Follow the leader with Dropout perturbations - Additive versus multiplicative noise

Manfred K. Warmuth
UCSC

Simon's Institute
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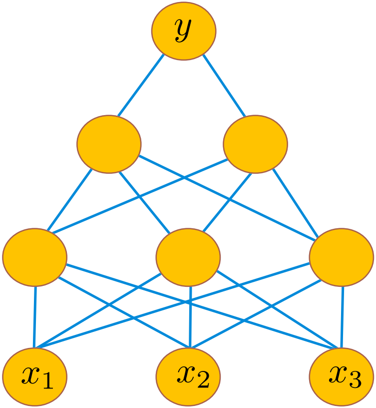
Joint work with Tim Van Erven and Wojciech Kotłowski
Universiteit Leiden

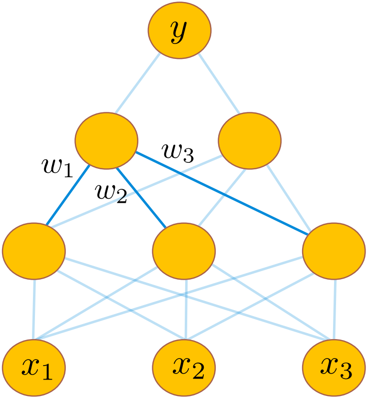
Major insights from [Devroye, Lugosi, Neu 2013]

Outline

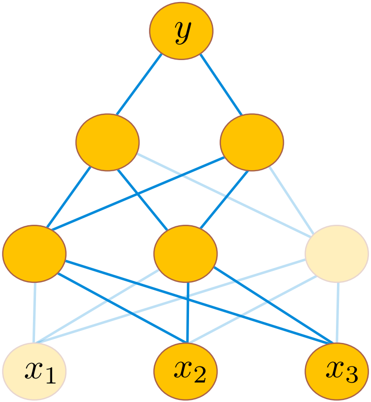
Feed forward neural net



Weights parameters - sigmoids at internal nodes



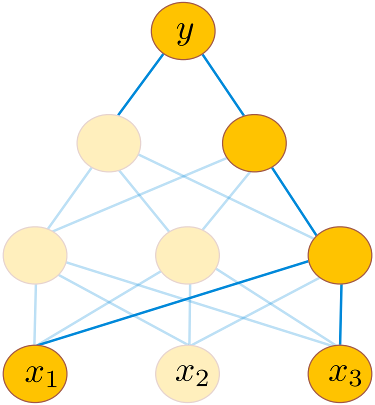
Dropout training



- Stochastic gradient descent
- Randomly remove every hidden/input node with prob. $\frac{1}{2}$ before each gradient descent update

[Hinton et al. 2012]

Dropout training



- Very successful in image recognition & speech recognition
- Why does it work?

  [Wagner, Wang, Liang 2013]
  [Helmbold, Long 2014]

What are we doing?

Prove bounds for dropout

- single neuron
- linear loss

Outline

Online learning with expert

	E_1	E_2	E_3	$\ldots$	E_n	$prediction$	label	loss
day 1	0	1	0	$\ldots$	0	0	1	1

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- Algorithm maintains probability vector $\boldsymbol{w}$:
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- Algorithm maintains probability vector $\boldsymbol{w}$:
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- Loss linear because label $y \in \{0,1\}$
- $\underbrace{|\overbrace{\boldsymbol{w} \cdot \boldsymbol{x}}^{\widehat{y}} - y|}_{\text{loss of alg.}} = \sum_i w_i \underbrace{|x_i - y|}_{\text{loss of expert } i}$

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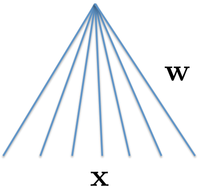
On-line learning

