ASSIGNMENT #1

(25K-0132) - Muhammad Zain

BAI - 1A

VECTORS

Q:01

Use components of the vectors given in figure: 1 to find the magnitude and direction of the following

(a) $\vec{A} - \vec{B}$

(b) $\vec{A} + \vec{B}$

(c) $\vec{B} - \vec{A}$

(d) $\vec{A} \times \vec{B}$

Let,

$$\vec{A} = A_x\hat{i} + A_y\hat{j}, \quad \vec{B} = B_x\hat{i} + B_y\hat{j}$$

(a) $\vec{A} - \vec{B}$

$$\vec{A} - \vec{B} = (A_x - B_x)\hat{i} + (A_y - B_y)\hat{j}$$

(b) $\vec{A} + \vec{B}$

$$\vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j}$$

(c) $\vec{B} - \vec{A}$

$$\vec{B} - \vec{A} = (B_x - A_x)\hat{i} - (B_y - A_y)\hat{j}$$

(d) $\vec{A} \times \vec{B}$

Since A and B lie in the xy plane, the cross product points in the z-direction

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & 0 \\ B_x & B_y & 0 \end{vmatrix} = (A_x B_x - A_y B_y$$

Date:

Q:02

(a) Solution (components)

⇒ Convert to consistent units (metres).

$$1.25\,km = 1250\,m,$$

⇒ Breaking each displacement into x (east) and y (south).

1. South:

$$\vec{d_1} = (0, -825)$$

2. 30° west of north:

→ x- component $= -1250 \sin 30° = -625$

→ y - component $= +1250 \cos 30° \overset{\text{(approx)}}{=} 1082$

$$\vec{d_2} = (-625, 1082)$$

3. 40° north of east:

→ x-component $= 1000 \cos 40° \approx 765$

→ y - component $= 1000 \sin 40° \approx 643$

$$\vec{d_3} = (766, 643)$$

⇒ Adding displacements.

$$x_{total} = 0 - 625 + 766 = 141$$

$$y_{total} = -825 + 1082 + 643 = 900$$

So,

$$\vec{R}_{net} = (141, 900)\,m$$

⇒ Returning displacement.

To return to the oak tree, we must go opposite:

$$\vec{R}_{return} = (-141, -900)\,m$$

Magnitude:

$$|\vec{R}_{return}| = \sqrt{(-141)^2 + (-900)^2} \approx 911\,m$$

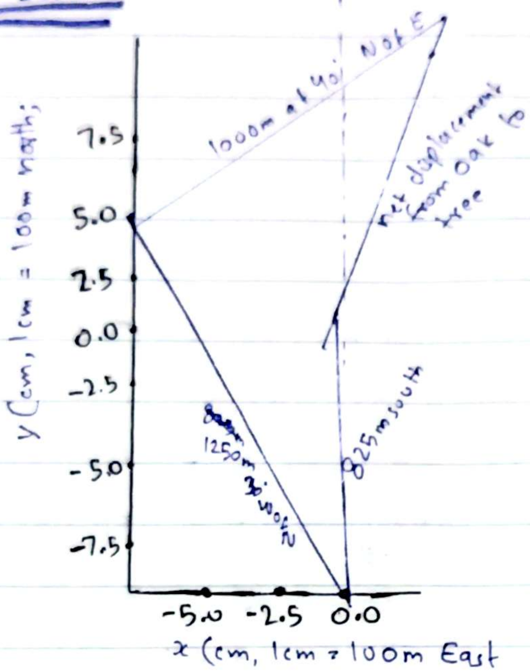
Direction:

$$\theta = \tan^{-1}\left(\frac{900}{141}\right) \approx 81° \text{ south of west}$$

Final Answer

$\boxed{|\vec{R}| \approx 911\text{ m}, \quad \text{direction} \approx 81^\circ \text{ S of W}}$

Graph Solution



Q:03:

⇒ Recalling Cross product magnitude relation.

