{BORG} \\ \hline 0\ 0\ 0 & 1 \end{array}\right]
$$

$${}^{A}P_{BORG} = [0 \quad 0 \quad 10]^T$$

$$
{}^{A}_{B}R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c30^\circ & -s30^\circ \\ 0 & s30^\circ & c30^\circ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{bmatrix}
$$

$$
{}^{A}_{B}R \ {}^{A}P_{BORG} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 5\sqrt{3} \end{bmatrix}
$$

$$
{}^{A}_{B}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 & -5 \\ 0 & 1/2 & \sqrt{3}/2 & 5\sqrt{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{A}P_{H} = {}^{A}_{B}T \cdot {}^{B}P_{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 & -5 \\ 0 & 1/2 & \sqrt{3}/2 & 5\sqrt{3} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 2 \\ \sqrt{3} - 1/2 - 5 \\ 1 + \sqrt{3}/2 + 5\sqrt{3} \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{3} - \frac{11}{2} \\ 1 + \frac{11\sqrt{3}}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3.77 \\ 10.5 \\ 1 \end{bmatrix}
$$

Exercise 4.

As rotations are performed with respect the moving frame, the Post-multiplication rule will be used

The first rotation is the rotation about a generic axis:

$$R_K(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix}$$

, where $v\theta = 1 - c\theta$

Given $\theta = 45°$ and $\hat{K} = [k_x \ k_y \ k_z]^T = V = [1 \ 1 \ 0]^T$,

$$R_K(45) = \begin{bmatrix} 1 & 1-\sqrt{2}/2 & \sqrt{2}/2 \\ 1-\sqrt{2}/2 & 1 & -\sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

$$R_x(-90) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-90) & -\sin(-90) \\ 0 & \sin(-90) & \cos(-90) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\Rightarrow R_K(45)\, R_x(-90) = \begin{bmatrix} 1 & 1-\sqrt{2}/2 & \sqrt{2}/2 \\ 1-\sqrt{2}/2 & 1 & -\sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & -\sqrt{2}/2 & 1-\sqrt{2}/2 \\ 1-\sqrt{2}/2 & \sqrt{2}/2 & 1 \\ -\sqrt{2}/2 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & -0{,}707 & 0{,}293 \\ 0{,}293 & 0{,}707 & 1 \\ -0{,}707 & -0{,}707 & 0{,}707 \end{bmatrix}$$