Predicting with expert advice

$\hat{y} = \mathbf{w} \cdot \mathbf{x} \qquad \text{loss } |\hat{y} - y|$



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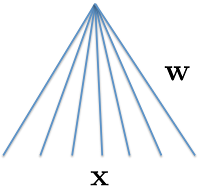

trial t

- get advice vector $\boldsymbol{x}_t$
- predict $\widehat{y}_t = \boldsymbol{w}_t \cdot \boldsymbol{x}_t$
- get label y_t
- exp. losses $|x_{t,i} - y_t|$
- alg. loss $|\widehat{y}_t - y_t|$
- update $\boldsymbol{w}_t \to \boldsymbol{w}_{t+1}$

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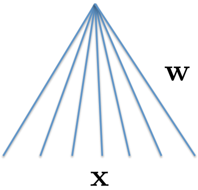

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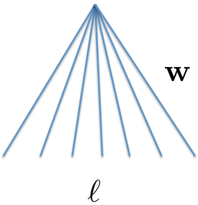
Hedge setting

loss $\mathbf{w} \cdot \ell$



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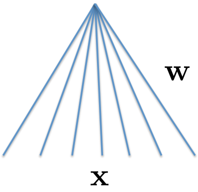

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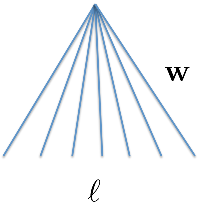

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weights are implicit

Only works for linear loss

How do we measure performance

Worst-case **regret**

$$\underbrace{\sum_{t=1}^{T} \boldsymbol{w}_t \cdot \boldsymbol{\ell}_t}_{\text{total expected loss of alg}} \quad - \quad \underbrace{\inf_i \ell_{\leq T,i}}_{\text{loss } \ell^* \text{ of best expert}}$$

Should be logarithmic in # of experts n

Outline

Main algorithms

	E_1	E_2	E_3	E_4	E_5
	0	1	0	0	1
	1	1	0	1	1
day $t-1$	0	0	1	1	1
$\ell_{\leq t-1,i}$					

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Hedge(η)	$w_i = \frac{e^{-\eta \ell_{\leq t-1,i}}}{Z}$	Weighted Majority algorithm for pred. with Expert Advice Soft min

Dropout

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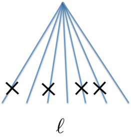
day $t-1$	0	0	$\cancel{1}$	$\cancel{1}$	1
$\widehat{\ell}_{\leq t-1,i}$	1	1	0	1	2

$\widehat{\ell}_{t,i} = \beta_{t,i}\ell_{t,i},$ where $\beta_{t,i}$ iid Bernoilli



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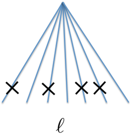

FL on dropout $\quad \widehat{i}_t = \operatorname*{argmin}_i \widehat{\ell}_{\leq t-1,i}$

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  - also no tuning, many other advantages

Our path to dropout

- Loss vectors $\boldsymbol{\ell}_t \quad \longrightarrow \quad$ loss matrices $\boldsymbol{L}_t$
- Prob. vectors $\boldsymbol{w}_t \quad \longrightarrow \quad$ density matrices $\mathbf{W}_t$
- Hedge $w_{t,i} = \frac{e^{-\eta \ell_{\leq t-1,i}}}{Z} \quad \longrightarrow \quad$ Matrix Hedge

$$\mathbf{W}_t = \frac{\mathbf{exp}\left(-\eta \boldsymbol{L}_{\leq t-1}\right)}{Z'}$$

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- ~~Follow the skipping leader~~ has linear regret [Lugosi,Neu2014]

What regularization?

