Chapter 6-2 - Orthogonal Sets

1) Determine if the set of vectors is orthogonal

$\vec{u}=\begin{bmatrix}-1\\4\\-3\end{bmatrix}, \vec{v}=\begin{bmatrix}5\\2\\1\end{bmatrix}, \vec{w}=\begin{bmatrix}3\\-4\\-7\end{bmatrix}$

Not orthogonal

$\vec{u}\cdot\vec{w}=(-1)(3)+(4)(-4)+(-7)(-3)$

$\vec{u}\cdot\vec{w}=-3-16+21$

$\vec{u}\cdot\vec{w}=\boxed{2}$

3) Determine if the set of vectors is orthogonal.

$\vec{u}=\begin{bmatrix}2\\-7\\-1\end{bmatrix}, \vec{v}=\begin{bmatrix}-6\\-3\\9\end{bmatrix}, \vec{w}=\begin{bmatrix}3\\1\\-1\end{bmatrix}$

Orthogonal

$\vec{u}\cdot\vec{v}=(2)(-6)+(-7)(-3)+(-1)(9)$

$\vec{u}\cdot\vec{v}=-12+21-9=\boxed{0}$

$\vec{u}\cdot\vec{w}=(2)(3)+(-7)(1)+(-1)(-1)$

$\vec{u}\cdot\vec{w}=6-7+1=\boxed{0}$

$\vec{v}\cdot\vec{w}=(-6)(3)+(-3)(1)+(9)(-1)$

5) Determine in the set of vectors is orthogonal.

$\vec{u}=\begin{bmatrix}3\\-2\\1\\3\end{bmatrix}, \vec{v}=\begin{bmatrix}-1\\3\\-3\\4\end{bmatrix}, \vec{w}=\begin{bmatrix}3\\8\\7\\0\end{bmatrix}$

$\vec{u}\cdot\vec{v}=(3)(-1)+(-2)(3)+(1)(-3)+(3)(4)$

$\vec{u}\cdot\vec{v}=-3-6-3+12=\boxed{0}$

$\vec{u}\cdot\vec{w}=(3)(3)+(-2)(8)+(1)(7)+(3)(0)$

$\vec{u}\cdot\vec{w}=9-16+7+0=\boxed{0}$

$\vec{v}\cdot\vec{w}=(-1)(3)+(3)(8)+(-3)(7)+(4)(0)$

$\vec{v}\cdot\vec{w}=-3+24-21+0=\boxed{0}$

Orthogonal

7) Show $\{\vec{u}_1,\vec{u}_2\}$ is an orthogonal basis for $\mathbb{R}^2$. Express $\vec{x}$ as a linear combination of $\vec{u}_1, \vec{u}_2$.

$\vec{u}_1=\begin{bmatrix}2\\-3\end{bmatrix}, \vec{u}_2=\begin{bmatrix}6\\4\end{bmatrix}, \vec{x}=\begin{bmatrix}9\\-7\end{bmatrix}$

$\vec{u}_1$ and $\vec{u}_2$ are linearly independent since not scalar multiples. Since they are in $\mathbb{R}^2$ and have dim $B=2$, they are a basis for $\mathbb{R}^2$.

$\vec{u}_1\cdot\vec{u}_2=(2)(6)+(-3)(4)=0$

orthognal.

$$C_1=\frac{\vec{x}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}$$

$$C_1=\frac{18+21}{13}=\frac{39}{13}=3$$

$$C_2=\frac{\vec{x}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}=\frac{54-28}{36+16}$$

$$=\frac{26}{52}=\frac{1}{2}$$

$\boxed{x=3\vec{u}_1+\frac{1}{2}\vec{u}_2}$

Show $\vec{u}_1,\vec{u}_2$, and $\vec{u}_3$ is an orthogonal basis for $\mathbb{R}^3$. Express $\vec{x}$ as a linear combination of $u_1,\ldots,\vec{u}_3$

$\vec{u}_1=\begin{bmatrix}1\\0\\1\end{bmatrix}, \vec{u}_2=\begin{bmatrix}-1\\4\\1\end{bmatrix}\ \vec{u}_3=\begin{bmatrix}2\\1\\-2\end{bmatrix}, \vec{x}=\begin{bmatrix}8\\-4\\-3\end{bmatrix}$

$$\begin{bmatrix}1&-1&2\\0&4&1\\1&1&-2\end{bmatrix}\underset{R_3-R_1}{\sim}\begin{bmatrix}1&-1&2\\0&4&1\\0&2&-4\end{bmatrix}\underset{\frac{1}{2}R_3}{\sim}\begin{bmatrix}1&-1&2\\0&4&1\\0&1&-2\end{bmatrix}$$

$$\underset{R_2\Leftrightarrow R_3}{\sim}\begin{bmatrix}1&-1&2\\0&1&-2\\0&4&1\end{bmatrix}\underset{R_3-4R_2}{\sim}\begin{bmatrix}1&-1&2\\0&1&-2\\0&0&9\end{bmatrix}$$

