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Assignment: CMPS218 Take Home Final Exam

Other Student Discussions: These problems were not discussed with any other students.

Problem Assessments: All problems are complete to the best of my understanding.

Chapter #4, Exercise #14

You are given 12 balls and the three-outcome balance of exercise 4.1; this time, two of the balls are odd; each odd ball may be heavy or light, and we don't know which. We want to identify the odd balls and in which direction they are odd.

(a) ***Estimate*** **how many weights are required by the optimal strategy. And what if there are three odd balls?**

When there was a single odd ball out of twelve, the size of the hypothesis class was $2 * \binom{12}{1} = 24$. To determine the number of weights required to determine the actual solution, we need to consider how each hypothesis would be represented on the balance. The six possibilities are shown in Table 1. As enumerated in Figure 4.2 of Mackay's text, we see that balance output has *roughly* equivalent probability making the number of required weights $\lceil \log_3 24 \rceil = 3$.

Table 1: Three-outcome balance position for a single odd ball that is light or heavy ball

Case #	Odd Ball		Balance
	Weight	Location	Position
1	Light	Left	¯|_
2	Heavy	Right	¯|_
3	Light	Right	_|¯
4	Heavy	Left	_|¯
5	Light	Neither	−|−
6	Heavy	Neither	−|−

The situation grows significantly more complex when there are three balls. For example, the size of the hypothesis class grows by more than an order of magnitude from 24 to $2 \cdot 2 \cdot \binom{12}{2} = 264$. In addition, interpreting the results of the balance also becomes more complex as shown in Table 2. Note first that the possible cases has increased six-fold from 6 to 36. Furthermore, as part of the problem, we are not provided any information about the relative or absolute weights of the balls. Therefore, we do not know if for example a heavy ball cancels out a light ball. Such additional uncertainty necessitates that we collect more require more weights to achieve certainty.

Table 2: Three-outcome balance position for two odd balls that are light and/or heavy Ball

Case #	Odd Ball #1		Odd Ball #2		Balance Position
	Weight	Location	Weight	Location	
1	Light	Left	Light	Left	$\overline{}\vert\underline{}$
2	Light	Left	Light	Neither	$\overline{}\vert\underline{}$
3	Light	Left	Heavy	Neither	$\overline{}\vert\underline{}$
4	Light	Neither	Light	Left	$\overline{}\vert\underline{}$
5	Heavy	Neither	Light	Left	$\overline{}\vert\underline{}$
6	Heavy	Right	Heavy	Right	$\overline{}\vert\underline{}$
7	Heavy	Right	Light	Neither	$\overline{}\vert\underline{}$
8	Heavy	Right	Heavy	Neither	$\overline{}\vert\underline{}$
9	Light	Neither	Heavy	Right	$\overline{}\vert\underline{}$
10	Heavy	Neither	Heavy	Right	$\overline{}\vert\underline{}$
11	Light	Right	Heavy	Left	$\overline{}\vert\underline{}$
12	Heavy	Left	Light	Left	$\overline{}\vert\underline{}$
13	Light	Right	Light	Right	$\underline{}\vert\overline{}$
14	Light	Right	Light	Neither	$\underline{}\vert\overline{}$
15	Light	Right	Heavy	Neither	$\underline{}\vert\overline{}$
16	Light	Neither	Light	Right	$\underline{}\vert\overline{}$
17	Heavy	Neither	Light	Right	$\underline{}\vert\overline{}$
18	Heavy	Left	Heavy	Left	$\underline{}\vert\overline{}$
19	Heavy	Left	Light	Neither	$\underline{}\vert\overline{}$
20	Heavy	Left	Heavy	Neither	$\underline{}\vert\overline{}$
21	Light	Neither	Heavy	Left	$\underline{}\vert\overline{}$
22	Heavy	Neither	Heavy	Left	$\underline{}\vert\overline{}$
23	Light	Left	Heavy	Right	$\underline{}\vert\overline{}$
34	Heavy	Right	Light	Right	$\underline{}\vert\overline{}$
25	Light	Right	Light	Left	$-\vert-$
26	Light	Left	Light	Right	$-\vert-$
27	Heavy	Right	Heavy	Left	$-\vert-$
28	Heavy	Left	Heavy	Right	$-\vert-$
29	Light	Neither	Light	Neither	$-\vert-$
30	Light	Neither	Heavy	Neither	$-\vert-$
31	Heavy	Neither	Light	Neither	$-\vert-$
32	Heavy	Neither	Heavy	Neither	$-\vert-$
33	Light	Left	Heavy	Left	**Unknown**
34	Heavy	Left	Right	Left	**Unknown**
35	Light	Right	Heavy	Right	**Unknown**
36	Heavy	Right	Light	Right	**Unknown**

