Solutions to Section 1.3 Homework Problems
Problems 1-25 (odd) and problem 24.
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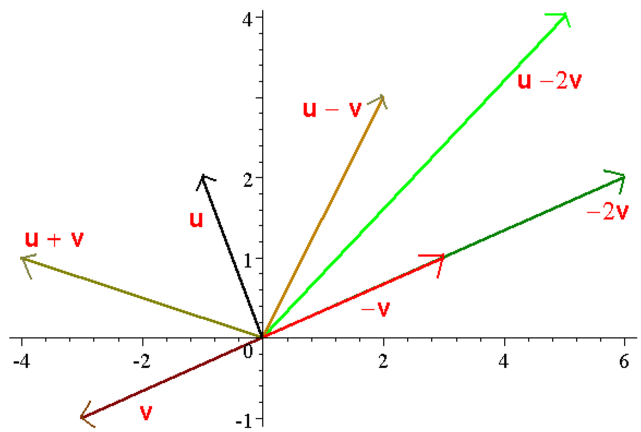
1. For $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$, we have

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

and

$$\mathbf{u} - 2\mathbf{v} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

3. For $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$, the vectors $\mathbf{u}$, $\mathbf{v}$, $-\mathbf{v}$, $-2\mathbf{v}$, $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$, and $\mathbf{u}-2\mathbf{v}$ are pictured below.



5.

$$\begin{aligned} 6x_1 - 3x_2 &= 1 \\ -x_1 + 4x_2 &= -7 \\ 5x_1 &= -5 \end{aligned}$$

7. $\mathbf{a} = \mathbf{u} - 2\mathbf{v}$, $\mathbf{b} = 2\mathbf{u} - 2\mathbf{v}$, $\mathbf{c} = 2\mathbf{u} - 3.5\mathbf{v}$, $\mathbf{d} = 3\mathbf{u} - 4\mathbf{v}$. Every vector in $\Re^2$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

9.

$$x_1 \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

11. We want to determine if there exist scalars (weights), x_1, x_2, and x_3 such that

$$x_1\begin{bmatrix}1\\-2\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\2\end{bmatrix}+x_3\begin{bmatrix}5\\-6\\8\end{bmatrix}=\begin{bmatrix}2\\-1\\6\end{bmatrix}.$$

This will be true if and only if the linear system with augmented matrix

$$\begin{bmatrix}1&0&5&2\\-2&1&-6&-1\\0&2&8&6\end{bmatrix}$$

is consistent. Using elementary row operations, we obtain

$$\begin{bmatrix}1&0&5&2\\-2&1&-6&-1\\0&2&8&6\end{bmatrix}\sim\begin{bmatrix}1&0&5&2\\0&1&4&3\\0&0&0&0\end{bmatrix}.$$

The latter matrix has echelon form and is the augmented matrix of a consistent linear system. Thus **b** is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. (Note that since there are free variables, there is more than one way to write **b** as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.)

13. To determine if **b** is a linear combination of the vectors that form the columns of the matrix A, we need to determine if the linear system whose augmented matrix is

$$\begin{bmatrix}1&-4&2&3\\0&3&5&-7\\-2&8&-4&-3\end{bmatrix}$$

is consistent. Since

$$\begin{bmatrix}1&-4&2&3\\0&3&5&-7\\-2&8&-4&-3\end{bmatrix}\sim\begin{bmatrix}1&-4&2&3\\0&3&5&-7\\0&0&0&3\end{bmatrix}$$

and the latter matrix represents an inconsistent linear system, we conclude that **b** is not a linear combination of the columns of A.

