Bayesian Statistics

Basic Distributions

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Numerical Characteristics
of Random Variables

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Joint and Conditional Distributions

Before We Begin…

In this unit:

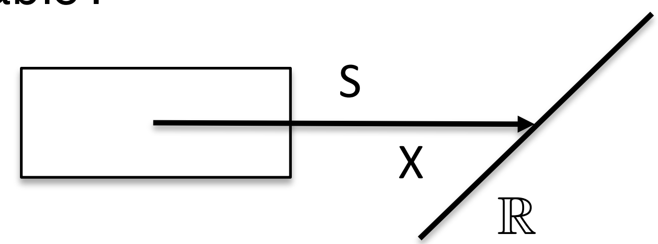
- Basic Distributions
- Bayes's Theorem
- Homework 2
- Conjugate Classes
- Prior Elicitation
- Empirical Bayes
- Homework 3

Basic Distributions

- Models and Parameters
- Numerical Characteristics
- Joint and Conditional Distributions
- Worked Examples

Models and Parameters

- Random variable is the simplest statistical model
- What is random variable?



X: Mapping from sample space S to real numbers

<u>Example</u>: Flip a fair coin. If head, pay a $1, if tail, get a $1. Your gain in one flip is a random variable.

- Random variables: Discrete and continuous (depending on realizations)
- Fully determined by distributions
  - **Discrete**: probability mass function (pmf)

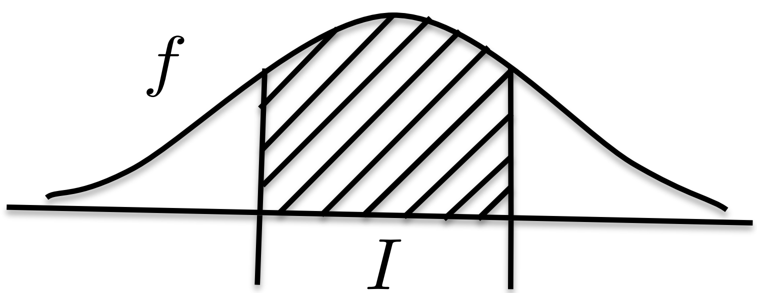
X	x_1	x_2	...	x_n	...
Probs	p_1	p_2	...	p_n	...

$$P(X = x_i) = p_i; \ \sum p_i = 1.$$

Example: Gain in one flip X:

X	-1	1
Prob	$\frac{1}{2}$	$\frac{1}{2}$

  - **Continuous**: Realizations are all points in an interval. Need density: $f(x)$



$$P(X \in I) = \int_I f(x)dx$$

Discrete Distributions

- Bernoulli: $\begin{array}{c|cc} X & 0 & 1 \\ \hline \text{Prob} & q & p \end{array}$, $q = 1 - p$, $0 \leq p \leq 1$

  p — Parameter; $\boxed{X \sim \text{Ber}(p)}$

- Binomial: $\begin{array}{c|cccc} X & 0 & 1 & \ldots & n \\ \hline \text{Prob} & p_0 & p_1 & \ldots & p_n \end{array}$, $p_k = P(X = k) = \binom{n}{k} p^k q^{n-k}, \ q = 1 - p, \ k = 0, 1, \ldots, n$

  $n, \ p$ — Parameters; $\boxed{X \sim \text{Bin}(n, p)}$

- Poisson: $\begin{array}{c|ccccc} X & 0 & 1 & \ldots & n & \ldots \\ \hline \text{Prob} & p_0 & p_1 & \ldots & p_n & \ldots \end{array}$, $p_k = P(X = k) = \dfrac{\lambda^k}{k!} e^{-\lambda}, \ \lambda > 0, \ k = 0, 1, 2, \ldots, n, \ldots$

  λ — Parameter; $\boxed{X \sim \text{Pois}(\lambda)}$

Continuous Distributions

- <u>Uniform:</u> $f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{else} \end{cases}, \quad a,\ b$ — Parameters; $\boxed{X \sim \mathrm{U}(a,b)}$
- <u>Exponential:</u> $f(x) = \lambda e^{-\lambda x},\ \ \lambda > 0,\ \ x \ge 0$, or $f(x) = \dfrac{1}{\mu} e^{-x/\mu},\ \ \mu > 0,\ \ x \ge 0$

  λ — Parameter (rate); $\boxed{X \sim \mathrm{Exp}(\lambda)}$, or μ — Parameter (scale); $\boxed{X \sim \mathrm{Exp}(\mu)}$
- <u>Beta:</u> $f(x) = \dfrac{1}{B(a,b)} x^{a-1}(1-x)^{b-1},\ \ 0 \le x \le 1$

  B: Beta function, defined via integral $\displaystyle\int_0^1 x^{a-1}(1-x)^{b-1}dx$;
  $a,\ b$ — Parameters; $\boxed{X \sim \mathrm{Be}(a,b)}$

Normal Distribution

- <u>Normal:</u> $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \ -\infty < x < +\infty, \ -\infty < \mu < +\infty, \ \sigma > 0$

  $\mu, \ \sigma^2$ — Parameters; $\boxed{X \sim N(\mu, \sigma^2)}$

- Let $X \sim f(x)$
  - $f(x-\mu) \leftarrow \mu$ is location shift parameter
  - $\frac{1}{\sigma} f(\frac{x}{\sigma}) \leftarrow \sigma$ is scale parameter
- If $f(x)$ depends on parameter θ, we write $f(x|\theta)$.

