Chapter 1-7 - Linear Independence

1) Determine if the vectors are linearly independent.

$$\begin{bmatrix}5\\0\\0\end{bmatrix}\begin{bmatrix}7\\2\\-6\end{bmatrix}\begin{bmatrix}9\\4\\-8\end{bmatrix}$$

Check if no free variables

$$\begin{bmatrix}5 & 7 & 9\\0 & 2 & 4\\0 & -6 & -8\end{bmatrix}$$

$R_3' = R_3 + 3R_2$

$\Downarrow$

$$\begin{bmatrix}5 & 7 & 9\\0 & 2 & 4\\0 & 0 & 4\end{bmatrix}$$

Since no free variables, linearly independent

3) Determine if the vectors are linearly independent.

$$\begin{bmatrix}1\\-3\end{bmatrix}\begin{bmatrix}-3\\9\end{bmatrix}$$

For two vectors, if at least one is a scalar multiple then linearly dependent

$$-3\begin{bmatrix}1\\-3\end{bmatrix} = \begin{bmatrix}-3\\9\end{bmatrix}$$

Linearly Dependent

5) Determine if the columns form a linearly independent set.

$$\begin{bmatrix}0 & -8 & 5\\3 & -7 & 4\\-1 & 5 & 4\\1 & -3 & 2\end{bmatrix}$$

Swap R_1 and R_4

$\Downarrow$

$$\begin{bmatrix}1 & -3 & 2\\3 & -7 & 4\\-1 & 5 & 4\\0 & -8 & 5\end{bmatrix}$$

$R_2' = R_2 - 3R_1$

$R_3' = R_3 + R_1$

$\Downarrow$

$$\begin{bmatrix}1 & -3 & 2\\0 & 2 & -2\\0 & 2 & 6\\0 & -8 & 5\end{bmatrix}$$

$R_3' = R_3 - R_2$

$R_4 = R_4 + 4R_2$

$\Downarrow$

$$\begin{bmatrix}1 & -3 & 2\\0 & 2 & -2\\0 & 0 & 8\\0 & 0 & -3\end{bmatrix}$$

No free variables

Linearly independent

7) Determine if the columns of the matrix form a linearly independent set.

$$\begin{bmatrix}1 & 4 & -3 & 0\\-2 & -7 & 5 & 1\\-4 & -5 & 7 & 5\end{bmatrix}$$

$R_2' = R_2 + 2R_1$

$R_3' = R_3 + 4R_1$

$\Downarrow$

$$\begin{bmatrix}1 & 4 & -3 & 0\\0 & 1 & -1 & 1\\0 & 11 & -5 & 5\end{bmatrix}$$

$R_3' = R_3 - 11R_2$

$\Downarrow$

$$\begin{bmatrix}1 & 4 & -3 & 0\\0 & 1 & -1 & 1\\0 & 0 & 6 & -6\end{bmatrix}$$

x_4 is free

Linearly dependent

Chapter 1-7 - Linear Independence

9) ☆

$V_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, V_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, V_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix}$

a) For what values of h is V_3 in Span $\{v_1, v_2\}$?

$\begin{bmatrix} 1 & -3 & 5 \\ -3 & 9 & -7 \\ 2 & -6 & h \end{bmatrix}$

$R_2' = R_2 + 3R_1, R_3' = R_3 - 2R_1$

$\Downarrow$

$\begin{bmatrix} 1 & -3 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & h+10 \end{bmatrix}$

Due to row [0 0 8], V_3 is never in Span$\{v_1, v_2\}$ for any h

b) For what values of h is $\{v_1, v_2, v_3\}$ linearly dependent?

In the reduced coefficient matrix, Column 2 is free.

For all h, this system is linearly independent.

11) For which values of "h" is the system linearly dependent

$\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \\ h \end{bmatrix}$

$\begin{bmatrix} 1 & 3 & -1 \\ -1 & -5 & 5 \\ 4 & 7 & h \end{bmatrix}$

$R_2' = R_2 + R_1$

$R_3' = R_3 - 4R_1$

$\Downarrow$

$\begin{bmatrix} 1 & 3 & -1 \\ 0 & -2 & 4 \\ 0 & -5 & h+4 \end{bmatrix}$

$R_2' = \frac{-1}{2} R_2$

$\Downarrow$

$\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -2 \\ 0 & -5 & h+4 \end{bmatrix}$

