MATH129A – Linear Algebra Midterm #1 Study Guide

Created By: Zayd Hammoudeh

Section 1.1 – Systems of Linear Equations

Linear Equation – An equation with **variables** $x_1, x_2, \dots, x_n$ that **can be written in the form**: $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ **Coefficients:** $a_1, a_2, \dots, a_n$ can be **real or complex**	**Linear System** or **System of Linear Equations** – Collection of **one or more linear equations**.	**Solution:** A **set of numbers** $(s_1, s_2, \dots, s_n)$ **that makes each equation a true statement** when substituted for variables $(x_1, x_2, \dots, x_n)$ respectively.	**Solution Set:** Set of **all possible solutions** for a linear system. **Possible Solution Sets:** • **No solution** • **One solution** • **Infinite solutions**

Consistent Linear System – Has **one or infinite solutions** **Inconsistent Linear System** – Has no solution	**Coefficient Matrix** – A matrix containing the **coefficients for each variable in each equation** in the linear system.	**Augmented Matrix** – A matrix of a system containing the **coefficient matrix and an added column containing the constants** from the ***right hand side*** of the equation.	**Techniques to Simplify a Linear System** **1.Replace** one equation with **sum of itself and the multiple of another linear system** (equation) **2.Interchange** two equations **3.Multiply all terms** in an equation by a **non-zero constant**

Row Equivalent Matrices – Any two matrices where a **series of elementary row operations** can transform one matrix into another.	**Row Operation Reversibility** – All row operations can be undone to get the previous matrix	**m x n Matrix** – Composed of m rows of n columns	**Equivalent Linear Systems** – Any two linear systems with the **same solution set**.

Approaches to Find the Solution Set of a Linear System 1. **Solve equations by substitution.** 2. **Multiply and add the equations** 3. **Graphically** a. Look at the intersection of the equations.			

Section 1.2 – Row Reduction and Echelon Forms

	Key Properties				
	If **two augmented matrices are row equivalent**, then the systems **have the same solution set**.	**Reduced echelon form is unique**	**Echelon form is not unique**	All linear systems have a reduced echelon form.	**Location of leading entries** is the **same between standard and reduced echelon form.**

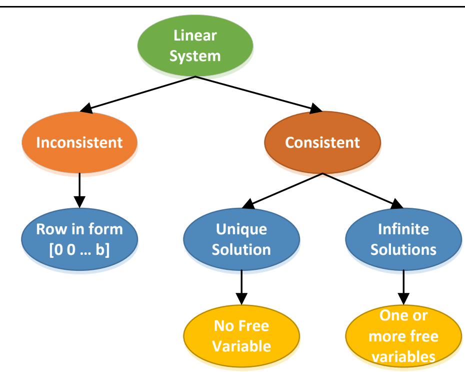
Echelon Matrix Criteria	Reduced (Row) Echelon Matrix Criteria	
1. All non-zero rows are above all zero rows. **2. The leading entries of lower rows are to the right of all those in upper rows.** a. Forms a "**step pattern**". **3. The entries in a column below a leading entry are zero.**	**1. All criteria of a standard echelon matrix.** **2. All leading entries equal 1.** **3. The entries in a column above a leading entry are 0.**	**Theorem #1: Any matrix has one and only one reduced echelon form**.

Pivot Position – A **position** in a given matrix that **corresponds to a "1" in reduced echelon form**.	**Pivot Position** – A column that contains a pivot position.	**Pivot** – A non-zero number in a pivot position that is used as needed to create zeros via row operations.	**Gaussian Elimination:** Same concept as row reduction.	**Non-zero row/column**: A row/column with **at least one non-zero entry**.	**Leading entry** – **Leftmost non-zero entry in a row**.

Row Reduction Algorithm (Gaussian Elimination) Forms an Echelon Matrix				Gauss-Jordan Elimination Forms the Reduced Echelon Matrix
1. Begin with the left most non-zero entry that is a pivot position. Move that position to the top of the matrix.	**2. Select a non-zero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.**	**3. Use row operations to create zeros in all positions below the pivot.**	**4. Cover (i.e. ignore) the rows containing the pivot position and cover (ignore) all rows above it. Apply steps 1-3 to the sub matrix that remains. Repeat the process until there are no more non-zero rows remaining.**	**5. Create zeros above each pivot and scale each pivot to 1.**

