**MATH129A – Linear Algebra Final Exam Study Guide**

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# Linear Equations in Linear Algebra

## Systems of Linear Equations

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| **Linear Equation** – An equation with **variables** that **can be written in the form**:  **Coefficients:** can be **real or complex** | **Linear System** or **System of Linear Equations** – Collection of **one or more linear equations**. | **Solution:** A **set of numbers** **that makes each equation a true statement** when substituted for variables respectively. | **Solution Set:** Set of **all possible solutions** for a linear system.    **Possible Solution Sets:**   * **No solution** * **One solution** * **Infinite solutions** |

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| **Consistent Linear System** – Has **one or infinite solutions** | **Coefficient Matrix** – A matrix containing the **coefficients for each variable in each equation** in the linear system. | **Augmented Matrix** – A matrix of a system containing the **coefficient matrix and** **an added column containing the constants** from the ***right hand side*** of the equation. | **Techniques to Simplify a Linear System**   1. **Replace** one equation with **sum of itself and the multiple of another linear system** (equation) 2. **Interchange** two equations 3. **Multiply all terms** in an equation by a **non-zero constant** |
| **Inconsistent Linear System** – Has no solution |

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| **Row Equivalent Matrices** – Any two matrices where a **series of elementary row operations** can transform one matrix into another. | **Row Operation Reversibility** – All row operations can be undone to get the previous matrix | **Matrix** – Composed of rows of columns | **Equivalent Linear Systems** – Any two linear systems with the **same solution set**. |

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| **Approaches to Find the Solution Set of a Linear System**   1. **Solve equations by substitution.** 2. **Multiply and add the equations** 3. **Graphically**    1. Look at the intersection of the equations. |  |  |  |

## Row Reduction and Echelon Forms

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|  | **Key Properties** | | | | |
| If **two augmented matrices are row equivalent**, then the systems **have the same solution set**. | **Reduced echelon form is unique** | **Echelon form is not unique** | All linear systems have a reduced echelon form. | **Location of leading entries** is the **same between standard and reduced echelon form.** |

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| **Echelon Matrix Criteria**   1. **All non-zero rows are above all zero rows.** 2. **The leading entries of lower rows are to the right of all those in upper rows.**    1. Forms a “**step pattern**”. 3. **The entries in a column below a leading entry are zero.** | **Reduced (Row) Echelon Matrix Criteria**   1. **All criteria of a standard echelon matrix.** 2. **All leading entries equal 1.** 3. **The entries in a column above a leading entry are 0.** | **Theorem #1: Any matrix has one and only one** **reduced echelon form**. |

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| **Pivot Position** – A **position** in a given matrix that **corresponds to a “1” in reduced echelon form**. | **Pivot Position** – A column that contains a pivot position. | **Pivot** – A non-zero number in a pivot position that is used as needed to create zeros via row operations. | **Gaussian Elimination:** Same concept as row reduction. | **Non-zero row/column**: A row/column with **at least one non-zero entry**. | **Leading entry** – **Leftmost non-zero entry** **in a row**. |

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| **Row Reduction Algorithm (Gaussian Elimination)**  **Forms an Echelon Matrix** | | | | **Gauss-Jordan Elimination**  **Forms the Reduced Echelon Matrix** |
| 1. **Begin with the left most non-zero entry that is a pivot position. Move that position to the top of the matrix.** | 1. **Select a non-zero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.** | 1. **Use row operations to create zeros in all positions below the pivot.** | 1. **Cover (i.e. ignore) the rows containing the pivot position and cover (ignore) all rows above it. Apply steps 1-3 to the sub matrix that remains. Repeat the process until there are no more non-zero rows remaining.** | 1. **Create zeros above each pivot and scale each pivot to 1.** |

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| **Basic Variable: Can only exist in a single solution set equation. Correspond to a pivot in the reduced echelon matrix.** | **Parametric Description of Solution Sets**   * Free variables act as parameters * Each basic variable is represented by an equation made up of constants and/or free variables. | **Theorem 1-2: Existence and Uniqueness Theorem**  A **linear system is consistent** if and only if the rightmost column of the augmented matrix is not a pivot column – that is, if and only if an echelon form of the **augmented matrix has no row in the form**:  withnon-zero  If a linear system is consistent, then the solution set contains either:   1. **A unique solution, when there is no free variable** 2. **Infinitely many solutions when there is at least one free variable.** |
| **Free Variable:** **Can be assigned to any number** since it has no pivot. Free variable **quantity dictates the resulting solution set’s shape** (e.g., plane, line etc.). |

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| **Example: Parametric Description of a Solution Set** | |  |
| **Augmented Matrix** | **Parametric Description of the Solution Set** |

## Vector Equations

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| **Vector** – An **ordered** list of numbers | **Column Vector** – A **matrix with only one column**. | **Vector Equality** – Two vectors of the **same size** with **all corresponding entries equal**. | **Vector Addition** – Sum obtained by **adding the corresponding entries of two vectors**. | **Scalar Multiple** –Given a number, , and a vector, , it is the vector obtained when is **multiplied by each element** in |

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| **Parallelogram Rule for Addition** – If and are in (i.e., points in the Cartesian plane), then corresponds to the **fourth point** in the **parallelogram whose vertices are** , and (the origin). | **Scalar Multiples of a Fixed Vector** – **Along a line between the vector point and the origin**. | **Zero Vector** () – A vector whose entries are all zeros. | **Linear Combination** – Given vectors and scalars , then the **linear combination vector**  is: |

