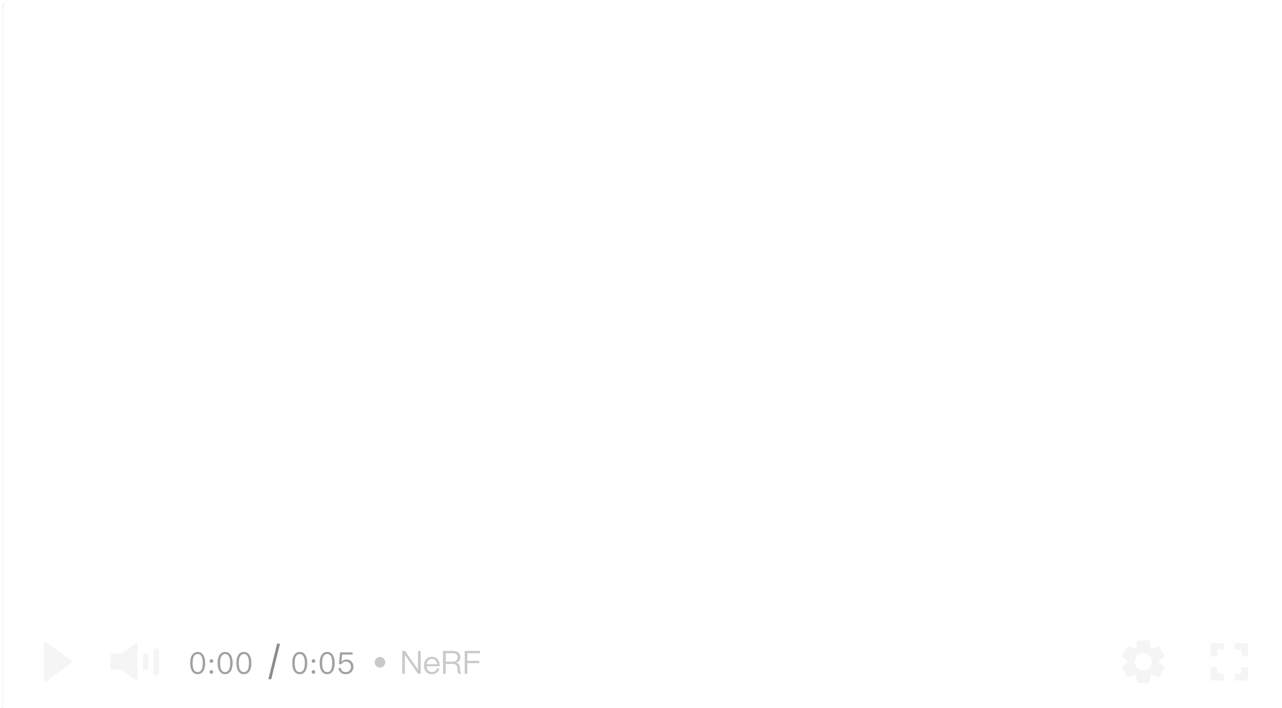
NeRF: A Volume Rendering Perspective

August 31, 2022 ML Graphics, Rendering, NeRF

Overview

NeRF **implicitly** represents a 3D scene with a multi-layer perceptron (MLP) $F : (\boldsymbol{x}, \boldsymbol{d}) \to (\boldsymbol{c}, \sigma)$ for some position $\boldsymbol{x} \in \mathbb{R}^3$, view direction $\boldsymbol{d} \in [0, \pi) \times [0, 2\pi)$, color $\boldsymbol{c}$, and "opacity" σ. Rendered results are spectacular.



There have been a number of articles introducing NeRF since its publication in 2020. While most posts mention general methods, few of them elaborate on *why* the volume rendering procedure

$$\mathbf{C}(\boldsymbol{r}) = \mathbf{C}(z; \boldsymbol{o}, \boldsymbol{d}) = \int_{z_n}^{z_f} T(z)\sigma\left(\boldsymbol{r}(z)\right)\boldsymbol{c}\left(\boldsymbol{r}(z), \boldsymbol{d}\right)\,dz,\ T(z) = \exp\left(-\int_{z_n}^{z} \sigma\left(\boldsymbol{r}(\right.\right.$$

for a ray $\boldsymbol{r} = \boldsymbol{o} + z\boldsymbol{d}$ works and *how* the equation is reduced to

$$\hat{\mathbf{C}}(\boldsymbol{r}) = \hat{\mathbf{C}}(z_1, z_2, \ldots, z_N; \boldsymbol{o}, \boldsymbol{d}) = \sum_{i=1}^{N} T_i\left(1 - e^{-\sigma_i\delta_i}\right)\boldsymbol{c}_i,\ T_i = \exp\left(-\sum_{j=1}^{i-1}\sigma_j\delta_j\right)$$

via numerical quadrature, let alone exploring its implementation via *Monte Carlo* method.

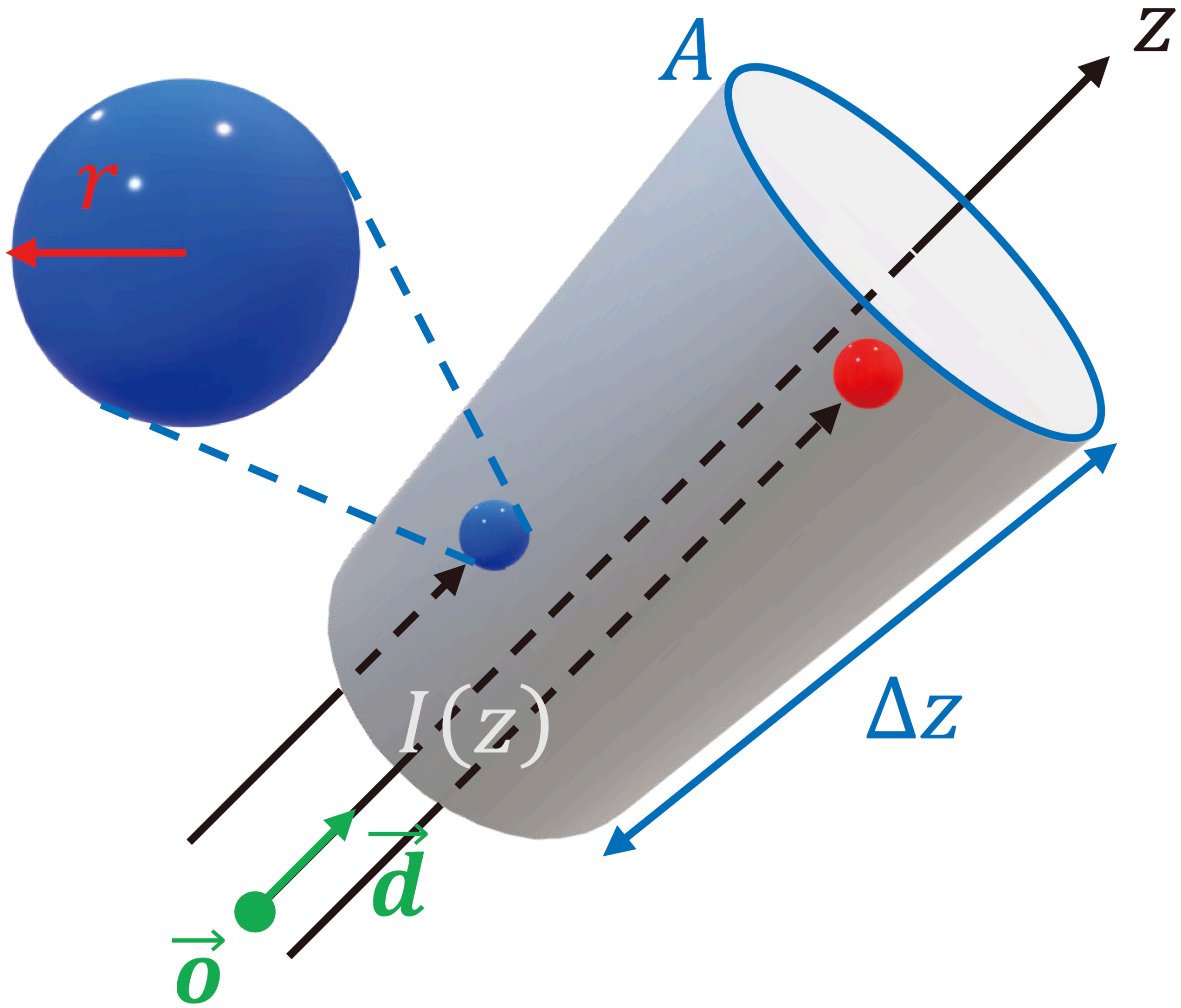
This post delves into the volume rendering aspect of NeRF. The equations will be derived; its implementation will be analyzed.

Prerequisites

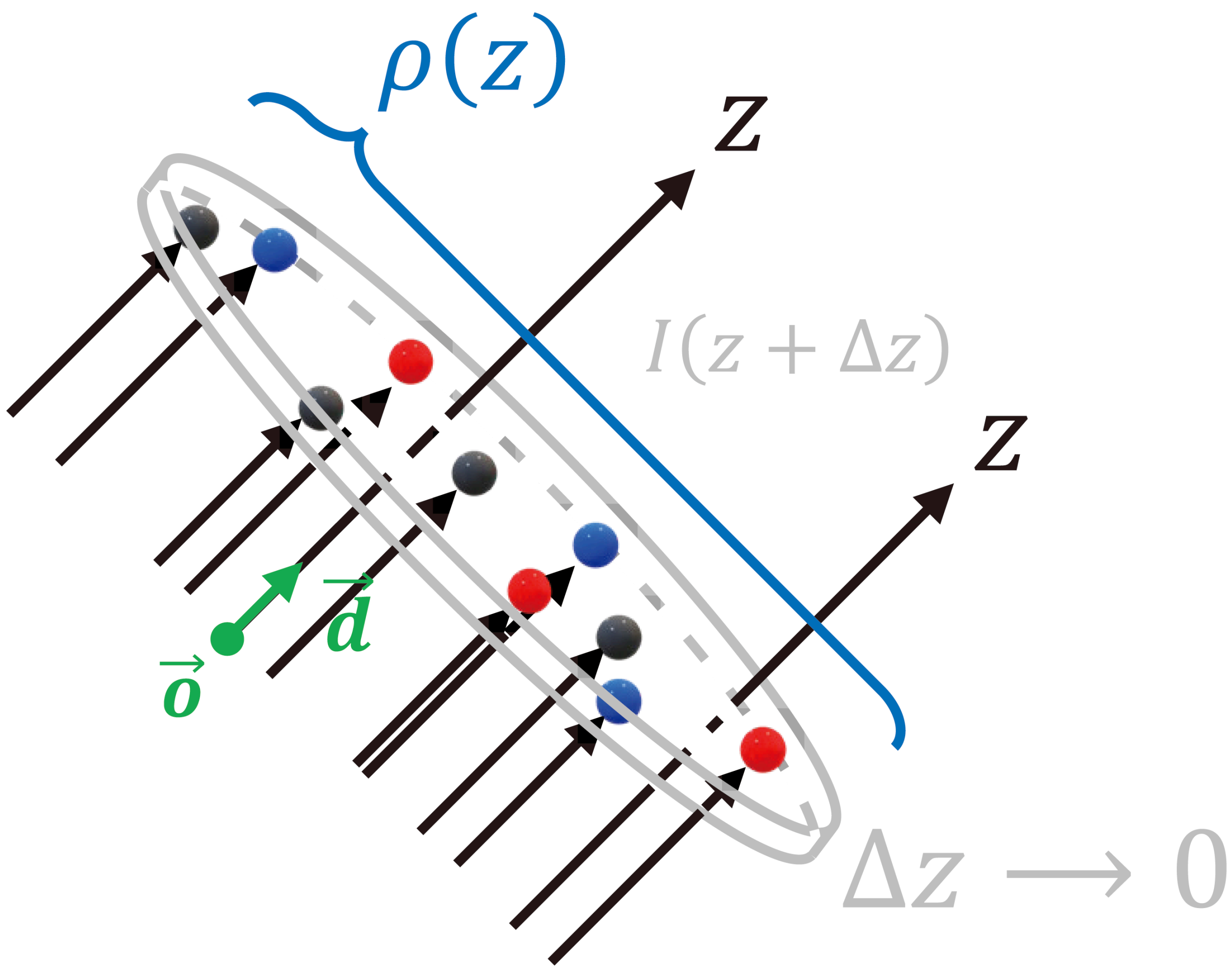
- Having read the NeRF paper
- Ability to solve ordinary differential equations (ODEs)
- Introductory probability theory
- Familiarity with PyTorch and NumPy

Background

The rendering formula



A ray with origin $\boldsymbol{o}$ and direction $\boldsymbol{d}$ casts to an arbitrary region of bounded space. Assume, for simplicity, that the cross section area A is uniform along the ray. Let's focus on a slice of the region with thickness Δz.



Occluding objects are modeled as spherical particles with radius r. Let $\rho(z; \boldsymbol{o}, \boldsymbol{d})$ denote the particle density — number of particles per unit volume — in that orientation. If Δz is small enough such that $\rho(z)$ is consistent on the slice, then there are

$$\underbrace{A \cdot \Delta z}_{\text{slice volume}} \cdot \rho(z)$$

particles contained in the slice. When $\Delta z \to 0$, solid particles do not overlap, then a total of

$$\underbrace{A \cdot \Delta z}_{\text{slice volume}} \cdot \rho(z) \cdot \underbrace{\pi r^2}_{\text{particle area}}$$

area is occluded, which amounts to a portion of $\frac{A \Delta z \cdot \rho(z) \cdot \pi r^2}{A} = \pi r^2 \rho(z) \Delta z$ of the cross section. Let $I(z; \boldsymbol{o}, \boldsymbol{d})$ denote the *light intensity* at depth z from origin $\boldsymbol{o}$ along direction $\boldsymbol{d}$. If a portion of $\pi r^2 \rho(z) \Delta z$ of all rays are occluded, then the intensity at depth $z + \Delta z$ decreases to

$$I(z+\Delta z) = (1 - \underbrace{\pi r^2 \rho(z) \Delta z}_{\text{occluded portion}}) I(z)$$

The difference in intensity is

$$\begin{aligned}\Delta I &= I(z+\Delta z) - I(z) \\ &= -\pi r^2 \rho(z) \Delta z \, I(z)\end{aligned}$$

Take a step from discrete to continuous, we have

$$dI(z) = -\underbrace{\pi r^2 \rho(z)}_{\sigma(z)} I(z) \, dz$$

Define volume density (or voxel "opacity") $\sigma(z; \boldsymbol{o}, \boldsymbol{d}) := \pi r^2 \rho(z; \boldsymbol{o}, \boldsymbol{d})$. This makes sense because the amount of ray reduction depends on both the number of occluding particles and the size of them, then the solution to the ODE

$$dI(z) = -\sigma(z) I(z) \, dz$$

is

$$I(z) = I(z_0) \, \underbrace{e^{\int_{z_0}^{z} -\sigma(s) \, ds}}_{T(z)}$$

Step-by-step solution

Exchange the terms at both sides of the ODE:

$$\frac{1}{I(z)}\,dI(z) = -\sigma(z)\,dz$$

which is a *separable* DE. Integrate both sides

$$\begin{aligned}\int \frac{1}{I(z)}\,dI(z) &= \int -\sigma(z)\,dz \\ \ln I(z) &= \int -\sigma(z)\,dz + C \\ I(z) &= Ce^{\int -\sigma(z)\,dz}\end{aligned}$$

