Chapter 1-5 – Solution Sets of Linear Equations

1) Determine if the system has a nontrivial solution

$$2x_1 - 5x_2 + 8x_3 = 0$$
$$-2x_1 - 7x_2 + x_3 = 0$$
$$4x + 2x_2 + 7x_3 = 0$$

Augmented Matrix

$$\begin{bmatrix} 2 & -5 & 8 & 0 \\ -2 & -7 & 1 & 0 \\ 4 & 2 & 7 & 0 \end{bmatrix}$$

$R_2' = R_2 + R_1$

$R_3' = R_3 - 2R_1$

$\Downarrow$

$$\begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 12 & -9 & 0 \end{bmatrix}$$

$R_3' = R_3 + R_2$

$\Downarrow$

$$\begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 is a free variable.

$\boxed{\text{Yes, since } x_3 \text{ is free, it has a nontrivial solution}}$

3) Determine if the system has a nontrivial solution

$$-3x_1 + 5x_2 - 7x_3 = 0$$
$$-6x_1 + 7x_2 + x_3 = 0$$

Without any row operations, it is trivial to see there there will be at least one free variable given 3 unknowns and only 2 equations for this homogeneous system.

$$\begin{bmatrix} -3 & 5 & -7 & 0 \\ -6 & 7 & 1 & 0 \end{bmatrix}$$

$R_2' = R_2 - 2R_1$

$\Downarrow$

$$\begin{bmatrix} -3 & 5 & -7 & 0 \\ 0 & -3 & 15 & 0 \end{bmatrix}$$

x_3 is free

$\boxed{\text{Yes, since } x_3 \text{ is free, it has a nontrivial solution}}$

5) Write the solution set of the given homogeneous system in parametric vector form.

$$x_1 + 3x_2 + x_3 = 0$$
$$-4x_1 - 9x_2 + 2x_3 = 0$$
$$-3x_2 - 6x_3 = 0$$

Augmented Matrix

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ -4 & -9 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix}$$

$R_2' = R_2 + 4R_1$

$\Downarrow$

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix}$$

$R_3' = R_3 + R_2$

$R_1' = R_1 - R_2$

$\Downarrow$

$$\begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2' = \frac{1}{3} R_2$

$\Downarrow$

$$\begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 5x_3$

$x_2 = -2x_3$

x_3 is free

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = \boxed{x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}}$$

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7) Describe the solution set for $A\vec{x}=\vec{0}$ in parametric vector form for:

$$A=\begin{bmatrix}1 & 3 & -3 & 7\\ 0 & 1 & -4 & 5\end{bmatrix}$$

$R_1' = R_1 - 3R_2$

$$\Downarrow$$

$$\begin{bmatrix}1 & 0 & 9 & -8\\ 0 & 1 & -4 & 5\end{bmatrix}$$

$x_1 = -9x_3 + 8x_4$

$x_2 = 4x_3 - 5x_4$

$$\vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-9x_3+8x_4\\4x_3-5x_4\\x_3\\x_4\end{bmatrix}=\boxed{x_3\begin{bmatrix}-9\\4\\1\\0\end{bmatrix}+x_4\begin{bmatrix}8\\-5\\0\\1\end{bmatrix}}$$

9) Describe all solutions in parametric vector form for $A\vec{x}=\vec{0}$ given

$$A=\begin{bmatrix}3 & -9 & 6\\ -1 & 3 & -2\end{bmatrix}$$

Swap $R_1 \Leftrightarrow R_2$

$$\begin{bmatrix}-1 & 3 & -2\\ 3 & -9 & 6\end{bmatrix}$$

$$\Downarrow$$

$R_1' = -1 \cdot R_1$

$$\Downarrow$$

$$\begin{bmatrix}1 & -3 & 2\\ 3 & -9 & 6\end{bmatrix}$$

$R_2' = R_2 - 3R_1$

$$\Downarrow$$

$$\begin{bmatrix}1 & -3 & 2\\ 0 & 0 & 0\end{bmatrix}$$

$x_1 = 3x_2 - 2x_3$

x_2 is free

x_3 is free

$$\vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}3x_2-2x_3\\x_2\\x_3\end{bmatrix}=\boxed{x_2\begin{bmatrix}3\\1\\0\end{bmatrix}+x_3\begin{bmatrix}-2\\0\\1\end{bmatrix}}$$

