Section 5-1- Eigenvectors and Eigen value

1) Is $\lambda=2$ an eigenvalue of $\begin{bmatrix}3&2\\3&8\end{bmatrix}$?

$$\begin{bmatrix}3&2\\3&8\end{bmatrix}-\begin{bmatrix}2&0\\0&2\end{bmatrix}=\begin{bmatrix}1&2\\3&6\end{bmatrix}$$

Yes as the matrix $\begin{bmatrix}1&2\\3&6\end{bmatrix}$ has one free variable.

2) Is $\lambda=-2$ an eigenvalue of $\begin{bmatrix}7&3\\3&-1\end{bmatrix}$?

$$\begin{bmatrix}7&3\\3&-1\end{bmatrix}-\begin{bmatrix}2&0\\0&2\end{bmatrix}=\begin{bmatrix}5&3\\3&-3\end{bmatrix}$$

No columns are linearly independent

3) Is $\begin{bmatrix}1\\4\end{bmatrix}$ an eigenvector of $\begin{bmatrix}-3&1\\-3&8\end{bmatrix}$?

$$\begin{bmatrix}-3&1\\-3&8\end{bmatrix}\begin{bmatrix}1\\4\end{bmatrix}=\begin{bmatrix}1\\29\end{bmatrix}$$

No the vectors are not scalar multiples.

5) Is $\begin{bmatrix}4\\-3\\1\end{bmatrix}$ an eigenvector of $\begin{bmatrix}3&7&9\\-4&-5&1\\2&4&4\end{bmatrix}$? If so find the eigenvalue.

$$\begin{bmatrix}3&7&9\\-4&-5&1\\2&4&4\end{bmatrix}\begin{bmatrix}4\\-3\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Yes $\boxed{\lambda=0}$

6) Is $\lambda=4$ a eigenvector of $\begin{bmatrix}3&6&7\\3&3&7\\5&6&5\end{bmatrix}$? If so find the eigenvalue.

$$\begin{bmatrix}3&6&7\\3&3&7\\5&6&5\end{bmatrix}\begin{bmatrix}1\\-2\\1\end{bmatrix}=\begin{bmatrix}-2\\4\\-2\end{bmatrix}$$

Yes $\boxed{\lambda=-2}$

7) Is $\lambda=4$ an eigenvalue of $\begin{bmatrix}3&0&-1\\2&3&1\\-3&4&5\end{bmatrix}$? If so find one corresponding eigenvector.

$$\begin{bmatrix}3&0&-1\\2&3&1\\-3&4&5\end{bmatrix}-\begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix}=\begin{bmatrix}-1&0&-1\\2&-1&1\\-3&4&1\end{bmatrix}$$

$$\begin{bmatrix}-1&0&-1\\2&-1&1\\-3&4&1\end{bmatrix}\underset{\substack{R_2'=R_2+2R_1\\R_3'=R_3-3R_1}}{\sim}\begin{bmatrix}-1&0&-1\\0&-1&-1\\0&4&4\end{bmatrix}$$

$$\underset{R_3'=R_3-4R_2}{\sim}\begin{bmatrix}-1&0&-1\\0&-1&-1\\0&0&0\end{bmatrix}$$

$$\underset{\substack{R_2'=-R_2\\R_1'=-R_1}}{\sim}\begin{bmatrix}1&0&1\\0&1&1\\0&0&0\end{bmatrix}$$

$$\vec{x}=\begin{cases}x_1=-x_3\\x_2=-x_3\\x_3 \text{ is free}\end{cases}$$

$$\vec{x}=x_3\begin{bmatrix}-1\\-1\\1\end{bmatrix},\ x_3\in\mathbb{R}$$

Yes $\boxed{\begin{bmatrix}-1\\-1\\1\end{bmatrix}}$

9) Find the eigen space where $A=\begin{bmatrix}5&0\\2&1\end{bmatrix}$, $\lambda=1,5$.

a) $\lambda=1$

$$\begin{bmatrix}5&0\\2&1\end{bmatrix}-\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}4&0\\2&0\end{bmatrix}$$

$$\vec{x}=\begin{cases}x_1=0\\x_2 \text{ is free}\end{cases}$$

$$\boxed{\text{Basis}_{\lambda=1}=\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}}$$

b) $\lambda=5$

$$\begin{bmatrix}5&0\\2&1\end{bmatrix}=\begin{bmatrix}5&0\\0&5\end{bmatrix}=\begin{bmatrix}0&0\\2&-4\end{bmatrix}$$

$$\vec{x}=\begin{cases}x_1=2x_2\\x_2 \text{ is free}\end{cases}$$

$$\boxed{\text{Basis } \lambda=5=\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}}$$

11) Find the basis for the eigenspace where $A=\begin{bmatrix}4&-2\\-3&9\end{bmatrix}$, $\lambda=10$.

$$\begin{bmatrix}4&-2\\-3&9\end{bmatrix}-\begin{bmatrix}10&0\\0&10\end{bmatrix}=\begin{bmatrix}-6&-2\\-3&-1\end{bmatrix}$$

$$\sim\begin{bmatrix}3&1\\0&0\end{bmatrix}$$

$$\vec{x}=\begin{cases}x_1=-\frac{1}{3}\\x_2 \text{ is free}\end{cases}$$

$$\boxed{\text{Basis}_{\lambda=10}=\left\{\begin{bmatrix}-\frac{1}{3}\\1\end{bmatrix}\right\}}$$

