# Chapter 4-2 - Null Spaces, Column Spaces, and Linear Transformations

**1)** Determine if $\vec{w} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$ is in Nul A where

$$A = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}$$

$$A\vec{w} = \begin{bmatrix} 3-15+12 \\ 6-6+0 \\ -8+12-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Yes since $A\vec{w} = \vec{0}$

**3)** Find an explicit description of Nullspace of A where:

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$$

$R_1' = R_1 - 3R_2$

$$A \sim \begin{bmatrix} 1 & 0 & -7 & 6 \\ 0 & 1 & 4 & -2 \end{bmatrix}$$

$$\vec{x} = \begin{cases} x_1 = 7x_3 - 6x_4 \\ x_2 = -4x_3 + 2x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \end{cases}$$

$$\text{Nul}A = a\begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$\forall a, b \in \mathbb{R}$

$$\boxed{\text{Nul}A = \text{span}\left\{\begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix}\right\}}$$

**5)** Find an explicit description of A of the null space of A as a span of vectors given

$$A = \begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\vec{x} = \begin{cases} x_1 = 2x_2 - 4x_4 \\ x_2 \text{ is free} \\ x_3 = 9x_4 \\ x_4 \text{ is free} \\ x_5 = 0 \end{cases}$$

$$\boxed{\text{Nul}A = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix}\right\}}$$

**7)** Determine if the set is a vector space. If not give an example to the contrary.

$$\left\{\begin{bmatrix} a \\ b \\ c \end{bmatrix} : a+b+c=0\right\}$$

Not a vector space

Since it does not contain the zero vector.

**9)** Determine if the set is a vector space. If not give an example to the contrary.

$$\left\{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{matrix} a-2b=4c \\ 2a=c+3d \end{matrix}\right\}$$

Yes it is a vector space

a) Contains the zero vector

b) Closed under vector addition

c) Closed under scalar multiplication.

$$a-2b-4c=0$$
$$2a-c-3d=0$$

**11)** Determine if the set is a vector space. If not, give an example to the contrary.

$$\left\{\begin{bmatrix} b-2d \\ 5+d \\ b+3d \\ d \end{bmatrix} : b, d \in \mathbb{R}\right\}$$

Not a vector space

as it does not contain the zero vector.

Only way to have $5+d=0$ is if $d=-5$. However that would make the fourth element nonzero.

**13)** Determine if the set is a vector space. Otherwise give a counter example.

$$\left\{\begin{bmatrix} c-6d \\ d \\ c \end{bmatrix} : d, c \in \mathbb{R}\right\}$$

Yes it is a vector space

a) Contains $\vec{0}$ when $d=0$ and $c=0$

b) Closed under addition:

$$\begin{bmatrix} c_1-6d_1 \\ d_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} c_2-6d_2 \\ d_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1+c_2-6d_1+d_2 \\ d_1+d_2 \\ c_1+c_2 \end{bmatrix} = \begin{bmatrix} c_3-6d_3 \\ d_3 \\ c_3 \end{bmatrix}$$

where $d_3 = d_1+d_2$ and $c_3 = c_1+c_2$

c) Closed under scalar multiplication

Using theorem 4-3, the set is also equal to the column space of the matrix

$$\begin{bmatrix} 1 & -6 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

making it a vector space.

**15)** F the matrix A given that

$$\text{Col}A = \left\{\begin{bmatrix} 2s+3t \\ r+s-2t \\ 4r+s \\ 3r-s-t \end{bmatrix} : r, s, t \in \mathbb{R}\right\}$$

$$\text{Col}A = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ -1 \end{bmatrix}\right\}$$

$$\boxed{A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -2 \\ 4 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}}$$

# Chapter 4-2 - Nullspaces, column spaces, and linear transformations

17) Given $A=\begin{bmatrix}2&-6\\-1&3\\-4&12\\3&-9\end{bmatrix}$

a) Find K such that Nul A is a subset of $\mathbb{R}^K$

$\boxed{K=2}$

b) Find K such that Col A is a subset of $\mathbb{R}^K$

$\boxed{K=4}$

1) Given $A=\begin{bmatrix}4&5&-2&6&0\\1&&0&1&0\end{bmatrix}$

a) Find K such that N A is a subset of $\mathbb{R}^K$

$\boxed{K=5}$

b) Find K such that Col A is a subset of $\mathbb{R}^K$

$\boxed{K=2}$

1) Given $A=\begin{bmatrix}2&-6\\-1&3\\-4&12\\3&-9\end{bmatrix}$

Find a nonzero vector in Col A and a nonzero vector in Nul A

a) $\begin{bmatrix}3\\1\end{bmatrix}\in \text{Nul}A$

b) $\begin{bmatrix}2\\-1\\-4\\3\end{bmatrix}\in \text{Col}A$

23) Let $A=\begin{bmatrix}-6&12\\-3&6\end{bmatrix}$ and $\vec{w}=\begin{bmatrix}2\\1\end{bmatrix}$.

a) Determine if $\vec{w}\in \text{Col}A$

Yes $\begin{bmatrix}-6&12\\-3&6\end{bmatrix}\begin{bmatrix}-\frac{1}{3}\\0\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$

b) Determine if $\vec{w}\in \text{Nul}A$

Yes $\begin{bmatrix}-6&12\\-3&6\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}-12+12\\-6+6\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

25)

a) True - The nullspace is all vectors $\vec{x}$ such that $A\vec{x}=\vec{0}$

b) False - For an $m\times n$ matrix the nullspace is in $\mathbb{R}^n$

c) True - Col A is all vectors $\vec{x}$ such that $x\mapsto Ax$

d) False - For Col A $=\mathbb{R}^m$, $A\vec{x}=\vec{b}$ must be true for all $\vec{b}$

e)

f) True - Col A is the set of all linear combinations of the columns of A or

$\text{Col}A=\{A\vec{x}:\vec{x}\in\mathbb{R}^n\}$

26)

a) True - All nullspaces are vector spaces

b) True - For an $m\times n$ matrix, Col A derives from $\mathbb{R}^m$

c) False - It is not the set of all solutions but all mappings.

d)

e)

f) True - Col A is:

$\text{Col}A=\{A\vec{x}:\vec{x}\in\mathbb{R}^n\}$

27) $\vec{x}=\begin{bmatrix}3\\2\\-1\end{bmatrix}$ is a solution to the linear system:

$$x_1-3x_2-3x_3=0$$
$$-2x_1+4x_2+2x_3=0$$
$$-x_1+5x_2+7x_3=0.$$

Given $\vec{x}\in \text{Nul}A$ for

$A=\begin{bmatrix}1&-3&-3\\-2&4&2\\-1&5&7\end{bmatrix}$

Since Nul A is a vector space it is closed under scalar multiplication. Hence:

$c\vec{x}\in \text{Nul}A$

if $c=10$, $c\vec{x}=\begin{bmatrix}30\\20\\-10\end{bmatrix}\in \text{Nul}A$

making $\begin{bmatrix}30\\20\\-10\end{bmatrix}$ a solution to the system.