1.(a) $M_{Y_n}(t) = E[e^{tY_n}]$

$$= E\left[e^{tSX_n}\right]$$

$$= \sum_{k=0}^{\infty} e^{tX_n} f_{S,X_n}(1,k) + \sum_{k=0}^{\infty} e^{-tX_n} f_{S,X_n}(-1,k)$$

$$= \sum_{k=0}^{\infty} e^{tX_n} P(S=1)P(X_n=k) + \sum_{k=0}^{\infty} e^{-tX_n} P(S=-1)P(X_n=k) \quad (S \perp X_n)$$

$$= \frac{1}{2}\sum_{k=0}^{\infty} e^{tX_n} P(X_n=k) + \frac{1}{2e}\sum_{k=0}^{\infty} e^{-tX_n} P(X_n=k)$$

$$= \frac{1}{2}\frac{(1+e^t)^n}{2^n} + \frac{1}{2}\frac{(1+e^{-t})^n}{2^n} \quad (\text{MGF of } X_n)$$

$$= \frac{(1+e^t)^n + (1+e^{-t})^n}{2^{n+1}}$$

(b) $M_{Y_n}(t) = \dfrac{(1+e^t)^n + e^{-nt}(1+e^t)^n}{2^{n+1}}$

$$= \frac{(1+e^{-nt})(1+e^t)^n}{2^{n+1}}$$

Assume $X_n \perp H_n$

Let $H_n \sim \text{Bernoulli}(1/2)$ that takes two values 0 and n.

$$P(H_n=0) = P(H_n=n) = \frac{1}{2}$$

$$M_{H_n}(t) = E[e^{tH_n}]$$

$$= e^{t(0)}P(H_n=0) + e^{t(n)}P(H_n=n)$$

$$= 1(1/2) + e^{nt}(1/2)$$

$$= \frac{1+e^{nt}}{2}$$

$$M_{X_n-H_n}(t) = E[e^{t(X_n-H_n)}]$$

$$= E[e^{tX_n + (-t)H_n}]$$

$$= E[e^{tX_n}]\,E[e^{-tH_n}] \quad (\because X_n \perp H_n)$$

$$= M_{X_n}(t)\; E_{H_n}(-t)$$

$$= \frac{(1+e^t)^n}{2^n} \cdot \frac{1+e^{-nt}}{2}$$

$$= \frac{(1+e^t)^n(1+e^{-nt})}{2^{n+1}}$$

$\therefore M_{Y_n}(t) = M_{X_n - H_n}(t)$

$\therefore Y_n$ and $X_n - H_n$ have the same distribution.

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2. Since $g = T \circ h$, we should have h to be part of formula for g.

$$g(u,v) = T(h_1(u,v), h_2(u,v))$$

$$= (h_1(u,v)\cos(h_2(u,v)),\; h_1(u,v)\sin(h_2(u,v))) \quad (\text{def of } T)$$

$$(\sqrt{-2\ln u}\cos((2v-1)\pi),\; \sqrt{-2\ln u}\sin((2v-1)\pi)) =$$
$$(h_1(u,v)\cos(h_2(u,v)),\; h_1(u,v)\sin(h_2(u,v)))$$
(by def of g)

$$\therefore h: (0,1)\times(0,1) \to (0,+\infty)\times(-\pi,\pi)$$
$$(u,v) \to (\sqrt{-2\ln u},\; (2v-1)\pi)$$

$$\det(J_{g^{-1}}(x,y)) = \frac{1}{\det(J_g(g^{-1}(x,y)))} = \frac{1}{\det(J_g(u,v))} = \frac{1}{\det(J_T(r,\theta))\det(J_h(u,v))}$$
(chain rule)

$$\det_T(r,\theta) = r = \sqrt{-2\ln u}$$

$$\det_h(u,v) = \det\left(\begin{bmatrix} \frac{\partial}{\partial u}\sqrt{-2\ln u} & \frac{\partial}{\partial v}\sqrt{-2\ln u} \\ \frac{\partial}{\partial u}(2v-1)\pi & \frac{\partial}{\partial v}(2v-1)\pi \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} \frac{1}{\cancel{2}\sqrt{-2\ln u}} \cdot \frac{-\cancel{2}}{u} & 0 \\ 0 & 2\pi \end{bmatrix}\right)$$

$$= \frac{-2\pi}{u\sqrt{-2\ln u}}$$

$$\therefore \det(J_{g^{-1}}(x,y)) = \frac{1}{\sqrt{-2\ln u} \cdot \frac{-2\pi}{u\sqrt{-2\ln u}}}$$

$$= \frac{u}{-2\pi}$$

We have to express u in terms of x and y.

By T^{-1}: $T^{-1}(x,y) = (t_1(x,y), t_2(x,y))$

$$t_1(x,y) = \sqrt{x^2+y^2}$$

By H^{-1}:

$$t_1(x,y) = \sqrt{-2\ln u}$$

$$\therefore \sqrt{x^2+y^2} = \sqrt{-2\ln u}$$

$$x^2+y^2 = -2\ln u$$

$$\frac{x^2+y^2}{-2} = \ln u$$

$$u = e^{\frac{x^2+y^2}{-2}}$$

$$\therefore \det(J_{g^{-1}}(x,y)) = \frac{e^{\frac{x^2+y^2}{-2}}}{-2\pi}$$

(b) Density of (X,Y) is

$$f_{X,Y}(x,y) = f_{U,V}(g^{-1}(x,y)) \left| \det J_{g^{-1}}(x,y) \right|$$

$$= f_{U,V}(u,v) \left| \frac{e^{\frac{x^2+y^2}{-2}}}{-2\pi} \right|$$

$$= (1) \frac{e^{\frac{x^2+y^2}{-2}}}{2\pi}$$

$$= \frac{e^{\frac{x^2+y^2}{-2}}}{2\pi} \qquad x,y \in \mathbb{R}^2 \setminus (-\infty, 0] \times \{0\}$$

(c) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$

$$= \int_{-\infty}^{\infty} \frac{e^{\frac{x^2+y^2}{-2}}}{2\pi}\,dy$$

$$= \int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,dy$$

$$= \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,dy$$

$$= \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}(1)$$

$$= \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

Since this is PDF of $N(0,1)$, $X \sim N(0,1)$

Similarly, $f_Y(y) = \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}$

$$f_X(x) \cdot f_Y(y) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \cdot \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}$$

$$= \frac{e^{-\frac{x^2+y^2}{2}}}{2\pi}$$

$$= f_{X,Y}(x,y)$$

$\therefore$ X and Y are independent.