Chapter 2-1 - Matrix Operations

1) Given:

$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}$

a) $-2A$

$$\boxed{\begin{bmatrix} -4 & 0 & 2 \\ -8 & 10 & -4 \end{bmatrix}}$$

b) $B - 2A$

$$\boxed{\begin{bmatrix} 3 & -5 & 3 \\ -7 & 6 & -7 \end{bmatrix}}$$

c) AC

UNDEFINED since number of columns in a does not match number of rows in C.

) CD

$$\begin{bmatrix} 1\cdot 3+(2)(-1) & 1\cdot 5+2\cdot 4 \\ -2\cdot 3-1\cdot 1 & -2\cdot 5+1\cdot 4 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 1 & 13 \\ -7 & -6 \end{bmatrix}}$$

3) Let $A = \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}$,

a) Compute $3I_2 - A$

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} -1 & 1 \\ -5 & 5 \end{bmatrix}}$$

b) Compute $(3I_2)A$

$$= 3(I_2 A)$$

$$= 3A$$

$$\boxed{= \begin{bmatrix} 12 & -3 \\ 15 & -6 \end{bmatrix}}$$

5) Given $A = \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$

a) Calculate AB via Ab_1 and Ab_2

$$\begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -3-4 \\ 15-8 \\ 6+6 \end{bmatrix} = \begin{bmatrix} -7 \\ 7 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+2 \\ -10+4 \\ -4-3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ -7 \end{bmatrix}$$

$$\boxed{AB = \begin{bmatrix} -7 & 4 \\ 7 & -6 \\ 12 & -7 \end{bmatrix}}$$

b) Use the row product rule.

$$\begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1\cdot 3+2\cdot(-2) & (-1)(-2)+2\cdot 1 \\ 5\cdot 3+(4)(-2) & (5)(-2)+(4)(1) \\ (2)(3)+(-3)(-2) & (-2)(2)+(-3)(1) \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} -7 & 4 \\ 7 & -6 \\ 12 & -7 \end{bmatrix}}$$

7) If matrix A is 5x3 and the product AB is 5x7, what is the size of B?

$$(5 \times 3)\ (m \times n) \Rightarrow 5 \times 7$$

$m = 3$

$n = 7$

B is size 3x7

9) What values of k (if any) will make $AB = BA$ (i.e. commute)?

$A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}$

$$AB = \begin{bmatrix} 23 & -10+5K \\ -9 & 15+K \end{bmatrix}$$

$$BA = \begin{bmatrix} 23 & 15 \\ 6-3K & 15+K \end{bmatrix}$$

$$-10+5K = 15$$

$$5K = 25$$

$$\boxed{K = 5}$$

$$6-3K = -9$$

$$-3K = -15$$

$$\boxed{K = 5}$$

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11) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

a) Find AD and DA

$$AD = \begin{bmatrix} 2 & 3 & 5 \\ 2 & 6 & 15 \\ 2 & 12 & 25 \end{bmatrix}$$

$$DA = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 6 & 9 \\ 5 & 20 & 25 \end{bmatrix}$$

b) Describe how the rows or columns of A change when D is multiplied on the right and on the left.

When D is on the right the values in column i of A are scaled by the diagonal entry d_{ii}

when D is on the left the values in row i are the values are scaled by the diagonal entry d_{ii}

c) Find a 3x3 matrix B (not the identity or zero matrix) such that AB=BA

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 4 & 6 \\ 2 & 8 & 10 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 4 & 6 \\ 2 & 8 & 10 \end{bmatrix}$$

13) Let $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_p$ be vectors in $\mathbb{R}^n$, and let Q be an mxn matrix. Write $[Q\vec{r}_1 \dots Q\vec{r}_p]$ as a product of two matrices (neither of which is an identity matrix).

$R = [\vec{r}_1 \ \vec{r}_2 \dots \vec{r}_p]$ (size $n \times p$)

QR is size $m \times p$

$QR = [Q\vec{r}_1 \ Q\vec{r}_2 \dots Q\vec{r}_p]$ by definition of matrix multiplication.

15)

a) False - The product of $AB = [A\vec{b}_1 \ A\vec{b}_2]$

b) False - Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

c) True - By the left distributive law.

d) True

e) False - The transpose of a product of matrices equals the product of the transposes in reverse order.