Basic Variable: Can only exist in a single solution set equation. Correspond to a pivot in the reduced echelon matrix. **Free Variable: Can be assigned to any number** since it has no pivot. Free variable **quantity dictates the resulting solution set's shape** (e.g., plane, line etc.).	**Parametric Description of Solution Sets** • Free variables act as parameters • Each basic variable is represented by an equation made up of constants and/or free variables.	**Theorem 1-2: Existence and Uniqueness Theorem** A **linear system is consistent** if and only if the rightmost column of the augmented matrix is not a pivot column – that is, if and only if an echelon form of the **augmented matrix has no row in the form**: $[\mathbf{0} \ldots \mathbf{0}\ \boldsymbol{b}]$ with b non-zero If a linear system is consistent, then the solution set contains either: i) **A unique solution, when there is no free variable** ii) **Infinitely many solutions when there is at least one free variable.**

Example: Parametric Description of a Solution Set	
Augmented Matrix $\begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$	**Parametric Description of the Solution Set** $\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 4x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases}$



Section 1.3 – Vector Equations

Vector – An **ordered** list of numbers	**Column Vector** – A **matrix with only one column**.	**Vector Equality** – Two vectors of the **same size** with **all corresponding entries equal**.	**Vector Addition** – Sum obtained by **adding the corresponding entries of two vectors**.	**Scalar Multiple** – Given a number, c, and a vector, $\vec{v}$, it is the vector obtained when c is **multiplied by each element** in $\vec{u}$

Parallelogram Rule for Addition – If u and v are in $\mathbb{R}^2$ (i.e., points in the Cartesian plane), then $\vec{u}+\vec{v}$ corresponds to the **fourth point** in the **parallelogram whose vertices are** $\vec{u}$, $\vec{v}$, and $\vec{0}$ (the origin).	**Scalar Multiples of a Fixed Vector** – **Along a line between the vector point and the origin**.	**Zero Vector** ($\vec{0}$) – A vector whose entries are all zeros.	**Linear Combination** – Given vectors $\vec{v_1}, \dots, \vec{v_n}$ and scalars $c_1, \dots, c_n$, then the **linear combination vector** $\vec{y}$ is: $\vec{y} = c_1\vec{v_1} + \cdots + c_n\vec{v_n}$

Algebraic Properties in $\mathbb{R}^n$

Commutative $\vec{u}+\vec{v}=\vec{v}+\vec{u}$	**Associative** $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$	**Identity** $\vec{u}+\vec{0}=\vec{u}$	**Inverse** $\vec{u}-\vec{u}=-\vec{u}+\vec{u}=\vec{0}$
Distributive $c(\vec{u}+\vec{v})=c\vec{u}+d\vec{v}$	**Distributive** $(c+d)\vec{v}=c\vec{v}+d\vec{v}$	**Associative** $c(d\vec{u})=(cd)\vec{u}$	**Identity** $1\vec{u}=\vec{u}$

The Vector Equation and the Augmented Matrix

A **vector equation**:

$$x_1\vec{a_1} + x_2\vec{a_2} + \cdots + x_n\vec{a_n} = \vec{b}$$

has the **same solution set as the linear system whose augmented matrix is**:

$$[\vec{a_1} \quad \vec{a_2} \quad \dots \quad \vec{a_n} \quad \vec{b}]$$

In particular, $\vec{b}$ can be generated by a linear combination of $\vec{a_1}, \dots, \vec{a_n}$ if and only if there exists a solution to the linear system corresponding to the augmented matrix.

Subset of $\mathbb{R}^n$ Spanned / Generated by $\vec{v_1}, \vec{v_2}, \dots \vec{v_p}$

If $\vec{v_1}, \vec{v_2}, \dots, \vec{v_p}$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\vec{v_1}, \vec{v_2}, \dots \vec{v_p}$ is denoted by:

$$span\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}\}$$

It is the set of all linear combination vectors that can be written in the form:

$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n}$$

Geometric Description of Spans of Vectors

$\vec{0}$ (Zero Vector)	A **point** – The origin
One non-zero vector $\vec{u}$	A **line** through the origin and $\vec{u}$
Two vectors $\vec{u}$ and $\vec{v}$ that **are scalar multiples**	A **line** through the origin and $\vec{u}$
Two vectors $\vec{u}$ and $\vec{v}$ **not scalar multiples**	A **plane** through the origin, $\vec{u}$, and $\vec{v}$

Members of All Spans

- **Zero vector** ($\vec{0}$)
- **Scalar multiples of the original vectors**

How to Determine if a Vector is in a Span

Check if the system is consistent

Important Notation

Vector $\vec{u}$ in $\mathbb{R}^3$			Not a vector	Set of Directions
$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$	(u_1, u_2, u_3)	$< u_1, u_2, u_3 >$	$[u_1, u_2, u_3]$ It is a 1x3 matrix	$\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}\}$

Section 1.4 – The Matrix Equation $A\vec{x} = \vec{b}$

Gaussian Elimination – Same concept as row reduction.

