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1. (10 points) Let $\Omega = \{\omega_1, \dots, \omega_N\}$ be a finite sample space and suppose $\mathbb{P}(\{\omega\}) > 0$ for all $\omega \in \Omega$. Consider X a real-valued random variable on $(\Omega, \mathbb{P})$ and $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ a function. Prove that X and $Y = f(X)$ are independent if and only if $f(X)$ is a constant random variable.

   Hint: Examine the conditional probability $\mathbb{P}(Y = y | X = x)$ when $y = f(x)$. Also, remember that an event with probability 0 or 1 is always independent from all other events.

   Note: You can choose to prove a more general result: remove the condition $\mathbb{P}(\{\omega\}) > 0$ for all ω. The equivalent condition for $X \perp\!\!\!\perp f(X)$ becomes "there exists $c \in \mathbb{R}$ such that $\mathbb{P}(f(X) = c) = 1$", i.e., $f(X)$ is *almost surely* constant.

2. (10 points) Let $X \sim \text{Geometric}(p)$ and $Y \sim \text{Geometric}(q)$ be two independent random variables, where $0 < p, q < 1$.

   Note: We use the convention (in the notes) that the geometric distribution is supported on $\{1, 2, 3, \dots\}$. See `https://en.wikipedia.org/wiki/Geometric_distribution` for the other convention where the distribution is supoprted on $\{0, 1, 2, \dots\}$.

   (a) (4 points) Compute $\mathbb{P}(X > Y)$.

   (b) (6 points) Assume $p = q$. For $k \in \{1, 2, \dots\}$, derive an explicit expression for

   $$\mathbb{P}(X \geq kY).$$

   Hint: Decompose the event based on the different values Y can take.

1. Prove $\Rightarrow$

Assume X and Y are independent.

$\therefore P(Y=y \mid X=x) = P(Y=y)$ (by independence)

$= P(f(X)=y)$ ($\because Y=f(X)$ given) — 1

However, $P(Y=y \mid X=x) = P(Y=f(x) \mid X=x) = 1$ — 2

This is because since we know $X=x$ and $Y=f(x)$ by definition, the conditional probability that $P(Y=f(x) \mid X=x)$ is certain.

$\therefore$ From 1 and 2, $P(f(X)=y)=1$

Thus, $f(X)$ is almost surely constant. (as wanted)

Prove $\Leftarrow$

Assume $f(X)$ is a constant random variable.

$\therefore \exists c \in \mathbb{R}$ such that $f(X)=c$.

$\therefore P(f(X)=c)=1$

Consider $P(X=x \mid Y=y) = P(X=x \mid f(X)=c)$ ($\because Y=f(X)$ is a constant random variable)

Since $P(f(X)=c)=1$, the conditional probability $P(X=x \mid f(X)=c)$ is just $P(X=x)$

$\therefore P(X=x \mid f(X)=c) = P(X=x)$

$\therefore X$ and $Y=f(X)$ are independent (as wanted)

2. (a)

$$P(X>Y) = \sum_{x=2}^{\infty} \sum_{y=1}^{x-1} (1-p)^{x-1} p \, (1-q)^{y-1} q$$ (Since X and Y are independent, we can just multiply two pmf)

$$= pq \sum_{x=2}^{\infty} (1-p)^{x-1} \sum_{y=1}^{x-1} (1-q)^{y-1}$$ (pq is constant)

$$= pq \sum_{x=2}^{\infty} (1-p)^{x-1} \frac{(1)(1-(1-q)^{x-1})}{1-(1-q)}$$ (Geometric sum)

$$= pq \sum_{x=2}^{\infty} (1-p)^{x-1} \frac{1-(1-q)^{x-1}}{q}$$

$$= \frac{pq}{q} \sum_{x=2}^{\infty} (1-p)^{x-1} (1-(1-q)^{x-1})$$ ($1/q$ is constant)

$$= p \sum_{x=2}^{\infty} (1-p)^{x-1} (1-(1-q)^{x-1})$$

$$= p \sum_{x=2}^{\infty} (1-p)^{x-1} - \sum_{x=2}^{\infty} (1-p)^{x-1} (1-q)^{x-1}$$

$$= p \left(\frac{1-p}{1-(1-p)} - \frac{(1-p)(1-q)}{1-(1-p)(1-q)} \right)$$ (Geometric sum)

$$= 1-p - \frac{p(1-p)(1-q)}{1-1+p+q-pq}$$

$$= \frac{\cancel{p}+q-\cancel{pq}-\cancel{p^2}-pq+\cancel{p^2q}-\cancel{p}+\cancel{pq}+\cancel{p^2}-\cancel{p^2q}}{p+q-pq}$$

$$= \frac{q-pq}{p+q-pq}$$

For $k \in \{1,2,\ldots\}$

(b) $$P(X \geq kY) = \sum_{i=1}^{\infty} P(Y=i)\, P(X \geq ki \mid Y=i)$$ (decompose based on different Y-values)

$$= \sum_{i=1}^{\infty} P(Y=i)\, P(X \geq ki)$$ (because X and Y are independent)

$$= \sum_{i=1}^{\infty} P(Y=i)\, P(X > ki-1)$$

$$= \sum_{i=1}^{\infty} (1-q)^{i-1} q \, (1-p)^{ki-1}$$ (tail probability of geometric distribution)

$$= \sum_{i=1}^{\infty} (1-p)^{i} p (1-p)^{-1} (1-p)^{ki} (1-p)^{-1}$$ ($\because q=p$)

$$= \frac{p}{(1-p)^2} \sum_{i=1}^{\infty} (1-p)^{(k+1)i}$$

$$= \frac{p}{(1-p)^2} \quad \frac{(1-p)^{k+1}}{1-(1-p)^{k+1}}$$

$$= \frac{p(1-p)^{k-1}}{1-(1-p)^{k+1}}$$