**MATH129A – Linear Algebra Final Exam Study Guide**

**Created By: Zayd Hammoudeh**

[Chapter 1 – Linear Equations in Linear Algebra 2](#_Toc481803189)

[Section 1.1 – Systems of Linear Equations 2](#_Toc481803190)

[Section 1.2 – Row Reduction and Echelon Forms 3](#_Toc481803191)

[Section 1.3 – Vector Equations 4](#_Toc481803192)

[Section 1.4 – The Matrix Equation 5](#_Toc481803193)

[Section 1.5 – Solution Sets of Linear Systems 6](#_Toc481803194)

[Section 1.6 – Linear Independence 7](#_Toc481803195)

[Section 1.7 – Introduction to Linear Transformations 8](#_Toc481803196)

[Chapter 2 – Matrix Algebra 9](#_Toc481803197)

[Section 2.1 – Matrix Operations 9](#_Toc481803198)

[Section 2.2 – The Inverse of a Matrix 10](#_Toc481803199)

[Section 2.3 – Characterizations of Invertible Matrices 11](#_Toc481803200)

[Chapter 3 – Determinants 12](#_Toc481803201)

[Section 3.1 – Introduction to Determinants 12](#_Toc481803202)

[Section 3.2 – Properties of Determinants 13](#_Toc481803203)

[Chapter 4 – Vector Spaces 14](#_Toc481803204)

[Section 4.1 – Vector Spaces and Subspaces 14](#_Toc481803205)

[Section 4.2 – Null Spaces, Column Spaces, and Linear Transformations 15](#_Toc481803206)

[Section 4.3 – Linearly Independent Sets and Bases 16](#_Toc481803207)

[Section 4.4 – Coordinate Systems 17](#_Toc481803208)

[Section 4.5 – The Dimension of a Vector Space 18](#_Toc481803209)

[Section 4.6 – Rank 19](#_Toc481803210)

[Chapter 5 – Eigenvalues and Eigenvectors 20](#_Toc481803211)

[Section 5.1 – Eigenvectors and Eigenvalues 20](#_Toc481803212)

[Section 5.2 – The Characteristic Equation 21](#_Toc481803213)

[Section 5.3 – Diagonalization 22](#_Toc481803214)

[Chapter 6 – Orthogonality and Least Squares 23](#_Toc481803215)

[Section 6.1 – Inner Product, Length, and Orthogonality 23](#_Toc481803216)

[Section 6.2 – Orthogonal Sets 24](#_Toc481803217)

[Linear Algebra Theorems 25](#_Toc481803218)

[Proofs for the Algebraic Properties of 28](#_Toc481803219)

# Linear Equations in Linear Algebra

## Systems of Linear Equations

|  |  |  |  |
| --- | --- | --- | --- |
| **Linear Equation** – An equation with **variables** that **can be written in the form**:  **Coefficients:** can be **real or complex** | **Linear System** or **System of Linear Equations** – Collection of **one or more linear equations**. | **Solution:** A **set of numbers** **that makes each equation a true statement** when substituted for variables respectively. | **Solution Set:** Set of **all possible solutions** for a linear system.    **Possible Solution Sets:**   * **No solution** * **One solution** * **Infinite solutions** |

|  |  |  |  |
| --- | --- | --- | --- |
| **Consistent Linear System** – Has **one or infinite solutions** | **Coefficient Matrix** – A matrix containing the **coefficients for each variable in each equation** in the linear system. | **Augmented Matrix** – A matrix of a system containing the **coefficient matrix and** **an added column containing the constants** from the ***right hand side*** of the equation. | **Techniques to Simplify a Linear System**   1. **Replace** one equation with **sum of itself and the multiple of another linear system** (equation) 2. **Interchange** two equations 3. **Multiply all terms** in an equation by a **non-zero constant** |
| **Inconsistent Linear System** – Has no solution |

|  |  |  |  |
| --- | --- | --- | --- |
| **Row Equivalent Matrices** – Any two matrices where a **series of elementary row operations** can transform one matrix into another. | **Row Operation Reversibility** – All row operations can be undone to get the previous matrix | **Matrix** – Composed of rows of columns | **Equivalent Linear Systems** – Any two linear systems with the **same solution set**. |

|  |  |  |  |
| --- | --- | --- | --- |
| **Approaches to Find the Solution Set of a Linear System**   1. **Solve equations by substitution.** 2. **Multiply and add the equations** 3. **Graphically**    1. Look at the intersection of the equations. |  |  |  |

## Row Reduction and Echelon Forms

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | **Key Properties** | | | | |
| If **two augmented matrices are row equivalent**, then the systems **have the same solution set**. | **Reduced echelon form is unique** | **Echelon form is not unique** | All linear systems have a reduced echelon form. | **Location of leading entries** is the **same between standard and reduced echelon form.** |

|  |  |  |
| --- | --- | --- |
| **Echelon Matrix Criteria**   1. **All non-zero rows are above all zero rows.** 2. **The leading entries of lower rows are to the right of all those in upper rows.**    1. Forms a “**step pattern**”. 3. **The entries in a column below a leading entry are zero.** | **Reduced (Row) Echelon Matrix Criteria**   1. **All criteria of a standard echelon matrix.** 2. **All leading entries equal 1.** 3. **The entries in a column above a leading entry are 0.** | **Theorem #1: Any matrix has one and only one** **reduced echelon form**. |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Pivot Position** – A **position** in a given matrix that **corresponds to a “1” in reduced echelon form**. | **Pivot Position** – A column that contains a pivot position. | **Pivot** – A non-zero number in a pivot position that is used as needed to create zeros via row operations. | **Gaussian Elimination:** Same concept as row reduction. | **Non-zero row/column**: A row/column with **at least one non-zero entry**. | **Leading entry** – **Leftmost non-zero entry** **in a row**. |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Row Reduction Algorithm (Gaussian Elimination)**  **Forms an Echelon Matrix** | | | | **Gauss-Jordan Elimination**  **Forms the Reduced Echelon Matrix** |
| 1. **Begin with the left most non-zero entry that is a pivot position. Move that position to the top of the matrix.** | 1. **Select a non-zero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.** | 1. **Use row operations to create zeros in all positions below the pivot.** | 1. **Cover (i.e. ignore) the rows containing the pivot position and cover (ignore) all rows above it. Apply steps 1-3 to the sub matrix that remains. Repeat the process until there are no more non-zero rows remaining.** | 1. **Create zeros above each pivot and scale each pivot to 1.** |

|  |  |  |
| --- | --- | --- |
| **Basic Variable: Can only exist in a single solution set equation. Correspond to a pivot in the reduced echelon matrix.** | **Parametric Description of Solution Sets**   * Free variables act as parameters * Each basic variable is represented by an equation made up of constants and/or free variables. | **Theorem 1-2: Existence and Uniqueness Theorem**  A **linear system is consistent** if and only if the rightmost column of the augmented matrix is not a pivot column – that is, if and only if an echelon form of the **augmented matrix has no row in the form**:  withnon-zero  If a linear system is consistent, then the solution set contains either:   1. **A unique solution, when there is no free variable** 2. **Infinitely many solutions when there is at least one free variable.** |
| **Free Variable:** **Can be assigned to any number** since it has no pivot. Free variable **quantity dictates the resulting solution set’s shape** (e.g., plane, line etc.). |

|  |  |  |
| --- | --- | --- |
| **Example: Parametric Description of a Solution Set** | |  |
| **Augmented Matrix** | **Parametric Description of the Solution Set** |