11) Describe all solutions of $A\vec{x}=\vec{0}$ given A is row equivalent to the matrix below:

$$\begin{bmatrix}1 & -4 & -2 & 0 & 3 & -5\\ 0 & 0 & 1 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 1 & -4\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$R_1' = R_1 + 2R_2$

$$\Downarrow$$

$$\begin{bmatrix}1 & -4 & 0 & 0 & 3 & -7\\ 0 & 0 & 1 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 1 & -4\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$R_1' = R_1 - 3R_3$

$$\Downarrow$$

$$\begin{bmatrix}1 & -4 & 0 & 0 & 0 & 5\\ 0 & 0 & 1 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 1 & -4\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$x_1 = 4x_2 - 5x_6$

x_2 is free

$x_3 = x_6$

x_4 is free

$x_5 = 4x_6$

x_6 is free

$$\vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix}=\begin{bmatrix}4x_2-5x_6\\x_2\\x_6\\x_4\\4x_6\\x_6\end{bmatrix}=\boxed{x_2\begin{bmatrix}4\\1\\0\\0\\0\\0\end{bmatrix}+x_4\begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}+x_6\begin{bmatrix}-5\\0\\1\\0\\4\\1\end{bmatrix}}$$

Chapter 1-5- Solution Sets of Linear Equations

13) Suppose the solution set of a certain system of linear equations can be described as:

$$x_1 = 5 + 4x_3$$
$$x_2 = -2 - 7x_3$$
$$x_3 \text{ is free}$$

Use vectors to describe this solution set in $\mathbb{R}^3$.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5+4x_3 \\ -2-7x_3 \\ x_3 \end{bmatrix} =$$

$$\boxed{\vec{x} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix} = \vec{p} + x_3\vec{q}}$$

Geometrically, this is the line through $\begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$ parallel to vector $\begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix}$

15) Give the solution set of the system in parametric vector form and give a geometric description:

$$x_1 + 3x_2 + x_3 = 1$$
$$-4x_1 - 9x_2 + 2x_3 \quad 1$$
$$-3x_2 - 6x_3 = -3$$

Augmented Matrix

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{bmatrix}$$

$$R_2' = R_2 + 4R_1$$
$$\Downarrow$$
$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & -3 & -6 & -3 \end{bmatrix}$$

$$R_3' = R_3 + R_2$$
$$R_1' = R_1 - R_2$$
$$\Downarrow$$
$$\begin{bmatrix} 1 & 0 & -5 & -2 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2' = \frac{1}{3}R_2$$
$$\Downarrow$$
$$\begin{bmatrix} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -2 + 5x_3$$
$$x_2 = 1 - 2x_3$$
$$x_3 \text{ is free}$$

$$\boxed{\vec{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}}$$

Geometrically, it is the line through $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ parallel to $\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$ which is a vector

17) Describe and compare the solution sets of:

$$x_1 + 9x_2 - 4x_3 = 0$$
and
$$x_1 + 9x_2 - 4x_2 = -2$$

First Solution Set

$$x_1 = -9x_2 + 4x_3$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -9x_2+4x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -9 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

Second Solution Set:

$$x_1 = -2 - 9x_2 + x_3$$

$$\vec{x} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -9 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

Both solution sets are planes with the first solution set is through the origin and vectors $\begin{bmatrix} -9 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$ (i.e. spans$\left\{\begin{bmatrix} -9 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}\right\}$)

The second solution set is parallel to the first (i.e. homogeneous) solution set and through $\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$

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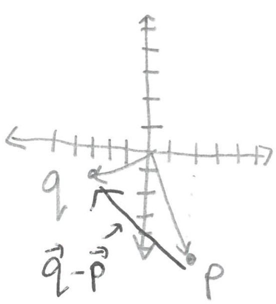
19) Find the parametric equation of the line through $\vec{a}$ and parallel to $\vec{b}$

$\vec{a} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$

$$\boxed{\begin{gathered} \vec{x} = \vec{a} + x\vec{b} \\ \vec{x} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} + x\begin{bmatrix} -5 \\ 3 \end{bmatrix} \end{gathered}}$$

21) Find a parametric equation of the line M through $\vec{p}$ and $\vec{q}$.

