CSE 546 Machine Learning, Autumn 2013
Homework 3

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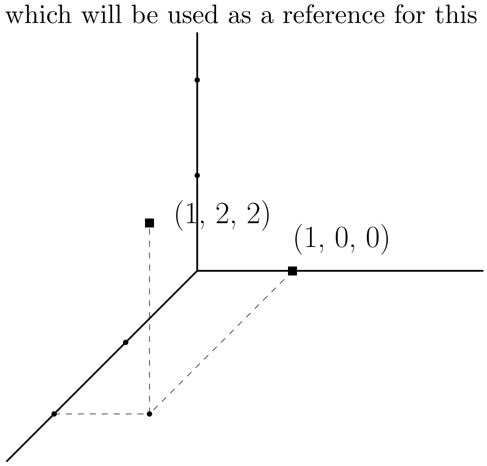
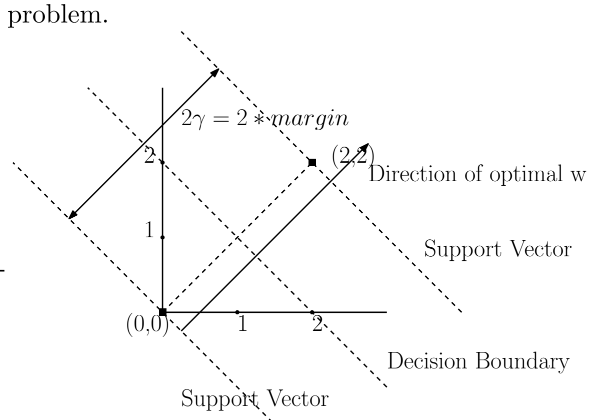
Collaborated with Nacho Cano

1 Fitting an SVM classifier by hand [25 Points]

1. $\phi(x) = [1, \sqrt{2}x, x^2]^T$. This gives $\phi(data) = [1, \sqrt{2} * 0, 0^2]^T, [1, \sqrt{2} * \sqrt{2}, \sqrt{2}^2]^T = [1, 0, 0]^T, [1, 2, 2]^T$. The following figures show the decision boundaries and margin which will be used as a reference for this problem.





Projection of the points on the 2^{nd} and 3^{rd} dimensions

From the figures it is clear that optimal w is parallel to the line joining the points $[1, 0, 0]^T \& [1, 2, 2]^T$. One such

$$w = [0, 2, 2] \;\; ...(Ans) \tag{1}$$

2. As shown in the second figure the margin is half of the distance between the points $[1, 0, 0]^T \& [1, 2, 2]^T$.

$$\gamma = \frac{1}{2}\sqrt{(1-1)^2 + (2-0) + (2-0)^2} = \sqrt{2} \;\; ...(Ans) \tag{2}$$

3. Let the optimal $w = [w_1, w_2, w_3]$

$$\gamma = \frac{1}{||w||} \Rightarrow ||w|| = \frac{1}{\sqrt{2}} \Rightarrow (w_1^2 + w_2^2 + w_3^2) = \frac{1}{2} \tag{3}$$

From figures we see that w is parallel to [0, 2, 2] and passes through origin. Hence,

$$w_1 = 0 \;\&\; w_2 = w_3 \tag{4}$$

Plugging into the previous equation we get:

$$2w_2^2 = \frac{1}{2} \Rightarrow [w_1, w_2, w_3] = [0, \frac{1}{2}, \frac{1}{2}] \;\; ...(Ans) \tag{5}$$

