Math 302 Report: Strategic Games and the Nash Equilibrium

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May 2020

1 Introduction

I have chosen this topic for my final project for the course since it is related to both Real Analysis and Financial Mathematics, which are two topics my mathematics courses for this semester are covering. I am also interested in the field of Game Theory because of its intersection with philosophy, politics and computer science, in the sense that it models rational decision-making and the individual-group conflict. Finally, I believe that having watched the movie "A Beautiful Mind" was a significant factor in my decision.

2 Strategic Games

2.1 Definition

A strategic game is an interaction between individuals, also called "players" in this framework. Their actions do not only impact their payoff in the game, but also other players'. Hence, there exists an interdependence of payoffs that can be explored mathematically in a way that generates a diversity of sub-types of strategic games.
Formally, an m-person strategic game is defined as any set ζ such that:

$$\zeta \coloneqq \{(X_i,\pi_i)_{i=1,..,m}\} \text{ where}$$

X_i is the action space of player i, $i.e$ all the actions that player i can perform.
Consequently, an outcome of the game can be represented as any m-vector x=(x_1,...,x_m) s.t:

$$\mathrm{x} \in \prod_{i=1}^{m} X_i = X.$$

π_i is the payoff function of player i, and is defined as follows:

$$\begin{aligned} \pi_i : X &\to \mathbb{R} \\ x &\mapsto \text{payoff of player i if the outcome of the game is x} \in X \\ &\quad i.e \text{ if player 1 took action } x_1\text{,..., and player } m \text{ took action } x_m \end{aligned}$$

Remark: We write $\pi_i \in \mathbb{R}^X, \forall i \in [\![1, m]\!]$.

Exercise 1. Represent **the Prisoner's Dilemma** as a strategic game.

Let ζ be the game of the Prisoner's Dilemma.
Let I be the set of individuals, X_i the action space of player i and π_i their payoff function. We know that in the literature, the game consists of two individuals. Let's denote them p_1 and p_2.
Therefore, I={p_1,p_2}.
Since $\forall i \in [\![1, 2]\!]$, p_i can either remain silent or betray the other player, the action space for p_i is defined as: X_i={S,B} for S and B denoting the action of remaining silent and the action of betraying the other, respectively.

The payoff can be easily represented as the following matrix $\begin{pmatrix} 1/2 & S & B \\ S & \text{-1/-1} & \text{-5/0} \\ B & \text{0/-5} & \text{-2/-2} \end{pmatrix}$ where the 1st row corresponds to the actions taken by p_1, and the 1st column to those taken by p_2, with the entries of the entirely numerical sub-matrix representing p_1's payoff and p_2's payoff respectively, whose absolute values can be interpreted as the length of the jail sentence.
Hence, we can define the following game outcome vectors:
x =(S,B), y =(S,S), z =(B,S), w =(B,B) $\in$ X = $\prod_{i=1}^{2} X_i$

Then $\forall i \in [\![1, 2]\!]$, $\pi_i(\lambda) = \begin{cases} \text{-5} & if\ (\lambda = x \wedge i = 1) \vee (\lambda = z \wedge i = 2) \\ \text{-1} & if\ \lambda = y \\ 0 & if\ (\lambda = z \wedge i = 1) \vee (\lambda = x \wedge i = 2) \\ \text{-2} & if\ \lambda = w \end{cases}$

Therefore, $\zeta = \{(X_1, \pi_1), (X_2, \pi_2)\}$ for:
$X_1 = X_2 =${S,B};
and $\pi_1 = \pi_2 = \pi_i(\lambda)$,$\forall \lambda \in X$, $\forall i \in [\![1, 2]\!]$.

Note that $\pi_i((x_i = B, x_j)) \geq \pi_i((x_i = S, x_j)), \forall x_j \in X_{j \neq i}, \forall i \in [\![1, 2]\!]$.
Therefore, B is considered a **dominant strategy**.

Remark: A possible application of the Prisoner's dilemma in finance would be a situation where two companies sell the same product and have to decide the advertising budget. The dilemma resides in the fact that if both companies approximately invest the same effort in advertising at the same time, each company is going to retain its original clientele and thus would have lost money in advertising. However, if one invests in advertising and the other does not, the first will attract the second's clientele. Note that while the prisoner's dilemma as defined in this paper has a dominant strategy, this situation does not; since the best strategy a company should adopt depends on the other's.

