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1. Suppose box 1 contains a white balls and b black balls, and box 2 contains c white balls and d black balls (a, b, c, d are positive integers). A randomly chosen ball is transferred from box 1 to box 2. Then, a randomly chosen ball from box 2 is transferred to box 1.

   (a) (10 points) Now, if a ball is randomly drawn from box 1, what is the probability that it will be white?

   (b) (5 points) Show that if $\frac{a}{b} = \frac{c}{d}$ then the answer in (a) reduces to $\frac{a}{a+b}$, i.e., it is equal to the probability of getting a white ball if we directly sample a ball from box 1 without transferring.

   (c) (5 points) Suppose the ball drawn in the end is white. Find the conditional probability that box 1 contains a white balls and b black balls (i.e., original composition) just after transferring the balls (and before sampling the last ball).

1.(a) Let A be event that we pick white ball from box 1

Let (B,W) be event that black ball is transferred from box 1 to 2 and white ball from box 2 to box 1.

Same applies for (W,W), (W,B) and (B,B).

by law of total probability,

$$P(A) = P(A \mid (B,B))P(B,B) + P(A \mid (B,W))P(B,W) + P(A \mid (W,W))P(W,W) + P(A \mid (W,B))P(W,B)$$

$$= \left(\frac{a}{a+b}\right)\left(\frac{b}{a+b}\right)\left(\frac{d+1}{c+d+1}\right) + \left(\frac{a+1}{a+b}\right)\left(\frac{b}{a+b}\right)\left(\frac{c}{c+d+1}\right) + \left(\frac{a}{a+b}\right)\left(\frac{a}{a+b}\right)\left(\frac{c+1}{c+d+1}\right)$$

$$+ \left(\frac{a-1}{a+b}\right)\left(\frac{a}{a+b}\right)\left(\frac{d}{c+d+1}\right)$$

$$= \frac{ab(d+1) + bc(a+1) + a^2(c+1) + ad(a-1)}{(a+b)^2 (c+d+1)}$$

(b) Assume $\frac{a}{b} = \frac{c}{d}$

$\therefore c = ak$, $d = bk$ for some $k \in \mathbb{N}$.

$$P(A) = \frac{ab(bk+1) + bak(a+1) + a^2(ak+1) + abk(a-1)}{(a+b)^2 (ak+bk+1)}$$

$$= \frac{ab^2k + ab + a^2bk + \cancel{abk} + a^3k + a^2 + a^2bk - \cancel{abk}}{(a+b)^2 (ak+bk+1)}$$

$$= \frac{ab^2k + ab + a^2bk + a^2bk + a^3k + a^2}{(a+b)^2 (ak+bk+1)}$$

$$= \frac{(ab)(bk+1+ak) + a^2(bk+ak+1)}{(a+b)^2 (ak+bk+1)}$$

$$= \frac{(ab+a^2)\cancel{(ak+bk+1)}}{(a+b)^2 \cancel{(ak+bk+1)}}$$

$$= \frac{a(a+b)}{(a+b)^2} = \frac{a}{a+b}$$

(c) Suppose A be event box 1 has a white balls and b black balls after transferring

Suppose W be event we pick white ball at the end

by Bayes' Theorem

$$P(A|W) = \frac{P(W|A)}{P(W)} P(A)$$

$$= \frac{P(W|A)}{P(W)} (P(B,B) + P(W,W))$$ (since we must have 'a' number of white and 'b' number of black at end)

$$= \frac{\frac{a}{a+b} \cdot \left(\frac{b}{a+b} \cdot \frac{d+1}{c+d+1} + \frac{a}{a+b} \cdot \frac{c+1}{c+d+1}\right)}{\frac{ab(d+1) + bc(a+1) + a^2(c+1) + ad(a-1)}{(a+b)^2 (c+d+1)}}$$

$$= \frac{\frac{a}{a+b} \left(\frac{b(d+1) + a(c+1)}{(a+b)(c+d+1)}\right)}{\frac{ab(d+1) + bc(a+1) + a^2(c+1) + ad(a-1)}{(a+b)^2 (c+d+1)}}$$

$$= \frac{abd + ab + a^2c + a^2}{abd + ab + abc + bc + a^2c + a^2 + a^2d - ad}$$