**MATH129A – Linear Algebra Midterm #2 Study Guide**

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**Section 1.1 – Systems of Linear Equations**

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| **Linear Equation** – An equation with **variables** that **can be written in the form**:  **Coefficients:** can be **real or complex** | **Linear System** or **System of Linear Equations** – Collection of **one or more linear equations**. | **Solution:** A **set of numbers** **that makes each equation a true statement** when substituted for variables respectively. | **Solution Set:** Set of **all possible solutions** for a linear system.    **Possible Solution Sets:**   * **No solution** * **One solution** * **Infinite solutions** |

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| **Consistent Linear System** – Has **one or infinite solutions** | **Coefficient Matrix** – A matrix containing the **coefficients for each variable in each equation** in the linear system. | **Augmented Matrix** – A matrix of a system containing the **coefficient matrix and** **an added column containing the constants** from the ***right hand side*** of the equation. | **Techniques to Simplify a Linear System**   1. **Replace** one equation with **sum of itself and the multiple of another linear system** (equation) 2. **Interchange** two equations 3. **Multiply all terms** in an equation by a **non-zero constant** |
| **Inconsistent Linear System** – Has no solution |

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| **Row Equivalent Matrices** – Any two matrices where a **series of elementary row operations** can transform one matrix into another. | **Row Operation Reversibility** – All row operations can be undone to get the previous matrix | **Matrix** – Composed of rows of columns | **Equivalent Linear Systems** – Any two linear systems with the **same solution set**. |

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| **Approaches to Find the Solution Set of a Linear System**   1. **Solve equations by substitution.** 2. **Multiply and add the equations** 3. **Graphically**    1. Look at the intersection of the equations. |  |  |  |

**Section 1.2 – Row Reduction and Echelon Forms**

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|  | **Key Properties** | | | | |
| If **two augmented matrices are row equivalent**, then the systems **have the same solution set**. | **Reduced echelon form is unique** | **Echelon form is not unique** | All linear systems have a reduced echelon form. | **Location of leading entries** is the **same between standard and reduced echelon form.** |

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| **Echelon Matrix Criteria**   1. **All non-zero rows are above all zero rows.** 2. **The leading entries of lower rows are to the right of all those in upper rows.**    1. Forms a “**step pattern**”. 3. **The entries in a column below a leading entry are zero.** | **Reduced (Row) Echelon Matrix Criteria**   1. **All criteria of a standard echelon matrix.** 2. **All leading entries equal 1.** 3. **The entries in a column above a leading entry are 0.** | **Theorem #1: Any matrix has one and only one** **reduced echelon form**. |

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| **Pivot Position** – A **position** in a given matrix that **corresponds to a “1” in reduced echelon form**. | **Pivot Position** – A column that contains a pivot position. | **Pivot** – A non-zero number in a pivot position that is used as needed to create zeros via row operations. | **Gaussian Elimination:** Same concept as row reduction. | **Non-zero row/column**: A row/column with **at least one non-zero entry**. | **Leading entry** – **Leftmost non-zero entry** **in a row**. |

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| **Row Reduction Algorithm (Gaussian Elimination)**  **Forms an Echelon Matrix** | | | | **Gauss-Jordan Elimination**  **Forms the Reduced Echelon Matrix** |
| 1. **Begin with the left most non-zero entry that is a pivot position. Move that position to the top of the matrix.** | 1. **Select a non-zero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.** | 1. **Use row operations to create zeros in all positions below the pivot.** | 1. **Cover (i.e. ignore) the rows containing the pivot position and cover (ignore) all rows above it. Apply steps 1-3 to the sub matrix that remains. Repeat the process until there are no more non-zero rows remaining.** | 1. **Create zeros above each pivot and scale each pivot to 1.** |

