Chapter 3-2 - Properties of Determinants

1) State the property shown.

$$\begin{vmatrix} 0 & 5 & -2 \\ 1 & -3 & 6 \\ 4 & -1 & 8 \end{vmatrix} = -\begin{vmatrix} 1 & -3 & 6 \\ 0 & 5 & -2 \\ 4 & -1 & 8 \end{vmatrix}$$

Interchanging two rows inverts the sign.

3) State the property shown.

$$\begin{vmatrix} 3 & -6 & 9 \\ 3 & 5 & -5 \\ 1 & 3 & 3 \end{vmatrix} = 3\begin{vmatrix} 1 & -2 & 3 \\ 3 & 5 & -5 \\ 1 & 3 & 3 \end{vmatrix}$$

Scaling a row means scaling the determinant by the same amount.

5) Find the determinant via row reduction.

$$A = \begin{bmatrix} 1 & 5 & -4 \\ -1 & -4 & 5 \\ -2 & -8 & 7 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 1 & 5 & -4 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \end{vmatrix} \quad \left(\begin{matrix} R_2' = R_2 + R_1 \\ R_3' = R_3 + 2R_1 \end{matrix} \right)$$

$$\det A = \begin{vmatrix} 1 & 5 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{vmatrix} \quad (R_3' = R_3 - 2R_1)$$

$$\det A = 1 \cdot 1 \cdot -3 = \boxed{-3}$$

7) Find the determinant via row reduction.

$$A = \begin{bmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 0 & -4 & 2 & -5 \end{vmatrix}$$

Last two rows equal so $\det A = 0$

9) Find the determinant via row reduction.

$$A = \begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 0 & 5 & 3 \\ 3 & -3 & -2 & 3 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 7 & 3 \end{vmatrix}$$

$R_3' = R_2 + R_3$

$$\det A = \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 7 & 7 \\ 0 & 0 & 0 & 3 \end{vmatrix}$$

$R_4' = R_4 - R_3$

$$\det A = \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 7 & 7 \\ 0 & 0 & 0 & -4 \end{vmatrix}$$

$$\det A = 1 \cdot 1 \cdot 7 \cdot -4 = \boxed{-28}$$

11) Combine row reduction and cofactor expansion to find the determinant.

$$\begin{vmatrix} 3 & 4 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 3 \\ 6 & 8 & -4 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 & -3 & -1 \\ 0 & -4 & 4 & -2 \\ 0 & 8 & -10 & 1 \\ 0 & 0 & 2 & 1 \end{vmatrix}$$

$$= 3\begin{vmatrix} -4 & 4 & -2 \\ 8 & -10 & 1 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= 3\left((-4)(-1)^2(-10-2) + (8)(-1)^3(4-(-4)) \right)$$

$$= 3\left((-4)(-12) + (-8)(8) \right)$$

$$= 3(48-64) = 3(-16) = \boxed{-48}$$

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13) Find the determinant via row reduction and cofactor expansion.

$$\begin{vmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 6 & 2 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix}$$

Use row reduction to create zeros in the fourth column

$$= \begin{vmatrix} 2 & 5 & 4 & 1 \\ 0 & -3 & -2 & 0 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix}$$

$$= 1(-1)^5 \begin{vmatrix} 0 & -3 & -2 \\ 6 & -2 & -4 \\ -6 & 7 & 7 \end{vmatrix}$$

$$= (-1)\left(6(-1)^3(-21+14) + (-6)(-1)^4(-12-4)\right)$$

$$= (-1)(-6)(1) = \boxed{6}$$

15) Find the determinant of matrix given

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ 3g & 3h & 3i \end{vmatrix} = 3 \cdot 7 = 21$$

17) Find the determinant of the matrix given

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$$

$$\begin{bmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{bmatrix} = 7$$

19) Find the determinant of the matrix given

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$$

$$\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix} = 2 \cdot 7 = 14$$

21) Use determinants to determine if the matrix is invertible.

