Chapter 1-4 - The Matrix Equation $A\vec{x}=\vec{b}$

1)

$$\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}$$

The matrix-vector product does not exist as the dimensions of matrix and vector do not correspond.

3)

$$\begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 12-15 \\ -8+9 \\ 14-18 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$$

5)

Matrix Equation

$$\begin{bmatrix} 5 & 1 & -8 & 4 \\ -2 & -7 & 3 & -5 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$$

Vector Equation

$$5\begin{bmatrix} 5 \\ -2 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ -7 \end{bmatrix} + 3\begin{bmatrix} -8 \\ 3 \end{bmatrix} + (-2)\begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$$

7)

Vector Equation

$$x_1\begin{bmatrix} 4 \\ -1 \\ 7 \\ -4 \end{bmatrix} + x_2\begin{bmatrix} -5 \\ 3 \\ -5 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 7 \\ -8 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$$

Matrix Equation

$$\begin{bmatrix} 4 & -5 & 7 \\ -1 & 3 & -8 \\ 7 & -5 & 0 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$$

Note: For problems 5, 7, and 9, when in vector form, the constant is on the right and vector on the left.

9)

System of Linear Equations

$$3x_1 + x_2 - 5x_3 = 9$$
$$x_2 + 4x_3 = 0$$

Vector Equation

$$x_1\begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

Matrix Equation

$$\begin{bmatrix} 3 & 1 & -5 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

11)

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ -2 & -4 & -3 \end{bmatrix}$$

$$b = \begin{bmatrix} -2 \\ 2 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ -2 & -4 & -3 & 9 \end{bmatrix}$$

$R_3' = R_3 + 2R_1$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 5 & 5 \end{bmatrix}$$

$R_3' = \frac{1}{5}R_3$

$$\rightarrow \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_2' = R_2 - 5R_3$

$R_1' = R_1 - 4R_3$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 2 & 0 & -6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_1' = R_1 - 2R_2$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\boxed{\vec{x} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}}$$

13) Let $\vec{u} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{bmatrix}$.

Is $\vec{u}$ in the plane spanned by the columns of A?

Determine if the system is consistent

$$\begin{bmatrix} 3 & -5 & 0 \\ -2 & 6 & 4 \\ 1 & 1 & 4 \end{bmatrix}$$

Swap R_1 & R_3

$$\begin{bmatrix} 1 & 1 & 4 \\ -2 & 6 & 0 \\ 3 & -5 & 4 \end{bmatrix}$$

$R_2' = R_2 + 2R_1$

$R_3' = R_3 - 3R_1$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 8 & 8 \\ 0 & -8 & -8 \end{bmatrix}$$

$R_3' = R_3 + R_2$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 8 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

Yes the augmented matrix is consistent

15) $A = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$

$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

Show that the matrix equation $A\vec{x} = \vec{b}$ does not have a solution for all b_1 and b_2

Augmented Matrix

$$\begin{bmatrix} 2 & -1 & b_1 \\ -6 & 3 & b_2 \end{bmatrix}$$

$R_2' = R_2 + 3R_1$

$$\Downarrow$$

$$\begin{bmatrix} 2 & -1 & b_1 \\ 0 & 0 & b_2+3b_1 \end{bmatrix}$$

Can be inconsistent when

$b_2 + 3b_1 \neq 0$

as then a contradictory row in the form

$[0 \ 0 \ c]$ where

$c \neq 0$

17) $A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$

How many pivot positions are in A?

$R_2' = R_2 + R_1$

$R_3' = R_3 - 2R_1$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & -4 & 2 & -8 \\ 0 & -6 & 3 & -7 \end{bmatrix}$$

$$\Downarrow$$

$R_3' = R_3 + 2R_2$

$R_4' = R_4 + 3R_2$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$\boxed{\text{Number of Pivot Rows: } 3}$

Are all $b \in \mathbb{R}^4$ a solution to the matrix equation $Ax=b$?

No – Since there is not a pivot in every row using theorem #4

Chapter 1-4

19)
$$A=\begin{bmatrix}1&3&0&3\\-1&-1&-1&1\\0&-4&2&-8\\2&0&3&-1\end{bmatrix}$$

Can each vector in $\mathbb{R}^4$ be written as a linear combination of A?

No - Based off the answer in question #17. Not every row has a pivot so A neither spans $\mathbb{R}^4$ nor is all of $\mathbb{R}^4$ a linear combination of A.

Proof: Theorem 4

21) Let $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ be defined as:

$$\vec{v}_1=\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \vec{v}_2=\begin{bmatrix}0\\-1\\0\\1\end{bmatrix}, \vec{v}_3=\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}$$

Does $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ span $\mathbb{R}^4$?

No - The matrix $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$ does not have a pivot in each row since it has more columns than rows and each pivot is in a single column.

Proof: Theorem 4

25) Note that:

$$\begin{bmatrix}4&-3&1\\5&-2&5\\-6&2&-3\end{bmatrix}\begin{bmatrix}3\\-1\\2\end{bmatrix}=\begin{bmatrix}-7\\-3\\10\end{bmatrix}$$

Find c_1, c_2, and c_3 such that

$$\begin{bmatrix}-7\\-3\\10\end{bmatrix}=c_1\begin{bmatrix}4\\5\\-6\end{bmatrix}+c_2\begin{bmatrix}-3\\-2\\2\end{bmatrix}+c_3\begin{bmatrix}1\\5\\-3\end{bmatrix}$$

without any row operations:

$$\begin{bmatrix}-7\\-3\\10\end{bmatrix}=\begin{bmatrix}4&-3&1\\5&-2&5\\6&2&-3\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$$

$$\boxed{\begin{array}{l}c_1=3\\c_2=-1\\c_3=2\end{array}}$$

23)

a) False - $A\vec{x}=\vec{b}$ is the matrix equation not the vector equation

b) True - This is based of Theorem #4.

c) False - Theorem #4 (d) refers to the coefficient matrix not the augmented matrix

d) True - Based on the row-vector rule.

e) True - Based on theorem #4

f) True - Based on theorem #4

29) Construct a 3x3 matrix not in echelon form whose colums span $\mathbb{R}^3$. Show that it has the desired property.

$$A=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$ - Not in echelon form since row #3 leading entry to the right of row 2

$$\Downarrow$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$ Row equivalent

Has a pivot in each row so it spans $\mathbb{R}^3$

31) For a matrix of size 3x2 to span $\mathbb{R}^3$, it must have a pivot in each row.

A pivot corresponds to a "1" in the reduced echelon form matrix

A pivot in a given row must be in a column to the right of all pivots above it.

In a 3x2 matrix with 3 rows and 2 columns, there are not enough columns to have a pivot position.

The same applies for any $m \times n$ matrix where $m > n$ as not every row can have a pivot column by the pigeon hole principle.