3 pivots makes it a basis for $\mathbb{R}^3$

$\vec{u}_1\cdot\vec{u}_2=(1)(-1)+(0)(4)+(1)(1)$

$\vec{u}_1\cdot\vec{u}_2=-1+0+1=\boxed{0}$

$\vec{u}_1\cdot\vec{u}_3=(1)(2)+(0)(1)+(1)(-2)$

$\vec{u}_1\cdot\vec{u}_3=2+0+-2=\boxed{0}$

$\vec{u}_2\cdot\vec{u}_3=(-1)(2)+(4)(1)+(1)(-2)$

$\vec{u}_2\cdot\vec{u}_3=-2+4-2=\boxed{0}$

Orthogonal so immediately linearly independent by theorem 6-4

$$C_1=\frac{\vec{x}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}=\frac{5}{1+1}=\frac{5}{2}$$

$$C_2=\frac{\vec{x}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}=\frac{-8-16-3}{18}=-\frac{3}{2}$$

$$C_3=\frac{\vec{x}\cdot\vec{u}_3}{\vec{u}_3\cdot\vec{u}_3}=\frac{16-4+6}{9}=2$$

$\boxed{\vec{x}=\frac{5}{2}\vec{u}_1-\frac{3}{2}\vec{u}_2+2\vec{u}_3}$

Chapter 6-2 – Orthonormal Sets

11) Compute the orthogonal projection of $\begin{bmatrix}1\\7\end{bmatrix}$ onto $\begin{bmatrix}-4\\2\end{bmatrix}$

$$\text{proj}_L \vec{y} = \frac{\begin{bmatrix}1\\7\end{bmatrix}\cdot\begin{bmatrix}-4\\2\end{bmatrix}}{\begin{bmatrix}-4\\2\end{bmatrix}\cdot\begin{bmatrix}-4\\2\end{bmatrix}}\begin{bmatrix}-4\\2\end{bmatrix}$$

$$\text{proj}_L \vec{y} = \frac{-4+14}{16+4}\begin{bmatrix}-4\\2\end{bmatrix}$$

$$\text{proj}_L \vec{y} = \frac{1}{2}\begin{bmatrix}-4\\2\end{bmatrix}$$

$$\boxed{\text{proj}_L \vec{y} = \begin{bmatrix}-2\\1\end{bmatrix}}$$

13) Let $\vec{y}=\begin{bmatrix}2\\3\end{bmatrix}$ and $\vec{u}=\begin{bmatrix}4\\7\end{bmatrix}$.

Write $\vec{y}$ as the sum of two orthogonal vectors, one in span$\{\vec{u}\}$ and one orthogonal to $\vec{u}$

$$\text{Proj}_L \vec{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$$

$$\text{proj}_L \vec{y} = \frac{8-21}{16+49}\vec{u}$$

$$\text{proj}_L \vec{y} = \frac{-13}{65}\vec{u}$$

$$\text{proj}_L \vec{y} = \begin{bmatrix}-\frac{4}{5}\\ \frac{7}{5}\end{bmatrix}$$

$$z = \vec{y} - \text{proj}_L \vec{y}$$

$$\vec{z} = \begin{bmatrix}\frac{14}{5}\\ \frac{8}{5}\end{bmatrix}$$

$$\boxed{\vec{y} = \begin{bmatrix}-\frac{4}{5}\\ \frac{7}{5}\end{bmatrix} + \begin{bmatrix}\frac{14}{5}\\ \frac{8}{5}\end{bmatrix}}$$

15) Let $\vec{y}=\begin{bmatrix}3\\1\end{bmatrix}$ and $\vec{u}=\begin{bmatrix}8\\6\end{bmatrix}$.