$p = \begin{bmatrix} 2 \\ -5 \end{bmatrix} \quad q = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$



$\vec{q} - \vec{p} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

$\vec{q} - \vec{p} = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$

$\vec{x} = \vec{p} + x(\vec{q} - \vec{p})$

$\vec{x} = \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x\begin{bmatrix} -5 \\ 6 \end{bmatrix}$

23)

a) True - It is not possible to get a row in the augmented matrix of a homogeneous system in the form

$[0 \ 0 \ldots 0 \ b]$ where $b \neq 0$

b) False - The equation $A\vec{x} = \vec{0}$ gives an implicit description of the solution set.

c) False - A homogeneous system always has a trivial solution

d) False - The equation $\vec{x} = \vec{p} + t\vec{v}$ describes a line through $\vec{p}$ and parallel to $\vec{v}$.

e) False - This statement is true if and only if $A\vec{x} = \vec{b}$ is consistent which may not be the case.

27) Suppose A is the 3x3 zero matrix. Describe the solution set of $A\vec{x} = \vec{0}$.

In this case all variables are free. This makes the solution set:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This means the solution set is all vectors in $\mathbb{R}^3$

29) For the given A,
a) Does $A\vec{x} = \vec{0}$ have a nontrivial solution?
b) Does the equation $A\vec{x} = \vec{b}$ have at least one solution for all $\vec{b}$

A is 3x3 matrix with 3 pivot positions

a) No - Since it has no free variables, it has only the trivial solution.

b) Yes - Since there is a pivot in each row, it is not possible to have a row in the augmented matrix with the form

$[0 \ 0 \ 0 \ 0 \ c]$ (where $c \neq 0$)

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31) For the given matrix A:

A is a 3x2 matrix with 2 pivot positions

a) Does the equation $A\vec{x} = \vec{0}$ have a non-trivial solution?

No - This equation has no free variable, so it cannot have a non-trivial solution.

b) Does the equation $A\vec{x} = \vec{b}$ have at least one solution for all $\vec{b}$?

No - By the Theorem 4 in section 1.4, if a matrix does not have a pivot in every row, then a solution does not exist for all $\vec{b}$

35) Construct a non-zero matrix of size 3x3 such that the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a solution to $A\vec{x} = \vec{0}$.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This example holds since the dot product for each row with $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ must be 0.

Chapter 1-5- Solution Sets of Linear Systems

1) Do the following planes intersect? If so, describe their intersection?

$$x_1 + 4x_2 - 5x_3 = 0$$
$$2x_1 - x_2 + 8x_3 = 9$$

Augmented Matrix

$$\begin{bmatrix} 1 & 4 & -5 & 0 \\ 2 & -1 & 8 & 9 \end{bmatrix}$$

$R_2' = R_2 - 2R_1$

$\Downarrow$

$$\begin{bmatrix} 1 & 4 & -5 & 0 \\ 0 & -9 & 18 & 9 \end{bmatrix}$$

$R_2' = -\frac{1}{9}R_2$

$\Downarrow$

$$\begin{bmatrix} 1 & 4 & -5 & 0 \\ 0 & 1 & -2 & -1 \end{bmatrix}$$

$R_1' = R_1 - 4R_2$

$$\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & -1 \end{bmatrix}$$

$x_1 = 4 - 3x_3$

$x_2 = -1 + 2x_3$

x_3 is free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4-3x_3 \\ -1+2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

Practice Problem