4. We have:

$$y_1(w^T\phi(x_1) + w_0) \geq 1 \tag{6}$$

$$y_2(w^T\phi(x_2) + w_0) \geq 1 \tag{7}$$

from these equations and the results from the previous parts, we get:

$$-1(0 * 1 + 1/2 * 0 + 1/2 * 0 + w_0) \geq 1 \Rightarrow w_0 \leq -1 \tag{8}$$

and

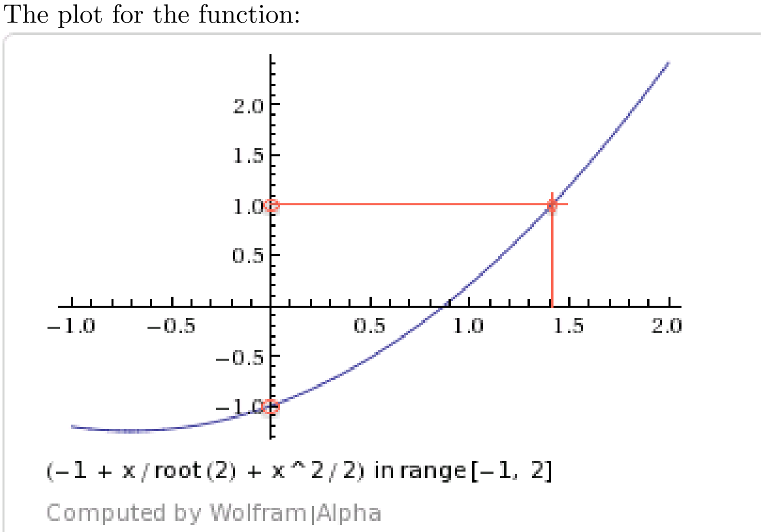
$$1(0 * 1 + 1/2 * 2 + 1/2 * 2 + w_0) \geq 1 \Rightarrow w_0 \geq -1 \tag{9}$$

This implies that $w_0 = -1$ ... (Ans)

5. Using the results from the previous parts:

$$f(x) = -1 + \frac{x}{\sqrt{2}} + \frac{x^2}{2} \tag{10}$$

The plot for the function:



2 Manual calculation of one round of EM for a GMM [30 points]

M step

1. The log likelihood function we are trying to optimize is:

$$Q(\theta, \theta^{(t-1)}) = \sum_i \sum_k r_{ic} log(\pi_c) + \sum_i \sum_k r_{ic} log(p(x_i|\theta_c)) \ \ ...(Ans) \tag{11}$$

taken from Murphy, page 351.

2. We know that:

$$\pi_c = \frac{1}{N} \sum_i r_{ic} = \frac{r_c}{N} \tag{12}$$

Hence,

$$\pi_1 = \frac{1}{3}(1 + 0.4 + 0) = \frac{1.4}{3} = \frac{7}{15} \ \ ...(Ans) \tag{13}$$

$$\pi_2 = \frac{1}{3}(0 + 0.6 + 1) = \frac{1.6}{3} = \frac{8}{15} \ \ ...(Ans) \tag{14}$$

3. We have:

$$\mu_c = \frac{\sum_i r_{ic} x_i}{r_c} \tag{15}$$

Plugging in the values we get:

$$\mu_1 = \frac{1 * 1 + 0.4 * 10 + 0 * 20}{1 + 0.4 + 0} = \frac{5}{1.4} = \frac{25}{7} \ \ ...(Ans) \tag{16}$$

$$\mu_2 = \frac{0 * 1 + 0.6 * 10 + 1 * 20}{1 + 0.6 + 0} = \frac{26}{1.6} = \frac{65}{4} \ \ ...(Ans) \tag{17}$$

4. We have,

$$\Sigma_c = \frac{\sum_i (r_{ic})(x_i - \mu_c)(x_i - \mu_c)^T}{r_c} \tag{18}$$

and

$$\sigma_c = \sqrt{\Sigma_c} \tag{19}$$

Plugging in the values:

$$\Sigma_1 = \frac{1(1 - \frac{25}{7})^2 + 0.4(10 - \frac{25}{7})^2 + 0(20 - \frac{25}{7})^2}{1 + 0.4 + 0} = 16.53 \Rightarrow \sigma_1 = 4.065 \ \ ...(Ans) \tag{20}$$

$$\Sigma_2 = \frac{0(1 - \frac{65}{4})^2 + 0.6(10 - \frac{65}{4})^2 + 1(20 - \frac{65}{4})^2}{0 + 0.6 + 1} = 23.4375 \Rightarrow \sigma_2 = 4.84 \ \ ...(Ans) \tag{21}$$

