74/A+ / 94/ A+

MATH 129A (8) **Midterm 1** **Name:** Zayd Hammoudeh

March 9, 2017 **75 pts.**

Show all your work!

1. 7 pts. Determine h and k such that the solution set of the system (i) is empty, (ii) contains a unique solution, and (iii) contains infinitely many solutions.

$$x_1 + hx_2 = 2$$
$$4x_1 + 8x_2 = k$$

Augmented Matrix

$$\begin{bmatrix} 1 & h & 2 \\ 4 & 8 & k \end{bmatrix}$$

$R_2' = R_2 - 4R_1$

$\Downarrow$

$$\begin{bmatrix} 1 & h & 2 \\ 0 & 8-4h & K-8 \end{bmatrix}$$

i) Solution set empty means last row is in form $[0\,0\,b]$ where $b \neq 0$

$8-4h=0 \quad K-8\neq 0$

$\boxed{h=2 \quad K\neq 8}$

ii) Unique solution means pivot in last row so

$8-4h \neq 0$

$\boxed{h\neq 2 \text{ for all } k}$

iii) Infinitely many solutions last row in form $[0\,0\,0]$

$8-4h=0 \quad K-8=0$

$\boxed{h=2 \quad K=8}$

7 pts

2. 10 pts. (a) Find the general solution of the following linear system:

$$2x_1 + 6x_2 - 9x_3 - 4x_4 = 0$$
$$-3x_1 - 11x_2 + 9x_3 - x_4 = 0$$
$$x_1 + 4x_2 - 2x_3 + x_4 = 0$$

10 pts

(b) Describe the solution of the given system in parametric vector form. Also, give a geometric description of the solution set.

Augmented Matrix

$$\begin{bmatrix} 2 & 6 & -9 & -4 & 0 \\ -3 & -11 & 9 & -1 & 0 \\ 1 & 4 & -2 & 1 & 0 \end{bmatrix}$$

Swap R_1 and R_3

$$\begin{bmatrix} 1 & 4 & -2 & 1 & 0 \\ -3 & -11 & 9 & -1 & 0 \\ 2 & 6 & -9 & -4 & 0 \end{bmatrix}$$

$R_2' = R_2 + 3R_1$
$R_3' = R_3 - 2R_1$

$$\begin{bmatrix} 1 & 4 & -2 & 1 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & -2 & -5 & -6 & 0 \end{bmatrix}$$

$R_3' = R_3 + 2R_2$

$$\begin{bmatrix} 1 & 4 & -2 & 1 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{bmatrix}$$

$R_2' = R_2 - 3R_3$
$R_1' = R_1 + 2R_3$

$\Downarrow$

$$\begin{bmatrix} 1 & 4 & 0 & -3 & 0 \\ 0 & 1 & 0 & 8 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{bmatrix}$$

$R_1' = R_1 - 4R_2$

$\Downarrow$

$$\begin{bmatrix} 1 & 0 & 0 & -35 & 0 \\ 0 & 1 & 0 & 8 & 0 \\ 0 & 0 & 1 & -2 & \end{bmatrix}$$

$x_1 = 35x_4$

$x_2 = -8x_4$

$x_3 = 2x_4$

x_4 is free

Parametric vector form

$$\vec{x} = \begin{bmatrix} 35x_4 \\ -8x_4 \\ 2x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 35 \\ -8 \\ 2 \\ 1 \end{bmatrix}$$

where x_4 any number in $\mathbb{R}$

Line

Solution Set is a $\boxed{\text{line}}$ through origin $(\vec{0})$ and $\begin{bmatrix} 35 \\ -8 \\ 2 \\ 1 \end{bmatrix}$

Checkwork

$35 - 4\cdot 8 - 2\cdot 2 + 1 = 35 - 32 - 4 + 1 = 0$

$-105 + 88 + 18 - 1 = 106 - 106 = 0$

$70 - 48 - 18 - 4 = 70 - 66 - 4 = 0$

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3. 8 pts. Let $B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$

(a) Can every vector in $\mathbb{R}^4$ be written as a linear combination of the columns of B?
(b) Do the columns of B span $\mathbb{R}^4$?

$\begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$

$R_3' = R_3 - R_1$
$R_4' = R_4 + 2R_1$

$\Downarrow$

$\begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & -1 & -1 & 5 \\ 0 & -2 & -2 & 3 \end{bmatrix}$

$R_3' = R_3 + R_2$
$R_4' = R_4 + 2R_2$

$\Downarrow$

$\begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 \end{bmatrix}$

Swap R_3 and R_4

$\begin{bmatrix} \boxed{1} & 3 & -2 & 2 \\ 0 & \boxed{1} & 1 & -5 \\ 0 & 0 & 0 & \boxed{-7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$ pivots

$\boxed{\text{No}}$ the columns neither span $\mathbb{R}^4$ nor can all of $\mathbb{R}^4$ be written as a linear combination of the columns of B since not a pivot in all rows. Only 3 pivots.