**Algebraic Properties in**

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| **Commutative** | **Associative** | **Identity** | **Inverse** |
| **Distributive** | **Distributive** | **Associative** | **Identity** |

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| **The Vector Equation and  the Augmented Matrix**  A **vector equation**:  has the **same solution set as the linear system whose augmented matrix is**:  In particular, can be generated by a linear combination of if and only if there exists a solution to the linear system corresponding to the augmented matrix. | **Subset of Spanned /  Generated by**  If are in , then the set of all linear combinations of is denoted by:  It is the set of all linear combination vectors that can be written in the form: | **Geometric Description of Spans of Vectors** | |
| **(Zero Vector)** | A **point** – The origin |
| **One non-zero vector** | A **line** through the origin and |
| **Two vectors**  and that **are scalar multiples** | A **line** through the origin and |
| **Two vectors**  and **not scalar multiples** | A **plane** through the origin, , and |

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| **Members of All Spans**   * **Zero vector** () * **Scalar multiples of the original vectors** | **How to Determine if a Vector is in a Span** **Check if the system is consistent** | **Important Notation** | | | | |
| **Vector in** | | | **Not a vector** | **Set of Vectors** |
|  |  |  | **It is a matrix** |  |

## The Matrix Equation

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| **Gaussian Elimination** – Same concept as row reduction. | **Theorem #1-4:** Let be an matrix. Then the following statements are equivalent meaning they are **either all true** **or all false**.   1. **For all , the equation has a solution.** 2. **Each is a linear combination of the columns of .** 3. **The columns of span** 4. **The (coefficient) matrix has a pivot position in every row.** | **Relationship between Spans and Free Variables**   |  |  | | --- | --- | |  | **General Solution Structure** | | **(Trivial Only)** |  | | **1 Free Variable** | – **Line** through and the origin | | **2 Free Variables** | – **Plane** through , , and the origin | | **Free Variables** | – **Multidimensional shape** through and the origin | |
| **Non-zero row/column** – A row/column with at least one non-zero entry. |
| **Theorem #1-3:** If is an matrix with columns and if , then **the matrix equation**:  **has the same solution set** as the **vector equation**:  which **has the same solution set as the system of linear equations whose augmented matrix is**: |

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| **Theorem #1-5:** If is an matrix, and are vectors in , and is a scalar, then: | **Row-Vector Rule for Computing**  If the product is defined (i.e., the sizes correspond), then the entry in is the sum of products of the corresponding entries from row of and from the vector . It is formally: | **Matrix Equation Definition**  If is an matrix with columns and if , then the **product**, of and , denoted by , **is the linear combination of the columns of** **using the corresponding entries in**  **as the weights**; that is: |

## Solution Sets of Linear Systems

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| **Homogeneous Linear System:**   * – Matrix * – Vector in * – Zero vector in   **Trivial Solution: (in )**   * **Exists for all homogeneous systems**   **Nontrivial Solution:** Any **non-zero vector solution to the equation:**   * If it exists, there are infinitely many. * Requires at least one free variable. | **Non-Homogenous System:** where is not the zero vector.   * May be inconsistent. * If its solution exists, it is in the form:   + If is , then   + – Particular solution for the specific non-homogenous system   + – Solution set for the homogenous system | **Theorem #1-6:** Suppose the equation is consistent for some given , and let be **any solution to that particular non-homogenous system**. Then the **solution set of** (**if it exists**)is **the set of all vectors of the form**:  where is **any solution of the homogenous equation** . |

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| **Solution Set Notations** | | | For to span all of , there **must be a pivot in every row of the echelon matrix** of . |
| **Augmented Matrix** | **Parametric Description of the Solution Set** | **Parametric Vector Form** |

## Linear Independence

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| **Span Review**   * **One Vector** – **At most** a **line**   + **Exception:** vector * **Two Vectors: At most** a **plane**   + **Exception:** Scalar multiples and the zero vector | **Linear Independence**  A set of vectors are **linearly independent** if:  has **only the trivial solution** | **Linear Dependence**  A set of vectors are **linearly dependent** if there exists a set of **non-zero weights** such that:  **Note:** This requires **at least one free variable**. |

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| **Linear Dependence Relation:** For vectors , it is defined as:  where are **not all zero.**  **Note:** The values of are not unique. **If one exists, an infinite number exist.** | **Procedure: Checking for Linear Independence**  **Step #1:** Create the coefficient matrix.  **Step #2:** Perform Gaussian elimination to find the echelon matrix.  **Step #3:** Check linear independence   * If **there is a pivot in every column**, the vectors are **linearly independent**. * If **there is a free variable**, the vectors are **linearly dependent**. | **Linear Independence of One Vectors**   * **Zero Vector** () – This is always **linearly dependent**. * **Any Non-Zero Vector** ( where ) then **linearly independent**. |

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| **Linear Independence of Two Vectors** – A set of two vectors is linearly dependent if **at least one** **of the vectors is a scalar multiple** of the other.   * “**At least one**” – Because of the case of the zero () vector. | For a set of vectors to be **linearly independent**, the echelon matrix made from those vectors must have no free variables.   * **There must be a pivot in every column.** | **Linear Dependence Summary:** If a set of  **vectors of -dimensions** are linearly independent, then they **span an dimensional shape**. |