The cross product magnitude is:

$$|\vec{A}\times\vec{B}| = AB\sin\theta$$

Since both vectors have magnitude 3:

$$|\vec{A}\times\vec{B}| = 3\cdot 3\cdot \sin\theta = 9\sin\theta$$

⇒ By Computing given magnitude

Given: $\vec{B}\times\vec{A} = -5\hat{k} + 2\hat{i}$

Magnitude:

$$|\vec{B}\times\vec{A}| = \sqrt{(-5)^2+(2)^2} = \sqrt{29} \approx \boxed{5.39} = \boxed{5.4}$$

⇒ Equating magnitudes:

$$9\sin\theta = 5.4$$

$$\sin\theta = \frac{5.4}{9} \approx \boxed{.6}$$

$\theta = \sin^{-1}(.6) \approx 37^\circ$

$\boxed{\theta = 37^\circ}$

(4)

Q:04

⇒ Representing Vectors in unit-vector notation Without the figure.

General rule:

$$\vec{V} = V\cos\theta\,\hat{i} + V\sin\theta\,\hat{j}$$

where θ is measured from $+x$-axis (east)

- If vector points left (west), $\cos\theta$ is negative.
- If vector points down (south), $\sin\theta$ is negative.

So:

$$\vec{B} = B_x\hat{i} + B_y\hat{j} = 59\cos\theta_B\hat{i} + 59\sin\theta_B\hat{j} \approx \boxed{55.8\hat{i} + 19.2\hat{j}}$$

$$\vec{C} = C_x\hat{i} + C_y\hat{j} = 25\cos\theta_C\hat{i} + 25\sin\theta_C\hat{j} \approx \boxed{15.4\hat{i} + 19.7\hat{j}}$$

⇒ Add "A" & "D"

:- $(A+D)_x = \boxed{146.03}$

:- $(A+D)_y = \boxed{78.81}$

In components:

$$\vec{A} + \vec{D} = (A_x + D_x)\hat{i} + (A_y + D_y)\hat{j}$$

Magnitude:

$$|\vec{A} + \vec{D}| = \sqrt{(A_x + D_x)^2 + (A_y + D_y)^2}$$

- $\vec{B} = 59\cos\theta_B\hat{i} + 59\sin\theta_B\hat{j} \approx \boxed{55.8\hat{i} + 19.2\hat{j}}$
- $\vec{C} = 25\cos\theta_C\hat{i} + 25\sin\theta_C\hat{j} \approx \boxed{15.4\hat{i} + 19.7\hat{j}}$
- $|\vec{A}+\vec{D}| = \sqrt{(A_x + D_x)\hat{i} + (A_y + D_y)\hat{j}} \approx \boxed{166}$

Date ⑤

Q:05

(a)

- Magnitude of the cross product of two vectors gives the area of the parallelogram.

$$|\vec{A} \times \vec{B}| = |\vec{A}|\,|\vec{B}| \sin\theta$$

- The area of the triangle formed by the same two vectors is half $(1/2)$ of that:

$$\text{Area of triangle} = \frac{1}{2}|\vec{A} \times \vec{B}|$$

(b)

Complete cross product

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 4 \\ -1 & -4 & 2 \end{vmatrix} = \hat{i}\big((-2)(2) - (4)(-4)\big) - \hat{j}\big((3)(2) - (4)(-1)\big) + \hat{k}\big((3)(-4) - (-2)(-1)\big)$$

$$\vec{A} \times \vec{B} = \boxed{12\hat{i} - 10\hat{j} - 14\hat{k}}$$

(c) Magnitude of cross product:

$$|\vec{A} \times \vec{B}| = \sqrt{12^2 + (-10)^2 + (-14)^2}$$

$$= \sqrt{144 + 100 + 196} = \sqrt{440} \approx \boxed{20.98}$$

(d) A.O.T

- Area $= \frac{1}{2}|\vec{A} \times \vec{B}| = \frac{1}{2}(20.98) = \boxed{10.49}$

- $\vec{A} \times \vec{B} = 12\hat{i} - 10\hat{j} - 14\hat{k}$
- $|\vec{A} \times \vec{B}| \approx 20.98$
- Area of triangle $= 10.49$

Q:6

$\Rightarrow$ Using dot product formula

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$$

since magnitudes are equal:

$$|\vec{A}| = |B| = M$$

So: $$\vec{A}\cdot\vec{B} = M^2\cos\theta$$

$\Rightarrow$ Applying given condition

$$\cancel{A^2}\cos\theta = \frac{1}{2}\cancel{A^2}$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\boxed{\theta = 60^\circ}$$