## Vector Equations

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Vector** – An **ordered** list of numbers | **Column Vector** – A **matrix with only one column**. | **Vector Equality** – Two vectors of the **same size** with **all corresponding entries equal**. | **Vector Addition** – Sum obtained by **adding the corresponding entries of two vectors**. | **Scalar Multiple** –Given a number, , and a vector, , it is the vector obtained when is **multiplied by each element** in |

|  |  |  |  |
| --- | --- | --- | --- |
| **Parallelogram Rule for Addition** – If and are in (i.e., points in the Cartesian plane), then corresponds to the **fourth point** in the **parallelogram whose vertices are** , and (the origin). | **Scalar Multiples of a Fixed Vector** – **Along a line between the vector point and the origin**. | **Zero Vector** () – A vector whose entries are all zeros. | **Linear Combination** – Given vectors and scalars , then the **linear combination vector**  is: |

**Algebraic Properties in**

|  |  |  |  |
| --- | --- | --- | --- |
| **Commutative** | **Associative** | **Identity** | **Inverse** |
| **Distributive** | **Distributive** | **Associative** | **Identity** |

|  |  |  |  |
| --- | --- | --- | --- |
| **The Vector Equation and  the Augmented Matrix**  A **vector equation**:  has the **same solution set as the linear system whose augmented matrix is**:  In particular, can be generated by a linear combination of if and only if there exists a solution to the linear system corresponding to the augmented matrix. | **Subset of Spanned /  Generated by**  If are in , then the set of all linear combinations of is denoted by:  It is the set of all linear combination vectors that can be written in the form: | **Geometric Description of Spans of Vectors** | |
| **(Zero Vector)** | A **point** – The origin |
| **One non-zero vector** | A **line** through the origin and |
| **Two vectors**  and that **are scalar multiples** | A **line** through the origin as well as and , which are all collinear. |
| **Two vectors**  and **not scalar multiples** | A **plane** through the origin, , and |

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **Members of All Spans**   * **Zero vector** () * **Scalar multiples of the original vectors** | **How to Determine if a Vector is in a Span** **Check if the system is consistent** | **Important Notation** | | | | |
| **Vector in** | | | **Not a vector** | **Set of Vectors** |
|  |  |  | **It is a matrix** |  |

## The Matrix Equation

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Gaussian Elimination** – Same concept as row reduction. | **Theorem #1-4:** Let be an matrix. Then the following statements are equivalent meaning they are **either all true** **or all false**.   1. **For all , the equation has a solution.** 2. **Each is a linear combination of the columns of .** 3. **The columns of span** 4. **The (coefficient) matrix has a pivot position in every row.** | **Relationship between Spans and Free Variables**   |  |  | | --- | --- | |  | **General Solution Structure** | | **(Trivial Only)** |  | | **1 Free Variable** | – **Line** through and the origin | | **2 Free Variables** | – **Plane** through , , and the origin | | **Free Variables** | – **Multidimensional shape** through and the origin | |
| **Non-zero row/column** – A row/column with at least one non-zero entry. |
| **Theorem #1-3:** If is an matrix with columns and if , then **the matrix equation**:  **has the same solution set** as the **vector equation**:  which **has the same solution set as the system of linear equations whose augmented matrix is**: |

|  |  |  |
| --- | --- | --- |
| **Theorem #1-5:** If is an matrix, and are vectors in , and is a scalar, then: | **Row-Vector Rule for Computing**  If the product is defined (i.e., the sizes correspond), then the entry in is the sum of products of the corresponding entries from row of and from the vector . It is formally: | **Matrix Equation Definition**  If is an matrix with columns and if , then the **product**, of and , denoted by , **is the linear combination of the columns of** **using the corresponding entries in**  **as the weights**; that is: |

## Solution Sets of Linear Systems

|  |  |  |
| --- | --- | --- |
| **Homogeneous Linear System:**   * – Matrix * – Vector in * – Zero vector in   **Trivial Solution: (in )**   * **Exists for all homogeneous systems**   **Nontrivial Solution:** Any **non-zero vector solution to the equation:**   * If it exists, there are infinitely many. * Requires at least one free variable. | **Non-Homogenous System:** where is not the zero vector.   * May be inconsistent. * If its solution exists, it is in the form:   + If is , then   + – Particular solution for the specific non-homogenous system   + – Solution set for the homogenous system | **Theorem #1-6:** Suppose the equation is consistent for some given , and let be **any solution to that particular non-homogenous system**. Then the **solution set of** (**if it exists**)is **the set of all vectors of the form**:  where is **any solution of the homogenous equation** . |

|  |  |  |  |
| --- | --- | --- | --- |
| **Solution Set Notations** | | | For to span all of , there **must be a pivot in every row of the echelon matrix** of . |
| **Augmented Matrix** | **Parametric Description of the Solution Set** | **Parametric Vector Form** |

## Linear Independence

|  |  |  |
| --- | --- | --- |
| **Span Review**   * **One Vector** – **At most** a **line**   + **Exception:** vector * **Two Vectors: At most** a **plane**   + **Exception:** Scalar multiples and the zero vector | **Linear Independence**  A set of vectors are **linearly independent** if:  has **only the trivial solution** | **Linear Dependence**  A set of vectors are **linearly dependent** if there exists a set of **non-zero weights** such that:  **Note:** This requires **at least one free variable**. |

|  |  |  |
| --- | --- | --- |
| **Linear Dependence Relation:** For vectors , it is defined as:  where are **not all zero.**  **Note:** The values of are not unique. **If one exists, an infinite number exist.** | **Procedure: Checking for Linear Independence**  **Step #1:** Create the coefficient matrix.  **Step #2:** Perform Gaussian elimination to find the echelon matrix.  **Step #3:** Check linear independence   * If **there is a pivot in every column**, the vectors are **linearly independent**. * If **there is a free variable**, the vectors are **linearly dependent**. | **Linear Independence of One Vectors**   * **Zero Vector** () – This is always **linearly dependent**. * **Any Non-Zero Vector** ( where ) then **linearly independent**. |

|  |  |  |
| --- | --- | --- |
| **Linear Independence of Two Vectors** – A set of two vectors is linearly dependent if **at least one** **of the vectors is a scalar multiple** of the other.   * “**At least one**” – Because of the case of the zero () vector. | For a set of vectors to be **linearly independent**, the echelon matrix made from those vectors must have no free variables.   * **There must be a pivot in every column.** | **Linear Dependence Summary:** If a set of  **vectors of -dimensions** are linearly independent, then they **span an dimensional shape**. |

|  |  |  |
| --- | --- | --- |
| **Theorem #1-7:** An indexed set of two or more vectors is **linearly dependent** **if and only if at least one of the vectors in is a linear combination of the others**.  In fact, if is linearly dependent and , then some (where ) is a linear combination of the preceding vectors: . | **Theorem #1-8:** If a set **contains more vectors than there are entries in each vector**, then the set is **linearly dependent**.  That is any set inis linearly dependent if:  **Proof:** More pivots than columns in an matrix so **at least one free variable**. | **Theorem #1-9:** If a set  **contains the zero vector**, then the set is **linearly dependent**.  **Proof:** If the zero vector is reordered to be , then |