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| **Theorem #1-7:** An indexed set of two or more vectors is **linearly dependent** **if and only if at least one of the vectors in is a linear combination of the others**.  In fact, if is linearly dependent and , then some (where ) is a linear combination of the preceding vectors: . | **Theorem #1-8:** If a set **contains more vectors than there are entries in each vector**, then the set is **linearly dependent**.  That is any set inis linearly dependent if:  **Proof:** More pivots than columns in an matrix so **at least one free variable**. | **Theorem #1-9:** If a set  **contains the zero vector**, then the set is **linearly dependent**.  **Proof:** If the zero vector is reordered to be , then |

## Introduction to Linear Transformations

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| **Function** – A **rule** that **assigns to each element** in a set **exactly one element** from set . | **Transformation () from to :** A rule that **assigns to each vector**  **a vector** | **Domain of :** | **Image of :** For a given , it is the **transformed value** | **Range: Set of all images** .  **Note:** The range may be (and often is) only a subset of the codomain. |
| **Codomain of :** |

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| **Linear Transformations:** **Preserve vector addition and scalar multiplication** |  |  |  |  |

**Properties of Matrix Transformations**

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| **Requirements of Linear Transformation** | | **Zero Vector Properties** | |  |
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**Types of Matrix Transformations**

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| **Contraction**  **,**  **where** | **Dilation**  **, where** | **Projection onto an Axis**  **Matrices in the form:**  **– Projection onto -plane**  **– Projection onto the -axis** | **Shear Transformations**  **Maps line segments onto line segments. Deforms the shape of the original input.** |

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| **Reflection through the Origin**  Matrix in the Form of: | **Reflection through the -axis**  Matrix in the Form of: | **Reflection through the -axis**  Matrix in the Form of: | **Reflection through the line**  Matrix in the Form of: |

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| **Twice as Long and 1.5 Times as High**  Matrix in the Form of: | **Rotation about Origin through 90­o**  Matrix in the Form of: |  |

# Matrix Algebra

## Matrix Operations

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| or – Given an matrix, it is the element in the  **row** and  **column** of | **Diagonal Entries** – In an matrix, it is the entries , etc. | **Diagonal Matrix** – A **square** matrix whose non-diagonal entries are 0.  **Example:** **Identity matrix** | **Equal Matrices** – Two matrices having **the same size** (i.e., equal number of rows and columns) and **whose corresponding entries are equal**. | **Matrix Sum** – Given two matrices of the same size, the matrix sum is equivalent to the **sum of the corresponding columns**. |

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| **Matrix Multiplication**  If is an matrix and if is an matrix composed of the columns , then the **product is an matrix** whose columns are: such that: | **Row-Product Rule**  **If the product of is defined**, then the entry in row and column of is the sum of the products of corresponding entries from row of and column of such that: | **Size of a Matrix Multiplication:** Given a matrix of size and a matrix of size , then the matrix multiplication is of size . |
| **Non-Commutativity of Matrix Multiplication**  In most cases, |
| **Cancelation Laws Do Not Hold For Matrix Multiplication**  Generally, if , it **cannot be assumed** that |
| **Zero Matrix Product**  If , you **cannot generally assume** or |

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| **Powers of a Matrix**  If is an matrix and is a positive integer, then denotes the product of copies of such that:  **Zero Power** – Given a square matrix , then **is the identity matrix**. | **Transpose of a Matrix**  Given an matrix , then transpose of is an **matrix whose columns are formed from the corresponding rows of A**. | **Transpose Multiplication Row:** The **transpose of the product of matrices** **equals** the **product of their transposes in reverse order**. |

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| **Theorem #2-1:** Let , , and be **matrices of the same size** and let and be scalars, then:   * 1. **(Commutative)**   2. **(Associative)**   3. **(Identity)** | **Theorem #2-2:** Given is the identity matrix of size and is an matrix, and that and have sizes for which the indicated sums and products exist, then   1. **(Associative)** 2. **(Left Distributive Law)** 3. **(Right Distributive Law)** 4. **, for any scalar** | **Theorem #2-3:** Let and be matrices whose sizes are appropriate for the following sums and products:   1. **, for any scalar**   **Note: The reverse order of the product.** |

## The Inverse of a Matrix

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| A matrix is **invertible** **if there exists** a matrix such that:  **and** | **Inverse Matrix Notation:**  The **inverse matrix is unique (if it exists)**. | **Singular Matrix** – Any matrix that is **non-invertible**.  **Nonsingular Matrix** – Any **invertible matrix**.  **Hint for memorization:** “Non’s” are swapped and do not match. | **Theorem 2-4:** Let . If , is invertible and: | **Determinant of** – For a  **matrix**, it is:  **The inverse exists only if** |

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| **Theorem 2.5** – If  **is an invertible matrix**, then for all , then equation has a **unique solution in the form**: | **Theorem 2-6**   1. If is an **invertible matrix**, then  **is invertible** and: 2. If and are invertible matrices, then so is . The **inverse of is the product of the inverses in reverse order**, that is: 3. If is an invertible matrix, then so is . **The inverse of is the transpose of** , that is: | **Elementary Row Operation**: the three types are:   1. **Interchange (swap)** 2. **Scale** 3. **Add** | **Theorem 2.7:** An matrix is **invertible** if and only if it is **row equivalent to the identity matrix** .  Hence, **any sequence of row operations that reduces to also transforms to** . |
| **Elementary Matrix** – A matrix **obtained by performing a single** **elementary row operation** **on the identity matrix**. |

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| **Algorithm for Finding the Inverse Matrix**   1. **Place matrix and side by side in an augmented matrix.** 2. **Row reduce the matrix to transform to .** 3. **Extract the inverse matrix where:** | **Example Elementary Matrices in** | | |  |
|  | **Interchange and** | **Scale** |

