Chapter 6-1 - Orthogonality and Least Squares

1) Given $\vec{u}=\begin{bmatrix}-1\\2\end{bmatrix}, \vec{v}=\begin{bmatrix}4\\6\end{bmatrix}$ find $\vec{u}\cdot\vec{u}$, $\vec{v}\cdot\vec{u}$ and $\frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}$

$\vec{u}\cdot\vec{u}=(-1)^2+(2)^2=\boxed{5}$

$\vec{v}\cdot\vec{u}=(-1)(4)+(2\cdot6)=\boxed{8}$

$\frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}=\boxed{\frac{8}{5}}$

3) Given $\vec{w}=\begin{bmatrix}3\\-1\\-5\end{bmatrix}$ find $\frac{1}{\vec{w}\cdot\vec{w}}\vec{w}$.

$\vec{w}\cdot\vec{w}=3^2+(-1)^2+(-5)^2=35$

$\frac{1}{\vec{w}\cdot\vec{w}}\vec{w}=\boxed{\begin{bmatrix}\frac{3}{35}\\-\frac{1}{35}\\-\frac{1}{7}\end{bmatrix}}$

5) Given $\vec{u}=\begin{bmatrix}-1\\2\end{bmatrix}, v=\begin{bmatrix}4\\6\end{bmatrix}$ find $\left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$.

$\vec{u}\cdot\vec{v}=(-4)+12=8$

$\vec{v}\cdot\vec{v}=16+36=52$

$\left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}=\frac{8}{52}\vec{v}=\frac{2}{13}\vec{v}=\boxed{\begin{bmatrix}\frac{8}{13}\\\frac{12}{13}\end{bmatrix}}$

7) Given $\vec{w}=\begin{bmatrix}3\\-1\\-5\end{bmatrix}$ find $||\vec{w}||$

$||\vec{w}||=\sqrt{(3)^2+(-1)^2+(-5)^2}$

$||\vec{w}||=\boxed{\sqrt{35}}$

9) Find a unit vector in the direction of $\begin{bmatrix}-30\\40\end{bmatrix}$.

To make the math easy use $\begin{bmatrix}-3\\4\end{bmatrix}$

$\left|\left|\begin{bmatrix}-3\\4\end{bmatrix}\right|\right|=\sqrt{(9)+16}=5$

$\boxed{\begin{bmatrix}-\frac{3}{5}\\\frac{4}{5}\end{bmatrix}}$

11) Find the unit vector for the vector $\begin{bmatrix}\frac{7}{4}\\\frac{1}{2}\\1\end{bmatrix}$

Use the $\begin{bmatrix}7\\2\\4\end{bmatrix}$

$\left|\left|\begin{bmatrix}7\\2\\4\end{bmatrix}\right|\right|=\sqrt{49+4+16}$

$\left|\left|\begin{bmatrix}7\\2\\4\end{bmatrix}\right|\right|=\sqrt{69}$

$\boxed{\begin{bmatrix}\frac{7}{\sqrt{69}}\\\frac{2}{\sqrt{69}}\\\frac{4}{\sqrt{69}}\end{bmatrix}}$

13) Find the distance between $\begin{bmatrix}10\\-3\end{bmatrix}$ and $\begin{bmatrix}-1\\5\end{bmatrix}$

$dist\left(\begin{bmatrix}10\\-3\end{bmatrix},\begin{bmatrix}-1\\5\end{bmatrix}\right)=\left|\left|\begin{bmatrix}10\\3\end{bmatrix}-\begin{bmatrix}-1\\5\end{bmatrix}\right|\right|$

$dist(\vec{u},\vec{v})=\left|\left|\begin{bmatrix}11\\-2\end{bmatrix}\right|\right|$

$dist(\vec{u},\vec{v})=\sqrt{(11^2+4)}=\sqrt{125}=\boxed{5\sqrt{5}}$

15) Determine if the vectors are orthogonal where:

$\vec{a}=\begin{bmatrix}8\\-5\end{bmatrix}, \vec{b}=\begin{bmatrix}-2\\-3\end{bmatrix}$