Cumulative Distribution Function (cdf)

pmf: Probability mass function (discrete)
pdf: Probability density function (continuous)

cdf: Probability Distribution (cumulative)

$$F(x) \stackrel{\text{def}}{=} P(X \leq x) = \begin{cases} \sum_{x_i \leq x} P(X = x_i) \\ \int_{-\infty}^{x} f(x)dx \end{cases}$$

<u>Example</u>: $X \sim \mathrm{Exp}(\lambda)$, $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$;

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{0}^{x} \lambda e^{-\lambda t} dt = -e^{-\lambda t}|_{0}^{x} = -e^{-\lambda x} - (-1) = 1 - e^{-\lambda x}, x \geq 0$$

Numerical Characteristics of Random Variables

Numerical Characteristics

- **Expectation, moments:**

Expectation

$$EX = x_1p_1 + x_2p_2 + ... + x_np_n + ... \text{ (for discrete)}$$
$$= \int_{\mathbb{R}} x \cdot f(x)dx \text{ (for continuous)}$$

k^{th} moment

$$EX^k = x_1^kp_1 + x_2^kp_2 + ... + x_n^kp_n + ... \text{ (for discrete)}$$
$$= \int_{\mathbb{R}} x^k \cdot f(x)dx \text{ (for continuous)}$$

General function φ

$$E\varphi(X) = \varphi(x_1)p_1 + \varphi(x_2)p_2 + ... + \varphi(x_n)p_n + ... \text{ (for discrete)}$$
$$= \int_{\mathbb{R}} \varphi(x) \cdot f(x)dx \text{ (for continuous)}$$

Variance and Standard Deviation

- **Variance:** $\mathrm{Var}(X) \stackrel{\text{def}}{=} E(X - EX)^2 = EX^2 - (EX)^2$
- **Standard deviation:** $\sigma_x = \sqrt{\mathrm{Var}(X)}$

Example: If $X \sim \mathrm{Exp}(\lambda)$, $\mathrm{Var}(X) = \dfrac{1}{\lambda^2};\ \sigma_x = \dfrac{1}{\lambda}.$

Quantiles

- ξ_p is p^{th} quantile of distribution F if $F(\xi_p) = p$.
    - $\xi_p = \inf\{x | \sum_{x_i \le x} p_i \ge p\}$ (discrete)
    - $\int_0^{\xi_p} f(x)dx = p$ or $F^{-1}(p) = \xi_p$ (continuous)

<u>Example:</u> If $X \sim \mathrm{Exp}(\lambda)$; $F(x) = 1 - e^{-\lambda x},\ \lambda > 0,\ x \ge 0$

$$F(\xi_p) = p,\ \text{and}\ \xi_p = -\frac{1}{\lambda}\log(1-p),\ 0 \le p \le 1.$$

- Median $\stackrel{\text{def}}{=} \xi_{1/2}$ (50% percentile, 0.5-quantile)
- $Q_1,\ Q_3 \stackrel{\text{def}}{=} \xi_{1/4},\ \xi_{3/4}$ (first and third quantiles)
- Mode: Most frequent/likely value
    - Continuous: Value that maximizes density
    - Discrete: Value x_i for which $p_i = P(X = x_i)$ is maximum.

<u>Example:</u> $X \sim \mathrm{Exp}(\lambda)$, median = $\dfrac{\log 2}{\lambda}$, mode = 0 .

Joint and Conditional Distributions

Vector r.v. $X = (X_1, X_2, ..., X_n)$

- **Joint distribution**

  pdf, cdf $f(x_1, x_2, ..., x_n), \; F(x_1, x_2, ..., x_n)$

  $X = (X_1, X_2) \leftarrow$ two dimensional case

- **Conditional distribution** $X_1, \;$ given X_2

$$f(x_1|x_2) \stackrel{\text{def}}{=} \frac{f(x_1, x_2)}{f(x_2)};$$

  here $f(x_2) \leftarrow$ marginal for X_2, defined as $\int_{-\infty}^{+\infty} f(x_1, x_2) dx_1$.