Section 5-1 - Eigenvalues and Eigenvectors

13) Find the basis for the eigenspace where $A=\begin{bmatrix}4&0&1\\-2&1&0\\-2&0&1\end{bmatrix}$ and $\lambda=1,2,3$.

a) $\lambda=1$

$$\begin{bmatrix}4&0&1\\-2&1&0\\-2&0&1\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}3&0&1\\-2&0&0\\-2&0&0\end{bmatrix}$$

$$\sim\begin{bmatrix}3&0&1\\1&0&0\\0&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&\frac{1}{3}\\1&0&0\\0&0&0\end{bmatrix}$$

$$\begin{bmatrix}1&0&\frac{1}{3}\\0&0&-\frac{1}{3}\\0&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&0\\0&0&-\frac{1}{3}\\0&0&0\end{bmatrix}$$

$$\vec{x}=\begin{cases}x_1=0\\x_2 \text{ is free}\\x_3=0\end{cases}\qquad \boxed{\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}}$$

b) $\lambda=2$

$$\begin{bmatrix}4&0&1\\-2&1&0\\-2&0&1\end{bmatrix}-\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}=\begin{bmatrix}2&0&1\\-2&-1&0\\-2&0&-1\end{bmatrix}$$

$$\sim\begin{bmatrix}2&0&1\\0&-1&1\\0&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&\frac{1}{2}\\0&1&-1\\0&0&0\end{bmatrix}$$

$$x=\begin{cases}x_1=-\frac{1}{2}x_3\\x_2=x_3\\x_3 \text{ is free}\end{cases}\qquad \boxed{\left\{\begin{bmatrix}-\frac{1}{2}\\1\\1\end{bmatrix}\right\}}$$

c) $\lambda=3$

$$\begin{bmatrix}4&0&1\\-2&1&0\\-2&0&1\end{bmatrix}-\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}=\begin{bmatrix}1&0&1\\-2&-2&0\\-2&0&-2\end{bmatrix}$$

$$\sim\begin{bmatrix}1&0&1\\1&1&0\\0&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&1\\0&1&-1\\0&0&0\end{bmatrix}$$

$$\vec{x}=\begin{cases}-1x_3\\x_3\\x_3 \text{ is free}\end{cases}$$

$$\boxed{\left\{\begin{bmatrix}-1\\1\\1\end{bmatrix}\right\}}$$

15) Find the basis for the eigenspace where $A=\begin{bmatrix}4&2&3\\-1&1&-3\\2&4&9\end{bmatrix}$ and $\lambda=3$.

$$\begin{bmatrix}4&2&3\\-1&1&-3\\2&4&9\end{bmatrix}-\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}=\begin{bmatrix}1&2&3\\-1&-2&-3\\2&4&6\end{bmatrix}$$

$$\sim\begin{bmatrix}1&2&3\\0&0&0\\0&0&0\end{bmatrix}$$

$$\vec{x}=\begin{cases}-2x_2-3x_3\\x_2 \text{ is free}\\x_3 \text{ is free}\end{cases}$$

$$\boxed{\text{Basis:}\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix},\begin{bmatrix}-3\\0\\1\end{bmatrix}\right\}}$$

17) Find the eigenvalues of the matrix $\begin{bmatrix}0&0&0\\0&2&5\\0&0&-1\end{bmatrix}$.

By theorem 5-1, they are $\{0,2,-1\}$

19) For $A=\begin{bmatrix}1&2&3\\1&2&3\\1&2&3\end{bmatrix}$, find one eigenvalue.

The columns of A are linearly dependent (scaler multiples). Hence one eigenvalue is $\boxed{0}$.

21)

a) False – $\vec{x}$ must be non-zero.

b) True – An eigenvalue is zero iff A is non-invertible.

c) True – This comes from the equation $A\vec{x}=\lambda\vec{x}$

d) True – It is found by checking $A\vec{x}=c\vec{x}$

e) False – Row reducing changes the eigenvalues and eigenvectors.

22)

a) False – $\vec{x}$ must be nonzero.

b) False – Linearly independent values can have the same eigenvalue. This is different from theorem 5-2

c) True – In a difference equation $\vec{x}_p=A\vec{x}_p$, where $\vec{x}_p$ is the steady state vector.

d) False – The eigenvalues are only on the diagonal of triangular matrices

e) True – It is the nullspace of $A-\lambda I$ where λ is an eigenvalue of A.

Section 5-1 - Eigenvalues and Eigenvectors

23) Explain why a 2x2 matrix can have at most two distinct eigenvalues. Why can an nxn matrix have at most n distinct eigenvalues.

An eigenvalue corresponds to a set of vectors linearly independent to other eigenvectors.

Hence, a 2x2 matrix can have at most 2 linearly independent sets of vectors. Any more and they would not be linearly independent. The same goes for a nxn matrix which can have at most n linearly independent sets.

15) Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}!

$$A\vec{x} = \lambda\vec{x}$$

$$A^{-1}(A\vec{x}) = A^{-1}(\lambda\vec{x})$$

$$(A^{-1}A)\vec{x} = \lambda(A^{-1}\vec{x})$$

$$I_n\vec{x} = \lambda(A^{-1}\vec{x})$$

$$\frac{1}{\lambda}\vec{x} = A^{-1}\vec{x}$$

$$\boxed{\lambda^{-1}\vec{x} = A^{-1}\vec{x}}$$