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1. (10 points)

   (a) (5 points) Use the identity $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B|A)$ and mathematical induction to prove that for any $n \geq 2$ and arbitrary events $A_1, ..., A_n$, the following identity holds:

   $$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1) \cdots \mathbb{P}(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1}).$$

   (b) (5 points) Consider the following experiment. You first roll a fair 6-sided die and obtain the value X. You then roll a fair X-sided die and obtain the value Y. Finally, you roll a fair Y-sided die and obtain the value Z. Use the result in (a) (with $n = 3$) to find the probability that you obtain $Z = 6$ with the last die.
   For example, if we first get $X = 4$, then Y is randomly chosen from $\{1, 2, 3, 4\}$ (each has equal probability $\frac{1}{4}$). Say we get $Y = 3$. Then Z is randomly chosen from $\{1, 2, 3\}$. In this case Z is at most 3.
   Hint: Formulate A_1, A_2 and A_3 suitably. The problem statement provides values of the (conditional) probabilities involved.

2. (10 points) Consider $N \geq 2$ empty boxes lined up from left to right. A coin is fliped where the probability of heads is p, $0 < p < 1$. If heads, a box is selected uniformly at random and a treasure is placed inside it. If tails, no treasure is placed.

   (a) (4 points) Construct a probability space $(\Omega, \mathbb{P})$ to model the above experiment. That is, define Ω and $\mathbb{P}$ suitably.
   Note: There are many possible constructions, but it is possible to choose Ω such that $|\Omega| = N + 1$.

   (b) (6 points) Suppose we open all but the rightmost box and find no treasure in the $N - 1$ opened boxes. What is the conditional probability there is a treasure in the remaining rightmost box?
   Note: If needed, you may argue using the probability space you construct in (a). The answer is independent of your construction in (a) (see the discussion in Example 3.2 of the lecture notes).

1.(a) Let $S(n)$ be the predicate that such that

$\forall n \geq 2, \quad P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1)P(A_2 | A_1)\dots P(A_n | A_1 \cap A_2 \cap \dots \cap A_{n-1})$

WTP: $S(n)$ holds.

by induction

Base case: $n = 2$

$$P(A_1 \cap A_2) = P(A_1)P(A_2 | A_1) \quad \text{(by product rule)}$$

$\therefore$ Base case holds.

Induction step:

Induction hypothesis: Suppose $S(n)$ holds i.e

$\forall n \geq 2, \quad P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1)P(A_2 | A_1)\dots P(A_n | A_1 \cap A_2 \cap \dots \cap A_{n-1})$

WTP: $S(n+1)$ holds.

$$\begin{aligned} P(A_1 \cap A_2 \dots \cap A_n \cap A_{n+1}) &= P(A_1 \cap A_2 \cap \dots \cap A_n)P(A_{n+1} | A_1 \cap A_2 \cap \dots A_n) \quad \text{(by product rule)} \\ &= P(A_1)P(A_2 | A_1)\dots P(A_n | A_1 \cap A_2 \cap \dots \cap A_{n-1})P(A_{n+1} | A_1 \cap A_2 \cap \dots \cap A_n) \quad \text{(by I.H)} \end{aligned}$$

$\therefore S(n+1)$ hold as wanted.

Thus, by mathematical induction, for any $n \geq 2$,

$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1)P(A_2 | A_1)\dots P(A_n | A_1 \cap A_2 \cap \dots \cap A_{n-1})$

(b) Let A_1 be event $X = 6$

Let A_2 be event $Y = 6$

Let A_3 be event $Z = 6$

$P(A_1) = P(X = 6) = \frac{1}{6}$

$P(A_2 | A_1) = P(Y = 6 | X = 6) = \frac{1}{6}$

$P(A_3 | A_1 \cap A_2) = P(Z = 6 | X = 6 \text{ and } Y = 6) = \frac{1}{6}$

Since for A_3 to occur, A_1 and A_2 must also occur.

Thus, $P(A_3) = P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2 | A_1)P(A_3 | A_1 \cap A_2)$ (by identity in 1(a))

$P(A_3) = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{216}$

2.(a) Let w_i for $0 \le i \le N$ represent the event that the treasure is placed at i^{th} box from left.

w_0 be no treasure is placed.

Example: $w_2 = (0, 1, 0, 0, \ldots, 0)$ of length N

Sample space, $\Omega = \{ w_0, w_1, w_2, \ldots, w_N \}$

$$= \{ w_i, \; 0 \le i \le N \}$$

$$\therefore \; |\Omega| = N + 1$$

Since placing treasure is decided on flipping a coin first, and probability of getting head is p

$$P(w_0) = 1 - p \;, \quad \sum_{i=1}^{N} P(w_i) = p$$

Since a box is uniformly selected at random over total N box,

$$P(w_i) = p \cdot \frac{1}{N} \;, \quad \text{for } 1 \le i \le N$$

$$P(w_i) = \frac{p}{N}$$

$$\therefore \text{ Probability measure, } P(w_i) = \begin{cases} 1 - p \;, & i = 0 \\ \dfrac{p}{N} \;, & 1 \le i \le N \end{cases} \quad \text{for, } w_i \in \Omega$$

2.(b) Let A be event that there is no treasure in first $N-1$ boxes.

$$P(A) = P(w_0 \cup w_N)$$

Since w_0 and w_N are disjoints,

$$P(A) = P(w_0) + P(w_N) \quad \text{(sum rule)}$$

$$= 1 - p + \frac{p}{N} \quad (\text{by } P \text{ defined in a)})$$

$$= \frac{N - Np + p}{N}$$

$$P(A \mid w_N) = 1$$

$$P(w_N) = \frac{p}{N} \quad (\text{by } P \text{ defined in a)})$$

by Bayes' rule,

$$P(w_N \mid A) = \frac{P(A \mid w_N) \, P(w_N)}{P(A)}$$

$$= \frac{1}{\frac{N - Np + p}{N}} \cdot \frac{p}{N}$$

$$= \frac{N}{N - Np + p} \cdot \frac{p}{N}$$

$$= \frac{p}{N - Np + p}$$

Since we've already checked $N-1$ boxes and there is only one remaining box, $N = 1$

$$\therefore P(w_n \mid A) = \frac{p}{1 - p + p} = p$$

The conditional probability that there is a treasure in remaining rightmost box is p.