Suppose I takes $I(z_0)$ at depth z_0, then

$$I(z) = I(z_0)\,e^{\int_{z_0}^{z} -\sigma(s)\,ds}$$

Define accumulated *transmittance* $T(z) := e^{\int_{z_0}^{z} -\sigma(s)\,ds}$, then $I(z) = I(z_0)T(z)$ means the **remainning** intensity after the rays travels from z_0 to z. $T(z)$ can also be viewed as the *cumulative density function* (CDF) that a ray does **not** hit any particles from z_0 to z. But **no** color will be observed if a ray passes empty space; radiance is "emitted" only when there is **contact** between rays and particles. Define

$$H(z) := 1 - T(z)$$

as the CDF that a ray hits particles from z_0 to z, then its *probability density function* (PDF) is

$$p_{\text{hit}}(z) = -\underbrace{e^{\int_{z_0}^{z} -\sigma(s)\,ds}}_{T(z)}\,\sigma(z)$$

CDF to PDF

Differentiate CDF $H(z)$ (w.r.t. z) to get $p_{\text{hit}}(z)$

$$\begin{aligned}
p_{\text{hit}}(z) &= \frac{dH}{dz} \\
&= -\frac{dT}{dz} \\
&= -\frac{d}{dz} e^{\int_{z_0}^{z} -\sigma(s)\, ds} \\
&= -e^{\int_{z_0}^{z} -\sigma(s)\, ds} \frac{d}{dz} \int_{z_0}^{z} -\sigma(s)\, ds \\
&= -e^{\int_{z_0}^{z} -\sigma(s)\, ds}\, \sigma(z)
\end{aligned}$$

Let a *random variable* a random variable $\mathbf{R}$ denote the emitted radiance, then

$$\begin{aligned}
p_{\mathbf{R}}(\boldsymbol{r}) &= p_{\mathbf{R}}(z; \boldsymbol{o}, \boldsymbol{d}) \\
&:= P[\mathbf{R} = \boldsymbol{c}(z)] \\
&= p_{\text{hit}}(z)
\end{aligned}$$

Hence, the color of a pixel is the *expectation* of emitted radiance:

$$\begin{aligned}
\mathbf{C}(\boldsymbol{r}) = \mathbf{C}(z; \boldsymbol{o}, \boldsymbol{d}) &= \mathbb{E}(\mathbf{R}) \\
&= \int_0^{\infty} \mathbf{R}\, p_{\mathbf{R}}\, dz \\
&= \int_0^{\infty} \boldsymbol{c}\, p_{\text{hit}}\, dz \\
&= \int_0^{\infty} T(z)\sigma(z)\boldsymbol{c}(z)\, dz
\end{aligned}$$

concluding the proof.

Integration bounds

In practice, $\boldsymbol{c}$, obtained from MLP query, is a function of both position z (or coordinate $\boldsymbol{x}$) and view direction $\boldsymbol{d}$. Also different are the integration bounds. A computer does not support an infinite range; the lower and upper bounds of integration are $z_{\text{near}} = \texttt{near}$ and $z_{\text{far}} = \texttt{far}$ within the range of floating point representation:

$$\mathbf{C}(\boldsymbol{r}) = \int_{\texttt{near}}^{\texttt{far}} T(z)\sigma(z)\boldsymbol{c}(z)\, dz$$

In NeRF, $\texttt{near} = \texttt{0.}$ and $\texttt{far} = \texttt{1.}$ for scaled **bounded** scenes and front facing scenes after conversion to normalized device coordinates (NDC).

Numerical quadrature

We took a step from discrete to continuous to derive the rendering integral. Nevertheless, integration on a continuous domain is not supported by computers. An alternative is numerical quadrature. Sample $\texttt{near} < z_1 < z_2 < \cdots < z_N < \texttt{far}$ along a ray, and define differences between adjacent samples as

$$\delta_i := z_{i+1} - z_i \ \forall i \in \{1, \ldots, N-1\}$$

then the transmittance is approximated by

$$\begin{aligned} T_i &:= T(z_i) \\ &\approx e^{-\sum_{j=1}^{i-1} \sigma_j \delta_j} \end{aligned}$$

where $T_1 = 1$ and $\sigma_j = \sigma(z_j; \boldsymbol{o}, \boldsymbol{d})$. Meanwhile, differentiation in $p_{\text{hit}}(z)$ is also substituted by **discrete difference**. That is,

$$\begin{aligned} p_i := p_{\text{hit}}(z_i) &= \left.\frac{dH}{dz}\right|_{z_i} \\ &\approx H(z_{i+1}) - H(z_i) \\ &= 1 - T(z_{i+1}) - (1 - T(z_i)) \\ &= T(z_i) - T(z_{i+1}) \\ &= T_i\left(1 - e^{-\sigma_i \delta_i}\right) \end{aligned}$$

Step-by-step solution

$$\begin{aligned} T(z_i) - T(z_{i+1}) &= T(z_i) \left(1 - \frac{T(z_{i+1})}{T(z_i)}\right) \\ &= T(z_i) \left(1 - \frac{e^{-\sum_{j=1}^{i} \sigma_j \delta_j}}{e^{-\sum_{j=1}^{i-1} \sigma_j \delta_j}}\right) \\ &= T(z_i) \left(1 - e^{-\sum_{j=1}^{i} \sigma_j \delta_j + \sum_{j=1}^{i-1} \sigma_j \delta_j}\right) \\ &= T_i \left(1 - e^{-\sigma_i \delta_i}\right) \end{aligned}$$

Hence, the discretized emmitted radiance is

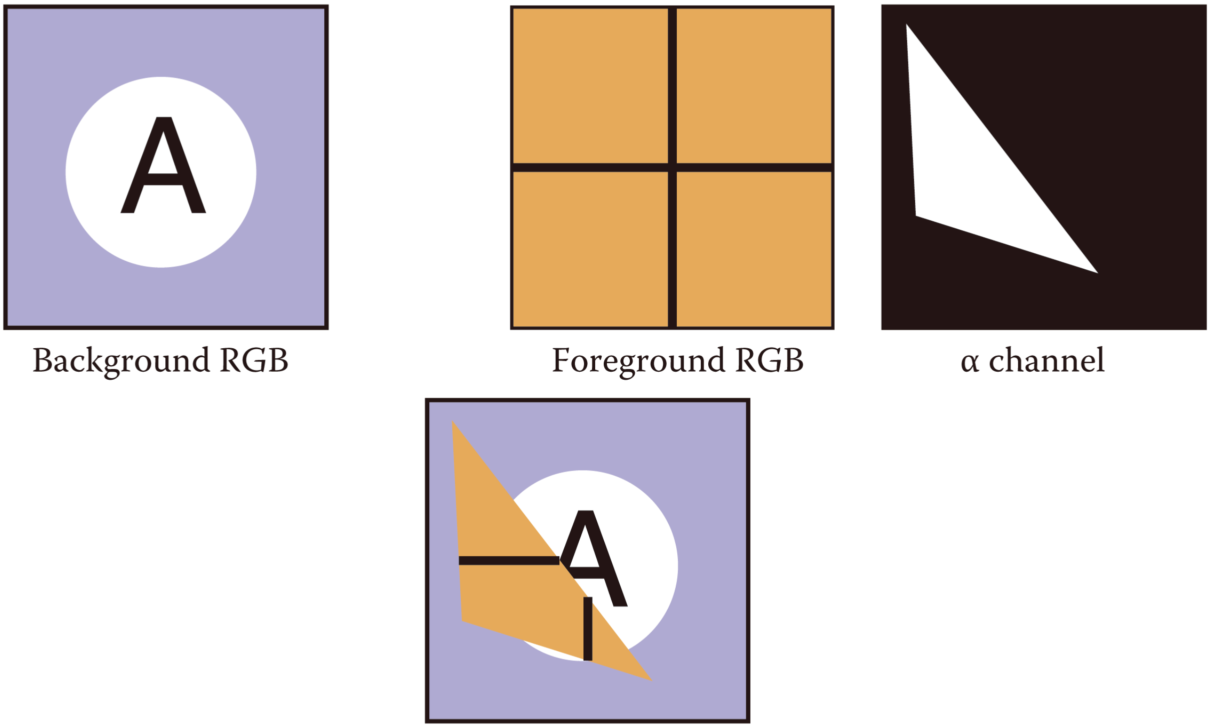
$$\begin{aligned} \hat{\mathbf{C}}(\boldsymbol{r}) = \hat{\mathbf{C}}(z; \boldsymbol{o}, \boldsymbol{d}) &= \mathbb{E}(\mathbf{R}) \\ &= \int_{\texttt{near}}^{\texttt{far}} \mathbf{R}\, p_{\mathbf{R}}\, dz \\ &\approx \sum_{i=1}^{N} \boldsymbol{c}_i T_i \left(1 - e^{-\sigma_i \delta_i}\right) \end{aligned}$$

where $\boldsymbol{c}_i := \boldsymbol{c}(z_i; \boldsymbol{o}, \boldsymbol{d})$ is the output RGB upon MLP query at $\{z_i \mid \boldsymbol{o}, \boldsymbol{d}\}$.

Note that if we denote $\alpha_i := 1 - e^{-\sigma_i \delta_i}$, then $\hat{\mathbf{C}}(\boldsymbol{r}) = \sum_{i=1}^{N} \alpha_i T_i \boldsymbol{c}_i$ resembles classical *alpha compositing*.

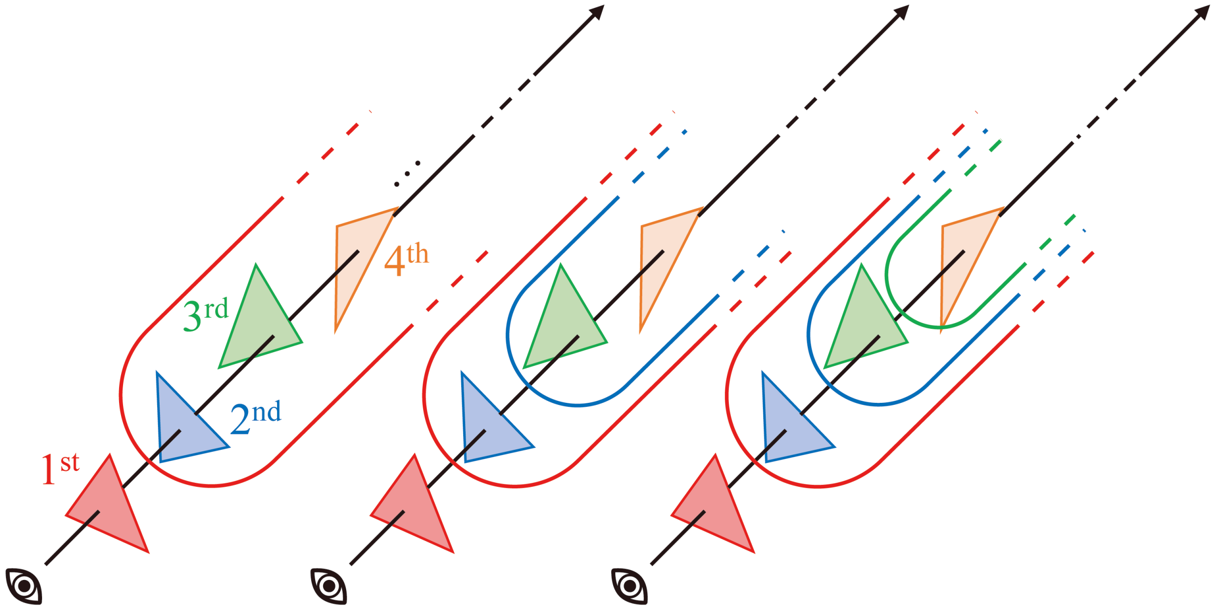
Alpha compositing

Consider the case where a foreground object is inserted ahead of the background. Now that a pixel displays a single color, we have to blend the colors of the two. *Compositing* is applied when there are **partially** transparent regions within the foreground object, or the foreground object **partially** covers the background.



In *alpha compositing*, a parameter α determines the extent to which each object contributes to what is displayed in a pixel. Let α denote the opacity (or *pixel coverage*) of the foreground object, then a pixel showing foreground color $\boldsymbol{c}_f$ over background color $\boldsymbol{c}_b$ is composited as

$$\boldsymbol{c} = \alpha \boldsymbol{c}_f + (1 - \alpha)\boldsymbol{c}_b$$



When blending colors of multiple objects, one can adopt a *divide-and-conquer* approach. Each time, cope with the unregistered object closest to the eye and treat the remaining objects as a **single** entity. Such a strategy is formulated by

$$
\begin{aligned}
\boldsymbol{c} &= \alpha_1 \boldsymbol{c}_1 + (1-\alpha_1)\Big(\alpha_2 \boldsymbol{c}_2 + (1-\alpha_2)\left(\alpha_3 \boldsymbol{c}_3 + (1-\alpha_3)\left(\alpha_4 \boldsymbol{c}_4 + (1-\alpha_4)(\cdots)\right)\right)\Big) \\
&= \alpha_1 \boldsymbol{c}_1 + (1-\alpha_1)\alpha_2 \boldsymbol{c}_2 \\
&\qquad + (1-\alpha_1)(1-\alpha_2)\Big(\alpha_3 \boldsymbol{c}_3 + (1-\alpha_3)\left(\alpha_4 \boldsymbol{c}_4 + (1-\alpha_4)(\cdots)\right)\Big) \\
&= \alpha_1 \boldsymbol{c}_1 + (1-\alpha_1)\alpha_2 \boldsymbol{c}_2 \\
&\qquad + (1-\alpha_1)(1-\alpha_2)\alpha_3 \boldsymbol{c}_3 \\
&\qquad + (1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\left(\alpha_4 \boldsymbol{c}_4 + (1-\alpha_4)(\cdots)\right) \\
&= \alpha_1 \boldsymbol{c}_1 + (1-\alpha_1)\alpha_2 \boldsymbol{c}_2 \\
&\qquad + (1-\alpha_1)(1-\alpha_2)\alpha_3 \boldsymbol{c}_3 \\
&\qquad + (1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\alpha_4 \boldsymbol{c}_4 \\
&\qquad + (1-\alpha_1)(1-\alpha_2)(1-\alpha_3)(1-\alpha_4)(\cdots) \\
&= \cdots
\end{aligned}
$$

which is essentially a *tail recursion*.

> Alpha compositing in NeRF
>
> There are N samples along each ray in NeRF. Consider the first $N-1$ samples as occluding foreground objects with opacity α_i and color $\boldsymbol{c}_i$, and the last sample as background, then the blended pixel value is
>
> $$\begin{aligned}\tilde{\mathbf{C}} = \alpha_1 \boldsymbol{c}_1 &+ (1-\alpha_1)\alpha_2 \boldsymbol{c}_2 \\ &+ (1-\alpha_1)(1-\alpha_2)\alpha_3 \boldsymbol{c}_3 \\ &+ (1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\alpha_4 \boldsymbol{c}_4 \\ &\dots \\ &+ (1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\cdots(1-\alpha_{N-1})\boldsymbol{c}_N \\ = \sum_{i=1}^{N} \left(\alpha_i \boldsymbol{c}_i \prod_{j=1}^{i-1} (1-\alpha_j) \right)&\end{aligned}$$
>
> where $\alpha_0 = 0$. Recall $\hat{\mathbf{C}} = \sum_{i=1}^{N} \alpha_i T_i \boldsymbol{c}_i$, then it remains to show that $\prod_{j=1}^{i-1} (1-\alpha_j) = T_i$:
>
> $$\begin{aligned}\prod_{j=1}^{i-1} (1-\alpha_j) &= \prod_{j=1}^{i-1} e^{-\sigma_j \delta_j} \\ &= \exp\left(\sum_{j=1}^{i-1} -\sigma_j \delta_j \right) \\ &= T_i\end{aligned}$$
>
> concluding the proof. This manifests the elegancy of differentiable volume rendering.