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| **Basic Variable: Can only exist in a single solution set equation. Correspond to a pivot in the reduced echelon matrix.** | **Parametric Description of Solution Sets**   * Free variables act as parameters * Each basic variable is represented by an equation made up of constants and/or free variables. | **Theorem 1-2: Existence and Uniqueness Theorem**  A **linear system is consistent** if and only if the rightmost column of the augmented matrix is not a pivot column – that is, if and only if an echelon form of the **augmented matrix has no row in the form**:  withnon-zero  If a linear system is consistent, then the solution set contains either:   1. **A unique solution, when there is no free variable** 2. **Infinitely many solutions when there is at least one free variable.** |
| **Free Variable:** **Can be assigned to any number** since it has no pivot. Free variable **quantity dictates the resulting solution set’s shape** (e.g., plane, line etc.). |

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| **Example: Parametric Description of a Solution Set** | |  |
| **Augmented Matrix** | **Parametric Description of the Solution Set** |

**Section 1.3 – Vector Equations**

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| **Vector** – An **ordered** list of numbers | **Column Vector** – A **matrix with only one column**. | **Vector Equality** – Two vectors of the **same size** with **all corresponding entries equal**. | **Vector Addition** – Sum obtained by **adding the corresponding entries of two vectors**. | **Scalar Multiple** –Given a number, , and a vector, , it is the vector obtained when is **multiplied by each element** in |

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| **Parallelogram Rule for Addition** – If and are in (i.e., points in the Cartesian plane), then corresponds to the **fourth point** in the **parallelogram whose vertices are** , and (the origin). | **Scalar Multiples of a Fixed Vector** – **Along a line between the vector point and the origin**. | **Zero Vector** () – A vector whose entries are all zeros. | **Linear Combination** – Given vectors and scalars , then the **linear combination vector**  is: |

**Algebraic Properties in**

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| **Commutative** | **Associative** | **Identity** | **Inverse** |
| **Distributive** | **Distributive** | **Associative** | **Identity** |

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| **The Vector Equation and  the Augmented Matrix**  A **vector equation**:  has the **same solution set as the linear system whose augmented matrix is**:  In particular, can be generated by a linear combination of if and only if there exists a solution to the linear system corresponding to the augmented matrix. | **Subset of Spanned /  Generated by**  If are in , then the set of all linear combinations of is denoted by:  It is the set of all linear combination vectors that can be written in the form: | **Geometric Description of Spans of Vectors** | |
| **(Zero Vector)** | A **point** – The origin |
| **One non-zero vector** | A **line** through the origin and |
| **Two vectors**  and that **are scalar multiples** | A **line** through the origin and |
| **Two vectors**  and **not scalar multiples** | A **plane** through the origin, , and |

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| **Members of All Spans**   * **Zero vector** () * **Scalar multiples of the original vectors** | **How to Determine if a Vector is in a Span** **Check if the system is consistent** | **Important Notation** | | | | |
| **Vector in** | | | **Not a vector** | **Set of Directions** |
|  |  |  | **It is a matrix** |  |

**Section 1.4 – The Matrix Equation**

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| **Gaussian Elimination** – Same concept as row reduction. | **Theorem #1-4:** Let be an matrix. Then the following statements are equivalent meaning they are **either all true** **or all false**.   1. **For all , the equation has a solution.** 2. **Each is a linear combination of the columns of .** 3. **The columns of span** 4. **The (coefficient) matrix has a pivot position in every row.** | **Relationship between Spans and Free Variables**   |  |  | | --- | --- | |  | **General Solution Structure** | | **(Trivial Only)** |  | | **1 Free Variable** | – **Line** through and the origin | | **2 Free Variables** | – **Plane** through , , and the origin | | **Free Variables** | – **Multidimensional shape** through and the origin | |
| **Non-zero row/column** – A row/column with at least one non-zero entry. |
| **Theorem #1-3:** If is an matrix with columns and if , then **the matrix equation**:  **has the same solution set** as the **vector equation**:  which **has the same solution set as the system of linear equations whose augmented matrix is**: |

