Chapter 4-4 - Coordinate Systems

1) Find $\vec{x}$ where

$B=\left\{\begin{bmatrix}3\\-5\end{bmatrix},\begin{bmatrix}-4\\6\end{bmatrix}\right\}, [\vec{x}]_B=\begin{bmatrix}5\\3\end{bmatrix}$

$$\begin{bmatrix}3&-4\\-5&6\end{bmatrix}\begin{bmatrix}5\\3\end{bmatrix}=\begin{bmatrix}15-12\\-25+18\end{bmatrix}$$

$$\boxed{\vec{x}=\begin{bmatrix}3\\-7\end{bmatrix}}$$

3) Find $\vec{x}$ where

$B=\left\{\begin{bmatrix}1\\-4\\3\end{bmatrix},\begin{bmatrix}5\\2\\-2\end{bmatrix},\begin{bmatrix}4\\-7\\0\end{bmatrix}\right\}, [\vec{x}]_B=\begin{bmatrix}3\\0\\-1\end{bmatrix}$

$$\begin{bmatrix}1&5&4\\-4&2&-7\\3&-2&0\end{bmatrix}\begin{bmatrix}3\\0\\-1\end{bmatrix}=\begin{bmatrix}3+0-4\\-12+0+7\\9+0+0\end{bmatrix}$$

$$\boxed{\vec{x}=\begin{bmatrix}-1\\-5\\9\end{bmatrix}}$$

5) Find x_B where

$\vec{b}_1=\begin{bmatrix}1\\-3\end{bmatrix}, b_2=\begin{bmatrix}2\\-5\end{bmatrix}, \vec{x}=\begin{bmatrix}-2\\1\end{bmatrix}$

$P_B=\begin{bmatrix}1&2\\-3&-5\end{bmatrix}, P_B^{-1}=\begin{bmatrix}-5&-2\\3&1\end{bmatrix}$

$P_B[\vec{x}]_B=\vec{x}$

$[\vec{x}]_B=P_B^{-1}\vec{x}$

$$[\vec{x}]_B=\begin{bmatrix}-5&-2\\3&1\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix}=\boxed{\begin{bmatrix}8\\-5\end{bmatrix}}$$

7) Find $[\vec{x}]_B$ where

$\vec{b}_1=\begin{bmatrix}1\\-1\\-3\end{bmatrix}, \vec{b}_2=\begin{bmatrix}-3\\4\\9\end{bmatrix}, \vec{b}_3=\begin{bmatrix}2\\-2\\4\end{bmatrix}, \vec{x}=\begin{bmatrix}8\\-9\\6\end{bmatrix}$

$$\left[\begin{array}{ccc|ccc}1&-3&2&1&0&0\\-1&4&-2&0&1&0\\-3&9&4&0&0&1\end{array}\right]\sim$$

$$\underset{\substack{R_2'=R_2+R_1\\R_3'=R_3+3R_1}}{\sim}\left[\begin{array}{ccc|ccc}1&-3&2&1&0&0\\0&1&0&1&1&0\\0&0&10&3&0&1\end{array}\right]$$

$$\underset{R_1'=R_1+3R_2}{\sim}\left[\begin{array}{ccc|ccc}1&0&2&4&3&0\\0&1&0&1&1&0\\0&0&10&3&0&1\end{array}\right]$$

$$\underset{R_1'=R_1-\frac{1}{5}R_3}{\sim}\left[\begin{array}{ccc|ccc}1&0&0&\frac{17}{5}&3&-\frac{1}{5}\\0&1&0&1&1&0\\0&0&10&3&0&1\end{array}\right]$$

$$\underset{R_3'=\frac{1}{10}R_3}{\sim}\left[\begin{array}{ccc|ccc}1&0&0&\frac{17}{5}&3&-\frac{1}{5}\\0&1&0&1&1&0\\0&0&1&\frac{3}{10}&0&\frac{1}{10}\end{array}\right]$$

$$P_B^{-1}=\begin{bmatrix}\frac{17}{5}&3&-\frac{1}{5}\\1&1&0\\\frac{3}{10}&0&\frac{1}{10}\end{bmatrix}$$