## Characterizations of Invertible Matrices

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| If , then is **onto (surjective)** if:  **Definition:** Every element has **at least one corresponding in element in set**  **: Domain**  **: Codomain**  In **onto (surjective) functions**, the **codomain equals the range.** | **Definition:** Let . is **one-to-one** (**injective**) if:  **Note:** A **bijective** function is **both onto and one-to-one**. | **Theorem 2.8 – The Invertible Matrix Theorem**  Let be a **square**, matrix, then the following statements are **equivalent**. That is for a given , they are either **all false or all true**.   1. **is an invertible matrix.** 2. **is row equivalent to the identity matrix.** 3. **has pivot positions.** 4. **The equation has only the trivial solution.** 5. **The columns of form linearly independent set.** 6. **The linear transformation is one-to-one.** 7. **The equation has a solution for all .** 8. **The columns of span .** 9. **The linear transformation maps to .** 10. **There is an matrix such that .** 11. **There is an matrix such that .** 12. **is an invertible matrix.** 13. **The columns of form a basis for .** 14. **The number 0 is not an eigenvalue of .**   **Note:** This **applies only to square matrices**. |

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| **Invertible Linear Transformation:** A linear transformation such that there **exists a linear transformation such that**:  and  is known as the **inverse** of (i.e., )  It is also true that: | **Theorem 2-9:** Let be a linear transformation and let be the **standard matrix** for . Then  **is invertible if and only if is an invertible matrix**. Then the linear transformation given by:  is the **unique function** such that:  and | **Standard Matrix** – Matrix used in a linear transformation. |

# Determinants

## **Introduction** to **Determinants**

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| **Determinant (** – **Cannot equal zero for invertible matrices**. | **Determinants for Simple Matrices** | | |
| **Matrix** | **Matrix** | **Matrix** |

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| **Determinant Notation**   * – Element in matrix at row and column * – **Submatrix** of formed by **deleting row and column**. * – -cofactor. It is defined as: * **–** Another notation for **determinant**, namely **two vertical lines.** | **Cofactor Expansion – Technique for Finding a Determinant**  Technique **to find the determinant of an matrix** where (example is along the row):  Given:  **Note:** Cofactor expansion **can be done along any row or down any column in a square matrix**. | **Transpose and the Determinant**  **Transpose has no effect on the value of the determinant**. Hence: |

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| **Theorem 3-1:** The determinant of an matrix can be computed via c**ofactor expansion across any row or column**. The expansion **across the row** is:  The expansion down the column is: | **Theorem 3-2:** If is a **triangular matrix**, then is the **product of the entries along the diagonal** of . |  |

## Properties of Determinants

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| **Theorem 3-3: Effect of Row Operations on the Determinant**   1. **If a multiple of one row of is added to another row to produce a matrix, then:** 2. **If two rows of are interchanged to form a new matrix , then** 3. **If one row of is multiplied (i.e., scaled) by to form a new matrix , then** | **Alternate Technique for Finding a Determinant**   1. **Reduce to echelon form.** 2. **Multiply the elements along the diagonal.** 3. **Use theorem 3-3 to adjust for the effect of the row operations on the determinant in step 2.** | **Theorem 3-4:** **A square matrix** is **invertible** if and only if:  **Corollary #1:** if and only if the **columns of**  are **linearly dependent**.  **Corollary #2:** if and only if the **rows of**  are **linearly dependent**. | **Linear Dependent Columns**  Columns of a matrix are linear dependent if:   * **Two columns are equivalent** * **Two rows are equivalent** * **One column is the zero vector** * **One column is a linear combination of the other columns.** |

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| **Theorem 3-5:** If is an  **matrix**, then: | **Theorem 3-6 – Multiplicative Property:** If and are matrices, then: | **Checking for Linear Independence with Determinants**  If the **dimensions** (and number of vectors) are **appropriate, check if the determinant equals zero**. **If not, they are linearly independent**. |  |

# Vector Spaces

## Vector Spaces and Subspaces

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| **Space** – A **set of vectors**  – Set of **all vectors of components**.  **Vector** – An **object** **in a vector space**. (*Need not be a column of numbers*)  **Zero Subspace** – A **vector space containing only the zero vector** | **Properties of a Subspace**   1. **Contains the zero vector ()** 2. **Closed under vector addition** 3. **Closed under scalar multiplication.** | **Definition:** A **vector space** is a **nonempty set**  of objects called **vectors** on which are defined two operations called **addition** and **subtraction** subject to the 10 axioms list below. They must hold and .   1. **The sum of and denoted by** 2. **There is a zero vector () in such that:** 3. **For all , there is a vector such that . is known as the negative of .** 4. **The scalar multiple of by denoted by** |

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| **Definition:** A **subspace** **of a** **vector space** is a **subset**  of that has **three properties**:   1. **The zero vector of is in .** 2. **is closed under vector addition. That is:** 3. **is closed under multiplication by scalars, that is:** | **Proving a Set is a Subspace**  If **is a subset of vector space** , then **sufficient to only prove three axioms**:   * **1 (addition closure)** * **4 (presence of zero vector)** * **6 (scalar multiplication closure)**   **Every vector space is a subspace** (of itself),and **every subspace is a vector space**. | **Linear Combination** – A **sum of all scalar multiplies** of a set of vectors  – Set of **all vectors that are linear combinations** of   * The set of vectors is known as the **subspace spanned/generated** by * – This is known as the **spanning set**. * are known **as spanning vectors**. | **Additional Derivable Properties of a Vector Space** |

