D.3 Snowshoe Hares and the Canadian Lynx

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Abstract

This report investigates predator-prey dynamics using the Lotka-Volterra model, specifically focusing on simulating the population interactions between Snowshoe Hares and Canadian Lynx. Both species have exhibited population oscillations with a cycle of approximately 10 years. The goal is to determine whether the predator-prey interaction between them is the primary factor driving these cycles. Our simulations reveal that predator-prey dynamics can indeed produce cyclical behavior, closely resembling the 10-year population cycles observed in nature. However, while these results align with real-world patterns, they should not be taken as conclusive evidence that predator-prey interactions are the sole cause of these cycles. There are subregions in North America that do not follow the same cycles as others and some islands still observe the 10 year cycle in Hare population even in the absence of Lynxes on those islands. As such further investigation should be done on the unique circumstances of those regions that may cause the different cycles.

Introduction

The Canada Lynx, *Lynx canadensis* — a mid-sized wild cat — preys upon the Snowshoe Hare, *Lepus americanus*, as a primary food source; reciprocally, the Lynx is the foremost predator of the Snowshoe Hare. Both these species inhabit the Boreal forest of northern North America and have nearly perfectly overlapping ranges of native presence. The populations of both species have been observed to oscillate with a cycle of about 10 years, making them an

ideal example of cyclic predator-prey interactions and a natural test case for predator-prey models. We will be modeling their population dynamics using the classical Lotka–Volterra model, which assumes exponential prey growth and linear predator response:

$$\frac{dU}{dt} = \alpha U - \gamma UV, \quad \frac{dU}{dt} = 0 : V = \frac{\alpha}{\gamma} \quad \text{or} \quad U = 0.$$

$$\frac{dV}{dt} = \varepsilon \gamma UV - \beta V, \quad \frac{dV}{dt} = 0 : U = \frac{\beta}{\varepsilon \gamma} \quad \text{or} \quad V = 0.$$

However, under real-world conditions, factors such as intraspecies competition between prey and limits on predation exist. Thus, a more accurate model would incorporate a logistic growth model for prey and a saturated predation rate (Holling's disk equation):

$$\frac{dU}{dt} = A(U)U - \Gamma(U)V,$$

$$\frac{dV}{dt} = \varepsilon \Gamma(U)V - \beta V,$$

where

$$A(U) = \alpha \left(1 - \frac{U}{K} \right),$$

$$\Gamma(U) = \frac{\gamma U}{1 + \gamma \kappa U}.$$

- U: Prey population (Snowshoe Hare).

- V: Predator population (Canada Lynx).

- α: Prey growth rate.

- γ: Predation rate, the rate at which predators capture prey.

- β: Predator death rate.

- ε: Efficiency of converting prey into predator offspring.

- κ: Handling time or the saturation constant in Holling's disk equation.

- K: Carrying capacity of the environment for the prey population.

- $A(U)$: Prey per capita growth rate, modified for logistic growth.

- $\Gamma(U)$: Saturated predation rate, reflecting the predator's limited ability to consume prey.

In addition to the above information, all our simulations look at population growth within a 40 year period.

Results and Discussion

First, we need to compute the parameters we will be using for our simulation, including $\alpha, \gamma, \beta, \epsilon, \kappa$, and K.

We know that single-species exponential growth, $\frac{dN}{dt} = kN$, has an analytical solution $N(t) = N(0)e^{kt}$. We can show that $k = \ln(R_a)$, where R_a is the annual per capita growth rate and t is measured in years. Given that R_a is the per capita annual growth rate, to get $N(1)$, we multiply $N(0)$ by the growth rate, such that $N(1) = N(0) \times R_a$. To get $N(1)$ using kt, we have $N(1) = N(0)e^{k(1)}$ where $t = 1$. Equating the two expressions for $N(1)$, we get:

$$N(0) \times R_a = N(0)e^k \quad \Rightarrow \quad R_a = e^k \quad \Rightarrow \quad \ln(R_a) = \ln(e^k) \quad \Rightarrow \quad k = \ln(R_a).$$

Inorder to start simulating Under optimal conditions, female Hares give birth to an average of 18 young per year, 33% of which survive the first month. The monthly survival rate for Hares older than one month is 95% per month, this phase lasts from the 2nd month to the 12th month which is a total of 11 months. Given that sexual maturity is reached at one year, We calculate the annual per capita reproduction rate, measured in terms of survival till reproductive age:

$$reproductionrate = (18/2 * .33 * (.95)^{11}) \approx 1.7064$$

We assume that the Hare population is 50/50 which is why we divide 18 by 2 to give us a per capita birth rate of 9. The per capita annual death rate of adult Hares is found by calculating the survival rate of adults for 12 months at a 95% monthly survival rate:

$$\text{Death Rate} = 1 - (.95)^{12} \approx .4596 \text{ per year}$$

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The overall per capita growth rate is reproduction rate - death rate:

$$R_a = 1.7064 - .4596 = 1.2468$$

To calculate α in the Lotka-Volterra model we can set $\alpha = $ k. We can do this because if we take the integral of $\frac{dU}{dt}$ in the absence of predation:

$$\frac{dU}{dt} = \alpha U,$$

we find the solution:

$$U(t) = U_0 e^{\alpha t}.$$

This is equivalent to the single-species exponential growth equation:

