L^p Estimates for Oscillatory Integral Operators and Hörmander's Theorem

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Abstract

We present a study of the boundedness properties of oscillatory integral operators. We provide detailed proofs for some previously known results, notably the Van Der Corput Lemma and other key theorems that will help us obtain L^p estimates of these operators. We also generalize the results of Hörmander to the class of oscillatory integral operators of the second kind with a degenerate C^∞ phase function following a procedure outlined in [3]. To that end, we present a comprehensive summary of notions from measure theory that are crucial to this study, mainly measure spaces, measurable functions and the Lebesgue spaces of $p-$integrable functions known as L^p spaces.

The first two chapters of this thesis are devoted to defining the notions that will be subsequently used in the third chapter, which presents a proof of Van Der Corput's lemma. We then present Hörmander's theorem and its extension by Ma in Chapter 4. Lastly, Chapter 5 presents and proves a new theorem that provides a generalization of Hörmander's results to oscillatory integral operators with a specific phase function.

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$\mathbb{N}$	set of natural numbers $\{1, 2, 3, ...\}$
$\mathbb{Z}$	set of integers $\{..., -2, -1, 0, 1, 2, ...\}$
$\mathbb{R}^+$	set of positive real numbers
C^∞	space of smooth (continuously differentiable) functions
C^k	space of functions f with $\partial^\alpha f$ continuous for all $\lvert\alpha\rvert \leq k$ where $k \in \mathbb{N}$
C^1	space of functions whose first derivative is continuous
C^0	space of continuous functions
C_0^∞	space of smooth functions with compact support
L^∞	space of bounded functions equipped with sup norm
L^p	space of p-power integrable functions for $0 < p < \infty$
L^1	space of integrable functions
$\phi^{(k)}$	k^{th} derivative of a function ϕ
$\lvert E\rvert$	the cardinality of the set E
$X \setminus E$	the complement of the set E with respect to the set X
$\mathcal{P}(E)$	the power set of the set E
$v \cdot w$	the dot product of v and w equal to $\sum_{i=1}^{n} v_i w_i$

Chapter 1

Introduction to Oscillatory Integral Operators

In this chapter, we shall present an overview of our study of oscillatory integral operators, as well as the motivation behind it.

Oscillatory integrals span multiple fields in engineering and physics, and are specifically crucial to the study of optics, image analysis and signal processing. Numerical solutions to oscillatory integrals are very complicated and challenging, as well as computationally intensive. Multiple algorithms have been proposed to that end, but they remain short of efficient. This is why we are interested in the study of their boundedness, especially that in practice, we often only need their order of magnitude.

In the field of mathematics, their main use is estimating the decay of Fourier transforms (see Definition 2.1) of measures carried on the surface. The conditions on their phase denoted ϕ translate the state of curvature of the surfaces in question. They can thus be applied in both maximal averages associated with curved surfaces, as well as restriction theorems for Fourier Transforms ([4],[5]). In order to study oscillatory integral operators, we shall draw a distinction between two types of such integrals:

Integrals of the first kind, and those of the second kind, borrowing the terminology introduced by Stein([4],[5]).

The first type of integrals is defined as follows:

Definition 1.1. [4] **Oscillatory Integrals of the First Kind.**

An oscillatory integral of the first kind is of the form

$$I(\lambda) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x)}\psi(x)dx$$

where ϕ is a real-valued smooth function (*the phase*), and ψ is a complex-valued smooth function (*the symbol*) that is also compactly supported ($i.e.$ ψ is nonzero on a compact set).

We are often interested in the behavior of $I(\lambda)$ for large values of λ.

One of the most heavily-studied oscillatory integrals of the first kind is the Fourier Transform defined as follows:

$$\mathscr{F}_f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)e^{-i\omega x}dx$$

for all real-valued functions f with a convergent improper integral.

Now, we shall introduce oscillatory integrals of the second kind.

Definition 1.2. [4] **Oscillatory Integrals of the Second Kind.**

Let $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$.

Oscillatory integrals of the second kind come in a variety of forms. The following is a form derived from the Fourier Transform:

$$(T_\lambda f)(\xi) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x,\xi)}\psi(x,\xi)f(x)dx,\ \xi \in \mathbb{R}^n$$

where ϕ is a real-valued smooth function (*the phase*) and ψ is a fixed smooth cut-off function that is also compactly supported in the neighborhood of the origin. Recall a smooth cutoff function is a smooth function equal to one on a neighbourhood of a given compact set S, and to zero at every point whose distance from S is greater than a given ϵ.

Similarly to how we evaluate the behavior of the operators of the first kind, we are interested in estimating the norm of the operator T_λ as $\lambda \to \infty$.

Chapter 2

Preliminaries

In this chapter, we shall present a summary of some fundamental concepts in Measure Theory that are crucial to studying the L^p estimates of oscillatory integral operators. The definitions we provide are collected from [6].

2.1 Measure Space

We shall begin by defining a measure space, starting with its three main components. The first component is a $\sigma-$algebra. We define it as follows:

Definition 2.1. A class $\mathcal{M}$ of subsets of X is a σ**-algebra**, provided the three following properties hold:

1. $\mathcal{M}$ is nonempty.
2. If $E \in \mathcal{M}$, then $X \setminus E \in \mathcal{M}$.
3. If (E_n) is a sequence of sets in $\mathcal{M}$, then $\displaystyle\bigcup_{k=1}^{\infty} E_k \in \mathcal{M}$.

The most economical $\sigma-$algebra on X is $\mathcal{M} = \{\emptyset, X\}$. It is trivial to prove that it satisfies the three properties above: $X \in \mathcal{M}$ implies $\mathcal{M} \neq \emptyset$, $\emptyset$ and X are each other's complements and any infinite union of elements of $\mathcal{M}$ is equal to $X \in \mathcal{M}$.

Now, we shall define the second component of a measure space: the measure. It represents a generalization on $\mathbb{R}^n$ of the concepts of length on $\mathbb{R}$, area on $\mathbb{R}^2$ and volume on $\mathbb{R}^3$. It was first defined by Borel in 1898 [6].

Definition 2.2. A set function μ defined on $\mathcal{M}$ is a **measure** provided the following three properties hold:

1. $\mu : \mathcal{M} \to [0, \infty)$.
2. $\mu(\emptyset) = 0$.
3. If (E_n) is a sequence of pairwise disjoint sets in $\mathcal{M}$, then $\mu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k)$.

A familiar example of a measure is the counting measure on $\mathbb{N}$. Let $X = \mathbb{N}$ and $\mathcal{M} = \mathcal{P}(\mathbb{N})$.

For all $E \in \mathcal{M}, \mu(E) = \begin{cases} |E| \text{ if } E \text{ is finite,} \\ +\infty \text{ if } E \text{ is infinite.} \end{cases}$

Lastly, the definition of a measure space follows directly from the definition of a measure.

Definition 2.3. $(X, \mathcal{M}, \mu)$ is a **measure space** if and only if μ is a measure on $(X, \mathcal{M})$ where $\mathcal{M}$ is a σ-algebra of X.

Now, let us look at some examples of measure spaces.

Example 1:

Consider the experiment of naming a random integer. We denote the outcomes as follows: E corresponds to the number being even and O corresponds to the odd case. Let $X = \{E, O\}$. We construct a $\sigma-$algebra $\mathcal{M} = \{\{E\}, \{O\}, \{E, O\}, \emptyset\}$, such that the empty set and $\{E, O\}$ denote the cases "neither even nor odd" and "even or odd" respectively.

We define our measure μ to correspond to the probability of each event. Hence, $\mu(\{E\}) = \mu(\{O\}) = 1/2$, $\mu(\emptyset) = 0$ and $\mu(\{E, O\}) = 1$.

It is therefore trivial to show $(X, \mathcal{M}, \mu)$ is a measure space. Furthermore, since $\mu(X) = 1$, this triplet is called a probability space.

Example 2:

One of the most commonly studied measures is the Lebesgue measure denoted m^*. It extends the notion of volume to $\mathbb{R}^n$ for $n \in \mathbb{N}$, by minimizing the sum of volumes of rectangles covering a given set. It is defined as follows for all sets $A \subseteq \mathbb{R}^n$:

$$m^*(A) = inf\{\sum_{k=1}^{\infty} V(R_k) : A \subseteq \bigcup_{k=1}^{\infty} R_k\}$$

where $\{R_k\}_{k=1}^{\infty}$ is the rectangle cover of the set A, and $inf\{.\}$ represents the infimum which corresponds to the largest lower bound of a given set.

From this definition follows that of measurable sets as characterized by Carathéodory. He states that a set $E \subseteq \mathbb{R}^n$ is measurable if for all $A \in \mathbb{R}^n$, we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \text{ where } E^c = \mathbb{R}^n \setminus E.$$

The collection of all such measurable sets forms the Lebesgue $\sigma-$algebra of $\mathbb{R}^n$ denoted $\mathcal{L}$, and the triplet $(\mathbb{R}^n, \mathcal{L}, m^*)$ forms a measure space.

2.2 Measurable functions

Measurable functions are integral to the discussion of L^p spaces in Chapter 2. In this section, we shall define measurability on functions and present some examples.

We define function measurability as follows:

Definition 2.4. Let X be a set and suppose $\mathcal{M}$ is a σ-algebra of subsets of X. Let f be an extended real-valued function defined on X. Then f is said to be **measurable** if and only if for all $\alpha \in \mathbb{R}$

$$A_\alpha = \{x \in X : \ f(x) > \alpha\} \in \mathcal{M}.$$

Measurable functions exhibit multiple interesting properties. We shall explore some simple properties through the following propositions.

Proposition 2.1. *All constant functions are measurable.*

Proof. Let $X = \mathbb{R}$, $\mathcal{M}$ a σ-algebra of subsets of X and $f(x) = c \in \mathbb{R}$ for all $x \in \mathbb{R}$. Let $\alpha \in \mathbb{R}$ be arbitrary.