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| **Polynomials as a Vector Space**  A **polynomial** in is defined as:  is referred to as the “**degree**”.  **It is a vector space**.  **Zero Polynomial** – A polynomial whose **coefficients** **are all zero**. **Its degree is undefined**. | **Important Notes**   * **is not a subspace of**  (it is not even a subset) | **Theorem 4-1:** If are in a vector space , then **is a subspace of** . |  |

## Null Spaces, Column Spaces, and Linear Transformations

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| **Homogeneous Linear System** has the form: | **Null Space for a matrix**  () – For an  **matrix**, , it is the **set of all solutions to the homogeneous system**:  In set notation:  **Alternate Definition:** Set of all **that can be mapped to the zero vector** of via the **linear transformation**: | **Theorem 4-2:** The **null space** of an  **matrix** is a **subspace of** .  Equivalently, the set of all solutions to a system:  of  **homogeneous linear equations in unknowns is a subspace of** | **Implicit Definition of the Null Space**  is an **implicit description** since **each element must be individually checked against** .  For a given vector and matrix , verify: |
| **Solution Set**: Set of all that **satisfies** (i.e., makes **consistent**) a linear system. | **Explicit Description of the Null Space**  **Solve** **to get the explicit description** of |

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| **Column Space** () – For an **matrix**, , the column space, , is the **set of all linear combination of the columns** **of** .  For a matrix **,** then:  The **column space** is the **range of the linear transformation**: | **Theorem 4-3:** The **column space** of an **matrix** is a subspace of .  In set notation: |  |  |

**Comparison of and**

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| is a subspace of | is a subspace of |
| is **implicitly defined**. **Only a condition is given** () that defines it. | is **explicitly defined**. The columns of describe how to build it. |
| It takes time to find the **explicit definition** of by row reducing **.** | It is easy to find vectors in **.** It **includes** **all columns of** and combinations of them**.** |
| There is no obvious relationship between and the entries in . | There is an obvious relationship between and the entries in since it includes the columns of . |
| A **typical entry**, , in has the property: | A **typical entry**, , in has the property of making consistent for some : |
| Given a **specific entry**, , it is easy to determine if by multiplying and checking if the result is the zero vector, . | Given a **specific entry**, , it is not easy to determine if and requires row reducing the augmented matrix . |
| if and only if has only the **trivial solution**. | if and only if the equation **has a solution for all** . |
| has if and only If the **linear transformation** is **one-to-one**. | if and only if the **linear transformation** maps  **onto** . |

## Linearly Independent Sets and Bases

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| **Linearly Independent Set:** **Key to “Efficiently” spanning a vector space**. | A **set of vectors** is said to be **linearly independent** if the vector equation:  has **only the trivial solution**. | **Linear Dependence Relation:** A set of **non-zero weights** that satisfy the equation: | **Theorem 4-4:** An indexed set of two or more vectors, with is **linearly dependent** if and only if some is a **linear combination of the preceding vectors** . |

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| **Definition:** Let be a **subspace** of a vector space . An indexed set of vectors in is a **basis** of if:   1. **is a linearly independent set** 2. **The subspace spanned by coincides with , that is:** | **Standard Basis for**  where are the **columns of the identity matrix**. | **Theorem 4-5 – The Spanning Set Theorem**  Let be a set of vectors in , and let , then:   1. **If one of the vectors in , say , is a linear combination of the remaining vectors, then the set formed by removing from still spans .** 2. **If *,* then some subset of is a basis of .** | **Standard Basis for** |

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| **Bases for and** | | **Two Views of a Basis** | |
| **Algorithm – Find the Basis of**   1. **Row reduce to echelon form.** 2. **Identify the columns with pivot positions in the echelon matrix.** 3. **Select the corresponding columns FROM THE ORIGINAL MATRIX as the basis of .** | **Theorem 4-6** – The **pivot columns of**  **form a basis for** . | A basis is a **spanning set** **that is as small as possible**. | A basis is a **linearly independent set** **that is as large as possible**. |

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| **No Free Variables Only Trivial Solution Linear independence** | If is an **invertible**  matrix, then  **spans**  by the **invertible matrix theorem**. |  |  |

## Coordinate Systems

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| **Theorem 4-7 – Unique Representation Theorem**  Let be a **basis** for a vector space . Then for each , there **exists a unique set of scalars** such that: | **Definition**: Suppose is a **basis** for a vector space and . The **coordinates of relative to the basis**  are the **weights** such that:  The **coordinate vector of**  (**relative to** ) or the **-coordinate vector of** is: | **Coordinate Mapping (Determined by )**  **Notation:**  **Creates a new coordinate system**  It is a one-to-one linear transformation from onto to R |
| **Standard Basis for**  where are the **columns of the identity matrix**.  **Note:** |
| **Standard Basic for**  **Note:** |

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| **Algorithm: Find the Coordinate Vector for**  For a given basis , **solve the linear system to get the -coordinate vector of** : | **Change of Coordinates Matrix** | | **Theorem 4-8:** Let be a basis for a vector space, . Then the **coordinate mapping**:  is a **one-to-one** **linear transformation** from to . |
| For a basis , the **change of coordinates matrix** is defined as:  where: | If the **change of coordinates matrix, , is square**, then it is also **invertible** by the **Invertible Matrix Theorem** **since the columns of the matrix (i.e., the basis) are linearly independent**.  In these **specific cases**: |

