# Chapter 4-3 - Linearly Independent Sets and Bases

1) Determine if the vectors are linearly independent and if they span $\mathbb{R}^3$ or both.

$$\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Both - They span $\mathbb{R}^3$ and are linearly independent. It is a basis

3) Determine if he vectors are linearly ndependent, span $\mathbb{R}^3$ r both

$$\begin{bmatrix}1\\0\\-2\end{bmatrix}\begin{bmatrix}3\\2\\-4\end{bmatrix}\begin{bmatrix}-3\\-5\\1\end{bmatrix}$$

$$\begin{bmatrix}1&3&-3\\0&2&-5\\-2&-4&1\end{bmatrix} \sim \begin{bmatrix}1&3&-3\\0&2&-5\\0&2&-5\end{bmatrix} \sim \begin{bmatrix}1&3&-3\\0&2&-5\\0&0&0\end{bmatrix}$$

Neither does not have 3 pivots and only spans a plane in $\mathbb{R}^3$

5) Determine if the vectors are linearly independent, span $\mathbb{R}^3$, or both.

$$\begin{bmatrix}1\\-3\\0\end{bmatrix}, \begin{bmatrix}-2\\9\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-3\\5\end{bmatrix}$$

Linearly dependent since it contains $\vec{0}$

$$\begin{bmatrix}1&-2&0\\-3&9&-3\\0&0&5\end{bmatrix} \sim \begin{bmatrix}1&2&0\\0&3&-3\\0&0&5\end{bmatrix}$$

Spans $\mathbb{R}^3$ since three pivots

7) Determine if the set of vectors are linearly independent span $\mathbb{R}^3$, neither, or both?

$$\begin{bmatrix}-2\\3\\0\end{bmatrix} \begin{bmatrix}6\\-1\\5\end{bmatrix}$$

Linearly independent since not scalar multiples

Does not span $\mathbb{R}^3$ as it only has two vectors.

9) Find a bases for the null space of $A$ where

$$A = \begin{bmatrix}1&0&-3&2\\0&1&-5&4\\3&-2&1&-2\end{bmatrix}$$

$R_3' = R_3 - 3R_1$

$$A \sim \begin{bmatrix}1&0&-3&2\\0&1&-5&4\\0&-2&10&-8\end{bmatrix}$$

$R_3' = R_3 + 2R_2$

$$A \sim \begin{bmatrix}1&0&-3&2\\0&1&-5&4\\0&0&0&0\end{bmatrix}$$

$$\vec{x}: \begin{cases} x_1 = +3x_3 - 2x_4 \\ x_2 = 5x_3 - 4x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free}\end{cases}$$

$$\vec{x} = x_3\begin{bmatrix}3\\5\\1\\0\end{bmatrix} + x_4\begin{bmatrix}-2\\-4\\0\\1\end{bmatrix}$$

Basis of Nul A

$$\left\{\begin{bmatrix}3\\5\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\-4\\0\\1\end{bmatrix}\right\}$$

11) Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane:

$x + 2y + z = 0$

$$\vec{u} = \begin{cases} x = -2y - z \\ y \text{ is free} \\ z \text{ is free}\end{cases}$$

$$\vec{u} = y\begin{bmatrix}-2\\1\\0\end{bmatrix} + z\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

Basis of the plane

$$\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$$

13) Assume $A$ is row equivalent to $B$. Find Nul A and Col A

$$A = \begin{bmatrix}-2&4&-2&-4\\2&-6&-3&1\\-3&8&2&-3\end{bmatrix}, \quad B = \begin{bmatrix}1&0&6&5\\0&2&5&3\\0&0&0&0\end{bmatrix}$$

Col A is vectors with pivots in echelon form

$$\text{Col}A = \text{span}\left\{\begin{bmatrix}-2\\2\\-3\end{bmatrix}, \begin{bmatrix}4\\-6\\8\end{bmatrix}\right\}$$

$$\text{Basis of Col}A = \left\{\begin{bmatrix}-2\\2\\-3\end{bmatrix}, \begin{bmatrix}4\\-6\\8\end{bmatrix}\right\}$$