Q:07

- $A = (30.0, -20.0)\,m$
- $B = (60.0, 80.0)\,m$
- $C = (-10.0, -10.0)\,m$
- $D = (40.0, -30.0)\,m$
- $E = (-70.0, 60.0)\,m$

Let $r_0 = A$. ~~after~~ ~~each~~

$$r_1 = r_0 + \frac{1}{2}(B - r_0) = \frac{1}{2}A + \frac{1}{2}B,$$

$$r_2 = r_1 + \frac{1}{3}(C - r_1) = \frac{2}{3}r_1 + \frac{1}{3}C = \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C,$$

$$r_3 = r_2 + \frac{1}{4}(D - r_2) = \frac{3}{4}r^2 + \frac{1}{4}D = \frac{1}{4}A + \frac{1}{4}B + \frac{1}{4}C + \frac{1}{4}D,$$

$$r_4 = r_3 + \frac{1}{5}(E - r_3) = \frac{4}{5}r^3 + \frac{1}{5}E = \frac{1}{5}A + \frac{1}{5}B + \frac{1}{5}C + \frac{1}{5}D + \frac{1}{5}E$$

$$r_{final} = \frac{1}{5}(A + B + C + D + E)$$

$x_{sum} = 30 + 60 - 10 + 40 - 70 = 50 \Rightarrow x_{final} = 50/5 = \boxed{10m}$

$y_{sum} = -20 + 80 - 10 - 30 + 60 = 80 \Rightarrow y_{final} = \frac{80}{5} = 16m.$

$\boxed{r_{final} = (10.0m, 16.0m)}$

Q:08

(a) and (b) - Direction of $\hat{i}\cdot\hat{j}/d$

- $\hat{i}\cdot\hat{j} = 0$ (since they are perpendicular)
- Dividing by d still gives zero,

therefore, the vector has no direction, regardless of being (+ve) or (−ve)

(c) and (d) — Magnitude of $\hat{i}\cdot\hat{i}/d$

- $\hat{i}\cdot\hat{i} = 1$
- $\div$ by d: $1/d$
- Magnitude $= [1/d]$

(e) and (f) — Direction of cross product.

- $\hat{i} \times \hat{j} = \hat{k}$ (points along $+z$, by the right hand rule).
- $\hat{j} \times \hat{i} = -\hat{k}$ (opposite direction)

thus;

- (e) direction is $+z$-axis
- (f) direction is $-z$-axis

(i) and (j) — Mag: & direc: of $\hat{i} \times (d\hat{j})$ if $d > 0$

- $\hat{i} \times (d\hat{j}) = d(\hat{i} \times \hat{j}) = d\hat{k}$
- Magnitude $= |d|$. since $d > 0$, its just d.
- Direction $=$ $+z$-axis (same as $\hat{k}$)

(g) and (h) - Mag: of cross products.

- Both $\hat{i}$ and $\hat{j}$ are unit vectors and perpendicular
- $|\hat{i} \times \hat{j}| = |\hat{j} \times \hat{i}| = |\hat{i}||\hat{j}| \sin 90° = 1$

So,

Magnitude = 1 for both.

Final answer:

(a) Direction of $\hat{i}.\hat{j}/d$: none (zero).

(b) same: none

(c) Magnitude of $\hat{i}.\hat{i}/d$: [1/d]

(d) same.

(e) Direction of $\hat{i} \times \hat{j}$: +z-axis

(f) Direction of $\hat{j} \times \hat{i}$: −z-axis

(g) Magnitude of (e): 1.

(h) Magnitude of (f): 1

(i) Magnitude of $\hat{i} \times (d\hat{j})$ (d>0): d

(j) Direction: + z axis

Q:09

(a) Magnitude of displacement

$$|\vec{d}| = \sqrt{(4.30)^2 + (3.70)^2 + (3.00)^2}$$
$$= \sqrt{18.49 + 13.69 + 9.00} = \sqrt{41.18} \approx \boxed{6.42\,m}$$

(b) Path length could be less than this magnitude?

→ ~~Distance~~ Displacement is the shortest possible straight-line distance, so, no.

(c) Could the path length be greater?

→ Yes. The fly could take a curve or zig-zag route which always be greater than 6.42m.