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| **Preservation of Vector Addition and Scalar Multiplication**  Due to the **change of coordinates matrix**, coordinate mapping is a **linear transformation** meaning that the operation **preserves vector addition and scalar multiplication**. Formally: | **Isomorphism**: A **map that preserves sets and relations among elements**.  **Requirements**:   * **Mapping is one to one.** * **Calculations in one space yield mapping values in the other set.**   **Example Isomorphism:** |  |  |

## The Dimension of a Vector Space

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| **Theorem 4-9**: If a vector space has a basis , then any set in **containing more than vectors must be** **linearly dependent**. | **Theorem 4.10** – If a vector space has a basis of vectors, then **every basis of** **must consist of exactly vectors**. | If a vector space is **spanned by a finite set**, then is said to be **finite dimensional**.  The **dimension of** , written as , is **the number of vectors in the basis of** .   * **Examples:**   + **– Zero dimensional**   + **– -dimensional**   + **- -dimensional** | If cannot be spanned by a finite set, then is said to be **infinite dimensional**.  **Example:**   * **– Set of all polynomials.** |

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| **Theorem 4.11:** Let be a **subspace of a finite-dimensional vector space** . **Any linearly independent set in** **can be expanded, if necessary, to a basis for** . Also, is **finite dimensional** where: | **Theorem 4.12: The Basis Theorem**  Let be a -dimensional vector space, where . **Any linearly independent set of exactly elements in is automatically a basis for** . **Any set of exactly elements that spans is automatically a basis for** . |  | **Dimensions of and**  = **Number of pivot columns in**  = **Number of free variables in the equation:** .  For an matrix: |

## Rank

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| **Row space** – Set of **all linear combinations of the rows of**  **Note:** | **Theorem 4-13:** If two matrices andare **row equivalent**, then their **row spaces are the same**. If is in **echelon form**, then the **non-zero rows** of form a **basis for the row space** of as well as that of . | **Algorithm: Finding the Row Space’s Basis**   1. **Row reduce the input matrix to another matrix in echelon form.** 2. **Select the non-zero rows of (i.e., the pivot rows) as the basis of the row spaces for and .** | The **rank** of a matrix is the **dimension** (i.e., **number of vectors in the basis**) of the **column space** (**or row space**) |

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| **Theorem 4-14 – The Rank Theorem**  The **dimensions** of the **column space** and **row space** are **equal** for an  **matrix**,. This common dimension, the **rank** of , also **equals the number of pivot columns** of and satisfies the equation: |  |  |  |

# Eigenvalues and Eigenvectors

## Eigenvectors and Eigenvalues

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| An **eigenvector** of an  **matrix** is a **nonzero vector** such that for some scalar . | The **scalar** is called an **eigenvalue** of if there is a **nontrivial solution**, , (i.e., ) such that ; such an is called an **eigenvector corresponding to**. | **Theorem 5-1:** The **eigenvalues** of a **triangular matrix** are the **entries on its main diagonal**. | **Theorem 5-2:** If are **eigenvectors** **that correspond to distinct eigenvalues** of an  **matrix** , then the set is **linearly independent**. |

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| **Checking if a Vector is an Eigenvector**  Given a vector, , **verify that the relationship holds**:  If it does, then it is an eigenvector. | **Checking if a Scalar is an Eigenvalue**  For an matrix and vector , **verify that the relationship holds**: | **Eigenspaces** | | |
| The **eigenspace** is the **subspace** **of**  corresponding to the **null space** of the matrix for a given eigenvalue, .  The **eigenspace** for a matrix is the **set of all solutions to the homogeneous equation**: | **Eigenspaces** are **-dimensional** which **corresponds to the number of free variables** in or the **number of vectors in the basis of its null space**. | Although the zero vector, , **can never be an eigenvector**,  **is in all eigenspaces**.  An **eigenspace** contains the  **plus all eigenvectors corresponding to** . |

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| **Difference Equations** | | **Note #1:** A matrix has an **eigenvalue** of “0” if and only if is **not invertible**.  **Note #2:** If is an matrix, then **and** **have the same** **eigenvalues**. |  |
| For a matrix , a **difference equation** is one where **each subsequent vector** **is** **based solely on the preceding vector**, , and such that: | A **solution** is an explicit description of the set whose **formula for each** **does not depend on**  **or on the sequence other than the initial seed vector,** , such that: |

## The Characteristic Equation

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| **Theorem 5-3:** Let and by matrices:  **is invertible if and only of**  **If is triangular, then is the product of the entries on the main diagonal of .**  **A row replacement operation on does not change its determinant. A row interchange changes the sign of the determinant. A row scaling scales the determinant by the same scaling factor.** | **Characteristic Equation**  **Used to find the eigenvalues** of a matrix.  For a matrix , the **characteristic equation** is defined as:  A scalar is an **eigenvalue** **if and only if it satisfies the characteristic equation**. | If matrix is size , then **is an -degree polynomial** called the **characteristic polynomial** of |
| The **algebraic multiplicity of an eigenvalue** is its **multiplicity as a root of the** **characteristic equation**.  **Example:**  “” has multiplicity of 2 |

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| **Complex Eigenvalues** – Any **eigenvalue** in the characteristic equation **with an imaginary term**. | **Similarity**  Given that and are matrices, then **is similar to** entails that an **invertible matrix**  **exists such that**:  Or Rewritten: | **Theorem 5-4** – If matrices and are **similar**, then they have the **same** **characteristic polynomial** and hence the **same eigenvalues**. | **Warning #1:** Two matrices having **the same eigenvalues** **are not necessarily similar**. |
| **Warning #2:** **Similarity** **does not entail row equivalence**.  **Note: Row operations generally change the eigenvalues.** |