Non-zero row/column – A row/column with at least one non-zero entry.

Theorem #1-3: If A is an m x n matrix with columns $\vec{a_1}, \dots, \vec{a_n}$ and if $\vec{b} \in \mathbb{R}^m$, then **the matrix equation**:

$$A\vec{x} = \vec{b}$$

has the same solution set as the **vector equation**:

$$x_1\vec{a_1} + x_2\vec{a_2} + \cdots + x_n\vec{a_n} = \vec{b}$$

which **has the same solution set as the system of linear equations whose augmented matrix is**:

$$[\vec{a_1} \quad \vec{a_2} \quad \dots \quad \vec{a_n} \quad \vec{b}]$$

Theorem #1-4: Let A be an m x n matrix. Then the following statements are equivalent meaning they are **either all true or all false**.

a) **For all $\vec{b} \in \mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.**

b) **Each $\vec{b} \in \mathbb{R}^m$ is a linear combination of the columns of A.**

c) **The columns of A span $\mathbb{R}^m$**

d) **The (coefficient) matrix A has a pivot position in every row.**

Relationship between Spans and Free Variables

$A\vec{x} = \vec{0}$	General Solution Structure
$\vec{x} = \vec{0}$ **(Trivial Only)**	$span\{\vec{0}\} = \vec{0}$
1 Free Variable	$span\{\vec{u}\}$ – **Line** through $\vec{u}$ and the origin
2 Free Variables	$span\{\vec{u}, \vec{v}\}$ – **Plane** through $\vec{u}$, $\vec{v}$, and the origin
p Free Variables	$span\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_p}\}$ – **Multidimensional shape** through $\vec{v_1}, \vec{v_2}, \dots, \vec{v_p}$ and the origin

Theorem #1-5: If A is an m x n matrix, $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^m$, and c is a scalar, then:

a) $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$

b) $A(c\vec{u}) = cA\vec{u}$

Row-Vector Rule for Computing $A\vec{x}$

If the product $A\vec{x}$ is defined (i.e., the sizes correspond), then the i^{th} entry in $A\vec{x}$ is the sum of products of the corresponding entries from row i of A and from the vector $\vec{x}$. It is formally:

$$b_i = \sum_{j=1}^{n} a_{i,j} \; b_j$$

Matrix Equation Definition

If A is an m x n matrix with columns $\vec{a_1}, \dots, \vec{a_n}$ and if $\vec{x} \in \mathbb{R}^n$, then the **product**, of A and $\vec{x}$, denoted by $A\vec{x}$, **is the linear combination of the columns of A using the corresponding entries in $\vec{x}$ as the weights**; that is:

$$A\vec{x} = [\vec{a_1} \quad \vec{a_2} \quad \dots \quad \vec{a_n}] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a_1} + x_2\vec{a_2} + \cdots + x_n\vec{a_n}$$

Section 1.5 – Solution Sets of Linear Systems

<table>
<tr>
<td>

Homogeneous Linear System: $A\vec{x} = \vec{0}$

- A – m x n Matrix
- $\vec{x}$ – Vector in $\mathbb{R}^n$
- $\vec{0}$ – Zero vector in $\mathbb{R}^m$

Trivial Solution: $\vec{x} = \vec{0}$ (in $\mathbb{R}^n$)

- **Exists for all homogeneous systems**

Nontrivial Solution: Any **non-zero vector solution to the equation: $A\vec{x} = \vec{0}$**

- If it exists, there are infinitely many.
- Requires at least one free variable.

</td>
<td>

Non-Homogenous System: $A\vec{x} = \vec{b}$ where $\vec{b}$ is not the zero vector.

- May be inconsistent.
- If its solution exists, it is in the form:

$$\vec{x} = \vec{p} + \vec{v_h}$$

  - If A is m x n, then $x \in \mathbb{R}^n$
  - $\vec{p}$ – Particular solution for the specific non-homogenous system
  - $\vec{v_h}$ – Solution set for the homogenous system

</td>
<td>

Theorem #1-6: Suppose the equation $A\vec{x} = \vec{b}$ is consistent for some given $\vec{b}$, and let $\vec{p}$ be **any solution to that particular non-homogenous system**. Then the **solution set of $A\vec{x} = \vec{b}$** (**if it exists**) is **the set of all vectors of the form**:

$$\vec{w} = \vec{p} + \vec{v_h}$$

where $\vec{v_h}$ is **any solution of the homogenous equation** $A\vec{x} = \vec{0}$.