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| **Theorem #1-5:** If is an matrix, and are vectors in , and is a scalar, then: | **Row-Vector Rule for Computing**  If the product is defined (i.e., the sizes correspond), then the entry in is the sum of products of the corresponding entries from row of and from the vector . It is formally: | **Matrix Equation Definition**  If is an matrix with columns and if , then the **product**, of and , denoted by , **is the linear combination of the columns of** **using the corresponding entries in**  **as the weights**; that is: |

**Section 1.5 – Solution Sets of Linear Systems**

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| **Homogeneous Linear System:**   * – Matrix * – Vector in * – Zero vector in   **Trivial Solution: (in )**   * **Exists for all homogeneous systems**   **Nontrivial Solution:** Any **non-zero vector solution to the equation:**   * If it exists, there are infinitely many. * Requires at least one free variable. | **Non-Homogenous System:** where is not the zero vector.   * May be inconsistent. * If its solution exists, it is in the form:   + If is , then   + – Particular solution for the specific non-homogenous system   + – Solution set for the homogenous system | **Theorem #1-6:** Suppose the equation is consistent for some given , and let be **any solution to that particular non-homogenous system**. Then the **solution set of** (**if it exists**)is **the set of all vectors of the form**:  where is **any solution of the homogenous equation** . |

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| **Solution Set Notations** | | | For to span all of , there **must be a pivot in every row of echelon matrix** of . |
| **Augmented Matrix** | **Parametric Description of the Solution Set** | **Parametric Vector Form** |

**Section 1.7 – Linear Independence**

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| **Span Review**   * **One Vector** – **At most** a **line**   + **Exception:** vector * **Two Vectors: At most** a **plane**   + **Exception:** Scalar multiples and the zero vector | **Linear Independence**  A set of vectors are **linearly independent** if:  has **only the trivial solution** | **Linear Dependence**  A set of vectors are **linearly dependent** if there exists a set of **non-zero weights** such that:  **Note:** This requires **at least one free variable**. |

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| **Linear Dependence Relation:** For vectors , it is defined as:  where are **not all zero.**  **Note:** The values of are not unique. **If one exists, an infinite number exist.** | **Procedure: Checking for Linear Independence**  **Step #1:** Create the coefficient matrix.  **Step #2:** Perform Gaussian elimination to find the echelon matrix.  **Step #3:** Check linear independence   * If **there is a pivot in every column**, the vectors are **linearly independent**. * If **there is a free variable**, the vectors are **linearly dependent**. | **Linear Independence of One Vectors**   * **Zero Vector** () – This is always **linearly dependent**. * **Any Non-Zero Vector** ( where ) then **linearly independent**. |

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| **Linear Independence of Two Vectors** – A set of two vectors is linearly dependent if **at least one** **of the vectors is a scalar multiple** of the other.   * “**At least one**” – Because of the case of the zero () vector. | For a set of vectors to be **linearly independent**, the echelon matrix made from those vectors must have no free variables.   * **There must be a pivot in every column.** | **Linear Dependence Summary:** If a set of  **vectors of -dimensions** are linearly independent, then they **span an dimensional shape**. |

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| **Theorem #1-7:** An indexed set of two or more vectors is **linearly dependent** **if and only if at least one of the vectors in is a linear combination of the others**.  In fact, if is linearly dependent and , then some (where ) is a linear combination of the preceding vectors: . | **Theorem #1-8:** If a set **contains more vectors than there are entries in each vector**, then the set is **linearly dependent**.  That is any set inis linearly dependent if:  **Proof:** More pivots than columns in an matrix so **at least one free variable**. | **Theorem #1-9:** If a set  **contains the zero vector**, then the set is **linearly dependent**.  **Proof:** If the zero vector is reordered to be , then |

**Section 1.8 – Introduction to Linear Transformations**

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| **Function** – A **rule** that **assigns to each element** in a set **exactly one element** from set . | **Transformation () from to :** A rule that **assigns to each vector**  **a vector** | **Domain of :** | **Image of :** For a given , it is the **transformed value** | **Range: Set of all images** .  **Note:** The range may be (and often is) only a subset of the codomain. |
| **Codomain of :** |