## Diagonalization

# Orthogonality and Least Squares

## Inner Product, Length, and Orthogonality

## Orthogonal Sets

# Linear Algebra Theorems

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| **Theorem #1: Any matrix has one and only one** **reduced echelon form**. | **Theorem 1-2: Existence and Uniqueness Theorem**  A **linear system is consistent** if and only if the rightmost column of the augmented matrix is not a pivot column – that is, if and only if an echelon form of the **augmented matrix has no row in the form**:  withnon-zero  If a linear system is consistent, then the solution set contains either:   1. **A unique solution, when there is no free variable** 2. **Infinitely many solutions when there is at least one free variable.** | **Theorem #1-3:** If is an matrix with columns and if , then **the matrix equation**:  **has the same solution set** as the **vector equation**:  which in turn **has the same solution set as the system of linear equations whose augmented matrix is**: |

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| **Theorem #1-4:** Let be an matrix. Then the following statements are equivalent meaning they are **either all true** **or all false**.   1. **For all in, the equation has a solution.** 2. **Each in is a linear combination of the columns of A.** 3. **The columns of A span** 4. **The (coefficient) matrix has a pivot position in every row.** | **Theorem #1-5:** If is an matrix, and are vectors in , and is a scalar, then: | **Theorem #1-6:** Suppose the equation is consistent for some given , and let be **any solution to that particular non-homogenous system**. Then the **solution set of** is **the set of all vectors of the form**:  where is **any solution of the homogenous equation** . |

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| **Theorem #1-7:** An indexed set of two or more vectors is **linearly dependent** **if and only if at least one of the vectors in is a linear combination of the others**.  In fact, if is linearly dependent and , then some (where ) is a linear combination of the preceding vectors: . | **Theorem #1-8:** If a set **contains more vectors than there are entries in each vector**, then the set is **linearly dependent**.  That is any set inis linearly dependent if:  **Proof:** More pivots than columns in an matrix so **at least one free variable**. | **Theorem #1-9:** If a set  **contains the zero vector**, then the set is **linearly dependent**.  **Proof:** If the zero vector is reordered to be , then |

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| **Theorem #2-1:** Let , , and be **matrices of the same size** and let and be scalars, then:   1. **(Commutative)** 2. **(Associative)** 3. **(Identity)** | **Theorem #2-2:** Given is the identity matrix of size and is an matrix, and that and have sizes for which the indicated sums and products exist, then   1. **(Associative)** 2. **(Left Distributive Law)** 3. **(Right Distributive Law)** 4. **, for any scalar** | **Theorem #2-3:** Let and be matrices whose sizes are appropriate for the following sums and products:   1. **, for any scalar**   **Note: The reverse order of the product.** |

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| **Theorem 2-4:** Let . If , is invertible and: | **Theorem 2-5:** If is an **invertible matrix**, then for all , there exists a **unique solution** . More formally: | **Theorem 2-6**   1. If is an **invertible matrix**, then **is invertible** and: 2. If and are invertible matrices, then so is . The **inverse of is the product of the inverses in reverse order**, that is: 3. If is an invertible matrix, then so is . **The inverse of is the transpose of** , that is: |

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| **Theorem 2.7:** An matrix is **invertible** if and only if it is **row equivalent to the identity matrix** .  Hence, **any sequence of row operations that reduces to also transforms to** . | **Theorem 2.8 – The Invertible Matrix Theorem**  Let be a **square**, matrix, then the following statements are **equivalent**. That is for a given , they are either **all false or all true**.   1. **is an invertible matrix.** 2. **is row equivalent to the identity matrix.** 3. **has pivot positions.** 4. **The equation has only the trivial solution.** 5. **The columns of form linearly independent set.** 6. **The linear transformation is one-to-one.** 7. **The equation has a solution for all .** 8. **The columns of span .** 9. **The linear transformation maps to .** 10. **There is an matrix such that .** 11. **There is an matrix such that .** 12. **is an invertible matrix.** 13. **The columns of form a basis for .** 14. **The number 0 is not an eigenvalue of .**   **Note:** This **applies only to square matrices**. | **Theorem 2-9:** Let be a linear transformation and let be the **standard matrix** for . Then  **is invertible if and only if is an invertible matrix**. Then the linear transformation given by:  is the **unique function** such that:  and |

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| **Theorem 3-1:** The determinant of an matrix can be computed via c**ofactor expansion across any row or column**. The expansion **across the row** is:  The expansion down the column is: | **Theorem 3-2:** If is a **triangular matrix**, then is the **product of the entries along the diagonal** of . | **Theorem 3-3: Effect of Row Operations on the Determinant**   1. **If a multiple of one row of is added to another row to produce a matrix, then:** 2. **If two rows of are interchanged to form a new matrix , then** 3. **If one row of is multiplied (i.e., scaled) by to form a new matrix , then** |

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| **Theorem 3-4:** **A square matrix** is **invertible** if and only if:  **Corollary #1:** if and only if the **columns of**  are **linearly dependent**.  **Corollary #2:** if and only if the **rows of**  are **linearly dependent**. | **Theorem 3-5:** If is an matrix, then: | **Theorem 3-6 – Multiplicative Property:** If and are matrices, then: |

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| **Theorem 4-1:** If are in a vector space , then **is a subspace of** . | **Theorem 4-2:** The **null space** of an  **matrix** is a **subspace of** .  Equivalently, the set of all solutions to a system:  of  **homogeneous linear equations in unknowns is a subspace of** | **Theorem 4-3:** The **column space** of an **matrix** is a subspace of .  In set notation: |