Nul A comes from echelon matrix

$$\vec{x} = \begin{cases} x_1 = -6x_3 - 5x_4 \\ x_2 = -\frac{5}{2}x_3 - \frac{3}{2}x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free}\end{cases}$$

$$\vec{x} = x_3\begin{bmatrix}-6\\-\frac{5}{2}\\1\\0\end{bmatrix} + x_4\begin{bmatrix}-5\\-\frac{3}{2}\\0\\1\end{bmatrix}$$

$$\text{Nul}A = \text{span}\left\{\begin{bmatrix}-6\\-\frac{5}{2}\\1\\0\end{bmatrix}, \begin{bmatrix}-5\\-\frac{3}{2}\\0\\1\end{bmatrix}\right\}$$

$$\text{Basis of Nul}A = \left\{\begin{bmatrix}-6\\-\frac{5}{2}\\1\\0\end{bmatrix}, \begin{bmatrix}-5\\-\frac{3}{2}\\0\\1\end{bmatrix}\right\}$$

# Chapter 4-3 - Linearly Independent Sets and Bases

15) Find a basis for the space spanned by the vectors:

$$\begin{bmatrix}1\\0\\-3\\2\end{bmatrix}\begin{bmatrix}0\\1\\2\\-3\end{bmatrix}\begin{bmatrix}-3\\-4\\1\\6\end{bmatrix}\begin{bmatrix}1\\-3\\-8\\7\end{bmatrix}\begin{bmatrix}2\\1\\-6\\9\end{bmatrix}$$

$$\begin{bmatrix}1&0&-3&1&2\\0&1&-4&-3&1\\-3&2&1&-8&-6\\2&-3&6&7&9\end{bmatrix}$$

$R_3' = R_3 + 3R_1$, $R_4' = R_4 - 2R_1$

$$\sim\begin{bmatrix}1&0&-3&1&2\\0&1&-4&-3&1\\0&2&-8&-5&0\\0&-3&12&5&5\end{bmatrix}$$

$R_3' = R_3 - 2R_1$
$R_4' = R_4 + 3R_1$

$$\sim\begin{bmatrix}1&0&-3&1&2\\0&1&-4&-3&1\\0&0&0&1&-2\\0&0&0&-4&8\end{bmatrix}$$

$R_4' = R_4 + 4R_3$

$$\sim\begin{bmatrix}1&0&-3&1&2\\0&1&-4&-3&1\\0&0&0&1&-2\\0&0&0&0&0\end{bmatrix}$$

3 pivots

$$\text{Basis} = \left\{\begin{bmatrix}1\\0\\-3\\2\end{bmatrix}, \begin{bmatrix}0\\1\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\-3\\-8\\7\end{bmatrix}\right\}$$

$$= \{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$$

19) Let $\vec{v}_1 = \begin{bmatrix}4\\-3\\7\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}1\\9\\-2\end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix}7\\11\\6\end{bmatrix}$ with $H = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

It can be verified $4\vec{v}_1 + 5\vec{v}_2 - 3\vec{v}_3 = \vec{0}$

Find a basis for $H$.

Set is linearly dependent since the linear dependence relation is given

$\vec{v}_1$ and $\vec{v}_2$ are linearly independent since not scalar multiples

Basis $= \{\vec{v}_1, \vec{v}_2\}$

21)

a) False - A single vector is linearly dependent only if it is the zero vector.

b) False - If $H = \text{span}\{\vec{b}_1, \ldots, \vec{b}_p\}$ it is not guaranteed $\{\vec{b}_1, \ldots, \vec{b}\}$ is linearly independent.

c) True - Since by Invertible matrix theorem the columns are linearly independent and have a pivot in every row.

d) False - A basis is a spanning set that is as small as possible.

c) False - Linear dependence relation is unchanged.

22)

a) False - Being linearly independent does not guarentee it spans $H$.

b) True - Vectors can be removed until it is linearly independent.

c) True - Basis is a linearly independent set that is as large as possible.

d) False - The approach always yields the basis of the null space of $A$ (Nul$A$).

e) False - The pivot columns of $A$ form the basis of Col$A$.

# Chapter 4-3 - Linearly Independent Sets and Bases

23) Suppose $\mathbb{R}^4 = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_4\}$

Explain why $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for $\mathbb{R}^4$.

If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ spans $\mathbb{R}^4$ then the matrix

$[\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4]$ must have a pivot in every row.

Since the matrix is square, it also has a pivot in every column, making them linearly independent.

If a set of vectors are linearly independent and span a vector space (e.g. $\mathbb{R}^4$) then it is a basis.

25) Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Let $H$ be the set of vectors whose second and third entries are equal.

Every vector in $H$ has a unique expression in terms of $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ such that

$$\begin{bmatrix} s \\ \cdot \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (t-s)\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a basis for $H$?

$$\begin{bmatrix} s \\ t \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (t-s)\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= s\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) + t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Not a basis for $H$

Basis contains two vectors maximum

$$\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$$