Rewrite $w_i := \alpha_i T_i$, then the expectation of emmitted radiance $\mathbf{C}(\boldsymbol{r}) = \sum_{i=1}^{N} w_i \boldsymbol{c}_i$ is weighted sum of colors.

Why (trivially) differentiable?

Given the above renderer, a coarse training pipeline is

$$(\boldsymbol{x}, \boldsymbol{d}) \xrightarrow{\text{MLP}} (\boldsymbol{c}, \sigma) \xrightarrow{\text{discrete rendering}} \left.\begin{matrix}\text{ground truth } \mathbf{C} \\ \text{prediction } \hat{\mathbf{C}}\end{matrix}\right\} \xrightarrow{\text{MSE}} \mathcal{L} = \|\hat{\mathbf{C}} - \mathbf{C}\|_2^2$$

If the discrete renderer is differentiable, then we can train the *end-to-end* model through gradient descent. No suprise, given a (sorted) sequence of random samples $\boldsymbol{t} = \{t_1, t_2, \ldots, t_N\}$, the derivatives are

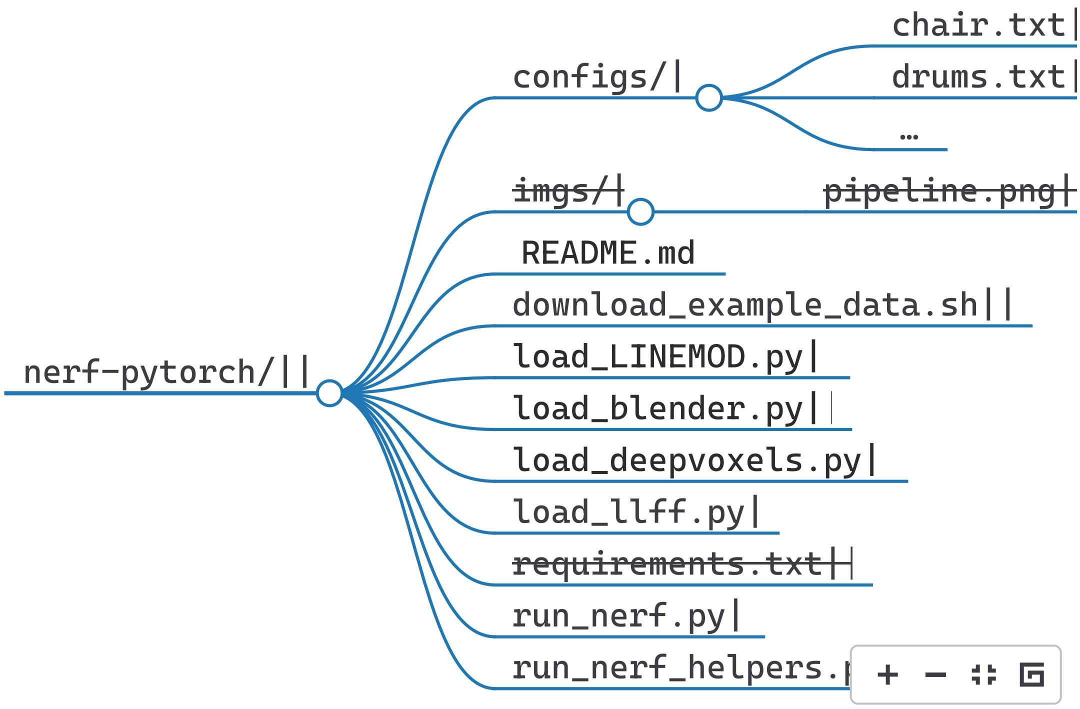
$$
\begin{aligned}
\left.\frac{d\hat{\mathbf{C}}}{d\boldsymbol{c}_i}\right|_{\boldsymbol{t}} &= T_i\left(1 - e^{-\sigma_i\delta_i}\right) \\
\left.\frac{d\hat{\mathbf{C}}}{d\sigma_i}\right|_{\boldsymbol{t}} &= \boldsymbol{c}_i\left(\frac{dT_i}{d\sigma_i}\left(1 - e^{-\sigma_i\delta_i}\right) + T_i\frac{d}{d\sigma_i}\left(1 - e^{-\sigma_i\delta_i}\right)\right) \\
&= \mathbf{c}_i\left(\left(1 - e^{-\sigma_i\delta_i}\right)\exp\left(-\sum_{j=1}^{i-1}\sigma_j\delta_j\right)\underbrace{\frac{d}{d\sigma_i}\left(-\sum_{j=1}^{i-1}\sigma_j\delta_j\right)}_{0} + T_i\left(-e^{-\sigma_i\delta_i}\right)\frac{d(\ldots\right. \\
&= \delta_i T_i \boldsymbol{c}_i e^{-\sigma_i\delta_i}
\end{aligned}
$$

Once the renderer is differentiable, weights and biases in an MLP can be updated via the chain rule.

> **Coarse-to-fine approach**
>
> NeRF jointly optimizes coarse and fine network.

Analysis

Whereas NeRF is originally implemented in Tensrorflow, code analysis is based on a faithful reproduction in PyTorch. The repository is organized as



Let's experiment with the LLFF dataset , which is comprised of front-facing scenes with camera poses. Pertinent directories and files are

Item	Type	Description
`configs`	directory	contains per scene configuration (`.txt`) for the LLFF dataset
`download_example_data.sh`	shell script	to download datasets
`load_llff.py`	Python script	data loader of the LLFF dataset
`run_nerf.py`	Prthon script	main procedures
`run_nerf_helpers.py`	Python script	utility functions

> **Modified identation and comments**
>
> Codes in this post deviate slightly from the [authentic version](). Dataflow and function calls remain intact whereas indentation and comments are modified for the sake of readibility.

The big picture

```python
if __name__ == '__main__':
    torch.set_default_tensor_type('torch.cuda.FloatTensor')
    train()
```

As shown, `train(…)` in `run_nerf.py` is the execution entry to the project. The entire training process is

```python
def train():

    parser = config_parser()
    args = parser.parse_args()

    # load data
    K = None
    if args.dataset_type == 'llff':
        images, poses, bds, render_poses, i_test =
load_llff_data(args.datadir, args.factor,

recenter=True, bd_factor=.75,

spherify=args.spherify)
        hwf = poses[0,:3,-1]
        poses = poses[:,:3,:4]
        print('Loaded llff', images.shape, render_poses.shape, hwf,
args.datadir)
        if not isinstance(i_test, list):
            i_test = [i_test]

        if args.llffhold > 0:
            print('Auto LLFF holdout,', args.llffhold)
            i_test = np.arange(images.shape[0])[ : :args.llffhold]

```

```
        i_val = i_test
        i_train = np.array([i for i in np.arange(int(images.shape[0]))
                                  if (i not in i_test and i not in
i_val)])

        print('DEFINING BOUNDS')
        if args.no_ndc:
            near = np.ndarray.min(bds) * .9
            far = np.ndarray.max(bds) * 1.

        else:
            near = 0.
            far = 1.
        print('NEAR FAR', near, far)

    elif args.dataset_type == 'blender':
        images, poses, render_poses, hwf, i_split =
load_blender_data(args.datadir, args.half_res, args.testskip)
        print('Loaded blender', images.shape, render_poses.shape, hwf,
args.datadir)
        i_train, i_val, i_test = i_split

        near = 2.
        far = 6.

        if args.white_bkgd:
            images = images[...,:3]*images[...,-1:] + (1.-
images[...,-1:])
        else:
            images = images[...,:3]

    elif args.dataset_type == 'LINEMOD':
        images, poses, render_poses, hwf, K, i_split, near, far =
load_LINEMOD_data(args.datadir, args.half_res, args.testskip)
        print(f'Loaded LINEMOD, images shape: {images.shape}, hwf:
{hwf}, K: {K}')
        print(f'[CHECK HERE] near: {near}, far: {far}.')
        i_train, i_val, i_test = i_split

        if args.white_bkgd:
            images = images[...,:3]*images[...,-1:] + (1.-
images[...,-1:])
        else:
```

```
            images = images[...,:3]

    elif args.dataset_type == 'deepvoxels':

        images, poses, render_poses, hwf, i_split =
load_dv_data(scene=args.shape,

basedir=args.datadir,

testskip=args.testskip)

        print('Loaded deepvoxels', images.shape, render_poses.shape,
hwf, args.datadir)
        i_train, i_val, i_test = i_split

        hemi_R = np.mean(np.linalg.norm(poses[:,:3,-1], axis=-1))
        near = hemi_R-1.
        far = hemi_R+1.

    else:
        print('Unknown dataset type', args.dataset_type, 'exiting')
        return

    # cast intrinsics to right types
    H, W, focal = hwf
    H, W = int(H), int(W)
    hwf = [H, W, focal]

    if K is None:
        K = np.array([
            [focal, 0     , 0.5*W],
            [0     , focal, 0.5*H],
            [0     , 0     , 1    ]
        ])

    if args.render_test:
        render_poses = np.array(poses[i_test])

    # create log dir and copy the config file
    basedir = args.basedir
    expname = args.expname
    os.makedirs(os.path.join(basedir, expname), exist_ok=True)
    f = os.path.join(basedir, expname, 'args.txt')
```

```
    with open(f, 'w') as file:
        for arg in sorted(vars(args)):
            attr = getattr(args, arg)
            file.write('{} = {}\n'.format(arg, attr))
    if args.config is not None:
        f = os.path.join(basedir, expname, 'config.txt')
        with open(f, 'w') as file:
            file.write(open(args.config, 'r').read())

    # create nerf model
    render_kwargs_train, render_kwargs_test, start, grad_vars,
optimizer = create_nerf(args)
    global_step = start

    bds_dict = {'near': near,
                'far' : far}
    render_kwargs_train.update(bds_dict)
    render_kwargs_test.update(bds_dict)

    # move test data to GPU
    render_poses = torch.Tensor(render_poses).to(device)

    # short circuit if rendering from trained model
    if args.render_only:
        print('RENDER ONLY')
        with torch.no_grad():
            if args.render_test:
                images = images[i_test] # switch to test poses
            else:
                # default is smoother render_poses path
                images = None

            testsavedir = os.path.join(basedir, expname,
'renderonly_{}_{:06d}'.format('test' if args.render_test else 'path',
start))
            os.makedirs(testsavedir, exist_ok=True)
            print('test poses shape', render_poses.shape)

            rgbs, _ = render_path(render_poses, hwf, K, args.chunk,
render_kwargs_test, gt_imgs=images, savedir=testsavedir,
render_factor=args.render_factor)
            print('Done rendering', testsavedir)
            imageio.mimwrite(os.path.join(testsavedir, 'video.mp4'),
```

```
to8b(rgbs), fps=30, quality=8)

            return

    # prepare raybatch tensor if batching random rays
    N_rand = args.N_rand
    use_batching = not args.no_batching
    if use_batching:
        # random ray batching
        print('get rays')
        rays = np.stack([get_rays_np(H, W, K, p) for p in poses[:,:3,:4]], 0) # (num_img, ro+rd, H, W, 3)
        print('done, concats')
        rays_rgb = np.concatenate([rays, images[:,None]], 1) # (num_img, ro+rd+rgb, H, W, 3)
        rays_rgb = np.transpose(rays_rgb, [0,2,3,1,4]) # (num_img, H, W, ro+rd+rgb, 3)
        rays_rgb = np.stack([rays_rgb[i] for i in i_train], 0) # training set only
        rays_rgb = np.reshape(rays_rgb, [-1,3,3]) # ((num_img-1)*H*W, ro+rd+rgb, 3)
        rays_rgb = rays_rgb.astype(np.float32)
        print('shuffle rays')
        np.random.shuffle(rays_rgb)

        print('done')
        i_batch = 0

    # move training data to GPU
    if use_batching:
        images = torch.Tensor(images).to(device)
    poses = torch.Tensor(poses).to(device)
    if use_batching:
        rays_rgb = torch.Tensor(rays_rgb).to(device)


    N_iters = 200000 + 1
    print('Begin')
    print('TRAIN views are', i_train)
    print('TEST views are', i_test)
    print('VAL views are', i_val)

    # summary writers
```

```
    #writer = SummaryWriter(os.path.join(basedir, 'summaries',
expname))

    start = start + 1
    for i in trange(start, N_iters):
        time0 = time.time()

        # sample random ray batch
        if use_batching:
            # random over all images
            batch = rays_rgb[i_batch:i_batch+N_rand] # (B, 2+1, 3*?)
            batch = torch.transpose(batch, 0, 1)
            batch_rays, target_s = batch[:2], batch[2]

            i_batch += N_rand
            if i_batch >= rays_rgb.shape[0]:
                print("Shuffle data after an epoch!")
                rand_idx = torch.randperm(rays_rgb.shape[0])
                rays_rgb = rays_rgb[rand_idx]
                i_batch = 0
        else:
            # random from one image
            img_i = np.random.choice(i_train)
            target = images[img_i]
            target = torch.Tensor(target).to(device)
            pose = poses[img_i, :3,:4]

            if N_rand is not None:
                rays_o, rays_d = get_rays(H, W, K, torch.Tensor(pose))
# (H, W, 3), (H, W, 3)

                if i < args.precrop_iters:
                    dH = int(H//2 * args.precrop_frac)
                    dW = int(W//2 * args.precrop_frac)
                    coords = torch.stack(
                        torch.meshgrid(
                            torch.linspace(H//2 - dH, H//2 + dH - 1,
2*dH),
                            torch.linspace(W//2 - dW, W//2 + dW - 1,
2*dW)
                        ), -1)
                    if i == start:
                        print(f"[Config] Center cropping of size {2*dH
```

```
x {2*dW} is enabled until iter {args.precrop_iters}")
                else:
                    coords =
torch.stack(torch.meshgrid(torch.linspace(0, H-1, H), torch.linspace(0
W-1, W)), -1)  # (H, W, 2)

                coords = torch.reshape(coords, [-1,2])  # (H * W, 2)
                select_inds = np.random.choice(coords.shape[0], size=
[N_rand], replace=False)  # (N_rand,)
                select_coords = coords[select_inds].long()  # (N_rand,
2)
                rays_o = rays_o[select_coords[:, 0], select_coords[:,
1]]  # (N_rand, 3)
                rays_d = rays_d[select_coords[:, 0], select_coords[:,
1]]  # (N_rand, 3)
                batch_rays = torch.stack([rays_o, rays_d], 0)
                target_s = target[select_coords[:, 0], select_coords[:
1]]  # (N_rand, 3)

        #####  core optimization loop  #####
        rgb, disp, acc, extras = render(H, W, K, chunk=args.chunk,
rays=batch_rays,
                                                verbose=i < 10,
retraw=True,
                                                **render_kwargs_train

        optimizer.zero_grad()
        img_loss = img2mse(rgb, target_s)
        trans = extras['raw'][...,-1]
        loss = img_loss
        psnr = mse2psnr(img_loss)

        if 'rgb0' in extras:
            img_loss0 = img2mse(extras['rgb0'], target_s)
            loss = loss + img_loss0
            psnr0 = mse2psnr(img_loss0)

        loss.backward()
        optimizer.step()

        # NOTE: IMPORTANT!
        ###   update learning rate   ###
        decay_rate = 0.1
        decay_steps = args.lrate_decay * 1000
```

```
        new_lrate = args.lrate * (decay_rate ** (global_step / decay_steps))
        for param_group in optimizer.param_groups:
            param_group['lr'] = new_lrate
        ################################

        dt = time.time()-time0
        # print(f"Step: {global_step}, Loss: {loss}, Time: {dt}")
        #####           end            #####

        # rest is logging
        if i%args.i_weights==0:
            path = os.path.join(basedir, expname, '{:06d}.tar'.format(i))
            torch.save({
                'global_step': global_step,
                'network_fn_state_dict': render_kwargs_train['network_fn'].state_dict(),
                'network_fine_state_dict': render_kwargs_train['network_fine'].state_dict(),
                'optimizer_state_dict': optimizer.state_dict(),
            }, path)
            print('Saved checkpoints at', path)

        if i%args.i_video==0 and i > 0:
            # test mode
            with torch.no_grad():
                rgbs, disps = render_path(render_poses, hwf, K, args.chunk, render_kwargs_test)
            print('Done, saving', rgbs.shape, disps.shape)
            moviebase = os.path.join(basedir, expname, '{}_spiral_{:06d}_'.format(expname, i))
            imageio.mimwrite(moviebase + 'rgb.mp4', to8b(rgbs), fps=30 quality=8)
            imageio.mimwrite(moviebase + 'disp.mp4', to8b(disps / np.max(disps)), fps=30, quality=8)

            #if args.use_viewdirs:
            #    render_kwargs_test['c2w_staticcam'] = render_poses[0][:3,:4]
            #    with torch.no_grad():
            #        rgbs_still, _ = render_path(render_poses, hwf, args.chunk, render_kwargs_test)
```

```
            #    render_kwargs_test['c2w_staticcam'] = None
            #    imageio.mimwrite(moviebase + 'rgb_still.mp4',
to8b(rgbs_still), fps=30, quality=8)

        if i%args.i_testset==0 and i > 0:
            testsavedir = os.path.join(basedir, expname,
'testset_{:06d}'.format(i))
            os.makedirs(testsavedir, exist_ok=True)
            print('test poses shape', poses[i_test].shape)
            with torch.no_grad():
                render_path(torch.Tensor(poses[i_test]).to(device),
hwf, K, args.chunk, render_kwargs_test, gt_imgs=images[i_test],
savedir=testsavedir)
            print('Saved test set')

        if i%args.i_print==0:
            tqdm.write(f"[TRAIN] Iter: {i} Loss: {loss.item()}  PSNR:
{psnr.item()}")
        """
            print(expname, i, psnr.numpy(), loss.numpy(),
global_step.numpy())
            print('iter time {:.05f}'.format(dt))

            with
tf.contrib.summary.record_summaries_every_n_global_steps(args.i_print)
                tf.contrib.summary.scalar('loss', loss)
                tf.contrib.summary.scalar('psnr', psnr)
                tf.contrib.summary.histogram('tran', trans)
                if args.N_importance > 0:
                    tf.contrib.summary.scalar('psnr0', psnr0)

            if i%args.i_img==0:

                # log a rendered validation view to Tensorboard
                img_i=np.random.choice(i_val)
                target = images[img_i]
                pose = poses[img_i, :3,:4]
                with torch.no_grad():
                    rgb, disp, acc, extras = render(H, W, focal,
chunk=args.chunk, c2w=pose,

**render_kwargs_test)
                psnr = mse2psnr(img2mse(rgb, target))
```

```
                with
tf.contrib.summary.record_summaries_every_n_global_steps(args.i_img):

                    tf.contrib.summary.image('rgb', to8b(rgb)
[tf.newaxis])
                    tf.contrib.summary.image('disp',
disp[tf.newaxis, ... ,tf.newaxis])
                    tf.contrib.summary.image('acc',
acc[tf.newaxis, ... ,tf.newaxis])

                    tf.contrib.summary.scalar('psnr_holdout', psnr)
                    tf.contrib.summary.image('rgb_holdout',
target[tf.newaxis])

                if args.N_importance > 0:

                    with
tf.contrib.summary.record_summaries_every_n_global_steps(args.i_img):
                        tf.contrib.summary.image('rgb0',
to8b(extras['rgb0'])[tf.newaxis])
                        tf.contrib.summary.image('disp0',
extras['disp0'][tf.newaxis, ... ,tf.newaxis])
                        tf.contrib.summary.image('z_std',
extras['z_std'][tf.newaxis, ... ,tf.newaxis])
        """
        global_step += 1
```

