Graph
True/False questions

Gilles
gilles.richard@irit.fr

1 Introduction

The following statements may or may not be correct. In each case, either prove it (if it is correct) or give a counterexample (if it is not correct). We assume that the graph $G = (X, U, \omega)$ is non directed connected weighted graph. We do not assume that edge weights are distinct unless this is specifically stated. We denote $\#X = n$ and $\#U = m$.

2 Statements to be checked

1. If G has more than $n-1$ edges, and there is a unique heaviest edge, then this edge cannot be part of a (MST) minimum spanning tree.

   FALSE The best option to understand is to consider a triangle a, b, c and d constituting the graph G such that $\omega(a, b) = \omega(b, c) = \omega(c, a) = 1$ and $\omega(c, d) = 2$. Necessarily, any MST will contain the edge $((c, d)$ which is the heaviest one. So, the heaviest edge can belong to a MST. In fact, any unique heaviest edge that is not part of a cycle must be in the MST.

2. If G has a cycle C with a unique heaviest edge u, then u cannot be part of any MST (or no MST contains u).

   TRUE Let T a MST for G and assume that $u \in T$. Necessarily T does not contain C because T is acyclic then there is an edge $u' \in C$ which is not in T (and which is not u of course). So $T = (X, U_T, \omega)$ and $u \in U_T, u' \notin U_T$. Let us then consider the graph $T' = (X, U_T - u \cup u', \omega)$ (draw a diagram on board). Obviously $\omega(T') < \omega(T)$ because we have just replaced u by u'; Obviously T is still connected because by adding u' we have reconnected the elements which would have been disconnected by removing u. Since the number of edges in T' is exactly the number of edges in T and equals to $n-1$, we are sure T' is acyclic. Then T' is a spanning tree for G and $\omega(T') < \omega(T)$ which contradicts the fact that T is a MST. Our assumption there exists a MST T such that u is in T is false so the opposite is true. And we are done.

3. Let u be any edge of minimum weight in G. Then u must be part of some MST for G.

   TRUE Let us consider Kruskal algorithm. We know that all the minimal edges will be at the beginning of the sorted list L of edges. Putting a given u of minimum edge at the beginning of the sorted list L shows that u will appear at least in one MST (the one build by Kruskal algo.).

4. If the lightest edge u in a graph is unique, then it must be part of every MST.

   TRUE Yes, due to the same reasoning as above: this edge u is at the very beginning of the ordered list L. Kruskal will necessarily use it to build a MST.

Please come back to me if you find some mistakes in this document or if this is not clear for you.