</td>
</tr>
</table>

<table>
<tr>
<th colspan="3">Solution Set Notations</th>
<td rowspan="2">

For $\vec{b}$ to span all of R^m, there **must be a pivot in every row of echelon matrix** of A.

</td>
</tr>
<tr>
<td>

Augmented Matrix

$$\begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

</td>
<td>

Parametric Description of the Solution Set

$$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 4x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases}$$

</td>
<td>

Parametric Vector Form

$$\vec{x} = \begin{bmatrix} -6x_2 - 3x_4 \\ x_2 \\ 5 + 4x_4 \\ x_4 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}$$

</td>
</tr>
</table>

Section 1.7 – Linear Independence

Span Review
- **One Vector** – **At most** a **line**
  - **Exception:** $\vec{0}$ vector
- **Two Vectors:** **At most** a **plane**
  - **Exception:** Scalar multiples and the zero vector

Linear Independence
A set of vectors $\boldsymbol{S} = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_p}\}$ are **linearly independent** if:

$$x_1\overrightarrow{v_1} + x_2\overrightarrow{v_2} + \cdots + x_p\overrightarrow{v_p} = \vec{0}$$

has **only the trivial solution**

Linear Dependence
A set of vectors $\boldsymbol{S} = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_p}\}$ are **linearly dependent** if there exists a set of **non-zero weights** $c_1, c_2, \dots, c_p$ such that:

$$c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + \cdots + c_p\overrightarrow{v_p} = \vec{0}$$

Note: This requires **at least one free variable**.

Linear Dependence Relation: For vectors $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_p}\}$, it is defined as:

$$c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + \cdots + c_p\overrightarrow{v_p} = \vec{0}$$

where $\{c_1, c_2, \dots, c_p\}$ are **not all zero.**

Note: The values of $\{c_1, c_2, \dots, c_p\}$ are not unique. **If one exists, an infinite number exist.**

Procedure: Checking for Linear Independence

Step #1: Create the coefficient matrix.

Step #2: Perform Gaussian elimination to find the echelon matrix.

Step #3: Check linear independence
- If **there is a pivot in every column**, the vectors are **linearly independent**.
- If **there is a free variable**, the vectors are **linearly dependent**.

Linear Independence of One Vectors
- **Zero Vector** ($\{\vec{0}\}$) – This is always **linearly dependent**.
- **Any Non-Zero Vector** ($\{\vec{v}\}$ where $\vec{v} \neq \vec{0}$) then **linearly independent**.

Linear Independence of Two Vectors – A set of two vectors $\{v_1, v_2\}$ is linearly dependent if **at least one of the vectors is a scalar multiple** of the other.
- "**At least one**" – Because of the case of the zero ($\vec{0}$) vector.

For a set of vectors to be **linearly independent**, the echelon matrix made from those vectors must have no free variables.
- **There must be a pivot in every column.**

Linear Dependence Summary: If a set of n **vectors of** m**-dimensions** are linearly independent, then they **span an** n **dimensional shape**.

Theorem #1-7: An indexed set $\boldsymbol{S} = \{v_1, v_2, \dots, v_p\}$ of two or more vectors is **linearly dependent if and only if at least one of the vectors in S is a linear combination of the others**.

In fact, if S is linearly dependent and $v_1 \neq 0$, then some v_j (where $j > 1$) is a linear combination of the preceding vectors: $v_1, \dots, v_{j-1}$.

Theorem #1-8: If a set **contains more vectors than there are entries in each vector**, then the set is **linearly dependent**.

That is any set $\{v_1, v_2, \dots, v_p\}$ in $\mathbb{R}^n$ is linearly dependent if:

$$p > n$$

Proof: More pivots than columns in an n x p matrix so **at least one free variable**.

Theorem #1-9: If a set $\boldsymbol{S} = \{v_1, v_2, \dots, v_p\}$ **contains the zero vector**, then the set is **linearly dependent**.