- If $c \leq \alpha$, then $A_\alpha = \emptyset \in \mathcal{M}$.
- If $c > \alpha$, then $A_\alpha = \mathbb{R} \in \mathcal{M}$.

Therefore, for all $\alpha \in \mathbb{R}$, $A_\alpha \in \mathcal{M}$.
We conclude thereby that all constant functions are measurable. $\square$

Proposition 2.2. *For any $\sigma-$algebra $\mathcal{M}$ on X, $f = \chi_A$ is measurable if and only if $A \in \mathcal{M}$.*

Proof. Recall χ_A is the characteristic function on the set A defined as follows:

$$\chi_A(x) := \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ if } x \in X \setminus A. \end{cases}$$

Let $\alpha \in \mathbb{R}$ be arbitrary.

Then $A_\alpha = \{x \in X: \ \chi_A(x) > \alpha\}$.

- If $\alpha < 0$ then $A_\alpha = \{\chi_A \geq 0\} = X \in \mathcal{M}$.
- If $\alpha \geq 1$ then $A_\alpha = \{\chi_A > 1\} = \ \emptyset \in \mathcal{M}$.
- If $0 \leq \alpha < 1$ then $A_\alpha = \{\chi_A \geq 1\} = \{\chi_A = 1\} = \{x | x \in A\} = A$.

Therefore, for χ_A to be measurable, it is necessary and sufficient that $A \in \mathcal{M}$ holds, *i.e.* $f = \chi_A$ is measurable if and only if $A \in \mathcal{M}$, for any $\sigma-$algebra $\mathcal{M}$ on X. $\square$

2.3 L^p Spaces

In this section, we shall define the space in which we study the boundedness of our class of oscillatory integral operators, notably the L^p space.

Definition 2.5. Let $0 < p < \infty$.

Let $(X, \mathscr{M}, \mu)$ be a measure space and f an extended real-valued measurable function defined on X.

Then $|f|^p$ is also measurable and $\|f\|_p = \left(\int_X |f|^p dx \right)^{1/p}$ is called the "p norm" of f and is well-defined.

We thereby define the space L^p as follows:

$$L^p(X) = \{f: \ f \text{ is measurable and } \|f\|_p < \infty\} \text{ for } \ 0 < p < \infty.$$

The following example illustrates computing a p norm of a function for $p = 2$.

Example: Consider the function $f(x) = \dfrac{1}{x}$ for all $x \in X = (0, 1]$.

Let us determine whether $f \in L^2(X)$.

We have $\displaystyle\int_{(0,1]} |f|^p dx = \lim_{c\to 0}\int_c^1 \left(\frac{1}{x}\right)^2 dx = \lim_{c\to 0}[-x^{-1}]_c^1 = \lim_{c\to 0} -1 + c^{-1} = \infty.$

Therefore, $\displaystyle\left(\int_{(0,1]} \left|\frac{1}{x}\right|^2 dx\right)^{\frac{1}{2}} = \infty$. We thereby conclude $f \notin L^2(X)$.

Let us now look at the $p-$norm of the same function when $p = \dfrac{1}{2}$.

We have $\displaystyle\int_{(0,1]} |f|^p dx = \lim_{c\to 0}\int_c^1 \left(\frac{1}{x}\right)^{1/2} dx = \lim_{c\to 0}[2\, x^{\frac{1}{2}}]_c^1 = \lim_{c\to 0}\, 2 - 2\, c^{\frac{1}{2}} = 2.$

Therefore, $\displaystyle\left(\int_{(0,1]} \left|\frac{1}{x}\right|^{\frac{1}{2}} dx\right)^2 = 4 < \infty.$

Therefore, we conclude $f \in L^{\frac{1}{2}}(X)$.

Note that $f \notin L^2(X)$ and $f \in L^{\frac{1}{2}}(X)$ implies $L^{\frac{1}{2}}(X) \not\subseteq L^2(X)$.

Now, let us present and prove one of the important results on L^p norms.

Theorem 2.3. *[6]* ***Minkowski's Inequality.*** *Suppose $(X, \mathcal{M}, \mu)$ is a measure space and the functions $f, g \in L^p(\mu)$, $1 \le p < \infty$. Then $\|f + g\|_p \le \|f\|_p + \|g\|_p$.*

In order to prove Minkowski's inequality, we shall use Hölder's inequality stated as follows:

Theorem 2.4. *[6]* ***Hölder's Inequality***. *Suppose $(X, \mathcal{M}, \mu)$ is a measure space, and let p and q be conjugate indices (i.e. $q = \frac{p}{p-1}$).*
If $f \in L^p(\mu)$ and $g \in L^q(\mu)$, then fg is integrable and $\|fg\| \le \|f\|_p\|g\|_q$.

Now, let us prove Minkowski's inequality.

Proof. Let $1 \le p < \infty$.

Let $(X, \mathcal{M}, \mu)$ be a measure space and $f, g \in L^p(\mu)$.

We shall show $\|f + g\|_p \le \|f\|_p + \|g\|_p$.

We know $f, g \in L^p(\mu)$ implies $\|f\|_p < \infty$ and $\|g\|_p < \infty$. We have

$$\begin{aligned}
\|f+g\|_p^p &= \int_X |f+g|^p d\mu \\
&= \int_X |f+g||f+g|^{p-1} d\mu \\
&\leq \int_X (|f|+|g|)|f+g|^{p-1} d\mu \qquad \text{(by the triangle inequality)} \\
&\leq \int_X |f||f+g|^{p-1} d\mu + \int_x |g||f+g|^{p-1} d\mu,
\end{aligned}$$

i.e. $\|f+g\|_p^p \leq \int_X |f||f+g|^{p-1} d\mu + \int_X |g||f+g|^{p-1} d\mu$.

Using Hölder's inequality, we get

$$\begin{aligned}
\|f+g\|_p^p &\leq \|f\|_p \|(f+g)^{p-1}\|_q + \|g\|_p \|(f+g)^{p-1}\|_q \\
&\leq \|(f+g)^{p-1}\|_q (\|f\|_p + \|g\|_p) \\
&\leq \left(\int_X |f+g|^{(p-1)\frac{p}{p-1}} d\mu \right)^{\frac{p-1}{p}} (\|f\|_p + \|g\|_p) \\
&\leq \left(\int_X |f+g|^p d\mu \right)^{\frac{p-1}{p}} (\|f\|_p + \|g\|_p) \\
&\leq \|f+g\|_p^{p-1} (\|f\|_p + \|g\|_p),
\end{aligned}$$

i.e. $\dfrac{\|f+g\|_p^p}{\|f+g\|_p^{p-1}} \leq (\|f\|_p + \|g\|_p)$ whenever $\|f+g\|_p^{p-1} \neq 0$,

implies $\|f+g\|_p \leq \|f\|_p + \|g\|_p$ whenever $\|f+g\|_p^{p-1} \neq 0$. $\square$

Chapter 3

Van Der Corput Lemma

In this chapter, we shall state and prove the Van Der Corput Lemma, which we will subsequently use to prove Hörmander's Theorem.

Lemma 3.1. ***(Van Der Corput):***

Let $a, b \in \mathbb{R}$ such that $a < b$.

Suppose ϕ is a real-valued function in $C^\infty([a,b])$ such that $|\phi^{(k)}(x)| \geq 1$ for all $x \in [a,b]$.

Then

$$\left| \int_a^b e^{i\lambda\phi(x)}\, dx \right| \leq c_k \lambda^{-1/k},$$

for some $c_k > 0$ independent of ϕ, whenever

i. $k = 1$ and $\phi'(x)$ is monotonic, or

ii. $k \geq 2$.

Proof. This proof is based on an outline of the proof provided in [4].

Let $k = 1 \in \mathbb{N}$.

We know $\dfrac{d}{dx}\left[e^{i\lambda\phi(x)}\right] = i\lambda\phi'(x) \cdot e^{i\lambda\phi(x)}$.

Then $e^{i\lambda\phi(x)} = (i\lambda\phi'(x))^{-1} \cdot \frac{d}{dx}\left[e^{i\lambda\phi(x)}\right]$.

Hence

$$\begin{aligned}
\int_a^b e^{i\lambda\phi(x)}\,dx &= \int_a^b (i\lambda\phi'(x))^{-1} \cdot \frac{d}{dx}\left[e^{i\lambda\phi(x)}\right] dx \\
&= (i\lambda)^{-1} \int_a^b \frac{1}{\phi'(x)} \cdot \frac{d}{dx}\left[e^{i\lambda\phi(x)}\right] dx \\
&= (i\lambda)^{-1}\left[\left[\frac{e^{i\lambda\phi(x)}}{\phi'(x)}\right]_a^b - \int_a^b \frac{d}{dx}\left[\frac{1}{\phi'(x)}\right] \cdot e^{i\lambda\phi(x)} dx\right] && (\textit{using integration by parts}) \\
&= (i\lambda)^{-1} \cdot \left[\frac{e^{i\lambda\phi(x)}}{\phi'(x)}\right]_a^b - (i\lambda)^{-1} \cdot \int_a^b \frac{d}{dx}\left[\frac{1}{\phi'(x)}\right] \cdot e^{i\lambda\phi(x)} dx.
\end{aligned}$$

Using the triangle inequality, we get

$$\left|\int_a^b e^{i\lambda\phi(x)}\right| \leq \left|(i\lambda)^{-1} \cdot \left[\frac{e^{i\lambda\phi(x)}}{\phi'(x)}\right]_a^b\right| + \left| - (i\lambda)^{-1} \cdot \int_a^b \frac{d}{dx}\left[\frac{1}{\phi'(x)}\right] \cdot e^{i\lambda\phi(x)} dx\right|. \tag{1}$$

