MATH 129A (8) **Midterm 1** **Name:**

March 9, 2017 **75 pts.**

Show all your work!

1. 7 pts. Determine h and k such that the solution set of the system (i) is empty, (ii) contains a unique solution, and (iii) contains infinitely many solutions.

$$x_1 + hx_2 = 2$$
$$4x_1 + 8x_2 = k$$

Reduce augmented matrix to echelon form:

$$\begin{bmatrix} 1 & h & 2 \\ 4 & 8 & k \end{bmatrix} \sim_{(-4)R_1+R_2} \begin{bmatrix} 1 & h & 2 \\ 0 & -4h+8 & -8+k \end{bmatrix}$$

(i) no solution / inconsistent system: if $-4h+8=0$ and $-8+k \neq 0$
$h=2$ and $k \neq 8$

(ii) unique solution (no free variables): if $-4h+8 \neq 0$ or $h \neq 2$

(iii) infinitely many solutions (one free variable):
if $-4h+8=0$ and $-8+k=0$ $\Rightarrow$ $h=2$ and $k=8$

2. 10 pts. (a) Find the general solution of the following linear system:
7pts

$$2x_1 + 6x_2 - 9x_3 - 4x_4 = 0$$
$$-3x_1 - 11x_2 + 9x_3 - x_4 = 0$$
$$x_1 + 4x_2 - 2x_3 + x_4 = 0$$

3 pts (b) Describe the solution of the given system in parametric vector form. Also, give a geometric description of the solution set.

Reduce augmented matrix to reduced echelon form:

$$\begin{bmatrix} 2 & 6 & -9 & -4 & 0 \\ -3 & -11 & 9 & -1 & 0 \\ 1 & 4 & -2 & 1 & 0 \end{bmatrix} \overset{R_1 \leftrightarrow R_3}{\sim} \begin{bmatrix} 1 & 4 & -2 & 1 & 0 \\ -3 & -11 & 9 & -1 & 0 \\ 2 & 6 & -9 & -4 & 0 \end{bmatrix} \sim \begin{array}{l} \\ 3R_1+R_2 \\ -2R_1+R_2 \end{array} \begin{bmatrix} \boxed{1} & 4 & -2 & 1 & 0 \\ 0 & \boxed{1} & 3 & 2 & 0 \\ 0 & -2 & -5 & -6 & 0 \end{bmatrix}$$

$$\sim_{2R_2+R_3} \begin{bmatrix} \boxed{1} & 4 & -2 & 1 & 0 \\ 0 & \boxed{1} & 3 & 2 & 0 \\ 0 & 0 & \boxed{1} & -2 & 0 \end{bmatrix} \sim \begin{array}{l} 2R_3+R_1 \\ (-3)R_3+R_2 \end{array} \begin{bmatrix} \boxed{1} & 4 & 0 & -3 & 0 \\ 0 & \boxed{1} & 0 & 8 & 0 \\ 0 & 0 & \boxed{1} & -2 & 0 \end{bmatrix} \sim$$

$$\sim_{-4R_2+R_1} \begin{array}{c} \begin{bmatrix} \boxed{1} & 0 & 0 & -35 & 0 \\ 0 & \boxed{1} & 0 & 8 & 0 \\ 0 & 0 & \boxed{1} & -2 & 0 \end{bmatrix} \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \end{array}$$

Basic variables: x_1, x_2, x_3
Free variable: x_4

$$x_1 - 35x_4 = 0$$
$$x_2 + 8x_4 = 0$$
$$x_3 - 2x_4 = 0$$

(a) General solution:

$$\begin{cases} x_1 = 35x_4 \\ x_2 = -8x_4 \\ x_3 = 2x_4 \\ x_4 \text{ is free} \end{cases}$$

(b) Parametric vector form:

$$\vec{y} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 35x_4 \\ -8x_4 \\ 2x_4 \\ x_4 \end{bmatrix} = x_4 \underbrace{\begin{bmatrix} 35 \\ -8 \\ 2 \\ 1 \end{bmatrix}}_{\vec{u}} = x_4 \vec{u}$$

$x_4 \in \mathbb{R}$

line through the origin and $\vec{u}$ in 4D space or
line spanned by $\vec{u}$ in 4D space

3. 8 pts. Let $B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$

6pts (a) Can every vector in $\mathbb{R}^4$ be written as a linear combination of the columns of B?

2pts (b) Do the columns of B span $\mathbb{R}^4$?