2.2 Aggregative Games

2.2.1 Definition

In the definition of strategic games, we mentioned an interdependence of payoffs. One of the possible mathematical models to express this interdependence is through making the sum of the actions played by all players a parameter in each player's payoff function, hence the nomenclature **aggregative games**.
Formally, a strategic game $\zeta \coloneqq \{(X_i,\pi_i)_{i=1,..,m}\}$ is aggregative if it satisfies the following two conditions:

i) $X_i \neq \varnothing \wedge X_i \subseteq \mathbb{R}, \forall i \in [\![1, m]\!]$

ii) $\pi_i(x) = H_i(x_i, \sum_{j=1}^{m} x_j)$ for:
$H_i : X_i \times \{\sum_{j=1}^{m} x_j : x \in X\} \to \mathbb{R}$ an arbitrary helper function.

Exercise 2. Represent **The tragedy of the commons** as an aggregative strategic game.

The tragedy of the commons conceptually describes a situation where for instance, m individuals have access to a common-pool resource. Because the resource is finite and all m individuals have access to it, selfish behavior can lead to over-exploitation, and incur loss on all the individuals, hence the tragedy.
Formally, let ζ be the game representing the tragedy of the commons.
Since the m individuals have access to the resource, they are inclined to extract it for profit. Hence, the action space of each individual will be the quantification of their extraction effort, $i.e$ any positive real number.
Then, $\forall i \in [\![1, m]\!], X_i = \mathbb{R}^+$.
Since ζ is an aggregative strategic game, each individual's payoff is a function of their extraction effort, as well as the sum of all the individuals' efforts ($i.e$ the total extraction effort). Therefore, we need a mapping from extraction effort to payoff or profit.
Then, let's consider a production function f with the following characteristics:

i) f should be strictly $\nearrow$ since {total extraction effort $\nearrow \implies$ production $\nearrow$}.

ii) $f(0) = 0$ since {no extraction $\implies$ no production}

Hence, $\forall x \in \mathbb{R}^+, f(x)$ will denote the production resulting from x extraction.
We can then establish a pricing scheme by using the unit-price valuation method.
Therefore, let p be the price per unit of production $s.t\, p \in \mathbb{R}^{+*}$.
Then, the profit from x extraction can be valued at $p \times f(x)$.
However, in order for the game to be fair, each individual's profit should be proportional to their contribution to the total extraction effort, hence:

{profit of player i}$= \dfrac{x_i}{\displaystyle\sum_{j=1}^{m} x_j} \times p \times f(\sum_{j=1}^{m} x_j)$.

Again, since the resource is finite, there is a cost proportional to each extraction effort.
Then
{opportunity cost borne by player i}$= w \times x_i$ for $w \in \mathbb{R}^{+*}$
Therefore
{payoff of player i}={profit of player i}−{opportunity cost borne by player i}

$$\implies \pi_i(x) = \frac{x_i}{\sum_{j=1}^{m} x_j} \times p \times f(\textstyle\sum_{j=1}^m x_j) - w \times x_i, \forall x \in X = \prod_{i=1}^m X_i.$$

We then conclude that $\zeta = \{(\mathbb{R}^+, \pi_i)_{i=1,...,m}\}$ for

$$\pi_i(x) = H(x_i, \textstyle\sum_{j=1}^m x_j) \text{ with } H_i(a,b) = \begin{cases} 0 & if \text{ b=0} \\ \dfrac{a}{b}\, pf(b) - wa & if \text{ b >0} \end{cases} \text{ for } w, p \in \mathbb{R}^{+*}$$

3 The Nash Equilibrium

3.1 Definition

Let $\zeta = \{(X_i,\pi_i)_{i=1,..,m}\}$ be a strategic game.
The outcome x^* is a Nash equilibrium $\iff x_i^* \in arg\ max\{\pi_i(x_i, x_{-i}^*) : x_i \in X_i\}, \forall i \in [\![1, m]\!]$
where:

1. $x_i^* \in X = \prod_{i=1}^m X_i$
2. $x_{-i}^* = (\omega_1, ..., \omega_{m-1}) \in X_{-i} = \prod_{j=1}^{m-1} X_{j \neq i}$ is the collection of actions taken by all players except player i.
3. $arg\ max_{x \in S} f(x) \coloneqq \{x | x \in S \wedge \forall y \in S : f(y) \leq f(x)\}$

Remark: The set of Nash Equilibria is denoted $NE(\zeta)$.

