# Applied Stochastic Processes (FIN 514) Midterm Exam

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**BM** stands for Brownian motion. Assume that $B_t$, $W_t$, and $Z_t$ are standard BMs if unless stated otherwise. **RN** and **RV** stand for random number and random variable, respectively. The PDF and CDF of the standard normal distribution are denoted by $n(z)$ and $N(z)$ respectively. You can use $n(z)$ and $N(z)$ in your answers without further evaluation.

1. (10 points) **[RN generation]** A gamma RV, $X \sim \text{Gamma}(k, \beta)$, is distrusted by the PDF,

$$f_X(x) = \frac{\beta^k}{\Gamma(k)} x^{k-1} e^{-\beta x} \quad \text{for} \quad \Gamma(k) = (k-1)\cdots 2\cdot 1 \; (\Gamma(1) = 1),$$

where $k$ is a positive integer and $X \ge 0$.

(a) (2 points) Find the mean and variance of $X$. **Hint:** $\int_0^\infty f_X(x)dx = 1$ for any $k$.

(b) (2 points) How can you generate the RV of $X \sim \text{Gamma}(1, \beta)$?

(c) (3 points) If $X \sim \text{Gamma}(1, \beta)$, $X' \sim \text{Gamma}(k, \beta)$, and $X$ and $X'$ are independent, find the PDF of $Y = X + X'$.

(d) (3 points) How can we generate the RV of $\text{Gamma}(k, \beta)$?

**Solution:**

(a)

$$E(X) = \int_0^\infty x \frac{\beta^k}{\Gamma(k)} x^{k-1} e^{-\beta x} dx = \frac{k}{\beta} \int_0^\infty \frac{\beta^{k+1}}{\Gamma(k+1)} x^k e^{-\beta x} dx = \frac{k}{\beta}$$

$$E(X^2) = \int_0^\infty x^2 \frac{\beta^k}{\Gamma(k)} x^{k-1} e^{-\beta x} dx = \frac{k(k+1)}{\beta^2} \int_0^\infty \frac{\beta^{k+2}}{\Gamma(k+2)} x^{k+1} e^{-\beta x} dx = \frac{k(k+1)}{\beta^2}$$

$$\text{Var}(X) = E(X^2) - E(X)^2 = \frac{k}{\beta^2}$$

(b) When $k = 1$, $X$ has the same PDF as the exponential distribution with $\lambda = \beta$:

$$f_X(x) = \beta e^{-\beta x}.$$

Therefore, we can generate $X$ by

$$X = -\frac{1}{\beta}\log U \quad \text{or} \quad -\frac{1}{\beta}\log(1-U),$$

where $U$ is a uniform RV.

(c) **Method 1:**

$$f_Y(y) = \int_{x=0}^{y} f_X(y-x) f_{X'}(x)dx = \int_{x=0}^{y} \beta e^{-\beta(y-x)} \frac{\beta^k}{\Gamma(k)} x^{k-1} e^{-\beta x} dx$$
$$= \frac{\beta^{k+1}}{\Gamma(k)} e^{-\beta y} \int_{x=0}^{y} x^{k-1} dx = \frac{\beta^{k+1}}{\Gamma(k)} e^{-\beta y} \frac{y^k}{k} = \frac{\beta^{k+1}}{\Gamma(k+1)} y^k e^{-\beta y}.$$

Therefore, $Y$ follows Gamma$(k+1, \beta)$.

**Method 2:** The MGF of $X'$ is

$$E\left(e^{-tX'}\right) = \int_0^\infty \frac{\beta^k}{\Gamma(k)} x^{k-1} e^{-(\beta+t)x} = \frac{\beta^k}{(\beta+t)^k} = (1+t/\beta)^{-k},$$

where we used the hint of (a) for $\beta' = \beta + t$. It follows that the MGF of $X$ is $(1+t/\beta)^{-1}$. Since $X$ and $X'$ are independent,

$$E\left(e^{-tY}\right) = E\left(e^{-tX}\right) E\left(e^{-tX'}\right) = (1+t/\beta)^{-1}(1+t/\beta)^{-k} = (1+t/\beta)^{-(k+1)}.$$

Therefore, we know that $Y \sim$ Gamma$(k+1, \beta)$.

**Method 3:** $X' \sim$ Gamma$(k, \beta)$ is the RV for the $k$-th arrival time of the Poisson-type events with intensity $\beta$. Because the events are memoryless, $X' + X$ is the $(k+1)$-th arrival time and it is Gamma$(k+1, \beta)$.

(d) From (b), Gamma$(k, \beta) \sim X_1 + \cdots X_k$, where $X_i$'s are independent Gamma variables following Gamma$(1, \beta)$. Therefore,

$$X = -\frac{1}{\beta} \log(U_1 \cdots U_k),$$

where $U_k$ are the sequence of uniform RVs.

2. (5 points) **[Simulation of correlated normal RVs]** The tri-variate normal variable $\boldsymbol{X}$ has the following mean and covariance. How can you simulate RNs for $X$?

$$\mu = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 16 & 4 \\ -2 & 4 & 9 \end{pmatrix}$$

**Solution:** First, we obtain the Cholesky decomposition of $\Sigma$. We find a lower triangular matrix $L$ such that $LL^T = \Sigma$. After some algebra, we get

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}.$$

Therefore, $X$ is simulated by $X = \mu + LZ$ where $Z$ is the independent standard normal random variable of size 3.

