Chapter 2-2 - The Inverse of a Matrix

1) Find the inverse of matrix

$$\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$$

$\det A = 8\cdot 4 - 5\cdot 6 = 2$

$$A^{-1} = \begin{bmatrix} \frac{4}{2} & -\frac{6}{2} \\ -\frac{5}{2} & \frac{8}{2} \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -\frac{5}{2} & 4 \end{bmatrix}$$

3) Find the inverse of matrix

$$\begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}$$

$\det A = (8)\cdot(-5) - (-7)(5) = -5$

$$A^{-1} = \begin{bmatrix} \frac{-5}{-5} & (-1)\left(\frac{5}{-5}\right) \\ (-1)\left(\frac{-7}{-5}\right) & \left(\frac{8}{-5}\right) \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 1 \\ \frac{-7}{5} & \frac{-8}{5} \end{bmatrix}$$

5) Use the inverse of problem #1 to solve the system

$8x_1 + 6x_2 = 2$

$5x_1 + 4x_2 = -1$

$$\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$$

$$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$(A^{-1})(A)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2 & -3 \\ -\frac{5}{2} & 4 \end{bmatrix} \text{ (from problem 1)}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -\frac{5}{2} & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2\cdot 2 + (-3)(-1) \\ -\frac{5}{2}\cdot 2 + (4)(-1) \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ -9 \end{bmatrix}$$

7) Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}$, $b_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $b_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, and $b_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

a) $$A^{-1} = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{bmatrix}$$

$$\vec{x}_1 = A^{-1}\vec{b}_1 = \begin{bmatrix} (6)(-1) + (3)(-1) \\ (-\frac{5}{2})(-1) + (3)(\frac{1}{2}) \end{bmatrix} = \boxed{\begin{bmatrix} -9 \\ 4 \end{bmatrix}}$$

$$\vec{x}_2 = A^{-1}\vec{b}_2 = \begin{bmatrix} 6\cdot 1 + (-1)(-5) \\ -\frac{5}{2}\cdot 1 - \frac{5}{2} \end{bmatrix} = \boxed{\begin{bmatrix} 11 \\ -5 \end{bmatrix}}$$

$$\vec{x}_3 = A^{-1}\vec{b}_3 = \begin{bmatrix} 6\cdot 2 + (-1)(6) \\ -\frac{5}{2}\cdot 2 + \frac{1}{2}\cdot 6 \end{bmatrix} = \boxed{\begin{bmatrix} 6 \\ -2 \end{bmatrix}}$$

$$\vec{x}_4 = A^{-1}\vec{b}_4 = \begin{bmatrix} 6\cdot 3 - 5 \\ -\frac{5}{2}\cdot 3 + \frac{5}{2} \end{bmatrix} = \boxed{\begin{bmatrix} 13 \\ -5 \end{bmatrix}}$$

b) Row reduce the augmented matrix $[A\ \vec{b}_1\ \vec{b}_2\ \vec{b}_3\ \vec{b}_4]$ to solve as above:

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 5 & 12 & 3 & -5 & 6 & 5 \end{bmatrix}$$

$R_2 = R_2 - 5R_1$

$\Downarrow$

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 0 & 2 & 8 & -10 & -4 & -10 \end{bmatrix}$$

$R_2' = \frac{1}{2}R_2$

$\Downarrow$

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 0 & 1 & 4 & -5 & -2 & -5 \end{bmatrix}$$

$R_1' = R_1 - 2R_2$

$\Downarrow$

$$\begin{bmatrix} 1 & 0 & -9 & 11 & 6 & 13 \\ 0 & 1 & 4 & -5 & -2 & -5 \end{bmatrix}$$

9)

a) **True** – By definition an inverse matrix B for a matrix A is

$BA = AB = I$

b) **False** – The inverse of the matrix product is

$(AB)^{-1} = B^{-1}A^{-1}$

c) **False** – For a 2x2 matrix the determinant must not equal 0

$\det(A) = ad - bc$

d) **True** – There must be a pivot in every row/column.

e) **True** – The elementary matrix can be converted back to the identity matrix.