which is lengthy and potentially obscure. Function calls are visualized below, which assists comprehension of the rendering pipeline.

As captioned, let's concentrate on implementating volume rendering in this post. Our journey starts from rays generation (`get_rays_np(…)` at line 144) and culminates in learning rate update (lines 242 to 246) in each iteration.

Prior to rendering is the data loader (lines 7 to 104) and network initialization (lines 107 to 116). We ommit the analysis of the data loader. Though loaded images and poses will be introduced upon their first appearance, feel free to peruse this post for details. Neither will we delve into lines 119 to 136, which terminates the project immediately after rendering a novel view or video. Testing NeRF is not our primary concern. Despite this, network creation and initialization will be covered in appendix.

> **Functions analyses are organized in horizontal tabs.**
>
> As you see in the function flow chart, procedure call is complex in NeRF. To facilitate clearity, there will be a few horizontal tabs in a section, each responsible for a single function.

Training set

Data preparation

```python
    # prepare raybatch tensor if batching random rays
    N_rand = args.N_rand
    use_batching = not args.no_batching
    if use_batching:
        # random ray batching
        print('get rays')
        rays = np.stack([get_rays_np(H, W, K, p) for p in poses[:,:3,:4]], 0) # (num_img, ro+rd, H, W, 3)
        print('done, concats')
        rays_rgb = np.concatenate([rays, images[:,None]], 1) # (num_img, ro+rd+rgb, H, W, 3)
        rays_rgb = np.transpose(rays_rgb, [0,2,3,1,4]) # (num_img, H, W, ro+rd+rgb, 3)
        rays_rgb = np.stack([rays_rgb[i] for i in i_train], 0) # training set only
        rays_rgb = np.reshape(rays_rgb, [-1,3,3]) # ((num_img-1)*H*W, ro+rd+rgb, 3)
        rays_rgb = rays_rgb.astype(np.float32)
```

```
print('shuffle rays')
np.random.shuffle(rays_rgb)

print('done')
i_batch = 0
```

There are 2 command line argument variables (CL args) in the above snippet:

Variable	Value	Description
`N_rand`	$32 \times 32 \times 4 = 4096$ by default	batch size: number of random rays per optimization loop
`no_batching`	`False` by default	whether or not adopt rays from a **single** image per iteration

`use_batching` , therefore, is asserted by default. The conditioned block contains in lines 7 to 11 a few alien variables, most of which are relevant to the dataloader (lines 7 to 104 in `train(…)`):

Variable	Type	Dimension	Description
`H`	`int`		height of image plane H in pixels
`W`	`int`		width of image plane W in pixles
`K`	NumPy array	$(3, 3)$	$\mathbf{K} = \begin{bmatrix} f_{\text{camera}} & 0 & \frac{W}{2} \\ 0 & f_{\text{camera}} & \frac{H}{2} \\ 0 & 0 & 1 \end{bmatrix}$, where f_{camera} is the focal length of the camera, is a calibration matrix, also the camera intrinsics. It is defined from line 82 to 87 in `train(…)` .

Variable	Type	Dimension	Description
`poses`	NumPy array	$(\texttt{num\_img}, 3, 5)$	all camera poses, where $\texttt{num\_img}$ is the number of images in a scene
`images`	NumPy array	$(\texttt{num\_img}, H, W, 3)$	all images
`i_train`	NumPy array	$(\texttt{num\_img} \times \frac{7}{8},)$	indices of training images, `i_train` $=$ $[0, \texttt{num\_img}) \setminus$ `i_test` `i_test` is initially provided by the dataloader (line 9 in `train(…)`); it is then overridden by lines 18 to 20 since `args.llffhold` is 8 by default.

Camera intrinsics

The calibration matrix takes a general form

$$\mathbf{K} = \begin{bmatrix} f_{\text{camera}} & s & c_x \\ 0 & a f_{\text{camera}} & c_y \\ 0 & 0 & 1 \end{bmatrix}$$

for some *aspect ratio* a, skew s, and *principle point* $\begin{bmatrix} c_x & c_y \end{bmatrix}^\top$.

$a = 1$ unless pixels are not square. "s encodes possible skew between the sensor axes due to the sensor not being mounted perpendicular to the optical axis." $\begin{bmatrix} c_x & c_y \end{bmatrix}^\top$ denotes the image center in pixel coordinates. In practice, $\mathbf{K}$ is simplified to

$$\mathbf{K} = \begin{bmatrix} f_{\text{camera}} & 0 & \frac{W}{2} \\ 0 & f_{\text{camera}} & \frac{H}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

`get_rays_np(…)` is then invoked at line 7 to generate rays (see right tab). Iterating all images, `rays` has shape $(\mathtt{num\_img}, 2, H, W, 3)$. Lines 9 and 10 packs `rays_o`, `rays_d`, and `images` together with their dimension changed to $(\mathtt{num\_img}, H, W, 3, 3)$. Lines 11 to 15 filter and shuffle rays in the training set, whose final result `rays_rgb` is of dimension $(\mathtt{num\_ray}, 3, 3)$.

> **Misleading comment**
>
> Training set dimension is commented to be $((\mathtt{num\_img} - 1) \times H \times W, 3, 3)$ at line 12, which implies only 1 image in a scene is for testing. This is not true for the LLFF dataset. Behavior of the dataloader is overridden by lines 18 to 20 in `train(…)`.

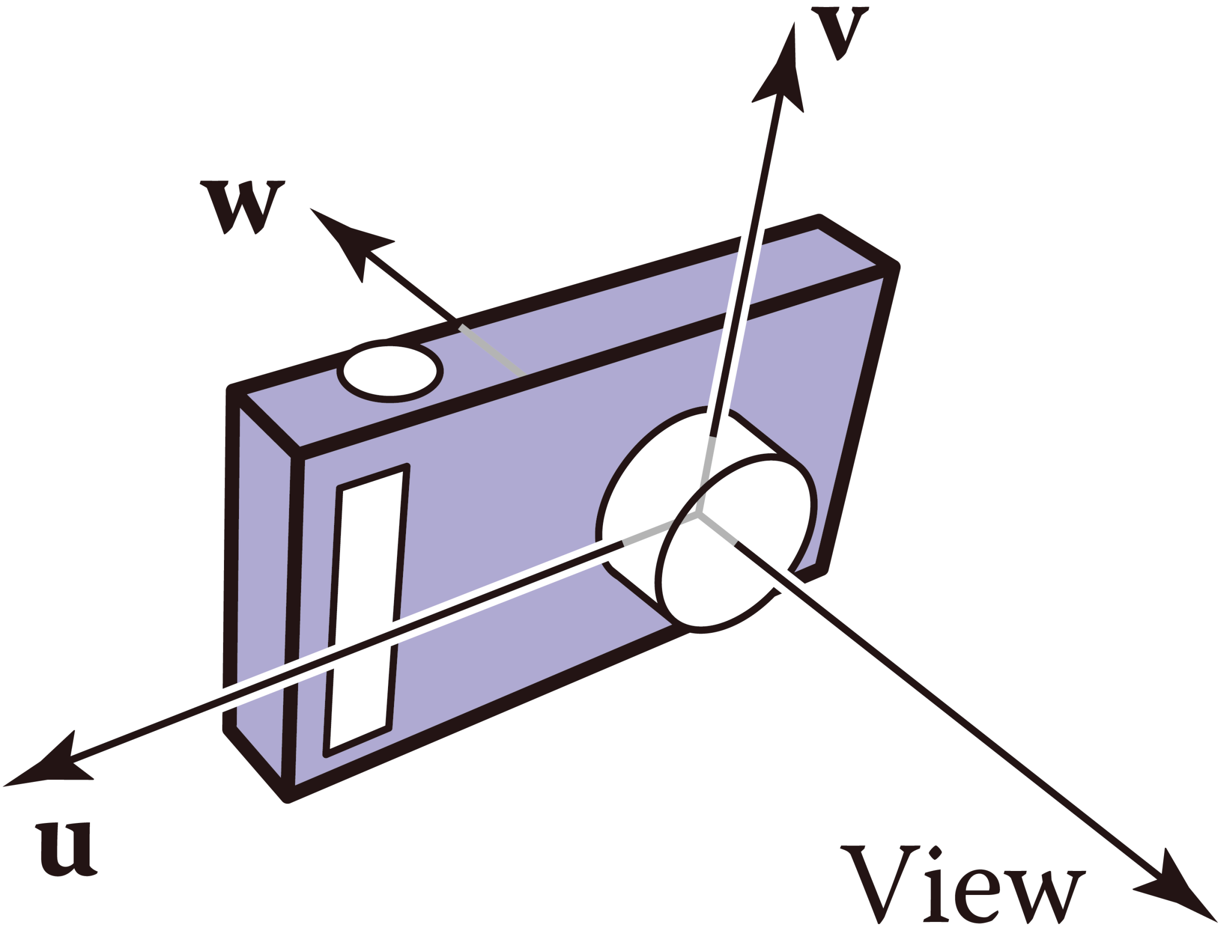
Rays generation

`get_rays_np(…)` is called by the line `rays = np.stack([get_rays_np(H, W, K, p) for p in poses[:,:3,:4]], 0)`, where `H` and `W` are respectively the height and width of the image plane, and `K` is the camera intrinsics. `p` is more physically involved, detailed below.

Suppose world frame (*canonical coordinates*) is characterized by an orthonormal basis $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}$ and an origin $\boldsymbol{o}$, and that camera space is defined by an orthonormal basis $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ and an origin $\boldsymbol{e}$. Denote camera space parameters w.r.t. canonical coordinates as

$$\begin{bmatrix} \boldsymbol{u}_{xyz} & \boldsymbol{v}_{xyz} & \boldsymbol{w}_{xyz} & \boldsymbol{e}_{xyz} \end{bmatrix} = \begin{bmatrix} x_u & x_v & x_w & x_e \\ y_u & y_v & y_w & y_e \\ z_u & z_v & z_w & z_e \end{bmatrix}$$

then $\mathrm{p} = \begin{bmatrix} \boldsymbol{u} & \boldsymbol{v} & \boldsymbol{w} & \boldsymbol{e} \end{bmatrix} \in \mathbb{R}^{3\times 4}$ is the frame-to-canonical matrix that maps rays in camera space to world coordinates

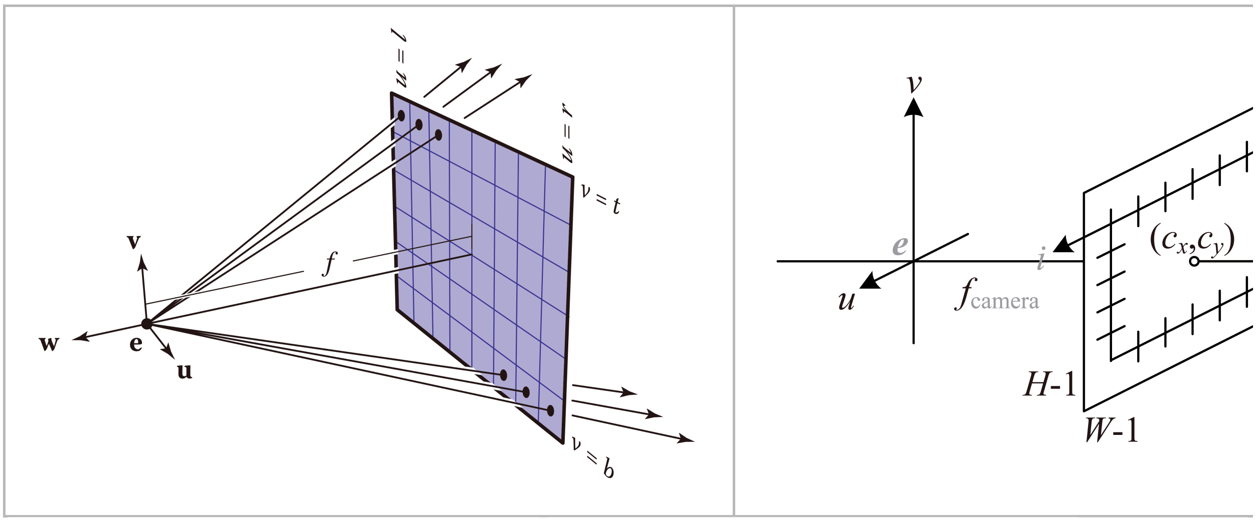



```python
def get_rays_np(H, W, K, c2w):
    i, j = np.meshgrid(np.arange(W, dtype=np.float32), np.arange(H,
dtype=np.float32), indexing='xy')
    dirs = np.stack([ (i-K[0][2]) / K[0][0],
                    -(j-K[1][2]) / K[1][1],
                    -np.ones_like(i)       ], -1)
    # rotate ray directions from camera frame to the world frame
    rays_d = np.sum(dirs[... , np.newaxis, :] * c2w[:3,:3], -1)  #
```

```
 9  dot product, equals to: [c2w.dot(dir) for dir in dirs]
10      # translate camera frame's origin to the world frame. It is the
    origin of all rays.
        rays_o = np.broadcast_to(c2w[:3,-1], np.shape(rays_d))
        return rays_o, rays_d
```