$P_B^{-1}\vec{x}=[\vec{x}]_B$

$$\begin{bmatrix}\frac{136}{5}-27-\frac{6}{5}\\8-9+0\\\frac{24+6}{10}\end{bmatrix}=[\vec{x}]_B$$

$$\boxed{\begin{bmatrix}-1\\-1\\3\end{bmatrix}=\vec{x}}$$

9) Find the change-of-coordinates matrix from B to $\mathbb{R}^n$ where $B=\left\{\begin{bmatrix}2\\-9\end{bmatrix},\begin{bmatrix}1\\8\end{bmatrix}\right\}$

$$\boxed{P_B=\begin{bmatrix}2&1\\-9&8\end{bmatrix}}$$

11) Use an inverse matrix to find $[\vec{x}]_B$ where $B=\left\{\begin{bmatrix}3\\-5\end{bmatrix},\begin{bmatrix}-4\\6\end{bmatrix}\right\}, \vec{x}=\begin{bmatrix}2\\-6\end{bmatrix}$.

$\det P_B=(3)(6)-(-4)(-5)=18-20=-2$

$$P_B^{-1}=\begin{bmatrix}-3&-\frac{4}{2}\\-\frac{5}{2}&-\frac{3}{2}\end{bmatrix}$$

$$[\vec{x}]_B=P_B^{-1}\vec{x}=\begin{bmatrix}-6+12\\-5+9\end{bmatrix}=\boxed{\begin{bmatrix}6\\4\end{bmatrix}}$$

13) The set $B=\{1+t^2, t+t^2, 1+2t+t^2\}$ is a basis for P_2. Find the coordinate vector $p(t)=1+4t+7t^2$ relative to B.

$$\boxed{P_B=\begin{bmatrix}1&0&1\\0&1&2\\1&1&1\end{bmatrix}}\quad\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&2&0&1&0\\1&1&1&0&0&1\end{array}\right]$$

$$\underset{R_3'=R_3-R_1}{\sim}\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&2&0&1&0\\0&1&0&-1&0&1\end{array}\right]\underset{\text{swap } R_2,R_3}{\sim}\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&0&-1&0&1\\0&1&2&0&1&0\end{array}\right]$$

$$\underset{R_3'=R_3-R_2}{\sim}\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&0&-1&0&1\\0&0&2&1&1&-1\end{array}\right]\underset{R_3'=\frac{1}{2}R_3}{\sim}\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&0&-1&0&1\\0&0&1&\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\end{array}\right]$$

$$\underset{R_1'=R_1-R_3}{\sim}\left[\begin{array}{ccc|ccc}1&0&0&\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\0&1&0&-1&0&1\\0&0&1&\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\end{array}\right]\quad P_B^{-1}=\begin{bmatrix}\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\-1&0&1\\\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\end{bmatrix}$$

$$P_B^{-1}\begin{bmatrix}1\\4\\7\end{bmatrix}=\begin{bmatrix}\frac{1}{2}-2+\frac{7}{2}\\-1+0+7\\\frac{1}{2}+2-\frac{7}{2}\end{bmatrix}$$

$$\boxed{[\vec{x}]_B=\begin{bmatrix}2\\6\\-1\end{bmatrix}}$$

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15)

a) True - The number of vectors in B defines the number of dimensions for $[\vec{x}]_B$

b) False - The change of coordinates matrix is defined as:

$$P_B[\vec{x}]_B = \vec{x}$$

c) False - P_3 is isomorphic with $\mathbb{R}^4$

16)

a) True - The standard basis maps $\mathbb{R}^n$ onto itself.

b) False - Coordinate mapping is $\vec{x} \mapsto [\vec{x}]_B$

c) True - If the plane passes through the origin.

17) Vectors $\vec{v}_1 = \begin{bmatrix}1\\-3\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}2\\-8\end{bmatrix}, v_3 = \begin{bmatrix}-3\\7\end{bmatrix}$ spans $\mathbb{R}^2$ but is not a basis. Find at least two ways to express $\begin{bmatrix}1\\1\end{bmatrix}$ as a linear combination of $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$.