E step

1. The probability of observation x_i belonging to cluster c:

$$r_{ic} = \frac{\pi_k p(x_i|\theta_k^{(t-1)})}{\sum_{c'} \pi_{c'} p(x_i|\theta_{c'}^{(t-1)})} \quad ...(Ans) \tag{22}$$

where probability follows a Gaussian distribution, i.e.:

$$p(x_i|\theta_c^{(t-1)}) = \frac{1}{\sigma_c\sqrt{2\pi}} \exp -\frac{1}{2}(\frac{x_i - \mu_c}{\sigma_c})^2 \tag{23}$$

taken from Murphy, page 351.

2. Using the two equations above and plugging in the values, we get:
$p(x_1|\theta_1) = \frac{1}{4.065\sqrt{2\pi}} \exp -\frac{1}{2}(\frac{1-\frac{25}{7}}{4.065})^2 = 0.0803$
$p(x_1|\theta_2) = \frac{1}{4.84\sqrt{2\pi}} \exp -\frac{1}{2}(\frac{1-\frac{65}{4}}{4.84})^2 = 0.000575$
$p(x_2|\theta_1) = \frac{1}{4.065\sqrt{2\pi}} \exp -\frac{1}{2}(\frac{10-\frac{25}{7}}{4.065})^2 = 0.028$
$p(x_2|\theta_2) = \frac{1}{4.84\sqrt{2\pi}} \exp -\frac{1}{2}(\frac{10-\frac{65}{4}}{4.84})^2 = 0.0358$
$p(x_3|\theta_1) = \frac{1}{4.065\sqrt{2\pi}} \exp -\frac{1}{2}(\frac{20-\frac{25}{7}}{4.065})^2 = 0.0000278$
$p(x_3|\theta_2) = \frac{1}{4.84\sqrt{2\pi}} \exp -\frac{1}{2}(\frac{20-\frac{65}{4}}{4.84})^2 = 0.061$

Hence, $r_{11} = \frac{\frac{7}{15}0.0803}{\frac{7}{15}0.0803+\frac{8}{15}0.000575} = 0.992$
$r_{12} = 1 - r_{11} = 1 - 0.992 = 0.008$
$r_{21} = \frac{\frac{7}{15}0.028}{\frac{7}{15}0.028+\frac{8}{15}0.0358} = 0.406$
$r_{22} = 1 - r_{21} = 1 - 0.406 = 0.594$
$r_{31} = \frac{\frac{7}{15}0.0000278}{\frac{7}{15}0.0000278+\frac{8}{15}0.061} = 0.00039$
$r_{32} = 1 - r_{31} = 1 - 0.00039 = 0.99961$

$$R_{new} = \begin{pmatrix} 0.992 & 0.008 \\ 0.406 & 0.594 \\ 0.00039 & 0.99961 \end{pmatrix}$$

...(Ans)

3 Programming Question [45 Points]

3.1 Dataset

No question in this part.

3.2 Perceptron

1. From the lecture on Oct 28^{th}, slide 7, we know the following:

$$\hat{y} = sign(\phi(x).w^{(t)}) = sign(\sum_{j \in M^{(t)}} y^{(j)} \phi(x).\phi(x^{(j)})) \tag{24}$$

and

$$sign(\sum_{j \in M^{(t)}} y^{(j)} \phi(x).\phi(x^{(j)})) = sign(\sum_{j \in M^{(t)}} y^{(j)} k(x, x^{(j)})) \tag{25}$$