`np.meshgrid(…)` at line 2 creates a 2D grid of points $[0, H) \times [0, W)$, where arrays `i` and `j` are respectively the x- and y coordinates of the grid points. However, the image plane is bounded by

$$\begin{aligned} \text{bottom: } v &= -\frac{H}{2} \\ \text{top: } v &= -\frac{H}{2} \\ \text{left: } u &= -\frac{W}{2} \\ \text{right: } u &= \frac{W}{2} \end{aligned}$$



Applying an offset, pixel coordinates are $\left[i - \frac{W}{2} \quad j - \frac{H}{2} \quad -f_{\text{camera}}\right]^\top$ in camera frame. A ray is defined by an origin $\boldsymbol{o}_{\text{ray}}$, which lies at the origin $\boldsymbol{e}$, and its direction $\boldsymbol{d}$ that connects the origin to a pixel. For every pixel $\boldsymbol{p}$ on the image plane, an ejective direction is

$$\begin{aligned} \boldsymbol{d} &= \boldsymbol{p} - \boldsymbol{o}_{\text{ray}} \\ &= \begin{bmatrix} i - \frac{W}{2} \\ j - \frac{H}{2} \\ -f_{\text{camera}} \end{bmatrix} - \mathbf{0} \end{aligned}$$

which is then "normalized" to $\hat{\boldsymbol{d}} = \frac{\boldsymbol{d}}{f_{\text{camera}}} = \begin{bmatrix} \frac{i-W/2}{f_{\text{camera}}} & \frac{j-H/2}{f_{\text{camera}}} & -1 \end{bmatrix}^\intercal$. This corresponds to lines 3 to 5, except that `dirs` for an **entire** image are generated concurrently. Note that j-axis is opposite to v-axis. Consequently, line 4 adopts $-j + \frac{H}{2}$ (instead of $j - \frac{H}{2}$) as v-coordinate of $\boldsymbol{d}$.

Rays are then mapped to world space. Apply a linear transformation `c2w` to directions $\boldsymbol{d}$ to obtain `rays_d` (line 7) . Ray origins `rays_o` are simply $\boldsymbol{e}$, the last column of `c2w` (line 9). It is broadcast to match the dimension of `rays_d` .

Coordinate transformation

A point $\boldsymbol{p}_{uvw} = \begin{bmatrix} u_p & v_p & w_p \end{bmatrix}^\intercal$ in camera coordinates is characterized by

$$\boldsymbol{p}_{uvw} = \boldsymbol{e}_{xyz} + u_p \boldsymbol{u}_{xyz} + v_p \boldsymbol{v}_{xyz} + w_p \boldsymbol{w}_{xyz}$$

alternatively, the same point $\boldsymbol{p}_{xyz} = \begin{bmatrix} x_p & y_p & z_p \end{bmatrix}^\intercal$ in world space is

$$\boldsymbol{p}_{xyz} = \boldsymbol{o} + x_p \boldsymbol{x} + y_p \boldsymbol{y} + z_p \boldsymbol{z}$$

then $\boldsymbol{p}_{xyz}$ and $\boldsymbol{p}_{uvw}$ are bridged by

$$\begin{aligned}
\begin{bmatrix} \boldsymbol{p}_{xyz} \\ 1 \end{bmatrix} = \begin{bmatrix} x_p \\ y_p \\ z_p \\ 1 \end{bmatrix} &= \underbrace{\begin{bmatrix} & x_e \\ \mathbf{I}_{3\times3} & y_e \\ & z_e \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix}}_{\text{translation: origins coincide}} \underbrace{\begin{bmatrix} x_u & x_v & x_w & \\ y_u & y_v & y_w & \mathbf{0}_{3\times1} \\ z_u & z_v & z_w & \\ & \mathbf{0}_{1\times3} & & 1 \end{bmatrix}}_{\text{rotation: axes align}} \begin{bmatrix} u_p \\ v_p \\ w_p \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} x_u & x_v & x_w & x_e \\ y_u & y_v & y_w & y_e \\ z_u & z_v & z_w & z_e \\ & \mathbf{0}_{1\times3} & & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_{uvw} \\ 1 \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} \boldsymbol{u} & \boldsymbol{v} & \boldsymbol{w} & \boldsymbol{e} \\ & \mathbf{0}_{1\times3} & & 1 \end{bmatrix}}_{\text{frame-to-canonical matrix}} \begin{bmatrix} \boldsymbol{p}_{uvw} \\ 1 \end{bmatrix}
\end{aligned}$$

Check 3B1B 's videos (linear transformation , matrix multiplication , and 3D trnsformation) on linear algebra to comprehend what the above linear transformations (matrices) physically mean.

The returned values are

Variable	Type	Dimension	Description
rays_o	NumPy array	$(H, W, 3)$	directions of rays in the image plane
rays_d	NumPy array	$(H, W, 3)$	origins of rays in the image plane

Training preparation

python

```python
def train_prepare(self, data):
    …
    # move training data to GPU
    if use_batching:
        images = torch.Tensor(images).to(device)
    poses = torch.Tensor(poses).to(device)
    if use_batching:
        rays_rgb = torch.Tensor(rays_rgb).to(device)


    N_iters = 200000 + 1
    print('Begin')
    print('TRAIN views are', i_train)
    print('TEST views are', i_test)
    print('VAL views are', i_val)

    # summary writers
    #writer = SummaryWriter(os.path.join(basedir, 'summaries',
expname))

    start = start + 1
    for i in trange(start, N_iters):
        time0 = time.time()

        # sample random ray batch
        if use_batching:
            # random over all images
```

```
28              batch = rays_rgb[i_batch:i_batch+N_rand] # [B, 2+1, 3*?]
29              batch = torch.transpose(batch, 0, 1)
30              batch_rays, target_s = batch[:2], batch[2]
31
32              i_batch += N_rand
33              if i_batch ≥ rays_rgb.shape[0]:
34                  print("Shuffle data after an epoch!")
35                  rand_idx = torch.randperm(rays_rgb.shape[0])
36                  rays_rgb = rays_rgb[rand_idx]
37                  i_batch = 0
38          else:
              …
```

Lines 2 to 6 convert the training set (from NumPy arrays) to PyTorch tensors and "transfer" them to GPU RAM, and `start = start + 1` (line 18) marks the commencement of training iterations. Training data were first divided into batches. Let B denote the batch size ($B =$ `N_rand`), then inputs `batch_rays` and ground truth `target_s` have shape $(2, B, 3)$ and $(B, 3)$ (line 27). Lines 30 to 34 handles out-of-bound cases where the index `i_batch` exceeds **`num_ray`**. We do not care about the `else` block starting from line 35 since `use_batching` is asserted by default.

Rendering

```python
    …
    for i in trange(start, N_iters):
        …
        #####  core optimization loop  #####
        rgb, disp, acc, extras = render(H, W, K, chunk=args.chunk,
rays=batch_rays,
                                                   verbose=i < 10,
retraw=True,

**render_kwargs_train)
        …
```

Ensuing is volume rendering. CL arg `chunk` defines the number of rays concurrently processed, which impacts performance rather than correctness. `render_kwargs_train` is a dictionary returned upon initiating a NeRF network (line 107 in [`train`]) with 2 more keys injected at line 112 (in [`train`]). Its internals are

Key	Element	Description
`network_query_fn`	a function	a subroutine that takes data and a network as input to perform query
`perturb`	`1.`	whether to adopt stratified sampling, `1.` for `True`
`N_importance`	128	number of addition samples N_F per ray in hierarchical sampling
`network_fine`	an object	the fine network
`N_samples`	64	number of samples N_C per ray to coarse network
`network_fn`	an object	the coarse network
`use_viewdirs`	`True`	whether to feed viewing directions to network, indispensible for view-dependent appearance
`white_bkgd`	`False`	whether to assume white background for rendering This applies to the synthetic dataset only, which contains images (`.png`) with transparent background.
`raw_noise_std`	`1.`	magnitude of noise to inject into volume density
`near`	`0.`	lower bound of rendering integration
`far`	`1.`	upper bound of rendering integration

> **Note**
>
> See appendix for how a NeRF model is implemented.

Basic procedure

```python
def render(H, W, K, chunk=1024*32, rays=None, c2w=None, ndc=True,
           near=0., far=1.,
           use_viewdirs=False, c2w_staticcam=None,
           **kwargs):
    if c2w is not None:
        # special case to render full image
        rays_o, rays_d = get_rays(H, W, K, c2w)
    else:
        # use provided ray batch
        rays_o, rays_d = rays
    # provide ray directions as input
    if use_viewdirs:
        viewdirs = rays_d
        if c2w_staticcam is not None:
            # special case to visualize effect of viewdirs
            rays_o, rays_d = get_rays(H, W, K, c2w_staticcam)
        viewdirs = viewdirs / torch.norm(viewdirs, dim=-1,
keepdim=True)
        viewdirs = torch.reshape(viewdirs, [-1,3]).float()

    sh = rays_d.shape # shape: … × 3
    # for forward facing scenes
    if ndc:
        rays_o, rays_d = ndc_rays(H, W, K[0][0], 1., rays_o,
rays_d)
    # create ray batch
    rays_o = torch.reshape(rays_o, [-1,3]).float()
    rays_d = torch.reshape(rays_d, [-1,3]).float()
    near, far = near * torch.ones_like(rays_d[ ... ,:1]), \
                far  * torch.ones_like(rays_d[ ... ,:1])
    rays = torch.cat([rays_o, rays_d, near, far], -1)
    if use_viewdirs:
        rays = torch.cat([rays, viewdirs], -1)
    # render and reshape
    all_ret = batchify_rays(rays, chunk, **kwargs)
```

```
35        for k in all_ret:
36            k_sh = list(sh[:-1]) + list(all_ret[k].shape[1:])
37            all_ret[k] = torch.reshape(all_ret[k], k_sh)
38
39        k_extract = ['rgb_map', 'disp_map', 'acc_map']
40        ret_list = [all_ret[k] for k in k_extract]
41        ret_dict = {k : all_ret[k] for k in all_ret
42                                   if k not in k_extract}
          return ret_list + [ret_dict]
```

Lines 5 to 7 and 14 to 16 are ignored because the conditions contradict the default setting. Rays are unpacked to origins and directions at line 10. Viewing directions are aliases to ray directions (line 13) except that they are normalized at line 17. Rays are then projected to NDC space at line 23; see my other post for details. `near` $= \mathbf{O}_{B\times 3}$ and `far` $= \mathbf{J}_{B\times 3}$ are initiated (lines 27 and 28) to match the shape of `rays_d`. All the above are concatenated (lines 29 to 31) such that input to the network `rays` have dimension $(B, \texttt{rays\_o}: 3 + \texttt{rays\_d}: 3 + \texttt{near}: 1 + \texttt{far}: 1 + \texttt{viewdirs}: 3) = (B, 11)$.

`batchify_rays(…)` at line 33 (see right tab) decomposes the input tensor into mini-batches to feed to the NeRF network sequentially. There are 8 elements in the `all_ret` dictionary at line 36:

Key	Description of element
`rgb_map`	output color map of the **fine** network
`disp_map`	output disparity map of the **fine** network
`acc_map`	output accumulated transmittance of the **fine** network
`raw`	raw output of the **fine** network
`rgb0`	output color map of the coarse network
`disp0`	output disparity map of the coarse network
`acc0`	output accumulated transmittance of the coarse network
`z_std`	standard variance of disparities of samples along each ray

The ensuing lines (line 38 onward) group and reorder the output such that what are returned can be properly unpacked by [`train(…)`]

Mini-batch operation

```python
1  def batchify_rays(rays_flat, chunk=1024*32, **kwargs):
2      """ render rays in smaller minibatches to avoid OOM
3      """
4      all_ret = {}
5      for i in range(0, rays_flat.shape[0], chunk):
6          ret = render_rays(rays_flat[i:i+chunk], **kwargs)
7          for k in ret:
8              if k not in all_ret:
9                  all_ret[k] = []
10             all_ret[k].append(ret[k])
11     all_ret = {k : torch.cat(all_ret[k], 0) for k in all_ret}
12
13     return all_ret
```

Mini-batches are sequentially passed to `render_rays(…)` (see below) at line 6, cached from line 7 to 10, and eventually concatenated at line 11. We may consider `batchify_rays(…)` as a broker connecting the high-level interface (`render(…)`) to actual rendering implementation.

> **Keyword arguments are passed from high-level interface to "worker" procedures.**
>
> By encapsulation with `batchify_rays(…)` and `render(…)` , training options `render_kwargs_train` [defined previously] are passed to the low-level "worker" `render_rays(…)` , which is to core of volume rendering.