$$\begin{bmatrix} 2 & 6 & 0 \\ 1 & 3 & 2 \\ 3 & 9 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 6 & 0 \\ 1 & 3 & 2 \\ 2 & 6 & 0 \end{bmatrix}$$

$$\Delta = (2)(-1)^5(12-12) = 0$$ Not invertible

23) Use determinants to determine if the matrix is invertible

$$\begin{bmatrix} 2 & 0 & 0 & 6 \\ 1 & -7 & -5 & 0 \\ 3 & 8 & 6 & 0 \\ 0 & 7 & 5 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 14 & 10 & 6 \\ 1 & -7 & -5 & 0 \\ 0 & 29 & 21 & 0 \\ 0 & 7 & 5 & 4 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 0 & 0 & 6 \\ 1 & -7 & -5 & 0 \\ 3 & 8 & 6 & 0 \\ 0 & 7 & 5 & 4 \end{vmatrix} = (1)(-1)^3 \begin{vmatrix} 14 & 10 & 6 \\ 29 & 21 & 0 \\ 7 & 5 & 4 \end{vmatrix}$$

$$= (-1)\left(6(-1)^4(29 \cdot 5 - 7 \cdot 21) + 4(-1)^6(14 \cdot 21 - 290)\right)$$

$$= -1\left(6(-2) + 4(4)\right)$$

$$= -4$$

Invertible

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25) Use determinants to decide if the set of vectors is linearly independent.

$$\begin{bmatrix}7\\-4\\-6\end{bmatrix} \begin{bmatrix}-8\\5\\7\end{bmatrix} \begin{bmatrix}7\\0\\-5\end{bmatrix}$$

$$\begin{vmatrix}7 & -8 & 7\\ -4 & 5 & 0\\ -6 & 7 & -5\end{vmatrix}$$

$= (7)(-1)^4(-28+30)$

$+(-5)(-1)^6(35-32)$

$= 7(2)+(-5)(3)$

$= 14-15 = -1$

Since $\det A = -1$ they are linearly dependent

27)

a) True - Only swap and scale affect the determinant

b) False - Any scaling affects it as well which could result in other echelon forms.

c) True - From the invertible matrix theorem

d) False - True for products

28)

a) False - It is the inverse of the old determinant.

b) False - Only true if A is triangular.

c) False - This need not be true in all cases.

d) False - See problem 31

29) Compute $\det B^4$ where:

$$B = \begin{vmatrix}1 & 0 & 1\\ 1 & 1 & 2\\ 1 & 2 & 1\end{vmatrix}$$

$$B \sim \begin{vmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 2 & 0\end{vmatrix}$$

$\det B = 0$

$\det B^4 = (\det B)^4$

$= 0^4 = 0$

31) Show that if A is invertible then $\det A^{-1} = \frac{1}{\det A}$

Given $\det I = 1$

By definition of invertible matrices

$AA^{-1} = I \Rightarrow \det(AA^{-1}) = 1$

By theorem 3-6

$\det(AB) = \det A \cdot \det B$

$(\det A)(\det A^{-1}) = 1$

$\therefore \det A^{-1} = \frac{1}{\det A}$

33) Show that AB may not equal BA that it is always true that $\det BA = \det AB$

By theorem 3-6

$\det AB = (\det A)(\det B)$

By commutative property of real number multiplication:

$\det(AB) = (\det B)(\det A)$

By theorem 3-6

$\det(AB) = \det(BA)$

35) Let U be a square matrix such that $U^T U = I$. Show that $\det U = 1$

By definition of determinant and the identity matrix:

$\det I = 1$

Given $U^T U = I$

$\det[U^T U] = \det I = 1$

By theorem 3-6

$(\det U^T)(\det U) = 1$

By theorem 3-5

$(\det U)(\det U) = 1$

By real number exponent definition

$(\det U)^2 = 1$

By definition of square root

$\det U = \pm 1$

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39) Let A and B be 3×3 matrices, with $\det A = -3$ and $\det B = 4$. Compute the following

a) $\det AB = (-3)(4)\ -12$

b) $\det 5A = 5^3 \cdot (-3) = -375$

c) $\det B^T = \det B = 4$

d) $\det A^{-1} = \frac{1}{\det A} = -\frac{1}{3}$

e) $\det A^3 = (\det A)^3 = -27$