$$N(t) = N(0)e^{kt},$$

where $\alpha = k$ and $N(0) = U_0$. we can use this information to set $\alpha = \ln(R_a)$ given that we previously established that $k = \ln(R_a)$. This gives us:

$$\alpha = ln(1.2468) \Rightarrow \alpha = .2206$$

To find β, if we assume 30% of Lynx die per month in absence of Hare, the monthly survival rate is 70%, as such the annual per capita death rate β is:

$$\beta = -ln(1 - (.7)^{12}) = 4.2801$$

We get the negative value of the natural log because we do not want our death rate to be negative. If a Lynx consumes 1 Hare per day and there are 1000 Hares per square km then the annual per capita predation rate, γ is:

$$\gamma = 365/1000 = .365$$

Finally to calculate ϵ we need to first establish that the mass of Lynx and Hares are 10 kg and 1.5 kg respectively, and 10% of consumed prey mass goes towards reproduction and rearing of kitten. Given this information we simplify our calculation for efficiency by assuming the entirety of the Hare is consumed and 10% of the consumed prey is used. This gives us:

$$\epsilon = \frac{1.5kg}{10kg} * .1 = .015$$

Now that we have our initial parameters, we can write the necessary code for our simulations. Figure 1 shows the code for our Lotka-Volterra model. Figure 2 shows the result of setting the starting Hare population to 400 and starting Lynx population to 1.

```matlab
%D3
global alpha gamma epsilon beta;
alpha = 0.2205; % per capita annual growth rate
beta = 4.2801; % per capita annual deatrh rate
gamma = .365; % per capita predation rate
epsilon = .015; % Efficiency

function dU = dU_dt(U, V) %Hare equation
    global alpha gamma
    dU = (alpha * U) - (gamma * U * V);
end

function dV = dV_dt(U, V) %Lynx equation
    global epsilon gamma beta
    dV = (epsilon * gamma * U * V) - (beta * V);
end

U0 = 400; %init hare pop
V0 = 1; %init lynx pop
h = .001; %Step Size
total_time = 40; %hrs
num_steps = total_time/h; %number of steps
Ut = zeros(1, num_steps+1); %initializing array to store our hare population at each step
Vt = zeros(1, num_steps+1); %initializing array to store our lynx population at each step
t_values = zeros(1, num_steps+1); %initializing array to store each step
Ut(1) = U0;
Vt(1) = V0;
t = 0;
% Euler method for population simulation
for i = 1:num_steps
    t = t + h;
    dU = dU_dt(Ut(i), Vt(i));
    dV = dV_dt(Ut(i), Vt(i));
    Ut(i+1) = Ut(i) + dU * h;
    Vt(i+1) = Vt(i) + dV * h;
    t_values(i+1) = t;
end


%code below plots population vs time
hold on
plot(t_values, Ut, 'DisplayName', 'Hare');
plot(t_values, Vt, 'DisplayName', 'Lynx');

xlabel('Time (hours)');
ylabel('Population per {km^2}');
title('Hare and Lynx growth over time');
legend(['U0 = ' num2str(U0) ''], ['V0 = ' num2str(V0)]);

hold off

% Calculate stationary points and nullclines
U_cline = beta / (epsilon * gamma); % Lynx nullcline (vertical)
V_cline = alpha / gamma;            % Hare nullcline (horizontal)

% Phase plane plot
figure;
plot(Ut, Vt, 'LineWidth', 1.5); % Hare vs Lynx trajectory
xlabel('Hare Population (per km^2)');
ylabel('Lynx Population (per km^2)');
title('Phase Plane of Hare vs Lynx Populations');
grid on;

% plots the stationary points
hold on;
plot(0, 0, 'ro', 'MarkerSize', 10, 'DisplayName', 'Stationary Point (0,0)'); % Trivial stationary point
plot(U_cline, V_cline, 'go', 'MarkerSize', 10, 'DisplayName', ['Stationary Point (', num2str(U_cline), ', ', num2str(V_cline), ')']);

% plots nullclines
yline(V_cline, 'b--', 'DisplayName', 'Hare Nullcline V* = \alpha/\gamma', 'LineWidth', 1.5);
xline(U_cline, 'm--', 'DisplayName', 'Lynx Nullcline U* = \beta/(\epsilon\gamma)', 'LineWidth', 1.5);

legend('Trajectory', 'Stationary Point (0,0)', ['Stationary Point (', num2str(U_cline), ', ', num2str(V_cline), ')']);
hold off;
```