15. For $\mathbf{v}_1=\begin{bmatrix}7\\1\\-6\end{bmatrix}$ and $\mathbf{v}_2=\begin{bmatrix}-5\\3\\0\end{bmatrix}$, we have

$$0\mathbf{v}_1+0\mathbf{v}_2=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\mathbf{v}_1 + \mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}$$

$$2\mathbf{v}_1 - 3\mathbf{v}_2 = \begin{bmatrix} 29 \\ -7 \\ -12 \end{bmatrix}$$

$$\mathbf{v}_1 - 0\mathbf{v}_2 = \begin{bmatrix} 7 \\ 1 \\ -6 \end{bmatrix}$$

$$\mathbf{v}_1 - \mathbf{v}_2 = \begin{bmatrix} 12 \\ -2 \\ -6 \end{bmatrix}$$

17. To find the values of h for which **b** is in the plane spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$, we must find the values of h for which the linear system with augmented matrix

$$\begin{bmatrix} 1 & -2 & 4 \\ 4 & -3 & 1 \\ -2 & 7 & h \end{bmatrix}$$

is consistent. Using elementary row operations, we obtain

$$\begin{bmatrix} 1 & -2 & 4 \\ 4 & -3 & 1 \\ -2 & 7 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 \\ 4 & -3 & 1 \\ 0 & 3 & 8+h \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 5 & -15 \\ 0 & 3 & 8+h \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 3 & 8+h \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 17+h \end{bmatrix}$$

The linear system whose augmented matrix is the last one shown is consistent if and only if $17 + h = 0$. Thus, **b** is in the plane spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$ if and only if $h = -17$.

19. Since $\mathbf{v}_2 = 1.5\mathbf{v}_1$, Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is a **line** through the origin in $\Re^3$. (If $\mathbf{v}_1$ and $\mathbf{v}_2$ were not scalar multiples of each other, then Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ would be a plane through the origin in $\Re^3$.)

21. Since the vectors

$$\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

are not scalar multiples of each other, then $Span\{\mathbf{u},\mathbf{v}\} = \Re^2$, meaning that every vector in $\Re^2$ is in $Span\{\mathbf{u},\mathbf{v}\}$. Another way to see this is to consider the *coefficient* matrix,

$$\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix},$$

of the vector equation

$$x_1\mathbf{u} + x_2\mathbf{v} = \begin{bmatrix} h \\ k \end{bmatrix}.$$

Since

$$\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we see that every column of this coefficient matrix is a pivot column – telling us that the above vector equation is consistent no matter what the values of h and k.

23.

- **a.** False. These are two different matrices. One is a 2x1 matrix and the other is a 1x2 matrix.
- **b.** False. If the points in the plane corresponding to these two vectors were on the same line through the origin, then the vectors would be scalar multiples of each other (but they are not).
- **c.** True (since $\frac{1}{2}\mathbf{v}_1 = \frac{1}{2}\mathbf{v}_1 + 0\mathbf{v}_2$).
- **d.** True.
- **e.** False. Span$\{\mathbf{u},\mathbf{v}\}$ is a line through the origin if **u** and **v** are scalar multiples of each other.

24.

- **a.** False (I guess), although this is sort of a technicality. According to the definition given on page 28 of the textbook, a vector is a **matrix** with only one column. Thus a vector is not a **list**. Thus for example, $1,2,3,4,5$ is not a vector in $\Re^5$, but

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

is a vector in $\Re^5$.

b. True.

c. False.

d. True. Any point, $\mathbf{p}$, on the line through $\mathbf{u}$ and the origin can be written as $\mathbf{p} = c\mathbf{u} + 0\mathbf{v}$ for some scalar c.

e. True.

25.

a. No. $\mathbf{b}$ is not in the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. This set contains only three vectors – $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

b. To determine whether or not $\mathbf{b}$ is in $W = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, we note that

$$\begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ -2 & 6 & 3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 6 & -5 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

and note that the latter echelon form matrix is the augmented matrix of a consistent system. Thus $\mathbf{b}$ is in $W = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. In fact, by further reducing the last matrix to

$$\begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -2 \end{bmatrix},$$

we see that $\mathbf{b} = -4\mathbf{a}_1 - \mathbf{a}_2 - 2\mathbf{a}_3$. (The set W contains infinitely many vectors.)

c. a_1 is in W because $\mathbf{a}_1 = 1\mathbf{a}_1 + 0\mathbf{a}_2 + 0\mathbf{a}_3$.