Exercise 3. Find the Nash Equilibrium of the Prisoner's Dilemma.
Let ζ be the strategic game of the Prisoner's Dilemma, with the parameters described in Exercise 1.
Let's suppose that (S,B) $\in NE(\zeta)$.
Then, it follows from the definition that:

1. $x^* = (S, B)$
2. S $\in arg\ max\{\pi_1(x_1, B) : x_1 \in X_1\}$
3. B $\in arg\ max\{\pi_2(x_2, S) : x_2 \in X_2\}$

However, if p_2 betrays p_1, we have: $\pi_1((S, B)) = -5 \wedge \pi_1((B, B)) = -2$
Therefore, S $\notin arg\ max\{\pi_1(x_1, B)\}$ since $-5 < -2$.

Hence, condition 2 is not valid.
Then, (S,B) $\notin NE(\zeta)$.
Note that the same discussion can be made for p_2 to conclude that:
(B,S) $\notin NE(\zeta)$.
Let's suppose that (S,S) $\in NE(\zeta)$.
Then, it follows from the definition that:

1. $x^* = (S, B)$
2. S $\in arg\ max\{\pi_1(x_1, S) : x_1 \in X_1\}$
3. S $\in arg\ max\{\pi_2(x_2, S) : x_2 \in X_2\}$

However, if p_2 choses to remain silent, we have: $\pi_1((S, S)) = -1 \wedge \pi_1((B, S)) = 0$
Therefore, S $\notin arg\ max\{\pi_1(x_1, S)\}$ since -1 <0.
Hence, condition 2 is not valid.
Then, (S,S) $\notin NE(\zeta)$.
Finally, let's assume (B,B) $\in NE(\zeta)$.
Then, it follows from the definition that:

1. $x^* = (S, B)$
2. B $\in arg\ max\{\pi_1(x_1, B) : x_1 \in X_1\}$
3. B $\in arg\ max\{\pi_2(x_2, B) : x_2 \in X_2\}$

We have $\pi_1(S, B) = -5 \wedge \pi_1(B, B) = -2 \implies B \in arg\ max\{\pi_1(x_1, B)\}$.
Also, $\pi_2(B, S) = -5 \wedge \pi_2(B, B) = -2 \implies B \in arg\ max\{\pi_2(x_2, B)\}$.
Therefore, (B,B) satsifies the definition of a Nash Equilibrium of ζ, thus $NE(\zeta)$ =(B,B).

Remark: Note that because the Nash Equilibrium occurs when both players take the same action, the set of Nash Equilibria is said to be symmetrical and is denoted $NE_{sym}(\zeta)$.

3.2 Nash's Existence Theorem

3.2.1 Definitions:

Let $\zeta = \{(X_i,\pi_i)_{i=1,..,m}\}$ be a strategic game.

1. **Compactness:**
   $X_{i,i=1,...,m}$ is a nonempty compact subset of a Euclidean space $\implies$ ζ is a compact Euclidean game.
2. **Continuity:**
   $\pi_i \in C(X), \forall i \in [\![1, m]\!] \implies \zeta$ is a continuous Euclidean game.
3. **Convexity:**
   $$\begin{cases} X_{i,i=1,...,m} \text{ is convex} \\ \pi_i(., x_{-i}) \text{ is quasi-concave, } \forall x_{-i} \in X_{-i} \end{cases} \implies \zeta \text{ is a convex Euclidean game.}$$

4. **Regularity:**
$$\begin{cases} \zeta \text{ is compact} \\ \zeta \text{ is continuous} \\ \zeta \text{ is convex} \end{cases} \implies \zeta \text{ is a regular Euclidean game.}$$