$\vec{a}\cdot\vec{b}=(8)\cdot(-2)+(-5)(-3)$

$\vec{a}\cdot\vec{b}=-16+15=\boxed{-1}$

Not orthogonal

17) Determine if the vectors are orthogonal where:

$\vec{u}=\begin{bmatrix}3\\2\\-5\\0\end{bmatrix}, \vec{v}=\begin{bmatrix}-4\\1\\-2\\6\end{bmatrix}$

$\vec{u}\cdot\vec{v}=-12+2+10+0$

$\vec{u}\cdot\vec{v}=\boxed{0}$

orthogonal

19)

a) True - By definition since $\vec{v}\cdot\vec{v}$ is always ≥ 0

b) True - By theorem 6-1

c) True - Definition of perpendicular

d) False - Vectors in Nul A^T are orthogonal to vectors in Col A.

e) True - If a vector $\vec{x}$ is orthogonal to the spanning set, it is also orthogonal to the span,

20)

a) True - Since $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$, then $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$

b) False - $||c\vec{v}||=|c|\,||\vec{v}||$.

c) True - If a vector $\vec{x}$ is perpendicular to all of a subspace W, then, $\vec{x}\in W^{\perp}$

d) True - By Pythagorean theorem.

e) True - By theorem 6-3.

Chapter 6-1 - Inner Product, Length, and Orthogonality

23) Let $\vec{u} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -7 \\ -4 \\ 6 \end{bmatrix}$.

Compute and compare $\vec{u}\cdot\vec{v}$, $||u||^2$, $||\vec{v}||^2$, and $||\vec{u}+\vec{v}||^2$

$\vec{u}\cdot\vec{v} = -14+20-6 = \boxed{0}$

$||\vec{u}||^2 = (2)^2+(-5)^2+(-1)^2 = \boxed{30}$

$||\vec{v}||^2 = (-7)^2+(-4)^2+(6)^2 = \boxed{101}$

$||\vec{u}+\vec{v}||^2 = (-5)^2+(-9)^2+(5)^2 = \boxed{131}$

$||\vec{u}||^2+||\vec{v}||^2 = ||\vec{u}+\vec{v}||^2$ meaning they are orthogonal as $\vec{u}\cdot\vec{v}=0$

25) Let $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$. Describe the set of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ that are orthogonal to $\vec{v}$.

To be orthogonal:

$ax+by=0$

Hence the vector is a scalar multiple of the vector $\begin{bmatrix} -b \\ a \end{bmatrix}$.

27) Suppose a vector $\vec{y}$ is orthogonal to vectors $\vec{u}$ and $\vec{v}$. Show $\vec{y}$ is orthogonal to $\vec{u}+\vec{v}$.

If orthogonal:

$\vec{y}\cdot\vec{u}=0$ and $\vec{y}\cdot\vec{v}=0$

$\vec{y}\cdot(\vec{u}+\vec{v}) = \vec{y}\cdot\vec{u}+\vec{y}\cdot\vec{v}$ by theorem 6-1 (b)

$y\cdot(\vec{u}+\vec{v}) = 0+0$

$y\cdot(\vec{u}+\vec{v}) = 0$ so orthogonal by definition.

29) Let $W=\text{span}\{\vec{v}_1,\ldots,\vec{v}_p\}$. Show that if $\vec{x}$ is orthogonal to each v_j where $1\le j\le p$, then $\vec{x}$ is orthogonal to every vector in W.

Any vector $\vec{u}\in W$ is a linear combination of $\vec{v}_1,\ldots,\vec{v}_p$.

$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p$

$\vec{x}\cdot\vec{u} = \vec{x}\cdot(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p)$

$= c_1(x\cdot\vec{v}_1) + c_2(x\cdot\vec{v}_2) + \cdots + c_p(\vec{x}\cdot\vec{v}_p)$

$= c_1\cdot 0 + c_2\cdot 0 + \cdots + c_p 0$

$\vec{x}\cdot\vec{u} = 0$

making $\vec{x}$ orthogonal to all vectors in W.