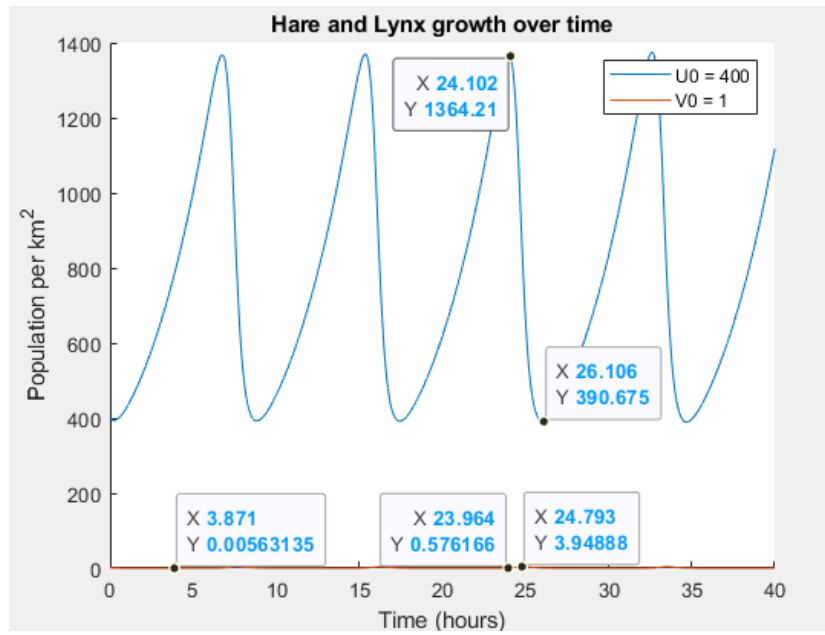Figure 1: Lotka-Volterra with logistical growth

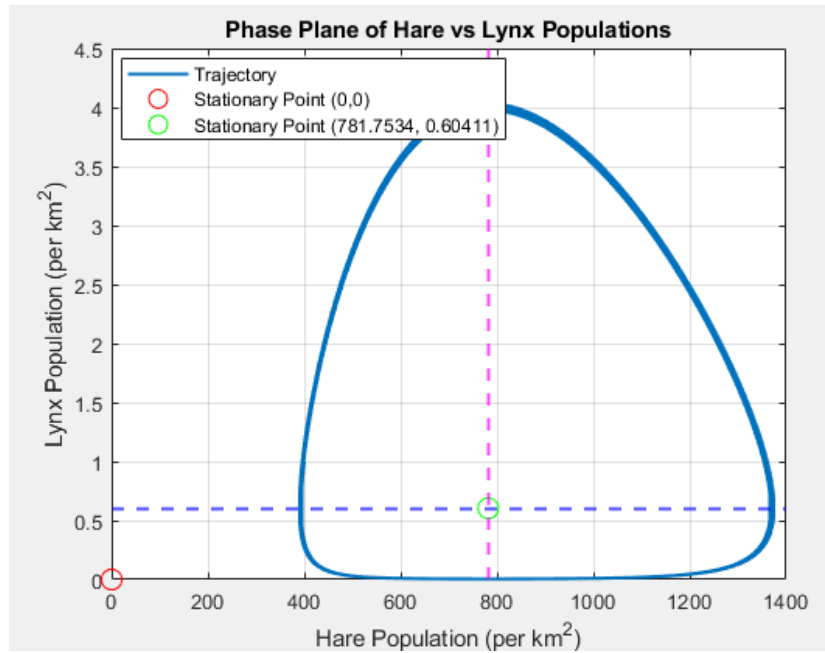Figure 2: Population over time graph, Init Hare and Lynx pop = 400, 1

Figure 3: Phase Plane of Figure 2

We see that our initial population choices cause dramatic shifts in cycles. This is because if we check the phase plane in figure 3, we see that our population is far away from the non Trivial Stationary point (781.7534, 0.60411). If the hare population was closer to 781 then the graph would have been more stable. Both species observe a less than a 10 year cycle, around 9 years. The maximum population density reached by Hares is almost 1365 and minimum is around 390. For Lynx the population reaches almost 4 and at minimum is about .0056. The amplitude for Hare growth is about 487 and Lynx amplitude is 2 which we calculate by taking half of the distance between the maximum and minimum values for each species. The Lynx population stays extremely low, only peaking to 3.9 Lynx per square km for a very short time and quickly within a year or 2 falling back to below 1 Lynx per square km. Infarct the Lynx population stays under 1 for about 2 years each cycle as seen in Figure 4.
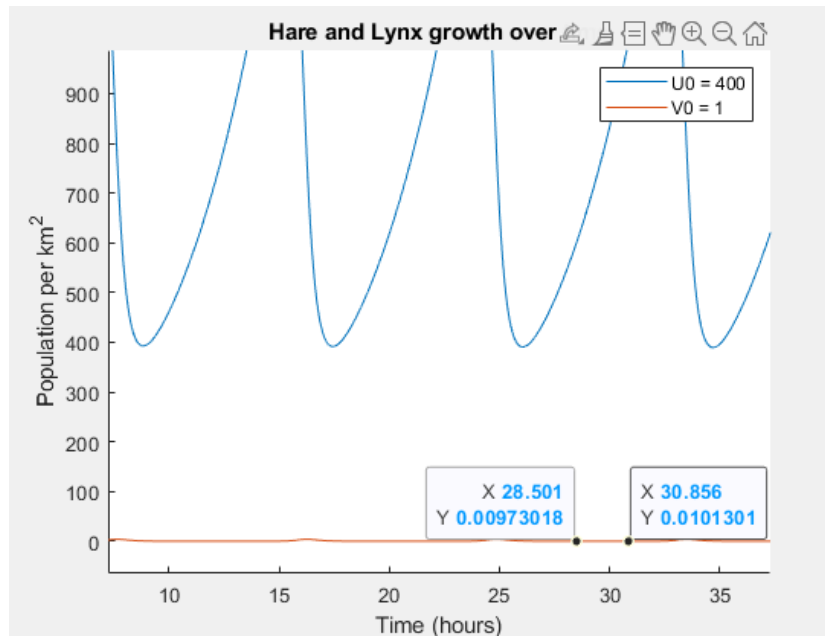