When there are three odd balls, the uncertainty increases further. First, the hypothesis class grows from 264 to $2 \cdot 2 \cdot 2 \cdot \binom{12}{3} = 1,760$. Next, the table of cases (not included) grows from 36 to 216. Furthermore, the cases where the balance output would correspond to “Unknown” grows from 4 to 24 as shown in Table 3.

Table 3: Three odd ball cases where the balance position is unknown if ball weights are not provided where an "X" corresponds to "do not care"

Case #	Odd Ball #A		Odd Ball #B		Odd Ball #C	
	Weight	Location	Weight	Location	Weight	Location
1–6	Light	Left	Heavy	Left	X	X
7–12	Heavy	Left	Light	Left	X	X
13–18	Light	Right	Heavy	Right	X	X
19–24	Heavy	Right	Light	Right	X	X

Now consider when there are two odd balls. There are substantially more possible outcomes. Likewise, since the relative weights of the light and heavy balls is unknown, there is also some ambiguity con

From a set of 12 balls, there are $\binom{12}{2}$ combinations of two "odd" balls. Similarly, each "odd" ball can be either light or heavy. Therefore, the number of unique solutions is:

$$2 \cdot 2 \cdot \binom{12}{2} = 4 \cdot 66 = 264,$$

all of which are equally likely. Ideally, the three-outcome balance roughly reduces the set of possible solutions by two-thirds with each measurement. A best case expected number of weights for two odd balls is:

$$\log_3 264 \approx 5.075.$$

Since 5.075 is much closer to 5 than 6, there is still slack for imbalance in the probabilities of the respective outcomes of a ball weighing. Therefore, I estimate $\boxed{6}$ weights are required for two odd balls.

When there are three odd balls, then there are $\binom{12}{3}$ combinations of "odd" balls. Again, since each odd ball is either light or heavy, the number of unique (equally likely) solutions is:

$$2^3 \cdot \binom{12}{3} = 8 \cdot 220 = 1760.$$

As such, the minimum expected number of weightings for three odd balls is:

$$\log_3 1760 \approx 6.802.$$

Unfortunately, 6.802 is close to 7 leaving less slack for any imbalance in the outcome probabilities. For that reason, I estimate the expected number of weights for three balls is $\boxed{\text{7 or 8}}$.

(b) How do your answers change if it is known that all the regular balls weigh 100g, that light balls weight 99g, and heavy ones weigh 110g?

As mentioned previously, in the multi-oddball case, uncertainty about the relative weights of the balls necessitates that additional measurements be made to clarify that uncertainty. Now that the relative ball weights are known, the Cases #33-34 and Cases #35-36 in Table 2 are assigned to “_|¯” and “¯|_” respectively.

Chapter #5, Exercise #24

Write a short essay describing how to play the game of twenty questions optimally. [In twenty questions, one player thinks of an object, and the other player has to guess the object using as few binary question as possible, preferably fewer than twenty.]

The naïve approach to answering this question is as follows:

> *There is a finite set of candidates, C, from which the first player will select the object. With each question, reduce the size of this set of possible objects by half.*

This strategy overlooks one key detail – each object in C does not have equal likelihood of selection. What is more, the likelihood of selection for each object varies based on the specific person doing the selection! For example, I am more likely to select my cat as the object than you are (whoever is reading this).