8 pts

4. 5 pts. Prove that $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ for each scalar c and d and any vector $\mathbf{u}$ in $\mathbb{R}^n$.

Let c, d be any real number and $\vec{u}$ be any real vector in $\mathbb{R}^n$
$\vec{u} = (u_1, u_2, \ldots, u_n)$
By definition of scalar multiplication for a vector
$(c+d)\vec{u} = ((c+d)u_1, (c+d)u_2, \ldots, (c+d)u_n)$

By distributive law for real number addition
$(c+d)\vec{u} = (cu_1 + du_1, cu_2 + du_2, \ldots, cu_n + du_n)$
By definition of vector addition
$(c+d)\vec{u} = (cu_1, cu_2, \ldots, cu_n) + (du_1, du_2, \ldots, du_n)$

By definition of $c\vec{u}$ and definition of scalar multiplication for a vector
$(c+d)\vec{u} = c\vec{u} + d\vec{u}$ QED

5 pts

5. 6 pts. Describe and compare the solution sets of $x_1 + 4x_2 - 3x_3 = 0$ and $x_1 + 4x_2 - 3x_3 = 6$.

First Linear (Homogeneous) System

$x_1 = -4x_2 + 3x_3$
x_2 is free
x_3 is free

$\vec{V}_h = \vec{x} = \begin{bmatrix} -4x_2 + 3x_3 \\ x_2 \\ x_3 \end{bmatrix}$

$\vec{V}_h = \vec{x} = x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

Call $\boxed{\vec{u} = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}}$

and $\boxed{\vec{v} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}}$

$\boxed{\vec{V}_h = \vec{x} = x_2\vec{u} + x_3\vec{v}}$

where x_2 and x_3 any number in $\mathbb{R}$

It is a plane through origin, $\vec{u}$ and $\vec{v}$.

Second (non-homogeneous) Linear System

$x_1 = 6 - 4x_2 + 3x_3$
x_2 is free
x_3 is free

$\vec{V}_{NH} = \vec{x} = \begin{bmatrix} 6 - 4x_2 + 3x_3 \\ x_2 \\ x_3 \end{bmatrix}$

$\vec{V}_{NH} = \vec{x} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ where x_2 and x_3 any number in $\mathbb{R}$

$\vec{V}_{NH} = \vec{x} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} + x_2\vec{u} + x_3\vec{v}_3 = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} + \vec{V}_h$

Second system is the plane through $\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ parallel to plane of first (homogeneous) system

6 pts

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6. 6 pts. Given $A = \begin{bmatrix} -2 & -6 \\ 7 & 21 \\ -3 & -9 \end{bmatrix}$, find one nontrivial solution of $A\mathbf{x} = \mathbf{0}$ by inspection.
[*Hint*: Think of the equation $A\mathbf{x} = \mathbf{0}$ written as a vector equation.]

$\vec{x} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

Since two vectors are scale multiples

Verification to check work

$$\begin{bmatrix} -2 & -6 \\ 7 & 21 \\ -3 & -9 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2\cdot -3 + (-6)\cdot 1 \\ 7\cdot(-3) + (21)\cdot 1 \\ (-3)(-3) + (-9)(1) \end{bmatrix} = \begin{bmatrix} 6-6 \\ -21+21 \\ 9-9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

6 pts

7. 8 pts. Determine by inspection whether the vectors are linearly independent. Justify your answer:

(a) $\begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 6 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ Dependent since more vectors (3) than elements per vector (2)

(b) $\begin{bmatrix} 1 \\ -8 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 1 \\ 12 \end{bmatrix}$ Dependent since contains the zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

7 pts

(c) $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}$ Dependent since $\begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}$ is a linear combination of the first two vector

(d) $\begin{bmatrix} -2 \\ 4 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix}$ Dependent since vector $\begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix}$ is a not Scaler multiple of the first vector $\begin{bmatrix} -2 \\ 4 \\ 6 \\ 10 \end{bmatrix}$ (i.e. -1.5 times)

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8. 7 pts. Find the value(s) of h for which the vectors are linearly dependent. Justify your answer:

$$\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ -3 \end{bmatrix}, \begin{bmatrix} 8 \\ h \\ 4 \end{bmatrix}$$

Coefficient matrix.

$$\begin{bmatrix} 2 & -6 & 8 \\ -4 & 7 & h \\ 1 & -3 & 4 \end{bmatrix}$$

Swap R_1 & R_3

$$\begin{bmatrix} 1 & -3 & 4 \\ -4 & 7 & h \\ 2 & -6 & 8 \end{bmatrix}$$

$R_2' = R_2 + 4R_1$

$R_3' = R_3 - 2R_1$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -3 & 4 \\ 0 & -5 & h+16 \\ 0 & 0 & 0 \end{bmatrix}$$

Only two pivots maximum

$$\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & 0 \end{bmatrix}$$

Linearly dependent for all h as no way to get a pivot in every column as one zero row in echelon form, leading to one free variable and a non-trivial solution.