Figure 4: closer look at Lynx growth

We can update our code to store and display the greatest rate of change for each species as seen in Figure 5. Running this code gives us a maximum rate of change of 1.0911e3 Hares per year and 5.7255 Lynx per year. The Lynx population reaches its peak population right before the Hare popula-

tion reaches its lowest. Hare population reaches its peak right after Lynx population reaches its lowest point.

```
max_dU = 0; % highest rate of change for hare
max_dV = 0; % highest rate of change for lynx

% Euler method for population simulation
for i = 1:num_steps
    t = t + h;

    dU = dU_dt(Ut(i), Vt(i));
    dV = dV_dt(Ut(i), Vt(i));

    Ut(i+1) = Ut(i) + dU * h;
    Vt(i+1) = Vt(i) + dV * h;

    t_values(i+1) = t;

    % update the rate of change
    if abs(dU) > max_dU
        max_dU = abs(dU);
    end
    if abs(dV) > max_dV
        max_dV = abs(dV);
    end
end
```

Figure 5: Code to find greatest rate of change

Lets repeat these simulations with an initial Hare population of 800 and Lynx population of 2. The results are displayed in Figure 6.
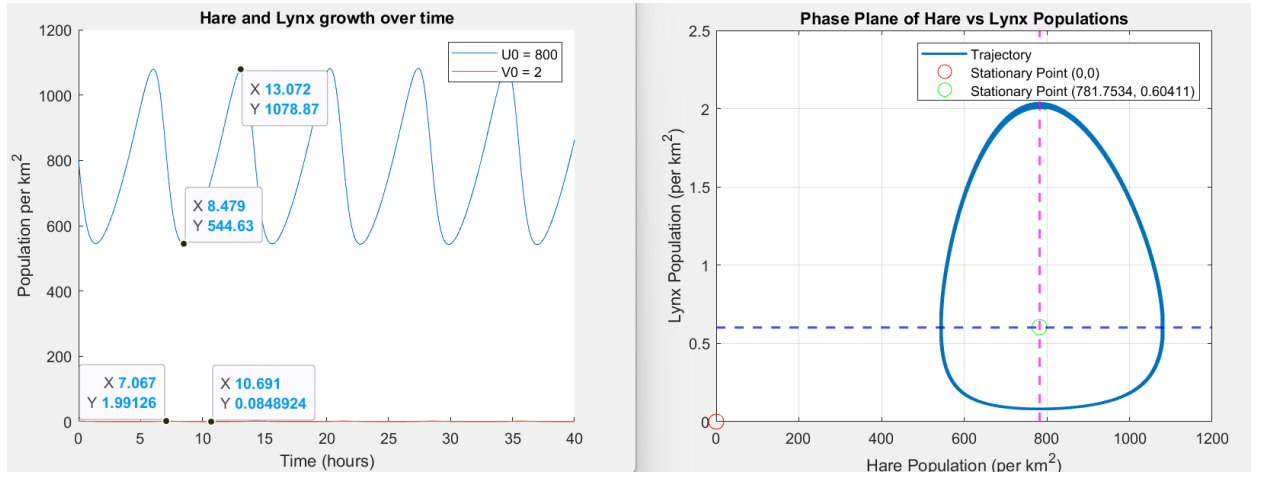
Figure 6: Initial populations: (800, 2)

There are many differences we see from changing our initial populations, however the most noticeable is that the cycles are much less dramatic as the amplitude is much lower at compared to our first graph. However this also came with shorter cycles, about 8 year cycles. We can also see that the maximum Hare population is lower at around 1080 which is almost 400 less than the maximum from our previous graph. This is likely due to having a more stable Lynx population which no longer goes below less than 1 Lynx per $100km^2$, lowest it goes is about 8 Lynx per $100km^2$. The Lynx population also doesn't exceed more than 2 Lynx per km square. This is likely due to predation rate being linearly tied to Hare population and Lynx growth rate is Linearly tied to predation rate. As such, since Hare population has a lower maximum so does the Lynx population. It is good to note that the stationary point is at (781.7354, 0.60411). Our initial Hare population is set closer to the stationary point but our initial Lynx population is now further. If our initial Lynx population stayed the same our graph would have been even more stable.