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| **Linear Transformations:** **Preserve vector addition and scalar multiplication** |  |  |  |  |

**Properties of Matrix Transformations**

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| **Requirements of Linear Transformation** | | **Zero Vector Properties** | |  |
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**Types of Matrix Transformations**

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| **Contraction**  **,**  **where** | **Dilation**  **, where** | **Projection onto an Axis**  **Matrices in the form:**  **– Projection onto -plane**  **– Projection onto the -axis** | **Shear Transformations**  **Maps line segments onto line segments. Deforms the shape of the original input.** |

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| **Reflection through the Origin**  Matrix in the Form of: | **Reflection through the -axis**  Matrix in the Form of: | **Reflection through the -axis**  Matrix in the Form of: | **Reflection through the line**  Matrix in the Form of: |

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| **Twice as Long and 1.5 Times as High**  Matrix in the Form of: | **Rotation about Origin through 90­o**  Matrix in the Form of: |  |

**Section 2.1 – Matrix Operations**

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| or – Given an matrix, it is the element in the  **row** and  **column** of | **Diagonal Entries** – In an matrix, it is the entries , etc. | **Diagonal Matrix** – A **square** matrix whose non-diagonal entries are 0.  **Example:** **Identity matrix** | **Equal Matrices** – Two matrices having **the same size** (i.e., equal number of rows and columns) and **whose corresponding entries are equal**. | **Matrix Sum** – Given two matrices of the same size, the matrix sum is equivalent to the **sum of the corresponding columns**. |

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| **Matrix Multiplication**  If is an matrix and if is an matrix composed of the columns , then the **product is an matrix** whose columns are: such that: | **Row-Product Rule**  **If the product of is defined**, then the entry in row and column of is the sum of the products of corresponding entries from row of and column of such that: | **Size of a Matrix Multiplication:** Given a matrix of size and a matrix of size , then the matrix multiplication is of size . |
| **Non-Commutativity of Matrix Multiplication**  In most cases, |
| **Cancelation Laws Do Not Hold For Matrix Multiplication**  Generally, if , it **cannot be assumed** that |
| **Zero Matrix Product**  If , you **cannot generally assume** or |

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| **Powers of a Matrix**  If is an matrix and is a positive integer, then denotes the product of copies of such that:  **Zero Power** – Given a square matrix , then **is the identity matrix**. | **Transpose of a Matrix**  Given an matrix , then transpose of is an **matrix whose columns are formed from the corresponding rows of A**. | **Transpose Multiplication Row:** The **transpose of the product of matrices** **equals** the **product of their transposes in reverse order**. |

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| **Theorem #2-1:** Let , , and be **matrices of the same size** and let and be scalars, then:   * 1. **(Commutative)**   2. **(Associative)**   3. **(Identity)** | **Theorem #2-2:** Given is the identity matrix of size and is an matrix, and that and have sizes for which the indicated sums and products exist, then   1. **(Associative)** 2. **(Left Distributive Law)** 3. **(Right Distributive Law)** 4. **, for any scalar** | **Theorem #2-3:** Let and be matrices whose sizes are appropriate for the following sums and products:   1. **, for any scalar**   **Note: The reverse order of the product.** |

**Section 2.2 – The Inverse of a Matrix**

**Section 2.3 – Characterizations of Invertible Matrices**

**Section 3.1 – Introduction to Determinants**

**Section 3.2 – Properties of Determinants**

**Section 4.1 – Vector Spaces and Subspaces**

**Section 4.2 – Null Spaces, Column Spaces, and Linear Transformations**

**Section 4.3 – Linearly Independent Sets and Bases**

**Linear Algebra Theorems**

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| **Theorem #1: Any matrix has one and only one** **reduced echelon form**. | **Theorem 1-2: Existence and Uniqueness Theorem**  A **linear system is consistent** if and only if the rightmost column of the augmented matrix is not a pivot column – that is, if and only if an echelon form of the **augmented matrix has no row in the form**:  withnon-zero  If a linear system is consistent, then the solution set contains either:   1. **A unique solution, when there is no free variable** 2. **Infinitely many solutions when there is at least one free variable.** | **Theorem #1-3:** If is an matrix with columns and if , then **the matrix equation**:  **has the same solution set** as the **vector equation**:  which in turn **has the same solution set as the system of linear equations whose augmented matrix is**: |