Proof: If the zero vector is reordered to be v_1, then

$$1\overrightarrow{v_1} + 0\overrightarrow{v_2} + \cdots + 0\overrightarrow{v_p} = \vec{0}$$

Section 1.8 – Introduction to Linear Transformations

Function – A **rule** that **assigns to each element** in a set A **exactly one element** from set B.	**Transformation (T) from $\mathbb{R}^n$ to $\mathbb{R}^m$:** A rule that **assigns to each vector** $\vec{x} \in \mathbb{R}^n$ **a vector** $T(\vec{x}) \in \mathbb{R}^m$	**Domain of T:** $\mathbb{R}^n$ **Codomain of T:** $\mathbb{R}^m$	**Image of $\vec{x}$:** For a given $x \in R^n$, it is the **transformed value** $T(\vec{x})$	**Range: Set of all images** $T(\vec{x})$. **Note:** The range may be (and often is) only a subset of the codomain.
Linear Transformations: Preserve vector addition and scalar multiplication				

Properties of Matrix Transformations

Requirements of Linear Transformation		Zero Vector Properties		
$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$	$T(c\vec{u}) = cT(\vec{u})$	$T(\vec{0}) = \vec{0}$	$T(\vec{0}) = T(0\vec{u})$	$T(c\vec{u}+d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$

Types of Matrix Transformations

Contraction	Dilation	Projection onto an Axis	Shear Transformations
$T(\vec{x}) = r\vec{x}$, where $0 \le r \le 1$	$T(\vec{x}) = r\vec{x}$, where $r > 1$	Matrices in the form: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ – Projection onto (x_1x_2)-plane $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ – Projection onto the x_2-axis	Maps line segments onto line segments. Deforms the shape of the original input. $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

Reflection through the Origin	Reflection through the x_1-axis	Reflection through the x_2-axis	Reflection through the line $x_1 = x_2$
Matrix in the Form of: $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	Matrix in the Form of: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	Matrix in the Form of: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Matrix in the Form of: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Twice as Long and 1.5 Times as High	Rotation about Origin through 90°	
Matrix in the Form of: $\begin{bmatrix} 2 & 0 \\ 0 & 1.5 \end{bmatrix}$	Matrix in the Form of: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	

Section 2.1 – Matrix Operations

a_{ij} or (i, j)-**entry** – Given A an m x n matrix, it is the element in the i^{th} **row** and j^{th} **column** of A	**Diagonal Entries** – In A an m x n matrix, it is the entries a_{11}, a_{22}, etc.	**Diagonal Matrix** – A **square** n x n matrix whose non-diagonal entries are 0. **Example: Identity matrix**	**Equal Matrices** – Two matrices having **the same size** (i.e., equal number of rows and columns) and **whose corresponding entries are equal**.	**Matrix Sum** – Given two matrices of the same size, the matrix sum is equivalent to the **sum of the corresponding columns**.

Matrix Multiplication If A is an m x n matrix and if B is an n x p matrix composed of the columns $\vec{b_1}, \vec{b_2}, \dots, \vec{b_p}$, then the **product AB is an m x p matrix** whose columns are: $A\vec{b_1}, A\vec{b_2}, \dots, A\vec{b_p}$ such that: $AB = A[\vec{b_1}\ \vec{b_2}\ \dots\ \vec{b_p}] = [A\vec{b_1}\ A\vec{b_2}\ \dots\ A\vec{b_p}]$	**Row-Product Rule** **If the product of AB is defined**, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B such that: $(AB)_{ij} = \sum_{k=1}^{n} a_{ik}\ b_{kj}$	**Size of a Matrix Multiplication:** Given a matrix A of size m x n and a matrix B of size n x p, then the matrix multiplication AB is of size n x p. **Non-Commutativity of Matrix Multiplication** In most cases, $AB \neq BA$ **Cancelation Laws Do Not Hold For Matrix Multiplication** Generally, if $AB = AC$, it **cannot be assumed** that $B = C$ **Zero Matrix Product** If $AB = 0$, you **cannot generally assume $A = 0$** or $B = 0$

Powers of a Matrix If A is an n x n matrix and k is a positive integer, then A^k denotes the product of k copies of A such that: $A^k = A \dots A$ **Zero Power** – Given a square matrix A, then A^0 **is the identity matrix**.	**Transpose of a Matrix** Given an m x n matrix A, then transpose of A is an **n x m matrix whose columns are formed from the corresponding rows of A**.	**Transpose Multiplication Row:** The **transpose of the product of matrices equals** the **product of their transposes in reverse order**.