Lets repeat these simulations one more time with an initial Hare population of 200 and Lynx population of .5. The results are displayed in Figure 7.
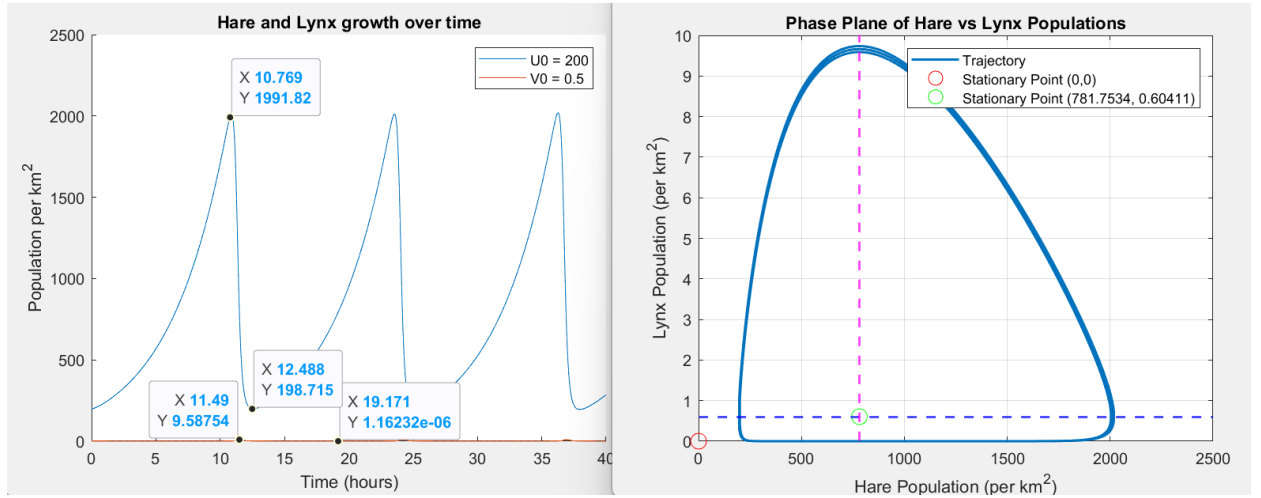
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Figure 7: Initial population: (200, .5)

We see that the populations are all over the place. The cycle was already barely stable at 400 starting Hares. Moving the initial population even further from the stationary point led to even more dramatic cycles. We observe that the cycles are now longer, between 11 to 12 years. The Lynx population reaches even more extreme lows with the lowest being around 1.1623E-06 per km squared. If we look at 100 km square area like before, we get .000116 Lynx per 100 km square which suggests extinction of the Lynx population. Yet, our Lynx population grows back up to 9.6 Lynx per 1 km squared at every cycle. The Hare population has an even higher maximum than both our previous graphs. This is likely because the death rate of Hare is dependant on predation and Lynx population. Since the Lynx population reaches such as extreme lows as we discussed before, it enables the Hare population reach greater maximum population, about 2000 Hares per square km. Similarly the higher maximum Lynx population leads to a steep decline in Hare population each cycle, at around 198.715 Hares per square km.

These unrealistic highs and lows shows the limitations of the Lotka-Volterra model stemming from its simplicity. One of the issues is that our model has an unrestricted exponential prey growth: αU. We can improve the Lotka-Volterra model by introducing a logistical prey growth model instead $\alpha \left(1 - \frac{U}{K}\right)$. We introduce a variable K which is our carrying capacity, which is the maximum number of individuals in a population that the environment

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can sustain. This variable accounts for any intraspecies competition and limited resources in the environment. Figure 8 shows the implementation of logistical growth in code.

```matlab
%D3
global alpha gamma epsilon beta K;
alpha = .2205; %
beta = 4.2801; %
gamma = .365; %
epsilon = .015; %Efficiency
K = 3000; %carrying capacity

function dU = dU_dt(U, V)
    global alpha gamma K
    dU = alpha * U * (1 - U / K) - gamma * U * V;
end

function dV = dV_dt(U, V)
    global epsilon gamma beta
    dV = epsilon * gamma * U * V - beta * V;
end

U0 = 400; %init hare pop
V0 = 1; %init lynx pop
h = .001; %Step Size
Ut = zeros(1, 40001);
Vt = zeros(1, 40001);
t_values = zeros(1, 40001);
Ut(1) = U0;
Vt(1) = V0;
t = 0;

for i = 1:40000
    t = t + h;
    Ut(i+1) = Ut(i) + dU_dt(Ut(i), Vt(i)) * h;
    Vt(i+1) = Vt(i) + dV_dt(Ut(i), Vt(i)) * h;
    t_values(i+1) = t; %we i + 1 to skip first index since we know that will 0

end
hold on
plot(t_values, Ut, 'DisplayName', 'Hare');
plot(t_values, Vt, 'DisplayName', 'Lynx');

xlabel('Time (hours)');
ylabel('Population per {km^2}');
title('Hare and Lynx growth over time');
legend('Hare', 'Lynx');

hold off

%discuss your observations, repeat sym with 800, 2 and 200, .5
% Calculate stationary points and nullclines
U_cline = beta / (epsilon * gamma); % Lynx nullcline (U*)
V_cline = (alpha / gamma) * (1 - (U_cline / K)); % Adjusted Hare nullcline (V*)

% Phase plane plot (Hare vs Lynx)
figure;
plot(Ut, Vt, 'LineWidth', 1.5); % Hare vs Lynx trajectory
xlabel('Hare Population (per km^2)');
ylabel('Lynx Population (per km^2)');
title('Phase Plane of Hare vs Lynx Populations');
grid on;

% Plot stationary points
hold on;
plot(0, 0, 'ro', 'MarkerSize', 10, 'DisplayName', 'Stationary Point (0,0)');
plot(U_cline, V_cline, 'go', 'MarkerSize', 10, 'DisplayName', ['Stationary Point (', num2str(U_cline), ', ', num2str(V_cline), ')']);

% Plot nullclines
fplot(@(U) (alpha / gamma) * (1 - U / K), [0, K], 'b--', 'DisplayName', 'Hare Nullcline V*', 'LineWidth', 1.5);
xline(U_cline, 'm--', 'DisplayName', 'Lynx Nullcline U*', 'LineWidth', 1.5);

legend('Trajectory', 'Stationary Point (0,0)', ['Stationary Point (', num2str(U_cline), ', ', num2str(V_cline), ')'], 'Hare Nullcline V*', 'Lynx Nullcline U*');
hold off;
```

