AMS 230: Numerical Optimization

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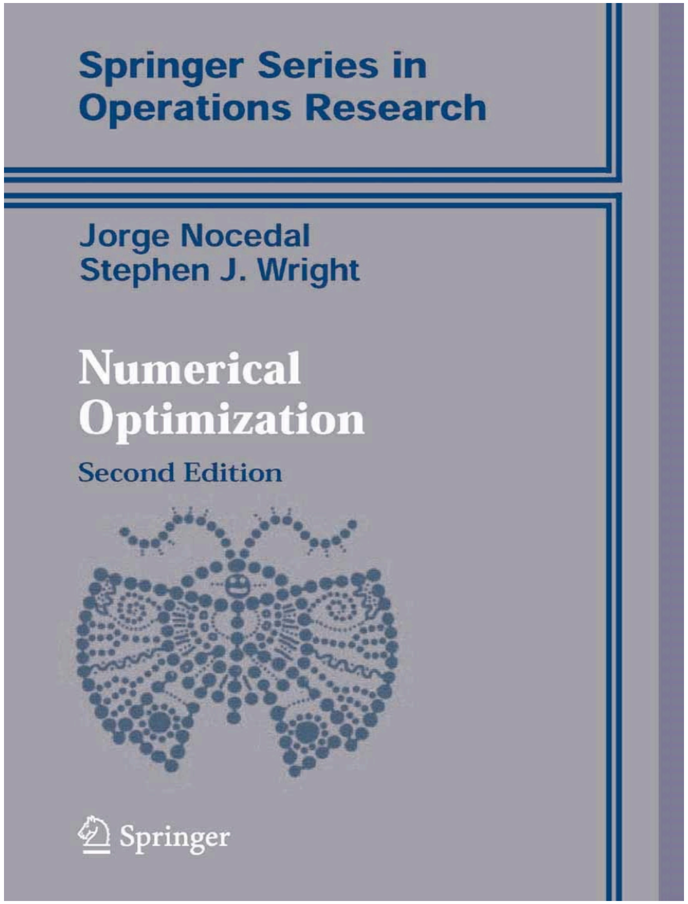
AMS 230: Numerical Optimization

This graduate course provides an introduction to a variety of widely used numerical algorithms for solving optimization problems. The course focuses on the derivation of numerical methods, mathematical performance analysis, and practical implementations of the computational algorithms for continuous optimization problems.

- **Instructor:** Qi Gong, qigong@soe.ucsc.edu), Office: BE 361A
- **Lectures:** Monday, Wednesday and Friday, 10:40am - 11:45am
- **Webcast:** https://webcast.ucsc.edu (password: UCSCAMS230)
- **Office Hours:** Wednesday 1:00pm – 3:00pm, BE 361A
- **Grading:** Homework 100%

Textbook and References

- **Textbook:** **"Numerical Optimization", J. Nocedal and S. Wright, Springer, 2nd edition, 2006**



Textbook and References

- **Textbook:** **"Numerical Optimization", J. Nocedal and S. Wright, Springer, 2nd edition, 2006**

- **References:**

  — "Nonlinear Programming" by Dimitri Bertsekas, Athena Scientific, 2nd edition, 1999

  — "Convex Optimization" by Stephen Boyd and Lieven Vandenberghe, Cambridge University Press, 2004

  — "Numerical Optimization: Theoretical and Practical Aspects" by J. Frederic Bonnans, Jean Charles Gilbert, Claude Lemarechal, Claudia A. Sagastizbal, Springer, 2006

Examples of Optimization Problems

A transportation (resource allocation) problem (Chapter 1 of Nocedal and Wright)

A company has two factories, F_1 and F_2, and a dozen retail outlets, R_1, R_2, …, R_{12}.