5. **Kakutani's Fixed Point Theorem:**
Let $f : A \rightrightarrows A$ be a correspondence satisfying:
.A is a non-empty compact and convex subset of a finite dimensional Euclidean space.
.$f(x) \neq \varnothing, \forall x \in A$.
.$f(x)$ is convex-valued.
.$f(x)$ has a closed graph $\iff$ $(x_n, y_n) \to (x, y)$ with $y_n \in f(x_n)$ implies $y \in f(x)$.
Then, f has a fixed point $\iff \exists x \in A\, s.t\, x \in f(x)$

6. **Weierstrass' Theorem:**
$$\begin{cases} X \text{ is a compact metric space.} \\ \varphi \in \mathbb{R}^X \text{ is a continuous function.} \end{cases} \implies \exists\, x, y \in X\, with \begin{cases} \varphi(x) = sup_{\varphi}(X) \\ \varphi(y) = inf_{\varphi}(X) \end{cases}$$

3.2.2 Statement:

$\zeta := \{(X_i,\pi_i)_{i=1,..,m}\}$is a regular Euclidean game $\implies NE(\zeta) \neq \varnothing$.

3.2.3 Proof:

Let $\zeta = \{(X_i,\pi_i)_{i=1,..,m}\}$ be a strategic game.
In order to prove Nash's Existence Theorem, let's suppose that ζ is a regular Euclidean game and let's define the following correspondences:

$$\begin{array}{ll} b_i : X_{-i} \rightrightarrows X_i & b : X \rightrightarrows X \\ b_i(x_{-i}) := \ arg\ max\ \{\pi_i(x_i, x_{-i}) : x_i \in X_i\} & b(x) := b_1(x_{-1}) \times ... \times b_m(x_{-m}) \end{array}$$

Let $x^* \in X$
By definition,

$$\begin{aligned} x^*\text{is a Nash Equilibrium} &\iff x_i^* \in \text{arg max}\{\pi_i(x_i, x_{-i}^*) : x_i \in X_i\}, \forall i \in [\![1, m]\!] \\ &\iff x_i^* \in b_i(x_{-i}^*), \forall i \in [\![1, m]\!] \\ &\iff x^* \in b(x^*) \end{aligned}$$

Therefore, in order to prove that $NE(\zeta) \neq \varnothing$, it suffices to prove that $\exists x \in X s.t\, x \in b(x)$.
That is, it suffices to prove that the correspondence b has a fixed point.
To that end, we shall use Kakutani's fixed-point theorem described in *section 3.2.1*.
First of all, let's prove that b is well-defined.
We shall used Weierstrass' Theorem as defined in *section 3.2.1*.

We have :

$$\begin{aligned} b \text{ is well-defined} &\iff \forall x \in X, \exists y \in X \; s.t \; y \in b(x) \\ &\iff \forall x \in X, \exists y \in X \; s.t \; y \in \prod_{i=1}^{m} b_i(x_{-i}) \\ &\iff \forall x \in X, \exists y \in X \; s.t \; y \in \prod_{i=1}^{m} arg\, max\{\pi_i(x_i, x_{-i}) : x_i \in X_i\} \end{aligned}$$

Then, to prove the existence of y as defined in the equivalence, it suffices to apply Weierstrass' Theorem on $\pi_i \in \mathbb{R}^{X_i}$ by assuming π_i is constant with respect to the parameter $x_{-i} \in X_{-i}$.
We have:

$$\begin{aligned} \zeta \text{ is a regular Euclidean game} &\implies \zeta \text{ is a compact Euclidean game } \textit{(By 3.2.1.4)} \\ &\implies X_i \text{ is a compact subset of a Euclidean space }, \forall i \in [\![1, m]\!] (*) \\ &\textit{(By 3.2.1.1)} \end{aligned}$$

and:

$$\begin{aligned} \zeta \text{ is a regular Euclidean game} &\implies \zeta \text{ is a continuous Euclidean game.} \textit{(By 3.2.1.4)} \\ &\implies \pi_i \in C(X), \forall i \in [\![1, m]\!] \; \textit{(By 3.2.1.2)} \\ &\implies \pi_i \in \mathbb{R}^X \text{ is a continuous function,} \forall i \in [\![1, m]\!]. \\ &\implies \pi_i \in \mathbb{R}^{X_i} \text{ is a continuous function,} \forall i \in [\![1, m]\!]. \\ &\text{(Since } \pi_i \text{ is cst with respect to the 2nd parameter and } X = X_i \times X_{-i})(**) \end{aligned}$$