Figure 8: Lotka-Volterra model with Logistic prey growth

We use a carrying capacity of 3000 per square km for our simulation. The results of the simulation are shown in Figure 9.
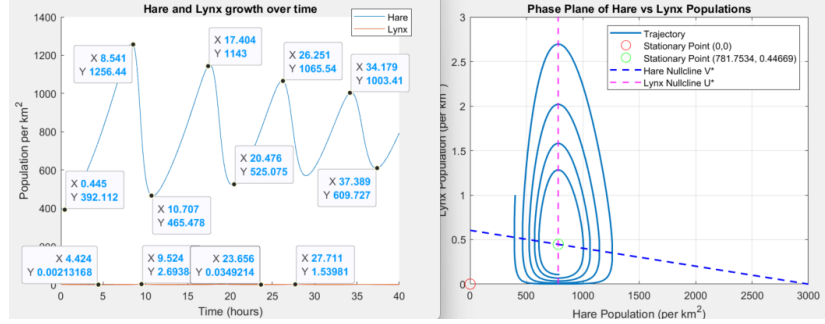
Figure 9: Results of Figure 8

Unlike our original Lotka-Volterra model, adding the logistical model is slowly shrinking the amplitude of our graphs. We see that the maximums and the minimums for both Lynx and Hare decrease each cycle. This demonstrates that the populations are moving towards an equilibrium. However within the 40 year timespan we set for ourselves, we are not able to see the populations reach equilibrium. There are two changes we can make to our model so we can see what it looks like when a predator-prey relationship reaches equilibrium. The first approach is to run the simulation for a longer period of time say 200 years or more to continue the trend of the graph until the relationship reaches equilibrium. The other approach is to lower the carrying capacity so that it has more of an impact on growth and potentially lead to the graph reaching equilibrium faster. In the latter approach, the carrying capacity needs to be higher than the null cline of the hare growth function or else it can lead to unrealistic growth behavior. Outside of the difference, the period of the cycles is about 9 years; the same case our unchanged Lotka-Volterra model when the initial population was also 400 and 1 for Hares and Lynx respectively.

Another modification we can make is to use Holling's disk equation. Currently our predation rate is linearly dependent on prey with no upperbound i.e γU. This is unrealistic as an individual predator cannot just keep consuming infinitely more food as the prey population increases. Both γUV in $\frac{dU}{dt}$, and $\varepsilon \gamma UV$ in $\frac{dV}{dt}$, depend on this predation rate. We can replace γU with the Holling's disk equation to account for the limits in predation, $\frac{\gamma U}{1+\gamma \kappa U}$. The code implementing these changes are shown in Figure 10.

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```matlab
%D3
global alpha gamma epsilon beta kappa;
alpha = .2205; %
beta = 4.2801; %
gamma = .365; %
epsilon = .015; %Efficiency
kappa = (1/6)/365; %handling time per prey caught

function dU = dU_dt(U, V)
    global alpha gamma kappa
    dU = alpha * U - ((gamma * U) / (1 + gamma * kappa * U)) * V;
end

function dV = dV_dt(U, V)
    global epsilon gamma beta kappa
    dV = epsilon * ((gamma * U) / (1 + gamma * kappa * U)) * V - beta * V;
end

U0 = 400; %init hare pop
V0 = 1; %init lynx pop
h = .001; %Step Size
Ut = zeros(1, 40001);
Vt = zeros(1, 40001);
t_values = zeros(1, 40001);
Ut(1) = U0;
Vt(1) = V0;
t = 0;

for i = 1:40000
    t = t + h;
    Ut(i+1) = Ut(i) + dU_dt(Ut(i), Vt(i)) * h;
    Vt(i+1) = Vt(i) + dV_dt(Ut(i), Vt(i)) * h;
    t_values(i+1) = t; %we i + 1 to skip first index since we know that will 0

end
hold on
plot(t_values, Ut, 'DisplayName', 'Hare');
plot(t_values, Vt, 'DisplayName', 'Lynx');

xlabel('Time (hours)');
ylabel('Population per {km^2}');
title('Hare and Lynx growth over time');
legend('Hare', 'Lynx');

hold off

%discuss your observations, repeat sym with 800, 2 and 200, .5

% Calculate stationary points and nullclines
U_cline = beta / ((epsilon - beta*kappa) * gamma); % Lynx nullcline (U*)
V_cline = (alpha / gamma) + (alpha * kappa * U_cline); % Hare nullcline (V*)

% Phase plane plot (Hare vs Lynx)
figure;
plot(Ut, Vt, 'LineWidth', 1.5); % Hare vs Lynx trajectory
xlabel('Hare Population (per km^2)');
ylabel('Lynx Population (per km^2)');
title('Phase Plane of Hare vs Lynx Populations');
grid on;