16)

a) False - AB is $[A\vec{b}_1 \ A\vec{b}_2 \ A\vec{b}_3]$

b) True

c) False - $(AB)C = A(BC)$ - Note no terms swapped.

d) False - $(AB)^T = B^T A^T$ - note reverse order

e) True

$(A+B)^T = A^T + B^T$

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17) If $A=\begin{bmatrix}1&-2\\-2&5\end{bmatrix}$ and $AB=\begin{bmatrix}-1&2&-1\\6&-9&3\end{bmatrix}$, determine the first and second columns of B.

$AB =$

$(2\times2)(2\times3) = (2\ 3)$

$1\cdot a_{11} + (-2)a_{21} = -1$

$(-2)a_{11} + 5a_{21} = 6$

$$\begin{bmatrix}1&-2&-1\\-2&5&6\end{bmatrix}$$

$R_2' = R_2 + 2R_1$

$$\Downarrow$$

$$\begin{bmatrix}1&-2&-1\\0&1&4\end{bmatrix}$$

$R_1' = R_1 + 2R_2$

$$\Downarrow$$

$$\begin{bmatrix}1&0&7\\0&1&4\end{bmatrix}$$

$\boxed{\text{First column} = \begin{bmatrix}7\\4\end{bmatrix}}$

$$\begin{bmatrix}1&-2&2\\-2&5&-9\end{bmatrix}$$

$R_2' = R_2 + 2R_1$

$$\Downarrow$$

$$\begin{bmatrix}1&-2&2\\0&1&-5\end{bmatrix}$$

$R_1' = R_1 + 2R_2$

$$\Downarrow$$

$$\begin{bmatrix}1&0&-8\\0&1&-5\end{bmatrix}$$

$\boxed{\text{Second column} \begin{bmatrix}-8\\-5\end{bmatrix}}$

19) Suppose the third column of B is the sum of the first two columns. What can you say about the third column of AB

$B = [\vec b_1\ \vec b_2\ \vec b_3]$

$B = [\vec b_1\ \vec b_2\ (\vec b_1 + \vec b_2)]$

$AB = [A\vec b_1\ \ A\vec b_2\ \ A(\vec b_1 + \vec b_2)]$

$= [\vec u\ \ \vec v\ \ A\vec b_1 + A\vec b_2]$

$= [\vec u\ \ \vec v\ \ (\vec u + \vec v)]$

The values in the third column are the sum of the values in the other two columns.

21) Suppose the last column of AB is entirely zero but B itself has no column of zero. What can you say about the columns of A?

The colums of A are linearly dependent

$[\vec a_1\ \vec a_2 \dots \vec a_n]\ \vec b_p = \vec 0$

Since $\vec b$ is not the zero vector meaning it has a nontrivial solution.

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29) Prove Theorem 2(b) and 2(c) using the row-column rule

a) 2(b) $A(B+C) = AB+AC$

By definition of matrix addition

$(B+C)_{ij} = b_{ij} + c_{ij}$

By definition of matrix multiplication

$$(A(B+C))_{ij} = \sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj})$$

By distribution of addition

$$(B+C)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}$$

By the row column rule.

$A(B+C)_{ij} = (ab)_{ij} + (ac)_{ij}$

b) 2(c) $(B+C)A = BA + BC$

By definition of row-column rule

$$((B+C)A)_{ij} = \sum_{k=1}^{n} (b_{ik} + c_{ik}) \cdot a_{kj}$$

By distributive property of addition

$$((B+C)A)_{ij} = \sum_{k=1}^{n} b_{ik}a_{kj} + \sum_{k=1}^{n} c_{ik}a_{kj}$$

By the row column rule

$((B+C)A)_{ij} = (ba)_{ij} + (ca)_{ij}$

31) Show that $I_m A = A$ when A is an $m \times n$ matrix. You can assume $I_m\vec{x} = \vec{x}$ for all $\vec{x}$ in $\mathbb{R}^m$

By definition of matrix multiplication

$I_m A = [I_m\vec{a}_1 \quad I_m\vec{a}_2 \quad \dots \quad I_m\vec{a}_n]$

By assumption given in problem

$= [\vec{a}_1 \quad \vec{a}_2 \quad \dots \quad \vec{a}_n]$

By matrix definition of A

$= A$