Note that

$$\begin{aligned}
\left|(i\lambda)^{-1} \cdot \left[\frac{e^{i\lambda\phi(x)}}{\phi'(x)}\right]_a^b\right| &= \left|(i\lambda)^{-1}\right| \cdot \left|\frac{e^{i\lambda\phi(b)}}{\phi'(b)} - \frac{e^{i\lambda\phi(a)}}{\phi'(a)}\right| \\
&\leq |\lambda^{-1}| \cdot \left[\left|\frac{e^{i\lambda\phi(b)}}{\phi'(b)}\right| + \left|\frac{e^{i\lambda\phi(a)}}{\phi'(a)}\right|\right] && (\textit{by the triangle inequality}) \\
&\leq |\lambda^{-1}| \cdot (1+1) \\
&\qquad (\textit{since } |\phi'| \geq 1 \textit{ by assumption and} |e^{i\theta}| = 1 \textit{ for all } \theta \in \mathbb{R}) \\
&\leq \frac{2}{|\lambda|} \\
&= \frac{2}{\lambda}. && (2)
\end{aligned}$$

Similarly, we show

$$\begin{aligned}
\left| -(i\lambda)^{-1} \cdot \int_a^b \frac{d}{dx}\left[\frac{1}{\phi'(x)}\right] \cdot e^{i\lambda\phi(x)} dx \right| &= |(i\lambda)^{-1}| \cdot \left| \int_a^b \frac{d}{dx}\left[\frac{1}{\phi'(x)}\right] \cdot e^{i\lambda\phi(x)} dx \right| \\
&= |\lambda^{-1}| \cdot \left| \int_a^b \frac{d}{dx}\left[\frac{1}{\phi'(x)}\right] \cdot e^{i\lambda\phi(x)} dx \right| \\
&\leq \lambda^{-1} \cdot \int_a^b \left| \frac{d}{dx}\left[\frac{1}{\phi'(x)}\right] \cdot e^{i\lambda\phi(x)} \right| dx \\
&= \lambda^{-1} \cdot \int_a^b \left| \frac{d}{dx}\left[\frac{1}{\phi'(x)}\right] \right| dx \\
&= \lambda^{-1} \cdot \left| \left[\frac{1}{\phi'(x)}\right]_a^b \right| \quad (\textit{by monotonicity of } \phi'(x)) \\
&= \lambda^{-1} \cdot \left| \frac{1}{\phi'(b)} - \frac{1}{\phi'(a)} \right| \\
&\leq \lambda^{-1}\left(\left| \frac{1}{\phi'(b)} \right| + \left| \frac{1}{\phi'(a)} \right| \right) \\
&\qquad\qquad (\textit{by the triangle inequality}) \\
&\leq \lambda^{-1}(1+1) \quad (\textit{since } |\phi'| \geq 1 \textit{ by assumption}) \\
&= \frac{2}{\lambda}. \qquad (3)
\end{aligned}$$

Combining (2) and (3), we conclude

$$\left| \int_a^b e^{i\lambda\phi(x)} dx \right| \leq \frac{2}{\lambda} + \frac{2}{\lambda} = \frac{4}{\lambda},$$

i.e.

$$\left| \int_a^b e^{i\lambda\phi(x)} dx \right| \leq 4\lambda^{-1}.$$

Hence, the lemma is verified for $k = 1$ and $c_1 = 4$.

Suppose now that $k \geq 2$.

We shall prove the results of the lemma using induction. To that end, we shall use the following proposition:

Proposition 3.2. *Let $k \in \mathbb{N}$.*

Let $a, b \in \mathbb{R}$ such that $a < b$.

Suppose the result of the lemma is true for k and $|\phi^{(k+1)}(x)| \geq 1$ on $[a, b]$. Then

$$\left| \int_a^b e^{i\lambda\phi(x)} dx \right| \leq 2(c_k + 1)\lambda^{-1/(k+1)}.$$

Proof. Let $k \in \mathbb{N}$ and assume $\phi^{(k+1)}(x) \geq 1$ on $[a, b]$.

We know $\phi \in C^\infty([a, b])$. Therefore $\phi^{(k)}$ is continuous over $[a, b]$. This implies $\phi^{(k)}$ assumes a minimum value over $[a, b]$.

Let $c \in [a, b]$ be that minimum value. We distinguish 2 cases:

case 1: $\phi^{(k)}(c) = 0$.

Then we can rewrite the integral as follows:

$$\int_a^b = \int_a^{c-\sigma} + \int_{c-\sigma}^{c+\sigma} + \int_{c+\sigma}^b, \qquad \textit{for } \sigma \to 0^+ \textit{ and } \sigma \neq 0. \tag{4}$$

Let $x \in [a, c - \sigma]$.

Then

$$\begin{aligned}
|\phi^{(k)}(x)| &= |\phi^{(k)}(c) - \phi^{(k)}(x)| && \textit{(since } \phi^{(k)}(c) = 0\textit{)} \\
&= \left| \int_x^c \phi^{(k+1)}(y) dy \right| \\
&= \int_x^c \left| \phi^{(k+1)}(y) \right| dy \\
&\geq \int_x^c dy && \textit{(since } |\phi^{(k+1)}| \geq 1 \textit{ on } [a, b]\textit{)} \\
&= (c - x) \\
&= \sigma,
\end{aligned}$$

i.e. $|\phi^{(k)}(x)| \geq \sigma$ for all $x \in [a, c-\sigma]$.

Similarly, we can prove $|\phi^{(k)}(x)| \geq \sigma$, for all $x \in [c+\sigma, b]$.

Hence, $\dfrac{|\phi^{(k)}(x)|}{\sigma} \geq 1$ for all $x \in [a,b] \setminus (c-\sigma, c+\sigma)$, and given that $\sigma \neq 0$,.

Let $\psi(x) = \dfrac{\phi(x)}{\sigma}$ for all $x \in [a,b]$.

Then $\psi^k(x) = \dfrac{\phi^{(k)}(x)}{\sigma} \geq 1$ for all $x \in [a,b]$.

By part (1) of the lemma, we conclude there exists $c_k > 0$ $s.t$ c_k is independent of ψ and

$$\begin{cases} \left| \displaystyle\int_a^{c-\sigma} e^{i\lambda\psi(x)} dx \right| \leq c_k \lambda^{-1/k}, \\ \\ \left| \displaystyle\int_{c+\sigma}^{b} e^{i\lambda\psi(x)} dx \right| \leq c_k \lambda^{-1/k}. \end{cases}$$

Then

$$\begin{cases} e^{\frac{1}{\sigma}} \left| \displaystyle\int_a^{c-\sigma} e^{i\lambda\phi(x)} dx \right| \leq c_k \lambda^{-1/k}, \\ \\ e^{\frac{1}{\sigma}} \left| \displaystyle\int_{c+\sigma}^{b} e^{i\lambda\phi(x)} dx \right| \leq c_k \lambda^{-1/k}. \end{cases}$$

Note $\sigma < 1$, which implies $\ln(\sigma) < 0 < \dfrac{k}{\sigma}$ and therefore $\dfrac{-1}{k} \ln(\sigma) > \dfrac{-1}{\sigma}$.

Hence $e^{-1/\sigma} \leq \sigma^{-1/k}$. This implies the following:

$$\begin{cases} \left| \displaystyle\int_a^{c-\sigma} e^{i\lambda\phi(x)} dx \right| \leq c_k (\lambda\sigma)^{-1/k}, \\ \\ \left| \displaystyle\int_{c+\sigma}^{b} e^{i\lambda\phi(x)} dx \right| \leq c_k (\lambda\sigma)^{-1/k}. \end{cases} \tag{5}$$

Now, let $x \in (c-\sigma, c+\sigma)$.

Then

$$\begin{aligned}\left|\int_{c-\sigma}^{c+\sigma} e^{i\lambda\phi(x)}dx\right| &\leq \int_{c-\sigma}^{c+\sigma} |e^{i\lambda\phi(x)}|dx \\ &= \int_{c-\sigma}^{c+\sigma} dx \\ &= 2\sigma,\end{aligned}$$

i.e.

$$\left|\int_{c-\sigma}^{c+\sigma} e^{i\lambda\phi(x)}dx\right| \leq 2\sigma. \tag{6}$$

Combining 5 and 6, we conclude

$$\left|\int_a^b e^{i\lambda\phi(x)}dx\right| \leq 2c_k(\sigma\lambda)^{-1/k} + 2\sigma.$$

Let $\sigma = \lambda^{-1/(k+1)} < 1$. Then

$$\left|\int_a^b e^{i\lambda\phi(x)}dx\right| \leq 2(c_k+1)\lambda^{-1/(k+1)}.$$

Therefore, the proposition holds when $\phi^{(k)}(c) = 0$.

case 2: $\phi^{(k)}(c) \neq 0$ for all $x \in [a,b]$.

Let $|\phi^{(k)}(c)| = \epsilon$.
Then $\dfrac{|\phi^{(k)}(x)|}{\epsilon} \geq 1$ for all $x \in [a,b]$.
Let $\Theta(x) = \dfrac{\phi(x)}{\epsilon}$ for all $x \in [a,b]$.
Then $\Theta^{(k)}(x) = \dfrac{\phi^{(k)}(x)}{\epsilon}$ for all $x \in [a,b]$.
By part (1) of the lemma, we conclude there exists $c_k > 0$ such that c_k is independent

of Θ, and

$$\left|\int_a^b e^{i\lambda\Theta(x)}dx\right| \leq c_k\lambda^{-1/k}.$$

This implies

$$\left|e^{\frac{1}{\epsilon}}\int_a^b e^{i\lambda\phi(x)}dx\right| \leq c_k\lambda^{-1/k}.$$

Therefore

$$\begin{aligned}\left|\int_a^b e^{i\lambda\phi(x)}dx\right| &\leq c_k\lambda^{-1/k}e^{\frac{-1}{\epsilon}}\\ &\leq c_k\lambda^{-1/(k+1)}e^{\frac{-1}{\epsilon}}\\ &\leq 2(c_k+1)\lambda^{-1/(k+1)}, \quad \text{since } \frac{-1}{\epsilon} < 0 \text{ implies } e^{\frac{-1}{\epsilon}} < 1 < 2(1+\frac{1}{c_k}),\\ &\qquad\qquad\qquad\qquad \text{and thereby } c_k e^{\frac{-1}{\epsilon}} < 2(c_k+1).\end{aligned}$$

Therefore, the proposition also holds when $\phi^{(k)}(c) \neq 0$.

