MATH 129A (8) **Midterm 2** **Name:**

April 27, 2017 **75 pts.**

Show all your work!

1. 8 pts. Find the inverse of the following matrix if it exists:

$$\begin{bmatrix} 1 & -2 & 1 \\ -3 & 7 & -6 \\ 2 & -3 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ -3 & 7 & -6 & 0 & 1 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 \end{array}\right] \sim \begin{matrix} \\ 3R_1+R_2 \\ -2R_1+R_3 \end{matrix} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 3 & 1 & 0 \\ 0 & 1 & -2 & -2 & 0 & 1 \end{array}\right] \sim \begin{matrix} \\ \\ (-1)R_2+R_3 \end{matrix} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 3 & 1 & 0 \\ 0 & 0 & 1 & -5 & -1 & 1 \end{array}\right]$$

$$\sim \begin{matrix} (-1)R_3+R_1 \\ 3R_3+R_2 \\ \end{matrix} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 6 & 1 & -1 \\ 0 & 1 & 0 & -12 & -2 & 3 \\ 0 & 0 & 1 & -5 & -1 & 1 \end{array}\right] \sim 2R_2+R_1 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -18 & -3 & 5 \\ 0 & 1 & 0 & -12 & -2 & 3 \\ 0 & 0 & 1 & -5 & -1 & 1 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} -18 & -3 & 5 \\ -12 & -2 & 3 \\ -5 & -1 & 1 \end{bmatrix}$$

Check: $AA^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -3 & 7 & -6 \\ 2 & -3 & 0 \end{bmatrix}\begin{bmatrix} -18 & -3 & 5 \\ -12 & -2 & 3 \\ -5 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \stackrel{\checkmark}{=} I$

2. 8 pts. Determine which of the matrices are invertible. Use as few calculations as possible. Justify your answers.

3pts (a) $\begin{bmatrix} 3 & -5 \\ 4 & 8 \end{bmatrix}$ $\det A = 3\cdot 8 - (-5)\cdot 4 = 44 \neq 0 \Rightarrow$ A is invertible
Or notice that columns are linearly independent (they are not scalar multiples), so by IMT (e) A is invertible

5pts (b) $\begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{bmatrix}$ Row reduce to echelon form:

$$\sim \begin{matrix} \\ \\ 3R_1+R_3 \end{matrix} \begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ 0 & -9 & -12 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & -5 & -4 \\ 0 & \boxed{3} & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

A is not row equivalent to I_3, $\Rightarrow$ A does not have 3 pivots $\Rightarrow$ By IMT (-b, c), A is not invertible

Also, notice that $\det A = 1\cdot 3\cdot 0 = 0$
$\Rightarrow$ A is not invertible!

3. 7 pts. Explain why the columns of A^2 span $\mathbb{R}^n$ whenever the columns of A are linearly independent. [*Hint:* $A^2 = AA$].

If the columns of A are linearly independent, then A is invertible by IMT (e). So, $A^2 = A \cdot A$ is also invertible because it is a product of two invertible matrices. Then, by IMT (h), the columns of A^2 span $\mathbb{R}^n$.

4. 8 pts. Combine the methods of row reduction and cofactor expansion to compute the following determinant:

$$\begin{vmatrix} -2 & 1 & 4 & -1 \\ 1 & 0 & -1 & 2 \\ 5 & -1 & 2 & 1 \\ 0 & 0 & 3 & -1 \end{vmatrix}$$

$$\begin{vmatrix} -2 & 1 & 4 & -1 \\ 1 & 0 & -1 & 2 \\ 5 & -1 & 2 & 1 \\ 0 & 0 & 3 & -1 \end{vmatrix} \underset{R_1+R_3 \to R_3}{=} \begin{vmatrix} -2 & 1 & 4 & -1 \\ 1 & 0 & -1 & 2 \\ 3 & 0 & 6 & 0 \\ 0 & 0 & 3 & -1 \end{vmatrix} \underset{\substack{\text{cofactor} \\ \text{expansion} \\ \text{down 2nd col.}}}{=} 1 \cdot (-1)^{1+2} \begin{vmatrix} 1 & -1 & 2 \\ 3 & 6 & 0 \\ 0 & 3 & -1 \end{vmatrix} \underset{(-3)R_1+R_2 \to R_2}{=}$$

$$= - \begin{vmatrix} 1 & -1 & 2 \\ 0 & 9 & -6 \\ 0 & 3 & -1 \end{vmatrix} \underset{\substack{\text{cofactor} \\ \text{expansion} \\ \text{down 1st col.}}}{=} (-1)(-1)^{1+1} \begin{vmatrix} 9 & -6 \\ 3 & -1 \end{vmatrix} = (-1)\big(9(-1) - (-6)3\big) =$$

$$= (-1)(-9+18) = -9$$

det $A = -9$

5. 10 pts. Let W be the set of all vectors of the form shown, where a, b, and c represent arbitrary real numbers. In each case, either find a set S of vectors that spans W or give an example to show that W is not a vector space.

(a) $\begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix}$

$$\begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix} \overset{?}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{cases} -a+1=0 \\ a-6b=0 \\ 2b+a=0 \end{cases} \quad \begin{matrix} a=1 \\ b=1/6 \\ b=-1/2 \end{matrix} \rightarrow$$

Contradiction (b cannot be 1/6 and -1/2 at the same time) There is no solution, the system is inconsistent!