Reduce augmented matrix to echelon form:

$$\begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix} \underset{\substack{(-1)R_1+R_3 \\ 2R_1+R_4}}{\sim} \begin{bmatrix} \boxed{1} & 3 & -2 & 2 \\ 0 & \boxed{1} & 1 & -5 \\ 0 & -1 & -1 & 5 \\ 0 & -2 & -2 & 3 \end{bmatrix} \underset{\substack{R_2+R_3 \\ 2R_2+R_4}}{\sim} \begin{bmatrix} \boxed{1} & 3 & -2 & 2 \\ 0 & \boxed{1} & 1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 \end{bmatrix} \sim$$

$$\underset{R_3 \updownarrow R_4}{\sim} \begin{bmatrix} \boxed{1} & 3 & -2 & 2 \\ 0 & \boxed{1} & 1 & -5 \\ 0 & 0 & 0 & \boxed{-7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) $\boxed{\text{No}}$ every vector in $\mathbb{R}^4$ cannot be written as a linear combination of the columns of B because B does not have a pivot in each row.

(b) $\boxed{\text{No}}$, because B does not have a pivot in each row, not every vector in $\mathbb{R}^4$ can be written as a linear combination of the columns of B.

4. 5 pts. Prove that $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ for each scalar c and d and any vector $\mathbf{u}$ in $\mathbb{R}^n$.

Proof: Suppose $\vec{u}$ is any vector in $\mathbb{R}^n$ and c, d are any scalars. By definition of scalar multiplication, $(c+d)\vec{u} = (c+d)\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} =$

$$= \begin{bmatrix} (c+d)u_1 \\ (c+d)u_2 \\ \vdots \\ (c+d)u_n \end{bmatrix} \underset{\substack{\text{by distributive law} \\ \text{of real numbers}}}{=} \begin{bmatrix} cu_1 + du_1 \\ \vdots \\ cu_n + du_n \end{bmatrix} \underset{\text{by vector addition}}{=} \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix} + \begin{bmatrix} du_1 \\ \vdots \\ du_n \end{bmatrix} = c\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + d\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = c\vec{u} + d\vec{u}$$

Q.E.D

5. 6 pts. Describe and compare the solution sets of $x_1 + 4x_2 - 3x_3 = 0$ and $x_1 + 4x_2 - 3x_3 = 6$.

General solution (homogeneous system):

$$\begin{cases} x_1 = -4x_2 + 3x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$$

In vector form: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4x_2 + 3x_3 \\ x_2 \\ x_3 \end{bmatrix} =$

$$= \begin{bmatrix} -4x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 3x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}}_{\vec{u}} + x_3 \underbrace{\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}} = x_2\vec{u} + x_3\vec{v}$$

— the plane through the origin spanned by $\vec{u}$ and $\vec{v}$.

General solution (nonhomogeneous):

$$\begin{cases} x_1 = 6 - 4x_2 + 3x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$$

In vector form

$$\vec{x} = \begin{bmatrix} 6 - 4x_2 + 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}}_{\vec{p}} + x_2 \underbrace{\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}}_{\vec{u}} + x_3 \underbrace{\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}} = \vec{p} + x_2\vec{u} + x_3\vec{v}$$

— plane through $\vec{p}$ parallel to the solution set of the homogeneous system (or parallel to the plane spanned by $\vec{u}$ and $\vec{v}$)

6. 6 pts. Given $A = \begin{bmatrix} -2 & -6 \\ 7 & 21 \\ -3 & -9 \end{bmatrix}$, find one nontrivial solution of $A\mathbf{x} = \mathbf{0}$ by inspection.

[*Hint*: Think of the equation $A\mathbf{x} = \mathbf{0}$ written as a vector equation.]

Find $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $\begin{bmatrix} -2 & -6 \\ 7 & 21 \\ -3 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ or,

equivalently, $x_1\begin{bmatrix} -2 \\ 7 \\ -3 \end{bmatrix} + x_2\begin{bmatrix} -6 \\ 21 \\ -9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ One possible solution: $\boxed{x_1 = -3 \text{ and } x_2 = 1}$

Another possible solution: $x_1 = 3$ and $x_2 = -1$

7. 8 pts. Determine by inspection whether the vectors are linearly independent. Justify your answer:

(a) $\begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 6 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ linearly dependent, because there are more vectors than components (entries) in each vector

(b) $\begin{bmatrix} 1 \\ -8 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 1 \\ 12 \end{bmatrix}$ linearly dependent, because the set contains the zero vector $\vec{0}$

(c) $\underset{\vec{u}}{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}}, \underset{\vec{v}}{\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}}, \underset{\vec{w}}{\begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}}$ Notice, that $\vec{w} = \vec{u} + \vec{v}$ - linearly dependent because $\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$

(d) $\underset{\vec{u}}{\begin{bmatrix} -2 \\ 4 \\ 6 \\ 10 \end{bmatrix}}, \underset{\vec{v}}{\begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix}}$

Check: $\vec{u} = a\vec{v}$?

$-2 = 3a$ $\quad a = -2/3$ works for these, but not for the last one $\Rightarrow$

$4 = -6a$

$6 = -9a$

$10 = 15a$ $\quad$ there is no such a !

linearly independent because they are not scalar multiples.