(d) can path length equal the displacement

→ If the fly flies in a straight line directly from start to end, then yes.

path length displacement = 6.42m

(e) Displacement vector components in unit-vector notation.

chosen coordinates:

$$\vec{d} = (4.30)\hat{i} + (3.70)\hat{j} + (3.00)\hat{k}$$

(f) shortest walking path along the walls.

→ Unfold the box

→ Imagine the fly walking along 2D surfaces. The shortest path is the diagonal of one unfolded rectangle.

→ Two options for unfolding:

1. Floor + one vertical wall: rectangle dimensions = 4.30+3.70 = 8.00 (one direction), height = 3.00
2. Floor + adjacent wall in other orientation: rectangle dimensions = 4.30 + 3.00 = 7.30, height = 3.70
3. Or width + height rectangle: 3.70 + 3.00 = 6.70 with length 4.30

Compute diagonals:

- Case 1: $\sqrt{(8.00)^2 + (3.00)^2} = \sqrt{64+4} = \sqrt{73} \approx \boxed{8.54\,m}$
- Case 2: $\sqrt{(7.30)^2 + (3.70)^2} = \sqrt{53.29 + (13.69)} = \sqrt{66.98} \approx \boxed{8.19\,m}$
- Case 3: $\sqrt{(6.70)^2 + (4.30)^2} = \sqrt{44.89 + 18.49} = \sqrt{63.38} \approx \boxed{7.96\,m}$

shortest walking path = 7.96 m

Q10.

A:

$$F_{Ax} = 100\cos 30^\circ = 100(0.866)\ \boxed{86.6}$$
$$F_{Ay} = 100\sin 30^\circ = 100(0.5) = \boxed{50.0}$$

B:

$$F_{Bx} = 40\cos 53^\circ = 40(0.602) \approx \boxed{24.1}$$
$$F_{By} = 40\sin 53^\circ = 40(0.799) \approx \boxed{32.0}$$

C (Below x-axis):

$$F_{Cx} = 80\cos 30^\circ = 80(0.866) = \boxed{69.3}$$
$$F_{Cy} = -80\sin 30^\circ = -80(0.5) = \boxed{-40.0}$$

$\Rightarrow$ Add components

$$R_x = 86.6 + 24.1 + 69.3 = \boxed{180.0}$$
$$R_y = 50.0 + 32.0 - 40.0 = \boxed{42.0}$$

$\Rightarrow$ Equilibrant force

$$\vec{F}_D = -\vec{R} = -180\hat{i} - 42\hat{j}$$

Magnitude:

$$|\vec{F}_D| = \sqrt{180^2 + 42^2} = \sqrt{32400 + 1764} = \sqrt{34164} \approx 185\,N$$

Direction (from +x-axis):

$$\theta = \tan^{-1}\left(\frac{+42}{+180}\right) = \tan^{-1}(0.233) \approx \boxed{13^\circ}$$

Since both components are negative, the vector points in the third quadrant, i.e

$$\theta \approx 180^\circ + 13^\circ = 193^\circ \text{ (below the } -x\text{-axis}$$

Q11:

(a) Express vectors OQ and OR

- A unit cube in the first octant has one vertex at the origin $O = (0,0,0)$.
- The opposite corner (Q) is $(1,1,1)$.

$$\vec{OQ} = (1-0)\hat{i} + (1-0)\hat{j} + (1-0)\hat{k} = \hat{i}+\hat{j}+\hat{k}$$

- A face center (R):

for example, the center of the square face in the xy-plane at $z=0$. its coordinates are $(0.5, 0.5, 0)$.

$$\vec{OR} = 0.5\hat{i} + 0.5\hat{j} + 0\hat{k}$$

(b) Cosine of the angle b/w OQ and OR

Formula:

$$\cos\theta = \frac{\vec{OQ}\cdot\vec{OR}}{|\vec{OQ}|\,|\vec{OR}|}$$

Dot product:

$$\vec{OQ}\cdot\vec{OR} = (1)(0.5) + (1)(0.5) + (1)(0) = 1.0$$

Magnitudes

$$|\vec{OQ}| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

$$|\vec{OR}| = \sqrt{(0.5)^2+(0.5)^2+0^2} = \sqrt{0.25+0.25} = \sqrt{0.5}$$

So,

$$\cos\theta = \frac{1.0}{\sqrt{3}\cdot\sqrt{0.5}} = \frac{1}{\sqrt{1.5}} \boxed{\approx 0.816}$$

$$\theta = \cos^{-1}(0.816) \approx 35^\circ$$

$$\boxed{\theta = 35^\circ}$$

Q:12

(a) $\vec{d_1} \times \vec{d_2}$

cross product will only have a ~~...~~ k-component in 2D

$$\vec{d_1} \times \vec{d_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 5 & 0 \\ -3 & 4 & 0 \end{vmatrix}$$