## Introduction to Linear Transformations

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Function** – A **rule** that **assigns to each element** in a set **exactly one element** from set . | **Transformation () from to :** A rule that **assigns to each vector**  **a vector** | **Domain of :** | **Image of :** For a given , it is the **transformed value** | **Range: Set of all images** .  **Note:** The range may be (and often is) only a subset of the codomain. |
| **Codomain of :** |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Linear Transformations:** **Preserve vector addition and scalar multiplication** |  |  |  |  |

**Properties of Matrix Transformations**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Requirements of Linear Transformation** | | **Zero Vector Properties** | |  |
|  |  |  |  |

**Types of Matrix Transformations**

|  |  |  |  |
| --- | --- | --- | --- |
| **Contraction**  **,**  **where** | **Dilation**  **, where** | **Projection onto an Axis**  **Matrices in the form:**  **– Projection onto -plane**  **– Projection onto the -axis** | **Shear Transformations**  **Maps line segments onto line segments. Deforms the shape of the original input.** |

|  |  |  |  |
| --- | --- | --- | --- |
| **Reflection through the Origin**  Matrix in the Form of: | **Reflection through the -axis**  Matrix in the Form of: | **Reflection through the -axis**  Matrix in the Form of: | **Reflection through the line**  Matrix in the Form of: |

|  |  |  |
| --- | --- | --- |
| **Twice as Long and 1.5 Times as High**  Matrix in the Form of: | **Rotation about Origin through 90­o**  Matrix in the Form of: |  |

# Matrix Algebra

## Matrix Operations

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| or – Given an matrix, it is the element in the  **row** and  **column** of | **Diagonal Entries** – In an matrix, it is the entries , etc. | **Diagonal Matrix** – A **square** matrix whose non-diagonal entries are 0.  **Example:** **Identity matrix** | **Equal Matrices** – Two matrices having **the same size** (i.e., equal number of rows and columns) and **whose corresponding entries are equal**. | **Matrix Sum** – Given two matrices of the same size, the matrix sum is equivalent to the **sum of the corresponding columns**. |

|  |  |  |
| --- | --- | --- |
| **Matrix Multiplication**  If is an matrix and if is an matrix composed of the columns , then the **product is an matrix** whose columns are: such that: | **Row-Product Rule**  **If the product of is defined**, then the entry in row and column of is the sum of the products of corresponding entries from row of and column of such that: | **Size of a Matrix Multiplication:** Given a matrix of size and a matrix of size , then the matrix multiplication is of size . |
| **Non-Commutativity of Matrix Multiplication**  In most cases, |
| **Cancelation Laws Do Not Hold For Matrix Multiplication**  Generally, if , it **cannot be assumed** that |
| **Zero Matrix Product**  If , you **cannot generally assume** or |

|  |  |  |
| --- | --- | --- |
| **Powers of a Matrix**  If is an matrix and is a positive integer, then denotes the product of copies of such that:  **Zero Power** – Given a square matrix , then **is the identity matrix**. | **Transpose of a Matrix**  Given an matrix , then the transpose of is an **matrix whose columns are formed from the corresponding rows of** . | **Transpose Multiplication Row:** The **transpose of the product of matrices** **equals** the **product of their transposes in reverse order**. |

|  |  |  |
| --- | --- | --- |
| **Theorem #2-1:** Let , , and be **matrices of the same size** and let and be scalars, then:   * 1. **(Commutative)**   2. **(Associative)**   3. **(Identity)** | **Theorem #2-2:** Given is the identity matrix of size and is an matrix, and that and have sizes for which the indicated sums and products exist, then   1. **(Associative)** 2. **(Left Distributive Law)** 3. **(Right Distributive Law)** 4. **, for any scalar** | **Theorem #2-3:** Let and be matrices whose sizes are appropriate for the following sums and products:   1. **, for any scalar**   **Note: The reverse order of the product.** |

## The Inverse of a Matrix

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| A matrix is **invertible** **if there exists** a matrix such that:  **and** | **Inverse Matrix Notation:**  The **inverse matrix is unique (if it exists)**. | **Singular Matrix** – Any matrix that is **non-invertible**.  **Nonsingular Matrix** – Any **invertible matrix**.  **Hint for memorization:** “Non’s” are swapped and do not match. | **Theorem 2-4:** Let . If , is invertible and: | **Determinant of** – For a  **matrix**, it is:  **The inverse exists only if** |

|  |  |  |  |
| --- | --- | --- | --- |
| **Theorem 2.5** – If  **is an invertible matrix**, then for all , then equation has a **unique solution in the form**: | **Theorem 2-6**   1. If is an **invertible matrix**, then  **is invertible** and: 2. If and are invertible matrices, then so is . The **inverse of is the product of the inverses in reverse order**, that is: 3. If is an invertible matrix, then so is . **The inverse of is the transpose of** , that is: | **Elementary Row Operation**: the three types are:   1. **Interchange (swap)** 2. **Scale** 3. **Add** | **Theorem 2.7:** An matrix is **invertible** if and only if it is **row equivalent to the identity matrix** .  Hence, **any sequence of row operations that reduces to also transforms to** . |
| **Elementary Matrix** – A matrix **obtained by performing a single** **elementary row operation** **on the identity matrix**. |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Algorithm for Finding the Inverse Matrix**   1. **Place matrix and side by side in an augmented matrix.** 2. **Row reduce the matrix to transform to .** 3. **Extract the inverse matrix where:** | **Example Elementary Matrices in** | | |  |
|  | **Interchange and** | **Scale** |

## Characterizations of Invertible Matrices

|  |  |  |
| --- | --- | --- |
| If , then is **onto (surjective)** if:  **Definition:** Every element has **at least one corresponding in element in set**  **: Domain**  **: Codomain**  In **onto (surjective) functions**, the **codomain equals the range.** | **Definition:** Let . is **one-to-one** (**injective**) if:  **Note:** A **bijective** function is **both onto and one-to-one**. | **Theorem 2.8 – The Invertible Matrix Theorem**  Let be a **square**, matrix, then the following statements are **equivalent**. That is for a given , they are either **all false or all true**.   1. **is an invertible matrix.** 2. **is row equivalent to the identity matrix.** 3. **has pivot positions.** 4. **The equation has only the trivial solution.** 5. **The columns of form linearly independent set.** 6. **The linear transformation is one-to-one.** 7. **The equation has a solution for all .** 8. **The columns of span .** 9. **The linear transformation maps to .** 10. **There is an matrix such that .** 11. **There is an matrix such that .** 12. **is an invertible matrix.** 13. **The columns of form a basis for .** 14. **The number 0 is not an eigenvalue of .**   **Note:** This **applies only to square matrices**. |

|  |  |  |
| --- | --- | --- |
| **Invertible Linear Transformation:** A linear transformation such that there **exists a linear transformation such that**:  and  is known as the **inverse** of (i.e., )  It is also true that: | **Theorem 2-9:** Let be a linear transformation and let be the **standard matrix** for . Then  **is invertible if and only if is an invertible matrix**. Then the linear transformation given by:  is the **unique function** such that:  and | **Standard Matrix** – Matrix used in a linear transformation. |

