Chapter 4-6 - Rank

1) A is row equivalent to B. Find rank A and dim Nul A. Find bases for Col A, Row A, and Nul A.

$$A=\begin{bmatrix}1 & -4 & 9 & 7\\ -1 & 2 & -4 & 1\\ 5 & -6 & 10 & 7\end{bmatrix},\quad B=\begin{bmatrix}1 & 0 & -1 & 5\\ 0 & -2 & 5 & -6\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$\boxed{\text{rank } A=2}$, $\boxed{\dim \text{Nul} A=2}$

$$\boxed{B_{\text{Col}A}=\left\{\begin{bmatrix}1\\-1\\5\end{bmatrix},\begin{bmatrix}-4\\2\\-6\end{bmatrix}\right\}}$$

$$\boxed{B_{\text{row}A}=\left\{\begin{bmatrix}1\\0\\-1\\5\end{bmatrix},\begin{bmatrix}0\\-2\\5\\-6\end{bmatrix}\right\}}$$

$$\vec{x}=\begin{cases}x_3-5x_4\\ \frac{5}{2}x_3-3x_4\\ x_3 \text{ is free}\\ x_4 \text{ is free}\end{cases}$$

$$\boxed{B_{\text{Nul}A}=\left\{\begin{bmatrix}1\\ \frac{5}{2}\\ 1\\ 0\end{bmatrix},\begin{bmatrix}-5\\-3\\0\\1\end{bmatrix}\right\}}$$

3) A is row equivalent to B. Find rank A, and dim Nul A. Find bases for Col A, row A, and Nul A.

$$A=\begin{bmatrix}2 & -3 & 6 & 2 & 5\\ -2 & 3 & -3 & -3 & -4\\ 4 & -6 & 9 & 5 & 9\\ -2 & 3 & 3 & -4 & 1\end{bmatrix}$$

$$B=\begin{bmatrix}2 & -3 & 6 & 2 & 5\\ 0 & 0 & 3 & -1 & 1\\ 0 & 0 & 0 & 1 & 3\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$\boxed{\text{rank} A=3}$, $\boxed{\dim \text{Nul} A=2}$

$$\boxed{B_{\text{Col}A}=\left\{\begin{bmatrix}2\\-2\\4\\-2\end{bmatrix},\begin{bmatrix}6\\-3\\9\\3\end{bmatrix},\begin{bmatrix}2\\-3\\5\\-4\end{bmatrix}\right\}}$$

$$\boxed{B_{\text{row}A}=\left\{\begin{bmatrix}2\\-3\\6\\2\\5\end{bmatrix},\begin{bmatrix}0\\0\\3\\-1\\1\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\\3\end{bmatrix}\right\}}$$

$$B=\begin{bmatrix}2 & -3 & 6 & 2 & 5\\ 0 & 0 & 3 & -1 & 1\\ 0 & 0 & 0 & 1 & 3\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$\underset{\substack{R_2'=R_2+R_3\\ R_1'=R_1-2R_3}}{\sim}\begin{bmatrix}2 & -3 & 6 & 0 & -1\\ 0 & 0 & 3 & 0 & 4\\ 0 & 0 & 0 & 1 & 3\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$\underset{R_1'=R_1-2R_2}{\sim}\begin{bmatrix}2 & -3 & 0 & 0 & -9\\ 0 & 0 & 3 & 0 & 4\\ 0 & 0 & 0 & 1 & 3\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$\vec{x}=\begin{cases}\frac{3}{2}x_2+\frac{9}{2}x_5\\ x_2 \text{ is free}\\ x_3=-\frac{4}{3}x_5\\ x_4=-3x_5\\ x_5 \text{ is free}\end{cases}$$

$$\boxed{B_{\text{Nul}A}=\left\{\begin{bmatrix}\frac{3}{2}\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}\frac{9}{2}\\0\\-\frac{4}{3}\\-3\\1\end{bmatrix}\right\}}$$

5) If a 3×8 matrix A has rank 3, find dim Nul A, dim Row A, and rank A^T.

$$\boxed{\begin{aligned}&\dim \text{Nul}A=5\\ &\dim \text{Row}A=3\\ &\dim \text{Rank } A^T=3\end{aligned}}$$

7) Suppose a 4×7 matrix A has four pivot columns. Is Col $A=\mathbb{R}^4$? Is Nul $A=\mathbb{R}^3$?

Yes Col $A=\mathbb{R}^4$ - Col $A\subseteq\mathbb{R}^4$ since 4 elements per vector. since it has 4 pivots, it spans $\mathbb{R}^4$ by the Invertible matrix theorem

No - Nul A is not even a subset of $\mathbb{R}^3$. It is a subset of $\mathbb{R}^7$

9) If the nullspace of a 5×6 matrix A is 4-dimensional, what is the dimension of the Column space of A?

For an $m\times n$ matrix, A

$\dim \text{Col}A+\dim \text{Nul}A=n$

$\dim \text{Col}A+4=6$

$\boxed{\dim \text{Col}A=2}$

11) The nullspace of an 8×5 matrix is 2-dimensional. What is the dimension of the row space.

$\dim \text{Row}A=\dim \text{Col}A$

$\dim \text{Col}A+\dim \text{Nul}A=n$

$\dim \text{Row}A+\dim \text{Nul}A=n$

$\dim \text{Row}A+2=5$

$\boxed{\dim \text{Row}A=3}$

Chapter 4-6 – Rank

13) If A is a 7x5 matrix. what is the largest possible rank of A? If A is a 5x7 matrix what is the largest possible rank of A?

In both cases, the largest possible rank is $\boxed{5}$ as the number of pivots cannot exceed the number of columns or rows.

15) If A is a 6x8 matrix, what is the smallest possible dimension of Nul A?

Largest possible rank is 6. Hence, smallest possible dimension of Nul A is:

$$n - \text{rank} = \dim \text{Nul} A$$

$$8 - 6 = \dim \text{Nul} A$$

$$\boxed{\dim \text{Nul} A = 2}$$

17)

a) True – The row space column space of the transpose are identical

b) False – The row space comes from the echelon matrix as A's rows may be linearly dependent.

c) True – By the Rank Theorem

d) False – The sum of the dimension of the row space and null space equal the number of columns.

18)

a) False – Pivot columns of A form the basis of col A.

b) False – Linear dependence relation can be affected by interchanges/swaps.

c) True – Number of free variables (i.e. number of non-pivot columns) is the dimension of the null space.

d) True – Rows of A^T become the columns in A.

e) True – This allows for finding the row space through row reduction.