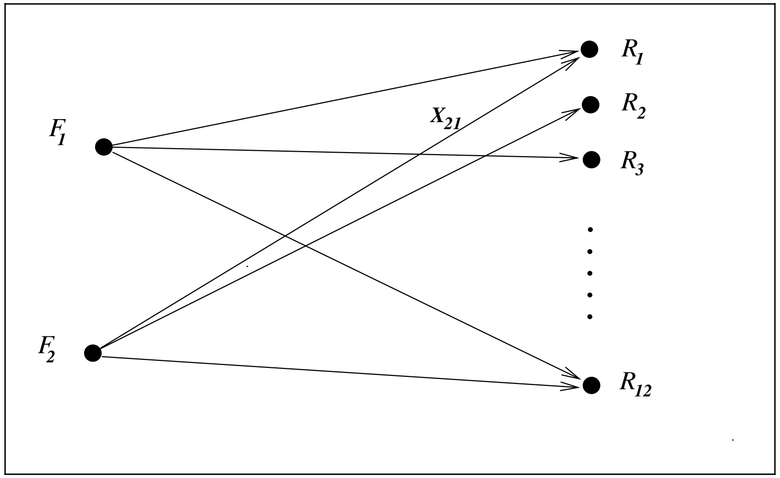
Denote $x_{ij},\ i = 1, 2,\ j = 1, \dots, 12,$ to be number of product shipped from factory F_i to retail outlet R_j.

Each factory produces a_i, i=1,2, unit of a certain product each week.

Each retail outlet has a weekly demand of b_j, j=1,2,…,12, unit of the product.

The cost of shipping one unit of product from factory F_i to retail outlet R_j is c_{ij}.

Question: determine how much of the product to ship from each factory to each outlet to satisfy all the requirements and minimize cost.



$$
\begin{aligned}
&\min \sum_{ij} c_{ij} x_{ij} \\
&\text{subject to} \sum_{j=1}^{12} x_{ij} \le a_i, \qquad i = 1, 2, \\
&\qquad\qquad \sum_{i=1}^{2} x_{ij} \ge b_j, \qquad j = 1, \dots, 12, \\
&\qquad\qquad x_{ij} \ge 0, \qquad i = 1, 2, \quad j = 1, \dots, 12.
\end{aligned}
$$

Examples of Optimization Problems

Markowitz' Portfolio selection problem:

An investor has a certain amount of money to be invested in n number of different securities (stocks, bonds, etc.) with random returns.

Let x_i , i =1, 2, … n, be the proportion of the total funds invested in the ith security.

Suppose the expected return of a portfolio, $x = (x_1, \ldots, x_n)$, is

$$E[x] = x_1\mu_1 + \ldots + x_n\mu_n = \mu^T x,$$

and the variance is $Var[x] = x^T Q x$, where μ and Q are known.

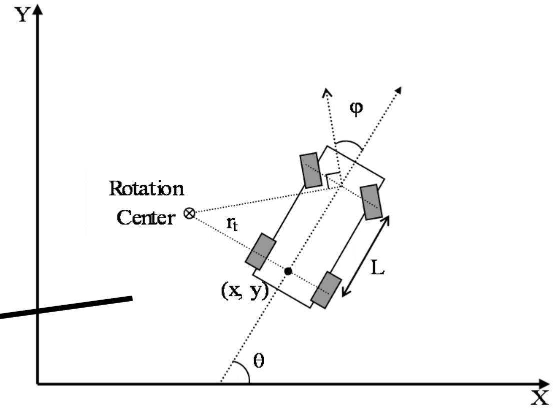
Question: among all possible portfolios that have at least a certain expected return, find the one with the minimum variance.

$$\begin{array}{ll} \text{Min} & x^T Q x \\ \text{Subject to} & \displaystyle\sum_{i=1}^{n} x_i = 1 \\ & \mu^T x \geq b \\ & x_i \geq 0, i = 1, 2, \ldots, n \end{array}$$

Examples of Optimization Problems

Path planning of autonomous vehicles:

Consider an autonomous vehicle moving through a set of obstacles.

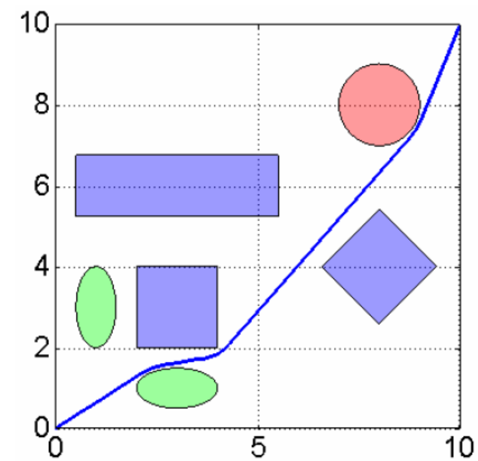


Dynamic of the vehicle is modeled as

$$\begin{aligned}
\frac{dx}{dt} &= v(t)\cos(\theta(t)) \\
\frac{dy}{dt} &= v(t)\sin(\theta(t)) \\
\frac{d\theta}{dt} &= \frac{v(t)}{L}\tan(\phi(t))
\end{aligned}$$