# Determinants

## **Introduction** to **Determinants**

|  |  |  |  |
| --- | --- | --- | --- |
| **Determinant (** – **Cannot equal zero for invertible matrices**. | **Determinants for Simple Matrices** | | |
| **Matrix** | **Matrix** | **Matrix** |

|  |  |  |
| --- | --- | --- |
| **Determinant Notation**   * – Element in matrix at row and column * – **Submatrix** of formed by **deleting row and column**. * – -cofactor. It is defined as: * **–** Another notation for **determinant**, namely **two vertical lines.** | **Cofactor Expansion – Technique for Finding a Determinant**  Technique **to find the determinant of an matrix** where (example is along the row):  Given:  **Note:** Cofactor expansion **can be done along any row or down any column in a square matrix**. | **Transpose and the Determinant**  **Transpose has no effect on the value of the determinant**. Hence: |

|  |  |  |
| --- | --- | --- |
| **Theorem 3-1:** The determinant of an matrix can be computed via c**ofactor expansion across any row or column**. The expansion **across the row** is:  The expansion down the column is: | **Theorem 3-2:** If is a **triangular matrix**, then is the **product of the entries along the diagonal** of . |  |

## Properties of Determinants

|  |  |  |  |
| --- | --- | --- | --- |
| **Theorem 3-3: Effect of Row Operations on the Determinant**   1. **If a multiple of one row of is added to another row to produce a matrix, then:** 2. **If two rows of are interchanged to form a new matrix , then** 3. **If one row of is multiplied (i.e., scaled) by to form a new matrix , then** | **Alternate Technique for Finding a Determinant**   1. **Reduce to echelon form.** 2. **Multiply the elements along the diagonal.** 3. **Use theorem 3-3 to adjust for the effect of the row operations on the determinant in step 2.** | **Theorem 3-4:** **A square matrix** is **invertible** if and only if:  **Corollary #1:** if and only if the **columns of**  are **linearly dependent**.  **Corollary #2:** if and only if the **rows of**  are **linearly dependent**. | **Linear Dependent Columns**  Columns of a matrix are linear dependent if:   * **Two columns are equivalent** * **Two rows are equivalent** * **One column is the zero vector** * **One column is a linear combination of the other columns.** |

|  |  |  |  |
| --- | --- | --- | --- |
| **Theorem 3-5:** If is an  **matrix**, then: | **Theorem 3-6 – Multiplicative Property:** If and are matrices, then: | **Checking for Linear Independence with Determinants**  If the **dimensions** (and number of vectors) are **appropriate, check if the determinant equals zero**. **If not, they are linearly independent**. |  |

# Vector Spaces

## Vector Spaces and Subspaces

|  |  |  |
| --- | --- | --- |
| **Space** – A **set of vectors**  – Set of **all vectors of components**.  **Vector** – An **object** **in a vector space**. (*Need not be a column of numbers*)  **Zero Subspace** – A **vector space containing only the zero vector** | **Properties of a Subspace**   1. **Contains the zero vector ()** 2. **Closed under vector addition** 3. **Closed under scalar multiplication.** | **Definition:** A **vector space** is a **nonempty set**  of objects called **vectors** on which are defined two operations called **addition** and **subtraction** subject to the 10 axioms list below. They must hold and .   1. **The sum of and denoted by** 2. **There is a zero vector () in such that:** 3. **For all , there is a vector such that . is known as the negative of .** 4. **The scalar multiple of by denoted by** |

|  |  |  |  |
| --- | --- | --- | --- |
| **Definition:** A **subspace** **of a** **vector space** is a **subset**  of that has **three properties**:   1. **The zero vector of is in .** 2. **is closed under vector addition. That is:** 3. **is closed under multiplication by scalars, that is:** | **Proving a Set is a Subspace**  If **is a subset of vector space** , then **sufficient to only prove three axioms**:   * **1 (addition closure)** * **4 (presence of zero vector)** * **6 (scalar multiplication closure)**   **Every vector space is a subspace** (of itself),and **every subspace is a vector space**. | **Linear Combination** – A **sum of all scalar multiplies** of a set of vectors  – Set of **all vectors that are linear combinations** of   * The set of vectors is known as the **subspace spanned/generated** by * – This is known as the **spanning set**. * are known **as spanning vectors**. | **Additional Derivable Properties of a Vector Space** |

|  |  |  |  |
| --- | --- | --- | --- |
| **Polynomials as a Vector Space**  A **polynomial** in is defined as:  is referred to as the “**degree**”.  **It is a vector space**.  **Zero Polynomial** – A polynomial whose **coefficients** **are all zero**. **Its degree is undefined**. | **Important Notes**   * **is not a subspace of**  (it is not even a subset) | **Theorem 4-1:** If are in a vector space , then **is a subspace of** . |  |

## Null Spaces, Column Spaces, and Linear Transformations

|  |  |  |  |
| --- | --- | --- | --- |
| **Homogeneous Linear System** has the form: | **Null Space for a matrix**  () – For an  **matrix**, , it is the **set of all solutions to the homogeneous system**:  In set notation:  **Alternate Definition:** Set of all **that can be mapped to the zero vector** of via the **linear transformation**: | **Theorem 4-2:** The **null space** of an  **matrix** is a **subspace of** .  Equivalently, the set of all solutions to a system:  of  **homogeneous linear equations in unknowns is a subspace of** | **Implicit Definition of the Null Space**  is an **implicit description** since **each element must be individually checked against** .  For a given vector and matrix , verify: |
| **Solution Set**: Set of all that **satisfies** (i.e., makes **consistent**) a linear system. | **Explicit Description of the Null Space**  **Solve** **to get the explicit description** of |

|  |  |  |  |
| --- | --- | --- | --- |
| **Column Space** () – For an **matrix**, , the column space, , is the **set of all linear combination of the columns** **of** .  For a matrix **,** then:  The **column space** is the **range of the linear transformation**: | **Theorem 4-3:** The **column space** of an **matrix** is a subspace of .  In set notation: |  |  |

**Comparison of and**

|  |  |
| --- | --- |
|  |  |
| is a subspace of | is a subspace of |
| is **implicitly defined**. **Only a condition is given** () that defines it. | is **explicitly defined**. The columns of describe how to build it. |
| It takes time to find the **explicit definition** of by row reducing **.** | It is easy to find vectors in **.** It **includes** **all columns of** and combinations of them**.** |
| There is no obvious relationship between and the entries in . | There is an obvious relationship between and the entries in since it includes the columns of . |
| A **typical entry**, , in has the property: | A **typical entry**, , in has the property of making consistent for some : |
| Given a **specific entry**, , it is easy to determine if by multiplying and checking if the result is the zero vector, . | Given a **specific entry**, , it is not easy to determine if and requires row reducing the augmented matrix . |
| if and only if has only the **trivial solution**. | if and only if the equation **has a solution for all** . |
| has if and only If the **linear transformation** is **one-to-one**. | if and only if the **linear transformation** maps  **onto** . |