7pts

9. 6 pts. Use a rectangular coordinate system to plot $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ and their images under the given transformation T. Describe geometrically what T does to each vector $\mathbf{x}$ in $\mathbb{R}^2$.

$$T(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\vec{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$T(\vec{u}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$T(\vec{u}) = \begin{bmatrix} 0 \cdot 5 + 0 \cdot 2 \\ 0 \cdot 5 + 1 \cdot 2 \end{bmatrix}$

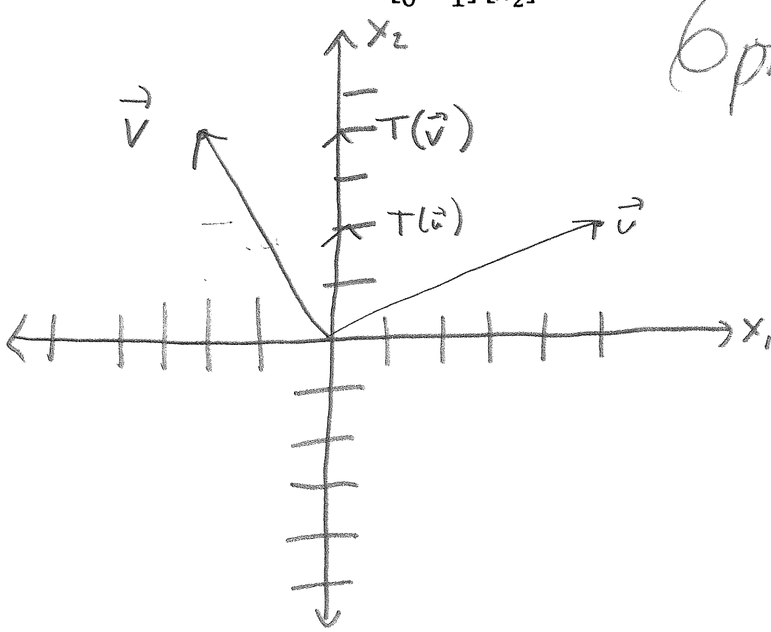
$T(\vec{u}) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$T(\vec{v}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

$T(\vec{v}) = \begin{bmatrix} 0 \cdot -2 + 0 \cdot 4 \\ 0 \cdot -2 + 1 \cdot 4 \end{bmatrix}$

$T(\vec{v}) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

$T(\vec{x}) = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$



6pts

Projection onto the x_2-axis

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10. 6 pts. Suppose that a $\boxed{\text{linear}}$ transformation T satisfies $T(\mathbf{u}_1) = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$ and $T(\mathbf{u}_2) = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$.

Find $T(3\mathbf{u}_1 - 2\mathbf{u}_2)$.

Since transformation is linear

$$T(3\vec{u}_1 - 2\vec{u}_2) = 3T(\vec{u}_1) - 2T(\vec{u}_2) = \begin{bmatrix} 9 \\ -3 \\ -6 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \\ -8 \end{bmatrix} = \boxed{\begin{bmatrix} 7 \\ -5 \\ -14 \end{bmatrix}}$$

$$3T(\vec{u}_1) = 3\begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 9 \\ -3 \\ -6 \end{bmatrix}$$

$$-2T(\vec{u}_2) = -2\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -8 \end{bmatrix}$$

6 pts

11. 6 pts. Compute the product AB in two ways: (1) by the definition, where $A\mathbf{b}_1$, $A\mathbf{b}_2$, and $A\mathbf{b}_3$ are computed separately, and (2) by the row-column rule for computing AB.

$$A = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 0 & 2 \\ 4 & -3 & 1 \end{bmatrix}$$

1) Compute columns separately

$$A\vec{b}_1 = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3\cdot(-1) + (1\cdot 4) \\ (-2)\cdot(-1) + (0\cdot 4) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A\vec{b}_2 = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 3\cdot 0 + 1\cdot(-3) \\ -2\cdot 0 + 0\cdot -3 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

$$A\vec{b}_3 = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3\cdot 2 + 1\cdot 1 \\ -2\cdot 2 + 0\cdot 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$

$$AB = \begin{bmatrix} (A\vec{b}_1) & (A\vec{b}_2) & (A\vec{b}_3) \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & -3 & 7 \\ 2 & 0 & -4 \end{bmatrix}$$

2) AB row-column rule

$$AB = \begin{bmatrix} 3\cdot(-1) + (1)(4) & (3)\cdot(0) + 1\cdot(-3) & 3\cdot 2 + 1\cdot 1 \\ (-2)\cdot(-1) + 0\cdot 4 & (-2)\cdot 0 + 0\cdot(-3) & -2\cdot 2 + 0\cdot 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} -3 + 4 & 0 - 3 & 6 + 1 \\ 2 + 0 & 0 + 0 & -4 + 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & -3 & 7 \\ 2 & 0 & -4 \end{bmatrix}$$

6 pts

Same as expected since equivalent operations