**MATH129A – Linear Algebra Midterm #1 Study Guide**

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**Section 1.1 – Systems of Linear Equations**

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| **Linear Equation** – An equation with **variables** that **can be written in the form**:  **Coefficients:** can be **real or complex** | **Linear System** or **System of Linear Equations** – Collection of **one or more linear equations**. | **Solution:** A **set of numbers** **that makes each equation a true statement** when substituted for variables respectively. | **Solution Set:** Set of **all possible solutions** for a linear system.    **Possible Solution Sets:**   * **No solution** * **One solution** * **Infinite solutions** |

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| **Consistent Linear System** – Has **one or infinite solutions** | **Coefficient Matrix** – A matrix containing the **coefficients for each variable in each equation** in the linear system. | **Augmented Matrix** – A matrix of a system containing the **coefficient matrix and** **an added column containing the constants** from the ***right hand side*** of the equation. | **Techniques to Simplify a Linear System**   1. **Replace** one equation with **sum of itself and the multiple of another linear system** (equation) 2. **Interchange** two equations 3. **Multiply all terms** in an equation by a **non-zero constant** |
| **Inconsistent Linear System** – Has no solution |

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| **Row Equivalent Matrices** – Any two matrices where a **series of elementary row operations** can transform one matrix into another. | **Row Operation Reversibility** – All row operations can be undone to get the previous matrix | **Matrix** – Composed of rows of columns | **Equivalent Linear Systems** – Any two linear systems with the **same solution set**. |

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| **Approaches to Find the Solution Set of a Linear System**   1. **Solve equations by substitution.** 2. **Multiply and add the equations** 3. **Graphically**    1. Look at intersection of the equations. |  |  |  |

**Section 1.2 – Row Reduction and Echelon Forms**

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|  | **Key Properties** | | | | |
| If **two augmented matrices are row equivalent**, then the systems **have the same solution set**. | **Reduced echelon form is unique** | **Echelon form is not unique** | All linear systems have a reduced echelon form. | **Location of leading entries** is the **same between standard and reduced echelon form.** |

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| **Echelon Matrix Criteria**   1. **All non-zero rows are above all zero rows.** 2. **The leading entries of lower rows are to the right of all those in upper rows.**    1. Forms a “**step pattern**”. 3. **The entries in a column below a leading entry are zero.** | **Reduced (Row) Echelon Matrix Criteria**   1. **All criteria of a standard echelon matrix.** 2. **All leading entries equal 1.** 3. **The entries in a column above a leading entry are 0.** | **Theorem #1: Any matrix has one and only one** **reduced echelon form**. |

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| **Pivot Position** – A **position** in a given matrix that **corresponds to a “1” in reduced echelon form**. | **Pivot Position** – A column that contains a pivot position. | **Pivot** – A non-zero number in a pivot position that is used as needed to create zeros via row operations. | **Gaussian Elimination:** Same concept as row reduction. | **Non-zero row/column**: A row/column with **at least one non-zero entry**. | **Leading entry** – **Leftmost non-zero entry** **in a row**. |

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| **Row Reduction Algorithm (Gaussian Elimination)**  **Forms an Echelon Matrix** | | | | **Gauss-Jordan Elimination**  **Forms the Reduced Echelon Matrix** |
| 1. **Begin with the left most non-zero entry that is a pivot position. Move that position to the top of the matrix.** | 1. **Select a non-zero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.** | 1. **Use row operations to create zeros in all positions below the pivot.** | 1. **Cover (i.e. ignore) the rows containing the pivot position and cover (ignore) all rows above it. Apply steps 1-3 to the sub matrix that remains. Repeat the process until there are no more non-zero rows remaining.** | 1. **Create zeros above each pivot and scale each pivot to 1.** |

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| **Basic Variable: Can only exist in a single solution set equation. Correspond to a pivot in the reduced echelon matrix.** | **Parametric Description of Solution Sets**   * Free variables act as parameters * Each basic variable is represented by an equation made up of constants and/or free variables. | **Theorem 1-2: Existence and Uniqueness Theorem**  A **linear system is consistent** if and only if the rightmost column of the augmented matrix is not a pivot column – that is, if and only if an echelon form of the **augmented matrix has no row in the form**:  withnon-zero  If a linear system is consistent, then the solution set contains either:   1. **A unique solution, when there is no free variable** 2. **Infinitely many solutions when there is at least one free variable.** |
| **Free Variable:** **Can be assigned to any number** since it has no pivot. Free variable **quantity dictates the shape of the resulting solution set’s shape**. |