Theorem #2-1: Let A, B, and C be **matrices of the same size** and let r and s be scalars, then: **a.** $A + B = B + A$ **(Commutative)** **b.** $(A + B) + C = A + (B + C)$ **(Associative)** **c.** $A + 0 = A$ **(Identity)** **d.** $r(A + B) = rA + rb$ **e.** $(r + s)A = rA + sA$ **f.** $r(sA) = (rs)A$	**Theorem #2-2:** Given I_m is the identity matrix of size m x m and A is an m x n matrix, and that B and C have sizes for which the indicated sums and products exist, then **a.** $A(BC) = (AB)C$ **(Associative)** **b.** $A(B + C) = AB + AC$ **(Left Distributive Law)** **c.** $(B + C)A = BA + CA$ **(Right Distributive Law)** **d.** $r(AB) = (rA)B = A(rB)$, **for any scalar r** **e.** $I_m A = A = AI_n$	**Theorem #2-3:** Let A and B be matrices whose sizes are appropriate for the following sums and products: **a.** $(A^T)^T = A$ **b.** $(A + B)^T = A^T + B^T$ **c.** $(rA)^T = rA^T$, **for any scalar r** **d.** $(AB)^T = B^T A^T$ **Note: The reverse order of the product.**

Midterm #1 Theorems

Theorem #1: Any matrix has one and only one reduced echelon form.

Theorem 1-2: Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column – that is, if and only if an echelon form of the augmented matrix has no row in the form:

$[0 \ldots 0\ b]$ with b non-zero

If a linear system is consistent, then the solution set contains either:

i) A unique solution, when there is no free variable

ii) Infinitely many solutions when there is at least one free variable.

Theorem #1-3: If A is an m x n matrix with columns $\vec{a_1}, \ldots, \vec{a_n}$ and if $\vec{b} \in \mathbb{R}^m$, then the matrix equation:

$$A\vec{x} = \vec{b}$$

has the same solution set as the vector equation:

$$x_1\vec{a_1} + x_2\vec{a_2} + \cdots + x_n\vec{a_n} = \vec{b}$$

which in turn has the same solution set as the system of linear equations whose augmented matrix is:

$$[\vec{a_1}\ \ \vec{a_2}\ \ \ldots\ \ \vec{a_n}\ \ \vec{b}]$$

Theorem #1-4: Let A be an m x n matrix. Then the following statements are equivalent meaning they are either all true or all false.

a) For all $\vec{b}$ in $\mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.

b) Each $\vec{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

c) The columns of A span $\mathbb{R}^m$

d) The (coefficient) matrix A has a pivot position in every row.

Theorem #1-5: If A is an m x n matrix, $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^m$, and c is a scalar, then:

a) $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$

b) $A(c\vec{u}) = cA\vec{u}$

Theorem #1-6: Suppose the equation $A\vec{x} = \vec{b}$ is consistent for some given $\vec{b}$, and let $\vec{p}$ be any solution to that particular non-homogenous system. Then the solution set of $A\vec{x} = \vec{b}$ is the set of all vectors of the form:

$$\vec{w} = \vec{p} + \vec{v_h}$$

where $\vec{v_h}$ is any solution of the homogenous equation $A\vec{x} = \vec{0}$.

Theorem #1-7: An indexed set $S = \{v_1, v_2, \ldots, v_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.

In fact, if S is linearly dependent and $v_1 \neq 0$, then some v_j (where $j > 1$) is a linear combination of the preceding vectors: $v_1, \ldots, v_{j-1}$.

Theorem #1-8: If a set contains more vectors than there are entries in each vector, then the set is linearly dependent.

That is any set $\{v_1, v_2, \ldots, v_p\}$ in $\mathbb{R}^n$ is linearly dependent if:

$$p > n$$

Proof: More pivots than columns in an n x p matrix so at least one free variable.

Theorem #1-9: If a set $S = \{v_1, v_2, \ldots, v_p\}$ contains the zero vector, then the set is linearly dependent.

Proof: If the zero vector is reordered to be v_1, then

$$1\vec{v_1} + 0\vec{v_2} + \cdots + 0\vec{v_p} = \vec{0}$$

Theorem #2-1: Let A, B, and C be matrices of the same size and let r and s be scalars, then:

a. $A + B = B + A$ **(Commutative)**
b. $(A + B) + C = A + (B + C)$ **(Associative)**
c. $A + 0 = A$ **(Identity)**
d. $r(A + B) = rA + rb$
e. $(r + s)A = rA + sA$
f. $r(sA) = (rs)A$

Theorem #2-2: Given I_m is the identity matrix of size m x m and A is an m x n matrix, and that B and C have sizes for which the indicated sums and products exist, then

a. $A(BC) = (AB)C$ **(Associative)**
b. $A(B + C) = AB + AC$ **(Left Distributive Law)**
c. $(B + C)A = BA + CA$ **(Right Distributive Law)**
d. $r(AB) = (rA)B = A(rB)$, **for any scalar r**
e. $I_mA = A = AI_n$

