Chapter 1-8 – Introduction to Linear Transformations

1) Let $A=\begin{bmatrix}2&0\\0&2\end{bmatrix}$ and define $T:\mathbb{R}^2\to\mathbb{R}^2$ by $T(\vec{x})=A(\vec{x})$

Find the images of

$\vec{u}=\begin{bmatrix}1\\-3\end{bmatrix}$

$$\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}1\\-3\end{bmatrix}=\begin{bmatrix}2\cdot1+(-3\cdot0)\\0\cdot1+(2)(-3)\end{bmatrix}=\boxed{\begin{bmatrix}2\\-6\end{bmatrix}}$$

$\vec{v}=\begin{bmatrix}a\\b\end{bmatrix}$

$$\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}2a+0b\\0a+2b\end{bmatrix}=\boxed{\begin{bmatrix}2a\\2b\end{bmatrix}}$$

3) Find a vector $\vec{x}$ whose image under T is $\vec{b}$.

$$A=\begin{bmatrix}1&0&-2\\-2&1&6\\3&-2&-5\end{bmatrix}\quad \vec{b}=\begin{bmatrix}-1\\7\\-3\end{bmatrix}$$

$$\begin{bmatrix}1&0&-2&-1\\-2&1&6&7\\3&-2&-5&-3\end{bmatrix}$$

$R_2'=R_2+2R_1$
$R_3'=R_3-3R_1$

$\Downarrow$

$$\begin{bmatrix}1&0&-2&-1\\0&1&2&5\\0&-2&1&0\end{bmatrix}$$

$R_3'=R_3+2R_2$

$\Downarrow$

$$\begin{bmatrix}1&0&-2&-1\\0&1&2&5\\0&0&5&10\end{bmatrix}$$

$R_3'=\frac{1}{5}R_3$

$\Downarrow$

$$\begin{bmatrix}1&0&-2&-1\\0&1&2&5\\0&0&1&2\end{bmatrix}$$

$R_2'=R_2-2R_3$
$R_1'=R_1+2R_2$

$\Downarrow$

$$\begin{bmatrix}1&0&0&3\\0&1&0&1\\0&0&1&2\end{bmatrix}$$

$$\boxed{\vec{x}=\begin{bmatrix}3\\1\\2\end{bmatrix}\ \text{It is unique}}$$

5) Find a vector $\vec{x}$ whose image under T is $\vec{b}$.

$$A=\begin{bmatrix}1&-5&-7\\-3&7&5\end{bmatrix}\quad \vec{b}=\begin{bmatrix}-2\\-2\end{bmatrix}$$

$$\begin{bmatrix}1&-5&-7&-2\\-3&7&5&-2\end{bmatrix}$$

$R_2'=R_2+3R_1$

$\Downarrow$

$$\begin{bmatrix}1&-5&-7&-2\\0&8&16&-8\end{bmatrix}$$

$R_2'=-\frac{1}{8}R_2$

$\Downarrow$

$$\begin{bmatrix}1&-5&-7&-2\\0&1&2&1\end{bmatrix}$$

$R_1'=R_1+5R_2$

$\Downarrow$

$$\begin{bmatrix}1&0&3&3\\0&1&2&1\end{bmatrix}$$

$x_1=3-3x_3$
$x_2=1-2x_3$
x_3 is free

$$\boxed{\vec{x}=\begin{bmatrix}3\\1\\0\end{bmatrix}\ \text{Not unique}}$$

7) Let A be a 6×5 matrix. What must a and b be in order to define $T:\mathbb{R}^a\to\mathbb{R}^b$ by $T(x)=Ax$

$$\boxed{\begin{matrix}a=n=5\\ b=m=6\end{matrix}}$$

9) Find all $\vec{x}$ in $\mathbb{R}^4$ that are mapped to the zero vector by the transformation $\vec{x}\mapsto A\vec{x}$ for

