75/A+ / 194/ A+

MATH 129A (8) **Midterm 2** **Name:** Zayd Hammoudeh

April 27, 2017 **75 pts.**

Show all your work!

1. 8 pts. Find the inverse of the following matrix if it exists:

$$\begin{bmatrix} 1 & -2 & 1 \\ -3 & 7 & -6 \\ 2 & -3 & 0 \end{bmatrix}$$

$$A^{-1} = \boxed{\begin{bmatrix} -18 & -3 & 5 \\ -12 & -2 & 3 \\ -5 & -1 & 1 \end{bmatrix}}$$

$[A \; I_3]$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ -3 & 7 & -6 & 0 & 1 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{\substack{R_2' = R_2 + 3R_1 \\ R_3' = R_3 - 2R_1}}{\sim} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 3 & 1 & 0 \\ 0 & 1 & -2 & -2 & 0 & 1 \end{array}\right]$$

$$\underset{R_3' = R_3 - R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 3 & 1 & 0 \\ 0 & 0 & 1 & -5 & -1 & 1 \end{array}\right]$$

$$\underset{\substack{R_2' = R_2 + 3R_3 \\ R_1' = R_1 - R_3}}{\sim} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 6 & 1 & -1 \\ 0 & 1 & 0 & -12 & -2 & 3 \\ 0 & 0 & 1 & -5 & -1 & 1 \end{array}\right]$$

$$\underset{R_1' = R_1 + 2R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -18 & -3 & 5 \\ 0 & 1 & 0 & -12 & -2 & 3 \\ 0 & 0 & 1 & -5 & -1 & 1 \end{array}\right]$$

See verification on loose sheet.

8 pts

2. 8 pts. Determine which of the matrices are invertible. Use as few calculations as possible. Justify your answers.

Verified on loose leaf with determinant as well.

(a) $\begin{bmatrix} 3 & -5 \\ 4 & 8 \end{bmatrix}$ These two vectors are linearly independent since they are not scalar multiples. By Invertible Matrix theorem, this square 2x2 matrix is $\boxed{\text{invertible}}$.

Also verified on loose leaf via determinant

(b) $\begin{array}{ccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ \end{array}$ $\begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{bmatrix}$

Only linearly dependent if:

$$c_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} -5 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 0 \end{bmatrix}$$

Check if row 1 holds

Columns 1 and 2 not linearly dependent since not scalar multiples. If column 3 is not a linear combination of columns 1 & 2, it is also linearly independent making it invertible.

only possible if $c_2 = \frac{4}{3}$ due to row 2. Hence

$$-3c_1 + \left(\tfrac{4}{3}\right)6 = 0$$

$$-3c_1 + 8 = 0$$

$$c_1 = \frac{8}{3}$$

$$\frac{8}{3}(1) + \left(\frac{4}{3}\right)(-5) = -4$$

$$\frac{8}{3} - \frac{20}{3} = -4$$

$$-4 = -4$$

8 pts

$\boxed{\text{Not invertible as } \vec{a}_3 \text{ is linearly dependent.}}$

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3. 7 pts. Explain why the columns of A^2 span $\mathbb{R}^n$ whenever the columns of A are linearly independent. [*Hint:* $A^2 = AA$].

This statement is only true if A is square (not given in the problem).

1) If the columns of A are linearly independent and A is square, then A is invertible by IMT.

2) If A is invertible, then A^2 is the product of two invertible matrices (A and itself) making A^2 also invertible.

3) If A^2 is invertible, then by IMT its columns span $\mathbb{R}^n$ (assuming A is $n \times n$).