Worked Examples

Example 1: Discrete 2-D

X \ Y	-1	0	1
0	0.1	0.3	0.1
1	0.1	0.05	0.1
2	0.05	0.1	0.1

X	Prob
0	0.5
1	0.25
2	0.25

Y	-1	0	1
Prob	0.25	0.45	0.30

X|Y=0	Prob
0	$\frac{0.3}{0.45} = \frac{2}{3}$
1	$\frac{0.05}{0.45} = \frac{1}{9}$
2	$\frac{0.1}{0.45} = \frac{2}{9}$

Y|X=2	-1	0	1
Prob	$\frac{0.05}{0.25} = 0.2$	$\frac{0.1}{0.25} = 0.4$	$\frac{0.1}{0.25} = 0.4$

Example 2: Continuous 2-D

- If $f(x,y) = \frac{1}{\pi} e^{-\frac{1}{2}(x^2 - 2xy + 5y^2)},\ x,\ y \in \mathbb{R}^2$, find marginal distributions for $X,\ Y$ and find Conditional distributions for $X|Y=y$ and $Y|X=x$.

$$
\begin{aligned}
f(x,y) &= \frac{1}{\pi} e^{-\frac{1}{2}(x^2 - 2xy + 5y^2)} \\
&= \frac{1}{\pi} e^{-2y^2} e^{-\frac{1}{2}(x-y)^2} \\
&= \frac{1}{\pi} \cdot e^{-2y^2} \cdot \sqrt{2\pi} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-y)^2}
\end{aligned}
$$

$$
\begin{aligned}
f(y) &= \int_{\mathbb{R}} f(x,y)dx = \sqrt{\frac{2}{\pi}} e^{-2y^2} \cdot \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-y)^2} dx \\
&= \frac{1}{\sqrt{2\pi \cdot \frac{1}{4}}} e^{-\frac{y^2}{2 \cdot \frac{1}{4}}} \cdot 1 \sim N\left(0, \left(\frac{1}{2}\right)^2\right)
\end{aligned}
$$

- $f(x) = \int f(x,y)dy$

$$f(x,y) = \frac{1}{\pi} e^{-\frac{5}{2}\left(y^2 - \frac{2x}{5}y + \frac{x^2}{25} - \frac{x^2}{25} + \frac{x^2}{5}\right)}$$

$$= \frac{1}{\pi} e^{-\frac{5}{2}\cdot\frac{4}{25}x^2} \cdot e^{-\frac{5}{2}(y-\frac{x}{5})^2}$$

$$= \frac{1}{\pi} \cdot e^{-\frac{2}{5}x^2} \cdot \sqrt{2\pi \cdot \frac{1}{5}} \cdot \frac{1}{\sqrt{2\pi \cdot \frac{1}{5}}} \cdot e^{-\frac{1}{2\cdot\frac{1}{5}}\left(y-\frac{x}{5}\right)^2}$$

$$f(x) = \frac{1}{\sqrt{2\pi \cdot \frac{5}{4}}} e^{-\frac{x^2}{e^{2\cdot\frac{5}{4}}}} \cdot \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\cdot\frac{1}{5}}} e^{-\frac{1}{2\cdot\frac{1}{5}}\left(y-\frac{x}{5}\right)^2} \quad \to X \sim N\left(0, \left(\frac{\sqrt{5}}{2}\right)^2\right)$$

Thus, marginals are normal $X \sim N\left(0, \left(\frac{\sqrt{5}}{2}\right)^2\right)$, and $Y \sim N\left(0, \left(\frac{1}{2}\right)^2\right)$.

- Conditionals: By definition $f(x|y) \stackrel{\text{def}}{=} \dfrac{f(x,y)}{f(y)}$

Recall: $P(A\ B) \stackrel{\text{def}}{=} P(A|B)P(B)$
$f(x,y) \stackrel{\text{def}}{=} f(x|y) \cdot f(y)$

$$f(x|y) = \frac{\frac{1}{\pi} e^{-\frac{1}{2}(x^2 - 2xy + 5y^2)}}{\frac{1}{\sqrt{2\pi \cdot \frac{1}{4}}} \cdot e^{-\frac{1}{2} \cdot 4y^2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-y)^2}, \ x \in \mathbb{R}$$

$$\to X|Y = y \sim N(y, 1)$$

$$f(y|x) = \frac{\frac{1}{\pi} e^{-\frac{1}{2}(x^2 - 2xy + 5y^2)}}{\frac{1}{\sqrt{2\pi \cdot \frac{5}{4}}} e^{-\frac{1}{2} \cdot x^2 \cdot \frac{4}{5}}} = \frac{1}{\sqrt{2\pi \cdot \frac{1}{5}}} e^{-\frac{1}{2 \cdot \frac{1}{5}}(y - \frac{x}{5})^2}, \ y \in \mathbb{R}$$

$$\to Y|X = x \sim N\left(\frac{x}{5}, \left(\frac{1}{\sqrt{5}}\right)^2\right)$$

Example 3: Independent 2-D

- If $f(x,y) = 2xe^{-x-2y}, x \geq 0, y \geq 0$, find conditional distributions for $f(x|y)$ and $f(y|x)$.

x and y are separates variables: $f(x,y) = xe^{-x} \cdot 2e^{-2y} = f(x) \cdot f(y)$
$\rightarrow (X,Y), X, Y$ are independent components; because of independence:

$$f(x|y) = f(x),\ f(y|x) = f(y)$$

Summary