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1. (10 points) Consider the following experiment. First, toss an unfair coin, with the probability $p \in (0, 1)$ for Heads and the probability $q = 1 - p$ for Tails, and record the result. Then, continue tossing the same coin until getting the complement of the recorded result. For example, if the outcome of the first toss is Heads, continue tossing the coin until getting the first Tails. If the random variable X represents the number of coin tossing in the above experiment, find the expected value of X.

Since we are interested in the first occurrence of the complement, we can think of X as geometric distribution.

$$E[X] = E[X \mid 1^{st} \text{ Head}]\, P(1^{st} \text{ Head}) + E[X \mid 1^{st} \text{ Tail}]\, P(1^{st} \text{ Tail})$$

For $E[X \mid 1^{st} \text{ Head}]$, since we know the first choice is head already, we will apply the geometric distribution $X \sim \text{Geometric}(q)$ for the rest of toss until Tail.

$$E[X \mid 1^{st} \text{ Head}] = 1 + \frac{1}{q}$$ (we add one as we already flip 1^{st} head)

Similarly,

$$E[X \mid 1^{st} \text{ Tail}] = 1 + \frac{1}{p}$$

$$\therefore E[X] = \left(1 + \frac{1}{q}\right)p + \left(1 + \frac{1}{p}\right)q$$

$$= \left(1 + \frac{1}{1-p}\right)p + \left(1 + \frac{1}{p}\right)(1-p) \quad (\because q = 1-p)$$

$$= p + \frac{p}{1-p} + (1-p) + \frac{1-p}{p}$$

$$= \frac{\cancel{p^2} - \cancel{p^3} + \cancel{p^2} + p - \cancel{p^2} - \cancel{p^2} + \cancel{p^3} + 1 + p^2 - 2p}{p - p^2}$$

$$= \frac{p^2 - p + 1}{p - p^2}$$

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2. (10 points) Let X and Y be two real-valued random variables that are independent and identically distributed and take values in an interval E of $\mathbb{R}$. Also, let f, g be two non-decreasing functions from E to $\mathbb{R}$ (i.e., $f(x) \leq f(y)$ if $x \leq y$ and same for g).
   (a) (2 points) For any x, y in E, find the sign of $(f(x) - f(y))(g(x) - g(y))$.
   (b) (4 points) Deduce from (a) that
   $$\mathbb{E}[f(X)g(X)] \geq \mathbb{E}[f(X)]\mathbb{E}[g(X)],$$
   studying $\mathbb{E}[(f(X) - f(Y))(g(X) - g(Y))]$.
   Note: We assume all expectations involved exist.
   (c) (4 points) Deduce from the previous parts that
   $$\mathbb{E}[X]\mathbb{E}[\frac{1}{X}] \geq 1,$$
   for any random variable X such that $X > 0$ almost surely.

2.(a) For any x, y in E,

case $x \leq y$,

$f(x) \leq f(y)$, $g(x) \leq g(y)$ (both f, g are non-decreasing)

$f(x) - f(y) \leq 0$, $g(x) - g(y) \leq 0$

Thus, $(f(x) - f(y))(g(x) - g(y)) \geq 0$ (since $(-) \times (-) = (+)$)

case $x > y$,

$\therefore f(x) \geq f(y)$, $g(x) \geq g(y)$ (both f, g are non-decreasing)

$f(x) - f(y) \geq 0$, $g(x) - g(y) \geq 0$

$\therefore$ Thus, $(f(x) - f(y))(g(x) - g(y)) \geq 0$ (since $(+) \times (+) = (+)$)

Thus, sign of $(f(x) - f(y))(g(x) - g(y))$ is $+$.

2.(b) Since X and Y takes values in interval E

$$(f(X)-f(Y))(g(X)-g(Y)) \geqslant 0 \qquad \text{(from a)}$$

$$\therefore E[(f(X)-f(Y))(g(X)-g(Y))] \geqslant 0$$

$$E[\, f(X)g(X)-f(X)g(Y)-f(Y)g(X)+f(Y)g(Y)] \geqslant 0$$

$$E[f(X)g(X)]-E[f(X)g(Y)]-E[f(Y)g(X)]+E[f(Y)g(Y)] \geqslant 0$$

(by linearity of E)

Since X and Y are independent,

$$E[f(X)g(X)]-E[f(X)]E[g(Y)]-E[f(Y)]E[g(X)]+E[f(Y)g(Y)] \geqslant 0$$

Since X and Y are identically distributed, $E[f(X)]=E[f(Y)]$ and $E[g(X)]=E[g(Y)]$

$$\therefore E[f(X)g(X)]-E[f(X)]E[g(X)]-E[f(X)]E[g(X)]+E[f(X)g(X)] \geqslant 0$$

$$2E[f(X)g(X)]-2E[f(X)]E[g(X)] \geqslant 0$$

$$2E[f(X)g(X)] \geqslant 2E[f(X)]E[g(X)]$$

$$E[f(X)g(X)] \geqslant E[f(X)]E[g(X)] \quad \#$$

For $x \in E$,

2.(c) Let $f(x)=x$ and $g(x)=-\frac{1}{x}$ which are both non-decreasing function.

$$\therefore f(X)=X \quad \text{and} \quad g(X)=-\frac{1}{X}$$

From part (b),

$$E[f(X)g(X)] \geqslant E[f(X)]E[g(X)]$$

$$E\left[X\left(-\frac{1}{X}\right)\right] \geqslant E[X]\,E\left[-\frac{1}{X}\right]$$

$$E[-1] \geqslant E[X]\,E\left[-(1)\frac{1}{X}\right]$$

$$-E[1] \geqslant -E[X]\,E\left[\frac{1}{X}\right] \quad \text{(by linearity of E)}$$

Since $X>0$ almost surely, $E[X]E[\frac{1}{X}] \geqslant 0$.

Thus, multiplying $(-)$ both sides flip the inequality.

$$E[1] \leqslant E[X]\,E\left[\frac{1}{X}\right]$$

$$1 \leqslant E[X]\,E\left[\frac{1}{X}\right] \quad \text{(Expectation of constant is constant)}$$

$$\therefore E[X]\,E\left[\frac{1}{X}\right] \geqslant 1$$