Theorem #2-3: Let A and B be matrices whose sizes are appropriate for the following sums and products:

a. $(A^T)^T = A$
b. $(A + B)^T = A^T + B^T$
c. $(rA)^T = rA^T$, **for any scalar r**
d. $(AB)^T = B^TA^T$

Note: **The reverse order of the product.**

Proofs for the Algebraic Properties of $\mathbb{R}^n$

Name	Description	Terms
Commutative (Vector Addition)	$\vec{u}+\vec{v}=\vec{v}+\vec{u}$	• $\vec{u},\vec{v},\vec{w}$ – Vectors • $\vec{0}$ – Zero vector • c,d – Real constants
Inverse (Vector Addition)	$\vec{u}+(-\vec{u})=(-u)+u=\vec{0}$	
Associative (Vector Addition)	$\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$	
Associative (Scalar Multiplication)	$c(d\vec{u})=(cd)\vec{u}$	
Distributive Law (Vector Addition)	$c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$	
Distributive Law (Scalar Multiplication)	$(c+d)\vec{u}=c\vec{u}+d\vec{u}$	
Identity (Vector Addition)	$\vec{u}+\vec{0}=\vec{u}$	
Identity (Scalar Multiplication)	$1\vec{u}=\vec{u}$	

Proof of the Commutative Property of Vector Addition

1. Suppose $\vec{u}$ and $\vec{v}$ are **any real vector in $\mathbb{R}^n$** in the form:
   $\vec{u}=(u_1,u_2,\dots,u_n)$ and $\vec{v}=(v_1,v_2,\dots,v_n)$
2. By the **definition of vector addition**:
   $\vec{u}+\vec{v}=(u_1+v_1,u_2+v_2,\dots,u_n+v_n)$
3. By the **commutative property of real number addition**:
   $(u_1+v_1,u_2+v_2,\dots,u_n+v_n)=(v_1+u_1,v_2+u_2,\dots,v_n+u_n)$
4. By the **definition of vector addition** and the **definition of $\vec{u}$ and $\vec{v}$**:
   $(v_1+u_1,v_2+u_2,\dots,v_n+u_n)=\vec{v}+\vec{u}$ **(QED)**

Proof of the Inverse Property of Vector Addition

1. Suppose $\vec{u}$ is **any real vector in $\mathbb{R}^n$** in the form:
   $\vec{u}=(u_1,u_2,\dots,u_n)$
2. By the **definition of scalar multiplication**:
   $-\vec{u}=(-1)u=(-u_1,-u_2,\dots,-u_n)$
3. By the **definition of vector addition**:
   $\vec{u}+(-\vec{u})=\big(u_1+(-u_1),u_2+(-u_2),\dots,u_2+(-u_n)\big)$
4. By the **inverse property of real number addition**:
   $\big(u_1+(-u_1),u_2+(-u_2),\dots,u_2+(-u_n)\big)=(0,0,\dots,0)$
5. By **definition of the zero vector**:
   $(0,0,\dots,0)=\vec{0}$ **(QED)**

Proof of the Associative Property for Vector Addition

1. Suppose $\vec{u}$, $\vec{v}$, and $\vec{w}$ are **any real vector in $\mathbb{R}^n$** in the form:
   $\vec{u}=(u_1,u_2,\dots,u_n)$ and $\vec{v}=(v_1,v_2,\dots,v_n)$ and $\vec{w}=(w_1,w_2,\dots,w_n)$
2. By the **definition of vector addition**:
   $\vec{v}+\vec{w}=(v_1+w_1,v_2+w_2,\dots,v_n+w_n)$
3. By the **definition of vector addition**:
   $\vec{u}+(\vec{v}+\vec{w})=\big(u_1+(v_1+w_1),u_2+(v_2+w_2),\dots,u_n+(v_n+w_n)\big)$
4. By the **associative property of real number addition**:
   $\big(u_1+(v_1+w_1),u_2+(v_2+w_2),\dots,u_n+(v_n+w_n)\big)$
   $=\big((u_1+v_1)+w_1,(u_2+v_2)+w_2,\dots,(u_n+v_n)+w_n\big)$
5. By the **definition of vector addition** and the **definition of $\vec{u}$ and $\vec{v}$**:
   $(\vec{u}+\vec{v})+\vec{w}=\big((u_1+v_1)+w_1,(u_2+v_2)+w_2,\dots,(u_n+v_n)+w_n\big)$ **(QED)**