## Linearly Independent Sets and Bases

|  |  |  |  |
| --- | --- | --- | --- |
| **Linearly Independent Set:** **Key to “Efficiently” spanning a vector space**. | A **set of vectors** is said to be **linearly independent** if the vector equation:  has **only the trivial solution**. | **Linear Dependence Relation:** A set of **non-zero weights** that satisfy the equation: | **Theorem 4-4:** An indexed set of two or more vectors, with is **linearly dependent** if and only if some is a **linear combination of the preceding vectors** . |

|  |  |  |  |
| --- | --- | --- | --- |
| **Definition:** Let be a **subspace** of a vector space . An indexed set of vectors in is a **basis** of if:   1. **is a linearly independent set** 2. **The subspace spanned by coincides with , that is:** | **Standard Basis for**  where are the **columns of the identity matrix**. | **Theorem 4-5 – The Spanning Set Theorem**  Let be a set of vectors in , and let , then:   1. **If one of the vectors in , say , is a linear combination of the remaining vectors, then the set formed by removing from still spans .** 2. **If *,* then some subset of is a basis of .** | **Standard Basis for** |

|  |  |  |  |
| --- | --- | --- | --- |
| **Bases for and** | | **Two Views of a Basis** | |
| **Algorithm – Find the Basis of**   1. **Row reduce to echelon form.** 2. **Identify the columns with pivot positions in the echelon matrix.** 3. **Select the corresponding columns FROM THE ORIGINAL MATRIX as the basis of .** | **Theorem 4-6** – The **pivot columns of**  **form a basis for** . | A basis is a **spanning set** **that is as small as possible**. | A basis is a **linearly independent set** **that is as large as possible**. |

|  |  |  |  |
| --- | --- | --- | --- |
| **No Free Variables Only Trivial Solution Linear independence** | If is an **invertible**  matrix, then  **spans**  by the **invertible matrix theorem**. |  |  |

## Coordinate Systems

|  |  |  |
| --- | --- | --- |
| **Theorem 4-7 – Unique Representation Theorem**  Let be a **basis** for a vector space . Then for each , there **exists a unique set of scalars** such that: | **Definition**: Suppose is a **basis** for a vector space and . The **coordinates of relative to the basis**  are the **weights** such that:  The **coordinate vector of**  (**relative to** ) or the **-coordinate vector of** is: | **Coordinate Mapping (Determined by )**  **Notation:**  **Creates a new coordinate system**  It is a one-to-one linear transformation from onto to R |
| **Standard Basis for**  where are the **columns of the identity matrix**.  **Note:** |
| **Standard Basic for**  **Note:** |

|  |  |  |  |
| --- | --- | --- | --- |
| **Algorithm: Find the Coordinate Vector for**  For a given basis , **solve the linear system to get the -coordinate vector of** : | **Change of Coordinates Matrix** | | **Theorem 4-8:** Let be a basis for a vector space, . Then the **coordinate mapping**:  is a **one-to-one** **linear transformation** from to . |
| For a basis , the **change of coordinates matrix** is defined as:  where: | If the **change of coordinates matrix, , is square**, then it is also **invertible** by the **Invertible Matrix Theorem** **since the columns of the matrix (i.e., the basis) are linearly independent**.  In these **specific cases**: |

|  |  |  |  |
| --- | --- | --- | --- |
| **Preservation of Vector Addition and Scalar Multiplication**  Due to the **change of coordinates matrix**, coordinate mapping is a **linear transformation** meaning that the operation **preserves vector addition and scalar multiplication**. Formally: | **Isomorphism**: A **map that preserves sets and relations among elements**.  **Requirements**:   * **Mapping is one to one.** * **Calculations in one space yield mapping values in the other set.**   **Example Isomorphism:** |  |  |

## The Dimension of a Vector Space

|  |  |  |  |
| --- | --- | --- | --- |
| **Theorem 4-9**: If a vector space has a basis , then any set in **containing more than vectors must be** **linearly dependent**. | **Theorem 4.10** – If a vector space has a basis of vectors, then **every basis of** **must consist of exactly vectors**. | If a vector space is **spanned by a finite set**, then is said to be **finite dimensional**.  The **dimension of** , written as , is **the number of vectors in the basis of** .   * **Examples:**   + **– Zero dimensional**   + **– -dimensional**   + **– -dimensional** | If cannot be spanned by a finite set, then is said to be **infinite dimensional**.  **Example:**   * **– Set of all polynomials.** |

|  |  |  |  |
| --- | --- | --- | --- |
| **Theorem 4.11:** Let be a **subspace of a finite-dimensional vector space** . **Any linearly independent set in** **can be expanded, if necessary, to a basis for** . Also, is **finite dimensional** where: | **Theorem 4.12: The Basis Theorem**  Let be a -dimensional vector space, where . **Any linearly independent set of exactly elements in is automatically a basis for** . **Any set of exactly elements that spans is automatically a basis for** . | **Dimensions of and**  = **Number of pivot columns in**  = **Number of free variables in the equation:** . | **Relationship between Rank, Row Space, and Column Space**  For an matrix:  Hence: |

## Rank

|  |  |  |  |
| --- | --- | --- | --- |
| **Row space** – Set of **all linear combinations of the rows of**  **Note:** | **Theorem 4-13:** If two matrices andare **row equivalent**, then their **row spaces are the same**. If is in **echelon form**, then the **non-zero rows** of form a **basis for the row space** of as well as that of . | **Algorithm: Finding the Row Space’s Basis**   1. **Row reduce the input matrix to another matrix in echelon form.** 2. **Select the non-zero rows of (i.e., the pivot rows) as the basis of the row spaces for and .** | The **rank** of a matrix is the **dimension** (i.e., **number of vectors in the basis**) of the **column space** (**or row space**) |

|  |  |  |  |
| --- | --- | --- | --- |
| **Theorem 4-14 – The Rank Theorem**  The **dimensions** of the **column space** and **row space** are **equal** for an  **matrix**,. This common dimension, the **rank** of , also **equals the number of pivot columns** of and satisfies the equation: |  |  |  |

# Eigenvalues and Eigenvectors

## Eigenvectors and Eigenvalues

|  |  |  |  |
| --- | --- | --- | --- |
| An **eigenvector** of an  **matrix** is a **nonzero vector** such that for some scalar . | The **scalar** is called an **eigenvalue** of if there is a **nontrivial solution**, , (i.e., ) such that ; such an is called an **eigenvector corresponding to**. | **Theorem 5-1:** The **eigenvalues** of a **triangular matrix** are the **entries on its main diagonal**. | **Theorem 5-2:** If are **eigenvectors** **that correspond to distinct eigenvalues** of an  **matrix** , then the set is **linearly independent**. |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Checking if a Vector is an Eigenvector**  Given a vector, , **verify that the relationship holds**:  If it does, then it is an eigenvector. | **Checking if a Scalar is an Eigenvalue**  For an matrix and vector , **verify that the relationship holds**: | **Eigenspaces** | | |
| The **eigenspace** is the **subspace** **of**  corresponding to the **null space** of the matrix for a given eigenvalue, .  The **eigenspace** for a matrix is the **set of all solutions to the homogeneous equation**: | **Eigenspaces** are **-dimensional** which **corresponds to the number of free variables** in or the **number of vectors in the basis of its null space**. | Although the zero vector, , **can never be an eigenvector**,  **is in all eigenspaces**.  An **eigenspace** contains the  **plus all eigenvectors corresponding to** . |

|  |  |  |  |
| --- | --- | --- | --- |
| **Difference Equations** | | **Note #1:** A matrix has an **eigenvalue** of “0” if and only if is **not invertible**.  **Note #2:** If is an matrix, then **and** **have the same** **eigenvalues**. |  |
| For a matrix , a **difference equation** is one where **each subsequent vector** **is** **based solely on the preceding vector**, , and such that: | A **solution** is an explicit description of the set whose **formula for each** **does not depend on**  **or on the sequence other than the initial seed vector,** , such that: |

