Chapter 2-3 - Characterizations of Invertible Matrices

1) Is the matrix invertible

$A=\begin{bmatrix}5 & 7\\ -3 & -6\end{bmatrix}$

$det(A)=(5)(-6)-(-3)(7)$

$det(A)=(-30)-(-21)$

$det(A)=-9$

Invertible

$\begin{bmatrix}5\\-3\end{bmatrix}\begin{bmatrix}7\\-6\end{bmatrix}$ are not scalar multiples making them linearly independent

3) Is this matrix invertible

$A=\begin{bmatrix}5 & 0 & 0\\ -3 & -7 & 0\\ 8 & 5 & -1\end{bmatrix}$

$A^T=\begin{bmatrix}5 & -3 & 8\\ 0 & -7 & 5\\ 0 & 0 & -1\end{bmatrix}$

3 pivots so invertible

5) Is this matrix invertible

$A=\begin{bmatrix}0 & 3 & -5\\ 1 & 0 & 2\\ -4 & -9 & 7\end{bmatrix}=[\vec{x}_1\ \vec{x}_2\ \vec{x}_3]$

$\vec{X}_2$ is not a scalar multiple of $\vec{x}_1$ making them linearly dependent

Determine if $\vec{x}_3$ is a linear combination of $\vec{x}_1$ and $\vec{x}_2$.

Only possible values of c_1 and c_2 as 2 and $-\frac{5}{3}$ respectively due to $a_{11}=0$ and $a_{21}=0$

$2(-4)+(\frac{-5}{3})(-9)=-8+15=7$

Columns of A are linearly dependent

$\boxed{\text{Not invertible}}$

7) Is the matrix invertible?

$\begin{bmatrix}-1 & -3 & 0 & 1\\ 3 & 5 & 8 & -3\\ -2 & -6 & 3 & 2\\ 0 & -1 & 2 & 1\end{bmatrix}$

$R_2'=R_2+3R_1$
$R_3'=R_3-2R_1$

$\Downarrow$

$\begin{bmatrix}-1 & -3 & 0 & 1\\ 0 & -4 & 8 & 0\\ 0 & 0 & 3 & 0\\ 0 & -1 & 2 & 1\end{bmatrix}$

$R_2'=-\frac{1}{4}R_2$

$\Downarrow$

$\begin{bmatrix}-1 & 3 & 0 & 1\\ 0 & 1 & -2 & 0\\ 0 & 0 & 3 & 0\\ 0 & -1 & 2 & 1\end{bmatrix}$

$R_4'=R_4+R_2$

$\Downarrow$

$\begin{bmatrix}-1 & 3 & 0 & 1\\ 0 & 1 & -2 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$

4 pivots so invertible

11) a) True - If A is square, then true by theorem 2-8 (d)

b) True - By the invertible matrix theorem (IMT)

c) False - A zero matrix does not have a solution for all $\vec{b}$. This statement would only be true if A is invertible.

d) True - To have a non-trivial solution, the matrix must have one free variable making it have less than n pivots.

e) True - By the Invertible Matrix Theorem (IMT)

12)

a) True - By the invertible matrix theorem, if (k) is true, then (j) is true.

b) True - IMT property (e) implies IMT property (h).

c) True - IMT property (g) implies IMT property (f) meaning the solution is unique.

d) False - The mapping of $\mathbb{R}^n$ to $\mathbb{R}^n$ gives no information regarding the invertibility of the transformation.

e) True - Invertible Matrix Theorem (IMT) property (g) being false implies IMT property (f) is also false.

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13) An $m \times n$ upper triangular matrix is one whose entries below the main diagonal are all zero. When is a square upper triangular matrix invertible?

When all elements along the main diagonal are non-zero because then it has n pivots for an $n \times n$ matrix making it invertible by the Invertible Matrix Theorem

15) Can a square matrix with two identical columns be invertible?

No, its columns would not be linearly independent meaning it is not invertible by the invertible matrix theorem

17) If A is invertible, then the columns of A^{-1} are linearly independent. Explain why.

If A is invertible, then A^{-1} is invertible by theorem 2-6(a).

If A^{-1} is invertible, then by the invertible matrix theorem the columns of A^{-1} are linearly independent by property (e)

19) If the columns of a 7×7 matrix are linearly independent, what can you say about solutions of $D\vec{x}=\vec{b}$? Why?

For all $\vec{b} \in \mathbb{R}^7$, a solution exists by the invertible matrix theorem

By theorem 2-5, the solution is unique for all $\vec{b}$.

21) If the equation $G\vec{x}=\vec{y}$ has more than one solution for some $\vec{y}$ in $\mathbb{R}^n$, can the columns of G span $\mathbb{R}^n$?

If a $\vec{b}$ has more than one solution, G is not invertible (by theorem 2-5).

That means the columns of G do not span $\mathbb{R}^n$ by the Inverse Matrix Theorem

23) If an $n \times n$ matrix K cannot be row reduced to I_n, what can you say about the columns of K? Why

By the Inverse Matrix Theorem the colums

a) Do not span $\mathbb{R}^n$

b) Are linearly dependent

25) Verify the statement "Let A and B be square matrices. If $AB=I$ then A and B are both invertible with $B=A^{-1}$ and $A=B^{-1}$."

If A is $n \times n$ and $AB=I$, then by the inverse Matrix theorem, A is invertible (by properties (k) and (a)).

Then:

$$AB=I$$
$$A^{-1}(AB)=A^{-1}I$$
$$B=A^{-1}$$

Same applies to B via properties (a) and (j) that B is invertible.

Then

$$AB=I$$
$$(AB)B^{-1}=IB^{-1}$$
$$A=B^{-1}$$

27) Show that if AB is invertible, then so is A.

If AB is invertible then:

$$(AB)W=I$$

Since matrix multiplication is Associative

$$A(BW)=I$$

Define $BW=D$ an $n \times n$ matrix

$$AD=I$$

By the Inverse Matrix Theorem A is invertible.

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33) Show that T is invertible and find T^{-1}.

$T(x_1, x_2) = (-5x_1 + 9x_2, 4x_1 - 7x_2)$

$$A = \begin{bmatrix} -5 & 9 \\ 4 & -7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{35 - 36} \begin{bmatrix} -7 & -9 \\ -4 & -5 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix}$$

$$\boxed{T^{-1}(x_1, x_2) = \begin{bmatrix} 7x_1 + 9x_2 \\ 4x_1 + 5x_2 \end{bmatrix}}$$