We hence conclude that if the result of the lemma is true for some $k \in \mathbb{N}$ and $|\phi^{(k+1)}(x)| \geq 1$ on $[a,b]$. Then $\left|\int_a^b e^{i\lambda\phi(x)}dx\right| \leq 2(c_k+1)\lambda^{-1/(k+1)}$. □

The proof of the second part of the lemma follows directly from the above proposition. Using induction with base case $k = 2$, and using part 1 of the lemma and the proposition with $k = 1$, we conclude

$$\left|\int_a^b e^{i\lambda\phi(x)}dx\right| \leq 2(c_1+1)\lambda^{-1/2}.$$

It suffices then to take $c_2 = 2(c_1+1) > 0$ to validate the base case.

Similarly, assuming the result of the lemma is verified for $k \in \mathbb{N}$ such that $k \geq 2$, the inductive step follows from the proposition by taking $c_{k+1} = 2(c_k + 1) > 0$. $\square$

Chapter 4

Hörmander's Theorem and Ma's extension

In this section, we present Hörmander's Theorem, which is the focus of the boundedness study undertaken by this thesis. To that end, we shall review the definition of a Hessian matrix. We recall a **Hessian matrix** is a square matrix of second order partial derivatives of a function. It is defined as follows:

Definition 4.1. Let f be a real-valued function defined on $\mathbb{R}^n$.

Then the Hessian matrix of f is given by:

$$H_{i,j} = \frac{\partial^2 f(x_1, .., x_n)}{\partial x_i \partial x_j} : \text{ with } 1 \leq i\,, j \leq n.$$

We apply this definition in the following example.

Example:

Consider the function $f(x, y) = x^3 + y^2$ for all $(x, y) \in \mathbb{R}^2$.

The Hessian matrix of f is given by:

$$\begin{bmatrix} \dfrac{\partial^2 f(x,y)}{\partial x \partial x} & \dfrac{\partial^2 f(x,y)}{\partial x \partial y} \\ \\ \dfrac{\partial^2 f(x,y)}{\partial y \partial x} & \dfrac{\partial^2 f(x,y)}{\partial y \partial y} \end{bmatrix} = \begin{bmatrix} 6x & 0 \\ 0 & 2 \end{bmatrix}.$$

Now, let us present the statement of Hörmander's theorem.

Theorem 4.1. *Consider the oscillatory integral operator T defined as follows:*

$$(T_\lambda f)(x) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x,y)}\psi(x,y)f(y)dy \tag{1}$$

such that $x \in \mathbb{R}^n$.

Suppose the mixed Hessian of ϕ is non-vanishing, i.e.

$$det\left(\frac{\partial^2\phi(x,y)}{\partial x_i \partial y_j}\right) \neq 0 \text{ with } 1 \leq i\,, j \leq n.$$

Then, we have the following estimate on the L^2 norm of the operator T_λ:

$$\|T_\lambda(f)\|_{L^2(\mathbb{R}^n)} \leq C\lambda^{-n/2}\|f\|_{L^2(\mathbb{R}^n)}.$$

Note that Hörmander restricted the bound stated above to oscillatory integral operators with non-degenerate phase functions (*i.e.* phase functions whose corresponding Hessian matrix has a nonzero determinant). However, we shall extend his result to operators with degenerate phase functions.

Let us look at an improvement suggested in [3].

We shall first clarify that the notation ϕ''_{xy} is equivalent to the mixed Hessian function $\partial_y\partial_x\phi(x,y)$, and that a set $S \subset \mathbb{R}^2$ is said to be *horizontally connected* if for all $c \in \mathbb{R}^1$, $\{(x,y)|y=c\} \cap S^c$ has at most two connected components.

Theorem 4.2. ***Ma's extension.***

For an oscillatory integral operator T_λ as in (1) with $x, y \in \mathbb{R}^1$, suppose that there exists a function $\Gamma(x,y) \in C^\infty(\mathbb{R}^2 \setminus \{xy = 0\})$ such that the mixed Hessian of its phase function $\phi(x,y)$ satisfies a degeneracy condition on the support of ψ as follows:

$$C_1 |xy\Gamma^2(x,y)|^\alpha \le |\phi''_{xy}(x,y)|, \tag{2}$$

$$|y\Gamma(x,y)|^l |\partial_y^l \phi''_{xy}(x,y)| \le C_2 |\phi''_{xy}(x,y)|, \tag{3}$$

where α, C_1, C_2 are positive constants and $l = 1, 2$. $\Gamma^2(x,y)$ in (3) denotes the square of the function $\Gamma^2(x,y)$.

The function Γ satisfies the following condition on the support of ψ except on the coordinate axes:

$$A_1|x-y| \le |x||\Gamma(x,z) - \Gamma(y,z)| \le A_2|x-y| \tag{4}$$

with A_1, A_2 being positive constants. The above inequality still holds if we substitute $\Gamma(z,x)$ and $\Gamma(z,y)$ for $\Gamma(x,z)$ and $\Gamma(y,z)$ respectively.

Also suppose that the support of ψ can be divided into finite parts such that each part is horizontally connected on each part we have either $\phi''_{xy} \ge 0$ or $\phi''_{xy} \le 0$.

Then we have the following estimate of the L^2 norm of the operator T_λ :

$$\|T_\lambda(f)\|_{L^2(\mathbb{R}^1)} \le C\lambda^{-1/[2(\alpha+1)]}\|f\|_{L^2(\mathbb{R}^1)} \tag{5}$$

where the constant C is independent of λ and α.

Note that this theorem extends Hörmander in the case of $x, y \in \mathbb{R}^1$ by eliminating the non-degeneracy condition related to the determinant of the Hessian matrix of the phase function. This is achieved by decomposing the support of T_λ.

Ma's extension covers Hörmander's non-degeneracy case when $n = 1$ and $\alpha = 0$ in (2) as the inequality becomes $C_1 \leq |\phi''_{xy}(x, y)|$ for $C_1 > 0$, implying the mixed Hessian is nonzero and therefore non-vanishing. Furthermore, for the same value of α, the result of the theorem (5) becomes the result of Hörmander when $x \in \mathbb{R}^1$.

Now, let us look at a corollary of Ma's theorem.

Corollary 4.3. *[3] For an oscillatory integral operator T_λ as in (1) with $x, y \in \mathbb{R}^1$, suppose that the mixed Hessian of its phase function $\phi(x, y)$ satisfies the following degeneracy condition on the support of ψ:*

$$C_1|xy|^\alpha \leq |\phi''_{xy}(x, y)|, \ |y^l \partial^l_y \phi''_{xy}(x, y)| \leq C_2|\phi''_{xy}(x, y)|, \tag{5}$$

where α, C_1, C_2 are positive constants and $l = 1, 2$.

Also suppose that the support of ψ can be divided into finite parts such that each part is horizontally connected and on each part we have either $\phi''_{xy} \geq 0$ or $\phi''_{xy} \leq 0$.

Then we have the following estimate of the L^2 norm of the operator T_λ :

$$\|T_\lambda(f)\|_{L^2(\mathbb{R}^1)} \leq C\lambda^{-1/[2(\alpha+1)]}\|f\|_{L^2(\mathbb{R}^1)} \tag{6}$$

where the constant C is independent of λ and α.

Note that when $|\Gamma(x, y)| \geq c > 0$ for $c \in \mathbb{R}$ on the compact support of ψ, the conditions (2) and (3) of Theorem 4.2 reduce to condition (5) of the corollary [3].

We shall present an example that applies Ma's extension and its corresponding corollary taken from the same paper.

Example: Consider the oscillatory integral operator T_λ such that

$$T_\lambda(f)(x) = \int_{-\infty}^{\infty} e^{-i\lambda(y-sinx)^{n+2}} \psi(x,y)f(y)dy, n \in \mathbb{N} \cup \{0\}, x \in \mathbb{R}^1,$$

with the support of ψ being the unit disk $\{(x,y)|x^2+y^2 \le 1\}$.

Then, the phase function is given by $\phi(x,y) = -(y-sinx)^{n+2}$, and its mixed Hessian by

$$\begin{aligned} \phi''_{xy}(x,y) &= \partial_y \partial_x \phi(x,y) \\ &= \partial_y[-(n+2)(-cosx)(y-sinx)^{n+2}] \\ &= (n+2)(n+1)(y-sinx)^n cosx. \end{aligned}$$

Therefore, the singularity lies in the neighborhood of the critical points $y = sinx$. To resolve it, we define the function $\Gamma(x,y) = \frac{y}{sinx} - 1 \in C^\infty(\mathbb{R}^2 \setminus \{x=0\})$.

In order to find the decay rate of the operator T_λ, we shall use Theorem 4.2 on the domain $D = \{(x,y) : |\Gamma(x,y)| \le \frac{1}{2}\}$ and its corollary on D^c, as outlined in [3].

