Chapter 4-1 – Vector Spaces and Subspaces

1) Let V be the first quadrant in xy-plane that is

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}$$

a) If $\vec{u}$ and $\vec{v}$ are in V, is $\vec{u} + \vec{v} \in V$? Why?

Yes, it is not possible to add two positive numbers and get anything other than a positive number.

b) Find a specific $\vec{u} \in V$ and a specific scalar c such that $c\vec{u} \notin V$

$c = -1$, $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$cu = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \notin V$

3) Let $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \leq 1 \right\}$ Find a specific example to show that H is not a subspace.

$\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $c = 10$

$c\vec{u} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$, $10^2 + 0^2 = 100 \geq 1$

5) Determine if p is a subspace

$p(t) = at^2$

$p_1(t) = at^2$, $p_2(t) = bt^2$

$p_1(t) + p_2(t) = at^2 + bt^2$

$= (a+b)t^2$

$cp(t) = c(at)^2$

$= (ca)t^2$

7) All polynomials of degree at most 3 with integers as coefficients

No, while this holds for addition, it does not hold for scalar multiplication. Example: If $c = \pi$, then $c\,p(t)$ must have non-integer coefficients if the initial coefficients were non-zero.

9) Let H be the set of all vectors in the form $\begin{bmatrix} s \\ 3s \\ 2s \end{bmatrix}$. Find a vector in $\mathbb{R}^3$ such that $H = \text{span}\{\vec{v}\}$. Why does this show that H is a subspace of $\mathbb{R}^3$?

$\vec{V} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

$c\vec{V} = \begin{bmatrix} c \\ 3c \\ 2c \end{bmatrix}$ which is H's definition.

By theorem 4-1, H is a vector space.

11) Let W be the set of all vectors of the form $\begin{bmatrix} 5b+2c \\ b \\ c \end{bmatrix}$ where b and c are arbitrary. Find vectors $\vec{u}$ and $\vec{v}$ such that $W = \text{span}\{\vec{u}, \vec{v}\}$. Why does this show that W is a subspace of $\mathbb{R}^3$

$\vec{u} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$W = \text{span}\{\vec{u}, \vec{v}\}$

By theorem 4-1, W is a subspace

13) Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

a) Is $\vec{w}$ in $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$? How many vectors are in $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

No. Only three vectors are in there.

b) How many vectors are in Span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

Infinite

c) Is W in the subspace spanned by $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$? Why?

Yes $\vec{w}$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$

$\vec{V}_1 + \vec{V}_2 = \vec{w}$

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15) If W is all vectors of the form $\begin{bmatrix} 3a+b \\ 4 \\ a-5b \end{bmatrix}$, is it a subspace?

No, the zero vector is not in the set W.

17) If W is all vectors of the form $\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix}$, is S a subspace and what are its vectors

Yes, $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

and $v_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

21) Determine if the set H of all matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ are a subspace of $M_{2\times2}$.

Yes this satisfies the conditions as H contains the zero vector and is closed under addition and scalar multiplication.

23)

a) False - To be the zero vector, it must hold for all t, not just "some."

b) False - While an arrow in $\mathbb{R}^3$ is an example, there are vectors that do not fulfill this definition.

c) False, the zero vector is a necessary, not sufficient condition.

d) True - Every subspace is a vector space.

d) False - Digital signals are used.

24)

a) True - This is the definition of a vector space.

b) True - By definition of the negative vector.

c) True - A vector space is a subspace of itself.

d) False - $\mathbb{R}^2$ is not even a subset of $\mathbb{R}^3$

e) False - Conditions (ii) and (iii) must hold for all $\vec{u}, \vec{v}$, and c.

25) Complete the following proof that the zero vector is unique.

Suppose that $\vec{w} \in V$ and has property $\vec{u} + \vec{w} = \vec{w} + \vec{u} = \vec{u}$

In particular: $\vec{0} + \vec{w} = \vec{0}$

But, $\vec{0} + \vec{w} = \vec{w}$ by $\boxed{\text{Axiom (iv)}}$ - identity over vector addition

Hence, $\vec{w} = 0 + \vec{w} = \vec{0}$

27) Fill in the missing axioms for the proof that $0\vec{u} = \vec{0}$

$0\vec{u} = (0+0)\vec{u} = 0\vec{u} + 0\vec{u}$ by Axiom 8 (a)

Add the negative of $0\vec{u}$ to both sides,

$0\vec{u} + (-0\vec{u}) = [0\vec{u} + 0\vec{u}] + (-0\vec{u})$

$0\vec{u} + (-0\vec{u}) = 0\vec{u} + (0\vec{u} + (-0\vec{u}))$ By axiom 3 (b)

$\vec{0} = 0\vec{u} + \vec{0}$ By axiom 5 (c)

$0 = 0\vec{u}$ By axiom 4 (d)

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31) Let $\vec{u}$ and $\vec{v}$ be vectors in a vector space V that contains both $\vec{u}$ and $\vec{v}$. Explain why H also contains Span$\{\vec{u}, \vec{v}\}$

If H is a vector space containing $\vec{u}$ and $\vec{v}$, then by theorem 4-1, it must contain span$\{\vec{u}, \vec{v}\}$