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1. (10 points) Suppose that X and Y are two independent random variables with the following probability density functions:

$$f_X(x) = \begin{cases} \frac{1}{x}, & \text{if } 1 \le x \le e \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} 2y, & \text{if } 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the density of the random variable $Z = X + Y$.

2. (10 points) Suppose that X is a random variable with the following probability distribution function,

$$f_X(x) = \begin{cases} \frac{2x}{\pi^2}, & \text{if } 0 \le x \le \pi; \\ 0, & \text{otherwise.} \end{cases}$$

Find the cdf and pdf of the random variable $Y = \sin(x)$.
Note: The function sin is not monotone on $[0, \pi]$.

1. By Convolution formula,

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$$

$$= \int_1^e f_X(x) f_Y(z-x)\,dx$$

$$= \int_1^e \frac{1}{x} f_Y(z-x)\,dx$$

We need $0 \le z-x \le 1$

$z-1 \le x \le z$

$\therefore x \in [1,e] \cap [z-1, z]$

Case: $1 \le z \le 2$

$x \in [1,e] \cap [z-1, z]$

$x \in [1,e] \cap [0, 2]$

$x \in [1, 2]$

$\therefore x \in [1, z]$

$$\int_1^z \frac{1}{x} 2(z-x)\,dx$$

$$= 2\int_1^z \frac{z}{x} - 1\,dx$$

$$= 2\left[\int_1^z \left(\frac{z}{x}\right) dx - \int_1^z 1\,dx\right]$$

$$= 2\left[z(\ln(z) - \ln(1)) - z + 1\right]$$

$$= 2z\ln(z) - 2z + 2$$

Case: $2 < z \le e$

$x \in [0,e] \cap [z-1, z]$

$x \in [0,e] \cap [1, e]$

$x \in [1, e]$

$\therefore x \in [z-1, z]$

$$\int_{z-1}^{z} \frac{1}{x} 2(z-x)\,dx$$

$$= 2\left[\int_{z-1}^{z} \frac{z}{x}\,dx - \int_{z-1}^{z} 1\,dx\right]$$

$$= 2\left[z(\ln(z) - z\ln(z-1)) - \cancel{z} + \cancel{z} - 1\right]$$

$$= 2z\ln(z) - 2z\ln(z-1) - 2$$

Case 3: $e < z \le e+1$

$x \in [0,e] \cap [z-1, z]$

$x \in [0,e] \cap [e-1, e+1]$

$x \in [e-1, e+1]$

$\therefore x \in [z-1, e]$

$$\int_{z-1}^{e} \frac{1}{x} 2(z-x)\,dx$$

$$= 2\left[\int_{z-1}^{e} \frac{z}{x}\,dx - \int_{z-1}^{e} 1\,dx\right]$$

$$= 2\left[z(\ln(e) - \ln(z-1)) - e + z - 1\right]$$

$$= 2z - 2\ln(z-1) - 2e + 2z - 2$$

$$= 4z - 2\ln(z-1) - 2e - 2$$

$$\therefore f_Z(z) = \begin{cases} 2z\ln(z) - 2z + 2, & 1 \le z \le 2 \\ 2z\ln(z) - 2z\ln(z-1) - 2, & 2 \le z \le e \\ 4z - 2\ln(z-1) - 2e - 2, & e < z \le e+1 \\ 0, & \text{otherwise} \end{cases}$$

2. $Y = T(x) = \sin x$

CDF: $F_Y(y) = P(Y \le y) = P(\sin(x) \le y)$

$= P(X \le \arcsin(y))$

Since $\sin(x)$ is not monotonic on $x \in [0, \pi]$, we have to consider two intervals $(0 \le x < \frac{\pi}{2})$ and $(\frac{\pi}{2} < x \le \pi)$

$$\therefore P(X \le \arcsin(y)) = \int_0^{\arcsin(y)} \frac{2x}{\pi^2}\,dx + \int_{\pi - \arcsin(y)}^{\pi} \frac{2x}{\pi^2}\,dx$$

$$= \frac{2}{\pi^2}\left[\frac{(\arcsin(y))^2}{2} + \frac{\pi^2}{2} - \frac{(\pi - \arcsin(y))^2}{2}\right]$$

$$= \frac{\cancel{\arcsin(y)^2} + \cancel{\pi^2} - \cancel{\pi^2} - \cancel{\arcsin(y)^2} + 2\pi\arcsin(y)}{\pi^2}$$

$$= \frac{2\arcsin(y)}{\pi}$$

Since $x \in [0, \pi)$, $y = \sin(x) \in [0, 1]$

$$\therefore \text{CDF}, \ F_Y(Y) = \begin{cases} 0, & y < 0 \\ \frac{2\arcsin(y)}{\pi}, & 0 \le y \le 1 \\ 1, & y > 1 \end{cases}$$

PDF, $f_Y(y) = \frac{d}{dy} F_Y(y)$

$$= \frac{d}{dy}\frac{2\arcsin(y)}{\pi}$$

$$= \frac{2}{\pi} \cdot \frac{1}{\sqrt{1-y^2}}$$

$$= \frac{2}{\pi\sqrt{1-y^2}}$$

$$f_Y(y) = \begin{cases} \frac{2}{\pi\sqrt{1-y^2}}, & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$