Then (*),(**) $\xrightarrow[Theorem]{Weierstrass'}$ π_i attains its maximum for a value in $X_i, \forall i \in [\![1, m]\!]$.
Thus $\forall x \in X, \exists y \in X \; s.t \; y \in \prod_{i=1}^{m} arg\, max\{\pi_i(x_i, x_{-i}) : x_i \in X_i\}$
That is, b is well-defined and thus $b(x) \neq \varnothing, \forall x \in X$. **(1)**
Let's show that X is a non-empty compact and convex subset of a Euclidean space.

1. Let's show that X is non-empty.
   We have

$$\begin{aligned} \zeta \text{ is a regular Euclidean game.} &\implies \zeta \text{ is a compact Euclidean game. } \textit{(By 3.2.1.4)} \\ &\implies X_i \text{ is non-empty }, \forall i \in [\![1, m]\!]. \; \textit{(By 3.2.1.1)} \\ &\implies \prod_{i=1}^{m} X_i \text{ is non-empty.} \\ &\implies X \text{ is non-empty.} \end{aligned}$$

   Thus, X is non-empty.

2. Let's show that X is compact.
   We have:

$$\begin{aligned}\zeta \text{ is a regular Euclidean game } &\implies \zeta \text{ is a compact Euclidean game } \textit{(By 3.2.1.4)}\\ &\implies X_i \text{ is a compact subset of a Euclidean space }, \forall i \in [\![1,m]\!]\\ &\textit{(By 3.2.1.1)}\end{aligned}$$

   By the Set Theory definition of compactness, we know that:
   $\forall i \in [\![1,m]\!], \forall$ sequence a_n in $X_i, \exists$ a subsequence a_{n_k} of a_n that converges to $a \in X_i$.
   Then, we shall show that $\forall$ sequence A_n in $X, \exists$ a subsequence A_{n_p} of A_n that converges to $A \in X$ for $p \in \mathbb{N}$.
   Let $A_n = (a_{1_n}, ..., a_{m_n})$ be a sequence of points in $\prod_{i=1}^m X_i$.
   Since X_1 is compact, $\exists\, n_{1_1}, n_{1_2}, n_{1_3}, ... \in \mathbb{N}\, s.t\, a_{1_{n_{1_k}}}$ converges to a point $a_1 \in X_1$.
   Since X_2 is compact, $\exists\, n_{2_1}, n_{2_2}, n_{2_3}, ... \in \{n_{1_1}, n_{1_2}, n_{1_3}, ...\}\, s.t\, a_{2_{n_{2_k}}}$ converges to a point $a_2 \in X_2$.
   Similarly, since X_m is compact, $\exists\, n_{m_1}, n_{m_2}, ... \in \{n_{(m-1)_1}, n_{(m-1)_2}, ...\}\, s.t\, a_{m_{n_{m_k}}}$ converges to a point $a_m \in X_m$.
   Therefore, $(a_{1_{n_{m_k}}}, ..., a_{m_{n_{m_k}}})$ converges to $(a_1, ..., a_m) \in \prod_{i=1}^m X_i$ for $k \in \mathbb{N}$.
   Hence, A_n has a subsequence $A_{n_{m_k}}$ that converges to $A = (a_1, ..., a_m) \in \prod_{i=1}^m X_i$ for $k \in \mathbb{N}$.
   Therefore, $X = \prod_{i=1}^m X_i$ is compact.