8. 7 pts. Find the value(s) of h for which the vectors are linearly dependent. Justify your answer:

$$\begin{bmatrix}2\\-4\\1\end{bmatrix}, \begin{bmatrix}-6\\7\\-3\end{bmatrix}, \begin{bmatrix}8\\h\\4\end{bmatrix}$$

Solve $A\vec{x}=\vec{0}$ and see for what values of h the solution is nontrivial

at least one free variable

$$\begin{bmatrix}2 & -6 & 8 & 0\\ -4 & 7 & h & 0\\ 1 & -3 & 4 & 0\end{bmatrix} \sim \begin{matrix}2R_1+R_2\\ -\tfrac{1}{2}R_1+R_3\end{matrix} \begin{bmatrix}\boxed{2} & -6 & 8 & 0\\ 0 & \boxed{-5} & 16+h & 0\\ 0 & 0 & \boxed{0} & 0\end{bmatrix}$$

basic variables x_1 and x_2 x_3 is free (always no matter what h is)

at least one free variable $\Rightarrow$ nontrivial solution for any h $\Rightarrow$ the set is linearly dependent

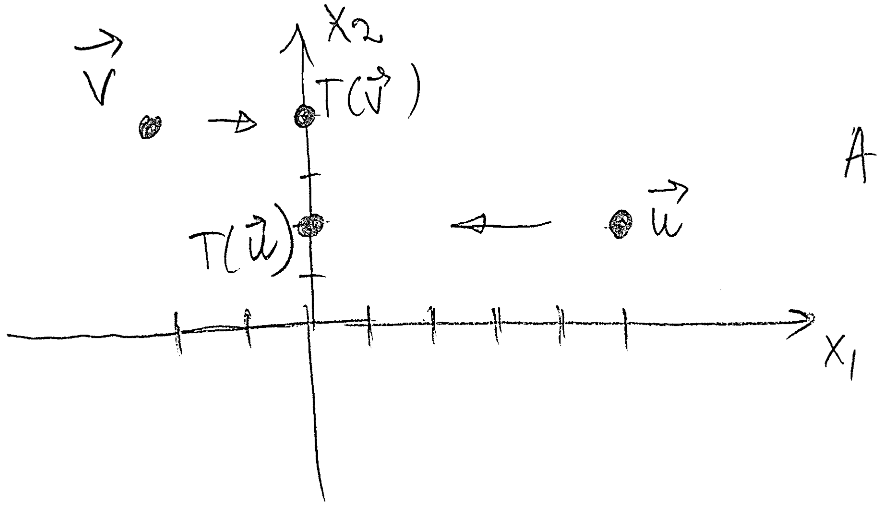
9. 6 pts. Use a rectangular coordinate system to plot $\mathbf{u} = \begin{bmatrix}5\\2\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}-2\\4\end{bmatrix}$ and their images under the given transformation T. Describe geometrically what T does to each vector $\mathbf{x}$ in $\mathbb{R}^2$.

$$T(\mathbf{x}) = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\x_2\end{bmatrix}$$

$$T(\vec{u}) = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}5\\2\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}$$

$$T(\vec{v}) = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-2\\4\end{bmatrix} = \begin{bmatrix}0\\4\end{bmatrix}$$



A projection onto the x_2 axis

10. 6 pts. Suppose that a linear transformation T satisfies $T(\mathbf{u}_1) = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$ and $T(\mathbf{u}_2) = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$.

Find $T(3\mathbf{u}_1 - 2\mathbf{u}_2)$.

$$T(3\vec{u}_1 - 2\vec{u}_2) = 3\,T(\vec{u}_1) - 2\,T(\vec{u}_2) =$$

$$= 3\begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ -3 \\ -6 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ -14 \end{bmatrix}$$

11. 6 pts. Compute the product AB in two ways: (1) by the definition, where $A\mathbf{b}_1$, $A\mathbf{b}_2$, and $A\mathbf{b}_3$ are computed separately, and (2) by the row-column rule for computing AB.

$$\vec{b}_1 \quad \vec{b}_2 \quad \vec{b}_3$$

$$A = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 0 & 2 \\ 4 & -3 & 1 \end{bmatrix}$$

(1) By definition:

$$A\vec{b}_1 = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad A\vec{b}_2 = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

$$A\vec{b}_3 = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$

$$AB = [A\vec{b}_1 \; A\vec{b}_2 \; A\vec{b}_3] = \begin{bmatrix} 1 & -3 & 7 \\ 2 & 0 & -4 \end{bmatrix}$$

(2) Row-Column rule:

$$\begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 2 \\ 4 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 7 \\ 2 & 0 & -4 \end{bmatrix}$$

AB_{11} = row 1 $\circ$ Col 1 $= [3 \quad 1]\begin{bmatrix} -1 \\ 4 \end{bmatrix} = -3 + 4 = 1$

AB_{12} = row 1 $\circ$ col 2 $= [3 \quad 1]\begin{bmatrix} 0 \\ -3 \end{bmatrix} = -3$ and so on...