Forward velocity v(t) and steering angle ϕ(t) are constrained as

$$\begin{aligned}
v_{min} &\leq v(t) \leq v_{max} \\
\phi_{min} &\leq \phi(t) \leq \phi_{max}.
\end{aligned}$$



Question: find forward velocity function v(t) and steering angle function $\phi(t)$ to steer the vehicle from a starting point (x_0,y_0) to a final point (x_f,y_f).

Mathematical Formulation

Mathematically, optimization is the minimization or maximization of a function subject to constraints on its variables. An optimization problem is essentially formulated based on

1. Decision variables, denoted by x;
2. Objective function, f(x), to be optimized; and
3. Constraints that decision variables must satisfy.

A general optimization problem formulation

$$\min_{x \in \mathrm{R}^n} f(x)$$

$$\text{subject to} \quad \begin{aligned} c_i(x) &= 0, \quad i \in \mathcal{E}, \\ c_i(x) &\geq 0, \quad i \in \mathcal{I}. \end{aligned}$$

- ❑ This formulation excludes optimization problems where decision variables are functions such as vehicle path planning problems. Optimization over functions is a subject of "optimal control" (the focus of AMS 232).
- ❑ We consider only minimization problems, since maximizing an objective function can be easily converted into a minimization problem.

Classification of Optimization Problems

Continuous versus Discrete Optimization

Continuous Optimization: decision variables are continuous (can take any real value)
Discrete Optimization: decision variables are discrete, for example, integers.

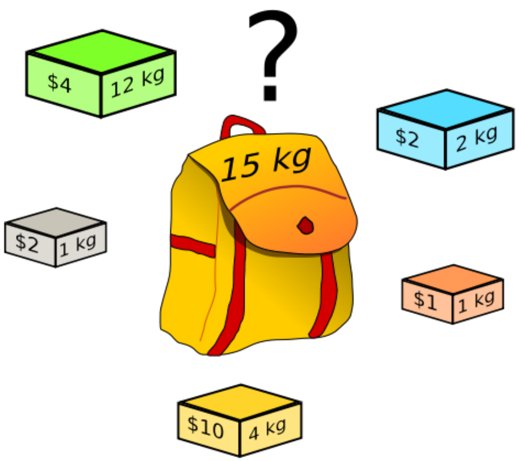
Example of discrete optimization: The Knapsack Problem

Given n number of items each with a weight, w, and a value, v.

Question: Determine the number of each item to include in a collocation so that:

1. The total weight is no more than a given limit;
2. The total value is as large as possible

$$
\begin{array}{ll}
\text{Max} & \sum_{i=1}^{n} v_i x_i \\
\text{Subject to} & \sum_{i=1}^{n} w_i x_i \leq W \\
 & x_i \in \{1, 2, \ldots, m, \ldots\}
\end{array}
$$



From https://en.wikipedia.org/wiki/Knapsack_problem

Classification of Optimization Problems

Deterministic versus Stochastic Optimization

Stochastic optimization: model involves uncertain (stochastic) quantities.

Example of stochastic optimization: An Inventory Model

Suppose a company needs to decide an order of x quantity of certain product to satisfy demand d.

1. The cost of ordering is $c > 0$ per unit.
2. If the demand d is bigger than x, a back order penalty of $b > c$ per unit incurred.
3. If the demand d is smaller than x, a holding cost of $h > 0$ per unit incurred.

The total cost is then $f(x,d) = cx + b[d-x]_+ + h[x-d]_+$
where $[y]_+$ equals y if y is non-negative, and equal 0 otherwise.