Rendering each ray

```python
def render_rays(ray_batch,
                network_fn,
                network_query_fn,
                N_samples,
                retraw=False,
                lindisp=False,
```

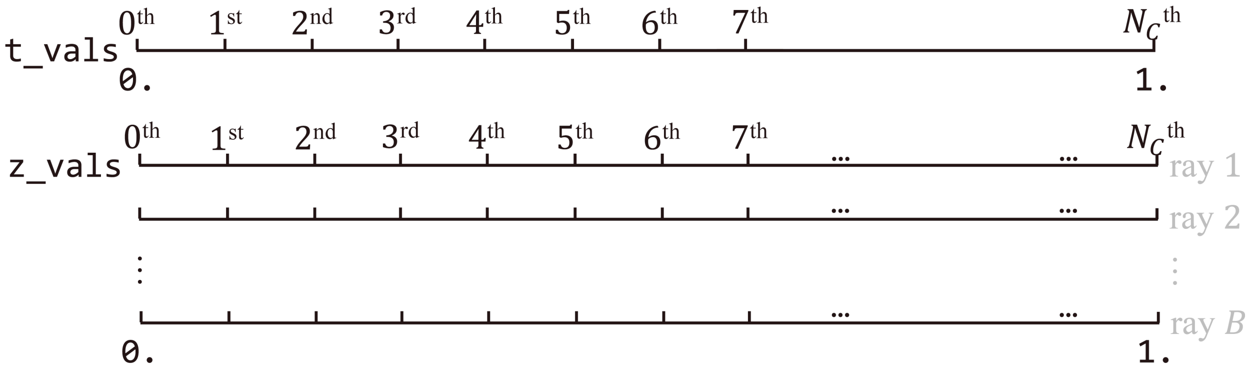
```
                perturb=0., # 1.0, overridden by input
                N_importance=0,
                network_fine=None,
                white_bkgd=False,
                raw_noise_std=0.,
                verbose=False,
                pytest=False):
    N_rays = ray_batch.shape[0]
    rays_o, rays_d = ray_batch[:,0:3], \
                     ray_batch[:,3:6] # (ray #, 3)
    viewdirs = ray_batch[:,-3:] if ray_batch.shape[-1] > 8 \
                                else None
    bounds = torch.reshape(ray_batch[...,6:8], [-1,1,2])
    near, far = bounds[...,0], \
                bounds[...,1] # (ray #, 1)
    t_vals = torch.linspace(0., 1., steps=N_samples)
    if not lindisp:
        z_vals = near * (1. - t_vals) + far * t_vals
    else:
        z_vals = 1. / (1./near * (1. - t_vals) +
                       1./far  * (     t_vals) )
    # copy sample distances of 1 ray to the others
    z_vals = z_vals.expand([N_rays, N_samples])

    if perturb > 0.:
        # get intervals between samples
        mids = .5 * (z_vals[...,1:] + z_vals[...,:-1])
        upper = torch.cat([mids, z_vals[...,-1:]], -1)
        lower = torch.cat([z_vals[...,:1], mids], -1)
        # stratified samples in those intervals
        t_rand = torch.rand(z_vals.shape)
        # pytest: overwrite U with fixed NumPy random numbers
        if pytest:
            np.random.seed(0)
            t_rand = np.random.rand(*list(z_vals.shape))
            t_rand = torch.Tensor(t_rand)

        z_vals = lower + (upper - lower) * t_rand

    pts = rays_o[..., None, :] + \
          rays_d[..., None, :] * z_vals[..., :, None] # (ray #, sample #, 3)
    #raw = run_network(pts)
```

```
    raw = network_query_fn(pts, viewdirs, network_fn)
    rgb_map, disp_map, acc_map, weights, depth_map =
raw2outputs(raw, z_vals, rays_d, raw_noise_std, white_bkgd,
pytest=pytest)
    # hierarchical sampling
    if N_importance > 0:
        # log outputs of coarse network
        rgb_map_0, disp_map_0, acc_map_0 = rgb_map, disp_map,
acc_map

        z_vals_mid = .5 * (z_vals[ ... , 1: ] + z_vals[ ... , :-1])
        z_samples = sample_pdf(z_vals_mid,
                               weights[ ... , 1:-1],
                               N_importance,
                               det=(perturb==0.), # FALSE by
default
                               pytest=pytest)
        z_samples = z_samples.detach()

        z_vals, _ = torch.sort(torch.cat([z_vals, z_samples], -1),
-1)
        pts = rays_o[ ... , None, :] + \
              rays_d[ ... , None, :] * z_vals[ ... , :, None] # (ray #,
coarse & fine sample #, 3)

        run_fn = network_fn if   network_fine is None \
                            else network_fine
        #raw = run_network(pts, fn=run_fn)
        raw = network_query_fn(pts, viewdirs, run_fn)

        rgb_map, disp_map, acc_map, weights, depth_map =
raw2outputs(raw, z_vals, rays_d, raw_noise_std, white_bkgd,
pytest=pytest)

    ret = {'rgb_map' :  rgb_map,
           'disp_map': disp_map,
           'acc_map' :  acc_map}
    if retraw:
        ret['raw'] = raw
    if N_importance > 0:
        ret['rgb0' ] =  rgb_map_0
        ret['disp0'] = disp_map_0
        ret['acc0' ] =  acc_map_0
```

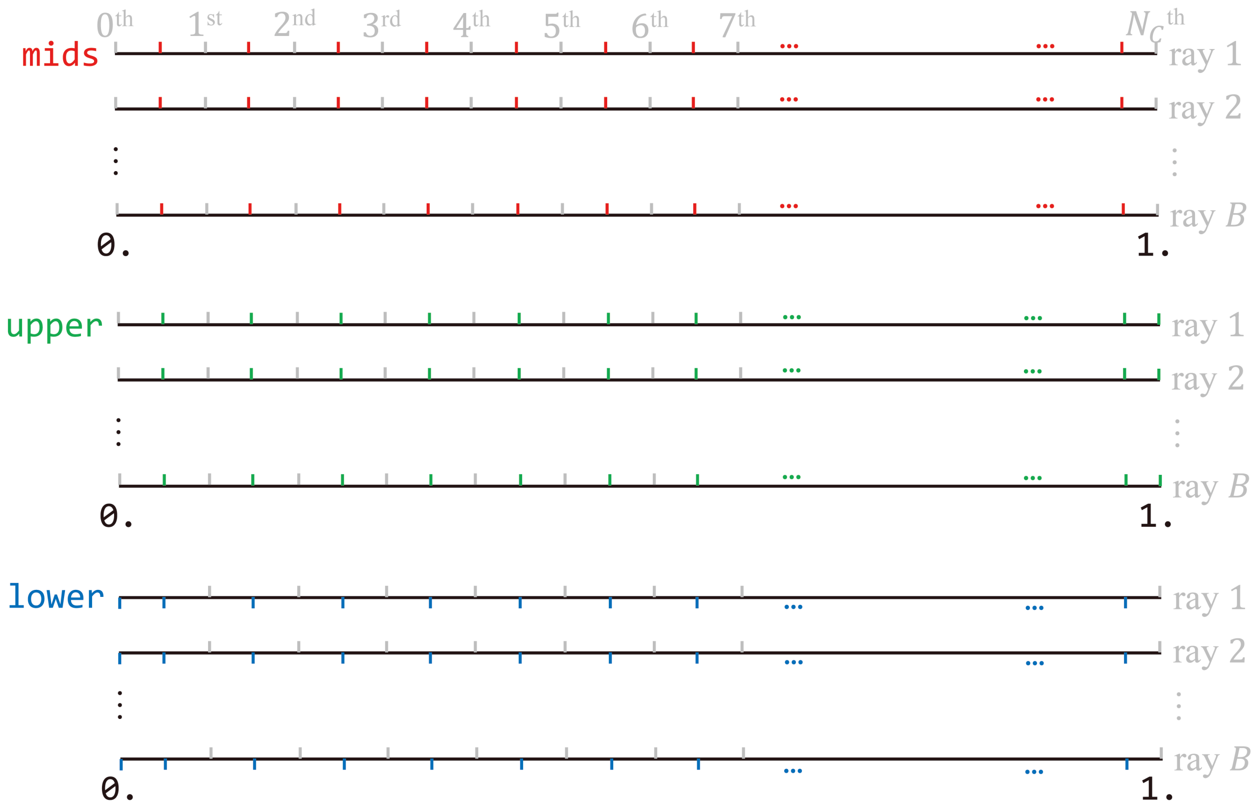
```
        ret['z_std'] = torch.std(z_samples, dim=-1, unbiased=False)
# (ray #)
    for k in ret:
        if (torch.isnan(ret[k]).any() or torch.isinf(ret[k]).any())
and DEBUG:
            print(f"! [Numerical Error] {k} contains nan or inf.")
    return ret
```

Lines 14 to 21 unpack each mini-batch to separate physical values. Ray origins `rays_o`, ray directions `rays_d`, and viewing directions `viewdirs` have shape $(B, 3)$. Integration boundaries `near` and `far` are both $B \times 1$ vectors.

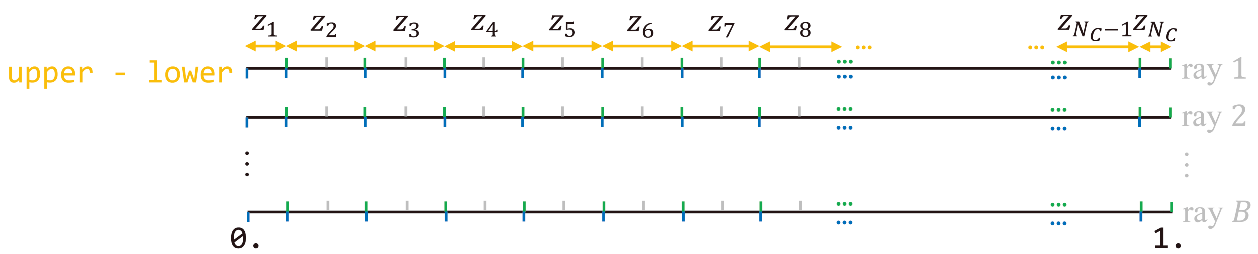
Lines 22 to 44 initialize the samples for ray marching. `torch.linspace(…)` at line 22 create a sequence of N_C points evenly scattered along unit length. Recall that rays are previously projected to NDC space. `False` by default, `lindisp` dictates the voxels are sampled linearly on disparity (inverse depth). `z_vals` simply replicates `t_vals` (lines 24 and 29) to modulate all points on a batch of rays.



There are $N_C - 1$ intervals along each ray. To pick N_C random points out of those intervals ("bins"), at least 2 of them should have length less than $\frac{1}{N_C-1}$. The authors cut the first and last "bin" in half so that all N_C "bins" fit to the interval $[0, 1]$. Line 33 determines of midpoints of `z_vals`, which are afterwards combined with the start (line 35) and endpoint bound (line 34) of each ray.



Subtracting the lower bound from the upper bound finalizes the length of "bins", and stratified sampling is achieved by uniformly sampling every interval. Now, sample z_i lies in bin i $\forall i \in \{1, 2, \dots, N_C\}$.



Let $z_i^{(b)}$ denote the i^{th} sample on the b^{th} ray in a batch, then `z_vals` at line 44 is

$$\texttt{z\_vals} = \begin{bmatrix} z_1^{(1)} & z_2^{(1)} & \cdots & z_{N_C}^{(1)} \\ z_1^{(2)} & z_2^{(2)} & \cdots & z_{N_C}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{(B)} & z_2^{(B)} & \cdots & z_{N_C}^{(B)} \end{bmatrix}$$

Sampling: slight deviation of practice from theory

Implementation of stratified sampling is **inconsistent** with what is described in the paper. Theoretically, a sample z_i is obtained via

$$t_i \sim \mathcal{U}\left[\texttt{near} + \frac{i-1}{N_C}(\texttt{far} - \texttt{near}), \texttt{near} + \frac{i}{N_C}(\texttt{far} - \texttt{near})\right]$$

which implies each "bin" is of equal length. The first and last bins, in practice, are **half** the size of the others. This does not harm the correctness of the algorithm.

Marching depth values `z_vals` along directions `rays_d`, inputs `pts` to `network_fn` at lines 46 and 47 are now

$$\boldsymbol{r}_{B\times N_C\times 3} = \begin{bmatrix} \boldsymbol{e}^{(1)} + z_1^{(1)}\boldsymbol{d}^{(1)} & \boldsymbol{e}^{(1)} + z_2^{(1)}\boldsymbol{d}^{(1)} & \cdots & \boldsymbol{e}^{(1)} + z_{N_C}^{(1)}\boldsymbol{d}^{(1)} \\ \boldsymbol{e}^{(2)} + z_1^{(2)}\boldsymbol{d}^{(2)} & \boldsymbol{e}^{(2)} + z_2^{(2)}\boldsymbol{d}^{(2)} & \cdots & \boldsymbol{e}^{(2)} + z_{N_C}^{(2)}\boldsymbol{d}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{e}^{(B)} + z_1^{(B)}\boldsymbol{d}^{(B)} & \boldsymbol{e}^{(B)} + z_2^{(B)}\boldsymbol{d}^{(B)} & \cdots & \boldsymbol{e}^{(B)} + z_{N_C}^{(B)}\boldsymbol{d}^{(B)} \end{bmatrix}$$

The coarse network `network_fn` is then queried at line 49 to predict raw output `raw` (see appendix for how NeRF is queried). `raw` has shape $(B, \textbf{rgb}: 3 + \sigma : 1) = (B, 4)$. "Shading " via `raw2outputs(…)` follows at line 50 to acquire sample weights and radiance of each ray (see middle tab).

To distinguish outputs of the fine network from those of the coarse one, prefixes `_0` are appended to initial outputs at line 54. Provided `z_vals_mid`, midpoints of coarse samples (line 56) and their weights $\mathbf{W}_{B\times N_C}$, lines 57 to 61 determine fine samples (see right tab). They are combined with coarse samples at line 64 to form a **sorted** tensor of disparities `z_vals`

$$\texttt{z\_vals} = \begin{bmatrix} z_1^{(1)} & z_2^{(1)} & \cdots & z_{N_C+N_F}^{(1)} \\ z_1^{(2)} & z_2^{(2)} & \cdots & z_{N_C+N_F}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{(B)} & z_2^{(B)} & \cdots & z_{N_C+N_F}^{(B)} \end{bmatrix}$$

Static compute graph

NeRF spans a **static** *computate graph*. The key is the `....detach()` call at line 62.

The coarse-to-fine passes are connected by hierarchical sampling, i.e., output of the coarse MLP is used to determine the input of the fine network. Bellow illustrates how the coarse samples are processed to for the fine ones:

$$\texttt{z\_vals}_{\text{coarse}} \xrightarrow[\text{lines 46 and 47}]{} \left.\begin{matrix}\texttt{pts}\\ \cdots\end{matrix}\right\} \xrightarrow[\text{line 49}]{\texttt{network\_query\_fn}} \texttt{raw} \xrightarrow[\text{line 50}]{\texttt{raw2outputs}}$$

$$\texttt{z\_vals}_{\text{coarse}} \xrightarrow[\text{line 5}]{}$$

Hierarchically sampled inputs are again fed to the network.

$$\texttt{z\_vals}_{\text{fine}} \xrightarrow[\text{lines 65 and 66}]{} \left.\begin{matrix}\texttt{pts}\\ \cdots\end{matrix}\right\} \xrightarrow[\text{line 71}]{\texttt{network\_query\_fn}} \cdots$$

Consequently, output of the MLP becomes part of its input. It is a **must** to cut the coarse-to-fine edge in the compute graph because it has to be a directed acyclic one . Otherwise, if the fine network shared with the coarse one an identical instance of class `NeRF` , there would be cyclic definition, and backpropagation would fail. This is exactly what `z_samples.detach()` does at line 62.

A corollary is that NeRF's compute graph is **static**.

New inputs `pts` to the fine network `network_fine` at lines 65 and 66 are now

$$\boldsymbol{r}_{B\times N_C\times 3} = \begin{bmatrix} \boldsymbol{e}^{(1)} + z_1^{(1)}\boldsymbol{d}^{(1)} & \boldsymbol{e}^{(1)} + z_2^{(1)}\boldsymbol{d}^{(1)} & \cdots & \boldsymbol{e}^{(1)} + z_{N_C+N_F}^{(1)}\boldsymbol{d}^{(1)} \\ \boldsymbol{e}^{(2)} + z_1^{(2)}\boldsymbol{d}^{(2)} & \boldsymbol{e}^{(2)} + z_2^{(2)}\boldsymbol{d}^{(2)} & \cdots & \boldsymbol{e}^{(2)} + z_{N_C+N_F}^{(2)}\boldsymbol{d}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{e}^{(B)} + z_1^{(B)}\boldsymbol{d}^{(B)} & \boldsymbol{e}^{(B)} + z_2^{(B)}\boldsymbol{d}^{(B)} & \cdots & \boldsymbol{e}^{(B)} + z_{N_C+N_F}^{(B)}\boldsymbol{d}^{(B)} \end{bmatrix}$$

Another mass network query is performed at line 71, whose raw outputs are converted to radiance `rgb_map` at 71.