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| **Theorem #2-1:** Let , , and be **matrices of the same size** and let and be scalars, then:   1. **(Commutative)** 2. **(Associative)** 3. **(Identity)** | **Theorem #2-2:** Given is the identity matrix of size and is an matrix, and that and have sizes for which the indicated sums and products exist, then   1. **(Associative)** 2. **(Left Distributive Law)** 3. **(Right Distributive Law)** 4. **, for any scalar** | **Theorem #2-3:** Let and be matrices whose sizes are appropriate for the following sums and products:   1. **, for any scalar**   **Note: The reverse order of the product.** |

**Proofs for the Algebraic Properties of**

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| **Name** | **Description** | **Terms** |
| **Commutative** (Vector Addition) |  | * – Vectors * – Zero vector * – Real constants |
| **Inverse** (Vector Addition) |  |
| **Associative** (Vector Addition) |  |
| **Associative** (Scalar Multiplication) |  |
| **Distributive** **Law** (Vector Addition) |  |
| **Distributive** **Law** (Scalar Multiplication) |  |
| **Identity** (Vector Addition) |  |
| **Identity** (Scalar Multiplication) |  |

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| **Proof of the Commutative Property of Vector Addition**   1. Suppose and are **any real vector in**  in the form:   and   1. By the **definition of vector addition**:      1. By the **commutative property of real number addition**:      1. By the **definition of vector addition** and the **definition of and** :   **(QED)** | **Proof of the Inverse Property of Vector Addition**   1. Suppose is **any** **real vector in**  in the form: 2. By the **definition of scalar multiplication**: 3. By the **definition of vector addition**:      1. By the **inverse property of real number addition**:      1. By **definition of the zero vector**:   **(QED)** |

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| **Proof of the Associative Property for Vector Addition**   1. Suppose , , and are **any real vector in**  in the form:   and and   1. By the **definition of vector addition**:      1. By the **definition of vector addition**: 2. By the **associative property of real number addition**:      1. By the **definition of** **vector addition** and the **definition of and** :   **(QED)** | **Proof of the Associative Property for Scalar Multiplication**   1. Suppose and are **any real number** and is **any real vector in**  in the form: 2. By the **definition of scalar multiplication**:      1. By the **definition of vector addition**: 2. By the **associative property of real number multiplication**: 3. By the **definition of** **scalar multiplication**:   **(QED)** |

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| **Proof of the Distributive Law for Vector Addition**   1. Suppose is **any real number** and and are **any real vector in**  in the form:   and   1. By the **definition of vector addition**:      1. By the **definition of scalar multiplication**:      1. By the **distributive law over real number addition**: 2. By the **defintition of vector addition**: 3. By the **definition of scalar multiplication**,and the **definition of and** :   **(QED)** | **Proof of the Distributive Law for Scalar Multiplication**   1. Suppose and are **any real number** and is **any real vector in**  in the form:      1. By the **definition of scalar multiplication**:      1. By the **distributive law over real number addition**:      1. By the **definition of vector addition**: 2. By the **definition of scalar multiplication** and the **definition of** :   **(QED)** |

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| **Proof of the Identity Property for Vector Addition**   1. Suppose is the **zero vector of length**  and is **any real vector in**  in the form: 2. By the **definition of vector addition**:      1. By the **identity property of real number addition**:      1. By the **definition of** :   **(QED)** | **Proof of the Identity Property for Scalar Multiplication**   1. Suppose is **any real vector in**  in the form: 2. By the **definition of scalar multiplication**:      1. By the **identity property of real number multiplication**:      1. By the **definition of** :   **(QED)** |