3. Let's show that X is convex. We have:

$$\begin{aligned}\zeta \text{ is a regular Euclidean game } &\implies \zeta \text{ is a convex Euclidean game } \textit{(By 3.2.1.4)}\\ &\implies X_i \text{ is a convex subset of a Euclidean space }, \forall i \in [\![1,m]\!]\\ &\textit{(By 3.2.1.3)}\end{aligned}$$

   Since X_i is convex $\forall i \in [\![1,m]\!]$, then $\forall i \in [\![1,m]\!]$, we have:
   $\forall x, y \in X_i, \forall \lambda \in [0,1], \lambda x + (1-\lambda)y \in X_i$
   In order to prove X is convex, we shall show the following:
   $\forall A, B \in X, \forall \lambda \in [0,1], \lambda A + (1-\lambda)B \in X$
   Let $A, B \in X \wedge \lambda \in [0,1]$
   Let's show that $\lambda A + (1-\lambda)B \in X$
   We have
   $A \in X \implies \exists\{a_1 \in X_1, ..., a_m \in X_m\}\, s.t\, A = (a_1, ..., a_m)$ and
   $B \in X \implies \exists\{b_1 \in X_1, ..., b_m \in X_m\}\, s.t\, B = (b_1, ..., b_m)$
   Then

$$\begin{aligned}\lambda A + (1-\lambda)B &= \lambda(a_1, ..., a_m) + (1-\lambda)(b_1, ..., b_m)\\ &= (\lambda a_1 + (1-\lambda)b_1, ..., \lambda a_m + (1-\lambda)b_m)\end{aligned}$$

Also

$$
\begin{aligned}
&\forall x, y \in X_i, \forall \lambda \in [0,1], \lambda x + (1-\lambda) y \in X_i \\
&\implies \lambda a_1 + (1-\lambda) b_1 \in X_1, ..., \lambda a_m + (1-\lambda) b_m \in X_m \\
&\implies (\lambda a_1 + (1-\lambda) b_1, ..., \lambda a_m + (1-\lambda) b_m) \in X_1 \times ... \times X_m \\
&\implies \lambda A + (1-\lambda) B \in X_1 \times ... \times X_m \\
&\implies \lambda A + (1-\lambda) B \in \prod_{i=1}^{m} X_i \\
&\implies \lambda A + (1-\lambda) B \in X
\end{aligned}
$$

Thus, $\forall A, B \in X, \forall \lambda \in [0,1], \lambda A + (1-\lambda) B \in X$
Then, X is convex.

Therefore, from 1, 2 and 3, we conclude that X is a non-empty compact and convex subset of Euclidean space. **(2)**
Now, let's prove that b is a convex-valued correspondence.
That is, let's prove that $\forall x \in X, b(x)$ is a convex set
$(i.e)\, \forall x \in X, \prod_{i=1}^{m} b_i(x_{-i})$ is convex.
We proved previously that the cartesian product of convex sets is convex, hence it suffices to prove that $b_i(x_{-i})$ is convex $\forall i \in [\![1, m]\!]$.
For that, we shall show the following $\forall i \in [\![1, m]\!]$:
$\forall x, y \in b_i(x_{-i}), \forall \lambda \in [0,1], \lambda x + (1-\lambda) y \in b_i(x_{-i})$.
Let $x_{-i} \in X_{-i}$
Let $i \in [\![1, m]\!], x, y \in b_i(x_{-i}) \wedge \lambda \in [0,1]$
let's prove that $\lambda x + (1-\lambda) y \in b_i(x_{-i})$:
We have

$$
\begin{aligned}
\begin{cases} \text{x} \in b_i(x_{-i}) \\ \text{y} \in b_i(x_{-i}) \end{cases}
&\implies \begin{cases} \text{x} \in arg\,max\{\pi_i(x_i, x_{-i}) : x_i \in X_i\} \\ \text{y} \in arg\,max\{\pi_i(x_i, x_{-i}) : x_i \in X_i\} \end{cases} \\
&\implies \begin{cases} \lambda x \in arg\,max\{\lambda \pi_i(x_i, x_{-i}) : x_i \in X_i\} \\ \text{(1-}\lambda\text{)} y \in arg\,max\{(1-\lambda)\pi_i(x_i, x_{-i}) : x_i \in X_i\} \end{cases} \\
&\implies \lambda x + (1-\lambda) y \in arg\,max\{\lambda \pi_i(x_i, x_{-i}) + (1-\lambda)\pi_i(x_i, x_{-i}) : x_i \in X_i\} \\
&\implies \lambda x + (1-\lambda) y \in arg\,max\{\pi_i(x_i, x_{-i}) : x_i \in X_i\} \\
&\implies \lambda x + (1-\lambda) y \in b_i(x_{-i})
\end{aligned}
$$