Shading

python

```python
def raw2outputs(raw, z_vals, rays_d, raw_noise_std=0, white_bkgd=False, pytest=False):
    raw2alpha = lambda raw, dists, act_fn=F.relu : \
                       1. - torch.exp(-act_fn(raw) * dists) # σ column of `raw`

    dists = z_vals[ ... , 1:] - z_vals[ ... , :-1]
    dists = torch.cat([dists, # (ray #, sample #)
                       torch.Tensor([1e10]).expand(dists[ ... , :1].shape)],
                       -1)
    dists = dists * torch.norm(rays_d[ ... , None, :], dim=-1)

    rgb = torch.sigmoid(raw[ ... , :3]) # (ray #, sample #, 3)

    noise = 0.
    if raw_noise_std > 0.:
        noise = torch.randn(raw[ ... , 3].shape) * raw_noise_std
        # overwrite randomly sampled data
        if pytest:
            np.random.seed(0)
            noise = np.random.rand(*list(raw[ ... ,3].shape)) * raw_noise_std
            noise = torch.Tensor(noise)

    alpha = raw2alpha(raw[ ... , 3] + noise, dists) # (ray #, sample #)
    #weights = alpha * tf.math.cumprod(1.-alpha + 1e-10, -1, exclusive=True)
    weights = alpha * torch.cumprod(torch.cat([torch.ones((alpha.shape[0], 1)),
                                               1. - alpha + 1e-10], -1),
                                    -1)[:, :-1]

    rgb_map   = torch.sum(weights[ ... , None] * rgb, -2)  # (ray #, 3)
    depth_map = torch.sum(weights * z_vals, -1)
    disp_map  = 1. / torch.max(1e-10 * torch.ones_like(depth_map),
                               depth_map / torch.sum(weights, -1))
```

```
    acc_map   = torch.sum(weights, -1)
    if white_bkgd:
        rgb_map = rgb_map + (1. - acc_map[ ... , None])

    return rgb_map, disp_map, acc_map, weights, depth_map
```

`dists` from line 5 to 9 calculates the difference between disparities $\delta_i := z_{i+1} - z_i$.

$$
\begin{aligned}
\boldsymbol{\Delta}_{B\times N_C} &= \begin{bmatrix} \delta_1^{(1)} & \delta_2^{(1)} & \cdots & \delta_{N_C-1}^{(1)} & \infty \\ \delta_1^{(2)} & \delta_2^{(2)} & \cdots & \delta_{N_C-1}^{(2)} & \infty \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta_1^{(B)} & \delta_2^{(B)} & \cdots & \delta_{N_C-1}^{(B)} & \infty \end{bmatrix} \\
&= \begin{bmatrix} z_2^{(1)} - z_1^{(1)} & z_3^{(1)} - z_2^{(1)} & \cdots & z_{N_C}^{(1)} - z_{N_C-1}^{(1)} & \infty \\ z_2^{(2)} - z_1^{(2)} & z_3^{(2)} - z_2^{(2)} & \cdots & z_{N_C}^{(2)} - z_{N_C-1}^{(2)} & \infty \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_2^{(B)} - z_1^{(B)} & z_3^{(B)} - z_2^{(B)} & \cdots & z_{N_C}^{(B)} - z_{N_C-1}^{(B)} & \infty \end{bmatrix}
\end{aligned}
$$

> **Purpose of appending a large vector**
>
> $\texttt{1e10} = 10^{10} \approx \infty$ is appended to the last column of $\boldsymbol{\Delta}_{B\times N_C}$ to (a) maintain the shape of `dists` as (B, N_C), and (b) to force the last column of "opacticy" `alpha` $\boldsymbol{\Delta}_{B\times N_C}$ to be 1 such that classic [alpha compositing] holds.

Line 11 forces RGB values `rgb` to lie in the range $(0, 1)$, that is,

$$
\mathbf{c}_{B\times N_C\times 3} = \begin{bmatrix} \boldsymbol{c}_1^{(1)} & \boldsymbol{c}_2^{(1)} & \cdots & \boldsymbol{c}_{N_C}^{(1)} \\ \boldsymbol{c}_1^{(2)} & \boldsymbol{c}_2^{(2)} & \cdots & \boldsymbol{c}_{N_C}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{c}_1^{(B)} & \boldsymbol{c}_2^{(B)} & \cdots & \boldsymbol{c}_{N_C}^{(B)} \end{bmatrix} \forall \boldsymbol{c}_i^{(b)} \in (0, 1)
$$

Random noise (line 15) is injected to volume density $\sigma_i^{(b)}$ (line 15) before it is rectified and raised to $\alpha_i^{(b)}$ (lines 2, 3, and 22). Let $\hat{\sigma}_i^{(b)} := \texttt{ReLU}\left(\sigma_i^{(b)} + \mathcal{U}[0, 1]\right)$ denote the recitified "opacity" of the i^{th} sample along the b^{th} ray, then `alpha` for alpha compositing are

$$
\begin{aligned}
\mathrm{A}_{B\times N_C} &= \mathbf{1}_{B\times N_C} - \exp\left(\boldsymbol{\sigma}_{B\times N_C} * \boldsymbol{\Delta}_{B\times N_C}\right) \\
&= \begin{bmatrix}
1-e^{-\sigma_1^{(1)}\delta_1^{(1)}} & 1-e^{-\sigma_2^{(1)}\delta_2^{(1)}} & \cdots & 1-e^{-\sigma_{N_C-1}^{(1)}\delta_{N_C-1}^{(1)}} & 1-e^{-\infty} \\
1-e^{-\sigma_1^{(2)}\delta_1^{(2)}} & 1-e^{-\sigma_2^{(2)}\delta_2^{(2)}} & \cdots & 1-e^{-\sigma_{N_C-1}^{(2)}\delta_{N_C-1}^{(2)}} & 1-e^{-\infty} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1-e^{-\sigma_1^{(B)}\delta_1^{(B)}} & 1-e^{-\sigma_2^{(B)}\delta_2^{(B)}} & \cdots & 1-e^{-\sigma_{N_C-1}^{(B)}\delta_{N_C-1}^{(B)}} & 1-e^{-\infty}
\end{bmatrix} \\
&= \begin{bmatrix}
\alpha_1^{(1)} & \alpha_2^{(1)} & \cdots & \alpha_{N_C}^{(1)} \\
\alpha_1^{(2)} & \alpha_2^{(2)} & \cdots & \alpha_{N_C}^{(2)} \\
\vdots & \ddots & \vdots & \vdots \\
\alpha_1^{(B)} & \alpha_2^{(B)} & \cdots & \alpha_{N_C}^{(B)}
\end{bmatrix}
\end{aligned}
$$

where $*$ denotes the *Hadamard product* . `torch.cumprod(…)` from line **24** to **26** calculates the cumulative transmittance

$$
\begin{aligned}
\mathbf{T}_{B\times(N_C+1)} &= \mathtt{cumprod}\left(\left[\mathbf{1}_{B\times 1} \mid \mathbf{1}_{B\times N_C} - \mathrm{A}\right]\right) \\
&= \mathtt{cumprod}\left(\begin{bmatrix}
1 & e^{-\sigma_1^{(1)}\delta_1^{(1)}} & e^{-\sigma_2^{(1)}\delta_2^{(1)}} & \cdots & e^{-\sigma_{N_C-1}^{(1)}\delta_{N_C-1}^{(1)}} & 0 \\
1 & e^{-\sigma_1^{(2)}\delta_1^{(2)}} & e^{-\sigma_2^{(2)}\delta_2^{(2)}} & \cdots & e^{-\sigma_{N_C-1}^{(2)}\delta_{N_C-1}^{(2)}} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & e^{-\sigma_1^{(B)}\delta_1^{(B)}} & e^{-\sigma_2^{(B)}\delta_2^{(B)}} & \cdots & e^{-\sigma_{N_C-1}^{(B)}\delta_{N_C-1}^{(B)}} & 0
\end{bmatrix}\right) \\
&= \begin{bmatrix}
1 & e^{-\sigma_1^{(1)}\delta_1^{(1)}} & e^{-\sigma_1^{(1)}\delta_1^{(1)}-\sigma_2^{(1)}\delta_2^{(1)}} & \cdots & \exp\left(-\sum_{j=1}^{N_C-1}\sigma_j^{(1)}\delta_j^{(1)}\right) \\
1 & e^{-\sigma_1^{(2)}\delta_1^{(2)}} & e^{-\sigma_1^{(2)}\delta_1^{(2)}-\sigma_2^{(2)}\delta_2^{(2)}} & \cdots & \exp\left(-\sum_{j=1}^{N_C-1}\sigma_j^{(2)}\delta_j^{(2)}\right) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & e^{-\sigma_1^{(B)}\delta_1^{(B)}} & e^{-\sigma_1^{(B)}\delta_1^{(B)}-\sigma_2^{(B)}\delta_2^{(B)}} & \cdots & \exp\left(-\sum_{j=1}^{N_C-1}\sigma_j^{(B)}\delta_j^{(B)}\right)
\end{bmatrix} \\
&= \begin{bmatrix}
T_1^{(1)} & T_2^{(1)} & \cdots & T_{N_C}^{(1)} & 0 \\
T_1^{(2)} & T_2^{(2)} & \cdots & T_{N_C}^{(2)} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
T_1^{(B)} & T_2^{(B)} & \cdots & T_{N_C}^{(B)} & 0
\end{bmatrix}
\end{aligned}
$$

The last column of $\mathbf{T}$ is discarded to match the shape of A. Rewriting weights (for points' colors) as $w_i^{(b)} := \alpha_i^{(b)} T_i^{(b)}$, `weights` is

$$\begin{aligned}\mathbf{W}_{B\times N_C} &= \mathrm{A} * \mathbf{T} \\ &= \begin{bmatrix} \alpha_1^{(1)}T_1^{(1)} & \alpha_2^{(1)}T_2^{(1)} & \cdots & \alpha_{N_C}^{(1)}T_{N_C}^{(1)} \\ \alpha_1^{(2)}T_1^{(2)} & \alpha_2^{(2)}T_2^{(2)} & \cdots & \alpha_{N_C}^{(2)}T_{N_C}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{(B)}T_1^{(B)} & \alpha_2^{(B)}T_2^{(B)} & \cdots & \alpha_{N_C}^{(B)}T_{N_C}^{(B)} \end{bmatrix} \\ &= \begin{bmatrix} w_1^{(1)} & w_2^{(1)} & \cdots & w_{N_C}^{(1)} \\ w_1^{(2)} & w_2^{(2)} & \cdots & w_{N_C}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ w_1^{(B)} & w_2^{(B)} & \cdots & w_{N_C}^{(B)} \end{bmatrix}\end{aligned}$$

Recall that radiance is a weighted sum of colors of samples along a ray. This corresponds to line 28, and the output `rgb_map` is

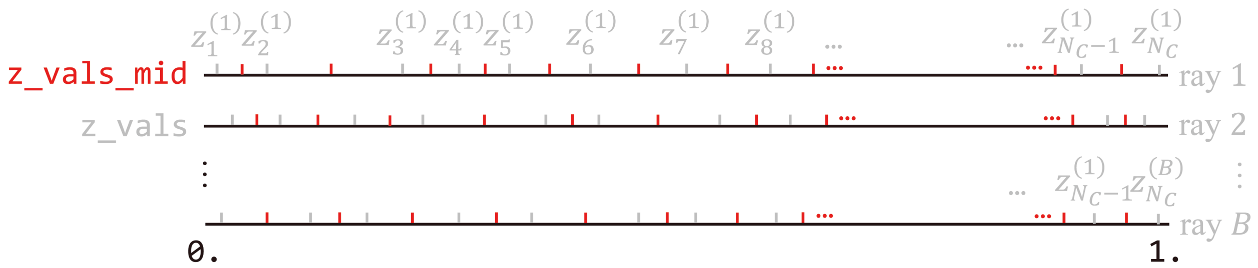
$$\begin{aligned}\mathbf{C}_{B\times 3} &= \texttt{sum}\left(\mathbf{W} * \mathbf{c}\right) \\ &= \begin{bmatrix} w_1^{(1)}\boldsymbol{c}_1^{(1)} + w_2^{(1)}\boldsymbol{c}_2^{(1)} + \cdots + w_{N_C}^{(1)}\boldsymbol{c}_{N_C}^{(1)} \\ w_1^{(2)}\boldsymbol{c}_1^{(2)} + w_2^{(2)}\boldsymbol{c}_2^{(2)} + \cdots + w_{N_C}^{(2)}\boldsymbol{c}_{N_C}^{(2)} \\ \vdots \\ w_1^{(B)}\boldsymbol{c}_1^{(B)} + w_2^{(B)}\boldsymbol{c}_2^{(B)} + \cdots + w_{N_C}^{(B)}\boldsymbol{c}_{N_C}^{(B)} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{C}^{(1)} \\ \mathbf{C}^{(2)} \\ \vdots \\ \mathbf{C}^{(B)} \end{bmatrix}\end{aligned}$$

`rgb_map` $\mathbf{C}$ and `weights` $\mathbf{W}$, along with other values, are returned to `render_rays(…)`.

What else are returned?

Content on the way. Stay tuned!

Hierarchical sampling



`sample_pdf(…)` in `run_nerf_helpers.py` performs hierarchical sampling via *Monte Carlo method*. It is invoked by

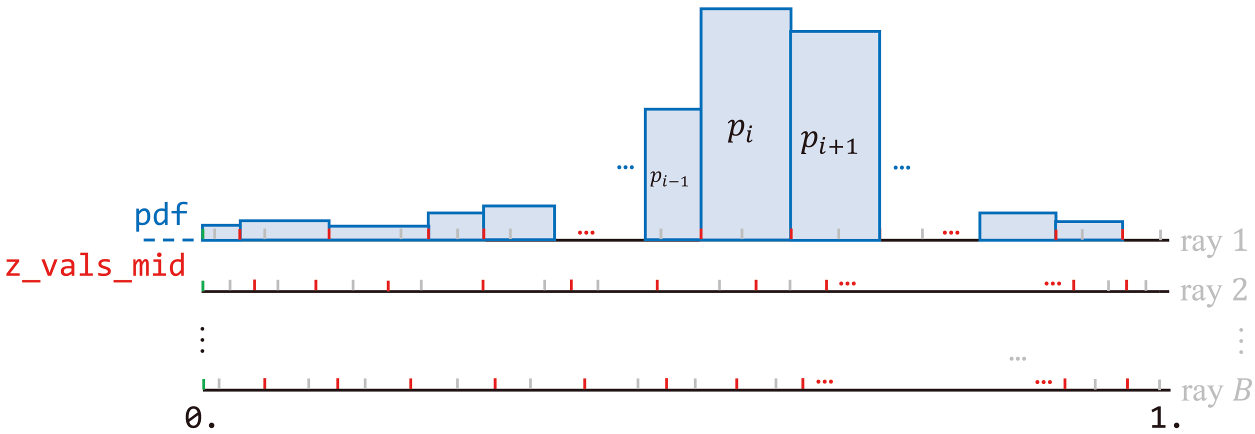
```python
        …
        z_samples = sample_pdf(z_vals_mid,
                               weights[ ... , 1:-1],
                               N_importance,
                               det=(perturb==0.), # FALSE by default
                               pytest=pytest)
        …
```

in `render_rays(…)`, where `z_vals_mid` is a tensor of midpoints of coarse sample disparities. Note that the leading and trailing columns of `weights` are excluded from the input such that