Let $E = \{(x,y)|x^2+y^2 \le 1\} \cap D$. One can verify the following inequalities on E:

$$|\phi''_{xy}(x,y)| > \frac{(n+2)(n+1)}{2}|\Gamma(x,y)sinx|^n > \frac{(n+2)(n+1)}{2(\sqrt{3})^n}|xy\Gamma^2(x,y)|^{n/2} \tag{7}$$

$$|y\Gamma\partial_y\phi''_{xy}(x,y)| \le \frac{3n}{2}|\phi''_{xy}(x,y)|, \;\; |(y\Gamma)^2\partial_y^2\phi''_{xy}(x,y)| \le \frac{9n(n-1)}{4}|\phi''_{xy}(x,y)|, \tag{8}$$

$$\frac{1}{2}|x-y| \le |x||\Gamma(z,x)-\Gamma(z,y)| = \left|\frac{x}{sinz}\right||x-y| \le \frac{3}{2}|x-y|, \tag{9}$$

$$\frac{1}{8}|x-y| \le |x||\Gamma(x,z)-\Gamma(y,z)| = 2|xz||sinx\, siny|^{-1}|cos\frac{x+y}{2}sin\frac{x-y}{2}| \le 3|x-y|. \tag{10}$$

Note that the inequality (7) satisfies condition 2 of Theorem 4.2, the inequalities in (8) satisfy condition 3 for $l = 1$ and $l = 2$, and the results in (9) and (10) satisfy condition 4 of the theorem.

Also, we know that when n is even, $\phi''_{xy}(x, y) \geq 0$ on E which is horizontally connected, and when n is odd, E can be divided into two parts such that each is horizontally connected and on each part we have either $\phi''_{xy}(x, y) \geq 0$ or $\phi''_{xy}(x, y) \leq 0$ [3]. Then, we can use Theorem 4.2 with $\alpha = n/2$ to conclude the decay rate of the operator T_λ for all $(x, y) \in E$ is

$$\|T_\lambda(f)\|_{L^2(\mathbb{R}^1)} \leq C\lambda^{-1/(n+2)}\|f\|_{L^2(\mathbb{R}^1)}.$$

Now, we shall use the corollary to Theorem 4.2 in order to verify the same decay rate on $E' = \{(x, y)|x^2 + y^2 \leq 1\} \cap D^c$.

We have $D = \{(x, y) : |\Gamma(x, y)| \leq \frac{1}{2}\}$ implies $D^c = \{(x, y) : |\Gamma(x, y)| > \frac{1}{2}\}$.

From the second part of (7) we have

$$\begin{aligned}
|\phi''_{xy}(x, y)| &> \frac{(n+2)(n+1)}{2(\sqrt{3})^n}|xy\Gamma^2(x, y)|^{n/2} \\
&> \frac{(n+2)(n+1)}{2(\sqrt{3})^n}|xy|^{n/2}|\Gamma(x, y)|^n \\
&> \frac{(n+2)(n+1)}{2(\sqrt{3})^n}|xy|^{n/2}\frac{1}{2}^n \qquad \textit{(since } |\Gamma| > 1/2 \textit{ on } D^c\textit{)} \\
&> \frac{(n+2)(n+1)}{2(2\sqrt{3})^n}|xy|^{n/2},
\end{aligned}$$

i.e. for $\alpha = n/2$ and $C_1 = \dfrac{(n+2)(n+1)}{2(2\sqrt{3})^n}|xy|^{n/2}$, we have $|\phi''_{xy}(x, y)| \geq C_1|xy|^\alpha$ thus satisfying the first part of condition (5) of the corollary.

Similarly, from the first part of (8) we have $|y\Gamma\partial_y\phi''_{xy}(x, y)| \leq \dfrac{3n}{2}|\phi''_{xy}(x, y)|$ implying $|y\partial_y\phi''_{xy}(x, y)| \leq \dfrac{3n}{2}\left|\dfrac{1}{\Gamma}\right||\phi''_{xy}(x, y)|$. Since $|\Gamma| > \frac{1}{2}$ implies $\left|\dfrac{1}{\Gamma}\right| \leq 2$, we conclude $|y\partial_y\phi''_{xy}(x, y)| \leq 3n|\phi''_{xy}(x, y)|$, which satisfies the second part of condition (5) of the corollary for $l = 1$ and $C_2 = 3n$.

Finally, from the second part of (8), we have $|(y\Gamma)^2\partial_y^2\phi''_{xy}(x,y)| \leq \dfrac{9n(n-1)}{4}|\phi''_{xy}(x,y)|$ implying $|y^2\partial_y^2\phi''_{xy}(x,y)| \leq \left|\dfrac{1}{\Gamma^2}\right|\dfrac{9n(n-1)}{4}|\phi''_{xy}(x,y)|$. Since $|\Gamma| > \frac{1}{2}$ implies $\left|\dfrac{1}{\Gamma^2}\right| \leq 4$, we conclude $|y^2\partial_y^2\phi''_{xy}(x,y)| \leq 9n(n-1)|\phi''_{xy}(x,y)|$, which satisfies the the second part of condition (5) of the corollary for $l = 2$ and $C_2 = 9n(n-1)$.

Therefore, by the corollary to Ma's extension, we conclude the decay rate of the operator T_λ for $\alpha = n/2$ and for all $(x,y) \in E'$ is

$$\|T_\lambda(f)\|_{L^2(\mathbb{R}^1)} \leq C\lambda^{-1/(n+2)}\|f\|_{L^2(\mathbb{R}^1)}.$$

Therefore, the same decay rate is found on both E and E'. We hence conclude it to be the decay rate of the operator on the entire domain.

Chapter 5

Generalization of Hörmander's Theorem to Operators with a Specific Phase Function

In this chapter, we shall present and prove a new theorem that generalizes the result from Hörmander's theorem to oscillatory integral operators with phase functions of the form $\phi(x,y) = |y - \eta(x)|^\alpha$.

We state the new theorem as follows:

Theorem 5.1. *Let $\eta \in C^\infty(\mathbb{R})$ and $\psi \in C_0^\infty(\mathbb{R}^2)$ such that:*

1. *$supp(\psi) \subseteq [a,b] \times [c,d]$ and*

2. *$\eta'(x) \neq 0$ for all $x \in [a,b]$*

where $a,b,c,d \in \mathbb{R}$ such that $a < b$ and $c < d$.

Let $\alpha > 1$ be an integer. Define T_λ by

$$T_\lambda(f)(x) = \int_{\mathbb{R}} e^{i\lambda|y-\eta(x)|^\alpha} \psi(x,y) f(y)dy \tag{1}$$

for $\lambda \in \mathbb{R}$. Then there exists a constant $C = C_\alpha$ independent of λ and f such that

$$\|T_\lambda(f)\|_{L^2(\mathbb{R}^1)} \leq C|\lambda|^{-1/\alpha}\|f\|_{L^2(\mathbb{R}^1)}. \tag{2}$$

Remark. We can reproduce the example from Ma's paper [3] by applying the theorem above.

We have

$$T_\lambda(f)(x) = \int_{-\infty}^{\infty} e^{-i\lambda(y-sinx)^{n+2}}\psi(x,y)f(y)dy, n \in \mathbb{N} \cup \{0\}, x \in \mathbb{R}^1,$$

with $supp(\psi) = \{(x,y)|x^2+y^2 \leq 1\}$.

Then $(x,y) \in supp(\psi)$ implies $x^2+y^2 \leq 1$, which further implies $|x| \leq 1$ and $|y| \leq 1$, *i.e.* $(x,y) \in ([-1,1])^2$.

Therefore $supp(\psi) \subseteq [-1,1] \times [-1,1]$, thereby satisfying condition (i) of the theorem for $a = c = -1$ and $b = d = 1$.

Let $\eta(x) = sinx$ and $\alpha = n+2 > 1$ since $n \in \mathbb{N} \cup \{0\}$, then for all $x \in [-1,1]$, we have $\eta'(x) = cosx \neq 0$ since $x \in [-1,1]$. Hence, condition (ii) of the theorem is also satisfied.

Using the theorem, we can therefore conclude the following bound

$$\|T_\lambda(f)\|_{L^2(\mathbb{R}^1)} \leq C|\lambda|^{-1/\alpha}\|f\|_{L^2(\mathbb{R}^1)} \text{ where } \alpha = n+2$$

i.e.

$$\|T_\lambda(f)\|_{L^2(\mathbb{R}^1)} \leq C|\lambda|^{-1/(n+2)}\|f\|_{L^2(\mathbb{R}^1)},$$

which is the same result as applying Ma's theorem.

5.1 Background Information

In this section, we compile notions and results we will need in the proof of Theorem 5.1. We start by presenting the following definitions:

Definition 5.1. [1] **The 1-dimensional Fourier Transform and Inverse.**
If f is a continuously differentiable function with $\int_{-\infty}^{\infty} |f(t)dt < \infty$, then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda x} d\lambda, \tag{3}$$

where $\hat{f}(\lambda)$ is the Fourier transform of f given by

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\lambda t} dt.$$

We can also write $\mathscr{F}[f](\lambda)$ to denote the Fourier transform.
Similarly, we define the inverse Fourier transform operator as

$$\mathscr{F}^{-1}[f](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\lambda) e^{i\lambda x} d\lambda.$$

Note $\mathscr{F}^{-1}[\mathscr{F}[f]](x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda x} d\lambda = f(x)$ by (3).
The following example illustrates the computation of a 1-dimensional Fourier transform.

Example. Let us find the Fourier transform of $f_s(x) = \sqrt{s} e^{-sx^2}$.
We know $\hat{f}_s(\lambda) = \frac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f_s(t) e^{-i\lambda t} dt$.

