Some Properties of Fourier Transform

1 Addition Theorem

If $g(x) \supset G(s)$ and $h(x) \supset H(s)$, and a and b are some scalars, then

$$ag(x) + bh(x) \supset aG(s) + bH(s). \tag{1}$$

Proof.

$$\begin{aligned}
\mathfrak{F}\{ag(x) + bh(x)\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [ag(x) + bh(x)]\mathrm{e}^{-\mathrm{j}2\pi sx}\,\mathrm{d}x \\
&= a\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)\mathrm{e}^{-\mathrm{j}2\pi sx}\,\mathrm{d}x + b\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x)\mathrm{e}^{-\mathrm{j}2\pi sx}\,\mathrm{d}x \\
&= aG(s) + bH(s).
\end{aligned}$$

□

2 Convolution Theorem

If $g(x) \supset G(s)$ and $h(x) \supset H(s)$, then

$$g(x) * h(x) \supset G(s)H(s). \tag{2}$$

Proof.

$$\begin{aligned}
\mathfrak{F}\{g(x) * h(x)\} &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} g(u)h(x-u)\,\mathrm{d}u\right)\mathrm{e}^{-\mathrm{j}2\pi sx}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} g(u)\left(\int_{-\infty}^{\infty} h(x-u)\mathrm{e}^{-\mathrm{j}2\pi sx}\,\mathrm{d}x\right)\mathrm{d}u \\
&= \int_{-\infty}^{\infty} g(u)\mathrm{e}^{-\mathrm{j}2\pi us}H(s)\,\mathrm{d}u \\
&= G(s)H(s).
\end{aligned}$$

□

3 Similarity Theorem

If $g(x) \supset G(s)$, then

$$g(ax) \supset \frac{1}{|a|}G\left(\frac{s}{a}\right). \tag{3}$$

Proof.

$$\begin{aligned}\int_{-\infty}^{\infty} g(ax)\mathrm{e}^{-\mathrm{j}2\pi sx}\,\mathrm{d}x &= \frac{1}{|a|}\int_{-\infty}^{\infty} g(ax)\mathrm{e}^{-\mathrm{j}2\pi(s/a)(ax)}\,\mathrm{d}(ax)\\ &= \frac{1}{|a|}G\left(\frac{s}{a}\right).\end{aligned}$$

□

This means a contraction in one domain gives rise to expansion in another domain, and vice versa.

4 Shift Theorem

If $g(x) \supset G(s)$, then

$$g(x-a) \supset \mathrm{e}^{-\mathrm{j}2\pi as}G(s). \tag{4}$$

Proof.

$$\begin{aligned}\int_{-\infty}^{\infty} g(x-a)\mathrm{e}^{-\mathrm{j}2\pi sx}\,\mathrm{d}x &= \int_{-\infty}^{\infty} g(x-a)\mathrm{e}^{-\mathrm{j}2\pi s(x-a)}\mathrm{e}^{-\mathrm{j}2\pi sa}\,\mathrm{d}(x-a)\\ &= \mathrm{e}^{-\mathrm{j}2\pi as}G(s).\end{aligned}$$

□

A shift in position in one domain gives rise to a phase change in another domain. In a similar fashion, we can show that

$$\mathrm{e}^{\mathrm{j}2\pi ax}g(x) \supset G(s-a). \tag{5}$$

5 Modulation Theorem

If $g(x) \supset G(s)$, then

$$g(x)\cos(2\pi ax) \supset \tfrac{1}{2}\left[G(s-a)+G(s+a)\right]. \tag{6}$$

Proof.

$$\begin{aligned}\mathfrak{F}\{g(x)\cos(2\pi ax)\} &= \tfrac{1}{2}\int_{-\infty}^{\infty} g(x)\mathrm{e}^{\mathrm{j}2\pi ax}\mathrm{e}^{-\mathrm{j}2\pi sx}\,\mathrm{d}x + \tfrac{1}{2}\int_{-\infty}^{\infty} g(x)\mathrm{e}^{-\mathrm{j}2\pi ax}\mathrm{e}^{-\mathrm{j}2\pi sx}\,\mathrm{d}x\\ &= \tfrac{1}{2}\int_{-\infty}^{\infty} g(x)\mathrm{e}^{-\mathrm{j}2\pi(s-a)x}\,\mathrm{d}x + \tfrac{1}{2}\int_{-\infty}^{\infty} g(x)\mathrm{e}^{-\mathrm{j}2\pi(s+a)x}\,\mathrm{d}x\\ &= \tfrac{1}{2}\left[G(s-a)+G(s+a)\right].\end{aligned}$$

□

Alternatively, if we make use of the fourier transform of a cosine and the convolution theorem in Equation 2,

$$\begin{aligned}\mathfrak{F}\{g(x)\cos(2\pi ax)\} &= \mathfrak{F}\{g(x)\} * \mathfrak{F}\{\cos(2\pi ax)\}\\ &= G(s) * \tfrac{1}{2}\left[\delta(s-a)+\delta(s+a)\right]\\ &= \tfrac{1}{2}\left[G(s-a)+G(s+a)\right].\end{aligned}$$

Multiplication with a cosine has the effect of shifting the spectrum to center on the frequency of the cosine.