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Hedge(η)	relative entropy
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FL on dropout	tricky

Feed forward NN	[Wagner, Wang, Liang 2013]
Logistic regression	[Helmbold, Long 2014]
Linear loss case	[ALST 2014]

Outline

Simple algorithms

Any deterministic alg. (such as FL) has huge regret

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- Loss of alg.: T loss of best: $\leq \frac{T}{n}$

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Recall optimum regret: $\sqrt{L^* \ln n} + \ln n$

FL with random ties

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FL with random ties

- Give every expert one unit of loss
  - iterate $L^* + 1$ times
- Loss per sweep $\frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{2} + 1 \approx \ln n$
- Loss of alg.: $(L^* + 1) \ln n$ loss of best: L^*

  regret: $L^* \ln n$

Our analysis of dropout

Unit rule

- Adversary forces more regret by splitting loss vectors into units

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

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Swapping rule

											$\ell_{\leq T,i}$
E_1	1	1	1	1	1	1	**1**	1	1		9
E_2	1	1	1	1	1	1	1	1			8
E_3	1	1	1	**1**	1	1	1	1	1	1	10
E_4	1	1	1	1	1	1					6

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Swapping rule

											$\ell_{\leq T,i}$
E_1	1	1	1	1	1	1	**1**	1	1		9
E_2	1	1	1	1	1	1	1	1			8
E_3	1	1	1	**1**	1	1	1	1	1	1	10
E_4	1	1	1	1	1	1					6

- 1's occur in some order
- Worst case: $\color{red}{1}$ before $\color{blue}{1}$
- Otherwise adversary benefits from swapping

Worst-case pattern

```
1    1    1    1    1
 1    1    1    1    1    1    1    1    1
  1    1    1    1    1    1    1    1    1
   1    1    1    1    1    1    1    1    1
    1    1    1    1    1    1    1    1    1
```

Cost per sweep

Assume we have s leaders

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$$s \text{ leader get unit} \atop \text{ignore non-leaders} \quad \left\{ \begin{array}{l} 1 \\ \;1 \\ \;\;1 \\ \;\;\;1 \\ \;\;\;\;1 \end{array} \right.$$

Cost per sweep

Assume we have s leaders

$$s \text{ leader get unit} \atop \text{ignore non-leaders} \quad \begin{cases} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{cases}$$

FL

$$\frac{1}{s} + \frac{1}{s - 1} + \frac{1}{s - 2} + \frac{1}{s - 3} + \ldots + \frac{1}{\underbrace{s - s - 2}_{2}} + \frac{1}{\underbrace{s - s - 1}_{1}}$$

$$\approx \ln s$$

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$$\approx \ln s$$

Dropout

$$\frac{1}{s} + \frac{1}{s-1/2} + \frac{1}{s-2/2} + \frac{1}{s-3/2} + \ldots + \frac{1}{s-(s-2)/2} + \frac{1}{s-(s-1)/2}$$

Cost per sweep

Assume we have s leaders

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FL

$$\frac{1}{s} + \frac{1}{s - 1} + \frac{1}{s - 2} + \frac{1}{s - 3} + \ldots + \frac{1}{\underbrace{s - s - 2}_{2}} + \frac{1}{\underbrace{s - s - 1}_{1}}$$

$$\approx \ln s$$

Dropout

$$\frac{2}{2s} + \frac{2}{2s - 1} + \frac{2}{2s - 2} + \frac{2}{s - 3} + \ldots + \frac{2}{2s - (s - 2)} + \frac{2}{2s - (s - 1)}$$

$$\approx 2\,(\ln 2s \ - \ \ln s) = 2 \ln 2$$

Cost per sweep

Assume we have s leaders

$$s \text{ leader get unit} \quad \text{ignore non-leaders} \quad \left\{ \begin{array}{l} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \right.$$

FL

$$\frac{1}{s} + \frac{1}{s-1} + \frac{1}{s-2} + \frac{1}{s-3} + \ldots + \frac{1}{\underbrace{s-s-2}_{2}} + \frac{1}{\underbrace{s-s-1}_{1}}$$

$$\approx \ln s$$

Dropout

$$\frac{2}{2s} + \frac{2}{2s-1} + \frac{2}{2s-2} + \frac{2}{s-3} + \ldots + \frac{2}{2s-(s-2)} + \frac{2}{2s-(s-1)}$$

$$\approx 2\,(\ln 2s \; - \; \ln s) = 2 \ln 2$$

$L^* = 0$ - one expert incurs no loss

FL

- One sweep

$$\frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{2} \cancel{+1} \approx (\ln n) - 1$$