3. (5 points) [**Euler and Milstein Scheme**] The stochastic differential equation for the constant-elasticity-of-variance (CEV) model is given by

$$dS_t = \sigma S_t^\beta dW_t \quad (0 \le \beta \le 1).$$

Find the Euler and Milstein schemes for obtaining $S_{t+\Delta t}$ from $S_t$.

**Solution:** For a standard normal RV, $W_1$, the Milstein scheme for the CEV model is given by

$$\begin{aligned} S_{t+\Delta t} &= S_t + \sigma S_t^\beta W_1 \sqrt{\Delta t} + \frac{\sigma S_t^\beta \cdot \sigma \beta S_t^{\beta-1}}{2}(W_1^2 - 1)\Delta t \\ &= S_t + \sigma S_t^\beta W_1 \sqrt{\Delta t} + \frac{1}{2}\sigma^2 \beta S_t^{2\beta-1}(W_1^2 - 1)\Delta t. \end{aligned}$$

4. (10 points) [**Conditional Monte Carlo Simulation**] We are going to formulate the conditional Monte Carlo simulation for the GARCH diffusion model. The SDEs for the price and volatility under the GARCH diffusion model are given by

$$\begin{aligned} \frac{dS_t}{S_t} &= \sqrt{v_t}(\rho dZ_t + \rho_* dX_t) \quad \text{for} \quad \rho_* = \sqrt{1-\rho^2}, \\ dv_t &= \kappa(\theta - v_t)dt + \nu v_t \, dZ_t \end{aligned}$$

where $X_t$ and $Z_t$ are independent standard BMs.

(a) (3 points) Derive the SDE for $\sigma_t = \sqrt{v_t}$.

(b) (3 points) Based on the answer of (a), express $S_T$ in terms of $\sigma_T, Y_T, U_T, V_T$, and a standard normal RV $X_1$, where $Y_T$, $U_T$ and $V_T$ are given by

$$Y_T = \int_0^T \frac{1}{\sigma_t}\, dt, \quad U_T = \int_0^T \sigma_t \, dt \quad \text{and} \quad V_T = \int_0^T \sigma_t^2 \, dt.$$

(c) (4 points) What are $E(S_T)$ and the BS volatility of $S_T$ conditional on the quadruplet $(\sigma_T, Y_T, U_T, V_T)$?

**Solution:**

(a) Using Itô's lemma,

$$\begin{aligned} d\sigma_t = d\sqrt{v_t} &= \frac{1}{2}\frac{dv_t}{\sqrt{v_t}} - \frac{1}{8}\frac{(dv_t)^2}{v_t\sqrt{v_t}} \\ &= \frac{1}{2}\kappa\left(\frac{\theta}{\sigma_t} - \sigma_t\right)dt + \frac{\nu}{2}\sigma_t dZ_t - \frac{\nu^2}{8}\sigma_t dt \\ &= \frac{1}{2}\left(\frac{\kappa\theta}{\sigma_t} - \left(\kappa + \frac{\nu^2}{4}\right)\sigma_t\right)dt + \frac{\nu}{2}\sigma_t dZ_t \end{aligned}$$

(b) Integrating the result of (a),

$$\sigma_T - \sigma_0 = \frac{1}{2}\left(\kappa\theta\, Y_T - \left(\kappa + \frac{\nu^2}{4}\right) U_T\right) + \frac{\nu}{2}\int_0^T \sigma_t dZ_t$$

$$\int_0^T \sigma_t dZ_t = \frac{2}{\nu}(\sigma_T - \sigma_0) - \left(\frac{\kappa\theta}{\nu}\, Y_T - \left(\frac{\kappa}{\nu} + \frac{\nu}{4}\right) U_T\right)$$

Therefore,

$$\begin{aligned}\log\left(\frac{S_T}{S_0}\right) &= \rho\int_0^T \sigma_t dZ_t + \rho_* \int_0^T \sigma_t dX_t - \frac{1}{2}V_T \\ &= \frac{2\rho}{\nu}(\sigma_T - \sigma_0) - \frac{\rho\kappa\theta}{\nu}\, Y_T + \rho\left(\frac{\kappa}{\nu} + \frac{\nu}{4}\right) U_T - \frac{1}{2}V_T + \rho_* \sqrt{V_T}\, X_1.\end{aligned}$$

(c) Accordingly, we obtain

$$\begin{aligned}E(S_T|\sigma_T, Y_T, U_T, V_T) &= S_0 \exp\left(E\left(\log\left(\frac{S_T}{S_0}\right)\right) + \frac{\rho_*^2}{2} V_T\right) \\ &= S_0 \exp\left(\frac{2\rho}{\nu}(\sigma_T - \sigma_0) - \frac{\rho\kappa\theta}{\nu}\, Y_T + \rho\left(\frac{\kappa}{\nu} + \frac{\nu}{4}\right) U_T - \frac{\rho^2}{2} V_T\right) \\ \sigma_{\text{BS}} &= \rho_* \sqrt{V_T/T}.\end{aligned}$$