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| **Theorem 4-4:** An indexed set of two or more vectors, with is **linearly dependent** if and only if some is a **linear combination of the preceding vectors** . | **Theorem 4-5 – The Spanning Set Theorem**  Let be a set of vectors in , and let , then:   1. **If one of the vectors in , say , is a linear combination of the remaining vectors, then the set formed by removing from still spans .** 2. **If *,* then some subset of is a basis of .** | **Theorem 4-6** – The **pivot columns of**  **form a basis for** . |

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| **Theorem 4-7 – Unique Representation Theorem**  Let be a **basis** for a vector space . Then for each , there **exists a unique set of scalars** such that: | **Theorem 4-8:** Let be a basis for a vector space, . Then the **coordinate mapping**:  is a **one-to-one** **linear transformation** from to . | **Theorem 4-9**: If a vector space has a basis , then any set in **containing more than vectors must be** **linearly dependent**. |

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| **Theorem 4.10** – If a vector space has a basis of vectors, then **every basis of** **must consist of exactly vectors**. | **Theorem 4.11:** Let be a **subspace of a finite-dimensional vector space** . **Any linearly independent set in** **can be expanded, if necessary, to a basis for** . Also, is **finite dimensional** where: | **Theorem 4.12: The Basis Theorem**  Let be a -dimensional vector space, where . **Any linearly independent set of exactly elements in is automatically a basis for** . **Any set of exactly elements that spans is automatically a basis for** . |

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| **Theorem 4-13:** If two matrices andare **row equivalent**, then their **row spaces are the same**. If is in **echelon form**, then the **non-zero rows** of form a **basis for the row space** of as well as that of . | **Theorem 4-14 – The Rank Theorem**  The **dimensions** of the **column space** and **row space** are **equal** for an  **matrix**,. This common dimension, the **rank** of , also **equals the number of pivot columns** of and satisfies the equation: |  |

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| **Theorem 5-1:** The **eigenvalues** of a **triangular matrix** are the **entries on its main diagonal**. | **Theorem 5-2:** If are **eigenvectors** **that correspond to distinct eigenvalues** of an matrix , then the set is **linearly independent**. | **Theorem 5-3:** Let and by matrices:  **is invertible if and only of**  **If is triangular, then is the product of the entries on the main diagonal of .**  **A row replacement operation on does not change its determinant. A row interchange changes the sign of the determinant. A row scaling scales the determinant by the same scaling factor.** |

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| **Theorem 5-4** – If matrices and are **similar**, then they have the **same** **characteristic polynomial** and hence the **same eigenvalues**. |  |  |

# Proofs for the Algebraic Properties of

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| **Name** | **Description** | **Terms** |
| **Commutative** (Vector Addition) |  | * – Vectors * – Zero vector * – Real constants |
| **Inverse** (Vector Addition) |  |
| **Associative** (Vector Addition) |  |
| **Associative** (Scalar Multiplication) |  |
| **Distributive** **Law** (Vector Addition) |  |
| **Distributive** **Law** (Scalar Multiplication) |  |
| **Identity** (Vector Addition) |  |
| **Identity** (Scalar Multiplication) |  |

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| **Proof of the Commutative Property of Vector Addition**   1. Suppose and are **any real vector in**  in the form:   and   1. By the **definition of vector addition**:      1. By the **commutative property of real number addition**:      1. By the **definition of vector addition** and the **definition of and** :   **(QED)** | **Proof of the Inverse Property of Vector Addition**   1. Suppose is **any** **real vector in**  in the form: 2. By the **definition of scalar multiplication**: 3. By the **definition of vector addition**:      1. By the **inverse property of real number addition**:      1. By **definition of the zero vector**:   **(QED)** |

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| **Proof of the Associative Property for Vector Addition**   1. Suppose , , and are **any real vector in**  in the form:   and and   1. By the **definition of vector addition**:      1. By the **definition of vector addition**: 2. By the **associative property of real number addition**:      1. By the **definition of** **vector addition** and the **definition of and** :   **(QED)** | **Proof of the Associative Property for Scalar Multiplication**   1. Suppose and are **any real number** and is **any real vector in**  in the form: 2. By the **definition of scalar multiplication**:      1. By the **definition of vector addition**: 2. By the **associative property of real number multiplication**: 3. By the **definition of** **scalar multiplication**:   **(QED)** |

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| **Proof of the Distributive Law for Vector Addition**   1. Suppose is **any real number** and and are **any real vector in**  in the form:   and   1. By the **definition of vector addition**:      1. By the **definition of scalar multiplication**:      1. By the **distributive law over real number addition**: 2. By the **defintition of vector addition**: 3. By the **definition of scalar multiplication**,and the **definition of and** :   **(QED)** | **Proof of the Distributive Law for Scalar Multiplication**   1. Suppose and are **any real number** and is **any real vector in**  in the form:      1. By the **definition of scalar multiplication**:      1. By the **distributive law over real number addition**:      1. By the **definition of vector addition**: 2. By the **definition of scalar multiplication** and the **definition of** :   **(QED)** |

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| **Proof of the Identity Property for Vector Addition**   1. Suppose is the **zero vector of length**  and is **any real vector in**  in the form: 2. By the **definition of vector addition**:      1. By the **identity property of real number addition**:      1. By the **definition of** :   **(QED)** | **Proof of the Identity Property for Scalar Multiplication**   1. Suppose is **any real vector in**  in the form: 2. By the **definition of scalar multiplication**:      1. By the **identity property of real number multiplication**:      1. By the **definition of** :   **(QED)** |