Then

$$
\begin{aligned}
\hat{f}_s(\lambda) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{s} e^{-st^2} e^{-i\lambda t} dt \\
&= \left(\frac{s}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-st^2 - i\lambda t} dt \\
&= \left(\frac{s}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-[(\sqrt{s}t)^2 + 2(\sqrt{s}t)(\frac{1}{2\sqrt{s}} i\lambda) - \frac{\lambda^2}{4s} + \frac{\lambda^2}{4s}]} dt \\
&= \left(\frac{s}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-(\sqrt{s}t + \frac{\lambda}{2\sqrt{s}})^2} e^{-\frac{\lambda^2}{4s}} dt \\
&= \left(\frac{s}{2\pi}\right)^{1/2} e^{-\frac{\lambda^2}{4s}} \int_{-\infty}^{\infty} e^{-(\sqrt{s}t + \frac{\lambda}{2\sqrt{s}})^2} dt \\
&= \left(\frac{s}{2\pi}\right)^{1/2} e^{-\frac{\lambda^2}{4s}} \int_{-\infty}^{\infty} e^{-(\sqrt{s}t + \frac{\lambda}{2\sqrt{s}})^2} \frac{\sqrt{s}}{\sqrt{s}} dt \\
&= \left(\frac{s}{2\pi}\right)^{1/2} e^{-\frac{\lambda^2}{4s}} \int_{-\infty}^{\infty} e^{-x^2} \frac{1}{\sqrt{s}} dx \qquad \textit{(for } x = \sqrt{s}t + \frac{\lambda}{2\sqrt{s}} \textit{ implies } dx = \sqrt{s}\, dt\textit{)} \\
&= \left(\frac{1}{2\pi}\right)^{1/2} e^{-\frac{\lambda^2}{4s}} \int_{-\infty}^{\infty} e^{-x^2} dx \\
&= \left(\frac{1}{2\pi}\right)^{1/2} e^{-\frac{\lambda^2}{4s}} \sqrt{\pi} \qquad \textit{(since } \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}\textit{)} \\
&= \left(\frac{1}{2}\right)^{1/2} e^{-\frac{\lambda^2}{4s}}.
\end{aligned}
$$

Therefore, the Fourier transform of f_s is given by $\hat{f}_s(t) = \left(\frac{1}{2}\right)^{1/2} e^{-\frac{\lambda^2}{4s}}$.

For the purposes of this paper, we shall study the extension of these Fourier notions to the 2-dimensional space. We provide below the general definition of the $n-$dimensional Fourier transform, which drops the $\frac{1}{\sqrt{2\pi}}$ in order to emphasize the unitary property of the transform.

Definition 5.2. The $n-$dimensional Fourier Transform and Inverse.

Let $n \in \mathbb{N}$ and let f be a real- or complex-valued function defined on $\mathbb{R}^n$ such that $f(\mathbf{x}) = f(x_1, ..., x_n)$.

The Fourier transform of $f(\mathbf{x})$ is given by

$$\mathscr{F}[f](\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \mathbf{x}\cdot\xi} f(\mathbf{x}) d\mathbf{x},$$

and its inverse by

$$\mathscr{F}^{-1}[f](\mathbf{x}) = \int_{\mathbb{R}^n} e^{2\pi i \mathbf{x}\cdot\xi} f(\xi) d\xi,$$

where $\xi \in \mathbb{R}^n$ and $\mathbf{x}\cdot\xi$ denotes the dot product of both vectors.

Therefore, the 2-dimensional Fourier Transform is defined as follows for $\mathbf{x}, \xi \in \mathbb{R}^2$ and f defined on $\mathbb{R}^2$:

$$\mathscr{F}[f](\xi_1, \xi_2) = \int_{\mathbb{R}^n} e^{-2\pi i(x_1\xi_1 + x_2\xi_2)} f(x_1, x_2) dx_1 dx_2,$$

and its inverse:

$$\mathscr{F}^{-1}[f](x_1, x_2) = \int_{\mathbb{R}^n} e^{2\pi i(x_1\xi_1 + x_2\xi_2)} f(\xi_1, \xi_2) d\xi_1 d\xi_2.$$

The proof of Theroem 5.1 will use Van Der Corput's lemma from Chapter 3 and the following theorems:

Theorem 5.2. ***Plancherel's Identity*** *(also known as Rayleigh's theorem or Parseval's formula).*
If $f \in L^2(\mathbb{R})$ then $\hat{f} \in L^2(\mathbb{R})$ and $\|f\|_{L^2(\mathbb{R})} = \|\hat{f}\|_{L^2(\mathbb{R})}$.

Theorem 5.3. *[6]* ***Fubini's Theorem.***
Let $f(x,y)$ be a measurable function defined on a measurable subset $E \subseteq \mathbb{R}^{n+m}$. Then,

i. For almost every $x \in \mathbb{R}^n$,$f(x,y)$ is a measurable function of y on $E_x = \{y \in \mathbb{R}^m : (x,y) \in E\}$.

ii. *For almost every* $y \in \mathbb{R}^m$,$f(x,y)$ *is a measurable function of* y *on* $E^y = \{x \in \mathbb{R}^n : (x,y) \in E\}$.

iii. *If* $f \in L(E)$, *then for almost every* $x \in \mathbb{R}^n$, $f(x,\cdot) \in L(E_x)$. *Moreover,* $\int_{E_x} f(x,y)dy$ *is an integrable function of* x *and* $\int_E f = \int_{\mathbb{R}^n} \int_{E_x} f(x,y)dydx$.

iv. *If* $f \in L(E)$, *then for almost every* $y \in \mathbb{R}^m$, $f(\cdot,y) \in L(E^y)$. *Moreover,* $\int_{E^y} f(x,y)dx$ *is an integrable function of* y *and* $\int_E f = \int_{\mathbb{R}^m} \int_{E^y} f(x,y)dxdy$.

Theorem 5.4. *[2]* ***Minkowski's Integral Inequality****. Let* $1 \leq p < \infty$. *Let* F *be a measurable function on the product space* $(X,\mu) \times (T,\nu)$. *Then*

$$\left[\int_T \left(\int_X |F(x,t)|d\mu\right)^p d\nu\right]^{\frac{1}{p}} \leq \int_X \left[\int_T |F(x,t)|^p d\nu\right]^{\frac{1}{p}} d\mu.$$

5.2 Proof of Theorem 5.1

In this section, we shall apply the previous results in order to prove the statement of Theorem 5.1. Note that we implicitly use the equivalence of the Riemann integral and the Lebesgue integral.

I. Let $\eta \in C^\infty(\mathbb{R})$ and $\psi \in C_0^\infty(\mathbb{R}^2)$ such that

   (a) $supp(\psi) \subseteq [a,b] \times [c,d]$ and

   (b) $\eta'(x) \neq 0$ for all $x \in [a,b]$

   where $a,b,c,d \in \mathbb{R}$ such that $a < b$ and $c < d$.

   Since $\eta \in C^\infty(\mathbb{R})$, η' is continuous on $\mathbb{R}$.

   By assumption (b), $\eta'(x) \neq 0$ for all $x \in [a,b]$. Since $[a,b]$ is a closed and bounded set in $\mathbb{R}$, it is compact.

Thus there exists $\nu > 0$ such that $|\eta'(x)| \geq \nu$ for all $x \in [a, b]$.

By continuity, there exist $\delta, \beta \in \mathbb{R}$ such that $[a, b] \subseteq (\delta, \beta)$ and $\eta'(x) \neq 0$ for all $x \in [\delta, \beta]$.

Then $\eta : [\delta, \beta] \to \eta([\delta, \beta])$ is a bijection. Thus $\eta^{-1} : \eta([\delta, \beta]) \to [\delta, \beta]$ is also in $C^{\infty}(\mathbb{R})$.

Let $\eta([\delta, \beta]) = J$. Then for all $t \in J$, we have $\eta^{-1}(t) \in [\delta, \beta]$ and

$T_{\lambda}(f)(\eta^{-1}(t)) = \int_{\mathbb{R}} e^{i\lambda|y-\eta(\eta^{-1}(t))|^{\alpha}} \psi(\eta^{-1}(t), y) f(y) dy.$

Note that when $x = \eta^{-1}(t)$, $\eta(x) = t$.

Let $h \in C_0^{\infty}(\mathbb{R})$ such that $h(t) = \begin{cases} 1 & \text{if } t \in \eta([\delta, \beta]), \\ 0 & \text{if } t \notin \eta([\delta, \beta]), \end{cases}$.

Let $F_{\lambda}(t) = h(t) \int_{\mathbb{R}} e^{i\lambda(y-t)^{\alpha}} \psi(\eta^{-1}(t), y) f(y) dy.$

Then $F_{\lambda}(t) = \int_{\mathbb{R}} e^{i\lambda(y-t)^{\alpha}} Q(t, y) f(y) dy$ for $Q(t, y) = h(t)\psi(\eta^{-1}(t), y) \in C_0^{\infty}(\mathbb{R}^2)$.

II. Let $M > 0$ such that $supp(Q) \subseteq [-M, M] \times [-M, M]$.

Let w be a function such that $w \in C_0^{\infty}(\mathbb{R})$ such that $w(x) = 1$

for all $x \in [-2M, 2M]$. (4)

By Definition 5.2, we get $Q(t, y) = \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{Q}(u, v) e^{2\pi i(tu+yv)} du\, dv$, where $\widehat{Q}$ is a $2-$dimensional Fourier Transform. (5)

III. Let us show $|F_{\lambda}(t)| \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |G_{\lambda}(t)||\widehat{Q}(u, v)| du\, dv$ where

$G_{\lambda}(t) = \int_{\mathbb{R}} e^{i\lambda(y-t)^{\alpha}} w(y - t) f_v(y) dy$ such that $f_v(y) = e^{2\pi i(vy)} f(y)$.