Compute the distance from $\vec{y}$ to the line through $\vec{u}$ and the origin.

$$\hat{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$$

$$\hat{y} = \frac{24+6}{64+36}\vec{u}$$

$$\hat{y} = \frac{3}{10}\vec{u}$$

$$\hat{y} = \begin{bmatrix}\frac{24}{10}\\ \frac{18}{10}\end{bmatrix}$$

$$\vec{z} = \begin{bmatrix}\frac{30}{10}-\frac{24}{10}\\ \frac{10}{10}-\frac{18}{10}\end{bmatrix} = \boxed{\begin{bmatrix}\frac{6}{10}\\ -\frac{8}{10}\end{bmatrix}}$$

$$\|z\| = \sqrt{.6^2+.8^2} = \boxed{1}$$

17) Determine which set of vectors are orthonormal. If a set is orthogonal only, normalize the vectors to produce an orthonormal set.

$$\vec{u}_1 = \begin{bmatrix}\frac{1}{3}\\ \frac{1}{3}\\ \frac{1}{3}\end{bmatrix},\ \vec{u}_2 = \begin{bmatrix}-\frac{1}{2}\\ 0\\ \frac{1}{2}\end{bmatrix}$$

$$\vec{u}_1\cdot\vec{u}_2 = -\frac{1}{6}+0+\frac{1}{6} = 0$$

$$\|\vec{u}_1\| = \sqrt{\frac{1}{3}^2+\frac{1}{3}^2+\frac{1}{3}^2}$$

$$\|\vec{u}_1\| = \sqrt{\frac{1}{3}}$$

$$\|\vec{u}_2\| = \sqrt{(-\tfrac{1}{2})^2+0+(\tfrac{1}{2})^2}$$

$$\|\vec{u}_2\| = \sqrt{\frac{1}{2}}$$

$$\boxed{\hat{u}_1 = \begin{bmatrix}\frac{\sqrt{3}}{3}\\ \frac{\sqrt{3}}{3}\\ \frac{\sqrt{3}}{3}\end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix}-\frac{\sqrt{2}}{2}\\ 0\\ \frac{\sqrt{2}}{2}\end{bmatrix}}$$

19) Determine if the set is orthonormal.

$$\begin{bmatrix}-.6\\ .8\end{bmatrix}\begin{bmatrix}.8\\ .6\end{bmatrix}$$

$$\begin{bmatrix}-.6\\ .8\end{bmatrix}\cdot\begin{bmatrix}.8\\ .6\end{bmatrix} = (-.6)(.8)+(.6)(.8) = 0$$

orthogonal

$$\left\|\begin{bmatrix}-.6\\ .8\end{bmatrix}\right\| = \sqrt{(-.6)^2+(.8)^2} = 1$$

$$\left\|\begin{bmatrix}.8\\ .6\end{bmatrix}\right\| = \sqrt{(.6)^2+(.8)^2} = 1$$

orthonormal

21) Determine if the set is orthonormal. If not, normalize it.

$$\vec{u}_1 = \begin{bmatrix}\frac{1}{\sqrt{10}}\\ \frac{3}{\sqrt{20}}\\ \frac{3}{\sqrt{20}}\end{bmatrix},\ \vec{u}_2 = \begin{bmatrix}\frac{3}{\sqrt{10}}\\ -\frac{1}{\sqrt{20}}\\ -\frac{1}{\sqrt{20}}\end{bmatrix},\ \vec{u}_3 = \begin{bmatrix}0\\ -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}$$

$$\vec{u}_1\cdot\vec{u}_2 = \frac{3}{10}-\frac{3}{20}-\frac{3}{20} = 0$$

$$\vec{u}_1\cdot\vec{u}_2 = 0 - \frac{3}{\sqrt{2}\sqrt{20}} + \frac{3}{\sqrt{2}\sqrt{20}} = 0$$

$$\vec{u}_2\cdot\vec{u}_3 = 0 + \frac{1}{\sqrt{2}\sqrt{20}} - \frac{1}{\sqrt{2}\sqrt{20}} = 0$$

Orthogonal

u_3 is orthonormal

$$\|u_1\| = \sqrt{\frac{1}{10}+\frac{9}{20}+\frac{9}{20}} = 1 \text{ (orthonormal)}$$

$$\|u_2\| = \sqrt{\frac{9}{10}+\frac{1}{20}+\frac{1}{20}} = 1 \text{ (orthonormal)}$$

Orthonormal

23)

a) True - $\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\}$ are linearly independent but not orthogonal.

b) True - This relies on theorem 6-5.

c) False - Normalizing affects only the magnitude not the direction.

d) False - To be an orthogonal matrix, it must also be square.

e) False - $\|\vec{y} - \hat{y}\|$ is the distance from $\vec{y}$ to L.

24)

a) False - All orthogonal sets are linearly independent.

b) False - To be orthonormal the vectors also have to be unit vectors.

c) True - By theorem 6-7

d) True - The orthogonal projection considers direction not magnitude of $\vec{u}$.

e) True - An orthogonal matrix is square and the columns are linearly independent since they are orthogonal.