A Better Strategy:

For each $c \in C$, the "other" player should define a probability which represents the likelihood that the first player would select c. As in the naïve strategy, the set of possible candidates shrinks with each question. However, rather than reducing the size of the candidate pool in half each time, the question posed should partition that pool such that the probability of all candidates that remain valid if the answer is "no" has **equal probability** to the set of candidates that remain if the answer is "yes." This approach maximizes the expected information content of each answer.

Once one remaining possible object, c, has greater probability than all other remaining objects combined, the "other" player should ask "Is your object c?"

Chapter #15, Exercise #5

In a magic trick, there are three participants: the magician, an assistant, and a volunteer. The assistant, who claims to have paranormal abilities, is in a soundproof room. The magician gives the volunteer six blank cards, five white and one blue. The volunteer writes a different integer from 1 to 100 on each card, as the magician is watching. The volunteer keeps the blue card. The magician arranges the white cards in some order and passes them to the assistant. The assistant then announces the number on the blue card.

How does this trick work?

For the sake of making this problem meaningful, I am going to *assume* the assistant does not actually have paranormal abilities.

Consider a set of five unique, arbitrary integers. We will represent these integers as a, b, c, d, and e. Assign these five unique integers to these letters satisfying the relationship $a \leq b \leq c \leq d \leq e$. At this point, forget what the original numbers were and only concentrate on the letters since it is only the letters that matter. It is elementary to see that there are ${}_5P_5 = 5! = 120$ permutations of five unique letters. Assign to each permutation a unique number from 1 to 120. This represents a unique decodable encoding of any number between 1 and 120.

In this trick, the five unique integers written on the white cards are bijectively mapped to the letters a through e as described above. The magician saw the number written on the blue card and encodes that number based on how he orders (permutes) the five white cards (letters) before passing them to the assistant. The assistant then decodes this permutation to get the volunteer's number on the blue card completing the trick – no paranormal abilities needed.

Chapter #15, Exercise #6
How does *this* trick work?

> *'Here's an ordinary pack of cards shuffled into random order. Please choose five cards from the pack, any you wish. Don't let me see their faces. No, don't give them to me: pass them to my assistant Esmerelda. She can look at them.*
>
> *'Now, Esmerelda, show me four of the cards. Hmm...nine of spades, six of clubs, four of hearts, ten of diamonds. The hidden card must be the queen of spades.*

This trick can be performed as described above for a pack of 52 cards. Use information theory to give an upper bound on the number of cards for which the trick can be performed.

Assumption:

To make this problem more meaningful, I am going to assume that when Esmerelda "shows" the four cards to the magician, she is not using any sleight of hand trickery to relay additional information to the magician (e.g., orient the cards in her hand in a certain way, show the face or backside first, etc.). Essentially, I consider that the magician is only provided numbers from a black box.

Answer:

Consider a collection of n unique objects. It is trivial to bijectively map these objects to the integers $\{1, \ldots, n\}$. If five integers from the set are selected of which four are revealed, then the previous problem's trick allows us to communicate the fifth object as long as n is less than or equal to $4! + 4 = 28$. Given a standard deck contains $n = 52$ cards, the number of permutations is insufficient to uniquely encode the hidden card. Therefore, we need a more creative encoding....

The key to this problem is that Esmerelda *chose* which of the five cards remained hidden. If she chose that card randomly, we are back to 24 permutations. However, Esmerelda was more cunning and chose a *specific* card to remain hidden. A scheme could be developed that uses this selection process to increase the upper bound on the number of encodable objects fivefold i.e., once for each card she could have chosen. When we combine that knowledge with the technique from the last question (exercise 15.5), we see that the maximum number of objects that can be uniquely encoded is $5 \cdot 4! = 120$. Therefore, combining the maximum size of the encodable objects pool with the four cards Esmerelda revealed, we see the upper bound for the maximum deck size is $4 + 120 = \boxed{124}$.

The problem does not require that we prove the tightness of this upper bound. However, after completing the problem, I did research online and found that this is in fact the true bound. What is more, the technique described above is not how this trick (known as the "Fitch Cheney Card Trick") is traditionally performed; that approach to the trick has a maximum deck size of 52.