$\vec{0}$ is not in W, which means that W is not a vector space

(b) $\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix}$

$$\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix} = \begin{bmatrix} a-b+0 \\ 0+b-c \\ -a+0+c \\ 0+b+0 \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}}_{\vec{u}} + b\underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}} + c\underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{w}}$$

So, $W = \text{Span}\{\vec{u}, \vec{v}, \vec{w}\}$ and, therefore, is a subspace of $\mathbb{R}^4$ (and, consequently, a vector space)

6. 10 pts. Find an explicit description of Nul A by listing vectors that span the null space.

$\text{Nul } A = \{\vec{x} : A\vec{x} = \vec{0}\}$

$$A = \begin{bmatrix} 1 & -6 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

① Reduce A to reduced echelon form:

$$\left[\begin{array}{cccc|c} 1 & -6 & 4 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{array}\right] \sim_{6R_2+R_1} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 16 & 0 & 0 \\ 0 & \boxed{1} & 2 & 0 & 0 \end{array}\right]$$

Basic var.: x_1, x_2
Free : x_3, x_4

$x_1 + 16x_3 = 0$
$x_2 + 2x_3 = 0$

General solution: $\begin{cases} x_1 = -16x_3 \\ x_2 = -2x_3 \\ x_3, x_4 \text{ are free} \end{cases}$

In vector form:

$$\vec{x} = \begin{bmatrix} -16x_3 \\ -2x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} -16 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Nul } A = \text{Span}\left\{\begin{bmatrix} -16 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$$\sim \begin{array}{l} \\ -2R_1+R_2 \\ -3R_1+R_3 \end{array}\begin{bmatrix} 1 & -2 & 3 & 0 & -1 \\ 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & -1 & 3 & -5 \end{bmatrix}.$$

$$\sim \begin{array}{l} \\ \\ R_2+R_3 \end{array}\begin{bmatrix} \boxed{1} & -2 & 3 & 0 & -1 \\ 0 & 0 & \boxed{1} & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7. 12 pts. Find bases for Nul A and Col A.
Also, find the dimensions of Nul A and Col A.

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 & -1 \\ 2 & -4 & 7 & -3 & 3 \\ 3 & -6 & 8 & 3 & -8 \end{bmatrix}$$

① Row reduce A to echelon form: Basis for Col A: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix} \right\}$.

4/15 (take pivot col. of A to form basis)

dim Col A = 2

② Row reduce A to reduced echelon form:

?/15

$$\begin{bmatrix} 1 & -2 & 3 & 0 & -1 \\ 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{array}{l} (-3)R_2+R_1 \\ \\ \\ \end{array}\begin{bmatrix} \boxed{1} & -2 & 0 & 9 & -16 \\ 0 & 0 & \boxed{1} & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solve $A\vec{x} = \vec{0}$
Basic var: x_1, x_3
Free var: x_2, x_4, x_5

$x_1 - 2x_2 + 9x_4 - 16x_5 = 0$
$x_3 - 3x_4 + 5x_5 = 0$

General solution: $\begin{cases} x_1 = 2x_2 - 9x_4 + 16x_5 \\ x_2 \text{ is free} \\ x_3 = 3x_4 - 5x_5 \\ x_4, x_5 \text{ are free} \end{cases}$

Nul A = Span$\{\vec{u}, \vec{v}, \vec{w}\}$

In vector form:

$$\vec{x} = \begin{bmatrix} 2x_2 - 9x_4 + 16x_5 \\ x_2 \\ 3x_4 - 5x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{u}} + x_4 \underbrace{\begin{bmatrix} -9 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}} + x_5 \underbrace{\begin{bmatrix} 16 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}}_{\vec{w}}$$

Basis for Nul A: $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 16 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$

dim Nul A = 3

8. 12 pts. Use coordinate vectors to test the linear independence of the set of polynomials. Explain your work.

$$1 + 3t^3, \qquad 2 + 4t - t^2, \qquad -t + 2t^2 - t^3$$

Standard basis for $\mathbb{P}_3$: $\{1, t, t^2, t^3\}$

Write coordinate vectors for each polynomial:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ -1 \end{bmatrix}$$

Vectors in $\mathbb{R}^4$ ⇒ Study $A\vec{x} = \vec{0}$ to test independence

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 3 & 0 & -1 & 0 \end{array}\right] \sim \begin{array}{l} \\ \\ \\ (-3)R_1+R_4 \end{array}\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -6 & -1 & 0 \end{array}\right] \sim$$

$$\sim \begin{array}{l} \\ \\ \\ R_2 \leftrightarrow R_3 \end{array}\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & -6 & -1 & 0 \end{array}\right] \sim \begin{array}{c} \left[\begin{array}{ccc|c} \boxed{1} & 2 & 0 & 0 \\ 0 & \boxed{-1} & 2 & 0 \\ 0 & 0 & \boxed{7} & 0 \\ 0 & 0 & -13 & 0 \end{array}\right] \\ x_1 \quad x_2 \quad x_3 \end{array}$$

⇒ no free variables ⇒ only trivial solution ⇒ vectors are independent in $\mathbb{R}^4$

By isomorphism between $\mathbb{P}_3$ and $\mathbb{R}^4$ ⇒ polynomials are independent.