## The Characteristic Equation

|  |  |  |
| --- | --- | --- |
| **Theorem 5-3:** Let and by matrices:  **is invertible if and only of**  **If is triangular, then is the product of the entries on the main diagonal of .**  **A row replacement operation on does not change its determinant. A row interchange changes the sign of the determinant. A row scaling scales the determinant by the same scaling factor.** | **Algorithm: Characteristic Equation**  **Used to find the eigenvalues** of a matrix.  For a matrix , the **characteristic equation** is defined as:  A scalar is an **eigenvalue** **if and only if it satisfies the characteristic equation**. | If matrix is size , then **is an -degree polynomial** called the **characteristic polynomial** of |
| The **algebraic multiplicity of an eigenvalue** is its **multiplicity as a root of the** **characteristic equation**.  **Example:**  “” has multiplicity of 2 |

|  |  |  |  |
| --- | --- | --- | --- |
| **Complex Eigenvalues** – Any **eigenvalue** in the characteristic equation **with an imaginary term**. | **Similarity**  Given that and are matrices, then **is similar to** entails that an **invertible matrix**  **exists such that**:  Or Rewritten: | **Theorem 5-4** – If matrices and are **similar**, then they have the **same** **characteristic polynomial** and hence the **same eigenvalues**. | **Warning #1:** Two matrices having **the same eigenvalues** **are not necessarily similar**. |
| **Warning #2:** **Similarity** **does not entail row equivalence**.  **Note: Row operations generally change the eigenvalues.** |
| **Note:** If  **is similar to** , then necessarily **is similar to** . Hence, it can be said that “ andare **similar**.” |

## Diagonalization

|  |  |  |
| --- | --- | --- |
| **Definition:** A **square,** **matrix**, , is **diagonalizable** if it is **similar to** an **diagonal matrix**, such that:  where is an **invertible**,  **matrix**.  **Note:** A **diagonal matrix** has **values along its main diagonal and zeros elsewhere**. | **Finding the Power of a Diagonal Matrix** | **Theorem 5-5 – Diagonalization Theorem**  An matrix is **diagonalizable** if and only if it has  **linearly independent eigenvectors**.  In fact,if and only if is a **diagonal matrix** **consisting of eigenvalues and**  **is an eigenvector matrix**. |
| **Finding the Power of a Diagonalizable Matrix** |

|  |  |  |
| --- | --- | --- |
| **Algorithm to Diagonalize a Matrix**  **Find the eigenvalues of**  **Find linearly independent eigenvectors of**  **Construct from the eigenvectors in step #2**  **Construct from the corresponding eigenvalues found in step #1** | **Theorem 5-6:** A matrix with **distinct** **eigenvalues is diagonalizable**.  **Note**: A matrix **can be diagonalizable with less than distinct eigenvalues**. | **Theorem 5-7:** Let be an  **matrix** who **distinct** **eigenvalues** are: .  **For , the dimension of the eigenspace of is less than or equal to the multiplicity of the eigenvalue .**  **The matrix is diagonalizable if and only if the sum of the dimensions of its eigenspaces equals. This happens only if (i) The characterstic polynomial factors complete into linear factors and (ii) the dimension of the eigenspace for each equals the multiplicity of**  **If is diagonalizable and is a basis for the eigenspace corresponding to for each , then the total collection of vectors in the sets forms an eigenvector basis for .** |

|  |  |  |
| --- | --- | --- |
| **Definition:** A matrix is **diagonalizable** if and only if there **are enough linearly independent eigenvectors to form a basis for** **.** This basis is known as the **eigenvector basis of** . | By **theorem 5-5**, **any matrix that is** **diagonalizable** is **also** **invertible**. |  |

# Orthogonality and Least Squares

## Inner Product, Length, and Orthogonality

|  |  |  |
| --- | --- | --- |
| **Definition**: If and are vectors of length, then the number:  is called the **inner product**. It is often written as:  It is also called the “**dot product**.” | **Theorem 6-1:** Let , , and be vectors in and let be a scalar in . Then:  **and if and only if** | **Length of a Vector**   * By **property (d)** of **theorem 6.1**, the **square root of the inner product is always defined** (i.e., **non-imaginary**).   **Definition:** The **length** (or **norm**) of a vector is a non-negative scalar‖ is defined by:  **Important Property:** Given a vector, , and a scalar, , then:   * Note the absolutely value on the scalar “” |

|  |  |  |  |
| --- | --- | --- | --- |
| **Definition**: A unit **vector** is any vector whose **length**/**norm** **equals 1** (i.e., ) | **Algorithm:** **Normalizing a Vector**  Given a vector, , it can be **normalized** to into the form of ***its*** **unit vector**, , **in the same direction as via**: | **Distance in**  **Definition:** For , the **distance between** **and** , written as is the **length of the vector**  such that: | **Orthogonal Vector**  Geometrically, two vectors are **perpendicular** if and only if:  And |
| **Definition**: Two vectors, and , are **orthogonal** (**to each other**) if: |

|  |  |  |
| --- | --- | --- |
| **Theorem 6-2: The Pythagorean Theorem**  Two vectors, and , are orthogonal if and only if: | **Orthogonal Complement** | **Theorem 6-3:** Let be an matrix. The **orthogonal complement** of the **row space of** is the **null space of** .  Also the **orthogonal complement** of the **column space of** is the **null space of** . |
| Given a **subspace**, , if any vector, , is **orthogonal to all vectors in** , then is **orthogonal to** . |
| **Definition:** The **set of all vectors of all vectors orthogonal** tois call the **orthogonal complement** of . |

|  |  |  |
| --- | --- | --- |
| **Angles in and**  If and  are **non-zero vectors** in or , then the **angle**, , **between the two vectors** is:  If **,** where , then the angle, , is referred to as the correlation coefficient **between the two vectors**. |  |  |

## Orthogonal Sets

|  |  |  |  |
| --- | --- | --- | --- |
| **Definition:** A set of vectors is said to be an **orthogonal set** if **each pair of distinct vectors from the set is orthogonal** such that: | **Theorem 6-4:** If is an orthogonal set of non-zero vectors in , then is **linearly dependent** and hence it is **a basis for the subspace spanned by** . | **Definition:** An **orthogonal basis** for a subspace of is a **basis for** that is an **orthogonal set**. | **Theorem 6-5:** Let be an **orthogonal basis** for a subspace . For each, the **weights in the linear combination**:  are for : |

|  |  |  |  |
| --- | --- | --- | --- |
| **Definition:** A set is an **orthonormal set** if it is an **orthogonal set of unit vectors**. | **Theorem 6-6:** An matrixhas **orthonormal columns** **if and only if** . | **Theorem 6-7:** Let be an matrix with **orthonormal columns**, and let , then:   1. **if and only if**   **Note: Properties (b) and (c)** mean that the **linear mapping** **preserves lengths and orthogonality**. | **Definition:** An **orthogonal matrix** is a **square, invertible** matrix **with orthonormal columns** where: |