7pts

4. 8 pts. Combine the methods of row reduction and cofactor expansion to compute the following determinant:

$$\begin{vmatrix} -2 & 1 & 4 & -1 \\ 1 & 0 & -1 & 2 \\ 5 & -1 & 2 & 1 \\ 0 & 0 & 3 & -1 \end{vmatrix}$$

8pts

$$\begin{bmatrix} -2 & 1 & 4 & -1 \\ 1 & 0 & -1 & 2 \\ 5 & -1 & 2 & 1 \\ 0 & 0 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 4 & -1 \\ -3 & 2 & 7 & 0 \\ 3 & 0 & 6 & 0 \\ 2 & -1 & -1 & 0 \end{bmatrix}$$

$R_2' = R_2 + 2R_1$

$R_3' = R_3 + R_1$

$R_4' = R_4 - R_1$

Cofactor expansion down the fourth column

$$\Delta = (-1)(-1)^5 \begin{vmatrix} -3 & 2 & 7 \\ 3 & 0 & 6 \\ 2 & -1 & -1 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} -3 & 2 & 7 \\ 3 & 0 & 6 \\ 2 & -1 & -1 \end{vmatrix}$$

Cofactor expansion across row 2

$$\Delta = (3)(-1)^{2+1} \begin{vmatrix} 2 & 7 \\ -1 & -1 \end{vmatrix} + 0 + (6)(-1)^{(2+3)} \begin{vmatrix} -3 & 2 \\ 2 & -1 \end{vmatrix}$$

$$\Delta = (-3)(-2+7) + (-6)(3-4)$$

$$\Delta = (-15) + 6$$

$$\boxed{\Delta = -9}$$

I calculated (-17) on my loose leaf. I have an error in one of them. Just grade this above derivation.

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5. 10 pts. Let W be the set of all vectors of the form shown, where $a, b,$ and c represent arbitrary real numbers. In each case, either find a set S of vectors that spans W or give an example to show that W is not a vector space.

(a) $\begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix}$ $\boxed{\text{Not a vector space as does not contain zero vector.}}$

$-a+1=0 \Rightarrow a=1$

$a-6b=0 \wedge a=1 \Rightarrow b=\frac{1}{6}$

$a+2b \wedge a=1 \wedge b=\frac{1}{6} \Rightarrow 1+\frac{1}{3} \neq 0$

So no way to get the zero vector.

(b) $\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix}$

$$\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ -a \\ 0 \end{bmatrix} + \begin{bmatrix} -b \\ b \\ 0 \\ b \end{bmatrix} + \begin{bmatrix} 0 \\ -c \\ c \\ 0 \end{bmatrix}$$

$$= a\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$W = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right\}$$ 10 pts

6. 10 pts. Find an explicit description of Nul A by listing vectors that span the null space.

$$A = \begin{bmatrix} 1 & -6 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$A = \begin{bmatrix} 1 & -6 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 16 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$

$R_1' = R_1 + 6R_2$

For the equation $A\vec{x} = \vec{0}$

$\vec{x} = \begin{cases} x_1 = -16x_3 \\ x_2 = -2x_3 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \end{cases}$

$$\text{Nul} A = \vec{x} = x_3\begin{bmatrix} -16 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad x_3, x_4 \in \mathbb{R}$$

$$\text{Nul} A = \text{span}\left\{\begin{bmatrix} -16 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

10 pts

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7. 12 pts. Find bases for Nul A and Col A.
Also, find the dimensions of Nul A and Col A.

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 & -1 \\ 2 & -4 & 7 & -3 & 3 \\ 3 & -6 & 8 & 3 & -8 \end{bmatrix}$$

$A = \begin{bmatrix} 1 & -2 & 3 & 0 & -1 \\ 2 & -4 & 7 & -3 & 3 \\ 3 & -6 & 8 & 3 & -8 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -2 & 3 & 0 & -1 \\ 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & -1 & 3 & -5 \end{bmatrix}$ $R_2' = R_2 - 2R_1$, $R_3' = R_3 - 3R_1$

$\sim \begin{bmatrix} \boxed{1} & -2 & 3 & 0 & -1 \\ 0 & 0 & \boxed{1} & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $R_3' = R_3 + R_2$

Two pivots in First and third column

$\text{Col} A = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix} \right\}$

$\boxed{\dim \text{Col} A = 2}$

$\boxed{\text{Basis col} A = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix} \right\}}$