Expand:

$$= \hat{i}\,(5(0)-0(4)) - \hat{j}\,(4(0)-0(-3)) + \hat{k}\,((4)(4)-(5)(-3))$$

$$= \boxed{31\hat{k}}$$

(b) $\vec{d_1} \cdot \vec{d_2}$

$$\vec{d_1} \cdot \vec{d_2} = (4)(-3) + (5)(4) = -12 + 20 = \boxed{8}$$

(c) $(\vec{d_1} + \vec{d_2}) \cdot \vec{d_2}$

$$(\vec{d_1} + \vec{d_2}) = (4-3)\hat{i} + (4+5)\hat{j} = \boxed{(1\hat{i} + 9\hat{j})}$$

$$(\vec{d_1} + \vec{d_2}) \cdot \vec{d_2} = (1)(-3) + (9)(4) = -3 + 36 = \boxed{33}$$

Q:13

(i) Sum In unit-vector notation

$$\vec{a} + \vec{b} = (4.0 + (-13.0))\hat{i} + (3.0 + 7.0)\hat{j}$$

$$\vec{a} + \vec{b} = -9.0\hat{i} + 10.0\hat{j}$$

(ii) Magnitude of $\vec{a} + \vec{b}$

$$|\vec{a} + \vec{b}| = \sqrt{(-9.0)^2 + (10.0)^2}$$

$$= \sqrt{81 + 100} = \sqrt{181} \approx \boxed{13.45\,m}$$

(iii) Direction of $\vec{a} + \vec{b}$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{10.0}{-9.0}\right)$$

$$= \tan^{-1}(-1.111) \approx \boxed{-48.0^\circ}$$

$x < 0,\ y > 0$

vector lies in 2nd Quadrant: $\theta = 180^\circ - 48^\circ = \boxed{132^\circ}$

Q:14

⇒ Property of cross product

For any vectors $\vec{P}$ and $\vec{A}$

$$\vec{P} \times \vec{A} \perp (90^\circ)\ \vec{P}$$

that is, Result of a cross product is always perpendicular to the first vector.

So if, $(2,-3,4) \times \vec{A} = (4,3,-1)$, then RHS must be perpendicular to $(2,-3,4)$

⇒ Testing perpendicularity using dot product

check:

$$(2,-3,4)\cdot(4,3,-1) = (2)(4)+(-3)(3)+(4)(-1)$$
$$= 8-9-4 = -5$$

not zero, RHS is not perpendicular to $(2,-3,4)$

⇒ Conclusion

- This violates the main rules of cross products.
- Therefore, no vector $\vec{A}$ can satisfy this equation.

Q:15

⇒ Recall perpendicularity condition

Two vectors are perpendicular if their dot product is zero:

$$(\vec{A}+\vec{B})\cdot(\vec{A}-\vec{B}) = 0$$

⇒ Expanding the dot product

$$(\vec{A}+\vec{B})\cdot(\vec{A}-\vec{B}) = \vec{A}\cdot\vec{A} - \vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{A} - \vec{B}\cdot\vec{B}$$

since $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$, the middle terms cancel:

$$= \vec{A}\cdot\vec{A} - \vec{B}\cdot\vec{B}$$

⇒ Simplify

$$= |\vec{A}|^2 - |\vec{B}|^2$$

since the condition says this $= 0$, so:

$$|\vec{A}|^2 = |\vec{B}|^2$$

$$\Rightarrow |\vec{A}| = |\vec{B}|$$

$$\boxed{(\vec{A} + \vec{B}) \cdot (\vec{A} - \vec{B}) = |\vec{A}|^2 - |\vec{B}|^2}$$

THE END