10)

a) **False** – The order is reversed meaning:

$(AB)^{-1} = B^{-1}A^{-1}$

b) **True** – By definition

$AA^{-1} = A^{-1}A = I_n$

c) **True** – The determinant of a 2x2 matrix must not equal zero

$\det A = ad - bc$ so $ad \neq bc$

d) **True** – The process of reducing A to I_n is how the inverse is found.

e) **False** – It reduces the identity matrix into the inverse matrix

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11) Let A be an invertible $n \times n$ matrix and let B be an $n \times p$ matrix. Show that the equation $AX=B$ has a unique solution $A^{-1}B$

$$AX=B$$

$$A^{-1}(AX)=A^{-1}B$$

$$(A^{-1}A)X=A^{-1}B$$

$$IX=A^{-1}B$$

$$X=A^{-1}B$$

13) Suppose $AB=AC$ where B and C are $n \times p$ matrices, and A is invertible. Show $B=C$. Is this true in general when A is not invertible?

$$AB=AC$$

$$A^{-1}(AB)=A^{-1}(AC)$$

$$(A^{-1}A)B=(A^{-1}A)C$$

$$B=C$$

$\boxed{\text{No this is not true in general}}$

15) Suppose A, B, and C are invertible $n \times n$ matrices. Show that ABC is also invertible by producing a matrix D such that $(ABC)D=I$ and $D(ABC)=I$

$(ABC)D=I$	$D(ABC)=I$
$(A^{-1}A)(BC)D=A^{-1}I$	$DAB(CC^{-1})=IC^{-1}$
$(B^{-1}B)CD=B^{-1}A^{-1}$	$DABB^{-1}=C^{-1}B^{-1}$
$C^{-1}CD=C^{-1}B^{-1}A^{-1}$	$DAA^{-1}=C^{-1}B^{-1}A^{-1}$
$D=C^{-1}B^{-1}A^{-1}$	$D=C^{-1}B^{-1}A^{-1}$

17) Solve the equation $AB=BC$ for A assuming A, B, and C are square and B is invertible

$$AB=BC$$

$$ABB^{-1}=BCB^{-1}$$

$$\boxed{A=BCB^{-1}}$$

19) If A, B, and C are $n \times n$ invertible matrices. Does the equation $C^{-1}(A+X)B^{-1}=I_n$ have a solution for X? If so, find it.

$$C^{-1}(A+X)B^{-1}=I_n$$

$$CC^{-1}(A+X)B^{-1}=CI_n$$

$$(A+X)B^{-1}B=CB$$

$$\boxed{X=CB-A}$$

Must back substitute

$$C^{-1}(\cancel{A}+CB\cancel{-A})B^{-1}=I_n$$

$$C^{-1}CBB^{-1}=I_n$$

$$\boxed{I_n=I_n}$$

21) Explain why the columns of an $n \times n$ matrix are linearly independent when A is invertible.

For an $n \times n$ square matrix to be invertible, it must row reduce to the identity matrix.

Hence the matrix has a pivot in every column making no free variables.

Hence it has only the trivial solution.

23) Suppose A is an $n \times n$ matrix and the equation $A\vec{x}=\vec{0}$ has only the trivial solution

a) Explain why A has n pivot columns

To have only the trivial solution, there must be a pivot in every column.

Since A is $n \times n$, there must be n pivots.

b) Explain why A is row reducible to I_n

If the reduced echelon matrix has a pivot in every column, it has a pivot in every row since A is square.

Hence, its reduced echelon form is the identity matrix.

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29) Find the inverse of the matrix:

$$\begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix}$$

$R_2' = R_2 - 4R_1$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & -4 & 1 \end{bmatrix}$$

$R_2' = (-1)R_2$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 4 & -1 \end{bmatrix}$$

$R_1' = R_1 - 2R_2$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 0 & -7 & 2 \\ 0 & 1 & 4 & -1 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix}}$$

31) Find the inverse of the matrix

$$\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{bmatrix}$$

$R_3' = R_3 - 2R_1$

$R_2' = R_2 + 3R_1$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{bmatrix}$$

$R_3' = R_3 + 3R_2$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{bmatrix}$$

$R_1' = R_1 + R_3$

$R_2' = R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{bmatrix}$$

$R_3' = \frac{1}{2} R_3$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 4 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 4 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}}$$

35) Let $A = \begin{bmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}$. Find the third column of A^{-1} without computing the other columns

$$A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Augmented Matrix

$$\begin{bmatrix} -2 & -7 & -9 & 0 \\ 2 & 5 & 6 & 0 \\ 1 & 3 & 4 & 1 \end{bmatrix}$$

$\Downarrow$ Swap R_1 & R_3

$$\begin{bmatrix} 1 & 3 & 4 & 1 \\ 2 & 5 & 6 & 0 \\ -2 & -7 & -9 & 0 \end{bmatrix}$$

$R_2' = R_2 - R_1$

$R_3' = R_3 + 2R_1$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 3 & 4 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

$R_3' = R_3 - R_2$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 3 & 4 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$R_1' = R_1 - 4R_3$

$R_2' = R_2 + 2R_1$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 3 & 0 & -15 \\ 0 & -1 & 0 & 6 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$R_1 = R_1 + 3R_2$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 6 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$R_2' = (-1)R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\boxed{\text{Third column of } A^{-1} = \begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix}}$$