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Assignment: CMPS218 Take Home Final Exam

Other Student Discussions: These problems were not discussed with any other students.

Problem Assessments: All problems are complete to the best of my understanding.

Chapter #4, Exercise #14

You are given 12 balls and the three-outcome balance of exercise 4.1; this time, two of the balls are odd; each odd ball may be heavy or light, and we don't know which. We want to identify the odd balls and in which direction they are odd.

(a) ***Estimate* how many weights are required by the optimal strategy. And what if there are three odd balls?**

From a set of 12 balls, there are $\binom{12}{2}$ combinations of two "odd" balls. Similarly, each "odd" ball can be either light or heavy. Therefore, the number of unique solutions is:

$$2 \cdot 2 \cdot \binom{12}{2} = 4 \cdot 66 = 264,$$

all of which are equally likely. Ideally, the three-outcome balance roughly reduces the set of possible solutions by two-thirds with each measurement. A best case expected number of weights for two odd balls is:

$$\log_3 264 \approx 5.075.$$

Since 5.075 is much closer to 5 than 6, there is still slack for imbalance in the probabilities of the respective outcomes of a ball weighing. Therefore, I estimate $\boxed{6}$ weights are required for two odd balls.

When there are three odd balls, then there are $\binom{12}{3}$ combinations of "odd" balls. Again, since each odd ball is either light or heavy, the number of unique (equally likely) solutions is:

$$2^3 \cdot \binom{12}{3} = 8 \cdot 220 = 1760.$$

As such, the minimum expected number of weightings for three odd balls is:

$$\log_3 1760 \approx 6.802.$$

Unfortunately, 6.802 is close to 7 leaving less slack for any imbalance in the outcome probabilities. For that reason, I estimate the expected number of weights for three balls is $\boxed{\text{7 or 8}}$.

(b) **How do your answers change if it is known that all the regular balls weigh 100g, that light balls weight 99g, and heavy ones weigh 110g?**

Chapter #5, Exercise #24

Write a short essay describing how to play the game of twenty questions optimally. [In twenty questions, one player thinks of an object, and the other player has to guess the object using as few binary question as possible, preferably fewer than twenty.]

The naïve approach to answering this question is as follows:

There is a finite set of candidates, C, from which the first player will select the object. With each question, reduce the size of this set of possible objects by half.

This strategy overlooks one key detail – each object in C does not have equal likelihood of selection. What is more, the likelihood of selection for each object varies based on the specific person doing the selection! For example, I am more likely to select my cat as the object than you are (whoever is reading this).

A Better Strategy:

For each $c \in C$, the "other" player should define a probability which represents the likelihood that the first player would select c. As in the naïve strategy, the set of possible candidates shrinks with each question. However, rather than reducing the size of the candidate pool in half each time, the question posed should partition that pool such that the probability of all candidates that remain valid if the answer is "no" has **equal probability** to the set of candidates that remain if the answer is "yes." This approach maximizes the expected information content of each answer.

Once one remaining possible object, c, has greater probability than all other remaining objects combined, the "other" player should ask "Is your object c?"

Chapter #15, Exercise #5

In a magic trick, there are three participants: the magician, an assistant, and a volunteer. The assistant, who claims to have paranormal abilities, is in a soundproof room. The magician gives the volunteer six blank cards, five white and one blue. The volunteer writes a different integer from 1 to 100 on each card, as the magician is watching. The volunteer keeps the blue card. The magician arranges the white cards in some order and passes them to the assistant. The assistant then announces the number on the blue card.

How does this trick work?

For the sake of making this problem meaningful, I am going to *assume* the assistant does not actually have paranormal abilities.

Consider a set of five unique, arbitrary integers. We will represent these integers as a, b, c, d, and e. Assign these five unique integers to these letters satisfying the relationship $a \leq b \leq c \leq d \leq e$. At this point, forget what the original numbers were and only concentrate on the letters since it is only the letters that matter.

It is elementary to see that there are ${}_5P_5 = 5! = 120$ permutations of five unique letters. Assign to each permutation a unique number from 1 to 120. This represents a unique decodable encoding of any number between 1 and 120.

In this trick, the five unique integers swritten on the white cards are mapped to the letters a through e as described above. The magician knows the number written on the blue card and encodes that number based on how he orders (permutes) the five white cards (letters) before passing them to the assistant. The assistant then decodes this permutation to get the volunteer's number on the blue card completing the trick.

Chapter #15, Exercise #6
How does *this* trick work?

'Here's an ordinary pack of cards shuffled into random order. Please choose five cards from the pack, any you wish. Don't let me see their faces. No, don't give them to me: pass them to my assistant Esmerelda. She can look at them.

'Now, Esmerelda, show me four of the cards. Hmm...nine of spades, six of clubs, four of hearts, ten of diamonds. The hidden card must be the queen of spades.

This trick can be performed as described above for a pack of 52 cards. Use information theory to give an upper bound on the number of cards for which the trick can be performed.

Consider a collection of n unique objects. It is trivial to bijectively map these objects to the integers $\{1, \ldots, n\}$. If five integers from the set are selected of which four are revealed, then the previous problem's trick would allow us to communicate the fifth object as long as n is less than or equal to $4! + 4 = 28$. Given a standard deck contains $n = 52$ cards, the number of permutations is insufficient to uniquely encode the hidden card. Therefore, we need a more creative encoding....

The key to this problem is that Esmerelda *chose* which of the five cards will remain hidden. If she chose that card randomly, we are back to 24 permutations. However, Esmerelda was more cunning and chose a *specific* card to remain hidden. A scheme could be developed that uses this selection process to increase the upper bound on the number of encodable objects fivefold i.e., once for each card the volunteer chose. When we combine that with the trick from the previous problem, we see that the maximum number of objects that can be uniquely encoded is now $5 \cdot 4! = 120$. Therefore, combining the maximum size of the encoded objects pool with the four cards revealed, we see the upper bound for the maximum deck size is $4 + 120 = \boxed{124}$.

The problem does not require that we prove the upper bound is the true bound. However, after completing the problem, I did research online and found that this is in fact the true bound.