$\boxed{\dim \text{Nul} A = 3}$ (as three free variables)

$\begin{bmatrix} 1 & -2 & 3 & 0 & -1 \\ 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -2 & 0 & 9 & -16 \\ 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $R_1' = R_1 - 3R_2$

$A\vec{x} = \vec{0}$ then

$\vec{x} = \begin{cases} x_1 = 2x_2 - 9x_4 + 16x_5 \\ x_2 \text{ is free} \\ x_3 = 3x_4 - 5x_5 \\ x_4 \text{ is free} \\ x_5 \text{ is free} \end{cases}$

$\vec{x} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -9 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 16 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$

Basis of Nul A: $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 16 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$

12 pts

8. 12 pts. Use coordinate vectors to test the linear independence of the set of polynomials. Explain your work.

$$1 + 3t^3, \qquad 2 + 4t - t^2, \qquad -t + 2t^2 - t^3$$

Standard basis for $P_3 = \{1, t, t^2, t^3\}$

Using this basis thes vectors can be mapped into $\mathbb{R}^4$. They would be:

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ -1 \end{bmatrix}$

To check for linear independence in $\mathbb{R}^4$ row reduce!

$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & -1 \\ 0 & -1 & 2 \\ 3 & 0 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & -1 \\ 0 & -1 & 2 \\ 0 & -6 & -1 \end{bmatrix}$ $R_4' = R_4 - 3R_1$

$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 2 \\ 0 & 4 & -1 \\ 0 & -6 & -1 \end{bmatrix}$ Swap R_2 and R_3

$\sim \begin{bmatrix} \boxed{1} & 2 & 0 \\ 0 & \boxed{-1} & 2 \\ 0 & 0 & \boxed{7} \\ 0 & 0 & -13 \end{bmatrix}$ $R_4' = R_4 - 6R_2$, $R_3' = R_3 + 4R_2$

This will have three pivots (i.e. one in each column). Hence This makes them $\boxed{\text{linearly independent}}$ in $\mathbb{R}^4$. By $\boxed{\text{isomorphism,}}$ they are also linearly independent in P_3.

12 pts

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1)

$$\begin{bmatrix} 1 & -2 & 1 \\ -3 & 7 & -6 \\ 2 & -3 & 0 \end{bmatrix}\begin{bmatrix} -18 & -3 & 5 \\ -12 & -2 & 3 \\ -5 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -18+24-5 & -3+4-1 & 5-6+1 \\ 54\ 84\ +30 & 9-14+6 & -15+21-6 \\ -36+36+0 & -6+6\ 0 & 10-9+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Verified.

2a) $\begin{bmatrix} 3 & -5 \\ 4 & 8 \end{bmatrix}$

$\Delta = (3\cdot 8 - (4\cdot -5))$

$\Delta = 24+20 = 44$

$\Delta \neq 0$ so invertible.

2b) $\begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{bmatrix}$

$\Delta = 0$

So not invertible

$\underset{R_3' = R_3 + 3R_1}{\sim} \begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ 0 & -9 & -12 \end{bmatrix}$

$\Delta = (1)(-1)^{1+1}\begin{vmatrix} 3 & 4 \\ -9 & -12 \end{vmatrix}$

$\Delta = (3)(-12) - (-9)(4)$

$\Delta\ -36 - (-36) = 0$

4)

$$\begin{bmatrix} -2 & 1 & 4 & -1 \\ 1 & 0 & -1 & 2 \\ 5 & -1 & 2 & 1 \\ 0 & 0 & 3 & -1 \end{bmatrix}$$

Swap R_1 and R_2

$$\Delta = (-1)\begin{vmatrix} 1 & 0 & -1 & 2 \\ -2 & 1 & 4 & -1 \\ 5 & -1 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{vmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ -2 & 1 & 4 & -1 \\ 5 & -1 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -3 & -9 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -6 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -6 \\ 0 & 0 & 0 & -17 \end{bmatrix}$$

$\det = (-1)(1)(-1)(-17)$

$\det = -17$

This conflicts with my test so one has a math error,