% Plot the stationary points
hold on;
plot(0, 0, 'ro', 'MarkerSize', 10, 'DisplayName', 'Stationary Point (0,0)'); % Trivial stationary point
plot(U_cline, V_cline, 'go', 'MarkerSize', 10, 'DisplayName', ['Stationary Point (', num2str(U_cline), ', ', num2str(V_cline), ')']);

% Plot nullclines
yline(V_cline, 'b--', 'DisplayName', 'Hare Nullcline V* = \alpha/\gamma', 'LineWidth', 1.5);
xline(U_cline, 'm--', 'DisplayName', 'Lynx Nullcline U* = \beta/(\epsilon\gamma)', 'LineWidth', 1.5);

legend('Trajectory', 'Stationary Point (0,0)', ['Stationary Point (', num2str(U_cline), ', ', num2str(V_cline), ')']);
hold off;
```

Figure 10: Lotka-Volterra model with Holling's disk equation

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Holling's disk uses the variable κ which is equivalent to food handling time. We assume that time to be 4 hours which is equivalent to $(1/6)/365$ of a year. We convert to years in order to stay consistent with our other variables. The results of our simulation is in Figure 11:
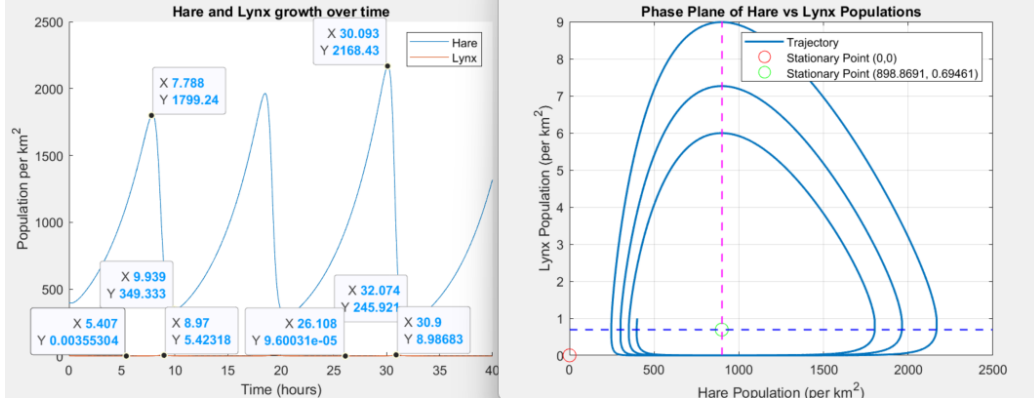

Figure 11: Results of Figure 10

Looking at Figure 11, we find that with the Holling's disk equation, the amplitudes for both functions keep increasing each cycle. The cycle has a period longer than 10 years and the Lynx population starts to effectively go extinct as we go further in time. Possibly Hares might reach extinction aswell if we run the simulation for a longer period of time. This is unexpected behavior and could be due to our choices for values for our parameters.

Inorder to get the most accurate model, we can incorporate both the logistic equation and the Holling's disk equation into the lotka-Volterra model. The code for this model is shown in Figure 12.

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```
global alpha gamma epsilon beta kappa K;
alpha = .2205; %
beta = 4.2801; %
gamma = .365; %
epsilon = .015; %Efficiency
kappa = (1/6)/365; %handling time per prey caught
K = 3000; %Carrying capacity

function dU = dU_dt(U, V)
    global alpha gamma kappa K
    dU = (alpha * U * (1 - U / K)) - ((gamma * U) / (1 + gamma * kappa * U)) * V;
end

function dV = dV_dt(U, V)
    global epsilon gamma beta kappa
    dV = (epsilon * ((gamma * U) / (1 + gamma * kappa * U)) * V) - (beta * V);
end

U0 = 400; %init hare pop
V0 = 1; %init lynx pop
h = .001; %Step Size
total_time = 40;
num_steps = floor(total_time/h);
Ut = zeros(1, num_steps + 1);
Vt = zeros(1, num_steps + 1);
t_values = zeros(1, num_steps + 1);
Ut(1) = U0;
Vt(1) = V0;
t = 0;

for i = 1:num_steps
    t = t + h;
    Ut(i+1) = Ut(i) + dU_dt(Ut(i), Vt(i)) * h;
    Vt(i+1) = Vt(i) + dV_dt(Ut(i), Vt(i)) * h;
    t_values(i+1) = t; %we i + 1 to skip first index since we know that will 0

end
hold on
plot(t_values, Ut, 'DisplayName', 'Hare');
plot(t_values, Vt, 'DisplayName', 'Lynx');

xlabel('Time (hours)');
ylabel('Population per {km^2}');
title('Hare and Lynx growth over time');
legend('Hare', 'Lynx');

hold off

%discuss your observations, repeat sym with 800, 2 and 200, .5

% Calculate stationary points and nullclines
U_cline = beta / ((epsilon - beta*kappa) * gamma); % Lynx nullcline (U*)
V_cline = (alpha / gamma) * (1 - ((1 - gamma * kappa * K) / K) * U_cline - (gamma * kappa / K) * U_cline^2); % Hare nullcline (V*)

% Phase plane plot (Hare vs Lynx)
figure;
plot(Ut, Vt, 'LineWidth', 1.5); % Hare vs Lynx trajectory
xlabel('Hare Population (per km^2)');
ylabel('Lynx Population (per km^2)');
title('Phase Plane of Hare vs Lynx Populations');
grid on;

% Plot the stationary points
hold on;
plot(0, 0, 'ro', 'MarkerSize', 10, 'DisplayName', 'Stationary Point (0,0)'); % Trivial stationary point
plot(U_cline, V_cline, 'go', 'MarkerSize', 10, 'DisplayName', ['Stationary Point (', num2str(U_cline), ', ', num2str(V_cline), ')']);

% Plot nullclines
yline(V_cline, 'b--', 'DisplayName', 'Hare Nullcline V* = \alpha/\gamma', 'LineWidth', 1.5);
xline(U_cline, 'm--', 'DisplayName', 'Lynx Nullcline U* = \beta/(\epsilon\gamma)', 'LineWidth', 1.5);

legend('Trajectory', 'Stationary Point (0,0)', ['Stationary Point (', num2str(U_cline), ', ', num2str(V_cline), ')']);
hold off;
```