$$\begin{bmatrix}1 & 2 & -3 & 1\\-3 & -8 & 7 & 1\end{bmatrix}$$

$$\underset{R_2' = R_2 + 3R_1}{\sim} \begin{bmatrix}1 & 2 & -3 & 1\\0 & -2 & -2 & 4\end{bmatrix}$$

$$\underset{R_1' = R_1 + R_2}{\sim} \begin{bmatrix}1 & 0 & -5 & 5\\0 & -2 & -2 & 4\end{bmatrix}$$

$$\underset{R_2' = -\frac{1}{2}R_2}{\sim} \begin{bmatrix}1 & 0 & -5 & 5\\0 & 1 & 1 & -2\end{bmatrix}$$

$$\vec{x} = \begin{bmatrix}5\\-2\\0\end{bmatrix} + x_3\begin{bmatrix}5\\-1\\1\end{bmatrix}$$

$$\boxed{\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}5\\-2\\0\end{bmatrix} \text{ or } \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}10\\-3\\1\end{bmatrix}}$$

21) Let $B = \left\{\begin{bmatrix}1\\-4\end{bmatrix}, \begin{bmatrix}-2\\9\end{bmatrix}\right\}$. Find a matrix A such that $A\vec{x} = [\vec{x}]_B$.

$$\begin{vmatrix}1 & -2\\-4 & 9\end{vmatrix} = (1)(9) - (-2)(-4) = 1$$

$$\boxed{P_B^{-1} = A = \begin{bmatrix}9 & 2\\4 & 1\end{bmatrix}}$$

27) Determine if the vectors are independent: $1 + 2t^3, 2 + t - 3t^2, -1 + 2t^2 - t^3$

Map to $\mathbb{R}^4$

$$\begin{bmatrix}1 & 2 & -1\\0 & 1 & 0\\0 & -3 & 2\\2 & 0 & -1\end{bmatrix}$$

$$\underset{R_4' = R_4 - 2R_1}{\sim} \begin{bmatrix}1 & 2 & -1\\0 & 1 & 0\\0 & -3 & 2\\0 & -4 & 1\end{bmatrix}$$

$$\underset{\substack{R_3' = R_3 + 3R_2\\R_4' = R_4 + 4R_2}}{\sim} \begin{bmatrix}1 & 2 & -1\\0 & 1 & 0\\0 & 0 & 2\\0 & 0 & 1\end{bmatrix}$$

$$\underset{R_4' = R_4 - \frac{1}{2}R_3}{\sim} \begin{bmatrix}1 & 2 & -1\\0 & 1 & 0\\0 & 0 & 2\\0 & 0 & 0\end{bmatrix}$$

Linearly independent as pivot in every column

29) Determine if the polynomials are linearly independent where $1 - 2t + t^2, t - 2t + t^3, 1 - 3t + 3t^2 - t^3$

$$\begin{bmatrix}1 & 0 & 1\\-2 & 1 & -3\\1 & -2 & 3\\0 & 1 & -1\end{bmatrix} = \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \vec{a}_3\end{bmatrix}$$

$$\vec{a}_3 = \vec{a}_1 - \vec{a}_2$$

Linear combination so linearly dependent

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31) Use coordinate vectors to determine if the polynomials span P_2

a) $1-3t+5t^2$, $-3+5t-7t^2$, $-4+5t-6t^2$, $1-t^2$

$$\begin{bmatrix} 1 & -3 & -4 & 1 \\ -3 & 5 & 5 & 0 \\ 5 & -7 & -6 & -1 \end{bmatrix}$$

$\sim$ $R_2' = R_2+3R_1$, $R_3' = R_3-5R_1$

$$\begin{bmatrix} 1 & -3 & -4 & 1 \\ 0 & -4 & -7 & 3 \\ 0 & 8 & 14 & -6 \end{bmatrix}$$

$\sim$ $R_3' = R_3+2R_2$

$$\begin{bmatrix} 1 & -3 & -4 & 1 \\ 0 & -4 & -7 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

No - There is not a pivot in every row.

b) $5t+t^2$, $1-8t-2t^2$, $-3+4t+2t^2$, $2-3t$

$$\begin{bmatrix} 0 & 1 & -3 & 2 \\ 5 & -8 & 4 & -3 \\ 1 & -2 & 2 & 0 \end{bmatrix}$$

$\sim$ Swap R_3 and R_1

$$\begin{bmatrix} 1 & -2 & 2 & 0 \\ 5 & -8 & 4 & -3 \\ 0 & 1 & -3 & 2 \end{bmatrix}$$

$\sim$ $R_2' = R_2-5R_1$

$$\begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 2 & -6 & -3 \\ 0 & 1 & -3 & 2 \end{bmatrix}$$

Yes - Last two rows are not scaler multiples meaning there will be a pivot in every row.