6 Time Reversal Theorem

If $g(x) \supset G(s)$, then

$$g(-x) \supset G(-s). \tag{7}$$

Proof.

$$\begin{aligned}
\mathfrak{F}\{g(-x)\} &= \int_{-\infty}^{\infty} g(-x) \mathrm{e}^{-\mathrm{j}2\pi s x}\, \mathrm{d}x \\
&= \int_{\infty}^{-\infty} g(\tilde{x}) \mathrm{e}^{-\mathrm{j}2\pi(-s)\tilde{x}} (-\,\mathrm{d}\tilde{x}) \qquad (\tilde{x} = -x) \\
&= G(-s).
\end{aligned}$$

□

7 Derivative Theorem

If $g(x) \supset G(s)$, then

$$\frac{\mathrm{d}}{\mathrm{d}x} g(x) \supset \mathrm{j}2\pi s G(s). \tag{8}$$

Proof.

$$\begin{aligned}
g(x) &= \int_{-\infty}^{\infty} G(s) \mathrm{e}^{\mathrm{j}2\pi s x}\, \mathrm{d}s \\
\frac{\mathrm{d}}{\mathrm{d}x} g(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \left(\int_{-\infty}^{\infty} G(s) \mathrm{e}^{\mathrm{j}2\pi s x}\, \mathrm{d}s \right) \\
&= \int_{-\infty}^{\infty} G(s) \frac{\mathrm{d}}{\mathrm{d}x} \left(\mathrm{e}^{\mathrm{j}2\pi s x} \right) \mathrm{d}s \\
&= \int_{-\infty}^{\infty} \left[\mathrm{j}2\pi s G(s) \right] \mathrm{e}^{\mathrm{j}2\pi s x}\, \mathrm{d}s \\
\text{Hence} \qquad \mathfrak{F}\left\{ \frac{\mathrm{d}}{\mathrm{d}x} g(x) \right\} &= \mathrm{j}2\pi s G(s).
\end{aligned}$$

□

By extension, we have

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} g(x) \supset (\mathrm{j}2\pi s)^n G(s). \tag{9}$$

8 2D rotation Theorem

If $g(x, y) \supset G(f_X, f_Y)$, then

$$g(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta) \supset G(f_X\cos\theta + f_Y\sin\theta, -f_X\sin\theta + f_Y\cos\theta). \tag{10}$$

In words, that means an anti-clockwise rotation of a function by an angle θ implies that its Fourier transform is also rotated anti-clockwise by the same angle.

Proof. We can define a new coordinate system $(\check{x}, \check{y})$, where

$$\begin{bmatrix} \check{x} \\ \check{y} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{11}$$

This is shown in Figure 1. We can also express (x, y) in terms of the new coordinate system, where

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \check{x} \\ \check{y} \end{bmatrix} = \begin{bmatrix} \check{x}\cos\theta - \check{y}\sin\theta \\ \check{x}\sin\theta + \check{y}\cos\theta \end{bmatrix}.$$

Thus,

$$\begin{aligned}
\mathfrak{F}\{g(\check{x}, \check{y})\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\check{x}, \check{y}) \mathrm{e}^{-\mathrm{j}2\pi(xf_X + yf_Y)} \,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\check{x}, \check{y}) \mathrm{e}^{-\mathrm{j}2\pi(\check{x}f_X\cos\theta - \check{y}f_X\sin\theta + \check{x}f_Y\sin\theta + \check{y}f_Y\cos\theta)} \,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\check{x}, \check{y}) \mathrm{e}^{-\mathrm{j}2\pi[\check{x}(f_X\cos\theta + f_Y\sin\theta) + \check{y}(-f_X\sin\theta + f_Y\cos\theta)]} \,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\check{x}, \check{y}) \mathrm{e}^{-\mathrm{j}2\pi[\check{x}(f_X\cos\theta + f_Y\sin\theta) + \check{y}(-f_X\sin\theta + f_Y\cos\theta)]} \,\mathrm{d}\check{x}\,\mathrm{d}\check{y} \\
&= G(f_X\cos\theta + f_Y\sin\theta, -f_X\sin\theta + f_Y\cos\theta).
\end{aligned}$$

In the derivation above, we have made use of the fact that

$$\mathrm{d}\check{x}\,\mathrm{d}\check{y} = \begin{vmatrix} \frac{\partial\check{x}}{\partial x} & \frac{\partial\check{y}}{\partial x} \\ \frac{\partial\check{x}}{\partial y} & \frac{\partial\check{y}}{\partial y} \end{vmatrix} \mathrm{d}x\,\mathrm{d}y = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} \mathrm{d}x\,\mathrm{d}y = \mathrm{d}x\,\mathrm{d}y.$$

$\square$

One consequence of the two-dimensional rotation theorem is that if the 2D function is circularly symmetric, its Fourier transform must also be circularly symmetric.

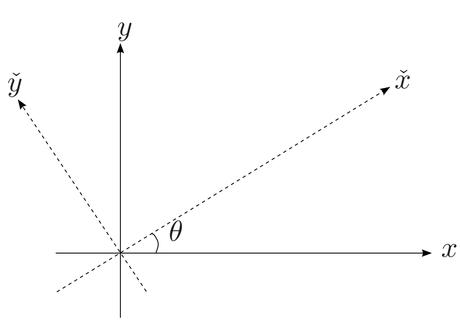

Figure 1: Illustration of a rotation in coordinates.