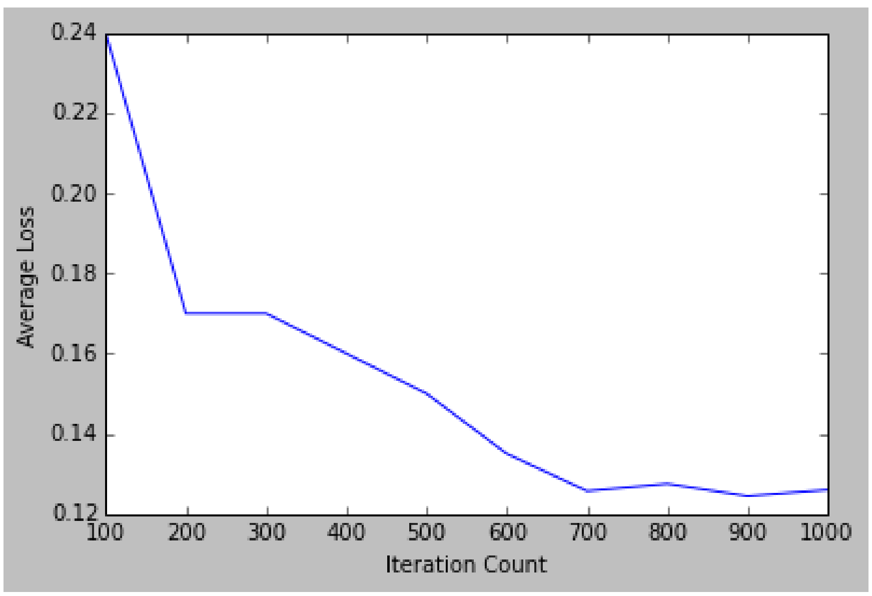
hence the prediction rule is:

$$\hat{y} = sign(\sum_{j \in M^{(t)}} y^{(j)} k(x, x^{(j)})) \;\; ...(Ans) \tag{26}$$

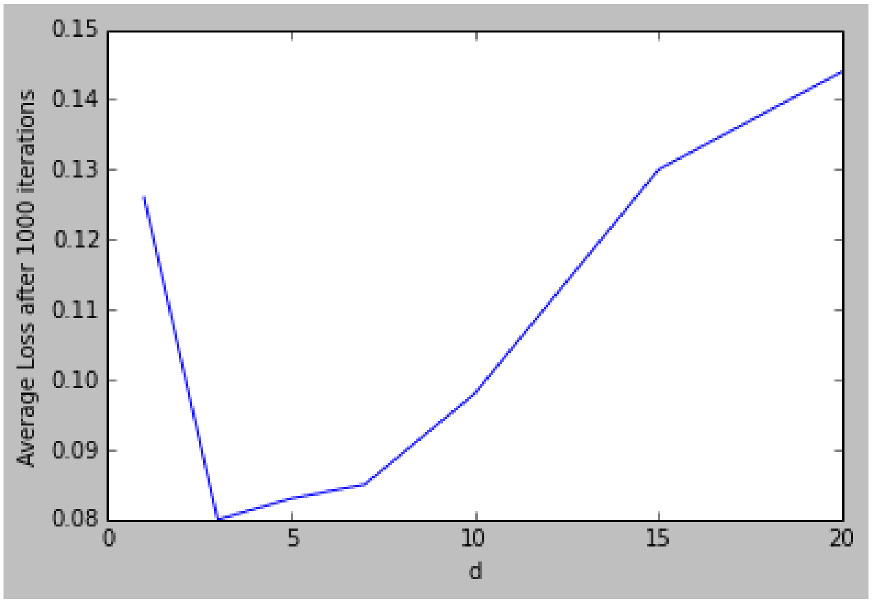
where $M^{(t)}$ = mistakes till iteration t.