$$A=\begin{bmatrix}1&-4&7&-5\\0&1&-4&3\\2&-6&6&-4\end{bmatrix}$$

Augmented Matrix

$$\begin{bmatrix}1&-4&7&-5&0\\0&1&-4&3&0\\2&-6&6&-4&0\end{bmatrix}$$

$R_3'=R_3-2R_1$

$\Downarrow$

$$\begin{bmatrix}1&-4&7&-5&0\\0&1&-4&3&0\\0&2&-8&6&0\end{bmatrix}$$

$R_3'=R_3-2R_2$

$\Downarrow$

$$\begin{bmatrix}1&-4&7&-5&0\\0&1&-4&3&0\\0&0&0&0&0\end{bmatrix}$$

$R_1'=R_1+4R_2$

$\Downarrow$

$$\begin{bmatrix}1&0&-9&7&0\\0&1&-4&3&0\\0&0&0&0&0\end{bmatrix}$$

$x_1=9x_3-7x_4$
$x_2=4x_3-3x_4$
x_3 is free
x_4 is free

$$\boxed{\vec{x}=x_3\begin{bmatrix}9\\4\\1\\0\end{bmatrix}+x_4\begin{bmatrix}-7\\-3\\0\\1\end{bmatrix}}$$

Chapter 1-8 - Introduction to Linear Transformations

11) Is $\vec{b}$ in the range of linear transformation $x \mapsto Ax$ if

$\vec{b}=\begin{bmatrix}-1\\1\\0\end{bmatrix}$, $A=\begin{bmatrix}1&-4&7&-5\\0&1&-4&3\\2&-6&6&-4\end{bmatrix}$

Augmented Matrix

$$\begin{bmatrix}1&-4&7&-5&-1\\0&1&-4&3&1\\2&-6&6&-4&0\end{bmatrix}$$

$R_3'=R_3-2R_1$

$\Downarrow$

$$\begin{bmatrix}1&-4&7&-5&-1\\0&1&-4&3&1\\0&2&-8&6&2\end{bmatrix}$$

$R_3'=R_3-2R_2$

$\Downarrow$

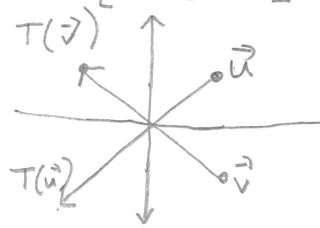
$$\begin{bmatrix}1&-4&7&-5&-1\\0&1&-4&3&1\\0&0&0&0&0\end{bmatrix}$$

Yes since the system is consistent

13) Describe geometrically what T does to each vector $\vec{x}$:

$T(\vec{x})=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$

$=\begin{bmatrix}-x_1\\-x_2\end{bmatrix}=-1\begin{bmatrix}x_1\\x_2\end{bmatrix}$

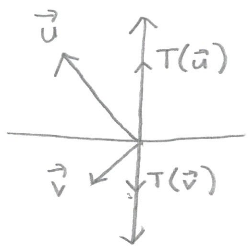


$\boxed{\text{A reflection through the origin}}$

15) Describe geometrically what T does to each vector $\vec{x}$:

$T(\vec{x})=\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$

$T(\vec{x})=\begin{bmatrix}0\\x_2\end{bmatrix}$



$\boxed{\text{A projection onto the } x_2 \text{ axis}}$

17) Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be a linear transformation that maps $\vec{u}=\begin{bmatrix}5\\2\end{bmatrix}$ into $\begin{bmatrix}2\\1\end{bmatrix}$ and maps $\vec{v}=\begin{bmatrix}1\\3\end{bmatrix}$ into $\begin{bmatrix}-1\\3\end{bmatrix}$. Use that T is linear to find the images under T of:

a) $3\vec{u}$

$T(3u)=3T(u)=3\begin{bmatrix}2\\1\end{bmatrix}=\boxed{\begin{bmatrix}6\\3\end{bmatrix}}$

b) $2\vec{v}$

$T(2\vec{v})=2T(\vec{v})=2\begin{bmatrix}-1\\3\end{bmatrix}=\boxed{\begin{bmatrix}-2\\6\end{bmatrix}}$

c) $3\vec{u}+2\vec{v}$

$T(3\vec{u}+2\vec{v})=T(3\vec{u})+T(2\vec{v})=\begin{bmatrix}6\\3\end{bmatrix}+\begin{bmatrix}-2\\6\end{bmatrix}$

