Chapter 5-3 – Diagonalization

1) Let $A = PDP^{-1}$. Compute A^4.

$$P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}, \quad D^4 = \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^4 = PD^4P^{-1}$$

$$A^4 = \begin{bmatrix} 80 & 7 \\ 32 & 3 \end{bmatrix} \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$$

$$\boxed{A^4 = \begin{bmatrix} 226 & -525 \\ 90 & -209 \end{bmatrix}}$$

3) Use the factorization $A = PDP^{-1}$ to find A^k for an arbitrary k.

$$\begin{bmatrix} a & 0 \\ 3(a-b) & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$A^k = PD^kP^{-1}$$

$$A^k = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^k & 0 \\ 3a^k & b^k \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$\boxed{A^k = \begin{bmatrix} a^k & 0 \\ 3a^k - 3b^k & b^k \end{bmatrix}}$$

Using the factorization $A = PDP^{-1}$, find the eigenvalues of A and a basis for each eigenspace.

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & -\frac{3}{4} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

$$\boxed{\lambda = 5, \ B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}}$$

$$\boxed{\lambda = 1, \ B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right\}}$$

7) Diagonalize the matrix if possible.

$$\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$(1-\lambda)(-1-\lambda) = 0$$

$$\lambda = -1, \quad \lambda = 1$$

$$(A + I)\vec{x} = \vec{0}$$

$$\begin{bmatrix} 1 & 0 \\ 6 & 0 \end{bmatrix} \vec{x} = \vec{0}$$

$$\vec{x} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$(A - I)\vec{x} = \vec{0}$$

$$\begin{bmatrix} 0 & 0 \\ 6 & -2 \end{bmatrix} \vec{x} = \vec{0}$$

$$\vec{x} = x_2 \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

$$\boxed{P = \begin{bmatrix} \frac{1}{3} & 0 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$$

9) Diagonalize the matrix if possible.

$$\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$(3-\lambda)(5-\lambda) + 1 = 0$$

$$\lambda^2 - 8\lambda + 16 = 0$$

$$(\lambda - 4)^2 = 0$$

$$\lambda = 4$$

$$(A - 4I)\vec{x} = 0$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \vec{x} = 0$$

$$\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}.$$

Not diagonalizable by theorem 5-7

Chapter 5-3 - Diagonalization

11) Diagonalize the matrix if possible.

$$\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$$

Given $\lambda = 1, 2, 3$

$\lambda = 1$

$$\begin{bmatrix} -2 & 4 & -2 \\ -3 & 3 & 0 \\ -3 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} -2 & 4 & -2 \\ 0 & -3 & 3 \\ 0 & -5 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda = 2$

$$\begin{bmatrix} -3 & 4 & -2 \\ -3 & 2 & 0 \\ -3 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} -3 & 4 & -2 \\ 0 & -2 & 2 \\ 0 & -3 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} -3 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x} = x_3 \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$$

$\lambda = 3$

$$\begin{bmatrix} -4 & 4 & -2 \\ -3 & 1 & 0 \\ -3 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -2 & 1 \\ -3 & 1 & 0 \\ -3 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -2 & 1 \\ 0 & -2 & 1.5 \\ 0 & 2 & 1.5 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -.5 \\ 0 & -2 & 1.5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 4 & 0 & -1 \\ 0 & 4 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x} = x_3 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$\boxed{P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}}$$

13) Diagonalize the matrix if possible.

$$\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$$

Given $\lambda = 5, 1$

$\lambda = 5$

$$\begin{bmatrix} -3 & 2 & -1 \\ 1 & -2 & -1 \\ -1 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ -3 & 2 & -1 \\ -1 & -2 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & -4 & -4 \\ 0 & -4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda = 1$

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ -1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{P = \begin{bmatrix} -1 & -2 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \; D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}$$

15) Diagonalize the matrix if possible.

$$\begin{bmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{bmatrix}$$

Given $\lambda = 3, 1$

$\lambda = 3$

$$\begin{bmatrix} 4 & 4 & 16 \\ 2 & 2 & 8 \\ -2 & -2 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 1$

$$\begin{bmatrix} 6 & 4 & 16 \\ 2 & 4 & 8 \\ -2 & -2 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 8 \\ 6 & 4 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 2 \\ 0 & -2 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

$$\boxed{P = \begin{bmatrix} -1 & -4 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Chapter 5-3 – Diagonalization

17) Diagonalize the matrix if possible.

$$\begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Since triangular, $\lambda = 5, 4$

$\lambda = 4$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda = 5$

$$\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Not diagonalizable

Sum of bases is $2 < n = 3$

19) Diagonalize the matrix if possible.

$$\begin{bmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Since triangular, then $\lambda = 5, 3, 2$

$\lambda = 5$

$$\begin{bmatrix} 0 & -3 & 0 & 9 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda = 3$

$$\begin{bmatrix} 2 & -3 & 0 & 9 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & 0 & 9 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\vec{x} = x_2 \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda = 2$

$$\begin{bmatrix} 3 & -3 & 0 & 9 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{P = \begin{bmatrix} 1 & 3 & -1 & -1 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}}$$

21)

a) False – D needs to be a diagonal matrix.

b) True – This is needed to construct P.

c) False – A square $n \times n$ matrix always has n eigenvalues, counting multiplicities.

d) False – A matrix with 0 as an eigenvalue can be diagonalizable but is not invertible.

22)

a) False – It is diagonalizable if it has n eigenvalues.

b) False – n distinct eigenvalues is a sufficient, not necessary, condition to be diagonalizable.

c) False – P has to be invertible

d) False – A matrix can be invertible but not diagonalizable (see problem 17).