**An Orthogonal Projection**

|  |  |  |
| --- | --- | --- |
| **Nomenclature**   * – A direction vector. * – A line through the origin and * – A scalar in * – A vector through insuch that:   This vector is known as the **orthogonal projection onto .**   * – A vector orthogonal to , , and. This vector is known as the **component of**  **orthogonal to** . * – A vector to be decomposed where: | **Algorithm: Finding**  **Define**  **such that** **is also orthogonal to**  **via**: **. Hence:** |  |
| **Algorithm: Finding the Distance from to**  **This distance is equivalent to** . It is found by:  **Step #1: Find**  **Step #2: Solve for** **via:** |

# Linear Algebra Theorems

|  |  |  |
| --- | --- | --- |
| **Theorem #1: Any matrix has one and only one** **reduced echelon form**. | **Theorem 1-2: Existence and Uniqueness Theorem**  A **linear system is consistent** if and only if the rightmost column of the augmented matrix is not a pivot column – that is, if and only if an echelon form of the **augmented matrix has no row in the form**:  withnon-zero  If a linear system is consistent, then the solution set contains either:   1. **A unique solution, when there is no free variable** 2. **Infinitely many solutions when there is at least one free variable.** | **Theorem #1-3:** If is an matrix with columns and if , then **the matrix equation**:  **has the same solution set** as the **vector equation**:  which in turn **has the same solution set as the system of linear equations whose augmented matrix is**: |

|  |  |  |
| --- | --- | --- |
| **Theorem #1-4:** Let be an matrix. Then the following statements are equivalent meaning they are **either all true** **or all false**.   1. **For all in, the equation has a solution.** 2. **Each in is a linear combination of the columns of A.** 3. **The columns of A span** 4. **The (coefficient) matrix has a pivot position in every row.** | **Theorem #1-5:** If is an matrix, and are vectors in , and is a scalar, then: | **Theorem #1-6:** Suppose the equation is consistent for some given , and let be **any solution to that particular non-homogenous system**. Then the **solution set of** is **the set of all vectors of the form**:  where is **any solution of the homogenous equation** . |

|  |  |  |
| --- | --- | --- |
| **Theorem #1-7:** An indexed set of two or more vectors is **linearly dependent** **if and only if at least one of the vectors in is a linear combination of the others**.  In fact, if is linearly dependent and , then some (where ) is a linear combination of the preceding vectors: . | **Theorem #1-8:** If a set **contains more vectors than there are entries in each vector**, then the set is **linearly dependent**.  That is any set inis linearly dependent if:  **Proof:** More pivots than columns in an matrix so **at least one free variable**. | **Theorem #1-9:** If a set  **contains the zero vector**, then the set is **linearly dependent**.  **Proof:** If the zero vector is reordered to be , then |

|  |  |  |
| --- | --- | --- |
| **Theorem #2-1:** Let , , and be **matrices of the same size** and let and be scalars, then:   1. **(Commutative)** 2. **(Associative)** 3. **(Identity)** | **Theorem #2-2:** Given is the identity matrix of size and is an matrix, and that and have sizes for which the indicated sums and products exist, then   1. **(Associative)** 2. **(Left Distributive Law)** 3. **(Right Distributive Law)** 4. **, for any scalar** | **Theorem #2-3:** Let and be matrices whose sizes are appropriate for the following sums and products:   1. **, for any scalar**   **Note: The reverse order of the product.** |

|  |  |  |
| --- | --- | --- |
| **Theorem 2-4:** Let . If , is invertible and: | **Theorem 2-5:** If is an **invertible matrix**, then for all , there exists a **unique solution** . More formally: | **Theorem 2-6**   1. If is an **invertible matrix**, then **is invertible** and: 2. If and are invertible matrices, then so is . The **inverse of is the product of the inverses in reverse order**, that is: 3. If is an invertible matrix, then so is . **The inverse of is the transpose of** , that is: |

|  |  |  |
| --- | --- | --- |
| **Theorem 2.7:** An matrix is **invertible** if and only if it is **row equivalent to the identity matrix** .  Hence, **any sequence of row operations that reduces to also transforms to** . | **Theorem 2.8 – The Invertible Matrix Theorem**  Let be a **square**, matrix, then the following statements are **equivalent**. That is for a given , they are either **all false or all true**.   1. **is an invertible matrix.** 2. **is row equivalent to the identity matrix.** 3. **has pivot positions.** 4. **The equation has only the trivial solution.** 5. **The columns of form linearly independent set.** 6. **The linear transformation is one-to-one.** 7. **The equation has a solution for all .** 8. **The columns of span .** 9. **The linear transformation maps to .** 10. **There is an matrix such that .** 11. **There is an matrix such that .** 12. **is an invertible matrix.** 13. **The columns of form a basis for .** 14. **The number 0 is not an eigenvalue of .**   **Note:** This **applies only to square matrices**. | **Theorem 2-9:** Let be a linear transformation and let be the **standard matrix** for . Then  **is invertible if and only if is an invertible matrix**. Then the linear transformation given by:  is the **unique function** such that:  and |

|  |  |  |
| --- | --- | --- |
| **Theorem 3-1:** The determinant of an matrix can be computed via c**ofactor expansion across any row or column**. The expansion **across the row** is:  The expansion down the column is: | **Theorem 3-2:** If is a **triangular matrix**, then is the **product of the entries along the diagonal** of . | **Theorem 3-3: Effect of Row Operations on the Determinant**   1. **If a multiple of one row of is added to another row to produce a matrix, then:** 2. **If two rows of are interchanged to form a new matrix , then** 3. **If one row of is multiplied (i.e., scaled) by to form a new matrix , then** |

|  |  |  |
| --- | --- | --- |
| **Theorem 3-4:** **A square matrix** is **invertible** if and only if:  **Corollary #1:** if and only if the **columns of**  are **linearly dependent**.  **Corollary #2:** if and only if the **rows of**  are **linearly dependent**. | **Theorem 3-5:** If is an matrix, then: | **Theorem 3-6 – Multiplicative Property:** If and are matrices, then: |

|  |  |  |
| --- | --- | --- |
| **Theorem 4-1:** If are in a vector space , then **is a subspace of** . | **Theorem 4-2:** The **null space** of an  **matrix** is a **subspace of** .  Equivalently, the set of all solutions to a system:  of  **homogeneous linear equations in unknowns is a subspace of** | **Theorem 4-3:** The **column space** of an **matrix** is a subspace of .  In set notation: |

|  |  |  |
| --- | --- | --- |
| **Theorem 4-4:** An indexed set of two or more vectors, with is **linearly dependent** if and only if some is a **linear combination of the preceding vectors** . | **Theorem 4-5 – The Spanning Set Theorem**  Let be a set of vectors in , and let , then:   1. **If one of the vectors in , say , is a linear combination of the remaining vectors, then the set formed by removing from still spans .** 2. **If *,* then some subset of is a basis of .** | **Theorem 4-6** – The **pivot columns of**  **form a basis for** . |

|  |  |  |
| --- | --- | --- |
| **Theorem 4-7 – Unique Representation Theorem**  Let be a **basis** for a vector space . Then for each , there **exists a unique set of scalars** such that: | **Theorem 4-8:** Let be a basis for a vector space, . Then the **coordinate mapping**:  is a **one-to-one** **linear transformation** from to . | **Theorem 4-9**: If a vector space has a basis , then any set in **containing more than vectors must be** **linearly dependent**. |