- Optimal

$L^* = 0$ - one expert incurs no loss

FL

- One sweep

$$\frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{2} \cancel{+1} \approx (\ln n) - 1$$

- Optimal

Dropout

- # of leaders reduced by half in each sweep
- $\approx \log_2 n$ sweeps times $\leq 2 \ln 2 = 1.386$
  $=========================$
  $2 \ln n$

Overview of proof for noisy case

- Focus on first L sweeps
- Only occurs constant regret if number of leaders > 1

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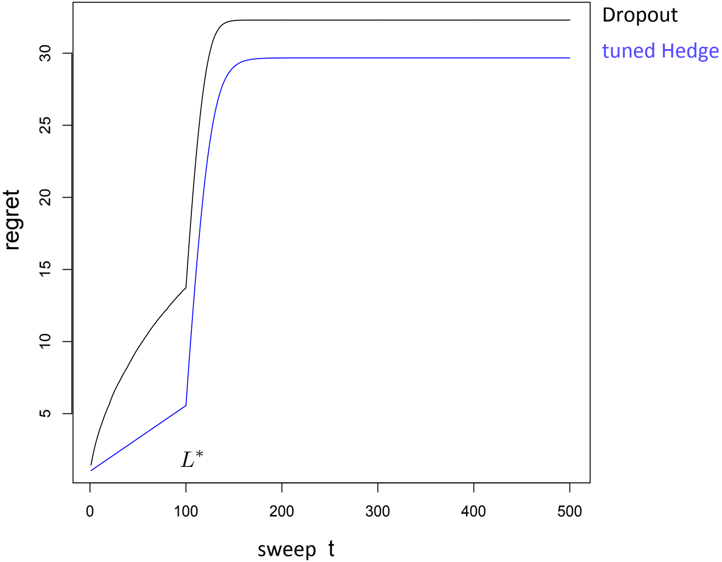
- For Hedge(η) and FPL(η) cost per sweep constant and dependent on η

Dropout versus Hedge



Outlook

- Combinatorial experts
- Matrix case
- Where else can dropout perturbations be used?
- Dropout for convex losses
- Dropout for neural nets

Outlook

- Combinatorial experts
- Matrix case
- Where else can dropout perturbations be used?
- Dropout for convex losses
- Dropout for neural nets
- Privacy

[Lugosi, Neu 2014] dense counter example

0	1^*
1	0
1	0
1	0
1	0
1	0
$\frac{n-1}{n}$	$\frac{1}{2}$

Iterate this pattern n times:

$$\sum_{i=1}^{n} \left(\frac{n-i}{n-i+1} + \frac{1}{2} \right)$$
$$\approx n - \ln n + \frac{n}{2}$$

$L^* = n$: Follow the Scipping Leader has linear regret

How does dropout ovoid this example?

0	1
1	0
1	0
1	0
1	0
1	0
$\frac{n-1}{n}$	$\frac{1}{\frac{n-1}{2}}$

How does dropout ovoid this example?

0	1
1	0
1	0
1	0
1	0
1	0
$\frac{n-1}{n}$	$\frac{1}{\frac{n-1}{2}}$

It leaves the adversary clueless as to who the leader is
i.e. privacy against adversary

sparse counter example

0	1^*
$\frac{1}{n-1}$	0
$\frac{1}{n-1}$	0
$\frac{1}{n-1}$	0
$\frac{1}{n-1}$	0
$\frac{1}{n-1}$	0
$\frac{n-1}{n^2}$	$\frac{1}{2}$

Iterate this pattern n times:

$$\sum_{i=1}^{n} \left(\frac{n-i}{(n-i+1)^2} + \frac{1}{2} \right)$$

$$= \sum_{i=1}^{n} \left(\frac{1}{n-i+1} - \frac{1}{(n-i+1)^2} + \frac{1}{2} \right)$$

$$\approx \ln n - O(1) + \frac{n}{2}$$

$L^* = \ln n$: Follow the Scipping Leader has linear regret