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| **Example Parametric Description of a Solution Set** | |  |
| **Augmented Matrix** | **Parametric Description of the Solution Set** |

**Section 1.3 – Vector Equations**

**Section 1.4 – The Matrix Equation**

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| **Gaussian Elimination** – Same concept as row reduction. | **Theorem #1-4:** Let be an matrix. Then the following statements are equivalent meaning they are **either all true** **or all false**.   1. **For all in , the equation has a solution.** 2. **Each in is a linear combination of the columns of A.** 3. **The columns of A span** 4. **The (coefficient) matrix has a pivot position in every row.** | **Relationship between Spans and Free Variables**   |  |  | | --- | --- | |  | **General Solution Structure** | | **(Trivial Only)** |  | | **1 Free Variable** | – **Line** through and the origin | | **2 Free Variables** | – **Plane** through , , and the origin | | **Free Variables** | – **Multidimensional shape** through and the origin | |
| Non-zero row/column – A row/column with at least one non-zero entry. |
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| **Theorem #1-5:** If is an matrix, and are vectors in , and is a scalar, then: |  |  |

**Section 1.5 – Solution Sets of Linear Systems**

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| **Homogeneous Equation:**  **Trivial Solution:**   * **Exists for all homogeneous systems**   **Nontrivial Solution:** Any **non-zero vector solution.**   * If it exists, there are infinitely many. * Requires at least one free variable. | **Non-Homogenous System:** where is not the zero vector.   * May be inconsistent. * If its solution exists, it is in the form:   + – Particular solution for the specific non-homogenous system   + – Solution set for the homogenous system |  |

**Section 1.7 – Linear Independence**

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| **Span Review**   * **One Vector** – **At most** a **line**   + **Exception:** vector * **Two Vectors: At most** a **plane**   + **Exception:** Scalar multiples | **Linear Independence**  A set of vectors are **linearly independent** if:  has **only the trivial solution** | **Linear Independence**  A set of vectors are **linearly dependent** if there exists a set of **non-zero weights** such that:  **Note:** This requires **at least one free variable**. |

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| **Linear Dependence Relation:** For vectors , it is defined as:  where are **not all zero.**  **Note:** The values of are not unique. | **Procedure: Checking for Linear Independence**  **Step #1:** Create the coefficient matrix.  **Step #2:** Perform Gaussian elimination to find the echelon matrix.  **Step #3:** Check linear independence   * If **there is a pivot in every row**, the vectors are **linearly independent**. * If **there is a free variable**, the vectors are **linearly dependent**. | **Linear Independence of One Vectors**   * **Zero Vector** () – This is always **linearly dependent**. * **Any Non-Zero Vector** ( where ) then **linearly independent**. |

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| **Linear Independence of Two Vectors** – A set of two vectors is linearly dependent if **at least one** **of the vectors is a scalar multiple** of the other.   * “**At least one**” – Because of the case of the zero vector. |  | **Linear Dependence Summary:** If a set of  **vectors of -dimensions** are linearly independent, then they **span an dimensional shape**. |

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| **Theorem 1-7:** An indexed set of two or more vectors is **linearly dependent** **if and only if at least one of the vectors in is a linear combination of the others**.  In fact, if is linearly dependent and , then some (where ) is a linear combination of the preceding vectors: . | **Theorem 1-8:** If a set **contains more vectors than there are entries in each vector**, then the set is **linearly dependent**.  That is any set inis linearly dependent if:  **Proof:** More pivots than columns in an matrix so **at least one free variable**. | **Theorem 1-9:** If a set  **contains the zero vector**, then the set is **linearly dependent**.  **Proof:** If the zero vector is reordered to be , then |

**Section 1.8 – Introduction to Linear Transformations**

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| **Function** – A **rule** that **assigns to each element** in a set A **exactly one element** from set B. | **Transformation () from to :** A rule that **assigns to each vector** **a vector** | **Domain of :** | **Image of :** For a given , it is the **transformed value** | **Range: Set of all images** .  **Note:** The range may (and often is) be only a subset of the codomain. |
| **Codomain of :** |

**Section 2.1 – Matrix Operations**

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**Theorems**

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