|  |  |  |
| --- | --- | --- |
| **Theorem 4.10** – If a vector space has a basis of vectors, then **every basis of** **must consist of exactly vectors**. | **Theorem 4.11:** Let be a **subspace of a finite-dimensional vector space** . **Any linearly independent set in** **can be expanded, if necessary, to a basis for** . Also, is **finite dimensional** where: | **Theorem 4.12: The Basis Theorem**  Let be a -dimensional vector space, where . **Any linearly independent set of exactly elements in is automatically a basis for** . **Any set of exactly elements that spans is automatically a basis for** . |

|  |  |  |
| --- | --- | --- |
| **Theorem 4-13:** If two matrices andare **row equivalent**, then their **row spaces are the same**. If is in **echelon form**, then the **non-zero rows** of form a **basis for the row space** of as well as that of . | **Theorem 4-14 – The Rank Theorem**  The **dimensions** of the **column space** and **row space** are **equal** for an  **matrix**,. This common dimension, the **rank** of , also **equals the number of pivot columns** of and satisfies the equation: |  |

|  |  |  |
| --- | --- | --- |
| **Theorem 5-1:** The **eigenvalues** of a **triangular matrix** are the **entries on its main diagonal**. | **Theorem 5-2:** If are **eigenvectors** **that correspond to distinct eigenvalues** of an matrix , then the set is **linearly independent**. | **Theorem 5-3:** Let and by matrices:  **is invertible if and only of**  **If is triangular, then is the product of the entries on the main diagonal of .**  **A row replacement operation on does not change its determinant. A row interchange changes the sign of the determinant. A row scaling scales the determinant by the same scaling factor.** |

|  |  |  |
| --- | --- | --- |
| **Theorem 5-4** – If matrices and are **similar**, then they have the **same** **characteristic polynomial** and hence the **same eigenvalues**. | **Theorem 5-5 – Diagonalization Theorem**  An matrix is **diagonalizable** if and only if it has  **linearly independent eigenvectors**.  In fact,if and only if is a **diagonal matrix** **consisting of eigenvalues and**  **is an eigenvector matrix**. | **Theorem 5-6:** A matrix with **distinct** **eigenvalues is diagonalizable**.  **Note**: A matrix **can be diagonalizable with less than distinct eigenvalues**. |

|  |  |  |
| --- | --- | --- |
| **Theorem 5-7:** Let be an  **matrix** who **distinct** **eigenvalues** are: .  **For , the dimension of the eigenspace of is less than or equal to the multiplicity of the eigenvalue .**  **The matrix is diagonalizable if and only if the sum of the dimensions of its eigenspaces equals. This happens only if (i) The characterstic polynomial factors complete into linear factors and (ii) the dimension of the eigenspace for each equals the multiplicity of**  **If is diagonalizable and is a basis for the eigenspace corresponding to for each , then the total collection of vectors in the sets forms an eigenvector basis for .** |  |  |

|  |  |  |
| --- | --- | --- |
| **Theorem 6-1:** Let , , and be vectors in and let be a scalar in . Then:  **and if and only if** | **Theorem 6-2: The Pythagorean Theorem**  Two vectors, and , are orthogonal if and only if: | **Theorem 6-3:** Let be an matrix. The **orthogonal complement** of the **row space of** is the **null space of** .  Also the **orthogonal complement** of the **column space of** is the **null space of** . |

|  |  |  |
| --- | --- | --- |
| **Theorem 6-4:** If is an orthogonal set of non-zero vectors in , then is **linearly dependent** and hence it is **a basis for the subspace spanned by** . | **Theorem 6-5:** Let be an **orthogonal basis** for a subspace . For each, the **weights in the linear combination**:  Are for : | **Theorem 6-6:** An matrixhas **orthonormal columns** **if and only if** . |

|  |  |  |
| --- | --- | --- |
| **Theorem 6-7:** Let be an matrix with **orthonormal columns**, and let , then:   1. **if and only if**   **Note: Properties (b) and (c)** mean that the **linear mapping** **preserves lengths and orthogonality**. |  |  |

# Proofs for the Algebraic Properties of

|  |  |  |
| --- | --- | --- |
| **Name** | **Description** | **Terms** |
| **Commutative** (Vector Addition) |  | * – Vectors * – Zero vector * – Real constants |
| **Inverse** (Vector Addition) |  |
| **Associative** (Vector Addition) |  |
| **Associative** (Scalar Multiplication) |  |
| **Distributive** **Law** (Vector Addition) |  |
| **Distributive** **Law** (Scalar Multiplication) |  |
| **Identity** (Vector Addition) |  |
| **Identity** (Scalar Multiplication) |  |

|  |  |
| --- | --- |
| **Proof of the Commutative Property of Vector Addition**   1. Suppose and are **any real vector in**  in the form:   and   1. By the **definition of vector addition**:      1. By the **commutative property of real number addition**:      1. By the **definition of vector addition** and the **definition of and** :   **(QED)** | **Proof of the Inverse Property of Vector Addition**   1. Suppose is **any** **real vector in**  in the form: 2. By the **definition of scalar multiplication**: 3. By the **definition of vector addition**:      1. By the **inverse property of real number addition**:      1. By **definition of the zero vector**:   **(QED)** |

|  |  |
| --- | --- |
| **Proof of the Associative Property for Vector Addition**   1. Suppose , , and are **any real vector in**  in the form:   and and   1. By the **definition of vector addition**:      1. By the **definition of vector addition**: 2. By the **associative property of real number addition**:      1. By the **definition of** **vector addition** and the **definition of and** :   **(QED)** | **Proof of the Associative Property for Scalar Multiplication**   1. Suppose and are **any real number** and is **any real vector in**  in the form: 2. By the **definition of scalar multiplication**:      1. By the **definition of vector addition**: 2. By the **associative property of real number multiplication**: 3. By the **definition of** **scalar multiplication**:   **(QED)** |

|  |  |
| --- | --- |
| **Proof of the Distributive Law for Vector Addition**   1. Suppose is **any real number** and and are **any real vector in**  in the form:   and   1. By the **definition of vector addition**:      1. By the **definition of scalar multiplication**:      1. By the **distributive law over real number addition**: 2. By the **defintition of vector addition**: 3. By the **definition of scalar multiplication**,and the **definition of and** :   **(QED)** | **Proof of the Distributive Law for Scalar Multiplication**   1. Suppose and are **any real number** and is **any real vector in**  in the form:      1. By the **definition of scalar multiplication**:      1. By the **distributive law over real number addition**:      1. By the **definition of vector addition**: 2. By the **definition of scalar multiplication** and the **definition of** :   **(QED)** |

|  |  |
| --- | --- |
| **Proof of the Identity Property for Vector Addition**   1. Suppose is the **zero vector of length**  and is **any real vector in**  in the form: 2. By the **definition of vector addition**:      1. By the **identity property of real number addition**:      1. By the **definition of** :   **(QED)** | **Proof of the Identity Property for Scalar Multiplication**   1. Suppose is **any real vector in**  in the form: 2. By the **definition of scalar multiplication**:      1. By the **identity property of real number multiplication**:      1. By the **definition of** :   **(QED)** |