$=\boxed{\begin{bmatrix}4\\9\end{bmatrix}}$

Chapter 1-8 – Introduction to Linear Transformation

19) Let $\vec{e_1} = \begin{bmatrix}1\\0\end{bmatrix}$ and $\vec{e_2} = \begin{bmatrix}0\\1\end{bmatrix}$ and $\vec{y_1} = \begin{bmatrix}2\\5\end{bmatrix}$ and $\vec{y_2} = \begin{bmatrix}-1\\6\end{bmatrix}$. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ such that $\vec{e_1}$ maps to $\vec{y_1}$ and $\vec{e_2}$ maps to $\vec{y_2}$.

a) Find the image for $\begin{bmatrix}5\\-3\end{bmatrix}$

$$a\vec{y_1} + b\vec{y_2} = \begin{bmatrix}5\\-3\end{bmatrix}$$

$$5\vec{y_1} - 3\vec{y_2} = \begin{bmatrix}5\\-3\end{bmatrix}$$

$$T\left(\begin{bmatrix}5\\-3\end{bmatrix}\right) = T(5\vec{y_1} - 3\vec{y_2})$$

$$= 5T(\vec{y_1}) - 3T(\vec{y_2})$$

$$= 5\begin{bmatrix}2\\5\end{bmatrix} - 3\begin{bmatrix}-1\\6\end{bmatrix}$$

$$= \begin{bmatrix}10+3\\25-18\end{bmatrix} = \boxed{\begin{bmatrix}13\\7\end{bmatrix}}$$

b) Find the image of $\begin{bmatrix}x_1\\x_2\end{bmatrix}$

$$a\vec{y_1} + b\vec{y_2} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1\vec{y_1} + x_2\vec{y_2}$$

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = T(x_1\vec{y_1} + x_2\vec{y_2})$$

$$= \boxed{\begin{bmatrix}2x_1 - x_2\\5x_1 + 6x_2\end{bmatrix}}$$

21)

a) True – It has properties $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(c\vec{u}) = cT(\vec{u})$.

b) False – The domain is $\mathbb{R}^5$ for a 3x5 matrix.

c) False – $\mathbb{R}^m$ is the codomain. The range is the actual values and maybe a subset of the codomain.

d) False – Every matrix transformation is a linear transformation but the converse may not be true.

e) True – This is the definition of a linear transform.

22)

a) True – All matrix transforms are linear transforms.

b) False – All linear combinations of the columns is the range.

c) False – "Is $\vec{c}$ in the range of T" is an existence question

d) True – This is the definition of a linear transformation.

e) True – Specified in the textbook on page 77

25) Given $\vec{v} \neq \vec{0}$ and $\vec{p}$ in $\mathbb{R}^n$, the line through $\vec{p}$ in the direction of $\vec{v}$ has a parametric equation $\vec{x} = \vec{p} + t\vec{v}$.

Show that a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ maps this line onto another line or onto a single point.

$$\vec{x} = \vec{p} + t\vec{v}$$

Define: $\vec{u} = T(\vec{p})$
$\vec{y} = T(\vec{v})$

$$T(\vec{x}) = T(\vec{p} + t\vec{v})$$

$$T(\vec{x}) = T(\vec{p}) + tT(\vec{v})$$

$$T(\vec{x}) = \vec{u} + t\vec{y}$$

If $\vec{y}$ (i.e. $T(\vec{v})$) is the zero vector, then $T(\vec{x})$ is a point. Otherwise $T(\vec{x})$ is a line given by the parametric equation

$$T(\vec{x}) = \vec{u} + t\vec{y}$$