Figure 12: Lotka-Volterra with both Logistic and Holling's Disk equations
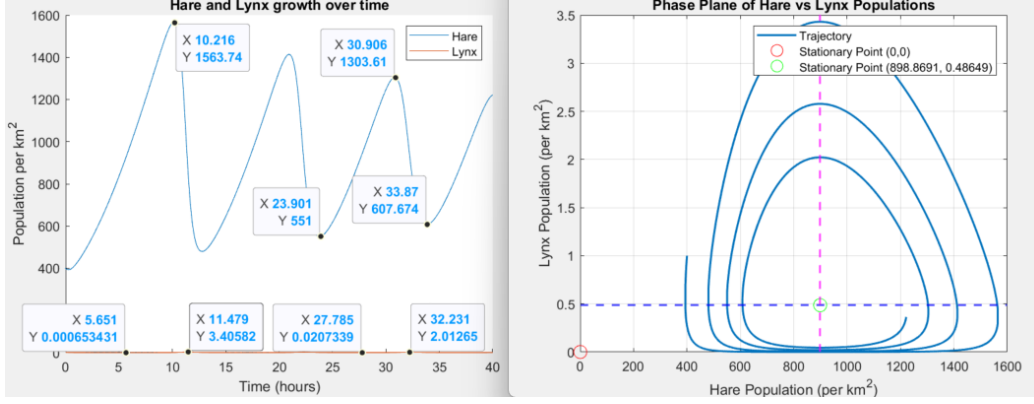
The results are shown in Figure 13.

Figure 13: Results of Figure 12

In Figure 13 we see with the addition of Logistic equation again into our model, our amplitudes starts to decrease over time for both the Lynx population and the Hare population. This behavior is the same as when we only used the Logsitic equation. The difference being the current graph inherited the longer periods we observed in the Lotka-Volterra model that only had the Holling's disk equation. Each cycle also has a greater maximum and minimum compared to their corresponding cycles in the logistic equation only model. We still see an overall converging behavor of the graphs, however the graph will likely take longer to reach equilibrium because of the longer cycle periods and the higher maximums and minimums of each cycle.

In the real world, we observe 10-year cyclic variations in lynx and rabbit populations. Our models match this cyclical behavior more or les depending on our starting populations. We find that the logistic equation only model with a starting population of 400 hares and 1 Lynx leads to an almost 10 year cycle period. Every other model comes close to the 10 year cyclic behavior when using the same starting population but not as close as the logistic only model. Accounting for any errors in calculating our parameters and possibly using better methods of choosing our parameter values, these models could perfectly match the 10 year cyclic behavior. Our models do correlate with the interpretation that predator-prey dynamics are the dominant cause of the observed 10-year cyclic variations in Lynx and Hare populations.

Interestingly the cycles over large swaths of the North American continent are synchronized, but there are multiple subregions out of phase with each

other. This behavior could be accounted for by local variations in resource availability, climate, or migration patterns, which may affect prey growth rates and predator efficiency. While our simulations do not capture multiple regions simultaneously to demonstrate these phase differences, obtaining parameters that more accurately reflect these local conditions could enable our models to better replicate the dynamics observed in individual regions.

There are also islands where lynx have gone locally extinct, yet the 10-year cycle persists in the local hare population. Island ecosystems often become self-contained, gradually diverging from the norms of mainland ecosystems. It is possible that the snowshoe hare has developed a different dominant predator on these islands, which maintains the 10-year cycle. Alternatively, the limited resources and increased intraspecies competition on the island could be driving this cyclic behavior. This isolation likely alters predator-prey dynamics, suggesting that factors other than lynx predation might be responsible for the cyclical population changes in these ecosystems.

Conclusion

In conclusion, while our models do correlate with the interpretation that predator-prey dynamics are the dominant cause of the observed 10-year cyclic variations in Lynx and Hare populations, further research is needed to make any definitive statement. Any variations in cycles in sub regions and islands can be attributed to local environmental factors such as differences in resources and existence other predators and preys in those regions. However, without actually observing and doing research into these differences in regions and learning the extent of their impacts on Lynx and Hare population, we cannot make a conclusive statement that predator-prey dynamics are the dominant cause of the observed 10-year cyclic variations in Lynx and Hare populations.