Question: Minimize the total cost f(x,d) with $\boxed{\text{an uncertain future demand, d (stochastic)}}$

Classification of Optimization Problems

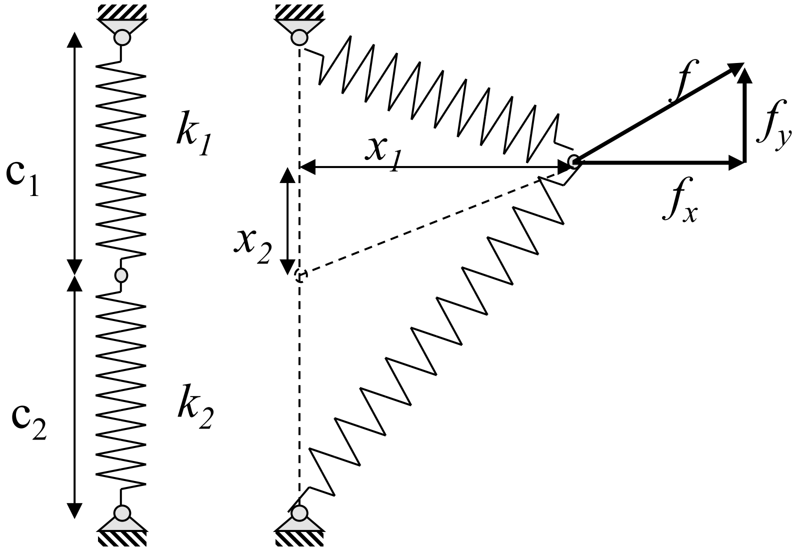
Unconstrained versus constrained Optimization

Unconstrained optimization: there is no requirement on the decision variable x

Example of unconstrained optimization: displacement of loaded structure

k1, k2: spring constants
C1, C2: equilibrium without external force

Question: find the equilibrium under external force f.



Potential energy:

$$
\begin{aligned}
P \;=\; & \frac{1}{2}k_1\left(\sqrt{x_1^2+(c_1-x_2)^2}-c_1\right)^2 \\
& +\frac{1}{2}k_2\left(\sqrt{x_1^2+(c_2+x_2)^2}-c_2\right)^2 \\
& -f_x x_1 - f_y x_2
\end{aligned}
$$

Equilibrium (x_1,x_2) minimizes the potential energy P.

Check
https://neos-guide.org/content/optimization-taxonomy
for more detailed classifications on optimization problems

This course focuses on numerical algorithms for continuous and deterministic optimization problems.

Nothing whatsoever takes place in the universe in which some form of optimization does not appear.

— Leonhard Euler

Numerical Optimization Algorithms

All practical optimization problems need to be solved numerically.

Goal: designing numerical optimization algorithms that are

- **Robust (work well on a wide variety of problems)**
- **Efficient (require little computational time or storage)**
- **Accurate**

The field of numerical optimization is a "fascinating blend of theory and computation, heuristics and rigor." -- Roger Fletcher

There is no universal algorithm that works for all problem!

Here is a list of algorithms for different types of optimization problems:
https://neos-guide.org/algorithms

Here is a list of some optimization solvers
http://plato.asu.edu/sub/nlores.html#general

Tentative Schedule

- Lecture 1-3: Introduction to numerical optimization and mathematical preliminaries. (Chapter 1 & 2 of Nocedal and Wright) Introduction: classification of optimization problems, application examples, and basic numerical strategies for unconstrained optimization. Mathematical preliminaries include: necessary/sufficient optimality conditions, convex functions/sets, sequence, rate of convergence, and descent directions.

- Lecture 4-8: Line search methods (Chapter 3 of Nocedal and Wright). Wolfe conditions and step-length selection algorithms; convergence of line search methods, steepest descent method, Newton's method, and Newton's method with Hessian modification.

- Lecture 9-12: Linear conjugate gradient method and its convergence properties; conjugate gradient method for nonlinear problems (Chapter 5 of Nocedal and Wright).

Tentative Schedule

- Lecture 13-16: Trust-region methods for unconstrained optimization (Chapter 4 of Nocedal and Wright).
- Lecture 17-20: Quasi-Newton methods: BFGS method, limited memory BFGS, symmetric-rank-1 method, convergence analysis of quasi-Newton methods (Chapter 6 of Nocedal and Wright).
- Lecture 21-23: Least square problems (Chapter 10 of Nocedal and Wright), and numerical algorithms for nonlinear equations (Chapter 11 of Nocedal and Wright).
- Lecture 24-26: Fundamental theory for constrained optimization. Constraint qualification; Karush-Kuhn-Tucker first-order optimality conditions; Lagrange multipliers and sensitivity (Chapter 12 of Nocedal and Wright).
- Lecture 27-30: Selected topics on constrained nonlinear programming, e.g., penalty methods, augmented Lagrangian methods, sequential quadratic programming.