Therefore, $b_i(x_{-i})$ is convex $\forall i \in [\![1, m]\!]$.
Thus, $b(x) = \prod_{i=1}^{m} b_i(x_{-i})$ is convex, $\forall x \in X$.
Consequently, b is a convex-valued correspondence. **(3)**
Finally, let's prove that $b(x)$ has a closed graph.
$(i.e)\, \forall$ sequences $(x^k), (y^k) \in X$,

$$
(\lim_{k \to \infty} x^k = \tilde{x}) \wedge (\lim_{k \to \infty} y^k = \tilde{y}) \wedge (y^k \in b(x^k)) \implies \tilde{y} \in b(\tilde{x}), \forall k \in \mathbb{N}
$$

Let $(x^k), (y^k)$ be two sequences in X $s.t$ $\begin{cases} \lim\limits_{k\to\infty} x^k = \tilde{x} \\ \lim\limits_{k\to\infty} y^k = \tilde{y} \\ \text{y}^k \in b(x^k) \end{cases}$ for $k \in \mathbb{N}$

Let's prove that $\tilde{y} \in b(\tilde{x})$.
We have the following:

$$
\begin{aligned}
y^k \in b(x^k) &\implies \forall i \in [\![1, m]\!], y_i^k \in b_i(x_{-i}^k) \\
&\implies \forall i \in [\![1, m]\!], \pi_i(y_i^k, x_{-i}^k) \geq \pi_i(\lambda_i, x_{-i}^k), \forall \lambda_i \in X_i \\
&\implies \forall i \in [\![1, m]\!], \lim_{k\to\infty} \pi_i(y_i^k, x_{-i}^k) \geq \lim_{k\to\infty} \pi_i(\lambda_i, x_{-i}^k), \forall \lambda_i \in X_i \\
&\xRightarrow{\pi_i \in C(X)} \forall i \in [\![1, m]\!], \pi_i(\lim_{k\to\infty} y_i^k, \lim_{k\to\infty} x_{-i}^k) \geq \pi_i(\lim_{k\to\infty} \lambda_i, \lim_{k\to\infty} x_{-i}^k), \forall \lambda_i \in X_i \\
&\implies \forall i \in [\![1, m]\!], \pi_i(\tilde{y}_i, \tilde{x}_{-i}) \geq \pi_i(\lambda_i, \tilde{x}_{-i}), \forall \lambda_i \in X_i \\
&\implies \forall i \in [\![1, m]\!], \tilde{y}_i \in b_i(\tilde{x}_i) \\
&\implies \tilde{y} \in b(\tilde{x})
\end{aligned}
$$

Therefore, $\forall$ sequences $(x^k), (y^k) \in X$,

$$(\lim_{k\to\infty} x^k = \tilde{x}) \wedge (\lim_{k\to\infty} y^k = \tilde{y}) \wedge (y^k \in b(x^k)) \implies \tilde{y} \in b(\tilde{x}), \forall k \in \mathbb{N}$$

Hence, b has a closed graph. **(4)**
Then, **(1),(2),(3) and (4)** $\xRightarrow[Theorem]{Kakutani's}$ b has a fixed point.
$(i.e)\, \exists x^* \in X\, s.t\, x^* \in b(x^*) \implies x^* \in NE(\zeta) \implies NE(\zeta) \neq \varnothing$
Then, ζ has a Nash equilibrium.
We conclude that: $\zeta \coloneqq \{(X_i, \pi_i)_{i=1,..,m}\}$ is a regular Euclidean game $\implies NE(\zeta) \neq \varnothing$.

References

[1] Ok, Efe *A. Real Analysis with Economic Applications.* World Publishing Corporation, 2013.

[2] `https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-254-game-theory-with-engineering-applications-spring-2010/lecture-notes/`