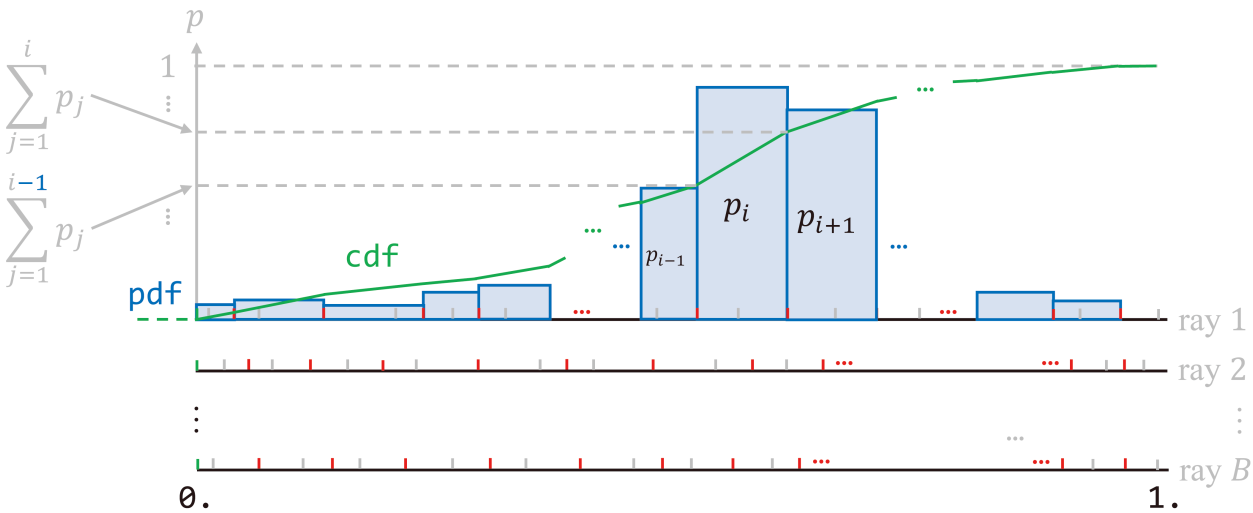
$$\mathbf{W}\texttt{[..., 1:-1]} = \begin{bmatrix} w_2^{(1)} & w_3^{(1)} & \cdots & w_{N_C-1}^{(1)} \\ w_2^{(2)} & w_3^{(2)} & \cdots & w_{N_C-1}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ w_2^{(B)} & w_3^{(B)} & \cdots & w_{N_C-1}^{(B)} \end{bmatrix}$$

```python
def sample_pdf(bins, weights, N_samples, det=False, pytest=False):
    # get PDF
    weights = weights + 1e-5 # prevent NaN
    pdf = weights / torch.sum(weights, -1, keepdim=True)
    cdf = torch.cumsum(pdf, -1)
    cdf = torch.cat([torch.zeros_like(cdf[ ... , :1]), cdf], -1)  # (ray #, bin #)
    # Here, `N_samples` refers to `N_importance`.
    if det:
        u = torch.linspace(0., 1., steps=N_samples)
        u = u.expand(list(cdf.shape[:-1]) + [N_samples])
    else:
        u = torch.rand(list(cdf.shape[ :-1]) + [N_samples])
```

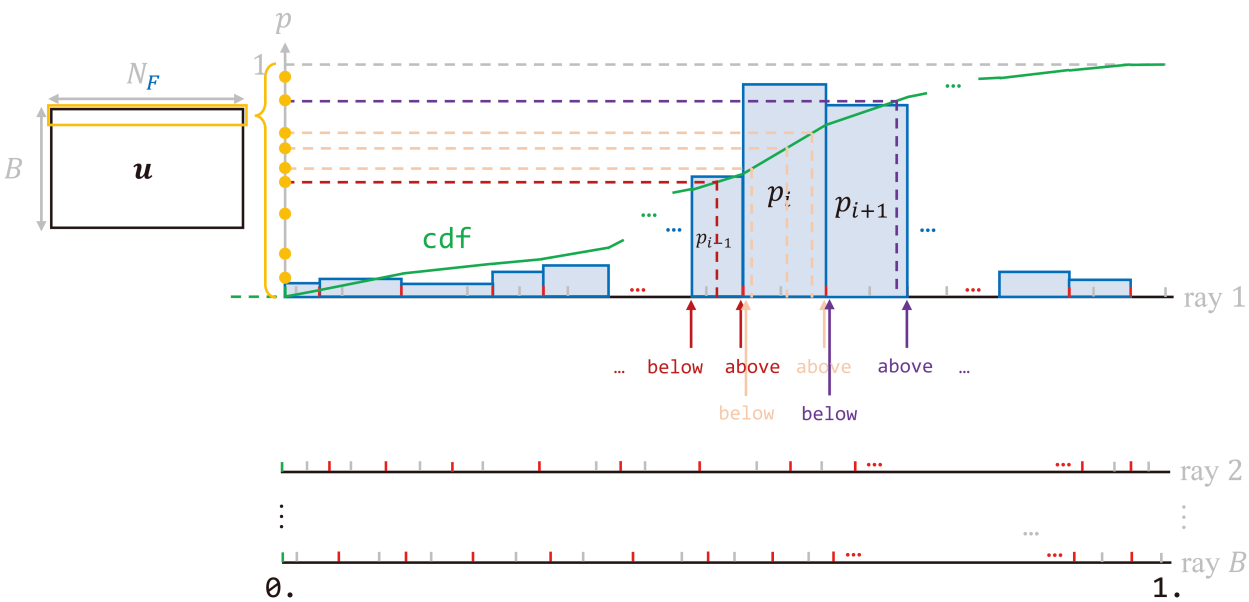
```
    # if pytest, overwrite u with NumPy fixed random numbers
    if pytest:
        np.random.seed(0)
        new_shape = list(cdf.shape[:-1]) + [N_samples]
        if det:
            u = np.linspace(0., 1., N_samples)
            u = np.broadcast_to(u, new_shape)
        else:
            u = np.random.rand(*new_shape)
        u = torch.Tensor(u)
    # invert CDF
    u = u.contiguous()
    inds   = torch.searchsorted(cdf, u, right=True)
    below  = torch.max(torch.zeros_like(inds-1), inds-1)
    above  = torch.min((cdf.shape[-1]-1) * torch.ones_like(inds),
inds)
    inds_g = torch.stack([below, above], -1)  # (ray #, sample #,
2)

    #cdf_g  = tf.gather(cdf , inds_g, axis=-1,
batch_dims=len(inds_g.shape)-2)
    #bins_g = tf.gather(bins, inds_g, axis=-1,
batch_dims=len(inds_g.shape)-2)
    matched_shape = [inds_g.shape[0], inds_g.shape[1],
cdf.shape[-1]]
    cdf_g  = torch.gather( cdf.unsqueeze(1).expand(matched_shape),
2, inds_g)
    bins_g = torch.gather(bins.unsqueeze(1).expand(matched_shape),
2, inds_g)

    denom = cdf_g[..., 1] - cdf_g[..., 0]
    denom = torch.where(denom<1e-5, torch.ones_like(denom),
                                    denom)
    t = (u - cdf_g[..., 0]) / denom
    samples =  bins_g[..., 0] + \
              (bins_g[..., 1] - bins_g[..., 0]) * t
    return samples # (ray #, sample #), unsorted along each ray
```



Line 4 defines the probability $p_i := \frac{w_i}{\sum_{j=2}^{N_C-1} w_j}$ that a ray is stopped by a particle at depth $\frac{z_i+z_{i+1}}{2}$. This corresponds to the **area** under the histogram, shown above (first ray only).



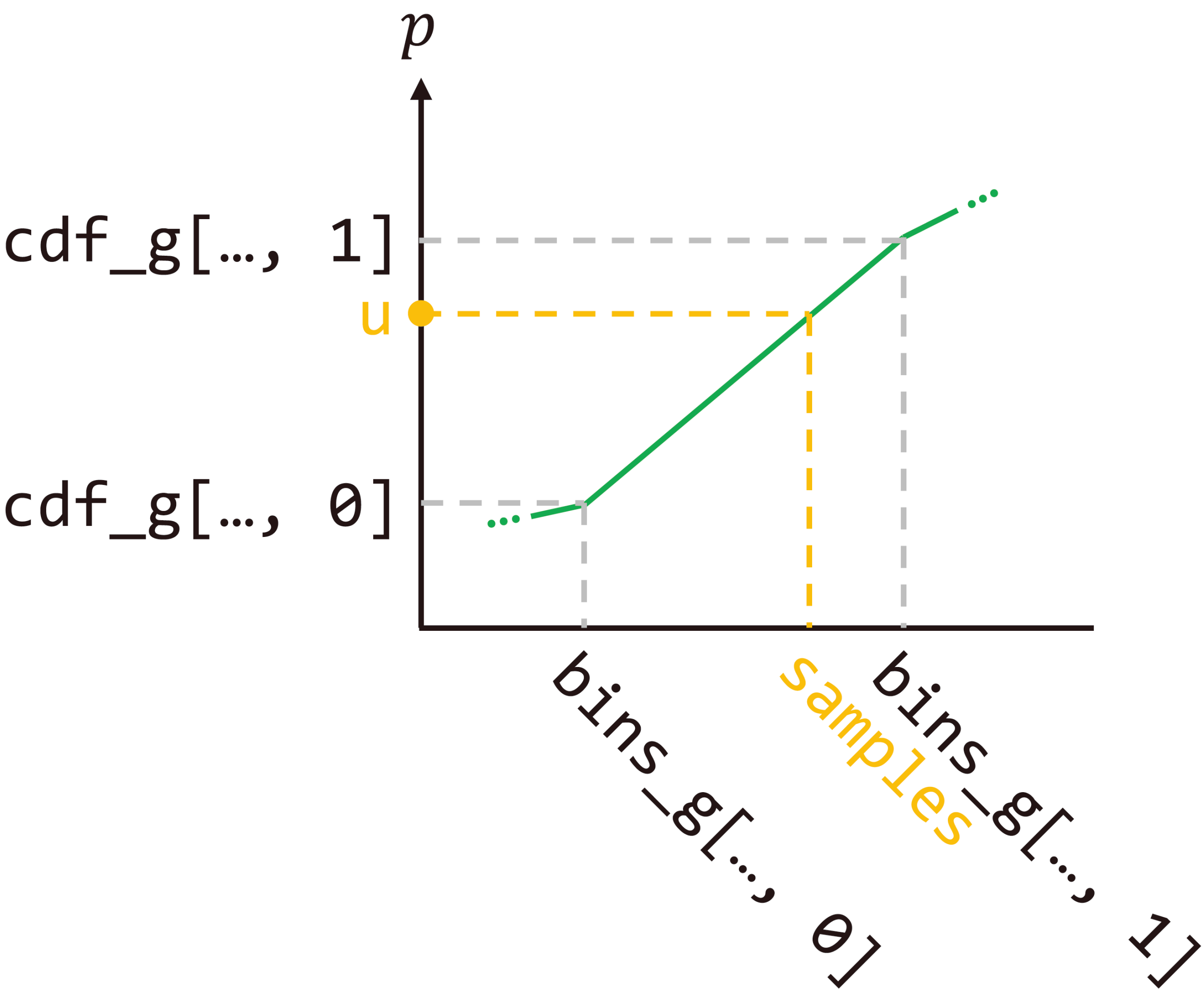
Lines 5 and 6 accumulate the area for the CDF $H(z)$.



What follows is key to *Monte Carlo sampling*. Line 12 generates a batch $\mathbf{u}_{B\times N_F}$ of random cumulative probabilities — "seeds". The above figure visualizes operations on `u[0, :]`, i.e., N_F seeds on the first ray. They fall into the "bins" at line 25 through comparison against the cumulative probabilities at the boundaries. [`torch.searchsorted(…)`](#) returns the positions (indices) `inds` $\in \mathbb{R}^{B\times N_F}$ of those random "seeds".

> **Quicker indicing**
>
> [`torch.Tensor.contiguous()`](#) returns a tensor with identical data but contiguous in memory. It is called before `torch.searchsorted(…)` for performance concern.

Lower bounds of the "bins" are collected as `below` at line 26, and upper bounds are gathered as `above` at line 27. `inds_g` $\in \mathbb{R}^{B\times N_F\times 2}$ combines `below` and `above` at line 28. [`torch.gather(…)`](#) at lines 33 and 34 determine how N_F points along each ray are distributed according to indices `inds_g`, or effectively the number of "seeds" in each "bin".



Finally, fine samples are found through *similarity*, whose concept is illustrated above. Indices `0` correspond to `below`, and indices `1` attach to `above`. Given cumulative probabilities `u`, there holds

$$\frac{\texttt{u} - \texttt{cdf[..., 0]}}{\texttt{cdf[..., 1]} - \texttt{cdf[..., 0]}} = \frac{\texttt{z\_samples} - \texttt{bins\_g[..., 0]}}{\texttt{bins\_g[..., 1]} - \texttt{bins\_g[..., 0]}}$$

Line 36 defines $\texttt{denom} := \texttt{cdf[..., 1]} - \texttt{cdf[..., 0]}$, and line 39 further denotes $\mathbf{t}_{B\times N_F} := \frac{\texttt{u}-\texttt{cdf[..., 0]}}{\texttt{denom}}$, then

$$\mathbf{t}_{B\times N_F} = \frac{\texttt{z\_samples} - \texttt{bins\_g[..., 0]}}{\texttt{bins\_g[..., 1]} - \texttt{bins\_g[..., 0]}}$$

Hence, lines 40 and 41 determine the unsorted output. Fine samples are expressed in offset from (midpoints of) coarse samples `bins_g[ … , 0]`.

$$\texttt{z\_samples} = \texttt{bins\_g[..., 0]} + \mathbf{t} * (\texttt{bins\_g[..., 1]} - \texttt{bins\_g[..., 0]}$$

Optimization

```python
…
for i in trange(start, N_iters):
    …
    optimizer.zero_grad()
    img_loss = img2mse(rgb, target_s)
    trans = extras['raw'][ … ,-1]
    loss = img_loss
    psnr = mse2psnr(img_loss)

    if 'rgb0' in extras:
        img_loss0 = img2mse(extras['rgb0'], target_s)
        loss = loss + img_loss0
        psnr0 = mse2psnr(img_loss0)

    loss.backward()
    optimizer.step()

    # NOTE: IMPORTANT!
    ###   update learning rate   ###
    decay_rate = 0.1
```

```
21          decay_steps = args.lrate_decay * 1000
22          new_lrate = args.lrate * (decay_rate ** (global_step /
23  decay_steps))
24          for param_group in optimizer.param_groups:
25              param_group['lr'] = new_lrate
26          ################################
            …
```

Radiance `rgb` $\in \mathbb{R}^{B\times 3}$ is compared against the ground truth `target_s` to obtain the MSE loss at line 5. The total loss also includes that of the coarse network (line 12). The coarse and fine network are jointly optimized at lines 15 and 16. Eventually, learning rate decays from line 20 to 24.

Summary

This post derives the volmue rendering integral and its numrical quadrature. Also explained is its connection with classical alpha compositing. The second part elaborates on the implementation of the rendering pipeline. Illustrations are included to assist understanding procedures such as rays generation and Monte Carlo sampling. Most importantly, the article clearly specifies the **physical meaning** of each variable and provides the **mathematical operation** for each statement. To sum, the blog functions as a **complete guide** for in-depth comprehension of NeRF.

References

Chapter 2.1 in Computer Vision: Algorithms and Applications
Foundamentals of Computer Graphics
NeRF: Representing Scenes as Neural Radiance Fields for View Synthesis
NeRF PyTorch implementation by Yen-Chen Lin
Neural Radiance Field's Volume Rendering 公式分析
Optical Models for Direct Volume Rendering
Part 1 of the SIGGRAPH 2021 course on Advances in Neural Rendering
深度解读yenchenlin/nerf-pytorch项目

Please use the following BibTeX to cite this post:

```tex
@misc{yyu2022nerfrendering,
    author = {Yu, Yue},
    title  = {NeRF: A Volume Rendering Perspective},
    year   = {2022},
    howpublished =
{\url{https://yconquesty.github.io/blog/ml/nerf/nerf_rendering.html}}
}
```

Appendix

Content on the way. Stay tuned!

Errata

Time	Modification
Aug 31 2022	Initial release
Nov 24 2022	Rectify reference list
Dec 2 2022	Add BibTeX for citation
Apr 20 2023	Elaborate on static compute graph