2. See attached code.

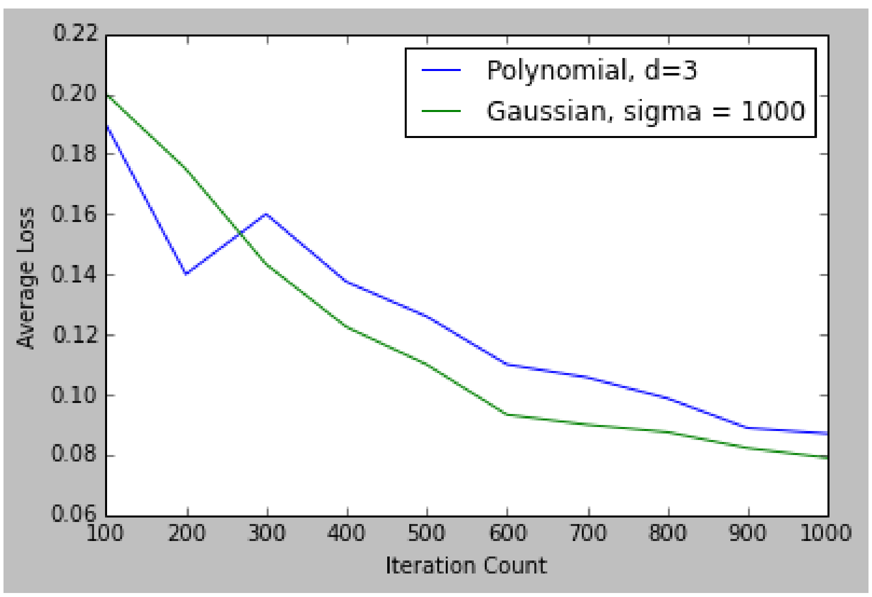
3. Plot for average loss at every 100 steps:



4. Plot for the average loss after 1000 iterations for the kernels with increasing degree of polynomials:



5. Plot for average loss at every 100 steps, polynomial kernel with d = 3 vs. gaussian kernel with $\sigma = 1000$:



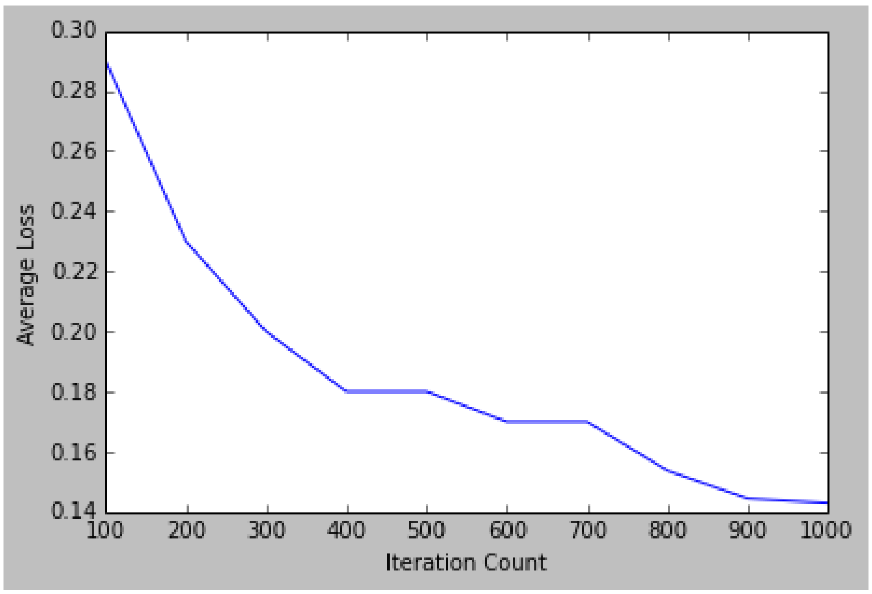
3.3 SVM

1. Update rules for linear SVM Stochastic gradient descent:

$$w^{(t+1)} = w^{(t)} + \eta(C\mathbb{I}((1 - y^{(t)}(w^{(t)}.x^{(t)} + w_0^{(t)}) > 0)) - 2w^{(t)}) \tag{27}$$

$$w_0^{(t+1)} = w_0^{(t)} + \eta C y^{(t)} \quad ...(Ans) \tag{28}$$

2. See attached code.
3. Average loss after every 100 steps for $\eta = 10^{-5}$ and $C = 1$:



4.
5.