We have

$$\begin{aligned}
F_\lambda(t) &= \int_{\mathbb{R}} e^{i\lambda(y-t)^\alpha} Q(t,y) f(y) dy \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda(y-t)^\alpha} \widehat{Q}(u,v) e^{2\pi i(tu+yv)}\, du\, dv\, f(y)\, dy \qquad \text{(by 5)} \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda(y-t)^\alpha} \widehat{Q}(u,v) e^{2\pi i(tu+yv)}\, f(y)\, du\, dv\, dy,
\end{aligned}$$

i.e. $F_\lambda(t) = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda(y-t)^\alpha} \widehat{Q}(u,v) e^{2\pi i(tu+yv)}\, f(y)\, du\, dv\, dy.$

Let $U = (u,v) \in \mathbb{R}^2$, and define $g(U,y) = e^{i\lambda(y-t)^\alpha} \widehat{Q}(u,v) e^{2\pi i(tu+yv)}\, f(y)$.

One can verify $g \in L(\mathbb{R}^3)$, and for almost every $y \in \mathbb{R}$, $g(U,y)$ is a measurable function of y on $\mathbb{R}^2$.

Therefore, by Fubini's theorem $(ii), (iii)$ and (iv), we conclude

$$\begin{aligned}
F_\lambda(t) &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda(y-t)^\alpha} \widehat{Q}(u,v) e^{2\pi i(tu+yv)}\, f(y)\, dy\, du\, dv \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda(y-t)^\alpha} e^{2\pi i(vy)} f(y)\, dy\, \widehat{Q}(u,v) e^{2\pi i(tu)}\, du\, dv,
\end{aligned}$$

i.e. $F_\lambda(t) = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda(y-t)^\alpha} e^{2\pi i(vy)} f(y)\, dy\, \widehat{Q}(u,v) e^{2\pi i(tu)}\, du\, dv.$

Note that $(t,y) \in supp(Q)$ implies $y, t \in [-M, M]$. Then $y - t \in [-2M, 2M]$.

By (4), we conclude $w(y-t) = 1$. (*)

We can then write $F_\lambda(t) = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda(y-t)^\alpha} w(y-t) e^{2\pi i(vy)} f(y)\, dy\, \widehat{Q}(u,v) e^{2\pi i(tu)}\, du\, dv.$

Using the definition of $G_\lambda(t)$, we conclude $F_\lambda(t) = \int_{\mathbb{R}}\int_{\mathbb{R}} G_\lambda(t)\, \widehat{Q}(u,v) e^{2\pi i(tu)}\, du\, dv.$

Then

$$\begin{aligned}
|F_\lambda(t)| &\leq \left| \int_{\mathbb{R}} \int_{\mathbb{R}} G_\lambda(t)\, \widehat{Q}(u,v) e^{2\pi i(tu)}\ du\, dv \right| \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}} \left| G_\lambda(t)\, \widehat{Q}(u,v) e^{2\pi i(tu)} \right| du\, dv \\
&\qquad\qquad\qquad \text{(by the absolute value property of integrals)} \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |G_\lambda(t)|\, |\widehat{Q}(u,v)| |e^{2\pi i(tu)}|\ du\, dv \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |G_\lambda(t)|\, |\widehat{Q}(u,v)|\ du\, dv, \qquad \text{(since } |e^{i\theta}| = 1 \text{ for all } \theta \in \mathbb{R})
\end{aligned}$$

i.e. $|F_\lambda(t)| \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |G_\lambda(t)|\, |\widehat{Q}(u,v)|\ du\, dv$ as desired.

IV. Let us prove $\widehat{G}_\lambda(x) = \hat{f}_v(x) \int_{-2M}^{2M} e^{i(2\pi xu + \lambda u^\alpha)} du$.

By definition, we have $\widehat{G}_\lambda(x) = \int_{\mathbb{R}} G_\lambda(t) e^{-2\pi ixt} dt = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\lambda(y-t)^\alpha} w(y-t) f_v(y)\, dy\, e^{-2\pi ixt} dt$.

By $(*)$ and since $e^{-2\pi ixt}$ is independent of y, we get $\widehat{G}_\lambda(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\lambda(y-t)^\alpha} f_v(y) e^{-2\pi ixt} dy\, dt$.

Let $u = y - t$. Then $t = y - u$.

Using variable substitution, we get $\widehat{G}_\lambda(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\lambda u^\alpha} f_v(y) e^{-2\pi ix(y-u)} |\mathcal{J}(u,y)| du\, dy$,

where $|\mathcal{J}(u,y)| = \begin{vmatrix} \frac{\delta y}{\delta u} & \frac{\delta y}{\delta y} \\ \frac{\delta t}{\delta u} & \frac{\delta t}{\delta y} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} = 1$.

Then

$$\begin{aligned}
\widehat{G}_\lambda(x) &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\lambda u^\alpha} f_v(y) e^{-2\pi ix(y-u)} du\, dy \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\lambda u^\alpha} f_v(y) e^{-2\pi ixy + 2\pi ixu} du\, dy \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f_v(y) e^{-2\pi ixy} dy\, e^{i(\lambda u^\alpha + 2\pi xu)} du \\
&= \int_{\mathbb{R}} \hat{f}_v(x)\, e^{i(\lambda u^\alpha + 2\pi xu)} du,
\end{aligned}$$

$i.e.\ \widehat{G}_\lambda(x) = \hat{f}_v(x) \int_{\mathbb{R}} e^{i(\lambda u^\alpha + 2\pi x u)} du.$
By $(*)$, $u = y - t \in [-2M, 2M]$.
Then $\widehat{G}_\lambda(x) = \hat{f}_v(x) \int_{-2M}^{2M} e^{i(\lambda u^\alpha + 2\pi x u)} du.$

V. Now, let us prove $\left|\widehat{G}_\lambda(x)\right| \leq C_\alpha |\lambda|^{-\frac{1}{\alpha}} |\hat{f}_v(x)|$ for some $C_\alpha > 0$.

Define $\phi(u) = \frac{2\pi}{\lambda} x u + u^\alpha$.
Then $\widehat{G}_\lambda(x) = \hat{f}_v(x) \int_{-2M}^{2M} e^{i\lambda\phi(u)} du.$ $(\diamond)$
We shall use Van Der Corput's lemma. To that end, we need to show

$\phi \in C^\infty([-2M, 2M])$ and $|\phi^{(k)}(u)| \geq 1$ for all $u \in [-2M, 2M]$ whenever $k \geq 2$.

We have $\phi^{(1)}(u) = \frac{2\pi}{\lambda} x + \alpha u^{\alpha - 1}$.

Similarly, we have $\phi^{(2)}(u) = \alpha(\alpha - 1) u^{\alpha - 2}$.

Therefore, whenever $k \geq 2$, we get $\phi^{(k)}(u) = \alpha(\alpha - 2)...(\alpha - k + 1) u^{\alpha - k}$.

Hence $\phi \in C^\infty([-2M, 2M])$ as desired.

Note that $|\phi^{(\alpha)}(u)| = |\alpha(\alpha - 1)...1 \times u^{\alpha - \alpha}| \geq |u^0| = 1$ since $\alpha > 1$. Then $|\phi^{(\alpha)}| \geq 1$.

Using Van Der Corput's lemma, we conclude there exists $C_\alpha > 0$ and independent of ϕ such that $\left| \int_{-2M}^{2M} e^{i\lambda\phi(u)} du \right| \leq C_\alpha |\lambda|^{-\frac{1}{\alpha}}$.
Multiplying both sides of the inequality by $|\hat{f}_v(x)|$, we get
$\left| \hat{f}_v(x) \int_{-2M}^{2M} e^{i\lambda\phi(u)} du \right| \leq C_\alpha |\lambda|^{-\frac{1}{\alpha}} |\hat{f}_v(x)|.$
By $(\diamond)$, we conclude there exists $C_\alpha > 0$ such that $\left|\widehat{G}_\lambda(x)\right| \leq C_\alpha |\lambda|^{-\frac{1}{\alpha}} |\hat{f}_v(x)|$.

VI. We shall prove that $||F_\lambda(t)||_2 \leq D_\alpha |\lambda|^{-\frac{1}{\alpha}} ||f||_2$ for some constant D_α.

We have

$$|F_\lambda(t)| \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |G_\lambda(t)||\hat{Q}(u, v)| du dv \leq \left| \int_{\mathbb{R}} \int_{\mathbb{R}} |G_\lambda(t)||\hat{Q}(u, v)| du dv \right| \text{ since } a \leq |a| \text{ for all } a \in \mathbb{R}.$$

$$\text{Then } |F_\lambda(t)| \leq \left| \int_{\mathbb{R}} \int_{\mathbb{R}} |G_\lambda(t)||\hat{Q}(u, v)| du dv \right|.$$

Squaring, we obtain $|F_\lambda(t)|^2 \leq \left| \int_{\mathbb{R}} \int_{\mathbb{R}} |G_\lambda(t)||\hat{Q}(u,v)|dudv \right|^2$.

Integrating both sides with respect to t, we get $\displaystyle\int_{\mathbb{R}} |F_\lambda(t)|^2 dt \leq \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \int_{\mathbb{R}} |G_\lambda(t)||\hat{Q}(u,v)|dudv \right|^2 dt$.