Proof of the Associative Property for Scalar Multiplication

1. Suppose c and d are **any real number** and $\vec{u}$ is **any real vector in $\mathbb{R}^n$** in the form:
   $\vec{u}=(u_1,u_2,\dots,u_n)$
2. By the **definition of scalar multiplication**:
   $d\vec{u}=(du_1,du_2,\dots,du_n)$
3. By the **definition of vector addition**:
   $c(d\vec{u})=\big(c(du_1),c(du_2),\dots,c(du_n)\big)$
4. By the **associative property of real number multiplication**:
   $\big(c(du_1),c(du_2),\dots,c(du_n)\big)=\big((cd)u_1,(cd)u_2,\dots,(cd)u_n\big)$
5. By the **definition of scalar multiplication**:
   $\big((cd)u_1,(cd)u_2,\dots,(cd)u_n\big)=(cd)\vec{u}$ **(QED)**

Proof of the Distributive Law for Vector Addition

1. Suppose c is **any real number** and $\vec{u}$ and $\vec{v}$ are **any real vector in $\mathbb{R}^n$** in the form:
   $\vec{u}=(u_1,u_2,\dots,u_n)$ and $\vec{v}=(v_1,v_2,\dots,v_n)$
2. By the **definition of vector addition**:
   $\vec{u}+\vec{v}=(u_1+v_1,u_2+v_2,\dots,u_n+v_n)$
3. By the **definition of scalar multiplication**:
   $c(\vec{u}+\vec{v})=\big(c(u_1+v_1),c(u_2+v_2),\dots,c(u_n+v_n)\big)$
4. By the **distributive law over real number addition**:
   $\big(c(u_1+v_1),c(u_2+v_2),\dots,c(u_n+v_n)\big)=(cu_1+cv_1,cu_2+cv_2,\dots,cu_n+cv_n)$
5. By the **defintition of vector addition**:
   $(cu_1+cv_1,cu_2+cv_2,\dots,cu_n+cv_n)=c(u_1,u_2,\dots u_n)+c(v_1,v_2,\dots,v_n)$
6. By the **definition of scalar multiplication**, and the **definition of $\vec{u}$ and $\vec{v}$**:
   $c\vec{u}+c\vec{v}=(cu_1+cv_1,cu_2+cv_2,\dots,cu_n+cv_n)$ **(QED)**

Proof of the Distributive Law for Scalar Multiplication

1. Suppose c and d are **any real number** and $\vec{u}$ is **any real vector in $\mathbb{R}^n$** in the form:
   $\vec{u}=(u_1,u_2,\dots,u_n)$
2. By the **definition of scalar multiplication**:
   $(c+d)\vec{u}=\big((c+d)u_1,(c+d)u_2,\dots,(c+d)u_n\big)$
3. By the **distributive law over real number addition**:
   $\big((c+d)u_1,(c+d)u_2,\dots,(c+d)u_n\big)=(cu_1+du_1,cu_2+du_2,\dots,cu_n+du_n)$
4. By the **definition of vector addition**:
   $(cu_1+du_1,cu_2+du_2,\dots,cu_n+du_n)=(cu_1,cu_2,\dots,cu_n)+(du_1,du_2,\dots,du_n)$
5. By the **definition of scalar multiplication** and the **definition of $\vec{u}$**:
   $(cu_1,cu_2,\dots,cu_n)+(du_1,du_2,\dots,du_n)=c\vec{u}+d\vec{u}$ **(QED)**

Proof of the Identity Property for Vector Addition	**Proof of the Identity Property for Scalar Multiplication**
1. Suppose $\vec{0}$ is the **zero vector of length n** and $\vec{u}$ is **any real vector in $\mathbb{R}^n$** in the form: $\vec{u} = (u_1, u_2, \dots, u_n)$ 2. By the **definition of vector addition**: $\vec{u} + \vec{0} = (u_1 + 0, u_2 + 0, \dots, u_n + 0)$ 3. By the **identity property of real number addition**: $(u_1 + 0, u_2 + 0, \dots, u_n + 0) = (u_1, u_2, \dots, u_n)$ 4. By the **definition of $\vec{u}$**: $\vec{u} = (u_1, u_2, \dots, u_n)$ **(QED)**	1. Suppose $\vec{u}$ is **any real vector in $\mathbb{R}^n$** in the form: $\vec{u} = (u_1, u_2, \dots, u_n)$ 2. By the **definition of scalar multiplication**: $1\vec{u} = (1u_1, 1u_2, \dots, 1u_n)$ 3. By the **identity property of real number multiplication**: $(1u_1, 1u_2, \dots, 1u_n) = (u_1, u_2, \dots, u_n)$ 4. By the **definition of $\vec{u}$**: $\vec{u} = (u_1, u_2, \dots, u_n)$ **(QED)**