$R_3' = R_3 + 5R_1$

$\Downarrow$

$\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & h-6 \end{bmatrix}$

Linearly dependent when

$h - 6 = 0$

$\boxed{h = 6}$

13) For which values of "h" is the system linearly dependent

$\begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix} \begin{bmatrix} -2 \\ -9 \\ 6 \end{bmatrix} \begin{bmatrix} 3 \\ h \\ -9 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 & 3 \\ -3 & 6 & -9 \\ 5 & -9 & h \end{bmatrix}$

$R_2' = R_2 + 3R_1$

$R_3' = R_3 - 5R_1$

$\Downarrow$

$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & h-15 \end{bmatrix}$

x_3 is free

Linearly dependent for all h

Chapter 1-7 - Linear Independence

15) Determine by inspection whether the vectors are linearly independent.

$$\begin{bmatrix}5\\1\end{bmatrix}\begin{bmatrix}2\\8\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}\begin{bmatrix}-1\\7\end{bmatrix}$$

Linearly dependent by Theorem 8 since more vectors than dimensions in the vectors.

17) Determine by inspection whether the vectors are linearly independent.

$$\begin{bmatrix}3\\5\\-1\end{bmatrix}\begin{bmatrix}0\\0\\0\end{bmatrix}\begin{bmatrix}-6\\5\\4\end{bmatrix}$$

Linearly dependent by theorem 9 since it contains the zero vector.

9) Determine by inspection whether the vectors are linearly independent.

$$\begin{bmatrix}-8\\12\\-4\end{bmatrix}\begin{bmatrix}2\\-3\\1\end{bmatrix}$$

Linearly dependent by theorem 7 since the second vector is a linear combination (i.e, scalar multiple) of the first.

21)

a) False - The linear solution always exists. It must be the only solution to be linearly independent.

b) False - All vectors need not be linear combinations of each other. Just one suffices.

c) True - By theorem 8, the set of vectors with more vectors than dimensions is linearly dependent.

d) True - If $\vec{x}$ and $\vec{y}$ are linearly independent and $\vec{z}$ is in span$\{\vec{x}, \vec{y}\}$, then $\vec{z}$ is a linear combination of $\vec{x}$ and $\vec{y}$. By theorem 7, then this makes linearly dependent.

23&25) Use the echelon matrix notation to indicate the following:

$\blacksquare$ - Pivot
$*$ - Any number
0 - Zero

23) A linearly independent 3x3 matrix

$$\begin{bmatrix}\blacksquare & * & *\\ 0 & \blacksquare & *\\ 0 & 0 & \blacksquare\end{bmatrix}$$

25) A 4x2 matrix $[\vec{a}_1\ \vec{a}_2]$ and $\vec{a}_2$ is not a multiple of $\vec{a}_1$.

$$\begin{bmatrix}0 & \blacksquare\\ 0 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix} \qquad \begin{bmatrix}\blacksquare & *\\ 0 & \blacksquare\\ 0 & 0\\ 0 & 0\end{bmatrix}$$

27) How many pivot columns must a 7x5 matrix have if it is linearly independent?

To be linearly independent, there must be no free variables.

To have no free variables, every column must have a pivot.

$\boxed{\text{This means there must be five pivot columns.}}$

Chapter 1-7 - Linear Independence

29) Construct 3x2 matrices A and B such that $A\vec{x}=\vec{0}$ has only the trivial solution while $B\vec{x}=\vec{0}$ has nontrivial solution.

$$A=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix} \qquad B=\begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix}$$

31) Given $A=\begin{bmatrix}2&3&5\\-5&1&-4\\-3&-1&-4\\1&0&1\end{bmatrix}$, observe that the third column is the sum of the first two columns. Find a nontrivial solution of $A\vec{x}=\vec{0}$

$\vec{a}_1+\vec{a}_2=\vec{a}_3$

$\vec{a}_1+\vec{a}_2-\vec{a}_3=\vec{0}$

$x_1=1,\ x_2=1,\ x_3=-1$

$$\vec{x}=\begin{bmatrix}1\\1\\-1\end{bmatrix}$$

33) If $\{\vec{v}_1,\vec{v}_2,\vec{v}_3,\vec{v}_4\}$ are in $\mathbb{R}^4$ and $\vec{v}_3=2\vec{v}_1+\vec{v}_2$ then $\{\vec{v}_1,\vec{v}_2,\vec{v}_3,\vec{v}_4\}$ are linearly dependent.

True - By theorem 7, if a vector is a linear combination of a set of other vectors, then they are linearly dependent.

35) If $\vec{v}_1$ and $\vec{v}_2$ are in $\mathbb{R}^4$ and v_2 is not a scalar multiple of v_1, then $\{v_1,v_2\}$ is linearly independent.

False - $\vec{v}_1$ (not $\vec{v}_2$) could be the zero vector.