Thus

$$\left[\int_{\mathbb{R}} |F_\lambda(t)|^2 dt \right]^{\frac{1}{2}} \leq \left[\int_{\mathbb{R}} \left| \int_{\mathbb{R}} \int_{\mathbb{R}} |G_\lambda(t)||\hat{Q}(u,v)|dudv \right|^2 dt \right]^{\frac{1}{2}}.$$

Then

$$\begin{aligned}
||F_\lambda(t)||_2 \leq & \left[\int_{\mathbb{R}} \left| \int_{\mathbb{R}} \int_{\mathbb{R}} |G_\lambda(t)||\hat{Q}(u,v)|dudv \right|^2 dt \right]^{\frac{1}{2}} \\
& \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |\ |G_\lambda(t)|\ |\hat{Q}(u,v)|\ |^2 dt \right]^{1/2} dudv \\
& \qquad \text{(using Minkowski's Integral Inequality)} \\
& = \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} [\ |G_\lambda(t)|\ |\hat{Q}(u,v)|\ ]^2 dt \right]^{\frac{1}{2}} dudv \\
& = \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |G_\lambda(t)|^2\ |\hat{Q}(u,v)|^2 dt \right]^{\frac{1}{2}} dudv \\
& = \int_{\mathbb{R}} \int_{\mathbb{R}} \left[|\hat{Q}(u,v)|^2 \int_{\mathbb{R}} |G_\lambda(t)|^2\ dt \right]^{\frac{1}{2}} dudv \\
& = \int_{\mathbb{R}} \int_{\mathbb{R}} \left[|\hat{Q}(u,v)|^2 \right]^{\frac{1}{2}} \left[\int_{\mathbb{R}} |G_\lambda(t)|^2\ dt \right]^{\frac{1}{2}} dudv \\
& = \int_{\mathbb{R}} \int_{\mathbb{R}} |\hat{Q}(u,v)|\ ||\ G_\lambda(t)\ ||_2\ dudv \\
& = \int_{\mathbb{R}} \int_{\mathbb{R}} |\hat{Q}(u,v)|\ ||\ \widehat{G}_\lambda(t)\ ||_2\ dudv \ , \qquad \text{(using Plancherel's Theorem)}
\end{aligned}$$

$$i.e.\ ||F_\lambda(t)||_2 \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |\hat{Q}(u,v)|\|\widehat{G}_\lambda(t)\|_2\, du\, dv. \qquad (**)$$

From (V), we know $\left|\widehat{G}_\lambda(x)\right| \leq C_\alpha |\lambda|^{-\frac{1}{\alpha}} |\hat{f}_v(x)|$ for some $C_\alpha > 0$.

We also know $|G_\lambda(t)| = \left| \displaystyle\int_{\mathbb{R}} \widehat{G}_\lambda(x) e^{2\pi i t x} dx \right|$.

Hence

$$
\begin{aligned}
|G_\lambda(t)| &\le \int_{\mathbb{R}} |\widehat{G}_\lambda(x) e^{2\pi i t x}| dx \\
&\le \int_{\mathbb{R}} |\widehat{G}_\lambda(x)|\,|e^{2\pi i t x}| dx \\
&\le \int_{\mathbb{R}} |\widehat{G}_\lambda(x)| dx && \text{(since } |e^{i\theta}| = 1 \text{ for all } \theta \in \mathbb{R}) \\
&\le \int_{\mathbb{R}} C_\alpha \lambda^{-\frac{1}{\alpha}} |\hat{f}_v(x)| dx \\
&\le C_\alpha \lambda^{-\frac{1}{\alpha}} \int_{\mathbb{R}} |f_v| dx && \text{(using Plancherel's Theorem)} \\
&\le C_\alpha \lambda^{-\frac{1}{\alpha}} \int_{\mathbb{R}} |e^{2\pi i v y} f| dx \\
&\le C_\alpha \lambda^{-\frac{1}{\alpha}} \int_{\mathbb{R}} |f| dx,
\end{aligned}
$$

i.e. $|G_\lambda(t)| \le C_\alpha \lambda^{-\frac{1}{\alpha}} \int_{\mathbb{R}} |f| dx.$

Squaring, we obtain $|G_\lambda(t)|^2 \le C_\alpha^2 \lambda^{-\frac{2}{\alpha}} \big(\int_{\mathbb{R}} |f| dx \big)^2.$

Integrating both sides with respect to t, we get $\int_{\mathbb{R}} |G_\lambda(t)|^2 \, dt \le C_\alpha^2 \lambda^{-\frac{2}{\alpha}} \int_{\mathbb{R}} \big(\int_{\mathbb{R}} |f| dx \big)^2 \, dt.$

Thus

$$\left[\int_{\mathbb{R}} |G_\lambda(t)|^2 \, dt \right]^{\frac{1}{2}} \le C_\alpha \lambda^{-\frac{1}{\alpha}} \left[\int_{\mathbb{R}} \big(\int_{\mathbb{R}} |f| dx \big)^2 \, dt \right]^{\frac{1}{2}}.$$

By Minkowski's Integral Inequality, we conclude

$$\|G_\lambda(t)\|_2 \le C_\alpha \lambda^{-\frac{1}{\alpha}} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f|^2 dt \right]^{\frac{1}{2}} dx,$$

i.e. $\|G_\lambda(t)\|_2 \le C_\alpha \lambda^{-\frac{1}{\alpha}} \|f\|_2.$ $(\diamond\diamond)$

Similarly, we know $\widehat{Q}(u,v)=\int_{\mathbb{R}}\int_{\mathbb{R}} Q(t,y)e^{-2\pi i(ut+vy)}dt\,dy$. Hence

$$
\begin{aligned}
|\widehat{Q}(u,v)| &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} |Q(t,y)||e^{-2\pi i(ut+yv)}|dt\,dy \\
&\qquad\qquad\qquad\qquad \text{(by the absolute value property of integrals)} \\
&\leq \int\limits_{[-M,M]}\int\limits_{[-M,M]} |Q(t,y)|dt\,dy \quad (\text{since } t,y\in supp(Q)\text{implies } t,y\in[-M,M]) \\
&\leq \int\limits_{[-M,M]}\int\limits_{[-M,M]} |h(t)||\psi(\eta^{-1}(t),y)|dt\,dy \\
&\leq \int\limits_{[-M,M]}\int\limits_{[-M,M]} |\psi(\eta^{-1}(t),y)|dt\,dy \qquad (\text{since } |h|\leq 1) \\
&\leq \int\limits_{[-M,M]}\int\limits_{[-M,M]} 1dt\,dy \qquad (\text{since } \psi\in C_0^\infty) \\
&\leq 4M^2,
\end{aligned}
$$

i.e. $|\widehat{Q}(u,v)|\leq 4M^2$. $\quad(\triangle)$

Therefore, from $(**)$, $(\diamond\diamond)$ and $(\triangle)$, we conclude $||F_\lambda(t)||_2\leq \int_{\mathbb{R}}\int_{\mathbb{R}} 4M^2C_\alpha\lambda^{-\frac{1}{\alpha}}\|f\|_2\,du\,dv$.

Then $||F_\lambda(t)||_2\leq \int\limits_{[-M,M]}\int\limits_{[-M,M]} 4M^2C_\alpha\lambda^{-\frac{1}{\alpha}}\|f\|_2\,du\,dv\leq 16M^4C_\alpha\lambda^{-\frac{1}{\alpha}}\|f\|_2$.

Let $D_\alpha=16M^4C_\alpha>0$. Then we conclude

$$||F_\lambda(t)||_2\leq D_\alpha|\lambda|^{-\frac{1}{\alpha}}\|f\|_2.$$

VII. Now we will reconsider $||T_\lambda(f)||_2$.

Recall that $T_\lambda f(x)=\int_{\mathbb{R}} e^{i\lambda[y-\eta(x)]^\alpha}\psi(x,y)f(y)dy$. Then

$$||T_\lambda(f)||_2=\left(\int_{\mathbb{R}}|T_\lambda f(x)|^2dx\right)^{1/2}.$$

Let $x = \eta^{-1}(t)$. Then $dx = \dfrac{1}{|\eta'(\eta^{-1}(t))|}dt$. So,

$$
\begin{aligned}
||T_\lambda f(x)||_2 &= \left(\int_J |T_\lambda f(\eta^{-1}(t))|^2 \frac{1}{|\eta'(\eta^{-1}(t))|}dt\right)^{1/2} && \text{(recall } J = \eta([\delta,\beta])\text{)} \\
&\le \left(\int_J |T_\lambda f(\eta^{-1}(t))|^2 \frac{1}{\nu}dt\right)^{1/2} && \text{(since } |\eta'(x)| \ge \nu\text{)} \\
&\le \left(\frac{1}{\nu}\right)^{1/2} \left(\int_J |T_\lambda f(\eta^{-1}(t))|^2 dt\right)^{1/2} \\
&\le \left(\frac{1}{\nu}\right)^{1/2} \left(\int_J \left|\frac{F_\lambda(t)}{h(t)}\right|^2 dt\right)^{1/2} && \text{(by the definition of } F_\lambda(t)\text{)} \\
&\le \left(\frac{1}{\nu}\right)^{1/2} \left(\int_{-2M}^{2M} |F_\lambda(t)|^2 dt\right)^{1/2} && \text{(because outside of } [-2M,2M] \ \frac{F_\lambda(t)}{h(t)} \\
& && \text{is undefined and } |h| = 1 \text{ on } J\text{)} \\
&= \left(\frac{1}{\nu}\right)^{1/2} ||F_\lambda(t)||_2 \\
&\le \left(\frac{1}{\nu}\right)^{1/2} D_\alpha |\lambda|^{-1/\alpha} ||f||_2 && \text{(by } VI\text{)} \\
&= \Omega_\alpha |\lambda|^{-1/\alpha} ||f||_2 && \text{where } \Omega_\alpha \text{ is the constant } \Omega_\alpha = \left(\frac{1}{\nu}\right)^{1/2} D_\alpha > 0.
\end{aligned}
$$

We have now proved that for some constant Ω_α,

$$||T_\lambda f(x)||_2 \le \Omega_\alpha |\lambda|^{-1/\alpha} ||f||_2$$

for all $f \in L^2(\mathbb{R})$, as desired.

Conclusion

This paper primarily presents and proves a generalization of Hörmander's theorem to oscillatory integral operators with a specific phase function. To that end, we present key notions and results in measure theory, and provide proofs for famous results such as Van Der Corput's lemma. As such, this thesis serves as a comprehensive introduction to the the study of the boundedness properties of oscillatory integral operators.

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